TUTORIAL



# A Beginner's Guide to 6-D Vectors (Part 1)

## *What They Are, How They Work, and How to Use Them*

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A rigid body has six degrees of motion freedom, so why not use six-dimensional (6-D) vectors to describe its motions and the forces acting upon it? In fact, some roboticists already do this, and the practice is becoming more common. The purpose of this tutorial is to present a beginner's guide to 6-D vectors in sufficient detail that a reader can begin to use them as a practical problem-solving tool right away. This tutorial covers the basics, and Part 2 will cover the application of 6-D vectors to a variety of robot kinematics and dynamics calculations.

6-D vectors come in various forms. The particular kind presented here is called *spatial vectors*. They are the tool that the author has been using for nearly 30 years to invent dynamics algorithms and write dynamics calculation software. Other kinds of 6-D vector include screws, motors, and Lie algebras. More will be said about them at the end of this tutorial. The differences between the various kinds of 6-D vector are relatively small. The more you understand any one of them, the easier it gets to understand the others.

The obvious advantage of 6-D vectors is that they cut the volume of algebra. Instead of having to define two three-dimensional (3-D) vectors to describe a force, another two to describe an acceleration, and writing two equations of motion for each body, a 6-D vector notation lets you pair up corresponding 3-D vectors and equations. The immediate result is a tidier, more compact notation involving fewer quantities and fewer equations. However, anyone who thinks that 6-D vectors are only a convenient notation for organizing 3-D vectors is missing half the point. 6-D vectors are tools for thought. They have their own physical meanings and mathematical properties. They let you solve a problem more directly, and at a higher level of abstraction, by letting you think in 6-D, which is easier than it sounds.

Using spatial vectors (and other kinds of 6-D vector) lets you formulate a problem more succinctly, solve it more quickly and in fewer steps, present the solution more clearly to others, implement it in fewer lines of code, and debug the software more easily. Furthermore, there is no loss of efficiency: spatial-vector software can be just as efficient as 3-D-vector software, despite the higher level of abstraction.

The rest of this tutorial is chiefly concerned with explaining what spatial vectors are and how to use them. It highlights the differences between solving a rigid-body problem using 3-D vectors and solving the same problem using spatial vectors, so that the reader can get an idea of what it means to think in 6-D.

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### A Note on Notation

When using spatial vectors, it is convenient to employ symbols like $\boldsymbol{f}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ (or $\dot{\boldsymbol{v}}$) to denote quantities like force, velocity, and acceleration. However, these same symbols are equally useful for 3-D vectors. Thus, whenever spatial and 3-D vectors are discussed together, there is a possibility of name clashes. To resolve these clashes, we shall use the following rule: in any context where a spatial symbol needs to be distinguished from a 3-D symbol, the spatial symbol is given a hat (e.g., $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{v}}$). These hats are dropped when they are no longer needed. An

## Solving a Two-Body Dynamics Problem Using 3-D Vectors

We are given a rigid-body system consisting of two bodies, $B_1$ and $B_2$, connected by a revolute joint [S1] The bodies have masses of $m_1$ and $m_2$, centers of mass located at the points $C_1$ and $C_2$, and rotational inertias of $I_1$ and $I_2$ about their respective centers of mass. Both bodies are initially at rest. The joint's axis of rotation passes through the point $P$ in the direction given by $s$. A system of forces acts on $B_1$, causing both bodies to accelerate. This system is equivalent to a single force $f$ acting on a line passing through $C_1$ together with couple $n$. These forces impart an angular acceleration of $\dot{\omega}_1$ to $B_1$ and a linear acceleration of $a_1$ to its center of mass. The problem is to express $a_1$ and $\dot{\omega}_1$ in terms of $f$ and $n$ (Figure S1).

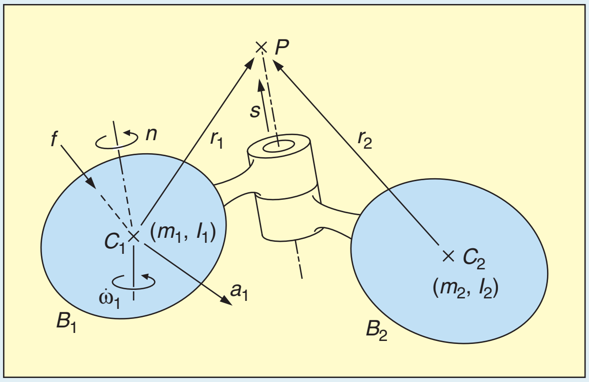


*Figure S1. Problem diagram using 3-D vectors.*

### Solution

Let $f_1$, $n_1$, $f_2$, and $n_2$ be the net forces and couples acting on $B_1$ and $B_2$, respectively, where the lines of action of $f_1$ and $f_2$ pass through $C_1$ and $C_2$, respectively; let $\dot{\omega}_2$ and $a_2$ be the angular acceleration of $B_2$ and the linear acceleration of its center of mass; let ${}^Pa_1$ , ${}^P\dot{\omega}_1$ , ${}^Pa_2$, and ${}^P\dot{\omega}_2$ be the linear and angular accelerations of $B_1$ and $B_2$ expressed at $P$; and let ${}^Pf_2$, ${}^Pn_2$, ${}^1f_2$, and ${}^1n_2$ be the net force and couple acting on $B_2$ expressed at $P$ and $C_1$, respectively. As the system of applied forces acts only on $B_1$, the net force and couple acting on $B_2$ are also the net force and couple transmitted through the joint. Let us also define $r_1 = \overrightarrow{C_1P}$ and $r_2 = \overrightarrow{C_2P}$ , and let $\alpha$ be the joint acceleration variable.

The equations of motion of the two bodies, expressed at their centers of mass, are

$$f_1 = m_1 a_1, \tag{S1}$$
$$n_1 = I_1 \dot{\omega}_1, \tag{S2}$$
$$f_2 = m_2 a_2, \tag{S3}$$

and

$$n_2 = I_2 \dot{\omega}_2. \tag{S4}$$

There are no velocity terms because the bodies are at rest. The rules for transferring forces and accelerations (of bodies at rest) from one point to another provide us with the following relationships between quantities referred to $C_1$, $C_2$, and $P$:

$${}^Pa_1 = a_1 - r_1 \times \dot{\omega}_1, \tag{S5}$$
$${}^Pa_2 = a_2 - r_2 \times \dot{\omega}_2, \tag{S6}$$
$${}^P\dot{\omega}_1 = \dot{\omega}_1, \tag{S7}$$
$${}^P\dot{\omega}_2 = \dot{\omega}_2, \tag{S8}$$
$${}^1f_2 = {}^Pf_2 = f_2, \tag{S9}$$
$${}^Pn_2 = n_2 - r_2 \times f_2, \tag{S10}$$

and

$${}^1n_2 = n_2 + (r_1 - r_2) \times f_2. \tag{S11}$$

If $B_1$ exerts ${}^1f_2$ and ${}^1n_2$ on $B_2$ then $B_2$ exerts $-{}^1f_2$ and $-{}^1n_2$ on $B_1$ (Newton's third law expressed at $C_1$); so, the net force and couple acting on $B_1$ are

$$f_1 = f - {}^1f_2,$$
$$n_1 = n - {}^1n_2,$$

from which we get [via (S9) and (S11)]

$$f = f_1 + f_2, \tag{S12}$$

and

$$n = n_1 + n_2 + (r_1 - r_2) \times f_2. \tag{S13}$$

The joint allows $B_2$ one degree of motion freedom relative to $B_1$ and imposes one constraint on the couple transmitted from $B_1$ to $B_2$. Expressed at $P$, the constraint equations are

$${}^Pa_2 = {}^Pa_1, \tag{S14}$$
$${}^P\dot{\omega}_2 = {}^P\dot{\omega}_1 + s\,\alpha, \tag{S15}$$

and

$$s^{\mathrm{T}}\,{}^Pn_2 = 0, \tag{S16}$$

where $\alpha$ is the unknown joint acceleration variable. There is no constraint on ${}^Pf_2$: (S16) is sufficient to ensure that the force and couple transmitted by the joint perform no work in the direction of relative motion permitted by the joint.

We are now ready to solve the problem. Let us start by calculating $a_2$ and $\dot{\omega}_2$ in terms of $a_1$, $\dot{\omega}_1$ and $\alpha$. From (S8), (S15), and (S7), we have

$$\begin{aligned}\dot{\omega}_2 &= {}^P\dot{\omega}_2 \\ &= {}^P\dot{\omega}_1 + s\,\alpha \\ &= \dot{\omega}_1 + s\,\alpha,\end{aligned} \tag{S17}$$

and from (S6), (S14), (S17), and (S5) we have

$$\begin{aligned}a_2 &= {}^Pa_2 + r_2 \times \dot{\omega}_2 \\ &= {}^Pa_1 + r_2 \times (\dot{\omega}_1 + s\,\alpha) \\ &= a_1 - r_1 \times \dot{\omega}_1 + r_2 \times (\dot{\omega}_1 + s\,\alpha) \\ &= a_1 + (r_2 - r_1) \times \dot{\omega}_1 + r_2 \times s\,\alpha.\end{aligned} \tag{S18}$$

Now let us calculate $\alpha$. From (S16), (S10), (S3), (S4), (S17), and (S18), we get

$$
\begin{aligned}
0 &= s^{\mathrm{T}}\,{}^{P}n_2 \\
&= s^{\mathrm{T}}(n_2 - r_2 \times f_2) \\
&= s^{\mathrm{T}}(I_2\,\dot{\omega}_2 - m_2\,r_2 \times a_2) \\
&= s^{\mathrm{T}}(I_2\,(\dot{\omega}_1 + s\,\alpha) - m_2\,r_2 \times \\
&\quad (a_1 + (r_2 - r_1) \times \dot{\omega}_1 + r_2 \times s\,\alpha))\,.
\end{aligned}
$$

Collecting terms in $\alpha$ gives

$$
\begin{aligned}
& s^{\mathrm{T}}(I_2\,s - m_2\,r_2 \times (r_2 \times s))\,\alpha \\
&\quad + s^{\mathrm{T}}(I_2\,\dot{\omega}_1 - m_2\,r_2 \times (a_1 + (r_2 - r_1) \times \dot{\omega}_1)) = 0\,,
\end{aligned}
$$

hence

$$
\alpha = -\frac{s^{\mathrm{T}}(I_2\,\dot{\omega}_1 - m_2\,r_2 \times (a_1 + (r_2 - r_1) \times \dot{\omega}_1))}{s^{\mathrm{T}}(I_2\,s - m_2\,r_2 \times (r_2 \times s))}\,. \tag{S19}
$$

This equation is only valid if the denominator is not equal to zero, so we must investigate the necessary conditions for it to be nonzero. This problem can be solved using the following trick. For any two vectors $u$ and $v$, the cross product $u \times v$ can be expressed in the form $u \times v = \tilde{u}\,v$, where $\tilde{u}$ is the skew-symmetric matrix:

$$
\tilde{u} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}.
$$

Using this trick, we can express the denominator in the form $s^{\mathrm{T}}J\,s$, where

$$
\begin{aligned}
J &= I_2 - m_2\,\tilde{r}_2\,\tilde{r}_2 \\
&= I_2 + m_2\,\tilde{r}_2^{\mathrm{T}}\,\tilde{r}_2\,.
\end{aligned} \tag{S20}
$$

$J$ is therefore the sum of an SPD matrix and an SPSD matrix, hence itself also SPD, so the denominator of (S19) is guaranteed to be strictly greater than zero. Substituting (S20) into (S19) gives us the following simplified expression for $\alpha$:

$$
\alpha = -\frac{s^{\mathrm{T}}(J\,\dot{\omega}_1 - m_2\,r_2 \times (a_1 - r_1 \times \dot{\omega}_1))}{s^{\mathrm{T}}J\,s}\,. \tag{S21}
$$

The next step is to express $f$ and $n$ in terms of $a_1$, $\dot{\omega}_1$, and $\alpha$, and then to eliminate $\alpha$ using (S21). Let us start with $f$. From (S12), (S1), (S3), and (S18), we get

$$
\begin{aligned}
f &= f_1 + f_2 \\
&= m_1\,a_1 + m_2\,a_2 \\
&= m_1\,a_1 + m_2(a_1 + (r_2 - r_1) \times \dot{\omega}_1 + r_2 \times s\,\alpha) \\
&= (m_1 + m_2)a_1 + m_2(r_2 - r_1) \times \dot{\omega}_1 + m_2\,r_2 \times s\,\alpha\,.
\end{aligned}
$$

Eliminating $\alpha$ using (S21) gives

$$
\begin{aligned}
f &= (m_1 + m_2)a_1 + m_2(r_2 - r_1) \times \dot{\omega}_1 \\
&\quad - m_2\,\frac{r_2 \times s\,s^{\mathrm{T}}(J\,\dot{\omega}_1 - m_2\,r_2 \times (a_1 - r_1 \times \dot{\omega}_1))}{s^{\mathrm{T}}J\,s}\,;
\end{aligned}
$$

and collecting terms in $a_1$ and $\dot{\omega}_1$ gives

$$
\begin{aligned}
f &= \left(m_1 + m_2 + m_2^2\,\frac{\tilde{r}_2\,s\,s^{\mathrm{T}}\,\tilde{r}_2}{s^{\mathrm{T}}J\,s}\right)a_1 \\
&\quad + \left(m_2(\tilde{r}_2 - \tilde{r}_1) - m_2\,\frac{\tilde{r}_2\,s\,s^{\mathrm{T}}(J + m_2\,\tilde{r}_2\,\tilde{r}_1)}{s^{\mathrm{T}}J\,s}\right)\dot{\omega}_1\,.
\end{aligned} \tag{S22}
$$

Repeating the procedure for $n$, (S13), (S2), (S3), (S4), (S17), and (S18) give

$$
\begin{aligned}
n &= n_1 + n_2 + (r_1 - r_2) \times f_2 \\
&= I_1\,\dot{\omega}_1 + I_2\,\dot{\omega}_2 + m_2(r_1 - r_2) \times a_2 \\
&= I_1\,\dot{\omega}_1 + I_2(\dot{\omega}_1 + s\,\alpha) + m_2(r_1 - r_2) \times \\
&\quad (a_1 + (r_2 - r_1) \times \dot{\omega}_1 + r_2 \times s\,\alpha) \\
&= (I_1 + I_2 - m_2(\tilde{r}_1 - \tilde{r}_2)^2)\dot{\omega}_1 \\
&\quad + m_2(\tilde{r}_1 - \tilde{r}_2)a_1 + Ks\,\alpha\,,
\end{aligned} \tag{S23}
$$

where

$$
\begin{aligned}
K &= I_2 + m_2(\tilde{r}_1 - \tilde{r}_2)\tilde{r}_2 \\
&= J + m_2\,\tilde{r}_1\,\tilde{r}_2\,.
\end{aligned} \tag{S24}
$$

Note that (S21) can now be simplified to

$$
\alpha = -\frac{s^{\mathrm{T}}(K^{\mathrm{T}}\dot{\omega}_1 - m_2\,\tilde{r}_2\,a_1)}{s^{\mathrm{T}}J\,s}\,. \tag{S25}
$$

Eliminating $\alpha$ from (S23) using (S25) gives

$$
\begin{aligned}
n &= (I_1 + I_2 - m_2(\tilde{r}_1 - \tilde{r}_2)^2)\dot{\omega}_1 \\
&\quad + m_2(\tilde{r}_1 - \tilde{r}_2)a_1 - \frac{Ks\,s^{\mathrm{T}}(K^{\mathrm{T}}\dot{\omega}_1 - m_2\,\tilde{r}_2\,a_1)}{s^{\mathrm{T}}J\,s}\,,
\end{aligned}
$$

and collecting terms in $\dot{\omega}_1$ and $a_1$ gives

$$
\begin{aligned}
n &= \left(I_1 + I_2 - m_2(\tilde{r}_1 - \tilde{r}_2)^2 - \frac{Ks\,s^{\mathrm{T}}K^{\mathrm{T}}}{s^{\mathrm{T}}J\,s}\right)\dot{\omega}_1 \\
&\quad + \left(m_2(\tilde{r}_1 - \tilde{r}_2) + m_2\,\frac{Ks\,s^{\mathrm{T}}\tilde{r}_2}{s^{\mathrm{T}}J\,s}\right)a_1\,.
\end{aligned} \tag{S26}
$$

The final step is to combine (S22) and (S26) into a single equation:

$$
\begin{bmatrix} f \\ n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} a_1 \\ \dot{\omega}_1 \end{bmatrix}, \tag{S27}
$$

where

$$
A = (m_1 + m_2)\,1_{3\times3} + m_2^2\,\frac{\tilde{r}_2\,s\,s^{\mathrm{T}}\,\tilde{r}_2}{s^{\mathrm{T}}J\,s}\,, \tag{S28}
$$

$$
B = m_2(\tilde{r}_2 - \tilde{r}_1) - m_2\,\frac{\tilde{r}_2\,s\,s^{\mathrm{T}}K^{\mathrm{T}}}{s^{\mathrm{T}}J\,s}\,, \tag{S29}
$$

$$
C = m_2(\tilde{r}_1 - \tilde{r}_2) + m_2\,\frac{Ks\,s^{\mathrm{T}}\tilde{r}_2}{s^{\mathrm{T}}J\,s}\,, \tag{S30}
$$

and

$$
D = I_1 + I_2 - m_2(\tilde{r}_1 - \tilde{r}_2)^2 - \frac{Ks\,s^{\mathrm{T}}K^{\mathrm{T}}}{s^{\mathrm{T}}J\,s}\,. \tag{S31}
$$

$1_{3\times3}$ is an identity matrix. The solution to the original problem is then

$$
\begin{bmatrix} a_1 \\ \dot{\omega}_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} f \\ n \end{bmatrix}. \tag{S32}
$$

## Reference


[S1] R. Featherstone. (2010). Spatial vector algebra [Online]. Available: http://users.cecs.anu.edu.au/~roy/spatial/



COURTESY OF R. FEATHERSTONE, 2010.

alternative strategy, which works just as well, is to distinguish the 3-D symbols from the spatial ones by marking them with arrows (e.g., $\vec{f}$ and $\vec{v}$). Rules like this provide a degree of flexibility to the user, and are preferable to simpler rules like "all spatial symbols have hats," which just lead to an unnecessary sea of hats—a nuisance to read and write.

It is sometimes useful to distinguish between a coordinate vector and the quantity it represents. This will be done by underlining the coordinate vector. Thus, you will occasionally see a symbol like $\underline{v}$ or $\underline{\hat{v}}$ used to denote the coordinate vector representing $v$ or $\hat{v}$. This notational device is used only where needed.

## A Worked Example

The main message of this tutorial is that spatial vectors are not merely a convenient way of pairing up 3-D vectors but are a problem-solving tool in their own right. Spatial vectors have their own physical interpretations, their own equations and formulae, and their own rules of use; and the best way to use them is to think directly in 6-D.

Perhaps the best way to explain is by means of a worked example, comparing the 3-D and 6-D approach to solving a rigid-body problem. "Solving a Two-Body Dynamics Problem Using 3-D Vectors" presents a detailed worked example of how to solve a simple two-body dynamics problem using

### Solving a Two-Body Dynamics Problem Using Spatial Vectors

We are given a rigid-body system consisting of two bodies, $B_1$ and $B_2$, connected by a revolute joint [S2]. The bodies have inertias of $I_1$ and $I_2$, respectively, and they are initially at rest. The joint's rotation axis is $s$. A force $f$ is applied to $B_1$, causing both bodies to accelerate. The problem is to calculate the acceleration of $B_1$ as a function of $f$ (Figure S2).

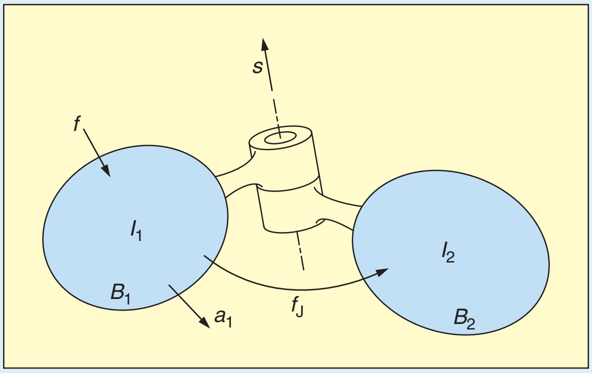


*Figure S2. Problem diagram using spatial vectors.*

### Solution

Let $a_1$ and $a_2$ be the accelerations of the two bodies, and let $f_J$ be the force transmitted from $B_1$ to $B_2$ through the joint. The net forces acting on the two bodies are therefore $f - f_J$ and $f_J$, respectively, and their equations of motion are

$$f - f_J = I_1 a_1 \tag{S33}$$

and

$$f_J = I_2 a_2 . \tag{S34}$$

There are no velocity terms because the bodies are at rest. The joint permits $B_2$ to accelerate relative to $B_1$ about the axis specified by $s$; so, $a_2$ can be expressed in the form

$$a_2 = a_1 + s\,\alpha, \tag{S35}$$

where $\alpha$ is the joint acceleration variable. Again, there are no velocity terms because the bodies are at rest. This motion constraint is implemented by $f_J$, which is the joint constraint force, so $f_J$ must satisfy

$$s^T f_J = 0, \tag{S36}$$

i.e., the constraint force does no work in the direction of motion allowed by the joint.

Given (S33)–(S36), the problem is solved as follows. First, substitute (S35) into (S34), giving

$$f_J = I_2 (a_1 + s\,\alpha) . \tag{S37}$$

Substituting (S37) into (S36) gives

$$s^T I_2 (a_1 + s\,\alpha) = 0 ,$$

from which we get the following expression for $\alpha$:

$$\alpha = -\frac{s^T I_2 a_1}{s^T I_2 s} . \tag{S38}$$

Substituting (S38) back into (S37) gives

$$f_J = I_2 \left( a_1 - \frac{s\,s^T I_2}{s^T I_2 s} a_1 \right),$$

and substituting this equation back into (S33) gives

$$\begin{aligned} f &= I_1 a_1 + I_2 a_1 - \frac{I_2 s\,s^T I_2}{s^T I_2 s} a_1 \\ &= \left( I_1 + I_2 - \frac{I_2 s\,s^T I_2}{s^T I_2 s} \right) a_1 . \end{aligned}$$

The expression in brackets is nonsingular and may therefore be inverted to express $a_1$ in terms of $f$:

$$a_1 = \left( I_1 + I_2 - \frac{I_2 s\,s^T I_2}{s^T I_2 s} \right)^{-1} f . \tag{S39}$$

### Reference


[S2] R. Featherstone. (2010). Spatial vector algebra [Online]. Available: http://users.cecs.anu.edu.au/~roy/spatial/

3-D vectors, and "Solving a Two-Body Dynamics Problem Using Spatial Vectors" shows how to solve the same problem using spatial vectors. We shall refer to them as the 3-D example and the spatial example, respectively. The 3-D example employs the methods of 3-D vectorial dynamics, and the spatial example employs the methods of spatial vector algebra. Even the briefest glance reveals that the spatial example is simpler in every aspect: the problem statement is shorter, the diagram is simpler, and the solution is much shorter. Let us now examine these examples in more detail.

Starting with the 3-D example, eight quantities are needed to describe the rigid-body system: the mass, center of mass, and rotational inertia of each body, plus the quantities $P$ and $\boldsymbol{s}$ to define the joint axis. A further two quantities are needed to describe the forces acting on $B_1$, and a further two to describe its resulting acceleration. Furthermore, it is not enough merely to state that $\boldsymbol{f}$, $\boldsymbol{n}$, $\boldsymbol{a}_1$, and $\dot{\boldsymbol{\omega}}_1$ describe the forces and accelerations—a complete description requires that we also identify the line of action of $\boldsymbol{f}$ and the particular point in $B_1$ to which $\boldsymbol{a}_1$ refers. Having introduced the three points $C_1$, $C_2$, and $P$ as a necessary part of describing the problem, it becomes desirable to show these points on the diagram, as they will play a major role in the solution process.

In the terminology of the 3-D-vector approach, $\boldsymbol{f}$, $\boldsymbol{n}$, $\boldsymbol{a}_1$, and $\dot{\boldsymbol{\omega}}_1$ are said to be referred to (or expressed at) $C_1$, meaning that $C_1$ serves as the reference point for these quantities. Equations (S1) and (S2) are likewise referred to (expressed at) $C_1$. This need to define various points in space, and to refer various vectors and equations to these points, is a characteristic feature of the 3-D-vector approach to solving a rigid-body problem. It accounts for a large part of the algebraic complexity, and it forces the analyst to think explicitly in terms of which point will be used to express which equation and which quantities will have to be transferred from one reference point to another. A poor choice of reference points can render a complicated solution procedure even more complicated.

In the 3-D example, we can see that the equations of motion of each body have been expressed at their respective centers of mass, and that the equations of constraint (S14)–(S16) have been expressed at $P$. These are good choices, but they require us to define an extra eight quantities (${}^{P}\boldsymbol{a}_1$ to ${}^{1}\boldsymbol{n}_2$) and an extra seven equations (S5)–(S11) to manage all the necessary transfers of vectors from one reference point to another.

At the highest level, the solution strategy is this: express the acceleration of $B_2$ in terms of $\boldsymbol{a}_1$, $\dot{\boldsymbol{\omega}}_1$, and $\alpha$, and then use the force-constraint equation (S16) to obtain an expression for $\alpha$ in terms of $\boldsymbol{a}_1$ and $\dot{\boldsymbol{\omega}}_1$. At this point, every force and acceleration in the system can be expressed in terms of $\boldsymbol{a}_1$ and $\dot{\boldsymbol{\omega}}_1$; so, the solution is obtained by expressing $\boldsymbol{f}$ and $\boldsymbol{n}$ in terms of $\boldsymbol{a}_1$ and $\dot{\boldsymbol{\omega}}_1$, and then inverting the equations to express the accelerations in terms of the forces.

Let us now examine the spatial-vector example. In this case, only three quantities ($\boldsymbol{I}_1$, $\boldsymbol{I}_2$, and $\boldsymbol{s}$) are required to describe the rigid-body system; only one quantity ($\boldsymbol{f}$) is required to describe the forces acting on $B_1$; and only one quantity ($\boldsymbol{a}_1$) is required to describe its acceleration. Furthermore, the solution procedure introduces only another three quantities ($\boldsymbol{a}_2$, $\boldsymbol{f}_{\mathrm{J}}$, and $\alpha$). So, the whole problem now involves only eight quantities.

Observe that there is no mention of any 3-D point anywhere in this example. The problem has been stated and solved without reference to any point in space. This absence of reference points is a characteristic feature of the spatial-vector approach (and some other 6-D formalisms) and is a key aspect of thinking in 6-D.

Referring back to the 3-D example, it is clearly possible to pair up corresponding 3-D vectors ($\boldsymbol{f}$ with $\boldsymbol{n}$, $\boldsymbol{a}_1$ with $\dot{\boldsymbol{\omega}}_1$, and so on) to make 6-D vectors, and this would result in some reduction in the volume of algebra. However, the points $C_1$, $C_2$, and $P$ would still be an essential part of the problem statement and the solution process. Thus, the stacking of pairs of 3-D vectors is purely a notational device: the resulting vectors are 6-D, but the concepts, methods, and thought processes are all still 3-D.

Returning to the spatial-vector example, the diagram is clearly simpler, but the arrows now have different meanings. The arrow associated with $\boldsymbol{f}$, which points from empty space to $B_1$, indicates only that $\boldsymbol{f}$ is an external force acting on $B_1$. It does not convey any geometrical information (such as the line of action of a force). Likewise, the arrow associated with $\boldsymbol{f}_{\mathrm{J}}$, which points from $B_1$ to $B_2$, indicates only that $\boldsymbol{f}_{\mathrm{J}}$ is a force transmitted from $B_1$ to $B_2$, whereas the arrow associated with $\boldsymbol{a}_1$, which points out of $B_1$, indicates only that $\boldsymbol{a}_1$ is the acceleration of $B_1$. From the directions of the arrows (and knowledge of spatial vectors), we can immediately deduce that the net force on $B_1$ is $\boldsymbol{f} - \boldsymbol{f}_{\mathrm{J}}$ and the net force on $B_2$ is $\boldsymbol{f}_{\mathrm{J}}$. The arrow associated with $\boldsymbol{s}$, which is aligned with the joint's rotation axis, is the only one with any geometrical significance.

The reason why there are no 3-D points in the spatial-vector example and why most of the arrows have no geometrical significance is because all the necessary positional information is intrinsic to the relevant spatial quantities. The inertias $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ implicitly locate the centers of mass of the two bodies; the line of action of $\boldsymbol{f}$ (if it has one) can be deduced from its value; and $\boldsymbol{s}$ defines both the direction and the location of an axis of rotation in 3-D space. Acceleration is a little more complicated and will be discussed in a later section. Nevertheless, $\boldsymbol{a}_1$ does provide a complete description of a body's acceleration and does not need to be referred to any point.

The high-level solution strategy in the spatial-vector example is the same as that in the 3-D example: express $\boldsymbol{a}_2$ in terms of $\boldsymbol{a}_1$ and $\alpha$; then, substitute into the force-constraint equation (S36) to obtain an expression for $\alpha$ in terms of $\boldsymbol{a}_1$; then, express $\boldsymbol{f}$ in terms of $\boldsymbol{a}_1$ and invert to express $\boldsymbol{a}_1$ in terms of $\boldsymbol{f}$. Using spatial vectors, the analyst is able to follow this high-level strategy directly, without having to think about the messy details associated with the 3-D-vector approach. Observe how the expression for $\alpha$ is obtained almost immediately, after just two simple substitutions, and the desired expression for $\boldsymbol{f}$ is obtained after just three more simple substitutions.

Another benefit of spatial vectors, which is not evident from this example, is that it is quite easy to prove that the expression in parentheses in (S39) is a symmetric, positive-definite matrix, and therefore invertible. The same is also true of the $6 \times 6$ matrix in (S27) of the 3-D example, but the proof (using 3-D vectors) is relatively complicated. Yet another advantage of spatial vectors is that a person who is new to the example problem is likely to get the correct answer quickly using spatial vectors but is likely to get lost and take wrong turns while trying to solve it using 3-D vectors.

## Some Formalities

To understand spatial vectors, it helps to review a few basic facts about vectors. First, there are many different types of vector, each having different mathematical properties. In fact, there are only two operations that are defined on all vectors: the addition of two vectors and the multiplication of a vector by a scalar (i.e., a real number). There are several more operators that are defined on particular types of vector and give them special properties. One important example is the Euclidean inner product, which is defined only on Euclidean vectors, and gives them the special properties of magnitude and direction.

Two types of vector are of special interest: Euclidean and coordinate vectors. Euclidean vectors have the special properties of magnitude and direction, and they are the elements of a Euclidean vector space, which we shall denote with the symbol $\mathsf{E}^n$. (The superscript indicates the dimension.) Coordinate vectors are $n$-tuples of real numbers, and they are elements of the vector space $\mathsf{R}^n$. The special property of coordinate vectors is that they have a first coordinate, a second coordinate, and so on.

Coordinate vectors are used to represent other kinds of vector via a basis. For example, if $V$ is a general vector space and $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\} \subset V$ is a basis on $V$, then any vector $\boldsymbol{v} \in V$ can be expressed in the form $\boldsymbol{v} = \sum_{i=1}^{n} v_i \boldsymbol{e}_i$, where $v_i$ are the coordinates of $\boldsymbol{v}$ in $\mathcal{E}$. If we assemble them into a coordinate vector $\underline{\boldsymbol{v}} = [v_1\, v_2 \cdots v_n]^{\mathrm{T}}$, then we can say that $\underline{\boldsymbol{v}} \in \mathsf{R}^n$ represents $\boldsymbol{v} \in V$ in the basis $\mathcal{E} \subset V$.

Some bases are more useful than others. For Euclidean vectors, the most useful is an orthonormal basis, which gives rise to a Cartesian coordinate system. The special property of Cartesian coordinates is this: if $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$ are coordinate vectors representing the Euclidean vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in a Cartesian coordinate system, then $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \underline{\boldsymbol{v}}_1^{\mathrm{T}} \underline{\boldsymbol{v}}_2$.

Spatial vectors are not Euclidean. They are the elements of two closely related vector spaces called $\mathsf{M}^6$ and $\mathsf{F}^6$. The former contains motion vectors, which describe the motions of rigid bodies, and the latter contains force vectors, which describe forces acting on rigid bodies. Formally, $\mathsf{F}^6$ is the dual vector space of $\mathsf{M}^6$ and vice versa. This notation can be extended to encompass other kinds of motion and force vector. For example, a generalized force vector could be described as an element of $\mathsf{F}^n$, and a vector describing the motion of a set of $N$ rigid bodies could be said to be an element of $\mathsf{M}^{6N}$.

There is no inner product defined on spatial vectors. Instead, there is a scalar product that takes one argument from each space. If $\boldsymbol{m} \in \mathsf{M}^6$ and $\boldsymbol{f} \in \mathsf{F}^6$, then the expressions $\boldsymbol{m} \cdot \boldsymbol{f}$ and $\boldsymbol{f} \cdot \boldsymbol{m}$ are defined (and are equal), but the expressions $\boldsymbol{m} \cdot \boldsymbol{m}$ and $\boldsymbol{f} \cdot \boldsymbol{f}$ are not. As we shall see later, if $\boldsymbol{m}$ is the velocity of a rigid body and $\boldsymbol{f}$ is the force acting on it, then $\boldsymbol{f} \cdot \boldsymbol{m}$ is the power delivered by $\boldsymbol{f}$.

A coordinate system for spatial vectors must span both $\mathsf{M}^6$ and $\mathsf{F}^6$. Thus, a total of 12 basis vectors are required: $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_6\} \subset \mathsf{M}^6$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_6\} \subset \mathsf{F}^6$. Again, some bases are more useful than others. For spatial vectors, the most useful is a dual basis, which defines a dual coordinate system on $\mathsf{M}^6$ and $\mathsf{F}^6$. To qualify as a dual basis, the vectors $\boldsymbol{d}_i$ and $\boldsymbol{e}_i$ must satisfy the following condition:

$$\boldsymbol{d}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The special property of a dual coordinate system is this: if $\underline{\boldsymbol{m}}$ and $\underline{\boldsymbol{f}}$ are coordinate vectors representing $\boldsymbol{m} \in \mathsf{M}^6$ and $\boldsymbol{f} \in \mathsf{F}^6$ in a dual coordinate system, then $\boldsymbol{m} \cdot \boldsymbol{f} = \underline{\boldsymbol{m}}^{\mathrm{T}} \underline{\boldsymbol{f}}$. Thus, dual coordinates are the spatial-vector equivalent of a Cartesian coordinate system on a Euclidean vector space. The Plücker coordinates that we shall meet later on are a special type of dual coordinate system having extra properties that make them especially convenient to use.

A general property of dual coordinate systems is that motion and force vectors obey different coordinate-transformation rules. If $\boldsymbol{X}$ is a coordinate transform for motion vectors, then the corresponding transform for force vectors is called $\boldsymbol{X}^*$, and the two are related by

$$\boldsymbol{X}^* = (\boldsymbol{X}^{-1})^{\mathrm{T}} = \boldsymbol{X}^{-\mathrm{T}}. \tag{2}$$

This relationship ensures that the scalar product is invariant with respect to any coordinate transformation, as can be seen from

$$(\boldsymbol{X}^* \boldsymbol{f})^{\mathrm{T}} (\boldsymbol{X} \boldsymbol{m}) = \boldsymbol{f}^{\mathrm{T}} \boldsymbol{X}^{-1} \boldsymbol{X} \boldsymbol{m} = \boldsymbol{f}^{\mathrm{T}} \boldsymbol{m}. \tag{3}$$

## What Is a Spatial Vector?

Let $P$ be a particle in 3-D space. If we say that $P$ has a velocity of $\boldsymbol{v}$, then we mean that $\boldsymbol{v}$ is a Euclidean vector ($\boldsymbol{v} \in \mathsf{E}^3$) whose magnitude and direction match the speed and direction of travel of $P$. Now suppose $B$ is a rigid body. If we are told that $B$ has a velocity of $\hat{\boldsymbol{v}}$, then we can immediately infer that $\hat{\boldsymbol{v}} \in \mathsf{M}^6$; but how exactly does $\hat{\boldsymbol{v}}$ describe the motion of $B$, and how is it possible for $\hat{\boldsymbol{v}}$ to describe the motion without using a reference point? An intuitive answer to these questions is provided by screw theory.

The most general motion of a rigid body, at any given instant, is a screwing motion along a directed line. (The body is behaving like a nut on a screw thread fixed somewhere in space.) In screw theory, such a motion is called a *twist*. A twist can be characterized by an angular magnitude, a linear magnitude, and a directed line. (The ratio of the two magnitudes is called the *pitch*.) These three quantities together define a twist, just as a magnitude and a direction together define a particle velocity. The two magnitudes describe the rate at which the body is rotating about and translating along the directed line, and the line itself describes the instantaneous screw axis of the motion. (As the name suggests, a rigid body will, in general, be screwing about two different axes at two different instants. Over a finite time interval, a body will have screwed about an infinity of axes, each one fixed in space and each one valid for a single instant.) If the linear magnitude is zero, then the motion is a pure rotation. If the angular magnitude is zero, then the motion is a pure translation, in which case the location of the line is irrelevant and only its direction matters.

We can now answer the question posed earlier. The elements of $\mathsf{M}^6$ are twists and are characterized by two magnitudes and a directed line. If we say that $\hat{\boldsymbol{v}} \in \mathsf{M}^6$ is the velocity of $B$, then we mean that the directed line and the linear and angular magnitudes of $\hat{\boldsymbol{v}}$ match the instantaneous screw axis of the body's motion and its linear and angular rates of progression along and about that axis.

Spatial force vectors can be explained in a similar manner. The most general force acting on a rigid body consists of a screwing force acting along a directed line: a linear force acting along the line, together with a turning force (a couple) acting about the line. In screw theory, such a quantity is called a *wrench*. A wrench can be characterized by a linear magnitude, an angular magnitude, and a directed line—exactly the same as a twist. The two magnitudes describe the intensities of the linear and angular components of the wrench, and the directed line is the instantaneous screw axis. If the angular magnitude is zero, then the wrench is a pure force. If the linear magnitude is zero, then the wrench is a pure couple, in which case the location of the line is irrelevant and only its direction matters.

If the elements of $\mathsf{F}^6$ are forces acting on rigid bodies, then they are wrenches, and they are characterized by two magnitudes and a directed line. If we say that a force $\hat{\boldsymbol{f}} \in \mathsf{F}^6$ is acting on body $B$, then we mean that the directed line and the linear and angular magnitudes of $\hat{\boldsymbol{f}}$ match the instantaneous screw axis and the linear and angular intensities of the wrench acting on body $B$.

If we introduce a reference point $O$, then it becomes possible to represent a spatial velocity $\hat{\boldsymbol{v}}$ by means of a pair of 3-D vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$, and to represent a spatial force $\hat{\boldsymbol{f}}$ by means of a pair of 3-D vectors $\boldsymbol{f}$ and $\boldsymbol{n}_O$. Having explained the nature of rigid-body motion and force by means of twists and wrenches, it should now be clear that reference points are not an intrinsic necessity but merely an artifact of the 3-D-vector representation. In other words, you only need reference points if you are using 3-D vectors. Choosing a reference point is like choosing a coordinate system in which a spatial vector is to be represented by a pair of vector-valued coordinates. (This idea is explored in [6].)

Having obtained $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$, there are two ways to view the velocity they describe. According to one view, the body is rotating with an angular velocity of $\boldsymbol{\omega}$ about an axis passing through $O$ while simultaneously translating with a linear velocity of $\boldsymbol{v}_O$. According to the other view, $\boldsymbol{\omega}$ defines the angular magnitude of the twist velocity and the direction of the instantaneous screw axis, while $\boldsymbol{v}_O$ (in combination with $\boldsymbol{\omega}$) defines the linear magnitude of the twist velocity and the location of the instantaneous screw axis relative to $O$. Both views are correct and useful. Similar comments apply to forces.

## Using Spatial Vectors

Spatial vectors are a tool for expressing and analyzing the physical properties and behavior of rigid-body systems. The vectors describe physical quantities, and the equations describe relationships between them. So what do the rules of classical mechanics look like in spatial-vector form? Here is a short list of basic facts and formulae. It mentions some quantities that we have not yet met, but will be described in the next section.

- *Relative velocity*: If bodies $B_1$ and $B_2$ have velocities of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively, then the relative velocity of $B_2$ with respect to $B_1$ is $\boldsymbol{v}_{\mathrm{rel}} = \boldsymbol{v}_2 - \boldsymbol{v}_1$. Obviously, this also means that $\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_{\mathrm{rel}}$.
- *Summation of forces*: If forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ act on the same rigid body, then they are equivalent to a single force, $\boldsymbol{f}_{\mathrm{tot}}$, given by $\boldsymbol{f}_{\mathrm{tot}} = \boldsymbol{f}_1 + \boldsymbol{f}_2$.
- *Action and reaction*: If body $B_1$ exerts a force $\boldsymbol{f}$ on body $B_2$, then $B_2$ exerts a force $-\boldsymbol{f}$ on $B_1$. This is Newton's third law in spatial form.
- *Scalar product*: If a force $\boldsymbol{f}$ acts on a body having a velocity of $\boldsymbol{v}$, then the power delivered by that force is $\boldsymbol{f} \cdot \boldsymbol{v}$.
- *Scalar multiplication*: This operation affects a spatial vector's magnitudes but not its directed line. If a spatial vector $\boldsymbol{s}$ is characterized by magnitudes $m_1$ and $m_2$ and line $\boldsymbol{l}$, then $\alpha \boldsymbol{s}$ is characterized by $\alpha m_1$, $\alpha m_2$, and $\boldsymbol{l}$. A body having a velocity of $\alpha \boldsymbol{v}$ makes the same infinitesimal motion over a period of $\delta t$ as a body with a velocity of $\boldsymbol{v}$ makes over a period of $\alpha\delta t$; and a force $\beta \boldsymbol{f}$ delivers $\beta$ times as much power as a force $\boldsymbol{f}$ acting on the same body. So, $(\alpha \boldsymbol{v}) \cdot (\beta \boldsymbol{f}) = \alpha\,\beta\,(\boldsymbol{v} \cdot \boldsymbol{f})$.
- *Differentiation*: Spatial vectors are differentiated just like any other vector. The derivative of a motion vector is a motion vector, and the derivative of a force vector is a force vector. If $\boldsymbol{m} \in \mathsf{M}^6$ and $\boldsymbol{f} \in \mathsf{F}^6$ are fixed in a body having a velocity of $\boldsymbol{v}$, then $\dot{\boldsymbol{m}} = \boldsymbol{v} \times \boldsymbol{m}$ and $\dot{\boldsymbol{f}} = \boldsymbol{v} \times^* \boldsymbol{f}$. [The operator $\times^*$ is defined in (18).]
- *Acceleration*: Spatial acceleration is the time derivative of spatial velocity ($\boldsymbol{a} = \dot{\boldsymbol{v}}$). For example, if $\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_{\mathrm{rel}}$, then $\boldsymbol{a}_2 = \boldsymbol{a}_1 + \boldsymbol{a}_{\mathrm{rel}}$. Spatial accelerations are elements of $\mathsf{M}^6$ and therefore obey the same coordinate-transformation rule as velocities.
- *Summation of inertias*: If bodies $B_1$ and $B_2$, having inertias of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, respectively, are rigidly connected to form a single composite rigid body, then the inertia of the composite is $\boldsymbol{I}_{\mathrm{tot}} = \boldsymbol{I}_1 + \boldsymbol{I}_2$.
- *Momentum*: If a rigid body has a velocity of $\boldsymbol{v}$ and an inertia of $\boldsymbol{I}$, then its momentum is $\boldsymbol{I}\boldsymbol{v}$.
- *Equation of motion*: The total force acting on a rigid body equals its rate of change of momentum. $\boldsymbol{f} = \mathrm{d}(\boldsymbol{I}\boldsymbol{v})/\mathrm{d}t = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{v} \times^* \boldsymbol{I}\boldsymbol{v}$.
- *Motion constraints*: If the relative velocity of two rigid bodies is constrained to lie in a subspace $S \subseteq \mathsf{M}^6$, then the motion constraint is implemented by a constraint force lying in the subspace $T = \{\boldsymbol{f} \in \mathsf{F}^6 \,|\, \boldsymbol{f} \cdot \boldsymbol{v} = 0 \;\forall\, \boldsymbol{v} \in S\}$. This is a statement of D'Alembert's principle of virtual work (or Jourdain's principle of virtual power) expressed using spatial vectors.

## Details

This section covers the practical details of coordinate systems, differentiation, inertia, the equation of motion, and motion constraints. By the end of this section, you should be able to understand the spatial-vector example clearly enough to be able to generalize it to the case where $B_1$ and $B_2$ have nonzero velocities.

## *Plücker Coordinates*

Plücker coordinates (pronounced plooker) are the coordinate systems of choice for 6-D vectors. To set up a Plücker coordinate system, all that is needed is a Cartesian coordinate frame $Oxyz$ placed anywhere. The position and orientation of this frame defines a Plücker coordinate system. In fact, there is a 1:1 correspondence between the set of all positions and orientations of a Cartesian frame and the set of all possible Plücker coordinate systems.

The frame $Oxyz$ defines the following items: a point $O$, three mutually orthogonal directions $x$, $y$, and $z$, and three directed lines $Ox$, $Oy$, and $Oz$. (They pass through $O$ in the $x$, $y$, and $z$ directions.) With this data, we can define three bases as follows:

$$\mathcal{C} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \subset \mathsf{E}^3, \tag{4}$$

$$\mathcal{D} = \{\boldsymbol{d}_{Ox}, \boldsymbol{d}_{Oy}, \boldsymbol{d}_{Oz}, \boldsymbol{d}_x, \boldsymbol{d}_y, \boldsymbol{d}_z\} \subset \mathsf{M}^6, \tag{5}$$

and

$$\mathcal{E} = \{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z, \boldsymbol{e}_{Ox}, \boldsymbol{e}_{Oy}, \boldsymbol{e}_{Oz}\} \subset \mathsf{F}^6. \tag{6}$$

The elements of $\mathcal{C}$ are unit Euclidean vectors in the $x$, $y$, and $z$ directions, and this basis defines a Cartesian coordinate system on $\mathsf{E}^3$. The elements of $\mathcal{D}$ and $\mathcal{E}$ are as follows: $\boldsymbol{d}_{Ox}$, $\boldsymbol{d}_{Oy}$, and $\boldsymbol{d}_{Oz}$ are unit pure rotations about the lines $Ox$, $Oy$, and $Oz$, respectively; $\boldsymbol{d}_x$, $\boldsymbol{d}_y$, and $\boldsymbol{d}_z$ are unit pure translations in the $x$, $y$, and $z$ directions, respectively; $\boldsymbol{e}_{Ox}$, $\boldsymbol{e}_{Oy}$, and $\boldsymbol{e}_{Oz}$ are unit pure forces acting along the lines $Ox$, $Oy$, and $Oz$, respectively; and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ are unit pure couples in the $x$, $y$, and $z$ directions. These vectors are illustrated in Figure 1. The bases $\mathcal{D}$ and $\mathcal{E}$ together define a Plücker coordinate system on $\mathsf{M}^6$ and $\mathsf{F}^6$.

Now that we have the bases, let us work out the Plücker coordinates of a spatial velocity $\hat{\boldsymbol{v}} \in \mathsf{M}^6$ and a spatial force $\hat{\boldsymbol{f}} \in \mathsf{F}^6$. Starting with $\hat{\boldsymbol{v}}$, the first step is to identify the two 3-D vectors, $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$, that represent $\hat{\boldsymbol{v}}$ at reference point $O$. Once this has been done, the motion of the body can be regarded as the sum of a pure rotation of $\boldsymbol{\omega}$ about $O$ (i.e., the axis of rotation passes through $O$) and a pure translation of $\boldsymbol{v}_O$. The next step is to express these vectors in basis $\mathcal{C}$:

$$\boldsymbol{\omega} = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k},$$

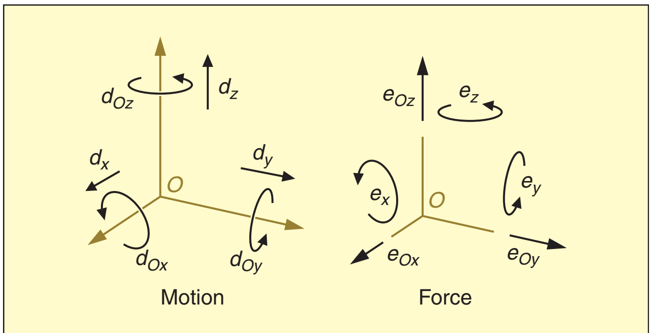


***Figure 1.*** *Plücker basis vectors.*

and

$$\boldsymbol{v}_O = v_{Ox} \boldsymbol{i} + v_{Oy} \boldsymbol{j} + v_{Oz} \boldsymbol{k}.$$

We can now describe the spatial velocity $\hat{\boldsymbol{v}}$ as the sum of six components: three rotations of magnitudes $\omega_x$, $\omega_y$, and $\omega_z$ about the lines $Ox$, $Oy$, and $Oz$, respectively, plus three translations of magnitudes $v_{Ox}$, $v_{Oy}$, and $v_{Oz}$ in the $x$, $y$, and $z$ directions. On comparing this description with the definitions of the basis vectors in $\mathcal{D}$, it follows immediately that

$$\hat{\boldsymbol{v}} = \omega_x \boldsymbol{d}_{Ox} + \omega_y \boldsymbol{d}_{Oy} + \omega_z \boldsymbol{d}_{Oz} + v_{Ox} \boldsymbol{d}_x + v_{Oy} \boldsymbol{d}_y + v_{Oz} \boldsymbol{d}_z. \tag{7}$$

Applying the same procedure to $\hat{\boldsymbol{f}}$, we identify $\boldsymbol{f}$ and $\boldsymbol{n}_O$ as the 3-D vectors representing $\hat{\boldsymbol{f}}$ at $O$ and express them in basis $\mathcal{C}$:

$$\boldsymbol{f} = f_x \boldsymbol{i} + f_y \boldsymbol{j} + f_z \boldsymbol{k},$$

and

$$\boldsymbol{n}_O = n_{Ox} \boldsymbol{i} + n_{Oy} \boldsymbol{j} + n_{Oz} \boldsymbol{k}.$$

The force $\hat{\boldsymbol{f}}$ can then be described as the sum of six components: pure forces of magnitudes $f_x$, $f_y$, and $f_z$ acting along the lines $Ox$, $Oy$, and $Oz$, respectively, and pure couples of magnitudes $n_{Ox}$, $n_{Oy}$, and $n_{Oz}$ in the $x$, $y$, and $z$ directions. On comparing this description with the definitions of the basis vectors in $\mathcal{E}$, it follows immediately that

$$\hat{\boldsymbol{f}} = n_{Ox} \boldsymbol{e}_x + n_{Oy} \boldsymbol{e}_y + n_{Oz} \boldsymbol{e}_z + f_x \boldsymbol{e}_{Ox} + f_y \boldsymbol{e}_{Oy} + f_z \boldsymbol{e}_{Oz}. \tag{8}$$

So, the Plücker coordinates of $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{f}}$ are none other than the Cartesian coordinates in $\mathcal{C}$ of the vectors $\boldsymbol{\omega}$, $\boldsymbol{v}_O$, $\boldsymbol{f}$, and $\boldsymbol{n}_O$. If $\underline{\hat{\boldsymbol{v}}}$ and $\underline{\hat{\boldsymbol{f}}}$ are the coordinate vectors representing $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{f}}$, then we can write them in full as

$$\underline{\hat{\boldsymbol{v}}} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_{Ox} \\ v_{Oy} \\ v_{Oz} \end{bmatrix}, \qquad \underline{\hat{\boldsymbol{f}}} = \begin{bmatrix} n_{Ox} \\ n_{Oy} \\ n_{Oz} \\ f_x \\ f_y \\ f_z \end{bmatrix}, \tag{9}$$

or we can write them in short form as

$$\underline{\hat{\boldsymbol{v}}} = \begin{bmatrix} \underline{\boldsymbol{\omega}} \\ \underline{\boldsymbol{v}}_O \end{bmatrix}, \qquad \underline{\hat{\boldsymbol{f}}} = \begin{bmatrix} \underline{\boldsymbol{n}}_O \\ \underline{\boldsymbol{f}} \end{bmatrix}, \tag{10}$$

where $\underline{\boldsymbol{\omega}} = [\omega_x\ \omega_y\ \omega_z]^{\mathrm{T}}$, etc. This short format is very convenient and popular, but it does have one small drawback: it looks like a stacked pair of 3-D vectors and is therefore capable of misleading readers into thinking that a spatial vector is a stacked pair of 3-D vectors; but in reality, it is only the coordinates that are being stacked. Equation (10) is nothing more nor less than a shorthand for (9).

Equations (9) and (10) list the angular coordinates above the linear ones. This order is not essential, and you will find examples in the literature where the linear coordinates precede the angular ones. From a mathematical point of view, the difference is purely cosmetic: it is simply a consequence of the order in which we have chosen to list the basis vectors in $\mathcal{D}$ and $\mathcal{E}$. However, if you want to use spatial arithmetic software, then you will have to comply with the order expected by the software.

The pattern of coordinate names appearing in (9) and (10) is not the general case because we have used some special symbols. In particular, we used the standard symbol $\boldsymbol{\omega}$ for angular velocity and the (not quite so) standard symbol $\boldsymbol{n}$ for moment. In the general case, the coordinates' names are derived from the name of the vector they describe. For a generic motion vector $\hat{\boldsymbol{m}}$, the coordinates would be called $m_x$, $m_y$, $m_z$, $m_{Ox}$, $m_{Oy}$, and $m_{Oz}$. Of course, you could also number the coordinates (and basis vectors) if you prefer.

## Plücker Transforms

Let $A$ and $B$ be two Cartesian frames defining two Plücker coordinate systems, which we shall also call $A$ and $B$. Let ${}^A\boldsymbol{m}, {}^B\boldsymbol{m}, {}^A\boldsymbol{f}, {}^B\boldsymbol{f} \in \mathsf{R}^6$ be coordinate vectors representing $\boldsymbol{m} \in \mathsf{M}^6$ and $\boldsymbol{f} \in \mathsf{F}^6$ in $A$ and $B$ coordinates, respectively. The coordinate transformation rules for these vectors are as follows:

$$ {}^B\boldsymbol{m} = {}^B\boldsymbol{X}_A \, {}^A\boldsymbol{m}, $$

and

$$ {}^B\boldsymbol{f} = {}^B\boldsymbol{X}_A^* \, {}^A\boldsymbol{f}, $$

where ${}^B\boldsymbol{X}_A$ is the coordinate transformation matrix from $A$ to $B$ coordinates for motion vectors, and ${}^B\boldsymbol{X}_A^*$ is the corresponding matrix for force vectors. The two are related by

$$ {}^B\boldsymbol{X}_A^* = ({}^B\boldsymbol{X}_A)^{-\mathrm{T}}, $$

[cf. (2)]. The formulae for ${}^B\boldsymbol{X}_A$ and ${}^B\boldsymbol{X}_A^*$ depend only on the location of $B$ relative to $A$ and are given by

$$ {}^B\boldsymbol{X}_A = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} \\ -\boldsymbol{r}\times & \boldsymbol{1} \end{bmatrix}, \tag{11} $$

and

$$ {}^B\boldsymbol{X}_A^* = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{1} & -\boldsymbol{r}\times \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}. \tag{12} $$

In these equations, $\boldsymbol{E}$ is the coordinate transform from $\mathcal{C}_A$ to $\mathcal{C}_B$ (the Cartesian coordinate systems defined by frames $A$ and $B$), and $\boldsymbol{r}$ locates the origin of frame $B$ in $\mathcal{C}_A$ coordinates (see Figure 2).

The symbols $\boldsymbol{0}$ and $\boldsymbol{1}$ denote zero and identity matrices of appropriate dimensions, and the expression $\boldsymbol{r}\times$ is the skew-symmetric matrix

$$ \boldsymbol{r}\times = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \times = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}, \tag{13} $$

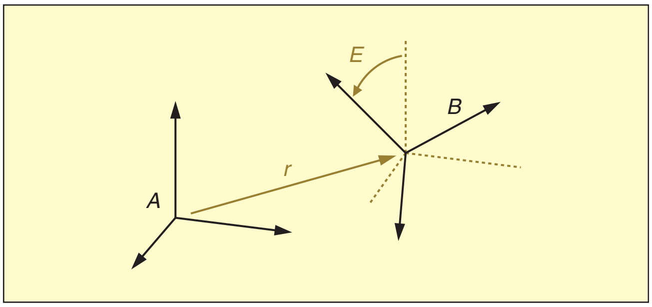


***Figure 2.*** *Location of frame B relative to A.*

which maps any Euclidean vector $\boldsymbol{v}$ to the vector product $\boldsymbol{r} \times \boldsymbol{v}$. (It is the same idea as the matrix $\tilde{\boldsymbol{u}}$ that we encountered in the 3-D example.)

## Differentiation

Spatial vectors are differentiated in the same way as any other vector, namely

$$ \frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{s}(x) = \lim_{\delta x \to 0} \frac{\boldsymbol{s}(x+\delta x) - \boldsymbol{s}(x)}{\delta x}. \tag{14} $$

The derivative of a motion vector is a motion vector, and the derivative of a force vector is a force vector. Unfortunately, the situation gets a little more complicated when we come to coordinate vectors because of the need to distinguish between the derivative of a coordinate vector and the coordinate vector representing a derivative.

To clarify this distinction, let $\boldsymbol{m} \in \mathsf{M}^6$ be a motion vector, and let ${}^A\boldsymbol{m} \in \mathsf{R}^6$ be a coordinate vector representing $\boldsymbol{m}$ in $A$ coordinates. We can now identify the following quantities:

- $d\boldsymbol{m}/dx$: the derivative of $\boldsymbol{m}$
- ${}^A(d\boldsymbol{m}/dx)$: the vector representing $d\boldsymbol{m}/dx$ in $A$ coordinates
- $d\,{}^A\boldsymbol{m}/dx$: the derivative of the coordinate vector ${}^A\boldsymbol{m}$.

The derivative of a coordinate vector is always its component-wise derivative. If the basis vectors do not vary with $x$, then we have ${}^A(d\boldsymbol{m}/dx) = d\,{}^A\boldsymbol{m}/dx$ (and similarly for forces); otherwise, these quantities will differ by a term depending on the derivatives of the basis vectors.

For the special case of a time derivative in a moving Plücker coordinate system, we have the following formulae:

$$ {}^A\!\left(\frac{d\boldsymbol{m}}{dt}\right) = \frac{d}{dt}{}^A\boldsymbol{m} + {}^A\boldsymbol{v}_A \times {}^A\boldsymbol{m}, \tag{15} $$

and

$$ {}^A\!\left(\frac{d\boldsymbol{f}}{dt}\right) = \frac{d}{dt}{}^A\boldsymbol{f} + {}^A\boldsymbol{v}_A \times^* {}^A\boldsymbol{f}. \tag{16} $$

In these equations, $A$ is both the name of a Plücker coordinate system and the name of the frame that defines it, while ${}^A\boldsymbol{v}_A$ is the velocity of frame $A$ expressed in $A$ coordinates. These equations introduce two new operators, $\times$ and $\times^*$, which are the spatial-vector equivalents of the cross-product operator

*If a rigid body has a velocity of $v$ and an inertia of $I$, then its momentum is $Iv$.*

for 3-D Euclidean vectors appearing in (13). They are defined (in Plücker coordinates) as follows:

$$\hat{\boldsymbol{v}}\times = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix}\times = \begin{bmatrix} \boldsymbol{\omega}\times & \boldsymbol{0} \\ \boldsymbol{v}_O\times & \boldsymbol{\omega}\times \end{bmatrix}, \tag{17}$$

and

$$\hat{\boldsymbol{v}}\times^* = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix}\times^* = \begin{bmatrix} \boldsymbol{\omega}\times & \boldsymbol{v}_O\times \\ \boldsymbol{0} & \boldsymbol{\omega}\times \end{bmatrix} = -\hat{\boldsymbol{v}}\times^{\mathrm{T}}. \tag{18}$$

These operators share many properties with their 3-D counterpart, which can be deduced from their definitions; e.g., $\hat{\boldsymbol{v}}\times\hat{\boldsymbol{v}} = \boldsymbol{0}$. The reason why there are two operators is because one ($\hat{\boldsymbol{v}}\times$) arises from the motion of the motion basis vectors, while the other ($\hat{\boldsymbol{v}}\times^*$) arises from the motion of the force basis vectors. The operator $\hat{\boldsymbol{v}}\times$ acts on a motion vector, producing a motion-vector result, and the operator $\hat{\boldsymbol{v}}\times^*$ acts on a force vector, producing a force-vector result.

A useful corollary of (15) and (16) is that if $\boldsymbol{m}$ and $\boldsymbol{f}$ are fixed in a body $B$ and are varying only because $B$ is in motion, then

$$\dot{\boldsymbol{m}} = \boldsymbol{v}\times\boldsymbol{m}, \tag{19}$$

and

$$\dot{\boldsymbol{f}} = \boldsymbol{v}\times^*\boldsymbol{f}, \tag{20}$$

where $\boldsymbol{v}$ is the velocity of $B$. A similar formula for rigid-body inertia appears in (27). If $\boldsymbol{s}\in\mathsf{M}^6$ denotes a revolute or prismatic joint axis that is fixed in body $B$, then $\dot{\boldsymbol{s}} = \boldsymbol{v}\times\boldsymbol{s}$.

### Acceleration

Spatial acceleration is just the time derivative of spatial velocity. However, that seemingly innocuous definition contains a surprise, as we shall now discover. Let $O$ be a fixed point in space, and let $B$ be a rigid body whose spatial velocity is given by $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$ at $O$. Let $O'$ be a body-fixed point that happens to coincide with $O$ at the current instant (time $t$), and let $\boldsymbol{r} = \overrightarrow{OO'}$. Thus, $\boldsymbol{r} = \boldsymbol{0}$ at time $t$, but $\boldsymbol{r}\neq\boldsymbol{0}$ in general. As $O$ is stationary, the velocity and acceleration of $O'$ are given by

$$\boldsymbol{v}_{O'} = \dot{\boldsymbol{r}},$$

and

$$\dot{\boldsymbol{v}}_{O'} = \ddot{\boldsymbol{r}}.$$

Now, the relationship between $\boldsymbol{v}_O$ and $\boldsymbol{v}_{O'}$ is

$$\boldsymbol{v}_O = \boldsymbol{v}_{O'} - \boldsymbol{\omega}\times\boldsymbol{r},$$

so,

$$\dot{\boldsymbol{v}}_O = \dot{\boldsymbol{v}}_{O'} - \dot{\boldsymbol{\omega}}\times\boldsymbol{r} - \boldsymbol{\omega}\times\dot{\boldsymbol{r}}.$$

Therefore, at the current instant (where $\boldsymbol{r} = \boldsymbol{0}$), we have

$$\begin{aligned} \boldsymbol{v}_{O'} &= \dot{\boldsymbol{r}}, & \boldsymbol{v}_O &= \dot{\boldsymbol{r}}, \\ \dot{\boldsymbol{v}}_{O'} &= \ddot{\boldsymbol{r}}, & \dot{\boldsymbol{v}}_O &= \ddot{\boldsymbol{r}} - \boldsymbol{\omega}\times\dot{\boldsymbol{r}}. \end{aligned} \tag{21}$$

The formula for spatial acceleration is therefore

$$\hat{\boldsymbol{a}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \ddot{\boldsymbol{r}} - \boldsymbol{\omega}\times\dot{\boldsymbol{r}} \end{bmatrix}. \tag{22}$$

We can explain this in words as follows. $\dot{\boldsymbol{r}}$ is the velocity of a particular body-fixed particle, but $\boldsymbol{v}_O$ is the velocity measured at $O$ of the stream of body-fixed particles passing through $O$. Likewise, $\ddot{\boldsymbol{r}}$ is the acceleration of a particular body-fixed particle, but $\dot{\boldsymbol{v}}_O$ is the rate of change in the velocity at which successive body-fixed particles stream through $O$.

Despite the slightly greater complexity of (22) compared with the classical description of rigid-body acceleration using $\dot{\boldsymbol{\omega}}$ and $\ddot{\boldsymbol{r}}$, spatial acceleration is significantly easier to use. For example, spatial accelerations can be summed like velocities, and they obey the same coordinate-transformation rule. To take another example, if two bodies $B_1$ and $B_2$ are connected by a joint such that the two velocities obey

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{s}\,\dot{q},$$

where $\boldsymbol{s}$ describes the joint axis and $\dot{q}$ is a joint velocity variable, then the relationship between their accelerations is obtained immediately by differentiating the velocity equation:

$$\boldsymbol{a}_2 = \boldsymbol{a}_1 + \dot{\boldsymbol{s}}\,\dot{q} + \boldsymbol{s}\,\ddot{q}.$$

If $\boldsymbol{s}$ describes a joint axis that is fixed in $B_2$, then $\dot{\boldsymbol{s}} = \boldsymbol{v}_2\times\boldsymbol{s}$, which implies that $\dot{\boldsymbol{s}}\,\dot{q} = \boldsymbol{v}_1\times\boldsymbol{v}_2$. (Can you prove this?) The equivalent 3-D-vector equations are significantly more complicated. For more on the topic of spatial acceleration see, [5] and [7].

### Inertia

The spatial inertia of a rigid body is a tensor that maps its velocity to its momentum (which is a force vector). If a body has an inertia of $\boldsymbol{I}$ and a velocity of $\boldsymbol{v}$, then its momentum, $\boldsymbol{h}\in\mathsf{F}^6$, is

$$\boldsymbol{h} = \boldsymbol{I}\boldsymbol{v}. \tag{23}$$

If rigid bodies $B_1\cdots B_N$ are rigidly connected together to form a single composite rigid body, then the inertia of the composite is

$$\boldsymbol{I}_{\mathrm{tot}} = \sum_{i=1}^{N}\boldsymbol{I}_i, \tag{24}$$

where $\boldsymbol{I}_i$ is the inertia of $B_i$.

Expressed in any dual coordinate system, spatial inertia is a symmetric, positive-definite matrix (or, in special circumstances, positive semidefinite). Expressed in Plücker coordinates, the spatial inertia of a rigid body is

$$\boldsymbol{I} = \begin{bmatrix} \bar{\boldsymbol{I}}_c + m\, \boldsymbol{c}\times \boldsymbol{c}\times^{\mathrm{T}} & m\, \boldsymbol{c}\times \\ m\, \boldsymbol{c}\times^{\mathrm{T}} & m\, \mathbf{1} \end{bmatrix}, \tag{25}$$

where $m$ is the body's mass, $\boldsymbol{c}$ is a 3-D vector locating the body's center of mass, and $\bar{\boldsymbol{I}}_c$ is the body's rotational inertia about its center of mass. Observe that a rigid-body inertia is a function of ten parameters: one in $m$, three in $\boldsymbol{c}$, and six in $\bar{\boldsymbol{I}}_c$. More general kinds of spatial inertia, such as articulated-body and operational-space inertia, do not have the special form shown in (25), and they can be functions of up to 21 independent parameters.

All spatial inertias, whether rigid or not, obey the following coordinate-transformation rule:

$${}^{B}\boldsymbol{I} = {}^{B}\boldsymbol{X}_A^{*}\, {}^{A}\boldsymbol{I}\, {}^{A}\boldsymbol{X}_B. \tag{26}$$

This formula is valid for any dual coordinate system, not only Plücker coordinates. If a rigid body has a velocity of $\boldsymbol{v}$ and an inertia of $\boldsymbol{I}$, then the time derivative of its inertia is

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{I} = \boldsymbol{v}\times^{*}\boldsymbol{I} - \boldsymbol{I}\boldsymbol{v}\times. \tag{27}$$

Another useful equation is

$$E = \frac{1}{2}\, \boldsymbol{v}\cdot \boldsymbol{I}\boldsymbol{v}, \tag{28}$$

which gives the kinetic energy of a rigid body.

### *Equation of Motion*

Expressed in spatial form, the equation of motion for a rigid body having a velocity of $\boldsymbol{v}$ and an inertia of $\boldsymbol{I}$ is

$$\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t}\, (\boldsymbol{I}\boldsymbol{v}) = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{v}\times^{*}\boldsymbol{I}\boldsymbol{v}, \tag{29}$$

where $\boldsymbol{f}$ is the total force acting on the body, and $\boldsymbol{a}$ is the resulting acceleration. [Can you verify this equation using (27)?] In words, it says that the total force acting on a rigid body equals its rate of change of momentum. This equation incorporates both Newton's equation applied to the center of mass and Euler's equation for the rotation of the body about its center of mass.

It is often useful to write the equation of motion in the following simplified form:

$$\boldsymbol{f} = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{p}, \tag{30}$$

where $\boldsymbol{p} \in \mathsf{F}^6$ is called a *bias force*. There are two main reasons why you might want to do this. First, the algebraic form of this equation is identical to the algebraic form of several other important equations of motion, such as the articulated-body equation of motion and the equation of motion of a rigid body expressed in generalized coordinates. Second, it offers the opportunity to split $\boldsymbol{f}$ into a known part and an unknown part, and incorporate the former into $\boldsymbol{p}$. For example, if the forces acting on the body consisted of an unknown force and a gravitational force, you could define $\boldsymbol{f}$ in (30) to be the unknown force and define $\boldsymbol{p}$ as follows:

$$\boldsymbol{p} = \boldsymbol{v}\times^{*}\boldsymbol{I}\boldsymbol{v} - \boldsymbol{f}_{\mathrm{g}},$$

where $\boldsymbol{f}_{\mathrm{g}}$ is the gravitational force. Incidentally, if $\boldsymbol{a}_{\mathrm{g}}$ is the acceleration due to gravity (in a uniform gravitational field), then the force of gravity acting on a rigid body with inertia $\boldsymbol{I}$ is

$$\boldsymbol{f}_{\mathrm{g}} = \boldsymbol{I}\boldsymbol{a}_{\mathrm{g}}.$$

**A constraint force does no work in any direction of motion permitted by the constraint.**

## Motion Constraints

In the simplest case, a motion constraint between two rigid bodies, $B_1$ and $B_2$, restricts their relative velocity to a vector subspace $S \subseteq \mathsf{M}^6$, which can vary with time. If $r$ is the dimension of $S$, then the constraint allows $r$ degrees of relative motion freedom between the two bodies, and consequently imposes $6 - r$ constraints. If $\boldsymbol{s}_1 \cdots \boldsymbol{s}_r$ are any set of vectors that span $S$ (i.e., they form a basis on $S$), then the relative velocity can be expressed in the form

$$\boldsymbol{v}_{\mathrm{rel}} = \boldsymbol{v}_2 - \boldsymbol{v}_1 = \sum_{i=1}^{r} \boldsymbol{s}_i\, \dot{q}_i,$$

where $\dot{q}_i$ are a set of velocity variables. However, we usually collect the vectors together into a single $6 \times r$ matrix $\boldsymbol{S}$, and express the relative velocity as follows:

$$\boldsymbol{v}_2 - \boldsymbol{v}_1 = \boldsymbol{S}\dot{\boldsymbol{q}}, \tag{31}$$

where $\dot{\boldsymbol{q}}$ is an $r$-dimensional coordinate vector containing the velocity variables. To obtain a constraint on the relative acceleration, we simply differentiate this equation, giving

$$\boldsymbol{a}_2 - \boldsymbol{a}_1 = \dot{\boldsymbol{S}}\dot{\boldsymbol{q}} + \boldsymbol{S}\ddot{\boldsymbol{q}}. \tag{32}$$

In a typical dynamics problem, the quantities $\dot{\boldsymbol{S}}$, $\dot{\boldsymbol{q}}$, and $\boldsymbol{S}$ would all be known, and $\ddot{\boldsymbol{q}}$ would be unknown. Expressions for $\boldsymbol{S}$ and $\dot{\boldsymbol{S}}$ will depend on the type of constraint. If the component vectors of $\boldsymbol{S}$ are fixed in body $i$ ($i = 1$ or 2), then $\dot{\boldsymbol{S}} = \boldsymbol{v}_i \times \boldsymbol{S}$.

Motion constraints are implemented by constraint forces, and constraint forces all have the following special property: a constraint force does no work in any direction of motion permitted by the constraint.

This is simply a statement of D'Alembert's principle of virtual work, or Jourdain's principle of virtual power, depending on whether you interpret motion to mean infinitesimal displacement or velocity.

Let $\boldsymbol{f}_c$ be the constraint force implementing the above motion constraint. As the relative motion has been defined to be the motion of $B_2$ relative to $B_1$, so we must define $\boldsymbol{f}_c$ to be a force transmitted from $B_1$ to $B_2$ (i.e., $\boldsymbol{f}_c$ acts on $B_2$ and $-\boldsymbol{f}_c$ acts on $B_1$). To comply with D'Alembert's principle, $\boldsymbol{f}_c$ must satisfy $\boldsymbol{f}_c \cdot \boldsymbol{s}_i = 0$ for all $i$. Therefore, $\boldsymbol{f}_c$ must satisfy

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}_c = \boldsymbol{0}. \tag{33}$$

Equations (32) and (33) together define the motion constraint between $B_2$ and $B_1$. To model a powered joint, we replace (33) with

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}_{\mathrm{J}} = \boldsymbol{\tau}, \tag{34}$$

where $\boldsymbol{f}_{\mathrm{J}}$ is the total force transmitted across the joint—the sum of an active force and a constraint force—and $\boldsymbol{\tau}$ is a vector of generalized force variables. The elements of $\boldsymbol{\tau}$ must be defined such that $\boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{q}}$ is the instantaneous power delivered by the joint to the system.

## Further Reading

Spatial vectors are described in detail in [7] and in somewhat less detail in [8]. An older version of spatial vectors is described in [4]. Web-based materials are available from [9], including a slide show, a set of exercises with answers, the two examples appearing at the beginning of this tutorial, and software for MATLAB and Octave that implements spatial vector arithmetic and a selection of the most important dynamics algorithms for robotics. Similar materials, plus a lot of materials from other authors on screw theory, are available at [17].

Spatial vectors are closely related to screw theory, to motor algebra, and the Lie algebra se(3). Materials on these topics can be found in [1]–[3], [10]–[12], and [15], [16]. Screw theory emphasizes geometrical aspects of 6-D vectors, expressed in terms of straight lines, pitches (of screws), and magnitudes. Lie algebra takes a more formal approach: se(3) is the tangent space at the identity of the Lie group se(3); so you can expect notions of group theory, manifolds, and tangent spaces to appear. Motor algebra has two forms: one based on real numbers [2], [10], [11] and the other on dual numbers [3]. The latter is not suitable for dynamics because dual numbers don't work with inertias.

One more notation that deserves a mention is the spatial operator algebra of [13], [14]. This notation shows obvious signs of 3-D-vector thinking, but its most important feature is the way the authors have stacked up the 6-D vectors to make $6N$-dimensional vectors and $6N \times 6N$ matrices that describe properties of a whole rigid-body system comprising $N$ bodies.

This concludes part 1 of this tutorial. Part 2 will show how spatial vectors are applied to several standard problems in robot kinematics and dynamics. In particular, it will show how a problem can be solved algebraically using spatial vectors, and the resulting solution translated directly into short, simple computer code for performing the desired calculations.

## Keywords

Robot dynamics, spatial vectors, rigid body dynamics, Plücker coordinates, screw theory.

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