ENPM 667 Final Project

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ENPM 667 Robot Control

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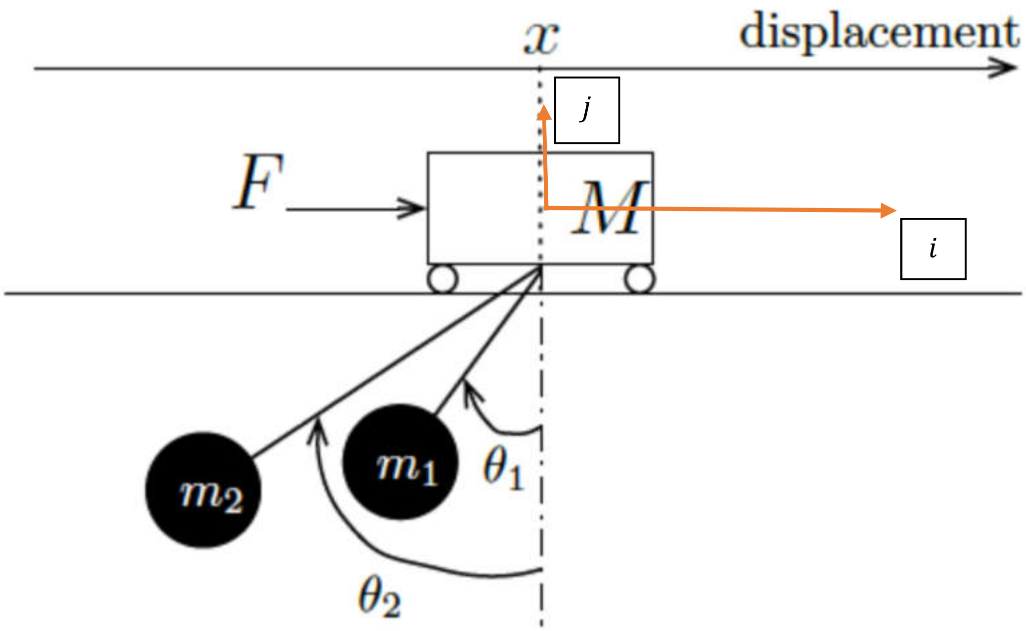


Let

Position of Cart = x,   Mass of Cart = M

Position of m1 = $\left(x - l_1 sin(\theta_1)\right)i + l_1\cos(\theta_1)j$,   Mass of Ball1 =$m_1$

Position of m2 = $\left(x - l_2 sin(\theta_2)\right)i + l_2\cos(\theta_2)j$,   Mass of Ball2 =$m_2$

Then Kinetic Energy of the System is:

$$\boldsymbol{K_M = \frac{1}{2}M\dot{x}^2}$$

$$K_{m_1} = \frac{1}{2}m_1\left(\frac{d}{dt}x - l_1 sin(\theta_1)\right)^2 + \frac{1}{2}m_1\left(\frac{d}{dt}l_1\cos(\theta_1)\right)^2$$

$$K_{m_1} = \frac{1}{2}m_1\left(\dot{x} - l_1\,\dot{\theta}_1\cos(\theta_1)\right)^2 + \frac{1}{2}m_1\left(-l_1\,\dot{\theta}_1\sin(\theta_1)\right)^2$$

$$K_{m_1} = \frac{1}{2}m_1(\dot{x}^2 - 2\dot{x}l_1\,\dot{\theta}_1\cos(\theta_1) + l_1^2\,\dot{\theta}_1^2\cos^2(\theta_1)) + \frac{1}{2}m_1(l_1^2\,\dot{\theta}_1^2\sin^2(\theta_1))$$

$$K_{m_1} = \frac{1}{2}m_1\dot{x}^2 - m_1\dot{x}l_1\,\dot{\theta}_1\cos(\theta_1) + \frac{1}{2}m_1 l_1^2\,\dot{\theta}_1^2(\cos^2(\theta_1) + \sin^2(\theta_1))$$

$$\boldsymbol{K_{m_1} = \frac{1}{2} m_1 \dot{x}^2 - m_1 \dot{x} l_1 \dot{\theta}_1 \cos(\theta_1) + \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2}$$

$$\boldsymbol{K_{m_2} = \frac{1}{2} m_2 \dot{x}^2 - m_2 \dot{x} l_2 \dot{\theta}_2 \cos(\theta_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2}$$

$P_M = Mg0 = 0$ (Center of Mass is supposed at zero height)

$\boldsymbol{P_{m1} = m_1 g l_1 (1 - \cos(\theta_1))}$ [1] (Energy datum is the lowest most point)

$\boldsymbol{P_{m2} = m_2 g l_2 (1 - \cos(\theta_2))}$

$$\begin{aligned} L &= (K_M + K_{m1} + K_{m2}) - (P_{m1} + P_{m2}) \\ &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m_1 \dot{x}^2 - m_1 \dot{x} l_1 \dot{\theta}_1 \cos(\theta_1) + \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{x}^2 \\ &\quad - m_2 \dot{x} l_2 \dot{\theta}_2 \cos(\theta_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 - m_1 g l_1 (1 - \cos(\theta_1)) - m_2 g l_2 (1 \\ &\quad - \cos(\theta_2)) \end{aligned}$$

$$\boldsymbol{\begin{aligned} L = {} & \frac{1}{2} \dot{x}^2 (M + m_1 + m_2) - m_1 \dot{x} l_1 \dot{\theta}_1 \cos(\theta_1) - m_2 \dot{x} l_2 \dot{\theta}_2 \cos(\theta_2) + \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \\ & + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 - m_1 g l_1 (1 - \cos(\theta_1)) - m_2 g l_2 (1 - \cos(\theta_2)) \end{aligned}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = \dot{x}(M + m_1 + m_2) - m_1 l_1 \dot{\theta}_1 \cos(\theta_1) - m_2 l_2 \dot{\theta}_2 \cos(\theta_2)$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \ddot{x}(M + m_1 + m_2) - m_1 l_1 \ddot{\theta}_1 \cos(\theta_1) + m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1)}$$
$$\boldsymbol{- m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)}$$

$$\frac{\partial L}{\partial \theta_1} = m_1 \dot{x} l_1 \dot{\theta}_1 \sin(\theta_1) - m_1 g l_1 \sin(\theta_1)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = -m_1 \dot{x} l_1 \cos(\theta_1) + m_1 l_1^2 \dot{\theta}_1$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = -m_1 \ddot{x} l_1 \cos(\theta_1) + m_1 \dot{x} l_1 \dot{\theta}_1 \sin(\theta_1) + m_1 l_1^2 \ddot{\theta}_1}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 \dot{x} l_2 \dot{\theta}_2 \sin(\theta_2) - m_2 g l_2 \sin(\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = -m_2 \dot{x} l_2 \cos(\theta_2) + m_2 l_2^2 \dot{\theta}_2$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = -m_2 \ddot{x} l_2 \cos(\theta_2) + m_2 \dot{x} l_2 \dot{\theta}_2 \sin(\theta_2) + m_2 l_2^2 \ddot{\theta}_2}$$

Eq1

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}$$
$$= \boldsymbol{\ddot{x}(M+m_1+m_2)-m_1 l_1\,\ddot{\theta}_1\cos(\theta_1)+m_1 l_1\,\dot{\theta}_1^2\sin(\theta_1)}$$
$$\boldsymbol{-\,m_2 l_2\,\ddot{\theta}_2\cos(\theta_2)+m_2 l_2\,\dot{\theta}_2^2\sin(\theta_2)=F}$$

Eq2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=$$
$$=\left(-m_1\ddot{x}l_1\cos(\theta_1)+m_1\dot{x}l_1\dot{\theta}_1\sin(\theta_1)+m_1 l_1^2\ddot{\theta}_1\right)$$
$$-\left(m_1\dot{x}l_1\,\dot{\theta}_1\sin(\theta_1)-m_1 g l_1\sin(\theta_1)\right)$$
$$=\boldsymbol{-m_1\ddot{x}l_1\cos(\theta_1)+m_1 l_1^2\ddot{\theta}_1+m_1 g l_1\sin(\theta_1)=0}$$

Eq3

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}$$
$$=\left(-m_2\ddot{x}l_2\cos(\theta_2)+m_2\dot{x}l_2\dot{\theta}_2\sin(\theta_2)+m_2 l_2^2\ddot{\theta}_2\right)$$
$$-\left(m_2\dot{x}l_2\,\dot{\theta}_2\sin(\theta_2)-m_2 g l_2\sin(\theta_2)\right)$$
$$=\boldsymbol{-m_2\ddot{x}l_2\cos(\theta_2)+m_2 l_2^2\ddot{\theta}_2+m_2 g l_2\sin(\theta_2)=0}$$

Isolate double derivatives on the left-hand side

Eq2

$$m_1 l_1^2\ddot{\theta}_1=-m_1 g l_1\sin(\theta_1)+m_1\ddot{x}l_1\cos(\theta_1)$$
$$\boldsymbol{\ddot{\theta}_1=\frac{-m_1 g l_1\sin(\theta_1)+m_1\ddot{x}l_1\cos(\theta_1)}{m_1 l_1^2}}$$

Eq3

$$m_2 l_2^2\ddot{\theta}_2=-m_2 g l_2\sin(\theta_2)+m_2\ddot{x}l_2\cos(\theta_2)$$
$$\boldsymbol{\ddot{\theta}_2=\frac{-m_2 g l_2\sin(\theta_2)+m_2\ddot{x}l_2\cos(\theta_2)}{m_2 l_2^2}}$$

Substitute Eq3 & Eq2 in Eq1

$$F = \ddot{x}(M + m_1 + m_2) - m_1 l_1 (\frac{-m_1 g l_1 \sin(\theta_1) + m_1 \ddot{x} l_1 \cos(\theta_1)}{m_1 l_1^2}) \cos(\theta_1) + m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 l_2 \frac{-m_2 g l_2 \sin(\theta_2) + m_2 \ddot{x} l_2 \cos(\theta_2)}{m_2 l_2^2} \cos(\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$

$$F = \ddot{x}(M + m_1 + m_2) - \frac{-m_1 g l_1 \sin(\theta_1) + m_1 \ddot{x} l_1 \cos(\theta_1)}{l_1} \cos(\theta_1) + m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - \frac{-m_2 g l_2 \sin(\theta_2) + m_2 \ddot{x} l_2 \cos(\theta_2)}{l_2} \cos(\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$

$$F = \ddot{x}(M + m_1 + m_2) - \cos(\theta_1)(-m_1 g \sin(\theta_1) + m_1 \ddot{x} \cos(\theta_1)) + m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - \cos(\theta_2)(-m_2 g l_2 \sin(\theta_2) + m_2 \ddot{x} l_2 \cos(\theta_2)) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$

$$-\ddot{x}(M + m_1 + m_2) + \cos(\theta_1)(-m_1 g \sin(\theta_1) + m_1 \ddot{x} \cos(\theta_1)) + \cos(\theta_2)(-m_2 g l_2 \sin(\theta_2) + m_2 \ddot{x} l_2 \cos(\theta_2)) = -F + m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$

$$\ddot{x}(M + m_1 + m_2) - \cos(\theta_1)(-m_1 g \sin(\theta_1) + m_1 \ddot{x} \cos(\theta_1)) - \cos(\theta_2)(-m_2 g l_2 \sin(\theta_2) + m_2 \ddot{x} l_2 \cos(\theta_2)) = +F - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$

$$\ddot{x}(M + m_1 + m_2 - m_1 \cos^2(\theta_1) - m_2 \cos^2(\theta_2)) = F - m_1 g \sin(\theta_1) cos(\theta_1) - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 g \sin(\theta_2) cos(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$

$$\boldsymbol{\ddot{x} = \frac{F - m_1 g \sin(\theta_1) cos(\theta_1) - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 g \sin(\theta_2) cos(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)}{M + m_1 + m_2 - m_1 \cos^2(\theta_1) - m_2 \cos^2(\theta_2)}}$$

Now substitute $\ddot{x}$ in Eq2 & Eq3 to get $\ddot{\theta}_1$ & $\ddot{\theta}_2$

$$m_1 l_1^2 \ddot{\theta}_1 = -m_1 g l_1 \sin(\theta_1) + m_1 \ddot{x} l_1 \cos(\theta_1)$$

$$l_1 \ddot{\theta}_1 = -g \sin(\theta_1) + \ddot{x} \cos(\theta_1)$$

$$\ddot{\theta}_1 = \frac{-g \sin(\theta_1) + \ddot{x} \cos(\theta_1)}{l_1}$$

$$\ddot{\theta}_1 = \frac{-\frac{F - m_1 g \sin(\theta_1) cos(\theta_1) - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 g \sin(\theta_2) cos(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)}{M + m_1 + m_2 - m_1 \cos^2(\theta_1) - m_2 \cos^2(\theta_2)} \cos(\theta_1)}{l_1} - \frac{g sin(\theta_1)}{l_1}$$

$$\boldsymbol{\ddot{\theta}_1 = -\frac{F \cos(\theta_1) - m_1 g \sin(\theta_1) cos^2(\theta_1) - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) \cos(\theta_1) - m_2 g \sin(\theta_2) cos(\theta_2) \cos(\theta_1) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2) \cos(\theta_1)}{l_1 (M + m_1 + m_2 - m_1 \cos^2(\theta_1) - m_2 \cos^2(\theta_2))} - \frac{g sin(\theta_1)}{l_1}}$$

$$\boldsymbol{\ddot{\theta}_2 = -\frac{F \cos(\theta_2) - m_2 g \sin(\theta_2) cos^2(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2) \cos(\theta_2) - m_1 g \sin(\theta_1) cos(\theta_1) \cos(\theta_2) - m_1 l_2 \dot{\theta}_1^2 \sin(\theta_1) \cos(\theta_2)}{l_2 (M + m_1 + m_2 - m_1 \cos^2(\theta_1) - m_2 \cos^2(\theta_2))} - \frac{g sin(\theta_2)}{l_2}}$$

Linearized Equations

Small angle approximations are true for the region around the equilibrium point where $\theta = 0$. Also, we ignore the higher order terms to achieve a linear system.

$$sin(\theta) \approx \theta, \qquad cos(\theta) \approx 1, \qquad sin^2(\theta) \approx 0, \qquad cos^2(\theta) \approx 1$$

$$\boldsymbol{\ddot{x} = \frac{F - m_1 g \theta_1 - m_2 g \theta_2}{M}}$$

$$\boldsymbol{\ddot{\theta}_1 = -\frac{F - m_1 g \theta_1 - m_2 g \theta_2 - M g \theta_1}{l_1 M}}$$

$$\boldsymbol{\ddot{\theta}_2 = -\frac{F - m_1 g \theta_1 - m_2 g \theta_2 - M g \theta_2}{l_2 M}}$$

State Space of Linearized System

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}, \qquad u = [F]$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{gm_1}{M} & 0 & -\frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g(m_1+M)}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{g(m_2+M)}{Ml_2} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

$$y = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

System is controllable if the controllability matrix's determinant is non-zero

$$\det[B\ AB\ A^2B\ A^3B\ A^4B\ A^5B] = -\frac{g^6(l_1^2 - 2l_1l_2 + l_2^2)}{M^6 l_1^6 l_2^6} \neq 0$$

$$l_1^2 + l_2^2 - 2l_1l_2 = (l_1 - l_2)^2$$

$$l_1 \neq l_2, \qquad l_1 \neq 0, \qquad l_2 \neq 0, \qquad M \neq 0$$

The system is also not controllable in outer space ($gravity = 0$)

Controllability of Linearized System

```
M=1000;
m1=100;
m2=100;
l1=20;
l2=10;
g=9.8;

A=[ 0 1 0 0 0 0
    0 0 (-g*m1)/M 0 (-g*m2)/M 0
    0 0 0 1 0 0
    0 0 (-g*(m1+M))/(M*l1) 0 (-g*m2)/(M*l1) 0
    0 0 0 0 0 1
    0 0 (-g*m1)/(M*l2) 0 (-g*(m2+M))/(M*l2) 0 ]
B=[ 0
    1/M
    0
    1/(M*l1)
    0
    1/(M*l2)]
C=[ 1 0 0 0 0 0
    0 1 0 0 0 0
    0 0 1 0 0 0
    0 0 0 1 0 0
    0 0 0 0 1 0
    0 0 0 0 0 1]
D=zeros(6,1);

controlM = [B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B]
controllability_check1 = det(controlM)
controllability_check2 = rank(controlM)
```

```
controllability_check1 =

  -1.3841e-24
```

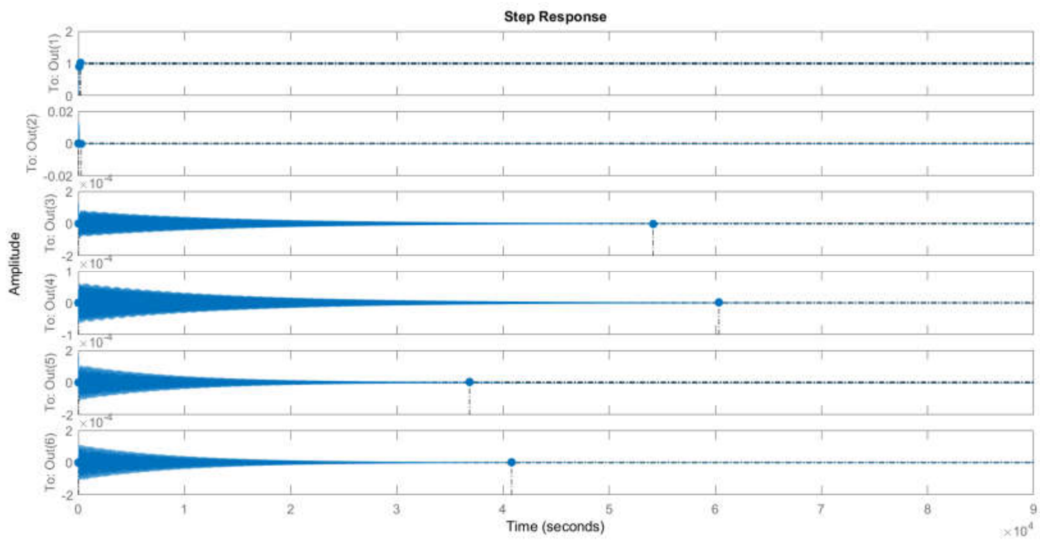
The controllability_check1 is the determinant check, where for a controllable system, the controllability matrix should have non-zero determinant. In this case it returns very small value which is probably due to floating point errors.

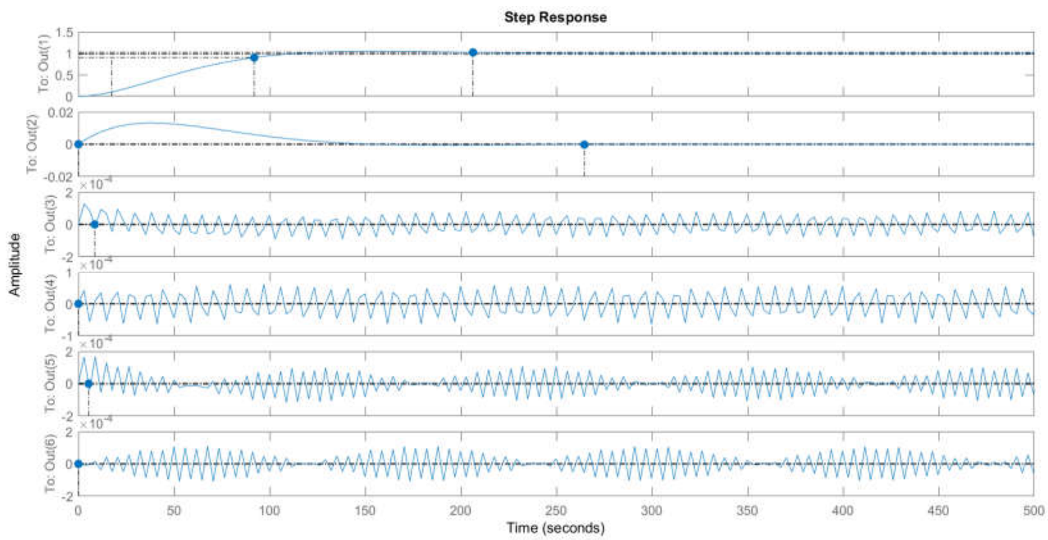
```
controllability_check2 =

     6
```

The controllability_check2 is the rank check for system controllability. It returns a rank 6 for the controllability matrix. This backs our previous conclusion that the system is controllable

LQR Controller – Linear System

```
Q = eye(6,6);                //State Cost Matrix
R = ones(1,1);               //Actuator Cost Matrix
[K,S,E]= lqr(A,B,Q,R)
step(A-B*K,B,C,D)
```

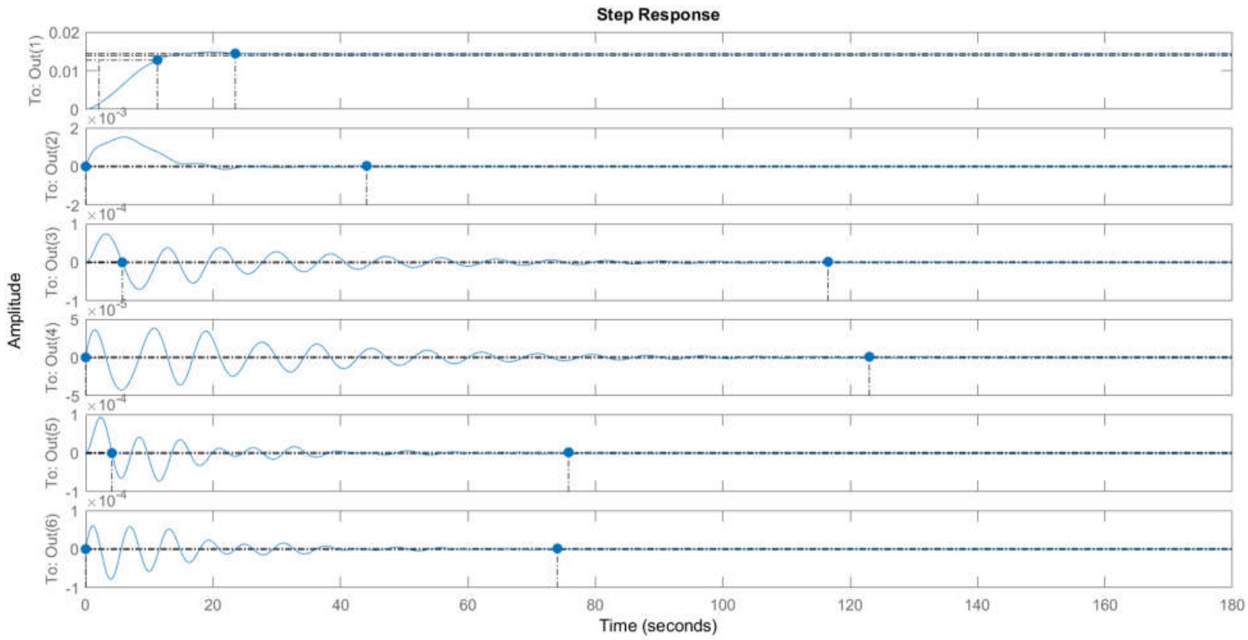


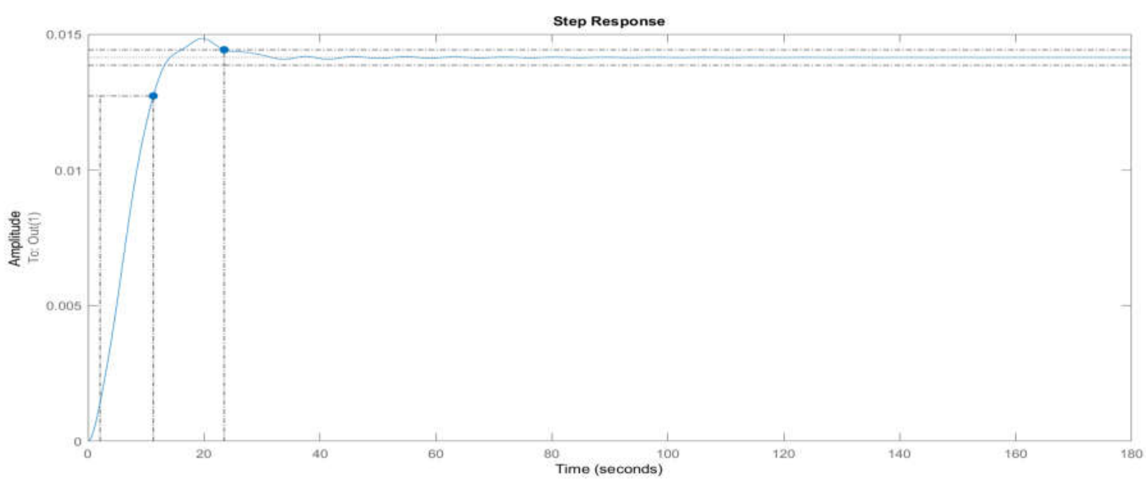


	Settling Time(s)	Rise8 Time(s)
x	208	85
θ_1	5.4×10^4	-
θ_2	3.7×10^4	-

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R = [1]$$

The state cost matrix can be tuned to give more weightage to states which are of more importance. After several iterations, we get the following system response:





	Settling Time(s)	Rise Time(s)
x	23	11
θ_1	107	-
θ_2	76	-

$$Q = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix}, \qquad R = [0.001]$$

LQR Controller – Non-Linear System

We can model the non-linear system in Simulink. This involves entering the non-linearized original equations in function blocks and connecting them appropriately using integrators. The plant model's state variables are fed back negatively with the gain K to the input forming the LQ Regulator. The Simulink diagrams below show the overview of the system.

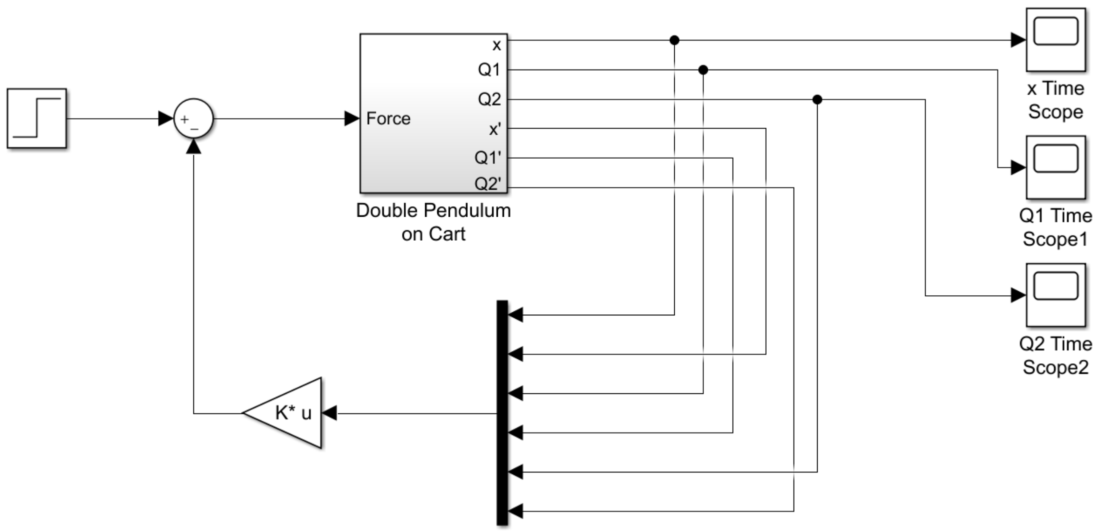


Figure 1 Non-Linear System

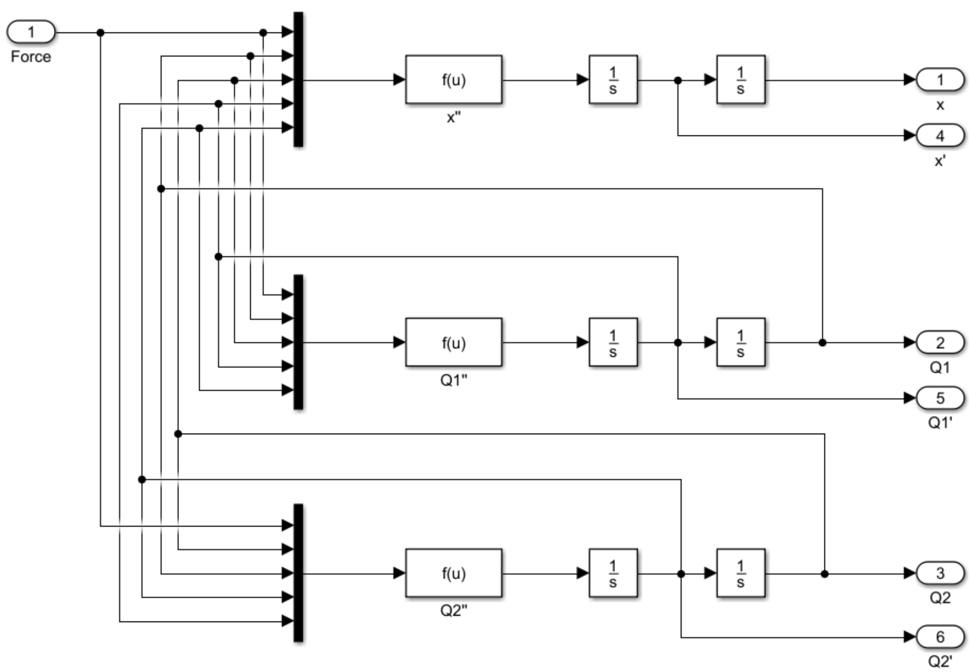


Figure 2 Double Pendulum on Cart - Plant

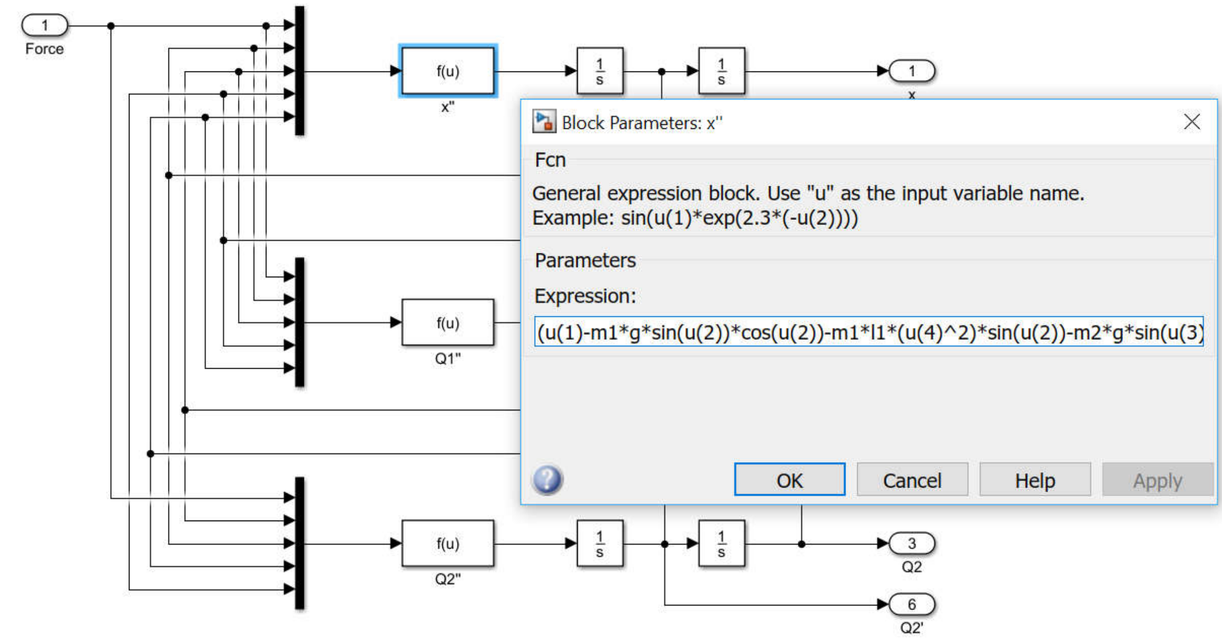
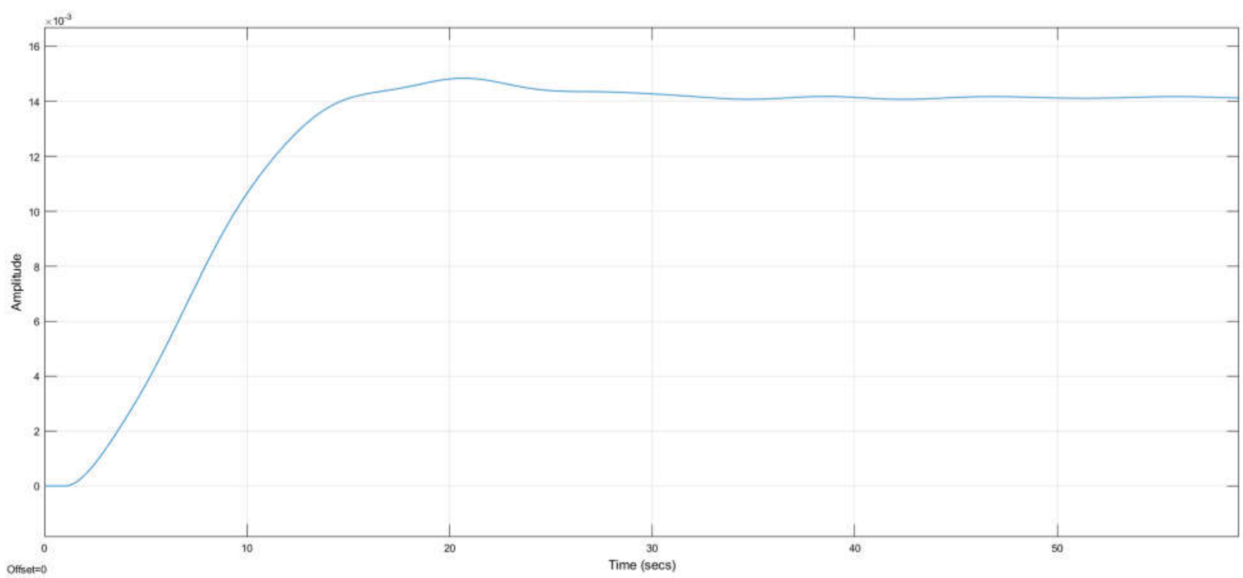
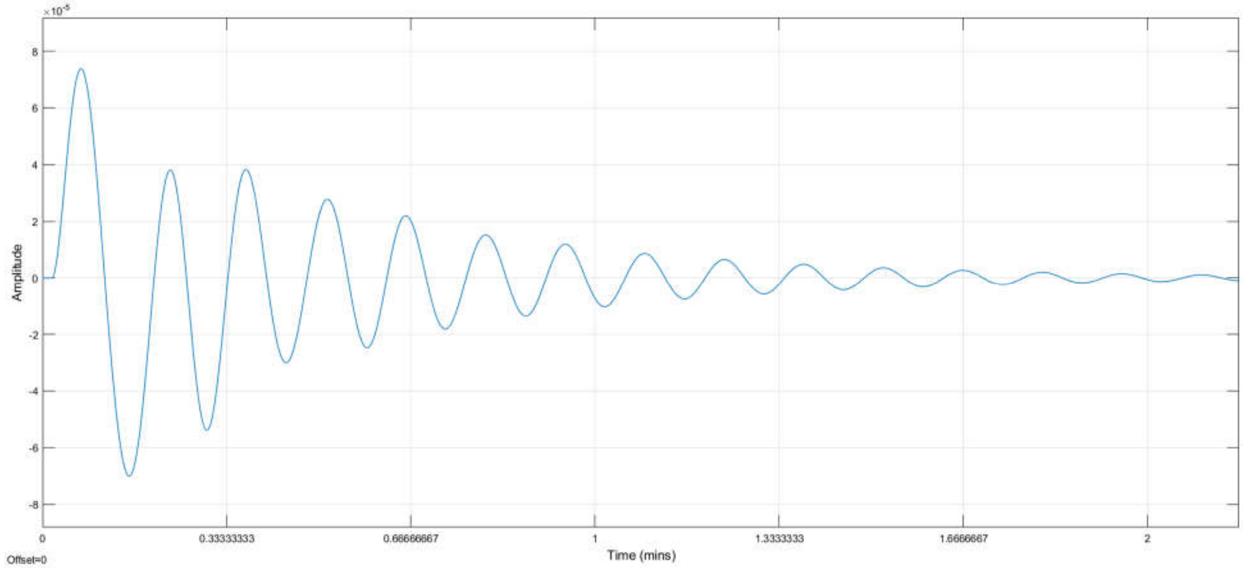
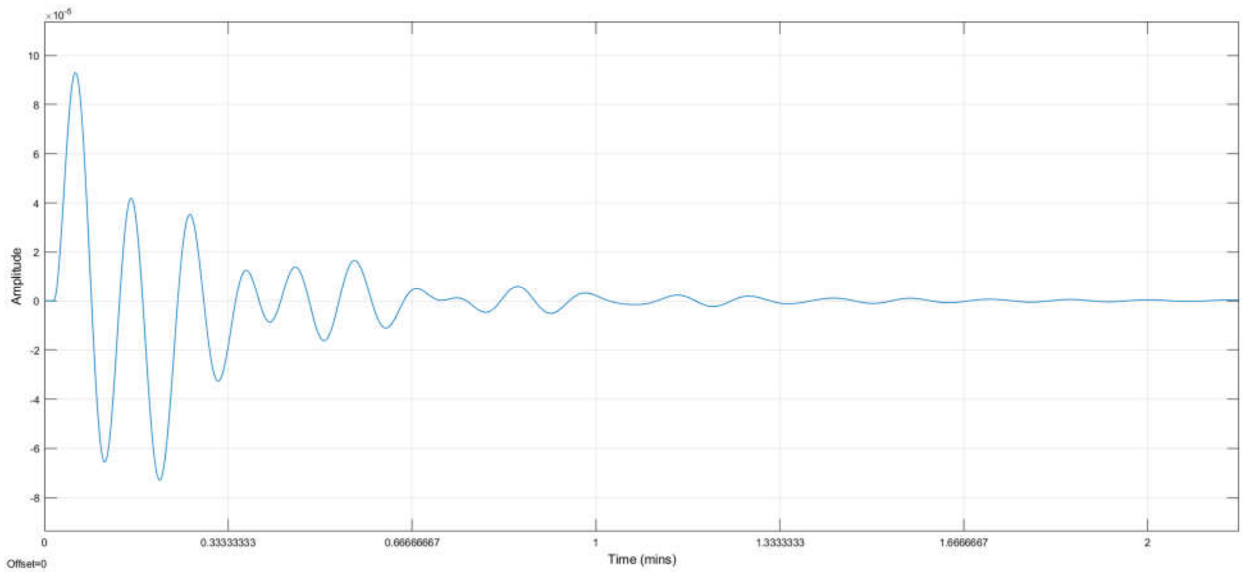


Figure 3 Double Pendulum on Cart - Plant







	Settling Time(s)	Rise Time(s)
x	37	9.2
θ_1	184	-
θ_2	92	-

$$Q = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix}, \qquad R = [0.001]$$

Lyapunov's Indirect Stability method states that if the system linearized at an equilibrium point is stable, then the non-linear original system is **locally stable** around that equilibrium point. Which holds true for this case, and so the system is locally stable

Vector Observability

$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}, \quad \boldsymbol{x} = \left[x\ \dot{x}\ \theta_1\ \dot{\theta}_1\ \theta_2\ \dot{\theta}_2\right]^T, \quad \boldsymbol{D} = \boldsymbol{0}$$

For Output Vector $\boldsymbol{x(t), \theta_1(t), \theta_2(t)}$

```
C = [ 1 0 0 0 0 0 %x Q1 Q2
      0 0 1 0 0 0
      0 0 0 0 1 0];
observe = [C; C*A; C*(A^2); C*(A^3); C*(A^4); C*(A^5)];
Rank = rank(observeM)
Rank =

     6 (Observable)
```

For Output Vector $\boldsymbol{x(t)}$

```
C = [ 1 0 0 0 0 0]; %x
observe = [C; C*A; C*(A^2); C*(A^3); C*(A^4); C*(A^5)] ;
Rank = rank(observeM)
Rank =

     6 (Observable)
```

For Output Vector $\boldsymbol{\theta_1(t), \theta_2(t)}$

```
C = [ 0 0 1 0 0 0 %Q1 Q2
      0 0 0 0 1 0];
observe = [C; C*A; C*(A^2); C*(A^3); C*(A^4); C*(A^5)];
Rank = rank(observeM)
Rank =

     4 (Not Observable)
```

For Output Vector $\boldsymbol{x(t), \theta_2(t)}$

```
C = [ 1 0 0 0 0 0 %x Q2
      0 0 0 0 1 0];
observe = [C; C*A; C*(A^2); C*(A^3); C*(A^4); C*(A^5)];
Rank = rank(observeM)
Rank =

     6 (Observable)
```

Luenberger State Observer – Linear System

The gain L is chosen such that the matrix $A - LC$ has eigenvalues in the left half-plane. Further, the exact eigenvalues of $A - LC$ govern the rate at which the state estimate (x) converges to the actual state (x) of the system. It is normally desired that the observer estimate of the state converges to the actual state at least an order of magnitude faster than the performance desired of the system. This helps the controller in obtaining a "good" estimate of the actual state of the system in relatively short time and thus it can take appropriate control action. [2]

So we need to make sure that the poles of the observer are considerably faster than the controller. Poles of the controller are:

```
>> eig(A-B*K)

=
  -0.1723 + 0.1701i
  -0.1723 - 0.1701i
  -0.0740 + 1.0423i
  -0.0740 - 1.0423i
  -0.0339 + 0.7275i
  -0.0339 - 0.7275i
```

Therefore, the poles of the observer are selected to be :

```
% Poles for Luenberger Observer
P=[-2 -3 -4 -5 -6 -7]

% Compute Observer Gain Matrix
L=place(A',C',P)'

% Closed Loop System with Luenberger Observer
A_Luen = [(A-B*K)          (B*K)
            zeros(size(A)) (A-L*C)];
B_Luen = [B
            zeros(size(B))];
C_Luen = [C zeros(size(C))];
D_luen = [zeros(3,1)];

% Step Response of Closed Loop System with Luenberger Observer
step(A_luen,B_Luen,C_Luen,D_Luen)
```

There is an alternative way to construct the linear system luenberger observer in a graphical mode using Simulink as follows (it utilizes the matrices from MATLAB workspace):

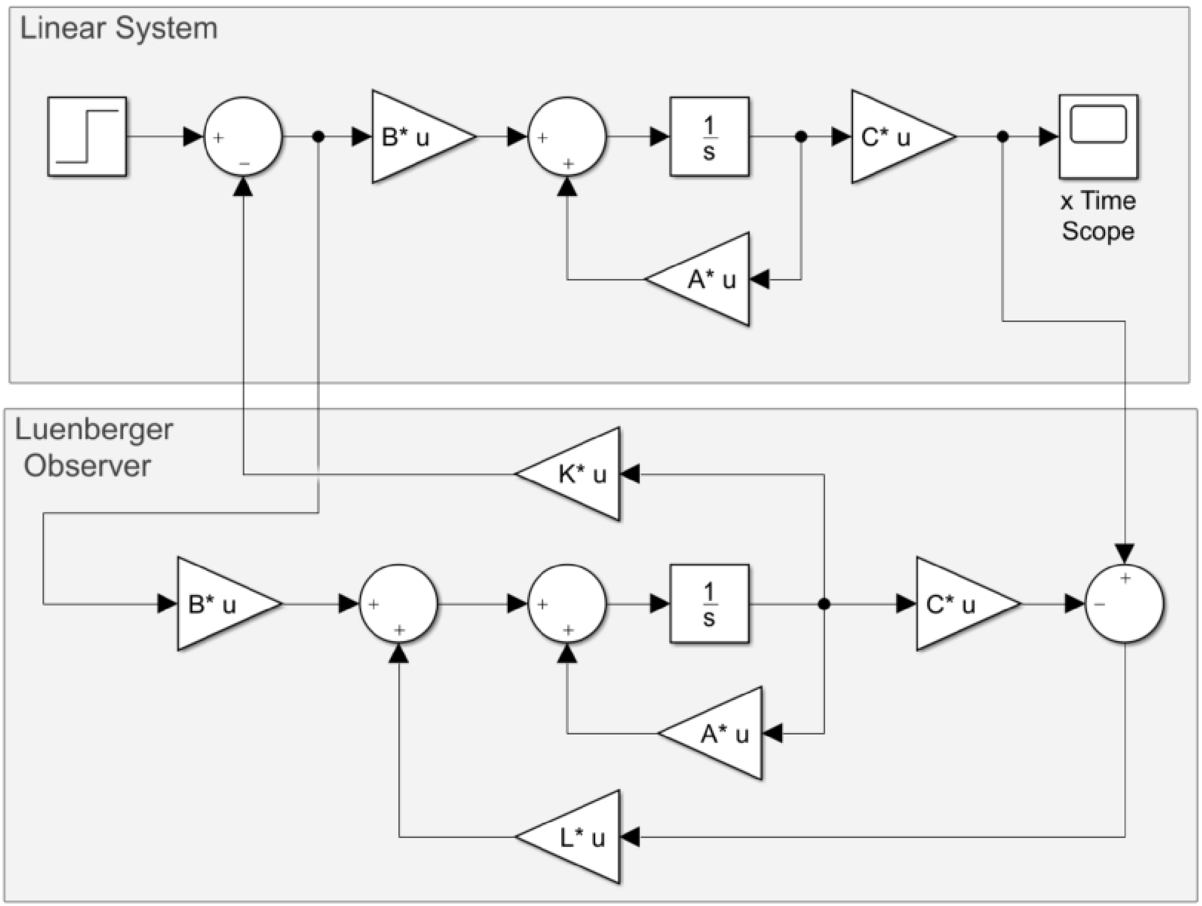
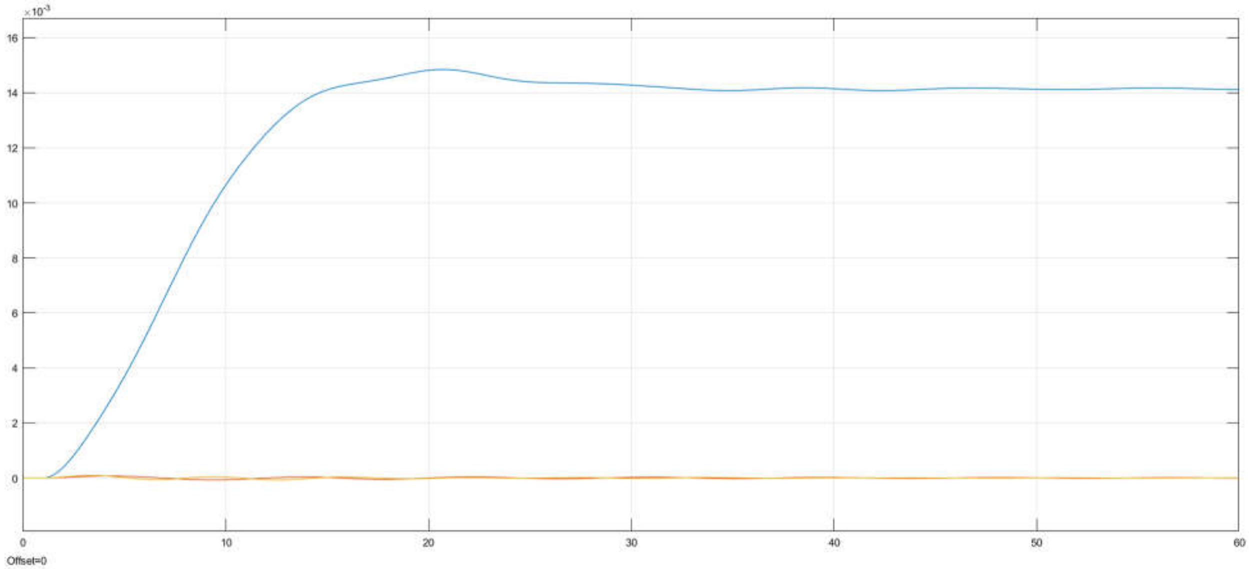
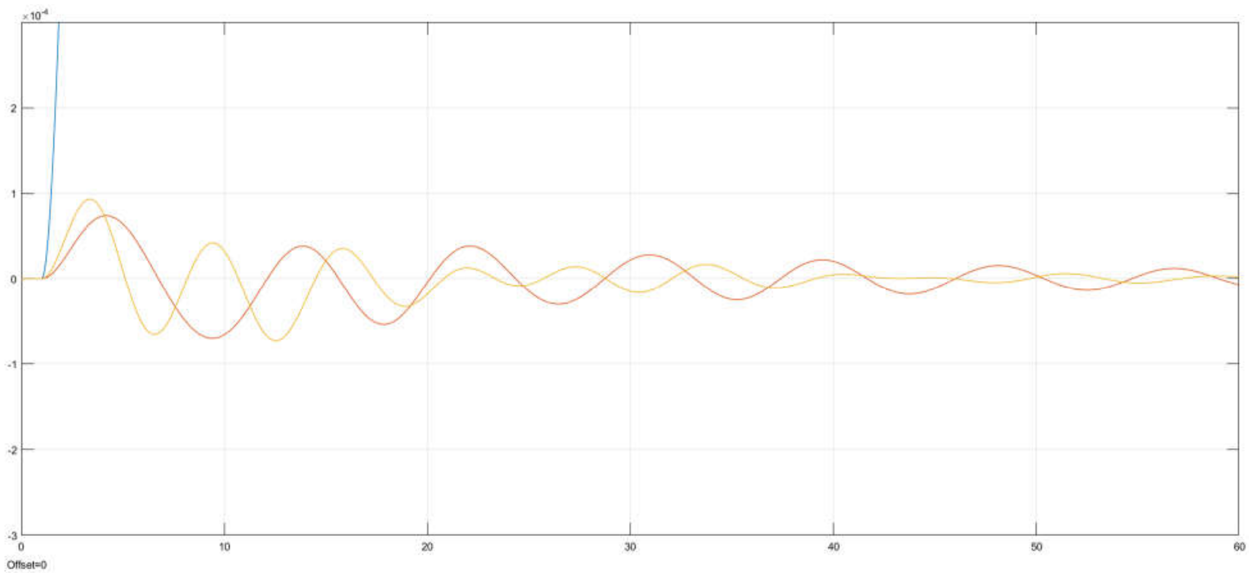


Figure 4 Linearized Model LQR Luenberger Observer

Response For vector $x(t), \theta_1(t), \theta_2(t)$

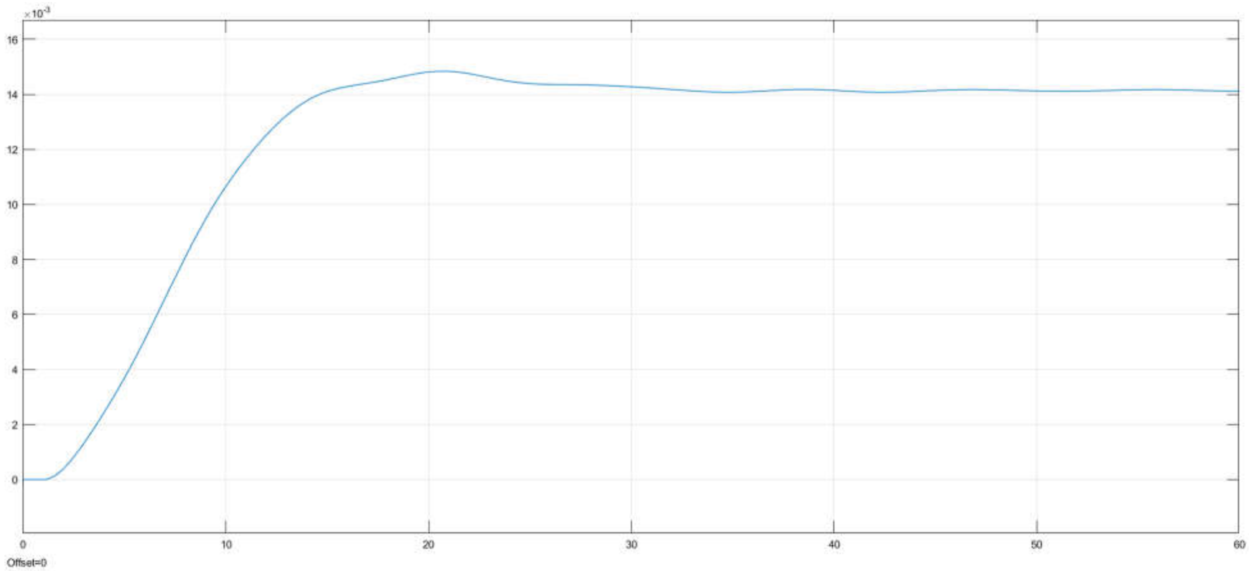
```
C = [ 1 0 0 0 0 0 %x Q1 Q2
      0 0 1 0 0 0
      0 0 0 0 1 0];
```





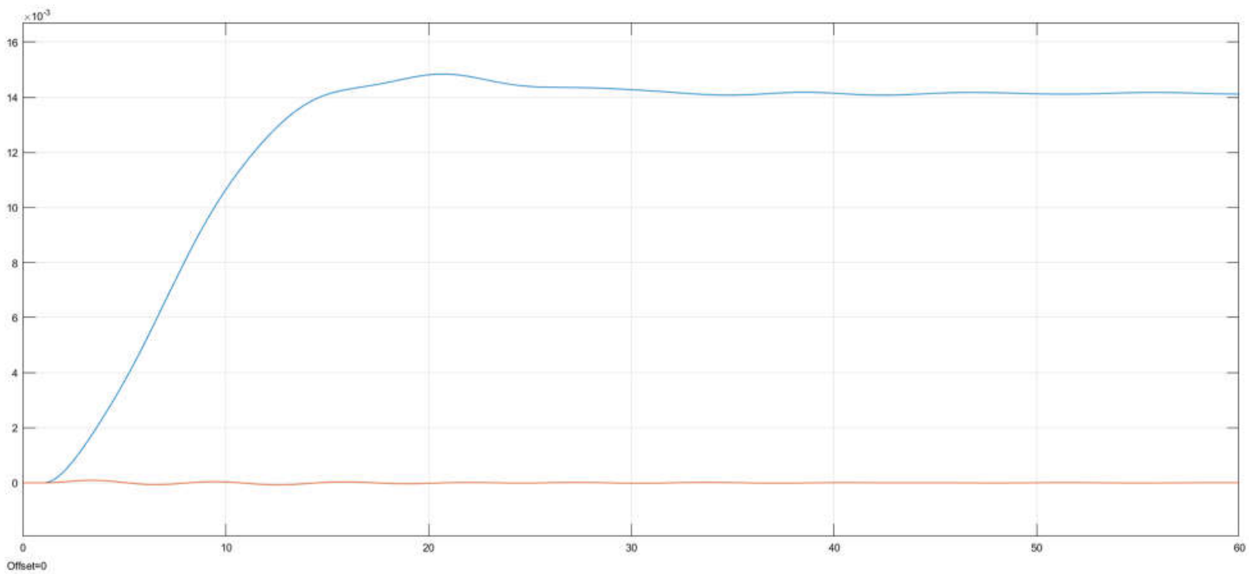
Response For vector $x(t)$

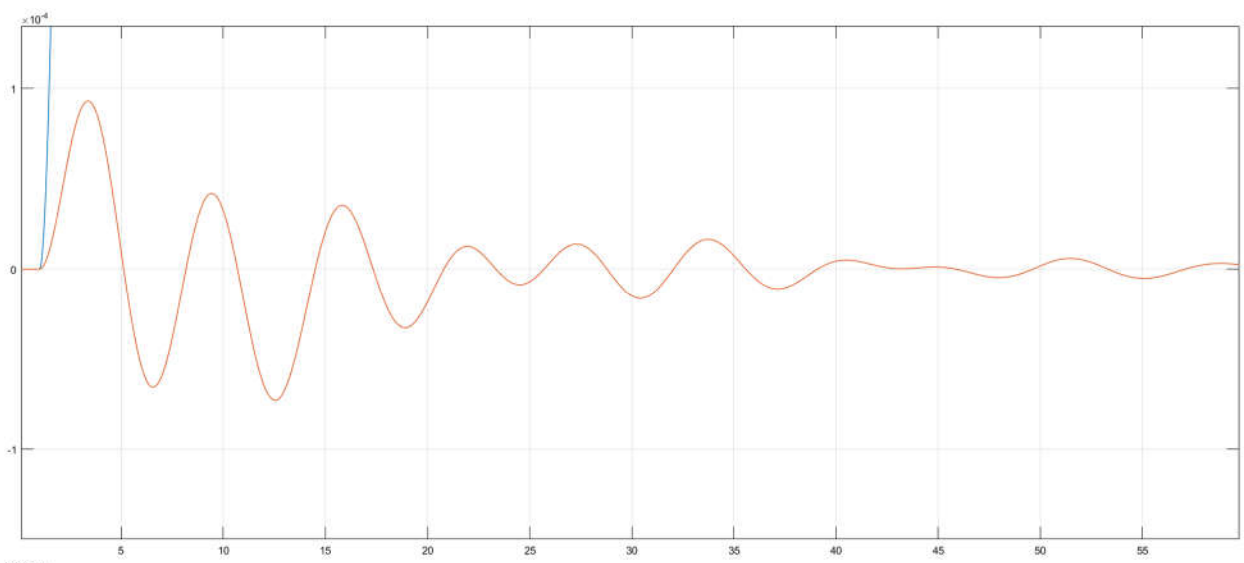
```
C = [ 1 0 0 0 0 0]; %x
```



Response For vector $x(t)$,θ_2

```
C = [ 1 0 0 0 0 0 %x Q2
      0 0 0 0 1 0];
```





Luenberger State Observer – Non-Linear System

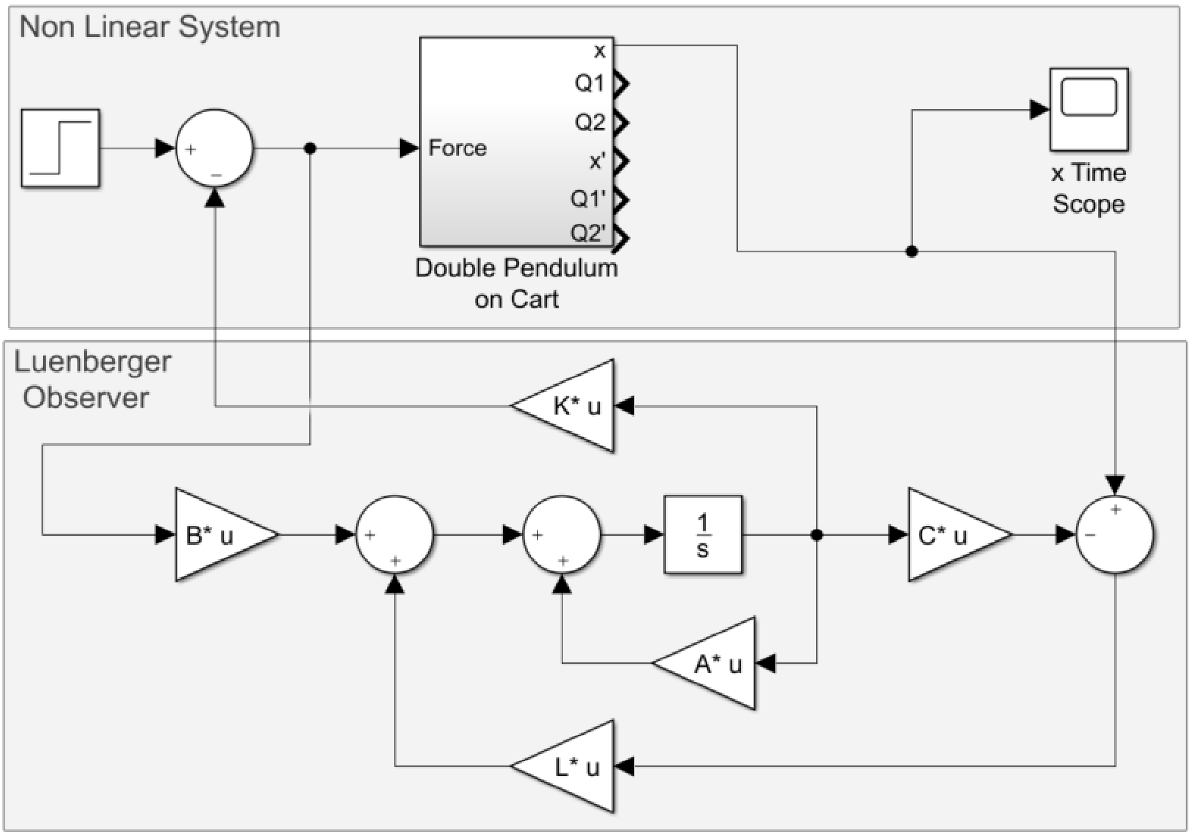
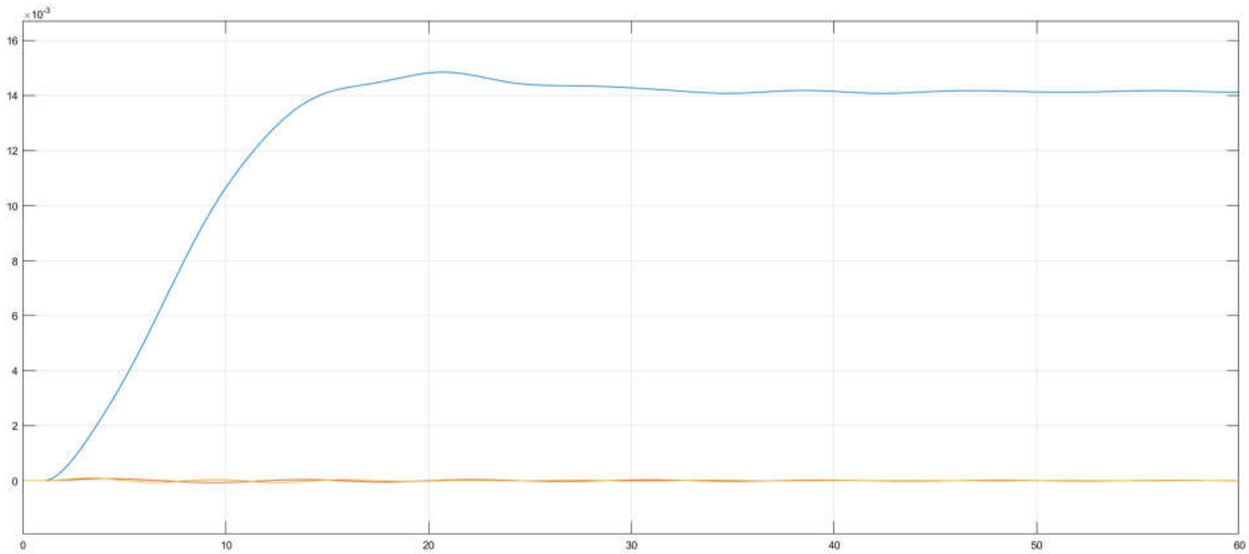
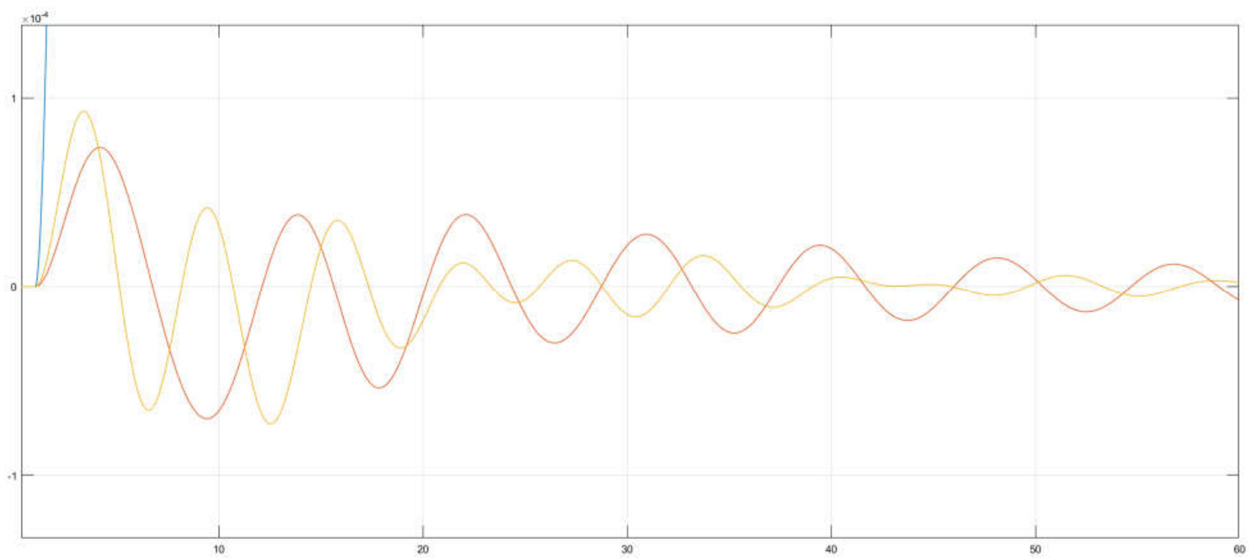


Figure 5 Non-Linearize Model LQR Luenberger Observer

Response For vector $x(t), \theta_1(t), \theta_2(t)$

```
C = [ 1 0 0 0 0 0 %x Q1 Q2
      0 0 1 0 0 0
      0 0 0 0 1 0];
```





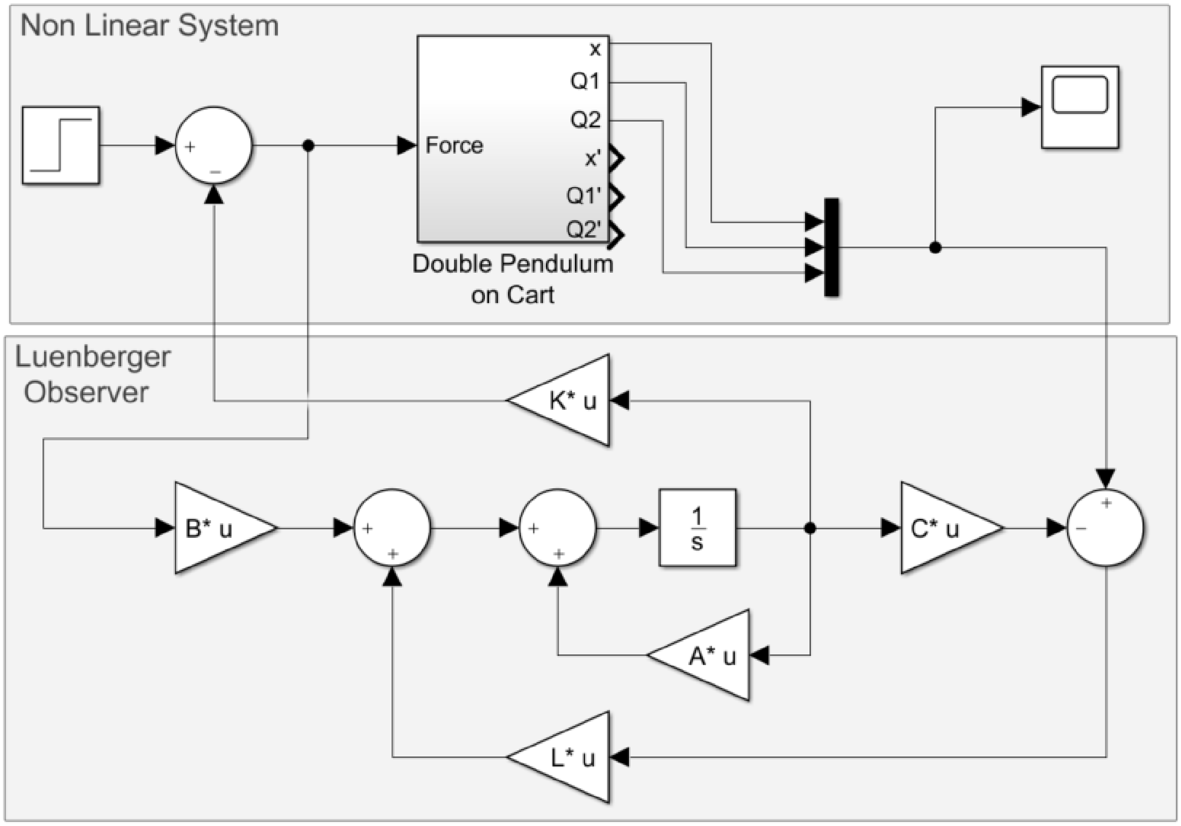
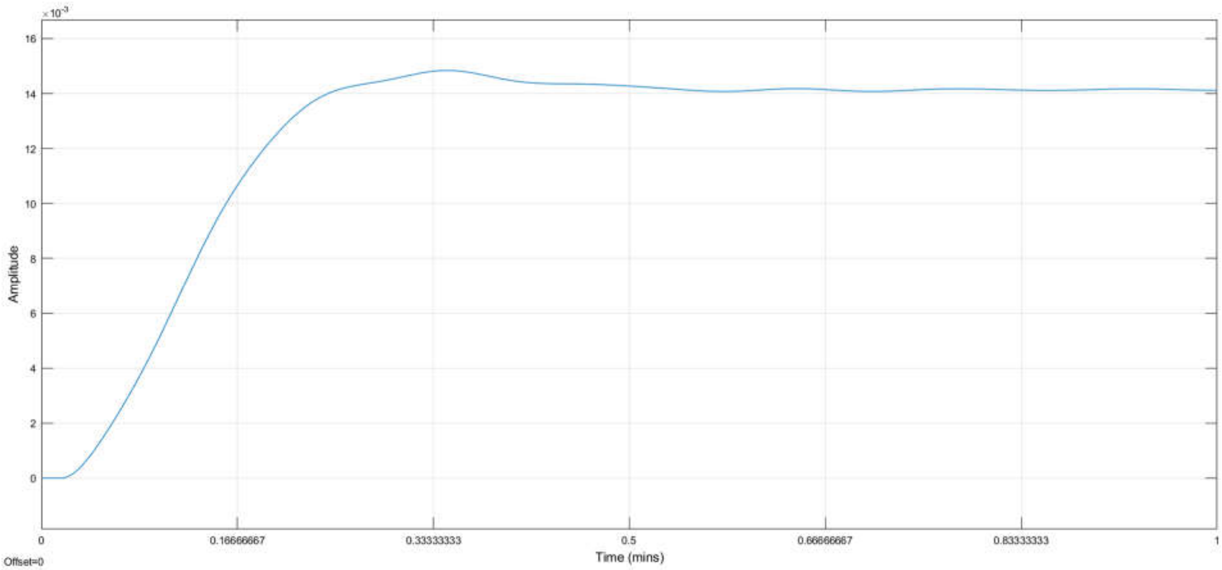


Figure 6 3 Output Vectors (x,Q1,Q2)

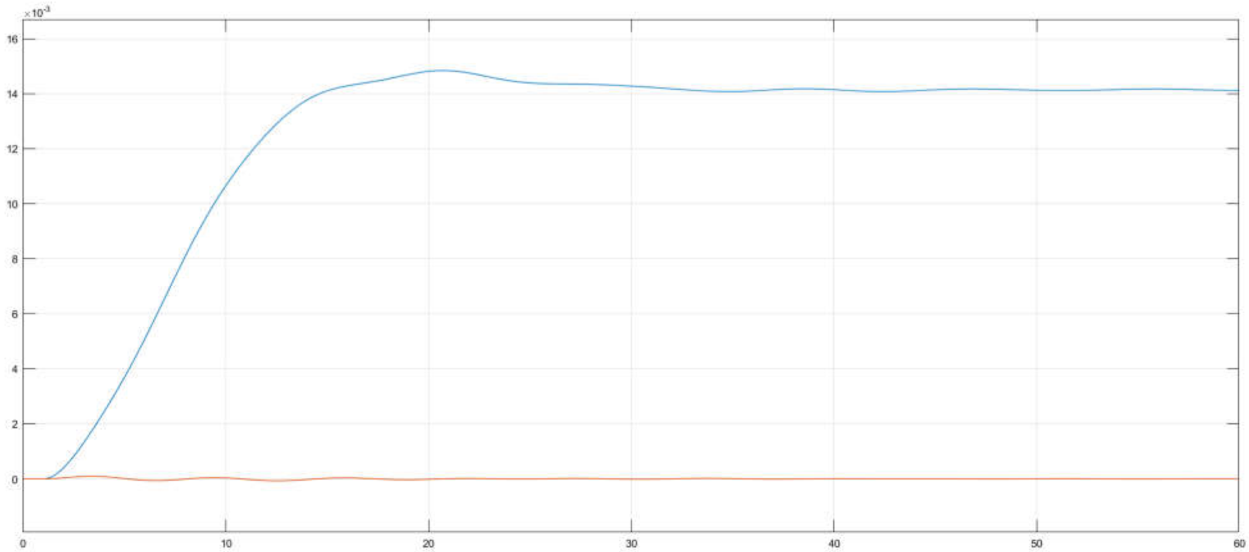
Response For vector $x(t)$

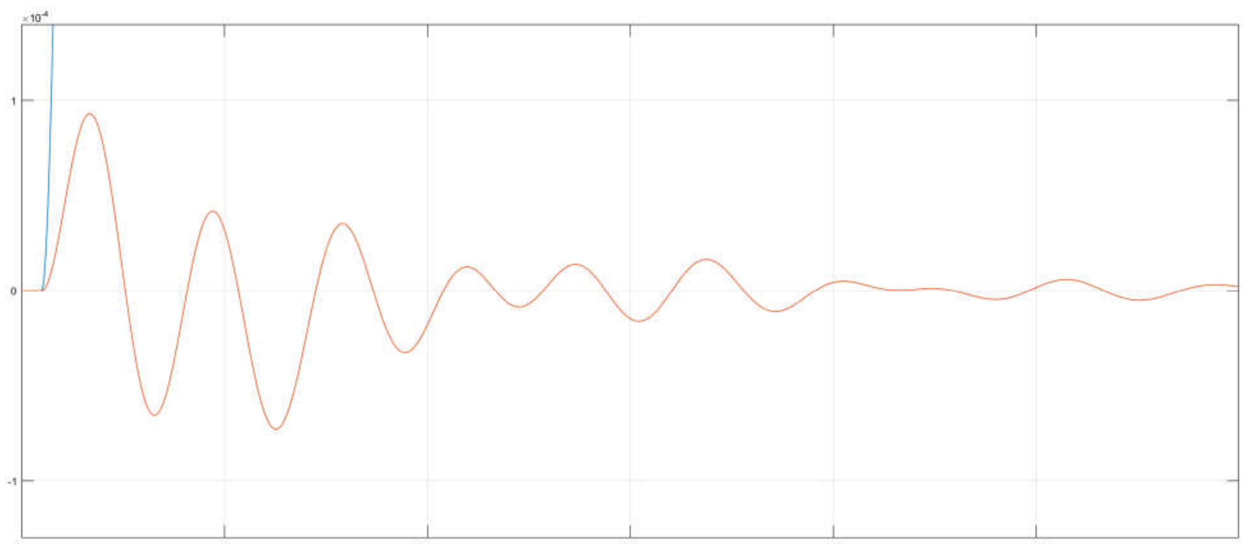
```
C = [ 1 0 0 0 0 0]; %x
```



Response For vector $x(t)$,θ_2

```
C = [ 1 0 0 0 0 0 %x Q2
      0 0 0 0 1 0];
```





LQG Regulator Design – Output Vector $x(t)$

The LQG controller is simply the combination of a Kalman filter, i.e. a linear–quadratic estimator (LQE), with a linear–quadratic regulator (LQR). We will use our previously derived LQR regulator.

Then we only must construct our Kalman filter and close the loop appropriately.

Kalman Filter accepts two inputs, first the output of our plant and second, the input to our plant.

$$\dot{x} = Ax + Bu + W_d$$

$$y = Cx + W_n$$

Where W_d is Gaussian disturbance, V_d is disturbance co-variance

and W_n is Gaussian sensor noise, V_n is sensor noise co-variance. Kalman filter balances the 'trust' between measurements and input to estimate the state and minimize the expected error.

```
%Wd %Gaussian disturbance
Vd= 0.1*eye(6); %Disturbance Co-Variance Matrix

%Wn %Gaussian noise
Vn=0.1; %Noise Co-Variance Matrix 1-by-1

%Construct Kalman Filter
[Kfilter,P,E] = lqe(A,Vd,C,Vd,Vn);

%Construct kalman Filter State Space Model
sysKF=ss(A-Kfilter*C,[B Kfilter],eye(6),0*[B Kfilter]);
```

[3] From Steve Brunton Lectures

We will then insert the constructed Kalman filter state space model into the non-linear system model in Simulink, with corresponding disturbance and sensor noises.

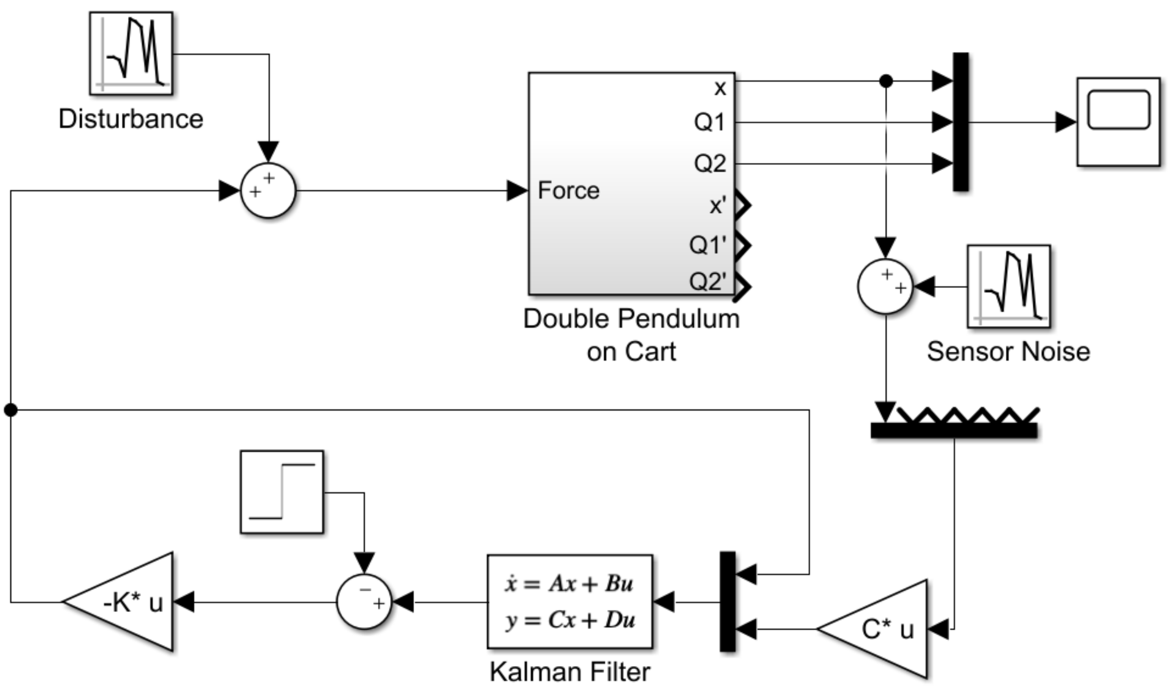


Figure 7 LQG Non-Linear Double Pendulum Cart

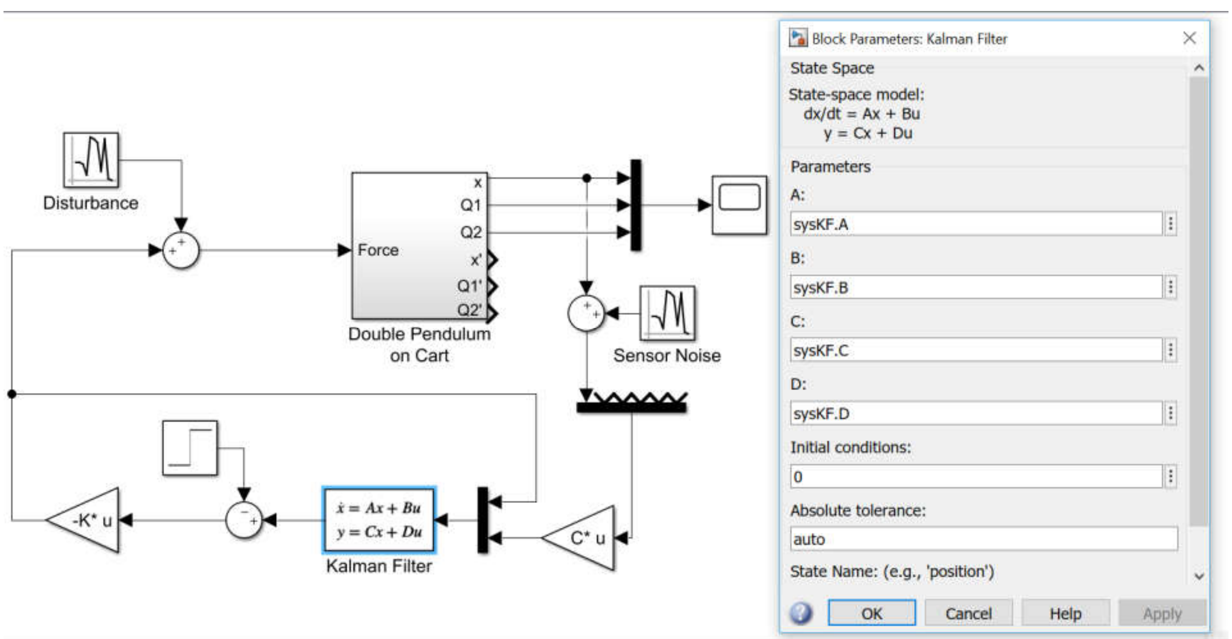
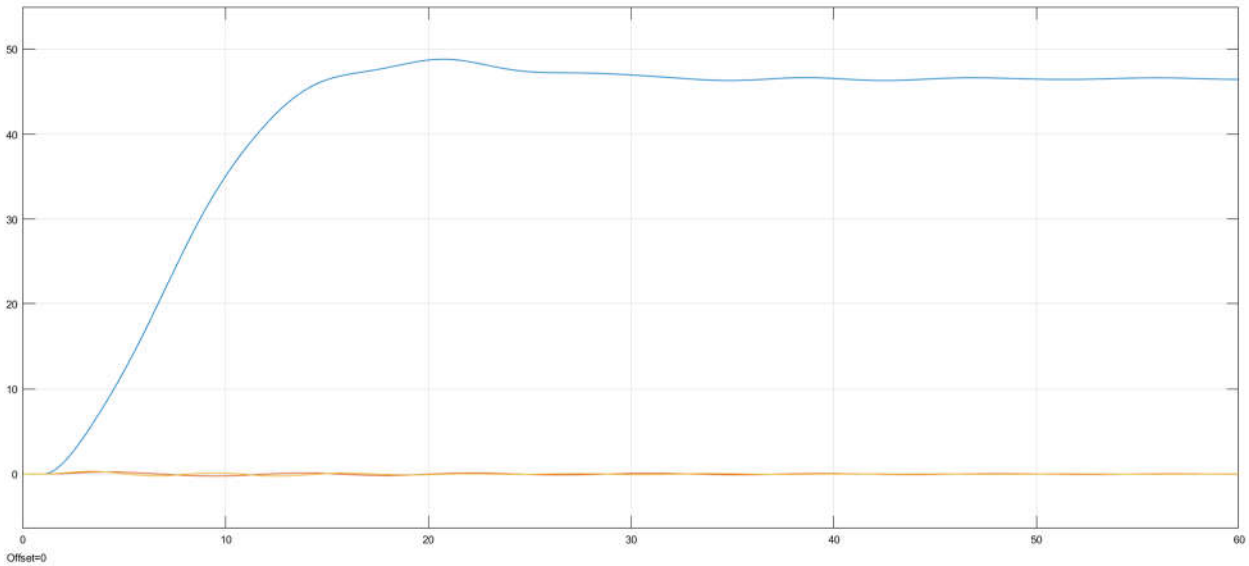
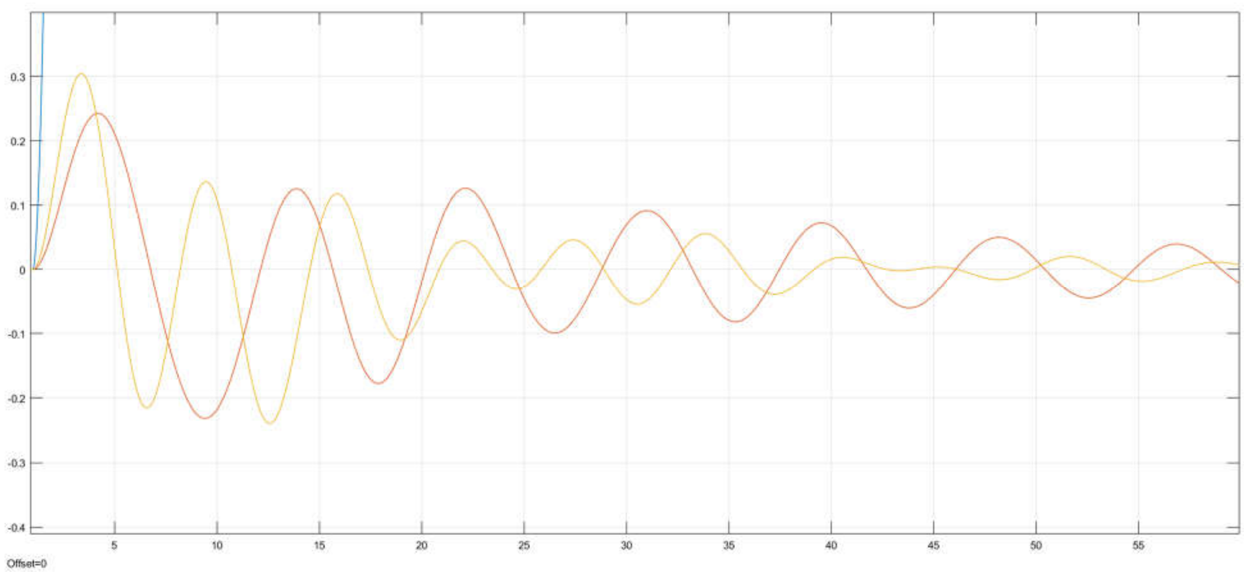
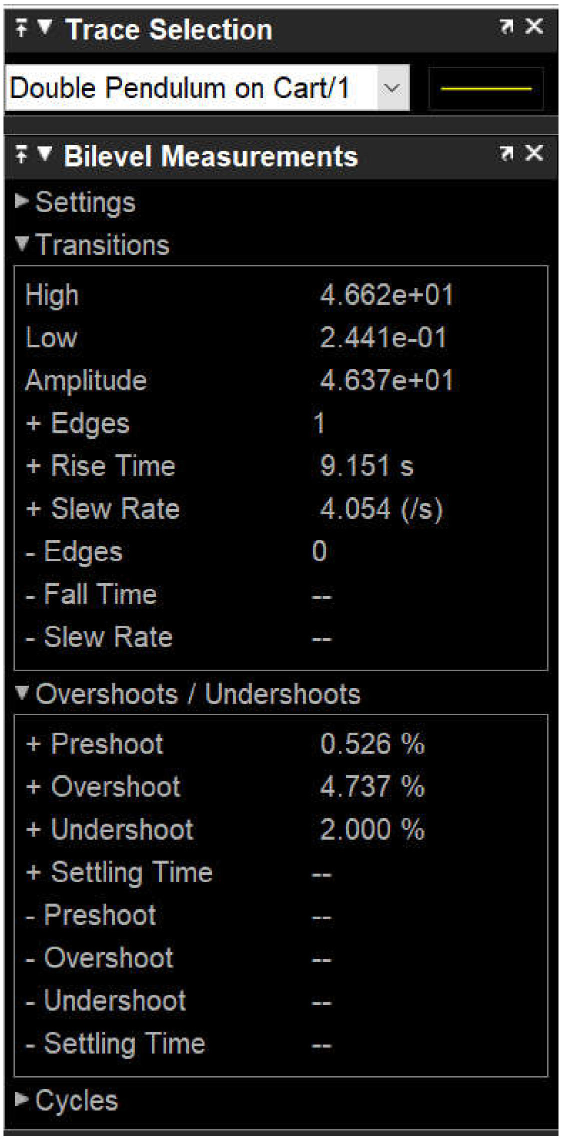


Figure 8 Kalman Filter State Space Matrices

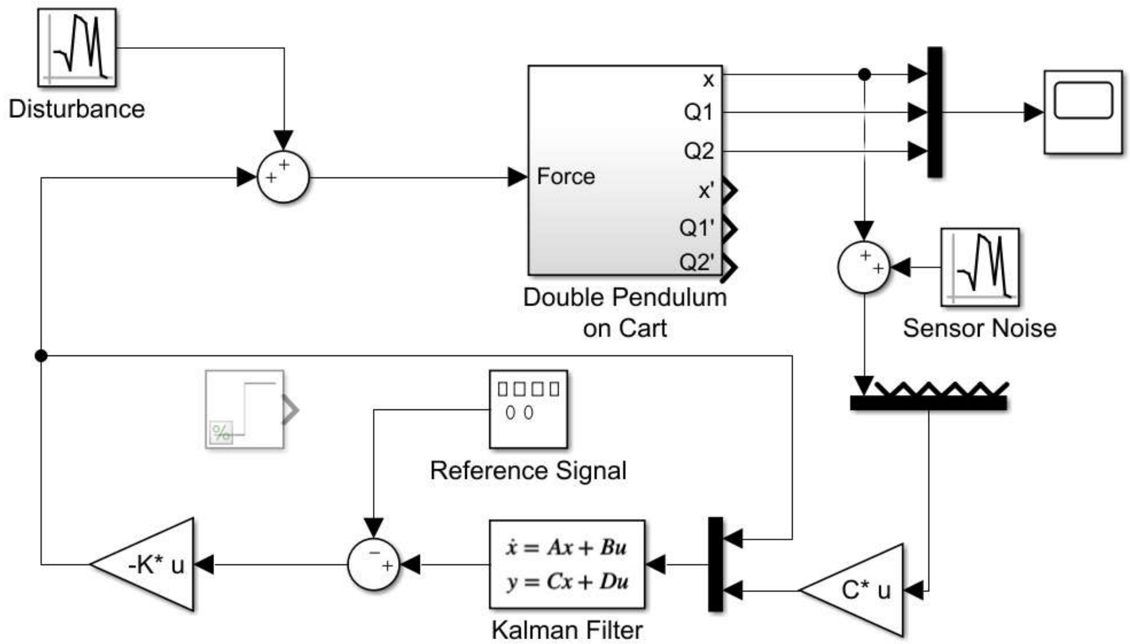
LQG Performance






Trace Selection
Double Pendulum on Cart/1
Bilevel Measurements
Settings
Transitions
High 4.662e+01
Low 2.441e-01
Amplitude 4.637e+01
+ Edges 1
+ Rise Time 9.151 s
+ Slew Rate 4.054 (/s)
- Edges 0
- Fall Time --
- Slew Rate --
Overshoots / Undershoots
+ Preshoot 0.526 %
+ Overshoot 4.737 %
+ Undershoot 2.000 %
+ Settling Time --
- Preshoot --
- Overshoot --
- Undershoot --
- Settling Time --
Cycles

The constant reference can be added before the K gain matrix is applied. In the same way the step input was previously being applied. The diagram below illustrates the appropriate Reference Signal placement:



A basic principle is: to be able to reject a disturbance, or track a reference, we need to incorporate a model of the disturbance in the controller. This is known as the Internal Model Principle. [4]. So, the system will not be able to reject any constant force disturbances since they are not modelled.

A relatively practical method of suppressing the effect of constant disturbances on nonlinear systems is by adding an integrator to the controller, it is possible to achieve both constant disturbance rejection and zero tracking error. [5]

[1] Gary Garber, “Energy in a Pendulum”, Dec 12, 2017 retrieved http://blogs.bu.edu/ggarber/interlace/pendulum/energy-in-a-pendulum/

[2] Electrical & Electronics Department “Luenberger Design for Inverted Pendulum”, Dec 12, 2017 retrieved http://www.inst.eecs.berkeley.edu/~ee128/fa10/Labs/Lab5b-Fa10.pdf

[3] Steve Brunton, “Control Bootcamp: Kalman Filter Example in Matlab”, Dec 12, 2017 retrieved https://www.youtube.com/watch?v=Lgq4R-F8SX8

[4] Julio H. Braslavsky , “Control Systems Design” . Dec 12, 2017 retrieved http://staff.uz.zgora.pl/wpaszke/materialy/spc/Lec23.pdf

[5] Steven W. SuBrian D. O. AndersonThomas S. Brinsmead, “Constant disturbance rejection and zero steady state tracking error for nonlinear systems design”. Dec 12, 2017 retrieved https://link.springer.com/chapter/10.1007/978-1-4615-1471-8_1