Robotics 2

Dynamic model of robots: Lagrangian approach

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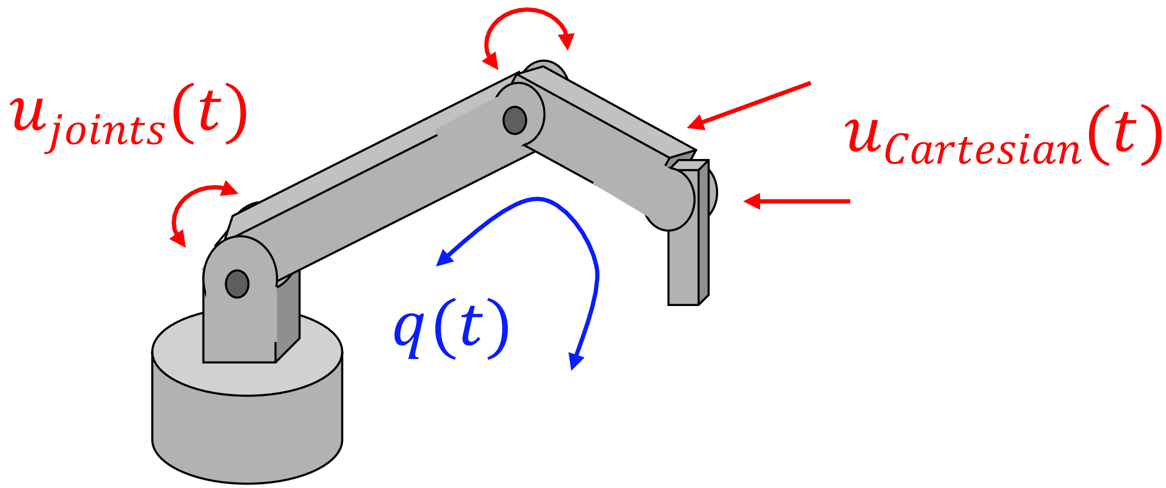
Dynamic model

- provides the relation between

generalized forces $u(t)$ acting on the robot

robot motion, i.e.,
assumed configurations $q(t)$ over time



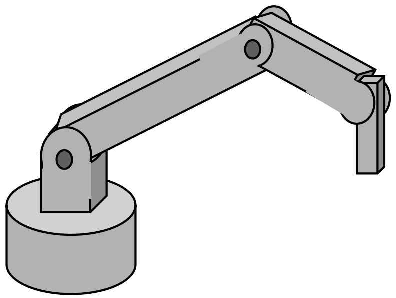
a system of 2nd order differential equations

$$\Phi(q, \dot{q}, \ddot{q}) = u$$

Direct dynamics

- direct relation

$$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$



$$q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

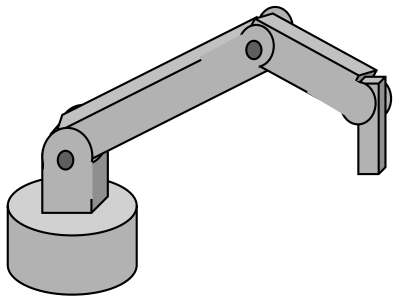
input for $t \in [0, T]$ + $q(0), \dot{q}(0)$ resulting motion

initial state at $t = 0$

- experimental solution
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time T_c)
- solution by simulation ⟷ $\boxed{\Phi(q, \dot{q}, \ddot{q}) = u}$
  - use dynamic model and integrate numerically the differential equations (with simulation step $T_s \leq T_c$)

Inverse dynamics

- inverse relation

$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ ➡  ➡ $u_d(t)$

desired motion
for $t \in [0, T]$

required input
for $t \in [0, T]$

- experimental solution
  - repeated motion trials of direct dynamics using $u_k(t)$, with iterative learning of nominal torques updated on trial $k+1$ based on the error in $[0, T]$ measured in trial k: $u_k(t) \Rightarrow u_d(t)$
- analytic solution ⟷ $\boxed{\Phi(q, \dot{q}, \ddot{q}) = u}$
  - use dynamic model and compute algebraically the values $u_d(t)$ at every time instant t

Approaches to dynamic modeling

Euler-Lagrange method (energy-based approach)	$\rightleftharpoons$	Newton-Euler method (balance of forces/torques)
dynamic equations in symbolic/closed form		dynamic equations in numeric/recursive form
best for study of dynamic properties and analysis of control schemes		best for implementation of control schemes (inverse dynamics in real time)

- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d'Alembert, of Hamilton, of virtual works, …

Euler-Lagrange method (energy-based approach)

basic assumption: the N links in motion are considered as **rigid bodies** (+ possibly, **concentrated elasticity** at the joints)

$q \in \mathbb{R}^N$ generalized coordinates (e.g., joint variables, but not only!)

Lagrangian $\boxed{L(q,\dot{q}) = T(q,\dot{q}) - U(q)}$

kinetic energy – potential energy

- least action principle of Hamilton
- virtual works principle

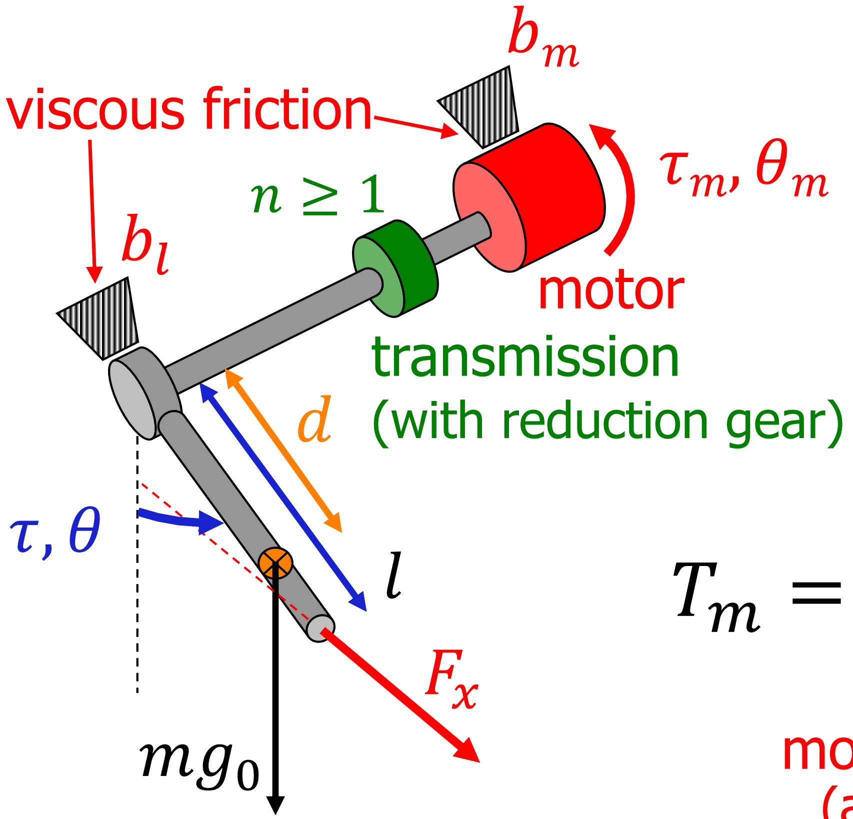
Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \qquad i = 1, \dots, N$$

non-conservative (external or dissipative) generalized forces performing work on q_i

Dynamics of actuated pendulum

a first example



$$\dot{\theta}_m = n\dot{\theta} \quad \Rightarrow \quad \theta_m = n\theta + \underbrace{\cancel{\theta_{m0}}}_{=0}$$

$$\tau = n\tau_m$$

$\boxed{q = \theta}$ (or $q = \theta_m$)

$$\boxed{T = T_m + T_l}$$

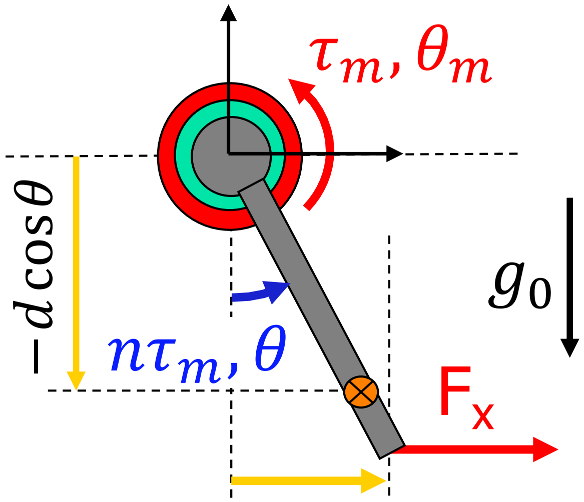
$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \qquad T_l = \frac{1}{2}(I_l + md^2)\dot{\theta}^2$$

motor inertia (around its spinning axis)

link inertia (around the z-axis through its center of mass)

kinetic energy $$\boxed{T = \frac{1}{2}(I_l + md^2 + n^2 I_m)\dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2}$$

Dynamics of actuated pendulum (cont)



$$\dot{p}_x = l\cos\theta \cdot \dot{\theta} = J_x\dot{\theta}$$

$$U = U_0 - mg_0 d\cos\theta$$ potential energy

$$L = T - U = \frac{1}{2}I\dot{\theta}^2 + mg_0 d\cos\theta - U_0$$

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = I\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mg_0 d\sin\theta$$

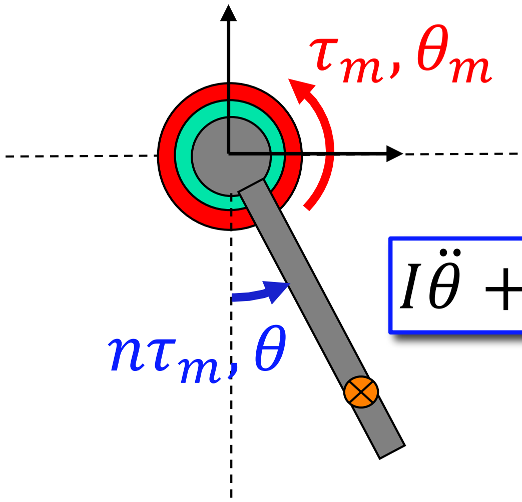
$$u = n\tau_m - b_l\dot{\theta} - nb_m\dot{\theta}_m + J_x^T F_x = n\tau_m - (b_l + n^2 b_m)\dot{\theta} + l\cos\theta\, F_x$$

applied or dissipated torques on motor side are multiplied by n when moved to the link side

equivalent joint torque due to force F_x applied to the tip at point p_x

"sum" of non-conservative torques

Dynamics of actuated pendulum (cont)



dynamic model in $q = \theta$

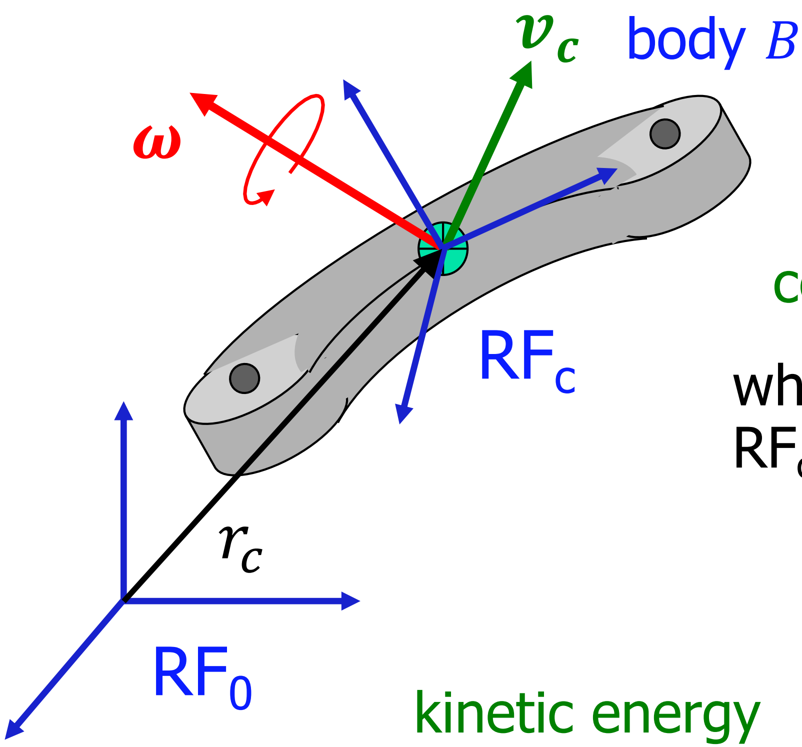
$$\boxed{I\ddot{\theta} + m g_0 d \sin\theta = n\tau_m - (b_l + n^2 b_m)\dot{\theta} + l\cos\theta \cdot F_x}$$

dividing by n and substituting $\theta = \theta_m/n$

$$\boxed{\frac{I}{n^2}\ddot{\theta}_m + \frac{m}{n} g_0 d \sin\frac{\theta_m}{n} = \tau_m - \left(\frac{b_l}{n^2} + b_m\right)\dot{\theta}_m + \frac{l}{n}\cos\frac{\theta_m}{n} \cdot F_x}$$

dynamic model in $q = \theta_m$

Kinetic energy of a rigid body



mass density

mass $m = \int_B \rho(x, y, z)\, dxdydz = \int_B dm$

position of center of mass (CoM) $r_c = \frac{1}{m} \int_B r\, dm$

when all vectors are referred to a body frame RF$_c$ attached to the CoM, then

$$r_c = 0 \quad \Rightarrow \quad \int_B r\, dm = 0$$

kinetic energy $T = \frac{1}{2} \int_B v^T(x, y, z)\, v(x, y, z)\, dm$

(fundamental) kinematic relation for a rigid body $v = v_c + \omega \times r = v_c + S(\omega)\, r$

skew-symmetric matrix

Kinetic energy of a rigid body (cont)

$$T = \frac{1}{2}\int_B (v_c + S(\omega)r)^T(v_c + S(\omega)r)\, dm$$

$$= \frac{1}{2}\int_B v_c^T v_c\, dm + \int_B v_c^T S(\omega)\, r dm + \frac{1}{2}\int_B r^T S^T(\omega) S(\omega) r\, dm$$

sum of elements on the diagonal of a matrix

$$a^T b = trace\{a b^T\}$$

$$\boxed{= \frac{1}{2} m\, v_c^T v_c}$$

$$= v_c^T S(\omega)\int_B r dm = 0$$

$$= \frac{1}{2}\int_B trace\{S(\omega) r\, r^T S^T(\omega)\} dm$$

$$= \frac{1}{2} trace\left\{S(\omega)\left(\int_B r\, r^T dm\right) S^T(\omega)\right\}$$

$$= \frac{1}{2} trace\{S(\omega)\, J_c\, S^T(\omega)\}$$

Euler matrix

translational kinetic energy (point mass in CoM)

+

rotational kinetic energy (of the whole body)

$$\boxed{= \frac{1}{2}\omega^T I_c\, \omega}$$

body inertia matrix (around the CoM)

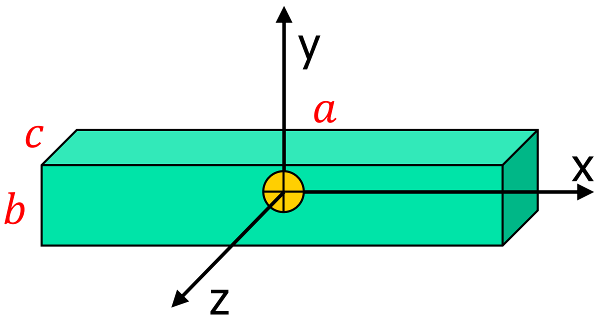
König theorem

Ex #1: provide the expressions of the elements of Euler matrix J_c

Ex #2: prove last equality and provide the expressions of the elements of inertia matrix I_c

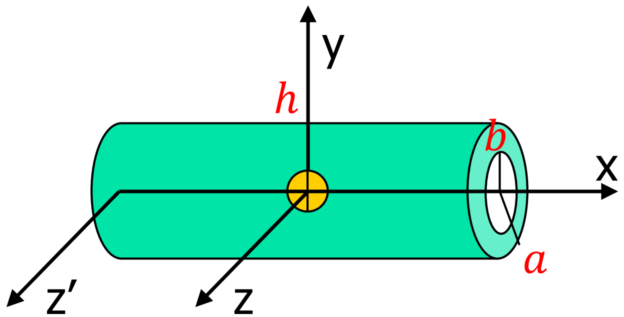
Examples of body inertia matrices

homogeneous bodies of mass m, with axes of symmetry



parallelepiped with sides
a (length/height), b, c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2+c^2) & & \\ & \frac{1}{12}m(a^2+c^2) & \\ & & \frac{1}{12}m(a^2+b^2) \end{pmatrix}$$



empty cylinder with length h,
and external/internal radius a, b

$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2+b^2) & & \\ & \frac{1}{12}m(3(a^2+b^2)^2+h^2) & \\ & & I_{zz} \end{pmatrix} \qquad I_{zz} = I_{yy}$$

$$I'_{zz} = I_{zz} + m\left(\frac{h}{2}\right)^2$$ (parallel) axis translation theorem

Steiner theorem

$$\boxed{I = I_c + m(r^T r \cdot E_{3\times 3} - r r^T) = I_c + m\, S^T(r) S(r)}$$

body inertia matrix relative to the CoM

identity matrix

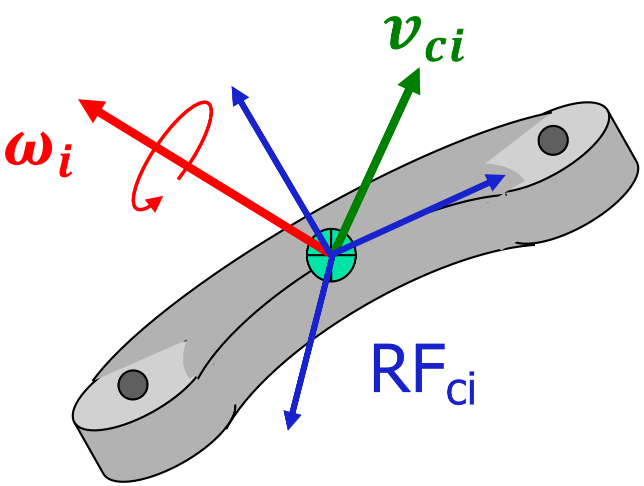
Ex #3: prove the last equality

its generalization:
changes on body inertia matrix due to a pure translation r of the reference frame

Robot kinetic energy

$$T = \sum_{i=1}^{N} T_i$$ ⇐ N rigid bodies (+ fixed base)

$$T_i = T_i(q_j, \dot{q}_j; \underbrace{j \le i})$$ ⇐ open kinematic chain



i-th link (body)
of the robot

König theorem

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2}\, \omega_i^T I_{ci}\, \omega_i$$

absolute velocity of the center of mass (CoM)

absolute angular velocity of whole body

Kinetic energy of a robot link

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2}\, \omega_i^T I_{ci}\, \omega_i$$

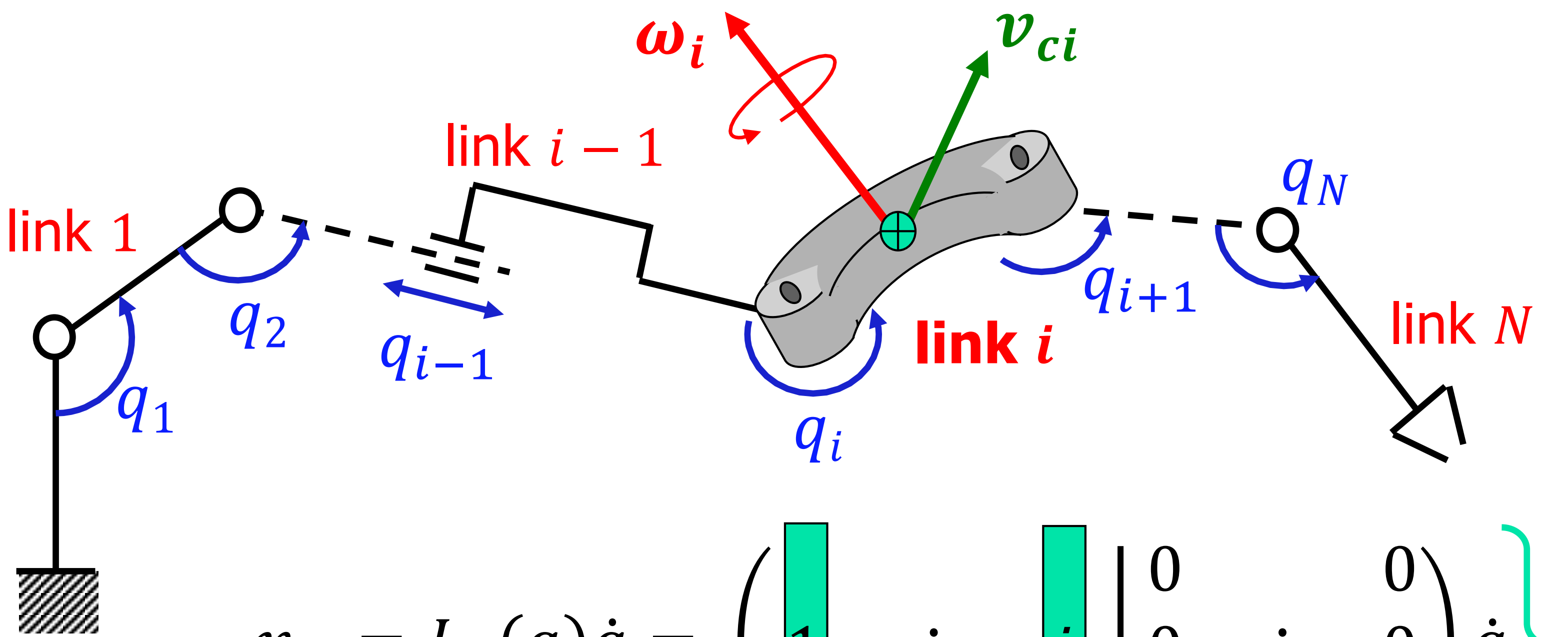
$\boldsymbol{\omega_i}, \boldsymbol{I_{ci}}$ should be expressed in the **same reference frame**, but the product $\boldsymbol{\omega_i}^T \boldsymbol{I_{ci}} \boldsymbol{\omega_i}$ is **invariant** w.r.t. any chosen frame

in frame RF_{ci} attached to (the center of mass of) link i

$$ {}^{i}I_{ci} = \begin{pmatrix} \int (y^2+z^2)dm & -\int xy\, dm & -\int xz\, dm \\ & \int (x^2+z^2)dm & -\int yz\, dm \\ symm & & \int (x^2+y^2)dm \end{pmatrix}$$

constant!

Dependence of T from q and $\dot{q}$



$$v_{ci} = J_{Li}(q)\dot{q} = \left(\begin{array}{ccc|ccc} & & & 0 & & 0 \\ 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \end{array}\right)\dot{q} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \text{3 rows}$$

(partial) Jacobians

typically expressed in RF_0

$$\omega_i = J_{Ai}(q)\dot{q} = \left(\begin{array}{ccc|ccc} & & & 0 & & 0 \\ 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \end{array}\right)\dot{q} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \text{3 rows}$$

Final expression of T

$$T = \frac{1}{2}\sum_{i=1}^{N}\left(m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci}\, \omega_i\right)$$

NOTE 1:
in practice, NEVER
use this formula
(or partial Jacobians)
for computing T;
a better method
is available...

$$= \frac{1}{2}\,\dot{q}^T \left(\sum_{i=1}^{N} m_i\, J_{Li}^T(q) J_{Li}(q) + J_{Ai}^T(q) I_{ci} J_{Ai}(q)\right)\dot{q}$$

constant if $\boldsymbol{\omega_i}$ is expressed in RF_{ci}

else

$${}^0I_{ci}(q) = {}^0R_i(q)\; {}^iI_{ci}\; {}^0R_i^T(q)$$

$$\boldsymbol{T = \frac{1}{2}\dot{q}^T M(q)\dot{q}}$$

NOTE 2:
in the past, I have
always used
the notation $\boldsymbol{B(q)}$
for the robot
inertia matrix...

robot (generalized) inertia matrix

- symmetric
- positive definite, $\forall q \Rightarrow$ **always invertible**

Robot potential energy

assumption: GRAVITY contribution only

$$U = \sum_{i=1}^{N} U_i$$ N rigid bodies (+ fixed base)

$$U_i = U_i(\underbrace{q_j; j \leq i})$$ open kinematic chain

$$\boxed{U_i = -m_i g^T r_{0,ci}}$$

gravity acceleration vector — position of the center of mass of link i — typically expressed in RF_0

dependence on q

$$\begin{pmatrix} r_{0,ci} \\ 1 \end{pmatrix} = {}^0A_1(q_1)\,{}^1A_2(q_2) \cdots {}^{i-1}A_i(q_i) \begin{pmatrix} r_{i,ci} \\ 1 \end{pmatrix}$$

constant in RF_i

NOTE: need to work with homogeneous coordinates

Summarizing ...

kinetic energy

$$T = \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j$$

positive definite quadratic form

$$\boxed{\begin{gathered} T \geq 0, \\ T = 0 \Leftrightarrow \dot{q} = 0 \end{gathered}}$$

potential energy

$$U = U(q)$$

Lagrangian

$$L = T(q,\dot{q}) - U(q)$$

Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k \qquad k = 1, \dots, N$$

non-conservative (active/dissipative) generalized forces **performing work** on $\boldsymbol{q_k}$ coordinate

Applying Euler-Lagrange equations
(the scalar derivation; see Appendix for vector format)

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}\dot{q}_j \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

(dependences of elements on q are not shown)

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial U}{\partial q_k}$$

LINEAR terms in ACCELERATION $\ddot{q}$

QUADRATIC terms in VELOCITY $\dot{q}$

NONLINEAR terms in CONFIGURATION q

k-th dynamic equation ...

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$\sum_j m_{kj}\ddot{q}_j + \sum_{i,j}\left(\frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

exchanging
"mute" indices i, j

$$\cdots + \sum_{i,j}\underbrace{\frac{1}{2}\left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right)}_{c_{kij} = c_{kji}}\dot{q}_i\dot{q}_j + \cdots$$

Christoffel symbols
of the first kind

… and interpretation of dynamic terms

$$\boxed{\sum_j m_{kj}(q)\ddot{q}_j} + \boxed{\sum_{i,j} c_{kij}(q)\,\dot{q}_i\dot{q}_j} + \boxed{\frac{\partial U}{\partial q_k}} = u_k \qquad k = 1,\dots,N$$

INERTIAL terms

CENTRIFUGAL $(i = j)$ and **CORIOLIS** $(i \neq j)$ terms

GRAVITY terms $g_k(q)$

$m_{kk}(q)$ = inertia at joint k when joint k accelerates ($m_{kk} > 0!!$)

$m_{kj}(q)$ = inertia "seen" at joint k when joint j accelerates

$c_{kii}(q)$ = coefficient of the centrifugal force at joint k when joint i is moving ($c_{iii} = 0, \forall i$)

$c_{kij}(q)$ = coefficient of the Coriolis force at joint k when joint i and joint j are both moving

Robot dynamic model
in vector formats

1. $$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q)\dot{q}$$

k-th component of vector c

k-th column of matrix $M(q)$

$$C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$$

symmetric matrix!

2. $$M(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = u$$

NOTE: the model is in the form

$\Phi(q,\dot{q},\ddot{q}) = u$

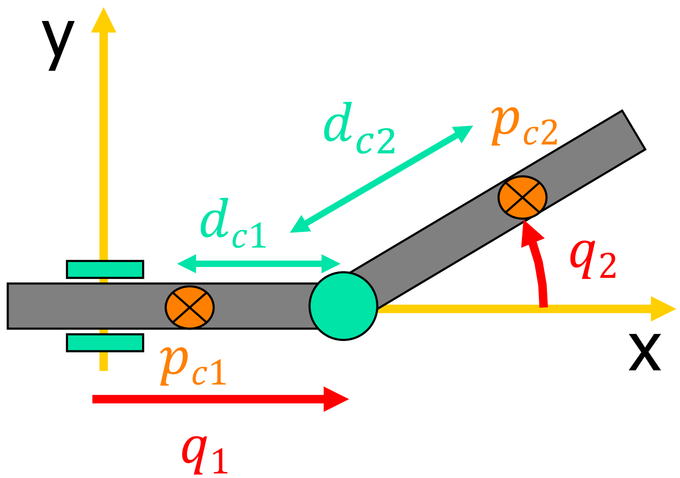
as expected

NOT a symmetric matrix in general

$$s_{kj}(q,\dot{q}) = \sum_i c_{kij}(q)\dot{q}_i$$

factorization of c by S is **not unique!**

Dynamic model of a PR robot



$T = T_1 + T_2 \qquad U = \text{constant} \Rightarrow g(q) \equiv 0$

(on horizontal plane)

$$p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|v_{c1}\|^2 = \dot{p}_{c1}^T \dot{p}_{c1} = \dot{q}_1^2$$

$$\boxed{T_1 = \frac{1}{2} m_1 \dot{q}_1^2}$$

$$T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2$$

$$p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \Rightarrow v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \, \dot{q}_2 \\ d_{c2} \cos q_2 \, \dot{q}_2 \\ 0 \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix}$$

$$\boxed{T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \, \dot{q}_2^2 - 2 d_{c2} \sin q_2 \, \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2}$$

Dynamic model of a PR robot (cont)

$$M(q) = \begin{pmatrix} \boxed{\begin{matrix} m_1 + m_2 \\ -m_2 d_{c2} \sin q_2 \end{matrix}} & \boxed{\begin{matrix} -m_2 d_{c2} \sin q_2 \\ I_{c2,zz} + m_2 d_{c2}^2 \end{matrix}} \end{pmatrix}$$

M_1 $\qquad$ M_2

$$c(q,\dot{q}) = \begin{pmatrix} c_1(q,\dot{q}) \\ c_2(q,\dot{q}) \end{pmatrix}$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q) \dot{q}$$

where $C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$

$$C_1(q) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$c_1(q,\dot{q}) = -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2$$

$$C_2(q) = \frac{1}{2}\left(\begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix}\right) = 0$$

$$c_2(q,\dot{q}) = 0$$

Dynamic model of a PR robot (cont)

$$\boxed{M(q)\ddot{q} + c(q,\dot{q}) = u}$$

$$\begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\ -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

NOTE: the m_{NN} element (here, for $N = 2$) of the robot inertia matrix is always constant!

Q1: why Coriolis terms are not present?

Q2: when applying a force u_1, does the second joint accelerate? … always?

Q3: what is the expression of a factorization matrix S? … is it unique?

Q4: which is the configuration with "maximum inertia"?

A structural property

Matrix $\dot{M} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix S)

Proof

$$\dot{m}_{kj} = \sum_i \frac{\partial m_{kj}}{\partial q_i}\dot{q}_i \qquad 2s_{kj} = \sum_i 2c_{kij}\,\dot{q}_i = \sum_i \left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i$$

$$\Rightarrow \quad \dot{m}_{kj} - 2s_{kj} = \sum_i \left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j}\right)\dot{q}_i = n_{kj}$$

$$n_{jk} = \dot{m}_{jk} - 2s_{jk} = \sum_i \left(\frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{ji}}{\partial q_k}\right)\dot{q}_i = -n_{kj}$$

using the symmetry of M

$$\Rightarrow \quad \boxed{x^T\left(\dot{M} - 2S\right)x = 0, \forall x}$$

Energy conservation

- total robot energy

$$E = T + U = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q)$$

- its evolution over time (using the dynamic model)

$$\dot{E} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{\partial U}{\partial q}\dot{q}$$

$$= \dot{q}^T (u - S(q,\dot{q})\dot{q} - g(q)) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T g(q)$$

$$= \dot{q}^T u + \frac{1}{2}\dot{q}^T \left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q}$$

here, any factorization of vector c by a matrix S can be used

- if $u \equiv 0$, total energy is constant (no dissipation or increase)

$$\dot{E} = 0 \;\Rightarrow\; \boxed{\dot{q}^T \left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q} = 0, \forall q, \dot{q}} \;\Rightarrow\; \dot{E} = \dot{q}^T u$$

weaker than skew-symmetry, as the external velocity is the same that appears in the internal matrices

in general, the variation of the total energy is equal to the work of non-conservative forces

Appendix: Vector format derivation of dynamic model

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)^T - \left(\frac{\partial L}{\partial q}\right)^T = u}$$

$$L = \frac{1}{2}\dot q^T M(q)\dot q - U(q)$$

$$M(q) = \begin{pmatrix} M_1(q) & \cdots & M_i(q) & \cdots & M_N(q) \end{pmatrix} = \sum_{i=1}^{N} M_i(q) e_i^T$$

$e_i^T = (0 \;\; \cdots \;\; 1 \;\; \cdots \;\; 0)$, with the 1 in the i-th position

dyadic expansion

$$\left(\frac{\partial L}{\partial \dot q}\right)^T = (\dot q^T M(q))^T = M(q)\dot q$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)^T = M(q)\ddot q + \dot M(q)\dot q = M(q)\ddot q + \sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q}\right)\dot q\,\dot q_i$$

$$\left(\frac{\partial L}{\partial q}\right)^T = \left(\frac{1}{2}\dot q^T \left(\sum_{i=1}^{N}\frac{\partial M_i(q)}{\partial q}\, e_i^T\right)\dot q - \frac{\partial U}{\partial q}\right)^T = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial M_i(q)}{\partial q}\right)^T \dot q_i\, \dot q - \left(\frac{\partial U}{\partial q}\right)^T$$

$$\Rightarrow \quad M(q)\ddot q + \underbrace{\left(\sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q} - \frac{1}{2}\left(\frac{\partial M_i(q)}{\partial q}\right)^T\right)\dot q_i\right)}_{S(q,\dot q)}\dot q + \underbrace{\left(\frac{\partial U}{\partial q}\right)^T}_{g(q)} = u$$

this construction gives to $\dot M - 2S$ skew-symmetry

k-th row of matrix S

$$S_k^T(q,\dot q) = \dot q^T C_k(q) \longrightarrow S(q,\dot q)$$