



Robotics 2

Dynamic model of robots: Lagrangian approach

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DIPARTIMENTO DI INGEGNERIA INFORMATICA
AUTOMATICA E GESTIONALE ANTONIO RUBERTI



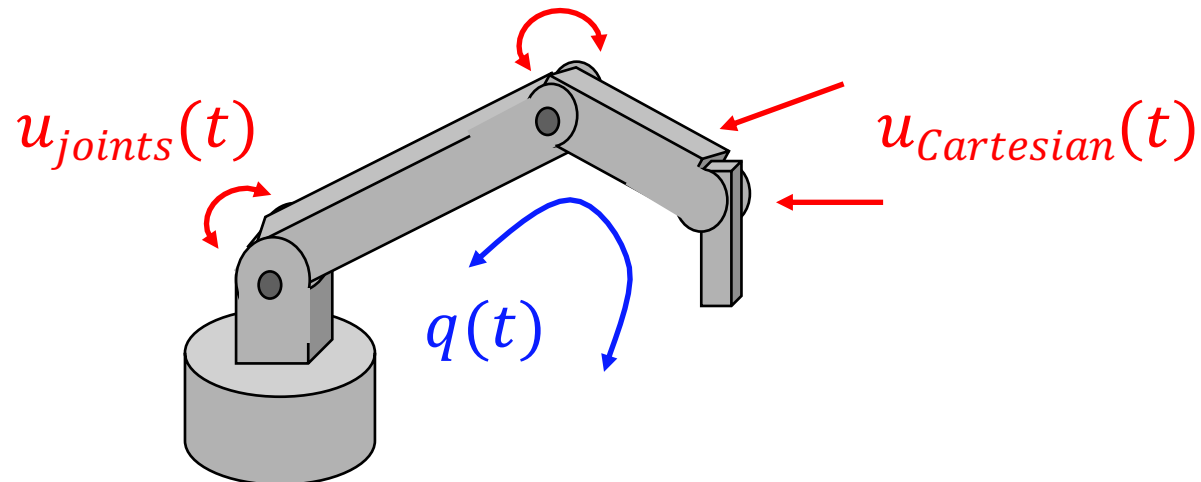
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Dynamic model

- provides the **relation** between
generalized forces $u(t)$ acting on the robot



robot motion, i.e.,
assumed configurations $q(t)$ over time

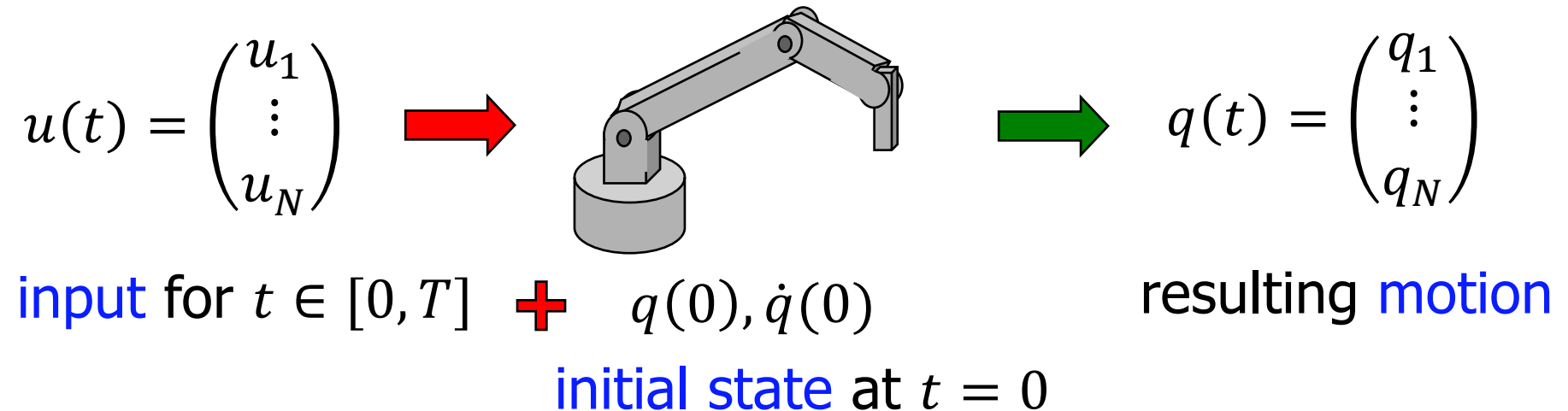


a system of 2nd order
differential equations

$$\Phi(q, \dot{q}, \ddot{q}) = u$$

Direct dynamics

- direct relation



- experimental solution

- apply torques/forces with motors and measure joint variables with encoders (with sampling time T_c)

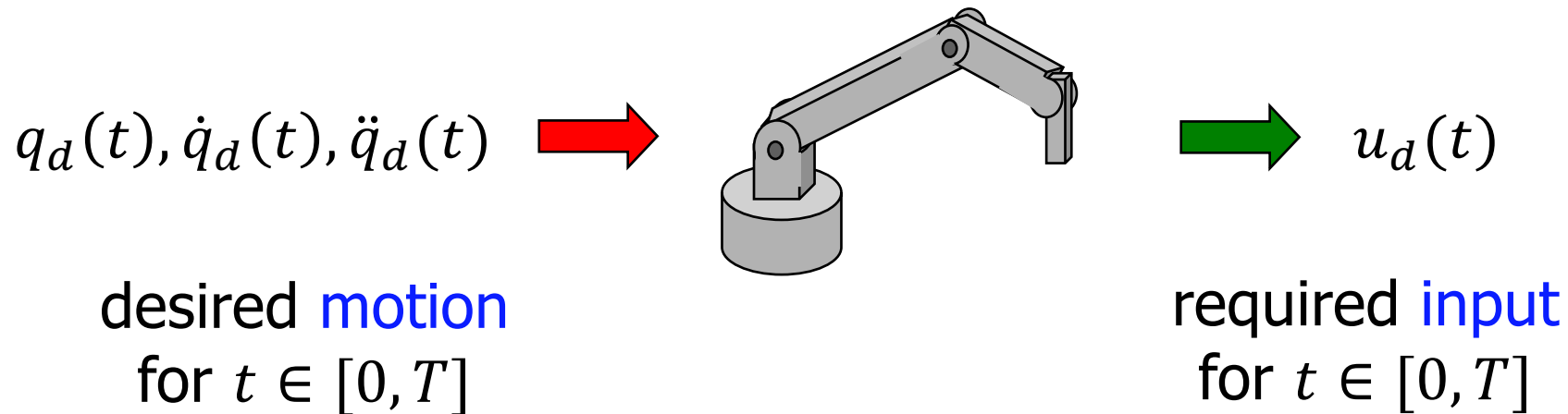
- solution by simulation

- use dynamic model and **integrate** numerically the differential equations (with simulation step $T_s \leq T_c$)

$$\longleftrightarrow \Phi(q, \dot{q}, \ddot{q}) = u$$

Inverse dynamics

- inverse relation



- experimental solution

- repeated motion trials of direct dynamics using $u_k(t)$, with **iterative learning** of nominal torques updated on trial $k + 1$ based on the error in $[0, T]$ measured in trial k : $\lim_{k \rightarrow \infty} u_k(t) \Rightarrow u_d(t)$

- analytic solution

- use dynamic model and **compute algebraically** the values $u_d(t)$ at every time instant t

 $\Phi(q, \dot{q}, \ddot{q}) = u$



Approaches to dynamic modeling

Euler-Lagrange method
(energy-based approach)



Newton-Euler method
(balance of forces/torques)

- dynamic equations in **symbolic**/closed form
- best for study of dynamic properties and analysis of control schemes
- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
 - principle of d'Alembert, of Hamilton, of virtual works, ...
- dynamic equations in **numeric**/recursive form
- best for implementation of control schemes (inverse dynamics in real time)



Euler-Lagrange method (energy-based approach)

basic assumption: the N links in motion are considered as **rigid bodies**
(+ later on, include also **concentrated elasticity** at the joints)

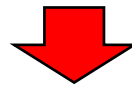
$q \in \mathbb{R}^N$ **generalized coordinates** (e.g., joint variables, but not only!)

Lagrangian

$$L(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

kinetic energy – potential energy

- principle of least action of Hamilton
- principle of virtual works



**Euler-Lagrange
equations**

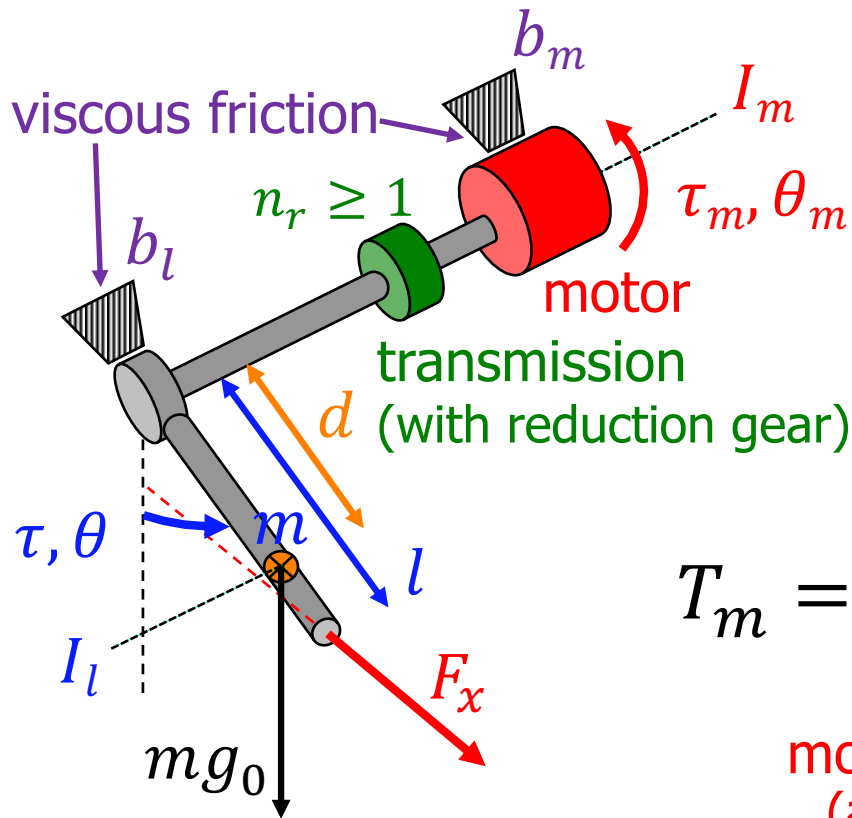
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i$$

$i = 1, \dots, N$

non-conservative (external or dissipative)
generalized forces performing work on q_i

Dynamics of an actuated pendulum

a first example



$$\dot{\theta}_m = n_r \dot{\theta} \Rightarrow \theta_m = n_r \theta + \theta_{m0}$$

$$\tau = n_r \tau_m$$

$$= 0$$

$$q = \theta \quad (\text{or } q = \theta_m)$$

$$T = T_m + T_l$$

$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2$$

motor inertia
(around its
spinning axis)

$$T_l = \frac{1}{2} (I_l + m d^2) \dot{\theta}^2$$

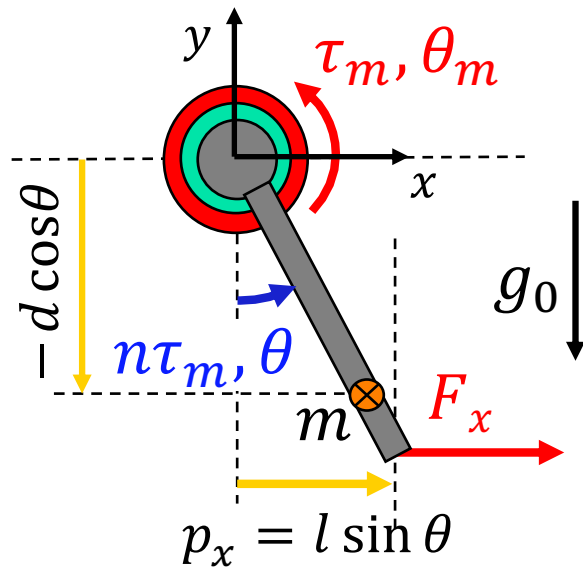
link inertia
(around the z-axis through
its center of mass...)

kinetic energy

$$T = \frac{1}{2} (I_l + m d^2 + I_m n_r^2) \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2$$



Dynamics of an actuated pendulum (cont)



$$\dot{p}_x = l \cos \theta \cdot \dot{\theta} = J_x \dot{\theta}$$

$$U = U_0 - m g_0 d \cos \theta \quad \text{potential energy}$$

$$L = T - U = \frac{1}{2} I \dot{\theta}^2 + m g_0 d \cos \theta - U_0$$

$$\frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = I \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -m g_0 d \sin \theta$$

$$u = n_r \tau_m - b_l \dot{\theta} - n_r b_m \dot{\theta}_m + J_x^T F_x = n_r \tau_m - (b_l + b_m n_r^2) \dot{\theta} + l \cos \theta F_x$$

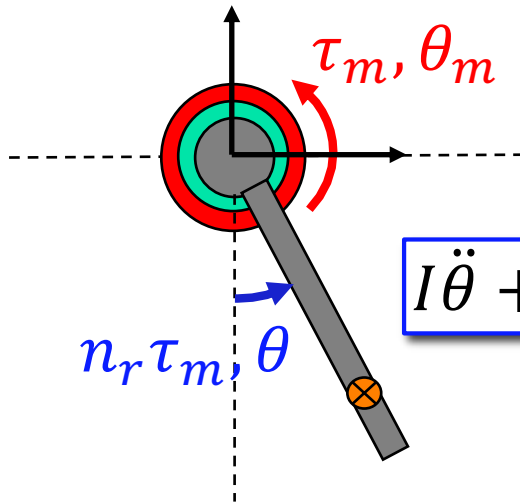
↑
applied or dissipated torques
on motor side are multiplied by n_r
when moved to the link side

↑
equivalent joint torque
due to force F_x applied to
the tip at point p_x

“sum” of
non-conservative
torques



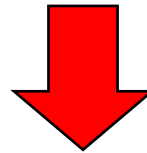
Dynamics of an actuated pendulum (cont)



dynamic model in $q = \theta$

$$I\ddot{\theta} + mg_0 d \sin \theta = n_r \tau_m - (b_l + b_m n_r^2) \dot{\theta} + l \cos \theta \cdot F_x$$

dividing by n_r and substituting $\theta = \theta_m / n_r$

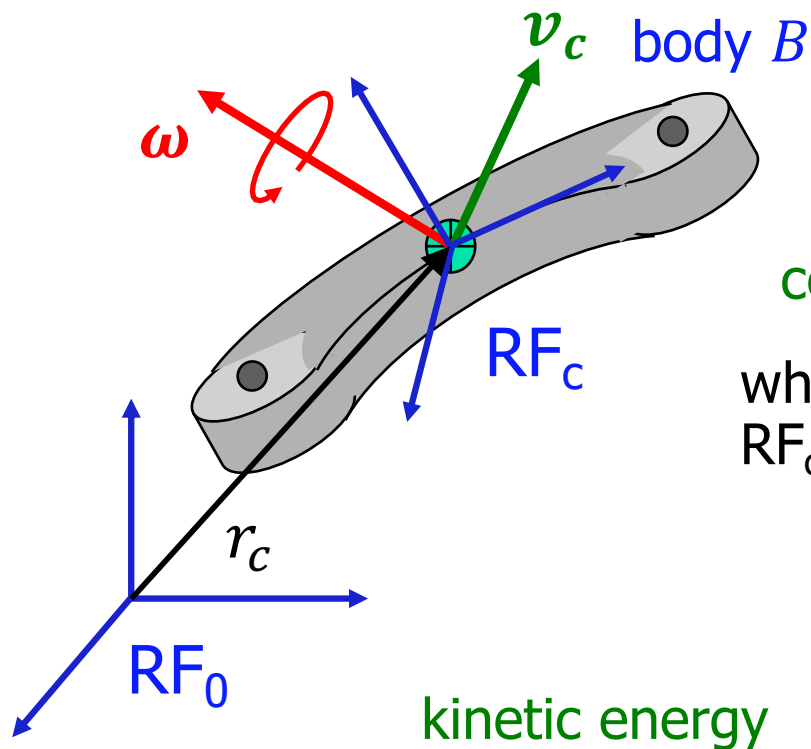


$$\frac{I}{n_r^2} \ddot{\theta}_m + \frac{m}{n_r} g_0 d \sin \frac{\theta_m}{n_r} = \tau_m - \left(\frac{b_l}{n_r^2} + b_m \right) \dot{\theta}_m + \frac{l}{n_r} \cos \frac{\theta_m}{n_r} \cdot F_x$$

dynamic model in $q = \theta_m$



Kinetic energy of a rigid body



mass density

$$\text{mass } m = \int_B \rho(x, y, z) dx dy dz = \int_B dm$$

position of center of mass (CoM)

$$r_c = \frac{1}{m} \int_B r dm$$

when all vectors are referred to a body frame RF_c attached to the CoM, then

$$r_c = 0 \Rightarrow \int_B r dm = 0$$

$$T = \frac{1}{2} \int_B v^T(x, y, z) v(x, y, z) dm$$

(fundamental)
kinematic relation
for a rigid body

$$v = v_c + \omega \times r = v_c + S(\omega) r$$

skew-symmetric matrix



Kinetic energy of a rigid body (cont)

$$T = \frac{1}{2} \int_B (v_c + S(\omega)r)^T (v_c + S(\omega)r) dm$$

$$= \frac{1}{2} \int_B v_c^T v_c dm + \int_B v_c^T S(\omega) r dm + \frac{1}{2} \int_B r^T S^T(\omega) S(\omega) r dm$$

$$\boxed{= \frac{1}{2} m v_c^T v_c}$$

translational
kinetic energy
(point mass
at CoM)

König theorem

$$= v_c^T S(\omega) \int_B r dm = 0$$

+

rotational
kinetic energy
(of the whole body)

$$\begin{aligned} &= \frac{1}{2} \int_B \text{trace}\{S(\omega)r r^T S^T(\omega)\} dm \\ &= \frac{1}{2} \text{trace}\left\{S(\omega) \left(\int_B r r^T dm\right) S^T(\omega)\right\} \\ &= \frac{1}{2} \text{trace}\{S(\omega) J_c S^T(\omega)\} \end{aligned}$$

$$\boxed{= \frac{1}{2} \omega^T I_c \omega}$$

body inertia matrix
(around the CoM)

sum of elements
on the diagonal
of a matrix

$$a^T b = \text{trace}\{ab^T\}$$

Euler matrix

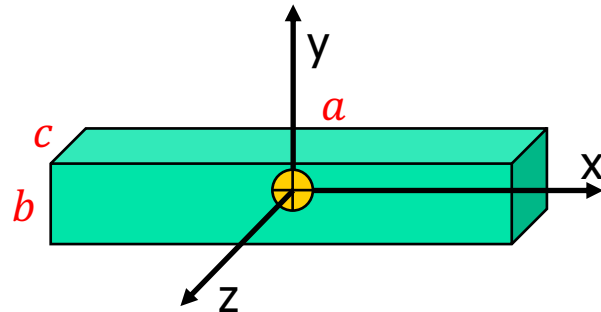
Homework #1:
provide the expressions
of the elements of Euler matrix J_c

Homework #2:
prove last equality and provide
the expressions of the elements
of inertia matrix I_c



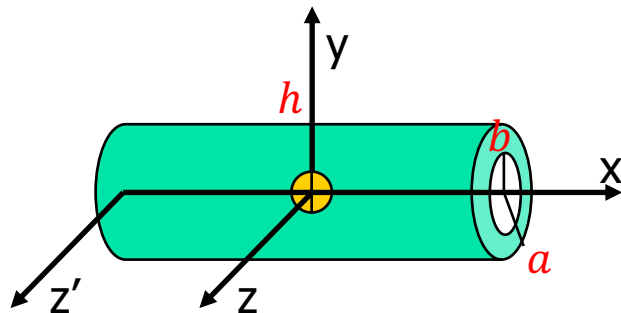
Examples of body inertia matrices

homogeneous bodies of mass m , with axes of symmetry



parallelepiped with sides
 a (length/height), b and c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12} m(b^2 + c^2) & & \\ & \frac{1}{12} m(a^2 + c^2) & \\ & & \frac{1}{12} m(a^2 + b^2) \end{pmatrix}$$



empty cylinder with length h ,
and external/internal radius a and b

$$I_c = \begin{pmatrix} \frac{1}{2} m(a^2 + b^2) & & \\ & \frac{1}{12} m(3(a^2 + b^2)^2 + h^2) & \\ & & I_{zz} \end{pmatrix} \quad I_{zz} = I_{yy}$$

$$I'_{zz} = I_{zz} + m \left(\frac{h}{2} \right)^2 \quad (\text{parallel}) \text{ axis translation theorem}$$

Steiner theorem

$$I = I_c + m(r^T r \cdot E_{3 \times 3} - r r^T) = I_c + m S^T(r) S(r)$$

body inertia matrix
relative to the CoM

identity
matrix

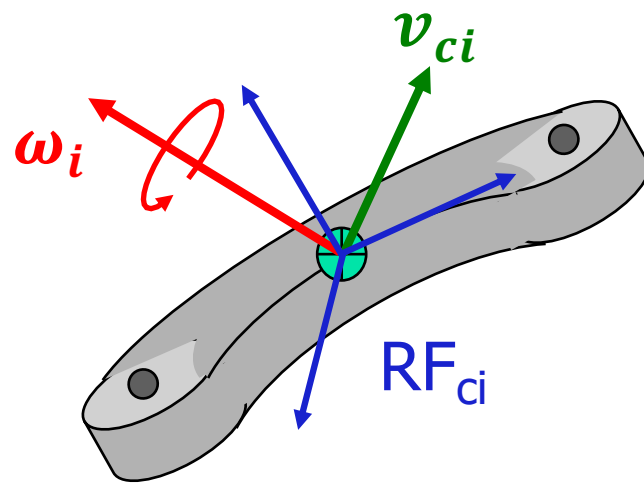
Homework #3:
prove the last equality

... its generalization:
changes on body inertia matrix
due to a pure translation r of
the reference frame

Robot kinetic energy

$$T = \sum_{i=1}^N T_i \quad \leftarrow N \text{ rigid bodies (+ fixed base)}$$

$$T_i = T_i(q_j, \dot{q}_j; j \leq i) \quad \leftarrow \text{open kinematic chain}$$



i-th link (body)
of the robot

König theorem

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

absolute velocity
of the center of mass
(CoM)

absolute
angular velocity
of whole body



Kinetic energy of a robot link

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

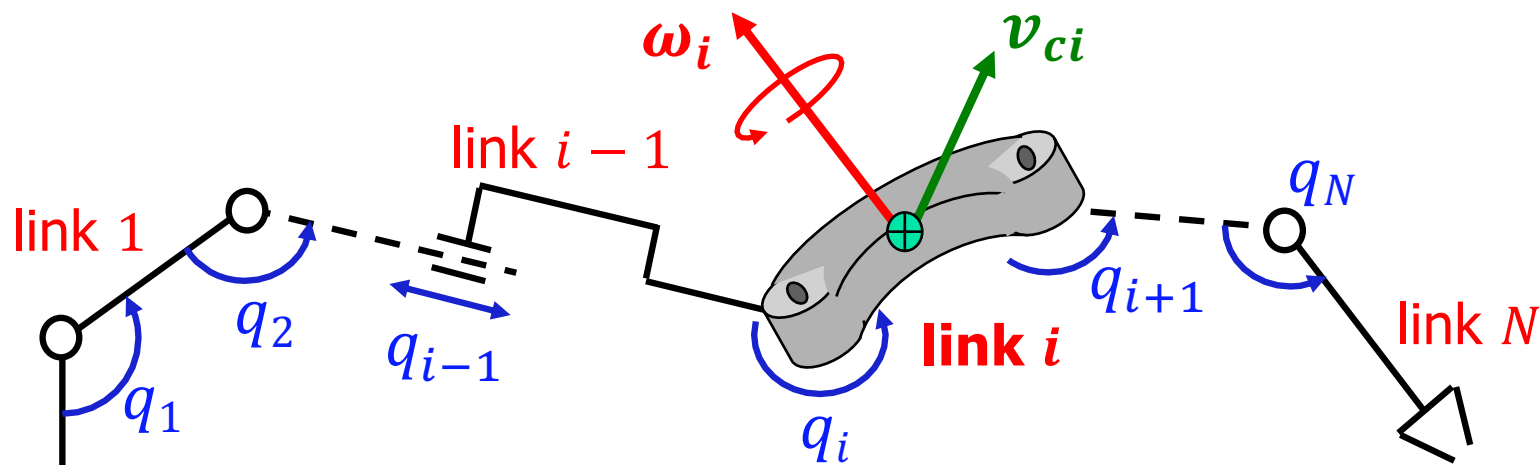
ω_i, I_{ci} should be expressed in the **same reference frame**,
but the product $\omega_i^T I_{ci} \omega_i$ is **invariant** w.r.t. any chosen frame

in frame RF_{ci} attached to (the center of mass of) link i

\uparrow
constant!

$${}^i I_{ci} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ & \int (x^2 + z^2) dm & -\int yz dm \\ \text{symm} & & \int (x^2 + y^2) dm \end{pmatrix}$$

Dependence of T from q and \dot{q}



$$v_{ci} = J_{Li}(q)\dot{q} = \begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \\ & & & 0 & & 0 \end{pmatrix} \dot{q} \left. \vphantom{\begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \\ & & & 0 & & 0 \end{pmatrix}} \right\} \text{3 rows}$$

(partial) Jacobians
typically expressed in RF_0

$$\omega_i = J_{Ai}(q)\dot{q} = \begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \\ & & & 0 & & 0 \end{pmatrix} \dot{q} \left. \vphantom{\begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \\ & & & 0 & & 0 \end{pmatrix}} \right\} \text{3 rows}$$



Final expression of T

$$T = \frac{1}{2} \sum_{i=1}^N (m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci} \omega_i)$$

NOTE 1:
in practice, **NEVER**
use this formula
(or partial Jacobians)
for computing T
 \Rightarrow a better method
is available...

$$= \frac{1}{2} \dot{q}^T \left(\sum_{i=1}^N m_i J_{Li}^T(q) J_{Li}(q) + J_{Ai}^T(q) I_{ci} J_{Ai}(q) \right) \dot{q}$$

constant if ω_i is
expressed in RF_{ci}
else

$$T = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

$${}^0I_{ci}(q) = {}^0R_i(q) {}^iI_{ci} {}^0R_i^T(q)$$

NOTE 2:
I used previously
the notation $B(q)$
for the robot
inertia matrix ...
(see past exams!)

robot (generalized) inertia matrix

- symmetric
- positive definite, $\forall q \Rightarrow$ **always invertible**



Robot potential energy

assumption: GRAVITY contribution only

$$U = \sum_{i=1}^N U_i \quad \leftarrow N \text{ rigid bodies (+ fixed base)}$$

$$U_i = U_i(q_j; \underbrace{j \leq i}) \quad \leftarrow \text{open kinematic chain}$$

$$U_i = -m_i g^T r_{0,ci}$$

{ gravity acceleration
vector

{ position of the
center of mass of link i

typically
expressed
in RF_0

dependence on q

$$\begin{pmatrix} r_{0,ci} \\ 1 \end{pmatrix} = {}^0A_1(q_1) {}^1A_2(q_2) \cdots {}^{i-1}A_i(q_i) \begin{pmatrix} r_{i,ci} \\ 1 \end{pmatrix}$$

constant
in RF_i

NOTE: need to work with **homogeneous** coordinates



Summarizing ...

kinetic
energy

$$T = \frac{1}{2} \dot{q}^T M(q) \dot{q} = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j$$

positive definite
quadratic form

$$T \geq 0, \\ T = 0 \iff \dot{q} = 0$$

potential
energy

$$U = U(q)$$

Lagrangian

$$L = T(q, \dot{q}) - U(q)$$

Euler-Lagrange
equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$k = 1, \dots, N$$

non-conservative (active/dissipative)
generalized forces **performing work** on q_k coordinate



Applying Euler-Lagrange equations

(the scalar derivation – see Appendix for vector format)

$$L(q, \dot{q}) = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \dot{q}_j \quad \rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j$$

(dependences of
elements on q
are not shown)

$$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial m_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial U}{\partial q_k}$$

LINEAR terms in ACCELERATION \ddot{q}

QUADRATIC terms in VELOCITY \dot{q}

NONLINEAR terms in CONFIGURATION q



k -th dynamic equation ...

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$\sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \left(\frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

exchanging
"mute" indices i, j

$$\dots + \sum_{i,j} \frac{1}{2} \left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \dots$$

$c_{kij} = c_{kji}$ Christoffel symbols
of the first kind



... and interpretation of dynamic terms

$$\boxed{\sum_j m_{kj}(q) \ddot{q}_j} + \boxed{\sum_{i,j} c_{kij}(q) \dot{q}_i \dot{q}_j} + \boxed{\frac{\partial U}{\partial q_k}} = u_k \quad k = 1, \dots, N$$

INERTIAL terms
CENTRIFUGAL ($i = j$) and **CORIOLIS** ($i \neq j$) terms
GRAVITY terms $g_k(q)$

$m_{kk}(q)$ = inertia at joint k when joint k accelerates ($m_{kk} > 0!!$)

$m_{kj}(q)$ = inertia "seen" at joint k when joint j accelerates

$c_{kii}(q)$ = coefficient of the centrifugal force at joint k when joint i is moving ($c_{iii} = 0, \forall i$)

$c_{kij}(q)$ = coefficient of the Coriolis force at joint k when joint i and joint j are both moving



Robot dynamic model in vector formats

1. $M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u$

k -th column
of matrix $M(q)$

$$c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q}$$

k -th component
of vector c

$$C_k(q) = \frac{1}{2} \left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q} \right)^T - \frac{\partial M}{\partial q_k} \right)$$

symmetric
matrix!

2. $M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) = u$

NOTE:

the model
is in the form

$$\Phi(q, \dot{q}, \ddot{q}) = u$$

as expected

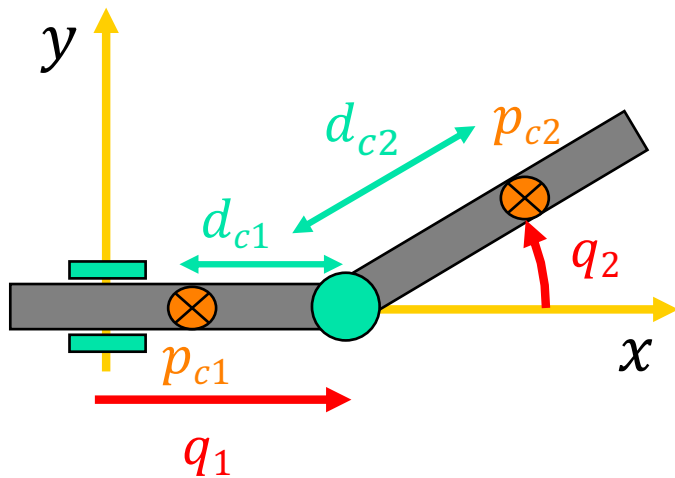
NOT a
symmetric
matrix
in general

$$s_{kj}(q, \dot{q}) = \sum_i c_{kij}(q) \dot{q}_i$$

factorization of c
by S is **not unique!**



Dynamic model of a PR robot



$$T = T_1 + T_2 \quad U = \text{constant} \Rightarrow g(q) \equiv 0$$

(on horizontal plane)

$$p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \rightarrow \|v_{c1}\|^2 = \dot{p}_{c1}^T \dot{p}_{c1} = \dot{q}_1^2$$

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2$$

$$T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2$$

$$p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \rightarrow v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \dot{q}_2 \\ d_{c2} \cos q_2 \dot{q}_2 \\ 0 \end{pmatrix} \quad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix}$$

$$T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2d_{c2} \sin q_2 \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2$$



Dynamic model of a PR robot (cont)

$$M(q) = \begin{pmatrix} \underbrace{m_1 + m_2}_{M_1} & \underbrace{-m_2 d_{c2} \sin q_2}_{M_2} \\ -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \quad c(q, \dot{q}) = \begin{pmatrix} c_1(q, \dot{q}) \\ c_2(q, \dot{q}) \end{pmatrix}$$
$$c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q}$$

where $C_k(q) = \frac{1}{2} \left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q} \right)^T - \frac{\partial M}{\partial q_k} \right)$

$$C_1(q) = \frac{1}{2} \left(\begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$c_1(q, \dot{q}) = -m_2 d_{c2} \cos q_2 \dot{q}_2^2$$

$$C_2(q) = \frac{1}{2} \left(\begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} \right) = 0$$

$$c_2(q, \dot{q}) = 0$$



Dynamic model of a PR robot (cont)

$$M(q)\ddot{q} + c(q, \dot{q}) = u$$



$$\begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\ -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} \cos q_2 \dot{q}_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

NOTE: the m_{NN} element (here, for $N = 2$) of $M(q)$ is always **constant**!

Q1: why does variable q_1 not appear in $M(q)$? ... this is a **general property**!

Q2: why Coriolis terms are not present?

Q3: when applying a force u_1 , does the second joint accelerate? ... always?

Q4: what is the expression of a factorization matrix S ? ... is it unique here?

Q5: which is the configuration with “maximum inertia”?



A structural property

Matrix $\dot{M} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix S)

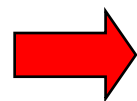
Proof

$$\dot{m}_{kj} = \sum_i \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \quad 2S_{kj} = \sum_i 2c_{kij} \dot{q}_i = \sum_i \left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i$$

$$\Rightarrow \dot{m}_{kj} - 2S_{kj} = \sum_i \left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j} \right) \dot{q}_i = n_{kj}$$

$$n_{jk} = \dot{m}_{jk} - 2S_{jk} = \sum_i \left(\frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{ji}}{\partial q_k} \right) \dot{q}_i = -n_{kj}$$

using the
symmetry of M



$$x^T (\dot{M} - 2S) x = 0, \forall x$$



Energy conservation

- total robot energy

$$E = T + U = \frac{1}{2} \dot{q}^T M(q) \dot{q} + U(q)$$

- its evolution over time (using the dynamic model)

$$\begin{aligned} \dot{E} &= \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + \frac{\partial U}{\partial q} \dot{q} \\ &= \dot{q}^T (u - S(q, \dot{q}) \dot{q} - g(q)) + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + \dot{q}^T g(q) \\ &= \dot{q}^T u + \frac{1}{2} \dot{q}^T (\dot{M}(q) - 2S(q, \dot{q})) \dot{q} \end{aligned}$$

here, any factorization of vector c by a matrix S can be used

- if $u \equiv 0$, **total energy is constant** (no dissipation or increase)

$$\dot{E} = 0 \quad \Rightarrow \quad \dot{q}^T (\dot{M}(q) - 2S(q, \dot{q})) \dot{q} = 0, \forall q, \dot{q} \quad \Rightarrow \quad \dot{E} = \dot{q}^T u$$

weaker property than skew-symmetry, as the external vector in the quadratic form is the same velocity \dot{q} that appears also inside the two internal matrices \dot{M} also S

in general, the variation of the total energy is equal to the work of non-conservative forces



Appendix:

Vector format derivation of dynamic model

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right)^T - \left(\frac{\partial L}{\partial q} \right)^T = u$$

$$L = \frac{1}{2} \dot{q}^T M(q) \dot{q} - U(q)$$

$$M(q) = \begin{pmatrix} M_1(q) & \cdots & M_i(q) & \cdots & M_N(q) \end{pmatrix} = \sum_{i=1}^N M_i(q) e_i^T$$

(0 ... 1 ... 0)
↑
i-th position

dyadic expansion

$$\left(\frac{\partial L}{\partial \dot{q}} \right)^T = (\dot{q}^T M(q))^T = M(q) \dot{q}$$

$$\rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right)^T = M(q) \ddot{q} + \dot{M}(q) \dot{q} = M(q) \ddot{q} + \sum_{i=1}^N \left(\frac{\partial M_i}{\partial q} \right) \dot{q} \dot{q}_i$$

$$\left(\frac{\partial L}{\partial q} \right)^T = \left(\frac{1}{2} \dot{q}^T \left(\sum_{i=1}^N \frac{\partial M_i(q)}{\partial q} e_i^T \right) \dot{q} - \frac{\partial U}{\partial q} \right)^T = \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial M_i(q)}{\partial q} \right)^T \dot{q}_i \dot{q} - \left(\frac{\partial U}{\partial q} \right)^T$$

this construction
gives to $\dot{M} - 2S$
skew-symmetry

$$\rightarrow M(q) \ddot{q} + \underbrace{\left(\sum_{i=1}^N \left(\frac{\partial M_i}{\partial q} - \frac{1}{2} \left(\frac{\partial M_i(q)}{\partial q} \right)^T \right) \dot{q}_i \right)}_{S(q, \dot{q})} \dot{q} + \underbrace{\left(\frac{\partial U}{\partial q} \right)^T}_{g(q)} = u$$

k-th row of matrix S

$$S_k^T(q, \dot{q}) = \dot{q}^T C_k(q)$$