Harmonic Motion

Duncan Beauch with Charlotte Hoelzl
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I. Introduction

Harmonic motion describes any type of periodic motion in which a restoring force is present that acts to return the object to its equilibrium position. Many types of harmonic motions are present in everyday phenomena such as pendulums, springs, and vibrating strings. In this experiment, we will look at the motion of a mass connected to springs on each side over a near-frictionless surface. The mass consists of a glider with two pairs of magnets on either side for dampening and a sonic reflector to be used by a PASCO ultrasonic motion sensor to record the position of the glider. The glider hovers on a pad of air provided by a tube of compressed air beneath a perforated aluminum track with notches to measure distance. The glider is attached to a spring fixed to one end of the track while the spring on the other side of the glider is connected to a variable speed motor in order to drive the oscillation in later runs. The goal for this experiment is to study the behavior of oscillations and use graphical analysis to determine physical constants and points of interest such as resonant frequency. This report covers three types of oscillations: free, damped, and driven.

II. Free Oscillation

For this setup, we want to see how the glider oscillates via the springs and no other external force such as the magnetic dampeners or driving motor. The motion is slightly dampened however due to air resistance. The glider was allowed to settle into its equilibrium position and the ultrasonic sensor was zeroed. The glider was pulled to one side by 10 mm, and 15 cycles of the oscillation was recorded. The graph of which is displayed below as Figure 1. The frequency was measured by taking the change in time over 10 cycles starting as the glider passed its equilibrium position and the average frequency is shown in Table 1, the uncertainty of which comes from variance in the ultrasonic sensor. The amplitude over these 10 cycles steadily declined by 16% of the initial amplitude due to air resistance. Then, using the curve fitting tool on Capstone, three of the cycles were fitted to a sine curve as shown in Figure 1 and their stats are shown in Table 1.

Table 1: Free Oscillation at 10 mm

Quantity	Value	Uncertainty
Average Frequency	0.8913 Hz	0.0008 Hz
Initial Amplitude	11.9 mm	0.1 mm
Amplitude Declination	1.9 mm	0.1 mm
Fitted Frequency	0.8920 Hz	0.0002 Hz
Fitted Amplitude	10.417 mm	0.016 mm

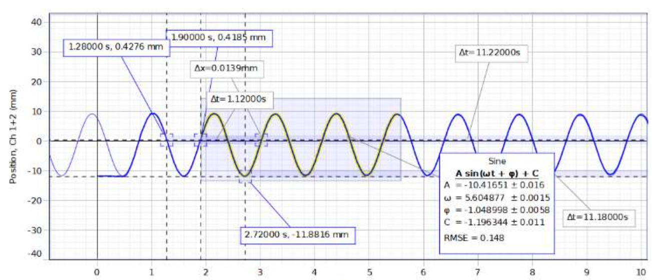

Figure 1: Graph of position vs. time for free oscillation at 10 mm

These free oscillations were then repeated for initial positions of 20 mm, 40 mm, and 6 mm. This is in order to determine any visible relation between amplitude and frequency. The average frequency does not appear to have a significant change as the amplitude varies across these oscillations as every measured frequency was nearly identical. Their data and graphs are shown in Tables 2-4.

Table 2: Free Oscillation at 20 mm		
Quantity	Value	Uncertainty
Average Frequency	0.8913 Hz	0.0008 Hz
Initial Amplitude	19.8 mm	0.1 mm
Amplitude Declination	2.1 mm	0.1 mm
Fitted Frequency	0.8917 Hz	0.0002 Hz
Fitted Amplitude	18.677 mm	0.021 mm

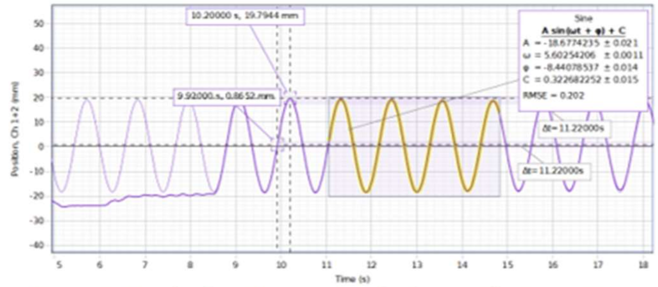

Figure 2: Graph of position vs. time for free oscillation at 20 mm

Table 3: Free Oscillation at 40 mm		
Quantity	Value	Uncertainty
Average Frequency	0.8913 Hz	0.0008 Hz
Initial Amplitude	40.6 mm	0.1 mm
Amplitude Declination	4.7 mm	0.1 mm
Fitted Frequency	0.8902 Hz	0.0001 Hz
Fitted Amplitude	39.20 mm	0.04 mm

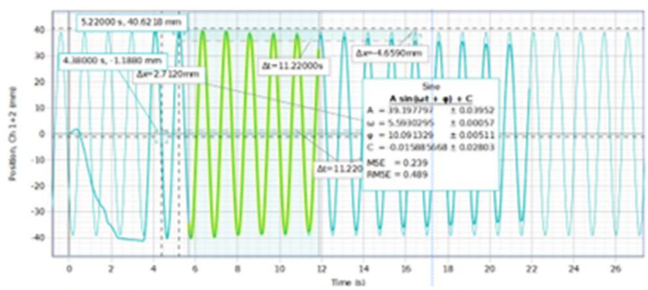

Figure 3: Position vs. time for free oscillation at 40 mm

Table 4: Free Oscillation at 60 mm		
Quantity	Value	Uncertainty
Average Frequency	0.8897 Hz	0.0008 Hz
Initial Amplitude	59.9 mm	0.1 mm
Amplitude Declination	8.1 mm	0.1 mm
Fitted Frequency	0.8898 Hz	0.0001 Hz
Fitted Amplitude	59.869 mm	0.063 mm

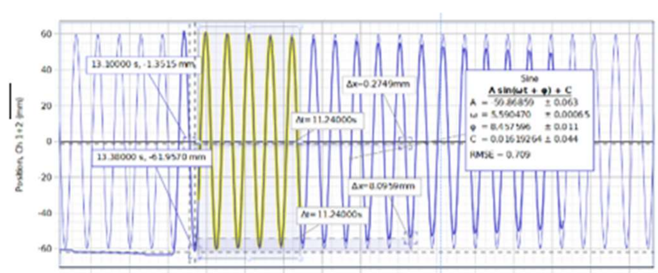

Figure 4: Position vs. time for free oscillation at 60 mm

III. Damped Oscillations

The setup and procedure from experiment 1 was continued for the following experiment except the magnets which were previously placed on top of the glider are now attached to the sides. Their positioning induces eddy currents in the aluminum track which effectively dampen the oscillation as a force roughly proportional to the velocity. This however is not perfectly achieved because the supports for the track are made from steel which also slows the glider in non-uniform quantities which results in slightly more uncertainty.

Now that a damping force has been added, the previous procedure was repeated, and the glider was pulled back to 100 mm from its starting position and released and position recorded. The resulting graph is shown in Figure 5 and using a damped sine fit the frequency F_d and decay time τ were recorded in Table 5. The damped frequency differs by only 0.2% from the frequency for a free oscillation F_0:

$$F_{diff} = \frac{F_0 - F_d}{F_0} = \frac{0.891 - 0.889}{0.891} = 0.00225$$

Decay time can also be calculated by finding the magnitude of peak amplitudes at each half cycle and creating a plot of ln(amplitude) vs. time which will fit a line with a slope of $-\frac{1}{\tau}$. This plot is shown in Figure 6. These values can then be used to calculate the oscillator's Q factor which represents the ratio of energy stored to energy lost in a single oscillation.

$$Q = \pi\tau F_d = 18.98$$

$$\frac{F_d}{F_0} = \sqrt{1 - \frac{1}{4Q^2}} = 0.99965$$

Table 5: Damped Oscillation at 100 mm

Quantity	Value	Uncertainty
Fitted Frequency	0.889 Hz	0.0005 Hz
Decay Time	6.888 s	0.017 s

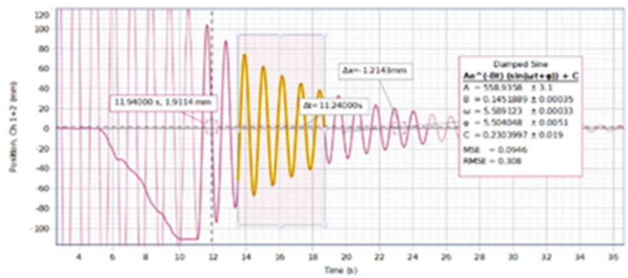

Figure 5: Position vs. time for damped oscillator at 100 mm

	Slope	Intercept
Value	-0.14729	4.641564
Error	0.001706	0.006235
r	0.998661	0.011492

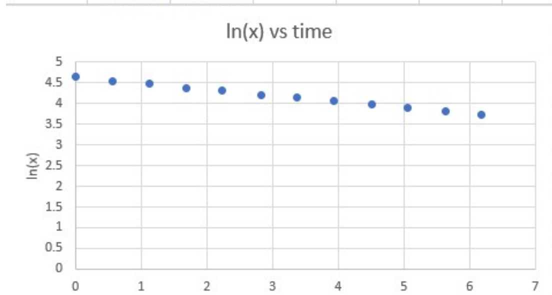

Figure 6: Graph of ln(amplitude) vs. time for damped oscillation

The magnitude of the maximum amplitudes of each half cycle plotted in Excel result in what appears to be a line with negative slope, resembling the line we would expect to see. Using the linest function to fit a least squares line to this data results in a correlation coefficient r of 0.9987, close to the maximum of 1 indicating strong correlation. The resulting decay time τ was 6.789 seconds. Using this to calculate the Q factor results in $Q \approx 19$ and a ratio of 0.99965 for damped frequency to free frequency oscillations. This confirms the theoretically expected difference between F_d and F_0 to be small.

IV. Driven Oscillations

Now we will use the damped oscillator in addition to the variable speed motor to drive one end of the glider at a given frequency. The motor was set at its lowest frequency 0.0833 Hz to create a baseline motion. The oscillation takes several cycles, in fact Q cycles, before it settles into a steady state motion. The motion achieved with the lowest frequency was fitted to a sinusoidal curve and its data can be seen in Table 6 and Figure 7.

Table 6: Lowest Frequency Driven Oscillation		
Quantity	Value	Uncertainty
Average Frequency	0.0833 Hz	0.0001Hz
Amplitude	5.93 mm	0.01 mm

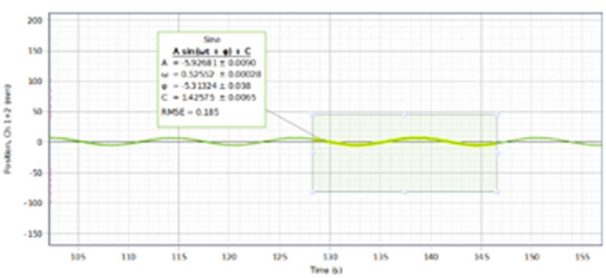

Figure 7: Position vs. time for lowest driven frequency

The driver was then set at its highest frequency of 1.566 Hz. Using the same procedure as for the lowest frequency, it was allowed to settle into consistent oscillatory motion and fitted to a sinusoidal curve. The data is shown in Table 7 and Figure 8.

Table 7: Highest Frequency Driven Oscillation		
Quantity	Value	Uncertainty
Average Frequency	1.566 Hz	0.001 Hz
Amplitude	2.846 mm	0.012 mm

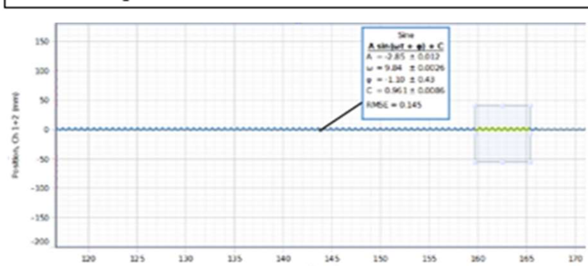

Figure 8: Position vs. time for highest driven frequency

Note that although the frequency was increased significantly, it lowered the amplitude achieved by the whole system. This is because amplitude is not directly proportional to frequency. There is a specific frequency, the resonant frequency, where the amplitude is at its maximum. This is the frequency it chooses to oscillate which we found in experiment 1 to be F_{res} = 0.890 Hz. The driver was set to this frequency and its data is recorded below in Table 8 and Figure 9.

Table 8: Resonant Frequency Driven Oscillation		
Quantity	Value	Uncertainty
Resonant Frequency	0.889 Hz	0.001 Hz
Resonant Amplitude	112.934 mm	0.035 mm

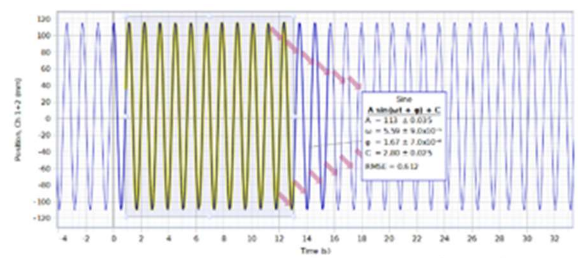

Figure 9: Position vs. time for resonant driven frequency

We can then find a different estimation of Q using the lowest frequency amplitude A_0 and the maximum amplitude A_{res}:

$$Q = \frac{A_{res}}{A_0} = 19.05$$

This closely resembles our previous calculation of Q using the decay time which found Q = 18.98 and supports our true $Q \approx 19$.

Now we want to gain insight to the relationship between driven frequency and amplitude. To do this we will record various frequencies and amplitudes for the system and plot the results. Theoretically, the curve will be symmetric around the resonant frequency which results in two frequencies giving the same amplitude, this is referred to as the upper and lower frequencies. A few key amplitudes will be the half power amplitude, when the power of the oscillation is half of the power for a resonant motion. This occurs at $A = \frac{A_{res}}{\sqrt{2}} = 79.9\ mm$ and is the inflection point of our graph for the upper and lower regions. Other amplitudes to aim for include 50% and 30% of the resonant amplitude which correspond to A = 56.5 mm and A=33.9 mm respectively. The resulting graph displayed in Figure 10 plots the normalized amplitude by the frequency at which it was achieved.

To achieve these frequencies, a marker is placed on the graph of the amplitude at the desired amplitude value and the driver frequency is adjusted until the oscillator settles into the desire motion. The frequencies and amplitudes recorded do not need to match perfectly because they will correctly match the theoretical graph either way. The reasoning for achieving these amplitudes is to spread out our datapoints for a more complete picture. The resulting data from the experiment is presented below in Table 9 and Figure 10. The plot was fitted to the equation:

Table 9: Driven Oscillations

Amplitude Type	Frequency	Normalized Amplitude A/A_0
Initial Amplitude	0.0833 Hz	1.00
30% Amplitude (Lower)	0.810 Hz	5.66
50% Amplitude (Lower)	0.845 Hz	9.33
Half-Power (Lower)	0.861 Hz	12.67
Resonant Amplitude	0.889 Hz	19.05
Half-Power (Upper)	0.913 Hz	13.22
50% Amplitude (Upper)	0.931 Hz	9.33
30% Amplitude (Upper)	0.967 Hz	5.36
Max Frequency	1.566 Hz	0.48

$$\frac{A(f)}{A_0} = \frac{Q(\frac{f_0}{f})}{\sqrt{1 + Q^2(\frac{f_0}{f} - \frac{f}{f_0})^2}}$$

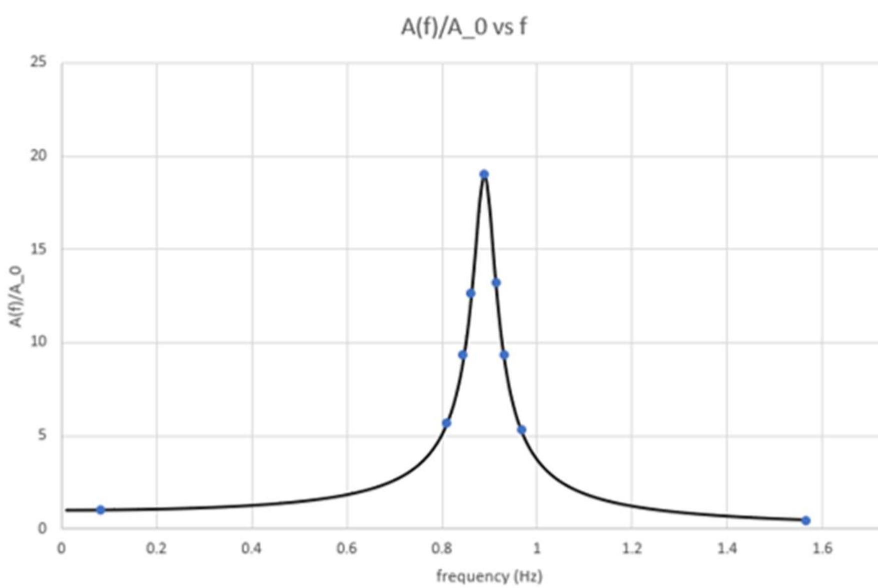

Figure 10: Normalized amplitude vs. frequency for the driven oscillator

V. Conclusion

Across all three experiments there appears to be a few couple characteristics of an oscillatory system: resonant frequency and Q factor. As shown in experiment 1, when left to oscillate on its own the system tends to run at its resonant frequency. Even in experiment 2 as the damping factor was introduced the frequency remained roughly equal to the resonant frequency. Finally, when driven at various frequencies the most efficient driving frequency for maximum amplitude occurred again at the resonant frequency. The resonant frequency of an oscillator is important to understanding its interaction with the world.

The Q factor is also related to the resonant frequency. As we saw in experiment 3, the Q factor is the ratio of the resonant frequency to its lowest frequency and when the normalized amplitude is plotted with respect to frequency, the Q factor of any given frequency is displayed as the graph itself. Q factor was even calculated in experiment 2 as a function of the decay time and damped frequency. Alternatively, we found it to also be roughly the number of cycles before the oscillator had settled into a constant rhythm for the driven frequency in experiment 3. Overall, Q factors and resonant frequencies play an important role in characterizing the motion of oscillators.