Planning and estimation using B-Splines with applications to autonomous vehicles

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Outline

- Motivation
- B-Spline Tutorial
- Applications to Planning
  - Collision Avoidance
  - Cooperative Temporal-Spatial Sequencing
  - Cooperative Informative Path Planning
- Applications to Estimation:
  - Moving horizon estimation exploiting Differential Flatness
  - Calibration

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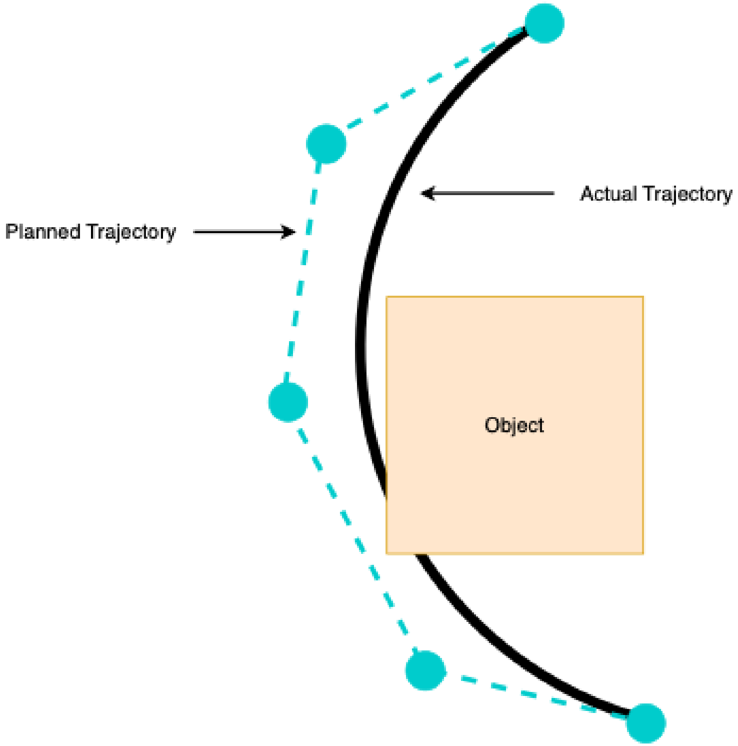
Motivation

Path planning:

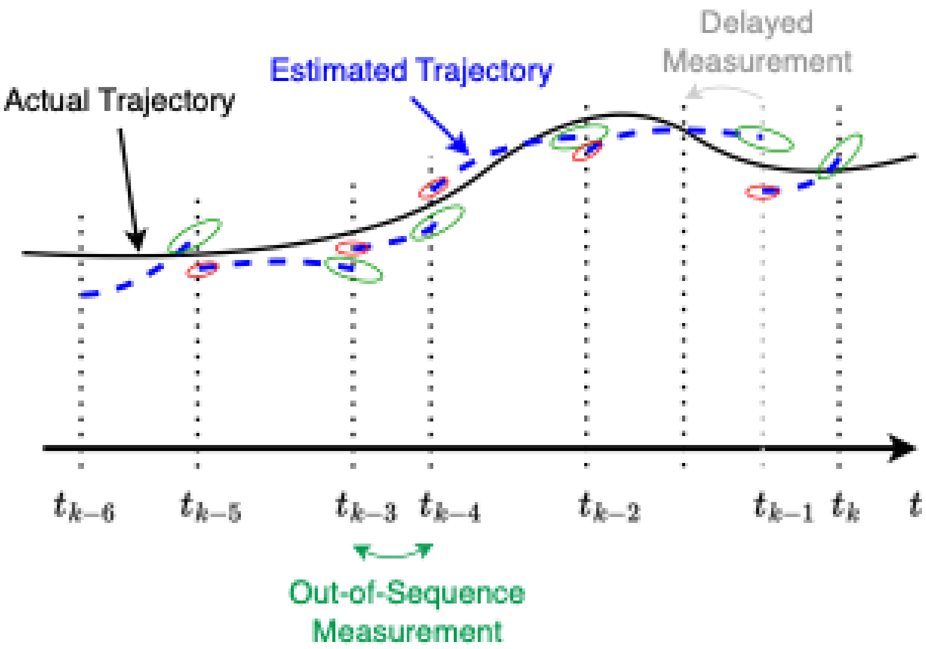
- Discrete parameterization:
    - Always a question of what happens in-between planning points.
    - Cannot guarantee collision avoidance or safety.
- Continuous parameterizations of path
    - Better chance to make provable statements with continuous paths



Motivation

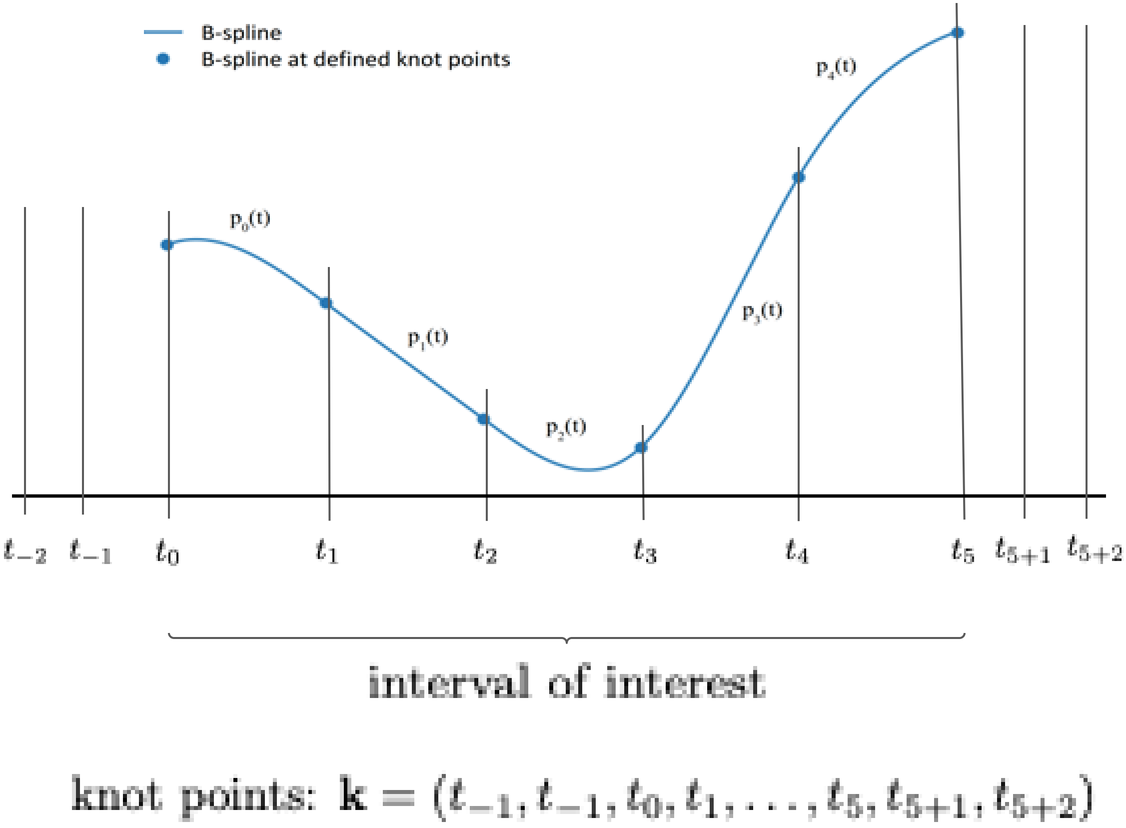
Estimation:

- Discrete parameterization:
  - Typical approach makes assumptions about clocks and sample rate.
- Continuous parameterizations
  - MHE: Easily handles
    - out of sequence measurement
    - delayed measurement
    - non-aligned clocks
    - inaccurate time stamping



What is a B-spline?

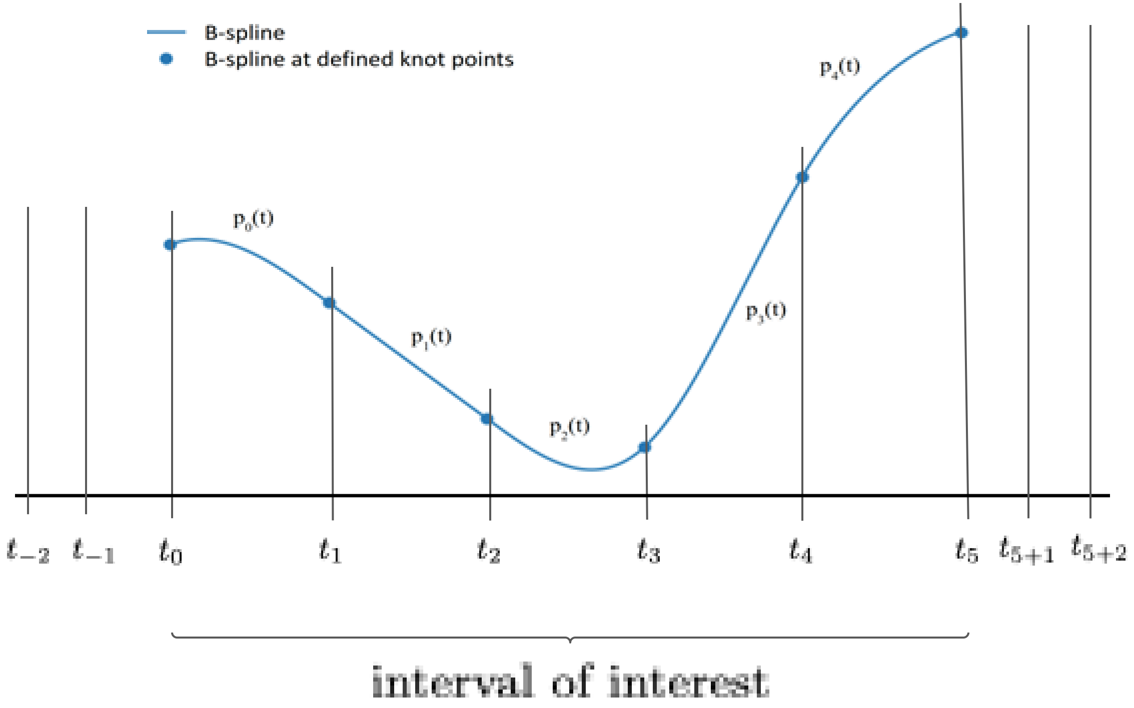
- "B" stands for basis
  - B-spline = Basis-spline
- The bases are Bernstein polynomials
  - So "B" could stand for Bernstein-splines.
- Each segment of B-spline is a Bezier curve
  - So "B" could stand for Bezier-splines



What is a B-spline?

Over each of M intervals of interest, the B-spline is a polynomial of degree d:

$$p(t) = \begin{cases} p_0(t) & \text{for} \quad t_0 \le t < t_1 \\ p_1(t) & \text{for} \quad t_1 \le t < t_2 \\ \quad \vdots & \\ p_{M-1}(t) & \text{for} \quad t_{M-1} \le t < t_M \end{cases}$$

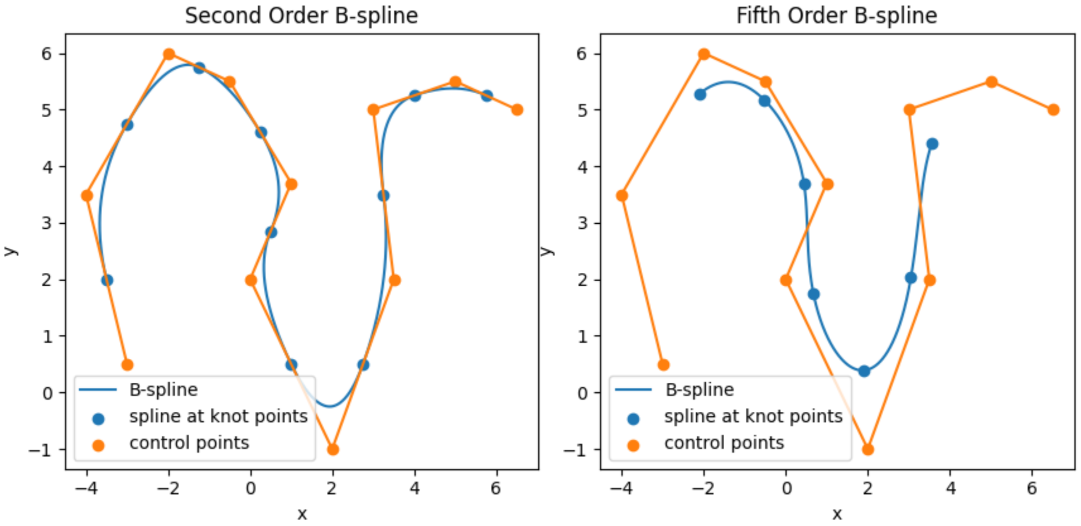


knot points: $\mathbf{k} = (t_{-1}, t_{-1}, t_0, t_1, \ldots, t_5, t_{5+1}, t_{5+2})$

What is a B-spline?

A B-spline of degree d, with M intervals of interest, is defined by

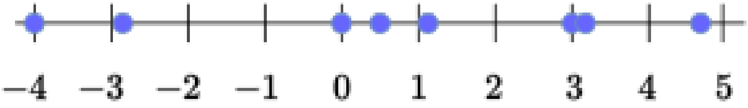
$$\mathbf{p}(t) = \sum_{m=0}^{M+d-1} \underbrace{\mathbf{c}_m}_{\text{Control points}} \underbrace{b_m^d(t, \mathbf{k})}_{\text{Basis functions}}$$



B-Spline: Knot Points

Natural knot vector: $\mathbf{k}_M^d = [\underbrace{t_{-d}, \dots, t_{-1},}_{\substack{d\text{-terms} \\ \text{non-decreasing}}} \quad \underbrace{t_0, t_1, \dots, t_M,}_{\substack{\text{interval of interest} \\ (M+1)\text{-terms} \\ \text{strictly increasing}}} \quad \underbrace{t_{M+1}, \dots, t_{M+d}}_{\substack{d\text{-terms} \\ \text{non-decreasing}}}].$

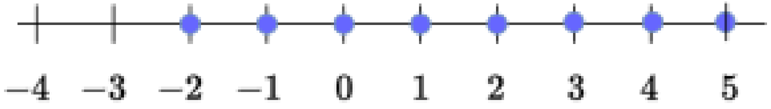
Example: $\mathbf{k}_{M=3}^{d=2} = [-4, -2.8, \vdots\, 0, 0.5, 1.1, 3.0, \vdots\, 3.1, 4.6]$



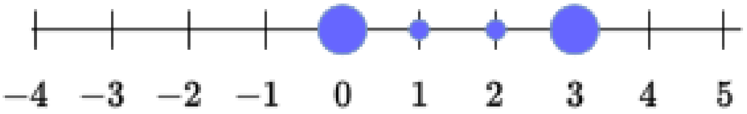
Uniform knot vector:

$$\mathbf{k}_M^d = [\underbrace{t_0 - d\Delta, \dots, t_0 - \Delta,}_{d\text{-terms}} \underbrace{t_0, t_0 + \Delta, t_0 + 2\Delta, \dots, t_0 + M\Delta,}_{\substack{\text{interval of interest} \\ (M+1)\text{-terms}}} \underbrace{t_0 + (M+1)\Delta, \dots, t_0 + (M+d)\Delta}_{d\text{-terms}}]$$

Example: $\mathbf{k}_{M=3}^{d=2} = [-2, -1, \vdots\, 0, 1, 2, 3, \vdots\, 4, 5], \quad \Delta = 1$



Clamped knot vector: $\mathbf{k}_M^d = [\underbrace{t_0, t_0, \dots, t_0,}_{d\text{-terms}} \quad \underbrace{t_0, t_1, t_2, \dots, t_M,}_{\substack{\text{interval of interest} \\ (M+1)\text{-terms} \\ \text{strickly increasing}}} \quad \underbrace{t_M, \dots, t_M}_{d\text{-terms}}],$



Example: $\mathbf{k}_{M=3}^{d=2} = [0, 0, \vdots\, 0, 1, 2, 3, \vdots\, 3, 3], \quad \Delta = 1$

B-Spline: Basis Functions

The B-spline basis function are defined by the recursive formula:

$$b_m^0(t, \mathbf{k}) = \begin{cases} 1 & \text{if } \tau_m \leq t \leq \tau_{m+1} \\ 0 & \text{otherwise} \end{cases}$$

$$b_m^d(t, \mathbf{k}) = w_m^d(t, \mathbf{k}) b_m^{d-1}(t, \mathbf{k}) + \left[1 - w_{m+1}^d(t, \mathbf{k})\right] b_{m+1}^{d-1}(t, \mathbf{k}),$$

$$w_m^d(t, \mathbf{k}) = \begin{cases} \frac{t - \tau_m}{\tau_{m+d} - \tau_m}, & \tau_{m+d} \neq \tau_m \\ 0, & \text{otherwise} \end{cases}.$$

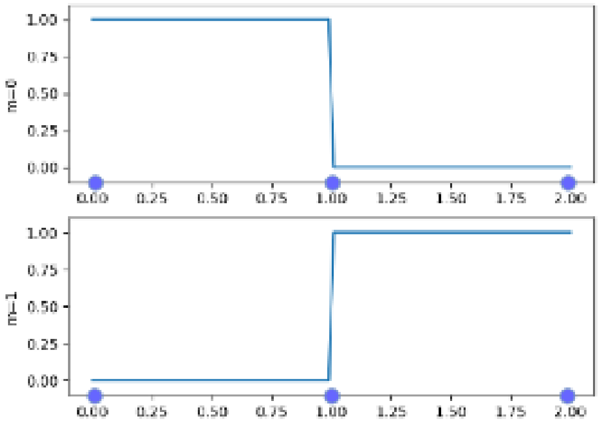
d=degree

m=index of basis

Depends on the knots $\mathbf{k}$, the degree d, and the index m

B-Spline: Degree-0 Basis Functions

$$d = 0$$
$$\mathbf{k} = (0, 1, 2)$$



$b_0^0(t, \mathbf{k})$

$b_1^0(t, \mathbf{k}) = b_0^0(t - t_1, \mathbf{k})$

For example, if the knot vector is given by

$$\mathbf{k}_{M=2}^{d=0} = [t_0, t_1, t_2] \doteq [0, 1, 2],$$

then there are two basis function of degree $d = 0$ given by

$$b_0^0(t, \mathbf{k}_2^0) = \begin{cases} 1 & \text{if } t_0 \leq t \leq t_1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1^0(t, \mathbf{k}_2^0) = \begin{cases} 1 & \text{if } t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{cases}$$

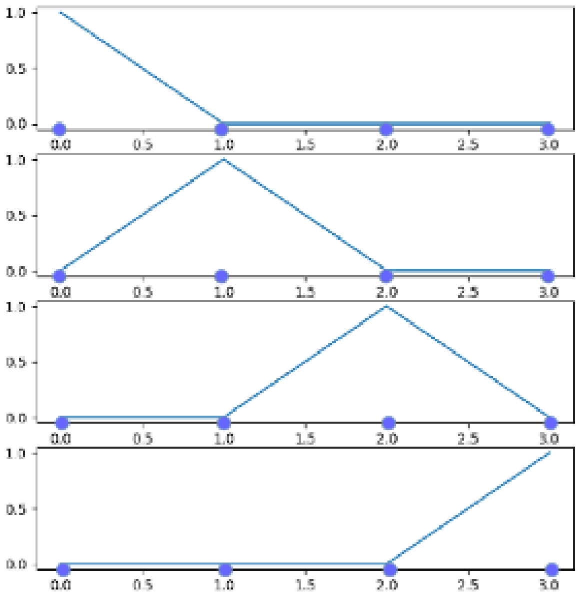
Clamped = Unclamped

All basis functions are shifted and scaled version of $b_0^0(t, \mathbf{k})$.

B-Spline: Degree-1 Basis Functions

$d = 1$

$\mathbf{k} = (-1, 0, 1, 2, 3, 4)$



$b_0^1(t,\mathbf{k})$

$b_1^1(t,\mathbf{k})$

$b_2^1(t,\mathbf{k})$

$b_3^1(t,\mathbf{k})$

Clamped = Unclamped

Knot vector:

$$\mathbf{k}^{d=1}_{M=3} = [t_{-1}, t_0, t_1, t_2, t_3, t_4] \doteq [-1, 0, 1, 2, 3, 4],$$

$$b_0^1(t,\mathbf{k}) = \frac{t - t_{-1}}{t_0 - t_{-1}} b_0^0(t,\mathbf{k}) + \frac{t_1 - t}{t_1 - t_0} b_1^0(t,\mathbf{k}) = \begin{cases} 0 & \text{if } t_{-1} \le t \le t_0 \\ 1 - t & \text{if } t_0 \le t \le t_1 \end{cases}$$

$$b_1^1(t,\mathbf{k}) = \frac{t - t_0}{t_1 - t_0} b_1^0(t,\mathbf{k}) + \frac{t_2 - t}{t_2 - t_1} b_2^0(t,\mathbf{k}) = \begin{cases} t & \text{if } t_0 \le t \le t_1 \\ 2 - t & t_1 \le t \le t_2 \\ 0 & \text{otherwise} \end{cases}$$

$$b_2^1(t,\mathbf{k}) = \frac{t - t_1}{t_2 - t_1} b_2^0(t,\mathbf{k}) + \frac{t_3 - t}{t_3 - t_2} b_2^0(t,\mathbf{k}) = \begin{cases} t - 1 & \text{if } t_1 \le t \le t_2 \\ 3 - t & t_2 \le t \le t_3 \\ 0 & \text{otherwise} \end{cases}$$

$$b_3^1(t,\mathbf{k}) = \frac{t - t_3}{t_4 - t_3} b_3^0(t,\mathbf{k}) + \frac{t_4 - t}{t_4 - t_3} b_4^0(t,\mathbf{k}) = \begin{cases} t - 3 & \text{if } t_2 \le t \le t_3 \\ 0 & \text{if } t_3 \le t \le t_4 \end{cases}$$

All basis functions are shifted and scaled version of $b_1^1(t,\mathbf{k})$.

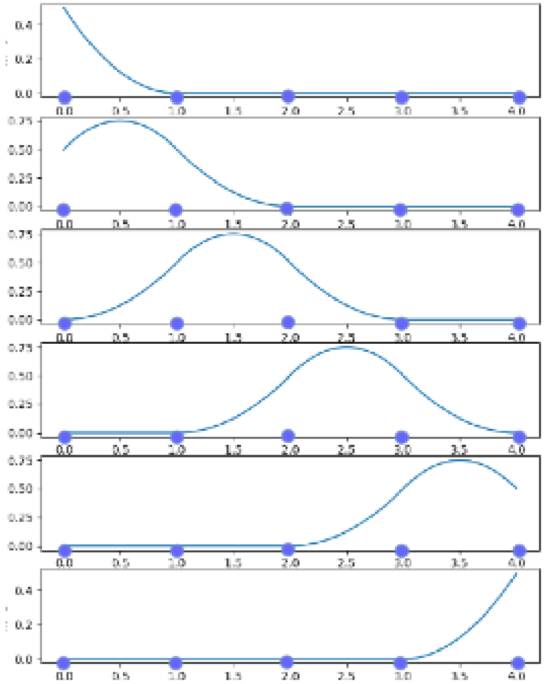
B-Spline: Degree-2 Basis Functions, unclamped

$d = 2$

$\mathbf{k} = (-2, -1, 0, 1, 2, 3, 4, 5, 6)$

Knot vector:

$\mathbf{k}_{M=4}^{k=2} = [t_{-2}, t_{-1}, t_0, t_1, t_2, t_3, t_4, t_5, t_6] \doteq [-2, -1, 0, 1, 2, 3, 4, 5, 6],$



$b_0^2(t, \mathbf{k})$

$b_1^2(t, \mathbf{k})$

$b_2^2(t, \mathbf{k})$

$b_3^2(t, \mathbf{k})$

$b_4^2(t, \mathbf{k})$

$b_5^2(t, \mathbf{k})$

$$b_0^2(t,\mathbf{k}) = \frac{t - t_{-2}}{t_0 - t_{-2}} b_0^1(t,\mathbf{k}) + \frac{t_1 - t}{t_1 - t_{-1}} b_1^1(t,\mathbf{k}) = \begin{cases} (1-t)^2 & \text{if } t_0 \le t \le t_1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1^2(t,\mathbf{k}) = \frac{t - t_{-1}}{t_1 - t_{-1}} b_1^1(t,\mathbf{k}) + \frac{t_2 - t}{t_2 - t_0} b_2^1(t,\mathbf{k}) = \begin{cases} t(2 - \frac{3}{2}t) & \text{if } t_0 \le t \le t_1 \\ \frac{(2-t)^2}{2} & t_1 \le t \le t_2 \\ 0 & \text{otherwise} \end{cases}$$

$$b_2^2(t,\mathbf{k}) = \frac{t - t_0}{t_2 - t_0} b_2^1(t,\mathbf{k}) + \frac{t_3 - t}{t_3 - t_1} b_3^1(t,\mathbf{k}) = \begin{cases} \frac{t^2}{2} & \text{if } t_0 \le t \le t_1 \\ -\frac{3}{2}t^2 + \frac{7}{2}t - \frac{3}{2} & \text{if } t_1 \le t \le t_2 \\ \frac{(3-t)^2}{2} & \text{if } t_2 \le t \le t_3 \\ 0 & \text{otherwise} \end{cases}$$

$$b_3^2(t,\mathbf{k}) = \frac{t - t_1}{t_3 - t_1} b_2^1(t,\mathbf{k}) + \frac{t_4 - t}{t_4 - t_2} b_4^1(t,\mathbf{k}) = \begin{cases} \frac{(t-1)^2}{2} & \text{if } t_1 \le t \le t_2 \\ -\frac{3}{2}(t-1)^2 + \frac{7}{2}(t-1) - \frac{3}{2} & \text{if } t_2 \le t \le t_3 \\ \frac{(4-t)^2}{2} & \text{if } t_3 \le t \le t_4 \\ 0 & \text{otherwise} \end{cases}$$

$$b_4^2(t,\mathbf{k}) = \frac{t - t_2}{t_4 - t_4} b_3^1(t,\mathbf{k}) + \frac{t_5 - t}{t_5 - t_6} b_4^1(t,\mathbf{k}) = \begin{cases} \frac{(t-2)^2}{2} & \text{if } t_2 \le t \le t_3 \\ -\frac{3}{2}(t-1)^2 + \frac{15}{2}(t-1) - \frac{15}{2} & \text{if } t_3 \le t \le t_4 \\ 0 & \text{otherwise} \end{cases}$$

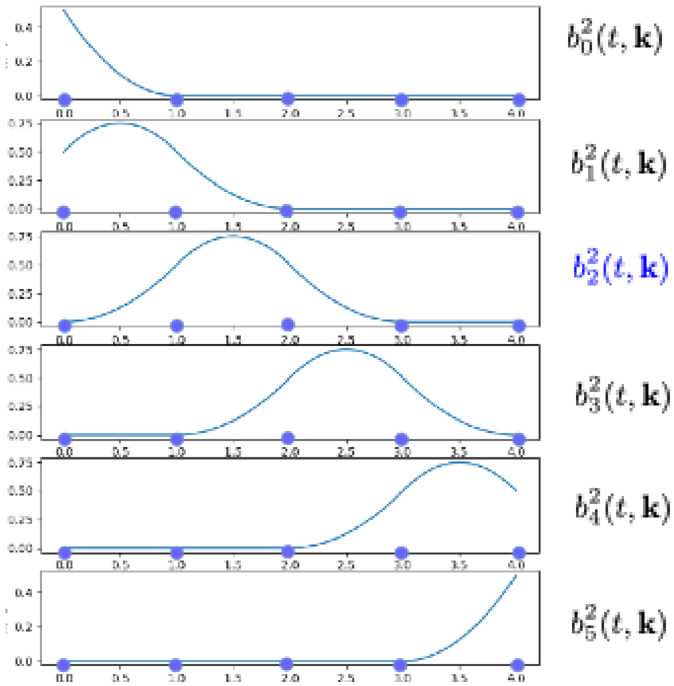
$$b_5^2(t,\mathbf{k}) = \frac{t - t_3}{t_5 - t_3} b_3^1(t,\mathbf{k}) + \frac{t_6 - t}{t_6 - t_4} b_4^1(t,\mathbf{k}) = \begin{cases} (t-3)^2 & \text{if } t_3 \le t \le t_4 \\ 0 & \text{otherwise} \end{cases}.$$

All basis functions are shifted and scaled version of $b_2^2(t, \mathbf{k})$.

B-Spline: clamped vs. unclamped

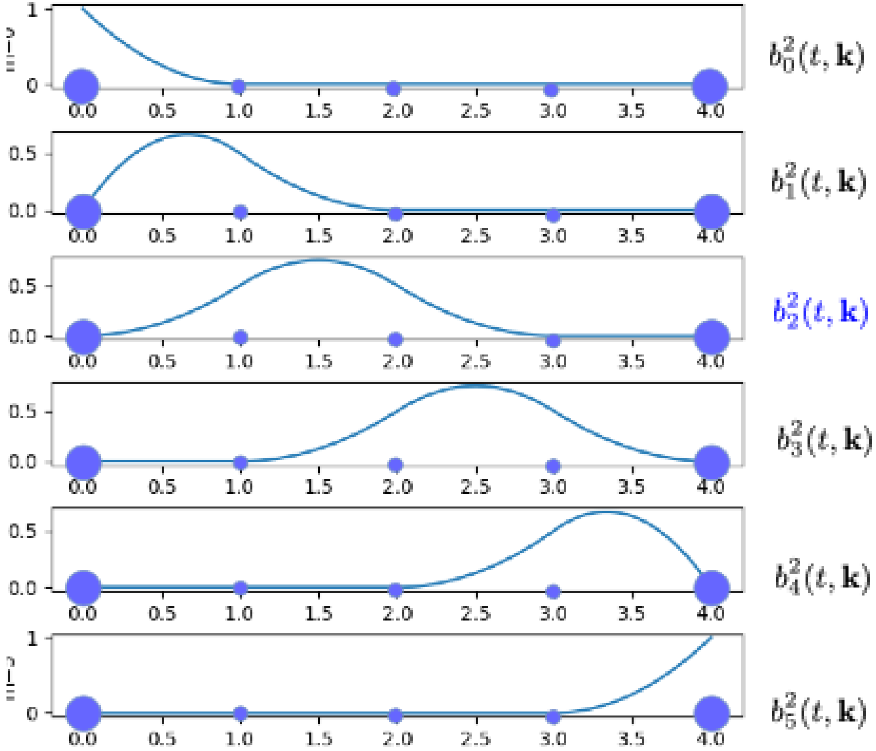
$d = 2$

$\mathbf{k} = (-2, -1, 0, 1, 2, 3, 4, 5, 6)$



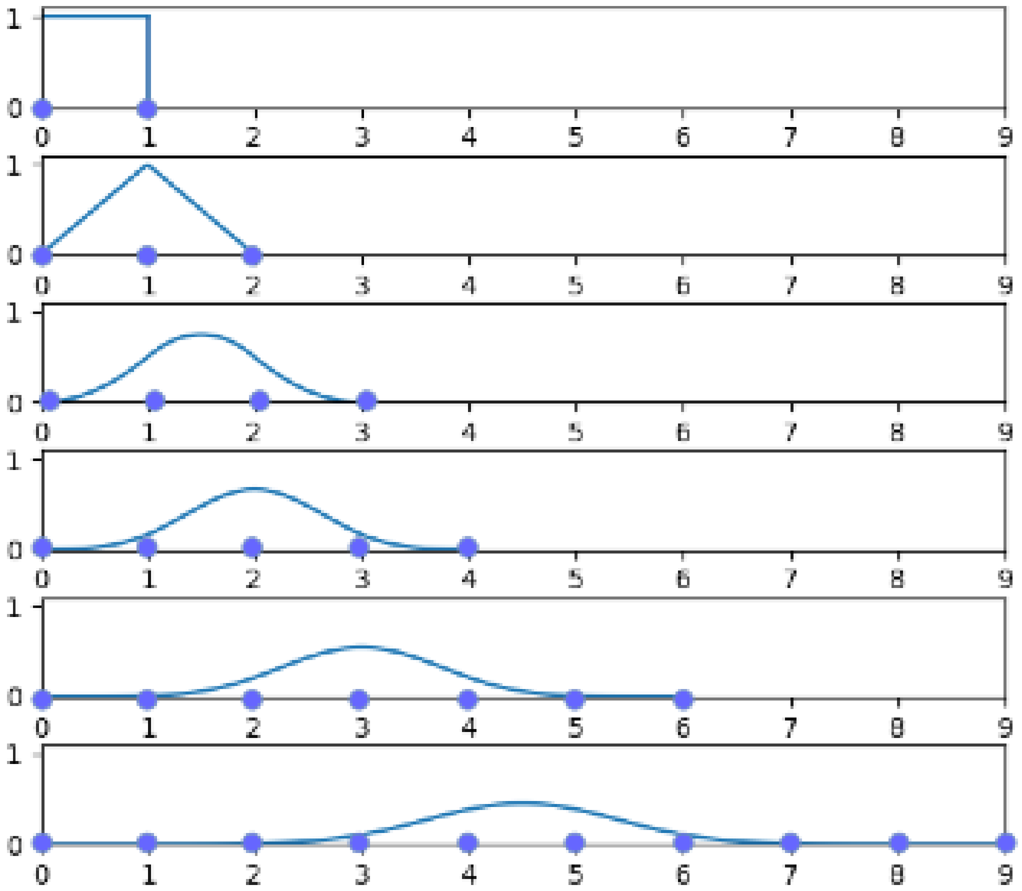
$d = 2$

$\mathbf{k} = (0, 0, 0, 1, 2, 3, 4, 4, 4)$



The basis function $b_2^2(t, \mathbf{k})$ still plays a central role.

B-Spline: Central Basis Functions



$b^{d=0}_{M=0}(t)$

$\mathbf{k} = [0, 1]$

$b^{d=1}_{M=1}(t)$

$\mathbf{k} = [-1, 0, 1, 2, 3]$

$b^{d=2}_{M=2}(t)$

$\mathbf{k} = [-2, -1, 0, 1, 2, 3, 4, 5]$

$b^{d=3}_{M=3}(t)$

$\mathbf{k} = [-3, -2, -1, 0, 1, 2, 3, 4,, 5, 6, 7]$

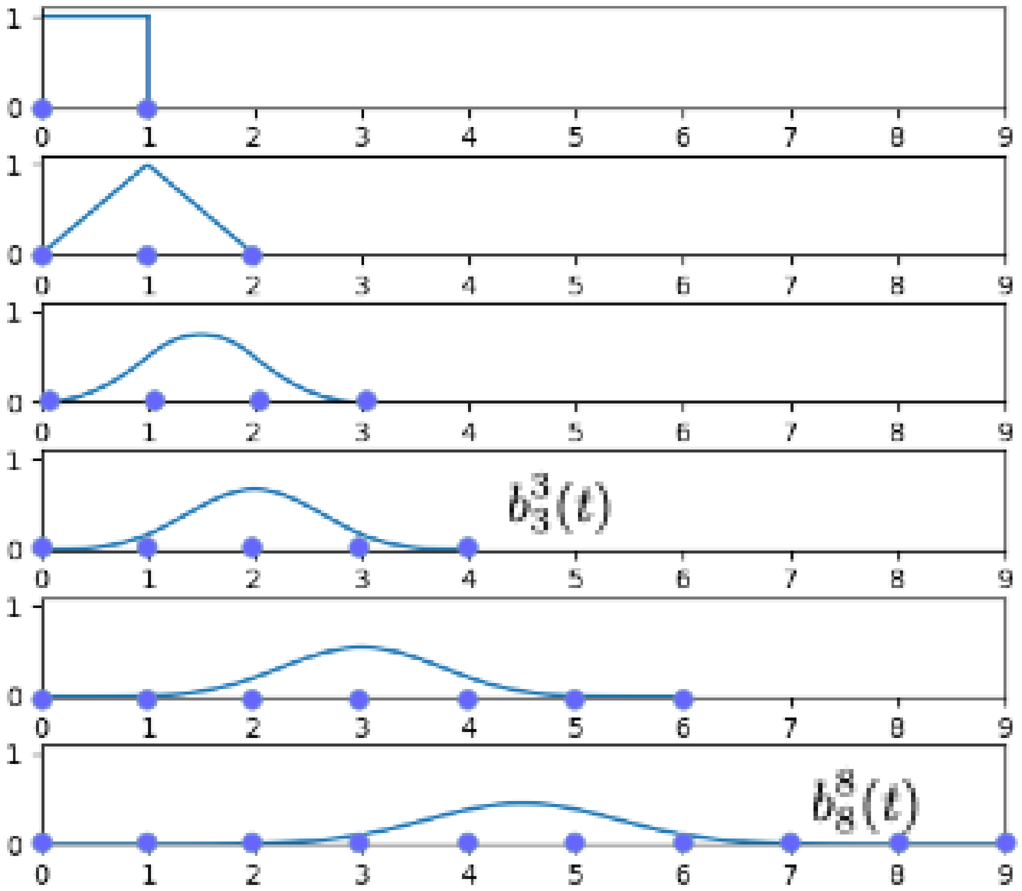
$b^{d=5}_{M=5}(t)$

$\mathbf{k} = [-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]$

$b^{d=8}_{M=8}(t)$

$$\mathbf{k} = [-8, -7, -6, -5, -4, -3, -2, -1, \\ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \\ 10, 11, 12, 13, 14, 15, 16, 17]$$

B-Spline: Central Basis Functions



Note how little difference there is between $b_3^3(t)$ and higher order bases like $b_8^8(t)$.

Higher order bases don't have a lot of addition expressive power.

But they do add degrees or smoothness!

B-Spline Properties: Shift and Scale Invariance

Lemma. (Shift Invariance)
Shifting the knot sequence:

$$\mathbf{k} = [\tau_0, \tau_1, \ldots \tau_T] \quad \longrightarrow \quad \mathbf{k} + \Delta\mathbf{1} = [\tau_0 + \Delta, \tau_1 + \Delta, \ldots \tau_T + \Delta]$$

is equivalent to shifting the basis functions (or spline):

$$b_j^d(t, \mathbf{k} + \Delta\mathbf{1}) = b_j^d(t - \Delta, \mathbf{k})$$

Lemma. (Scale Invariance)
Scaling the knot sequence:

$$\mathbf{k} = [\tau_0, \tau_1, \ldots \tau_T] \quad \longrightarrow \quad \alpha\mathbf{k} = [\alpha\tau_0, \alpha\tau_1, \ldots \alpha\tau_T]$$

is equivalent to scaling the basis functions (or spline):

$$b_j^d(t, \alpha\mathbf{k}) = b_j^d(t/\alpha, \mathbf{k})$$

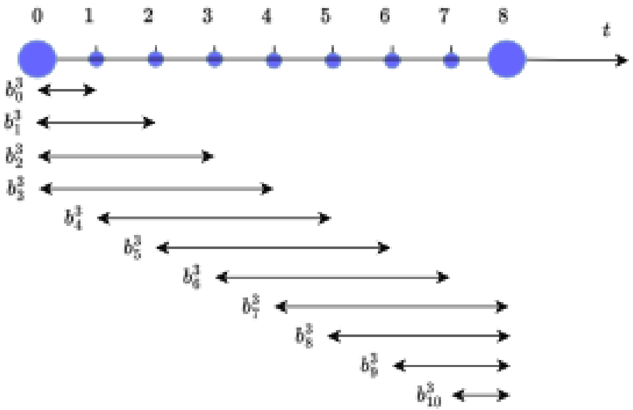
B-Spline Properties: Finite Support

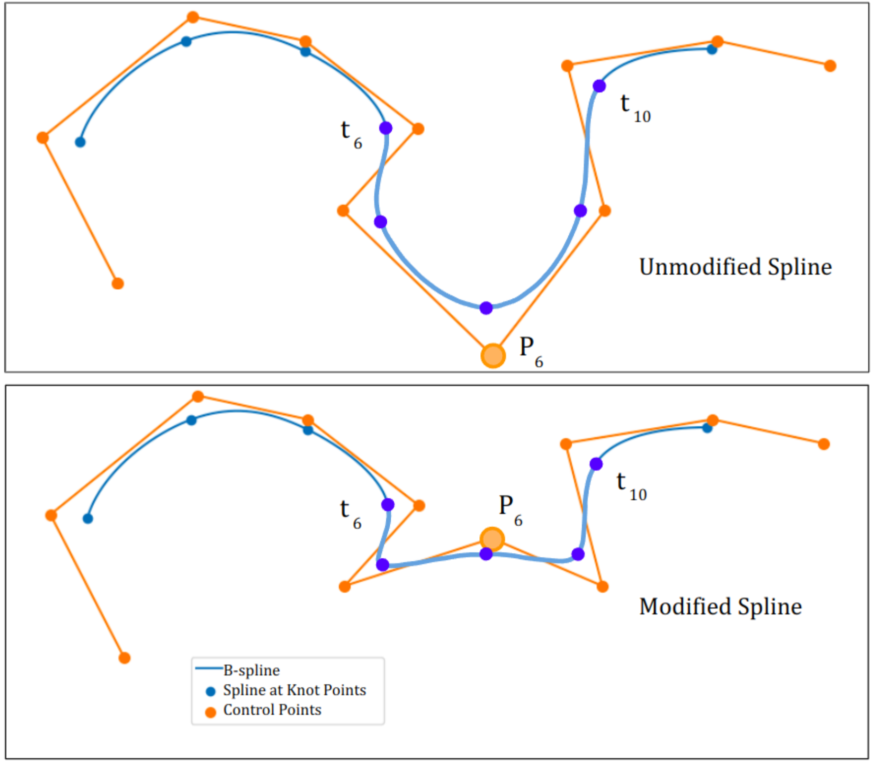
Lemma. (Finite Support)
For any time $t \in [t_s, t_{s+1}]$, there are only $d+1$ non-zero basis functions:

$$p(t) = \sum_{m=0}^{M+d-1} c_m b_m^d(t, k_M^d) = \sum_{m=s}^{s+d} c_m b_m^d(t, k_M^d).$$

Implication: There are only $d+1$ control points that influence $p(t)$ at any instance of time.

$$\mathbf{t}_8^3 = [0, 0, 0, \vdots 0, 1, 2, 3, 4, 5, 6, 7, 8, \vdots 8, 8, 8]$$

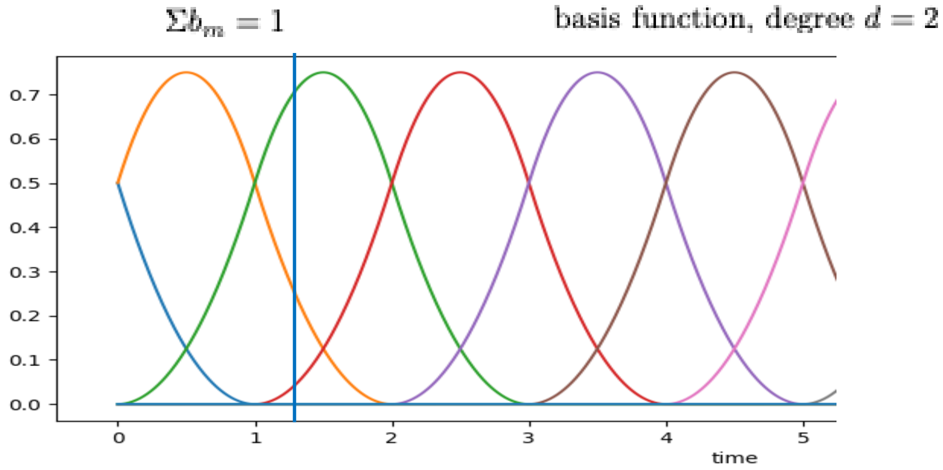


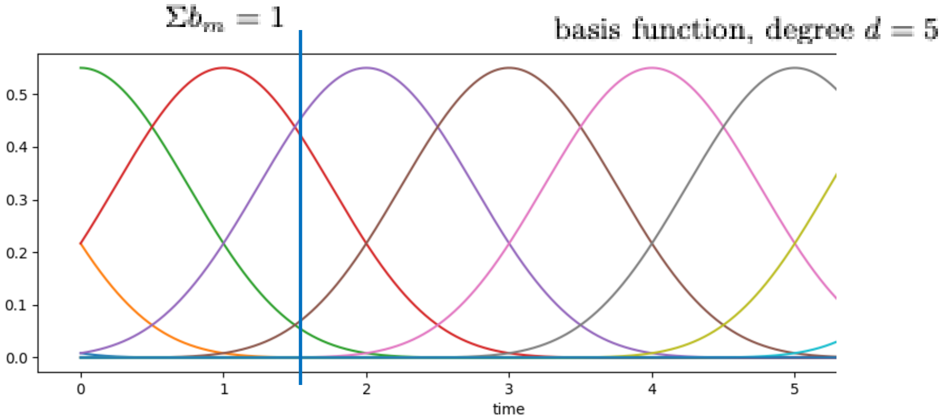


B-Spline Properties: Convex Hull

Lemma. (Partition of Unity)
For any time $t \in [t_s, t_{s+1}]$, the $d+1$ basis functions sum to unity:

$$\sum_{m=0}^{M+d-1} b_m^d(t, \mathbf{k}_M^d) = \sum_{m=s}^{s+d} b_m^d(t, \mathbf{k}_M^d) = 1$$



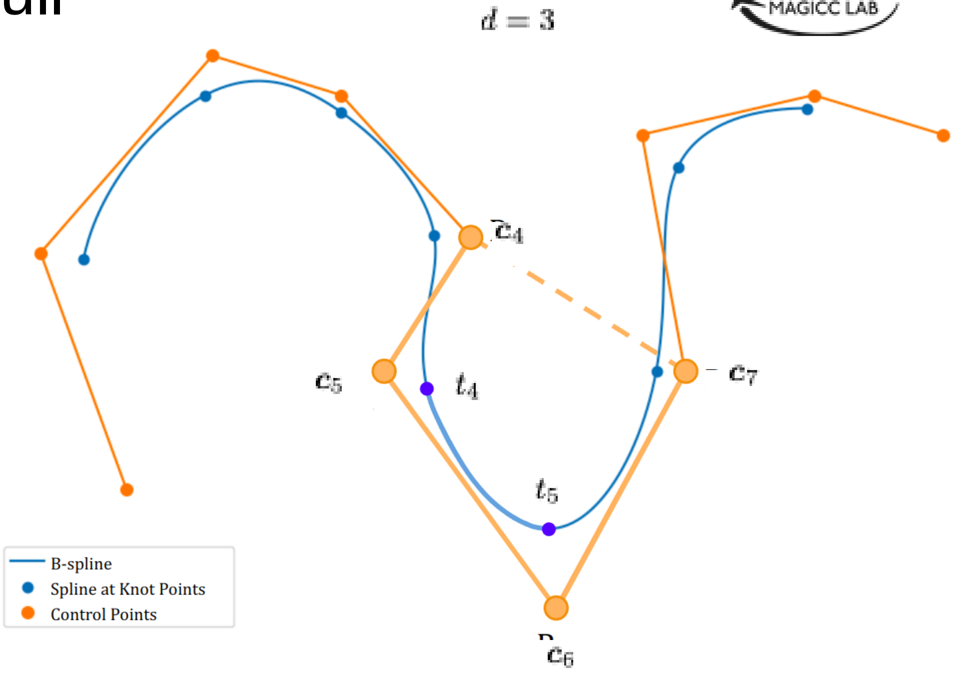


B-Spline Properties: Convex Hull

Corollary. (Convex Hull)
For any time $t \in [t_s, t_{s+1}]$, the spline

$$\mathbf{p}(t) = \sum_{m=0}^{M+d-1} c_m b_m^d(t, \mathbf{k}_M^d) = \sum_{m=s}^{s+d} c_m b_m^d(t, \mathbf{k}_M^d)$$

is contained in the convex hull of $\{c_s, \ldots, c_{s+d}\}$

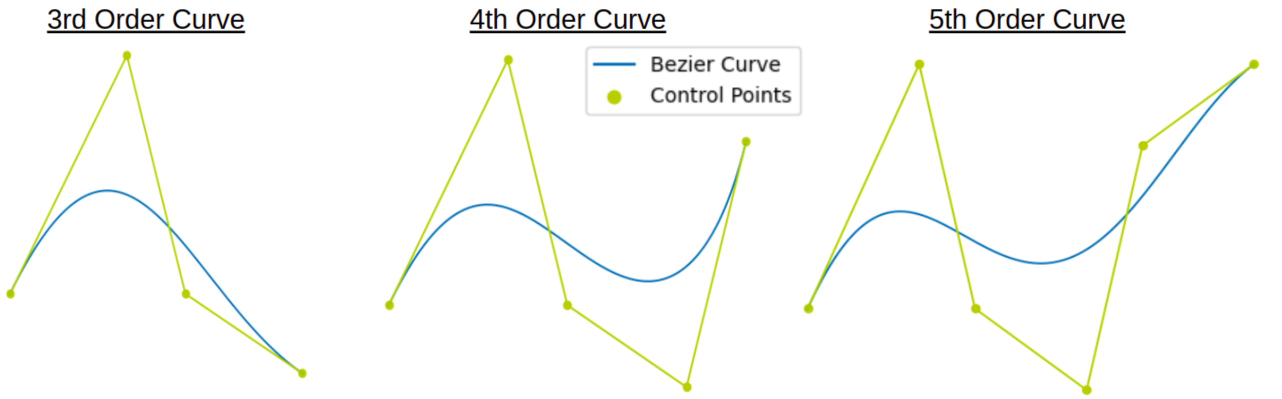


The spline segment from $[t_4, t_5]$ is contained in

$\text{conv}\{c_4, c_5, c_6, c_7\}$

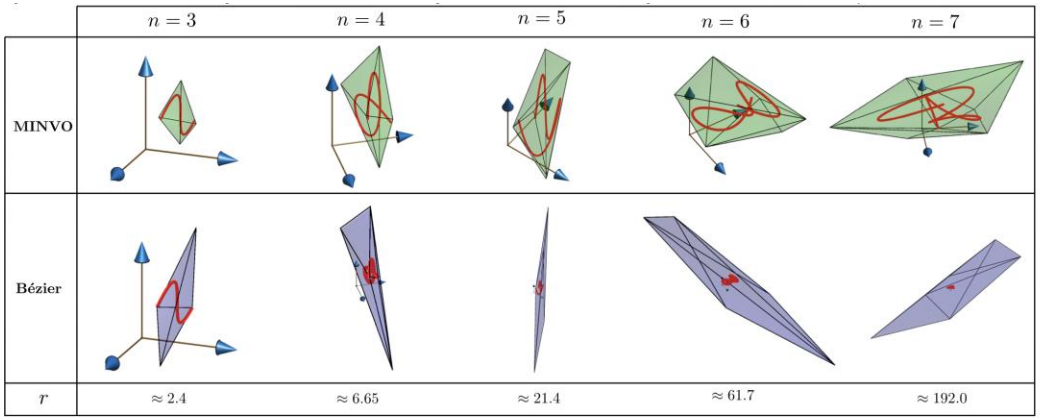
Bezier Curves: Convex Hull

- Every segment of a B-spline is a Bezier curve, but with different control points: the curve always starts and ends on a control point
- Bezier curves also enjoy a convex hull property.
- Often the convex hull is smaller for Bezier curves.

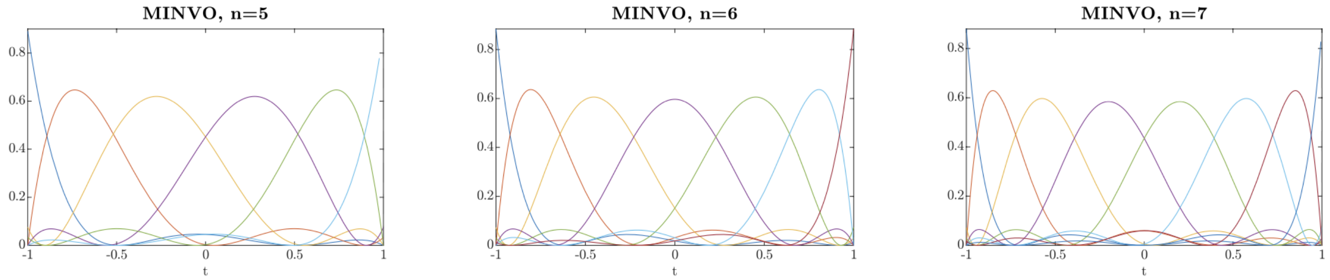


MINVO Curves: Convex Hull

MINVO basis:
Given a polynomial curve $\boldsymbol{p}(t)$ of degree d defined on $[t_0, t_1]$, find the minimum volume simplex (control points) so that $\boldsymbol{p}(t) \subseteq \text{conv}(\boldsymbol{C})$.

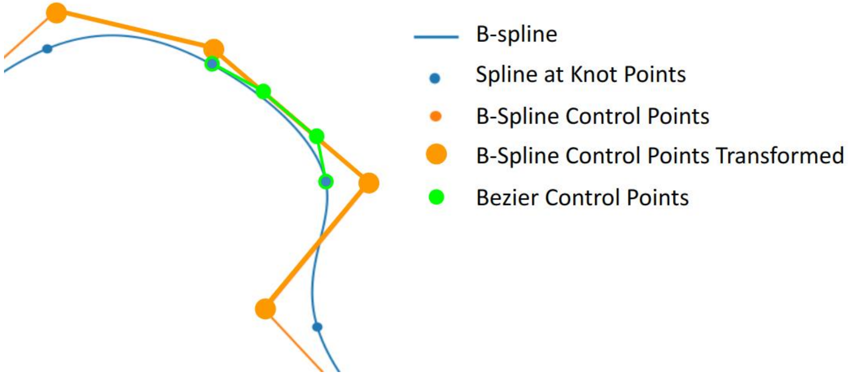


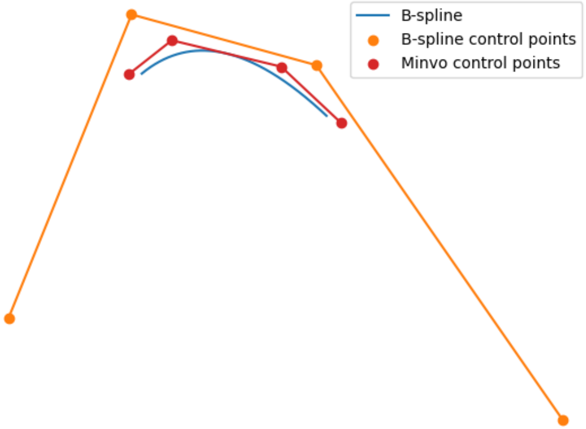
Generates MINVO basis functions, similar to Bezier curves.



MINVO Basis: Finding Simplexes with Minimum Volume Enclosing Polynomial Curves, Jesus Tordesillas and Jonathan P. How, Computer Aided Design, vol 151, 2022

B-Spline vs. Bezier vs MINVO Bounds





- The bounding simplexes for Bezier and MINVO are much tighter than for B-splines
- Bezier and MINVO are only valid over *one* time interval (not splines)
- For each time interval, there are well-defined matrix transformations between control points for B-splines, Bezier, and MINVO

Therefore, for applications like path planning, collision checks can be performed using MINVO simplexes instead of B-spline simplexes.

Derivatives of B-Spline

Stack the basis functions as

$$\boldsymbol{b}_M^j(t, \boldsymbol{k}_M^d) \doteq \begin{pmatrix} b_0^j(t, \boldsymbol{k}_M^d) \\ b_1^j(t, \boldsymbol{k}_M^d) \\ \vdots \\ b_{M+2d-j-1}^j(t, \boldsymbol{k}_M^d) \end{pmatrix}, \qquad 0 \leq j \leq d,$$

Collect the control points as

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_0 & \boldsymbol{c}_1 & \dots & \boldsymbol{c}_{M+d-1} \end{pmatrix}$$

Therefore B-spline can be written as

$$\boldsymbol{p}(t) = \boldsymbol{C}\boldsymbol{b}_M^d(t, \boldsymbol{k}_M^d), \qquad t \in \mathrm{span}(\boldsymbol{k}_M^d).$$

Derivatives of B-Spline

Lemma Given spline of degree d

$$\boldsymbol{p}(t) = \boldsymbol{C}\boldsymbol{b}_M^d(t)$$

its time derivative is given by

$$\frac{d\boldsymbol{p}}{dt}(t) = \boldsymbol{C}D_M^d\boldsymbol{b}_M^{d-1}(t)$$

where

$$D_M^d = -\begin{bmatrix} \bar{D}_M^d \\ \mathbf{0}_{1\times(M+d-1)} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{1\times(M+d-1)} \\ \bar{D}_M^d \end{bmatrix},$$

and where

$$\bar{D}_M^d = \operatorname{diag}\left(\frac{d}{t_{-d+1} - t_1}, \frac{d}{t_{-d+2} - t_2}, \dots, \frac{d}{t_{M+2d-2} - t_{M+d-2}}\right)$$

Take Away: To differentiate, control points changed by matrix multiply and reducing the degree of the spline basis.

Derivatives of B-Spline

For uniform clamped the control points for the derivative are

$$C' \doteq CD_M^k = \begin{pmatrix} \frac{d}{1}(\mathbf{c}_1 - \mathbf{c}_0)^\top \\ \vdots \\ \frac{d}{d-1}(\mathbf{c}_{d-2} - \mathbf{c}_{d-3})^\top \\ (\mathbf{c}_{d-1} - \mathbf{c}_{d-2})^\top \\ \vdots \\ (\mathbf{c}_{M+1} - \mathbf{c}_M)^\top \\ \frac{d}{d-1}(\mathbf{c}_{M+2} - \mathbf{c}_{M+1})^\top \\ \vdots \\ \frac{d}{1}(\mathbf{c}_{M+d-1} - \mathbf{c}_{M+d-2})^\top \end{pmatrix}^\top$$

For uniform unclamped the control points for the derivative are

$$C' \doteq CD_M^k = \begin{pmatrix} (\mathbf{c}_1 - \mathbf{c}_0)^\top \\ \vdots \\ (\mathbf{c}_{d-1} - \mathbf{c}_{d-2})^\top \\ \vdots \\ (\mathbf{c}_{M+1} - \mathbf{c}_M)^\top \\ \vdots \\ (\mathbf{c}_{M+d-1} - \mathbf{c}_{M+d-2})^\top \end{pmatrix}^\top$$

Implication for Path Planning: Clamped, uniform

If the desired B-spline trajectory with $d \geq 3$ has the following desired endpoint conditions:

$$\begin{aligned}
\text{Initial position:} \quad & \boldsymbol{p}(0) \stackrel{des}{=} \boldsymbol{p}_0 \\
\text{Final position:} \quad & \boldsymbol{p}(M) \stackrel{des}{=} \boldsymbol{p}_f \\
\text{Initial velocity:} \quad & \frac{d\boldsymbol{p}}{dt}(0) \stackrel{des}{=} \mathbf{v}_0 \\
\text{Final velocity:} \quad & \frac{d\boldsymbol{p}}{dt}(M) \stackrel{des}{=} \mathbf{v}_f, \\
\text{Initial acceleration:} \quad & \frac{d^2\boldsymbol{p}}{dt^2}(0) \stackrel{des}{=} \mathbf{a}_0 \\
\text{Final acceleration:} \quad & \frac{d^2\boldsymbol{p}}{dt^2}(M) \stackrel{des}{=} \mathbf{a}_f,
\end{aligned}$$

then the first and last three control points satisfy

$$\begin{aligned}
\mathbf{c}_0 &= \boldsymbol{p}_0 \\
\mathbf{c}_1 &= \boldsymbol{p}_0 + \frac{1}{d}\mathbf{v}_0 \\
\mathbf{c}_2 &= \boldsymbol{p}_0 + \frac{3}{d}\mathbf{v}_0 + \frac{2}{(d)(d-1)}\mathbf{a}_0 \\
\mathbf{c}_{M+d-3} &= \boldsymbol{p}_f - \frac{3}{d}\mathbf{v}_f + \frac{2}{(d)(d-1)}\mathbf{a}_f \\
\mathbf{c}_{M+d-2} &= \boldsymbol{p}_f - \frac{1}{d}\mathbf{v}_f \\
\mathbf{c}_{M+d-1} &= \boldsymbol{p}_f.
\end{aligned}$$

Similar techniques for unclamped and non-uniform B-spline trajectories

Implication for Velocity Constraints

Velocity constraints:

$$\left\| \frac{\mathbf{c}_{i+1} - \mathbf{c}_i}{\alpha} \right\|^2 \leq V_{\max}^2$$

$$\|\mathbf{c}_{i+1} - \mathbf{c}_i\|^2 \leq \alpha^2 V_{\max}^2$$

where α is the knot vector scale (from integers $1 : M$).

Use MINVO for tighter bounds:
For each time interval, transform control points as

$$\begin{bmatrix} \boldsymbol{\nu}_i & \cdots & \boldsymbol{\nu}_{i+k} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_i & \cdots & \mathbf{c}_{i+k} \end{bmatrix} F_{\text{b-spline}}^{\text{MINVO}}$$

where $F_{\text{b-spline}}^{\text{MINVO}}$ transforms B-spline control points to MINVO control points.

Constraint becomes

$$\|\boldsymbol{\nu}_{i+1} - \boldsymbol{\nu}_i\|^2 \leq \alpha^2 V_{\max}^2$$

Implication for Acceleration Constraints

Acceleration constraints:

$$\left\|\frac{\boldsymbol{c}_{i+2} - 2\boldsymbol{c}_{i+1} - 2\boldsymbol{c}_i}{\alpha^2}\right\|^2 \leq a_{\max}^2$$

$$\|\boldsymbol{c}_{i+2} - 2\boldsymbol{c}_{i+1} - 2\boldsymbol{c}_i\|^2 \leq \alpha^2 a_{\max}^2$$

Use MINVO for tighter bounds:

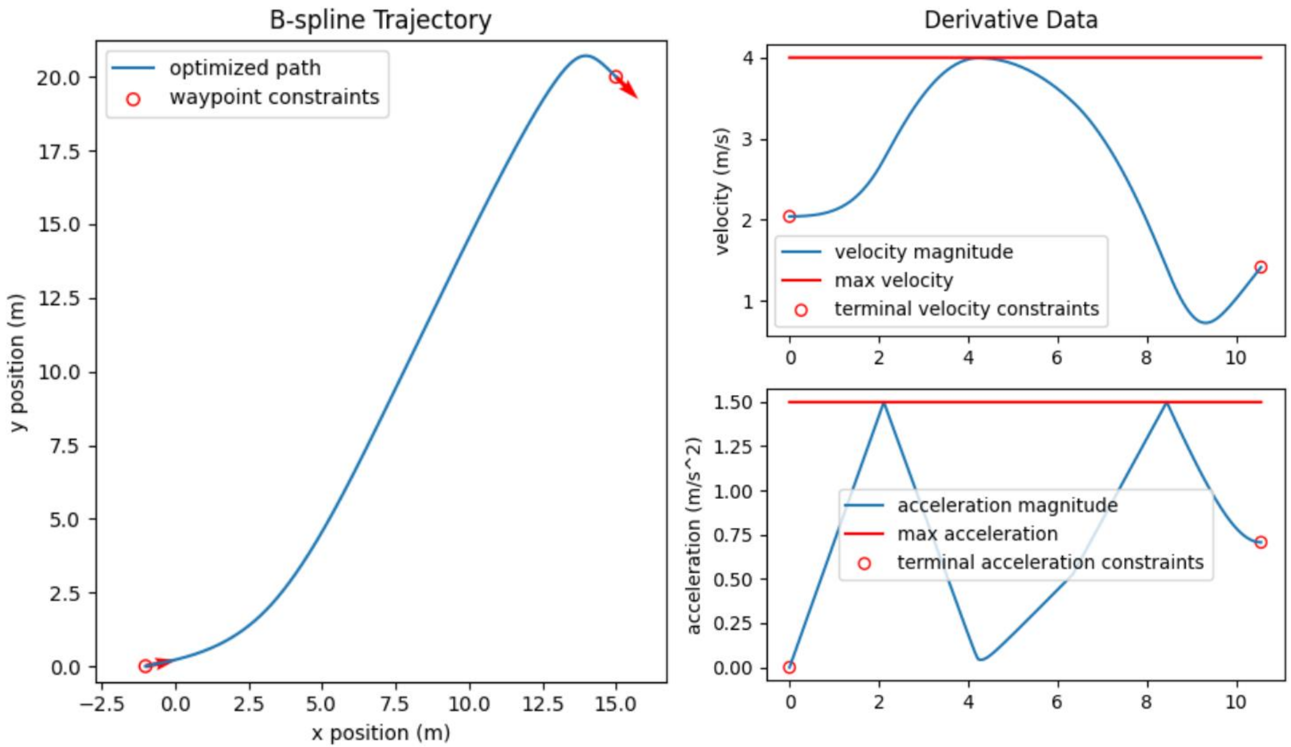
$$\begin{bmatrix}\boldsymbol{\nu}_i & \cdots & \boldsymbol{\nu}_{i+k}\end{bmatrix} = \begin{bmatrix}\boldsymbol{c}_i & \cdots & \boldsymbol{c}_{i+k}\end{bmatrix} F_{\text{b-spline}}^{\text{MINVO}}$$

Constraint becomes

$$\|\boldsymbol{\nu}_{i+2} - 2\boldsymbol{\nu}_{i+1} - 2\boldsymbol{\nu}_i\|^2 \leq \alpha^2 a_{\max}^2$$

Similar (but more complicated) techniques can be derived for constraining curvature


Trajectory Optimization Example



KEY POINT: velocity, acceleration, and velocity constraints at *every time* are guaranteed by checking constraints on the control points, NOT at discrete points along the trajectory.

Trajectory Optimization Example:
Tiltrotor aircraft

Initial and final:
- Position
- Velocity
- Acceleration

Acceleration and velocity constraints.

https://www.youtube.com/watch?v=nQ5Z9ri2-AY&t=3s

Nonlinear, Low-Energy-Actuator-Prioritizing Control Allocation for Winged eVTOL UAVs, Mason B. Peterson, Randal W. Beard, Jacob B. Willis, Intermountain Engineering, Technology, and Computing Conference, 2022

Outline

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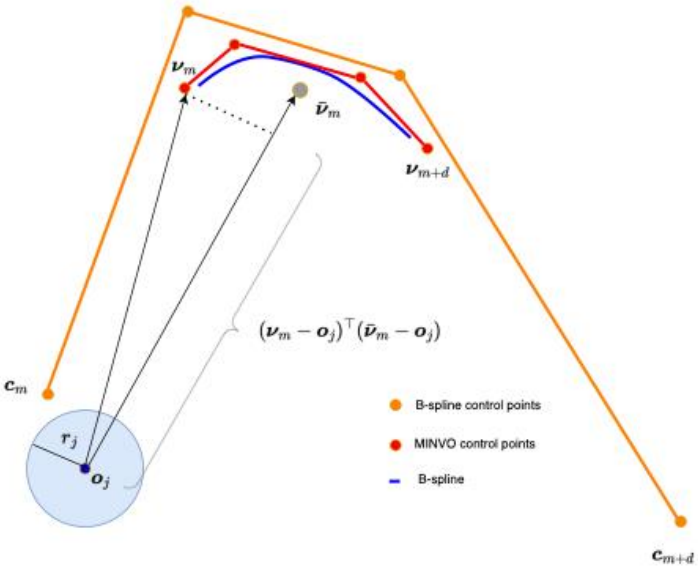
- B-Spline Tutorial
- Applications to Planning
  - Collision Avoidance
  - Cooperative Temporal-Spatial Sequencing
  - Cooperative Informative Path Planning
- Applications to Estimation:
  - Moving horizon estimation exploiting Differential Flatness
  - Calibration

Applications to Planning: Collision Avoidance



- For the m^{th} spline segment, convert to MINVO control points:

$$\underbrace{\begin{pmatrix}\boldsymbol{\nu}_m & \cdots & \boldsymbol{\nu}_{m+d}\end{pmatrix}}_{3\times d} = \underbrace{\begin{pmatrix}\boldsymbol{c}_m & \cdots & \boldsymbol{c}_{m+d}\end{pmatrix}}_{3\times d} \underbrace{F_{\text{b-spline}}^{\text{Minvo}}}_{d\times d}$$

- Define geometric center:

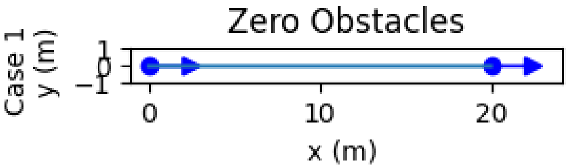
$$\bar{\boldsymbol{\nu}}_m \doteq \frac{1}{d}\sum_{i=m}^{m+d} \boldsymbol{\nu}_i$$

- Projection of relative MINVO control point onto the relative geometric center must be larger than r_j:

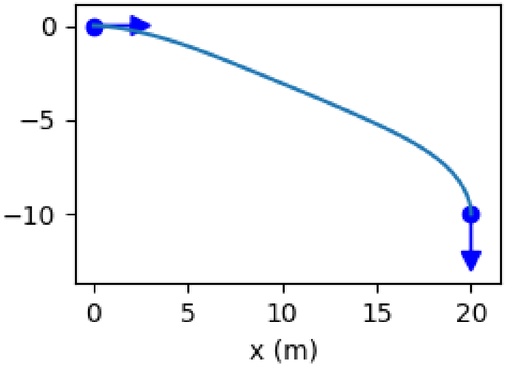
$$(\boldsymbol{\nu}_m - \boldsymbol{o}_j)^\top(\bar{\boldsymbol{\nu}}_m - \boldsymbol{o}_j) > r_j$$

Convex constraint in control points $\boldsymbol{c}_m, \ldots, \boldsymbol{c}_{m+d}$
...but nonlinear

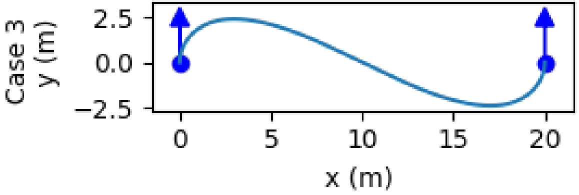
Applications to Planning: Collision Avoidance



evaluation time: 0.1
path length: 20.0
num ctrl pts: 8



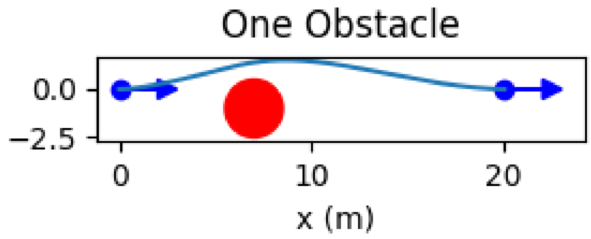
evaluation time: 0.26
path length: 23.22
num ctrl pts: 8



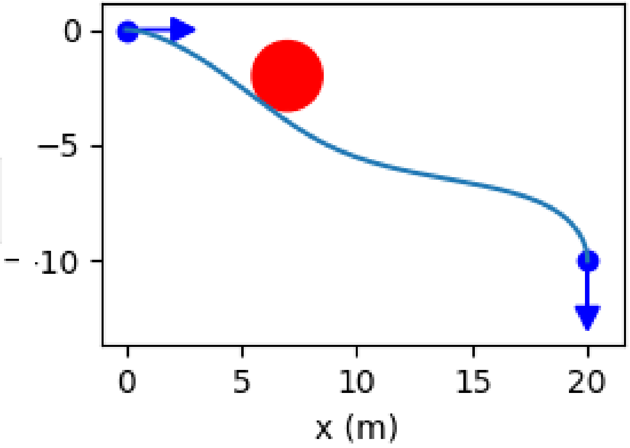
evaluation time: 0.76
path length: 23.46
num ctrl pts: 8

Zero Obstacles

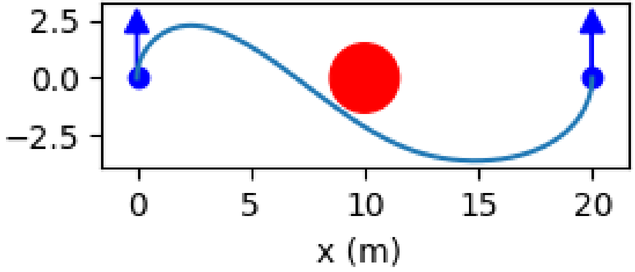
Applications to Planning: Collision Avoidance



evaluation time: 0.26
path length: 20.27
num ctrl pts: 8



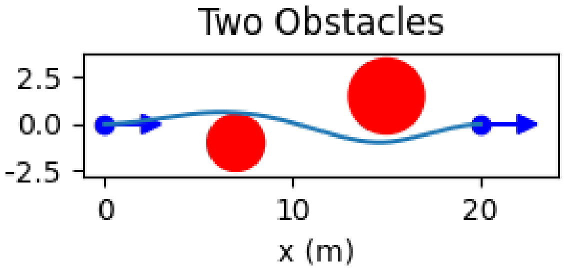
evaluation time: 0.94
path length: 23.18
num ctrl pts: 8



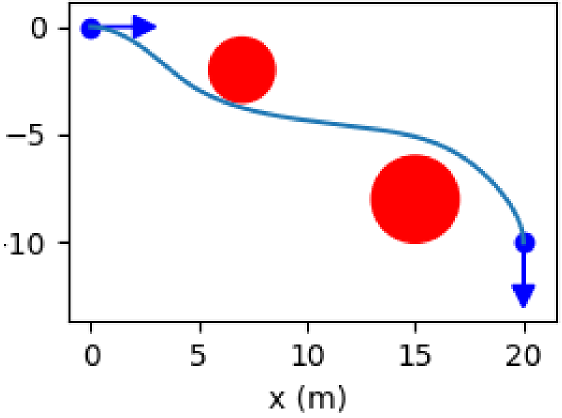
evaluation time: 1.46
path length: 24.73
num ctrl pts: 8

One Obstacle

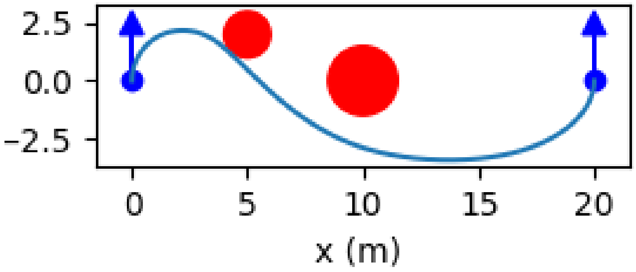
Applications to Planning: Collision Avoidance



evaluation time: 0.65
path length: 20.33
num ctrl pts: 10



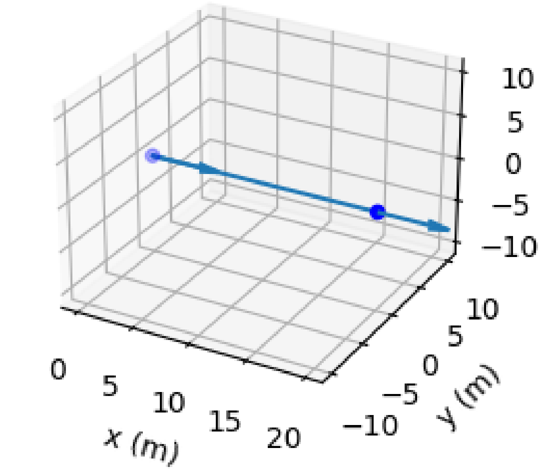
evaluation time: 2.19
path length: 23.59
num ctrl pts: 10



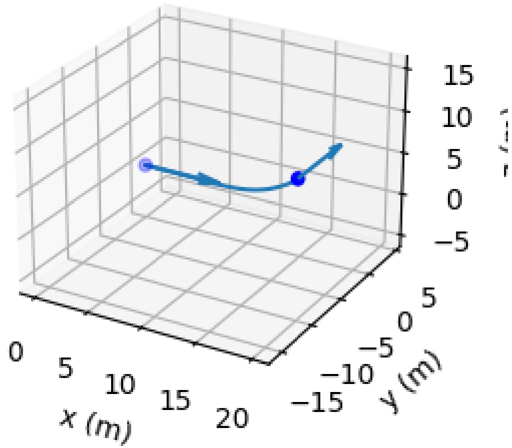
evaluation time: 2.25
path length: 24.52
num ctrl pts: 10

Two Obstacles

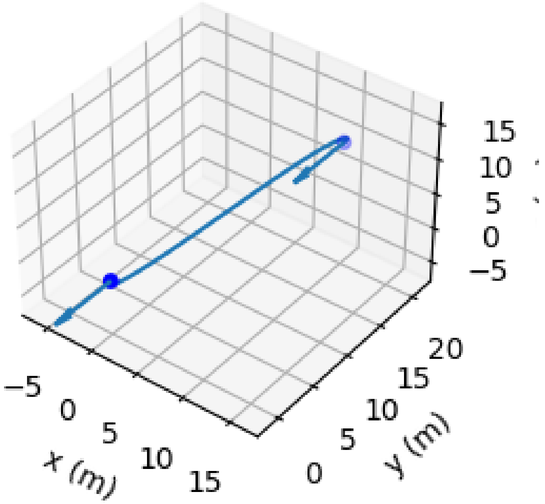

Applications to Planning: Collision Avoidance



evaluation time: 0.17
path length: 20.0
num ctrl pts: 8



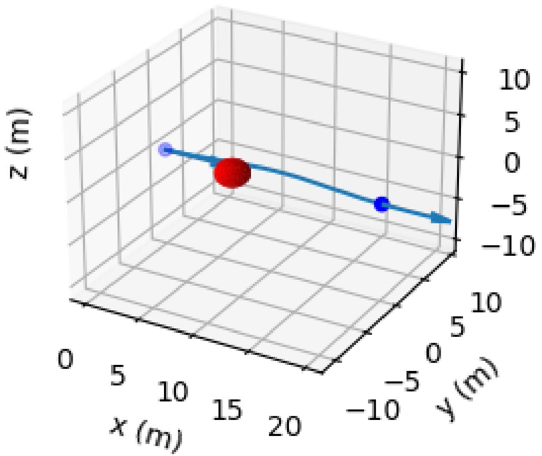
evaluation time: 0.63
path length: 26.25
num ctrl pts: 8



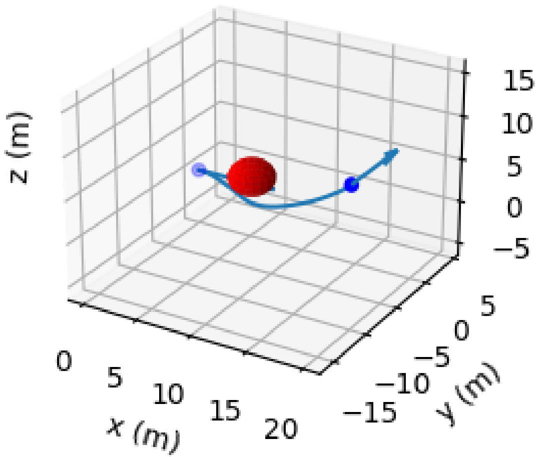
evaluation time: 1.62
path length: 30.52
num ctrl pts: 8

Zero Obstacles

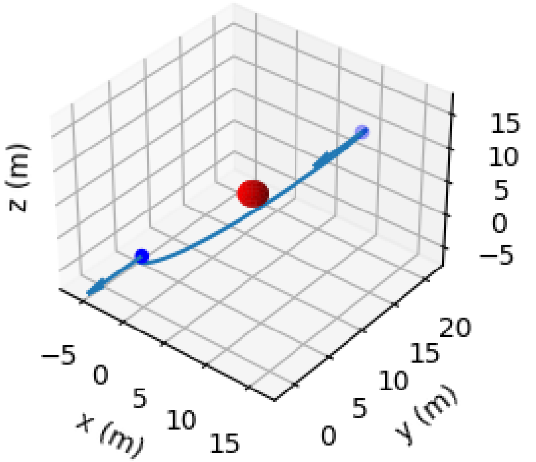
Applications to Planning: Collision Avoidance



evaluation time: 0.54
path length: 20.13
num ctrl pts: 8



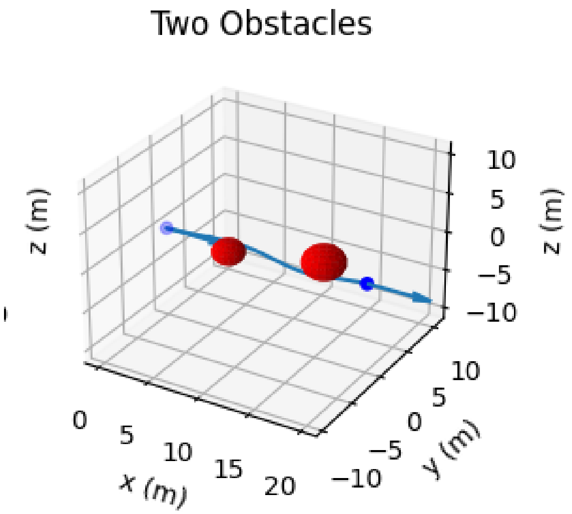
evaluation time: 2.03
path length: 26.73
num ctrl pts: 8



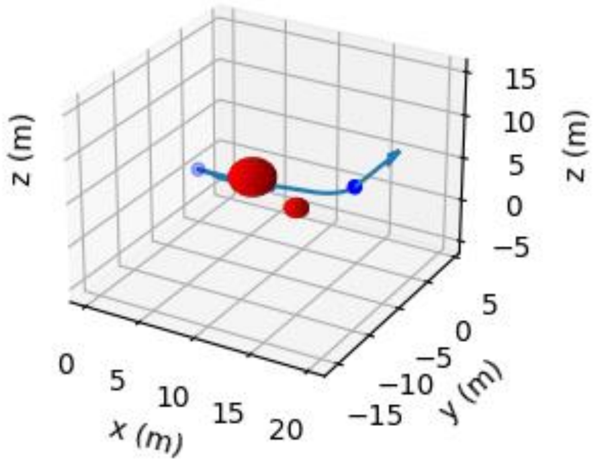
evaluation time: 2.07
path length: 30.72
num ctrl pts: 8

One Obstacle

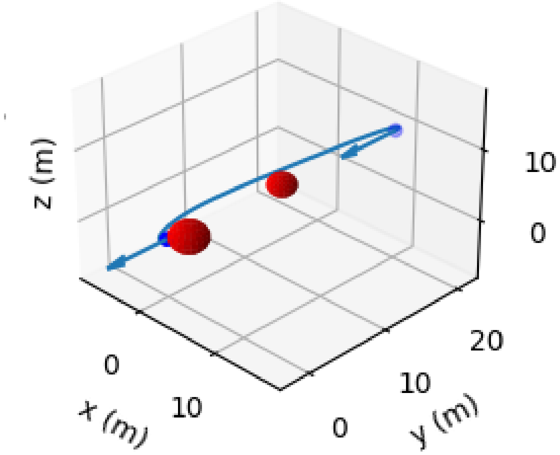
Applications to Planning: Collision Avoidance



evaluation time: 1.31
path length: 20.33
num ctrl pts: 10



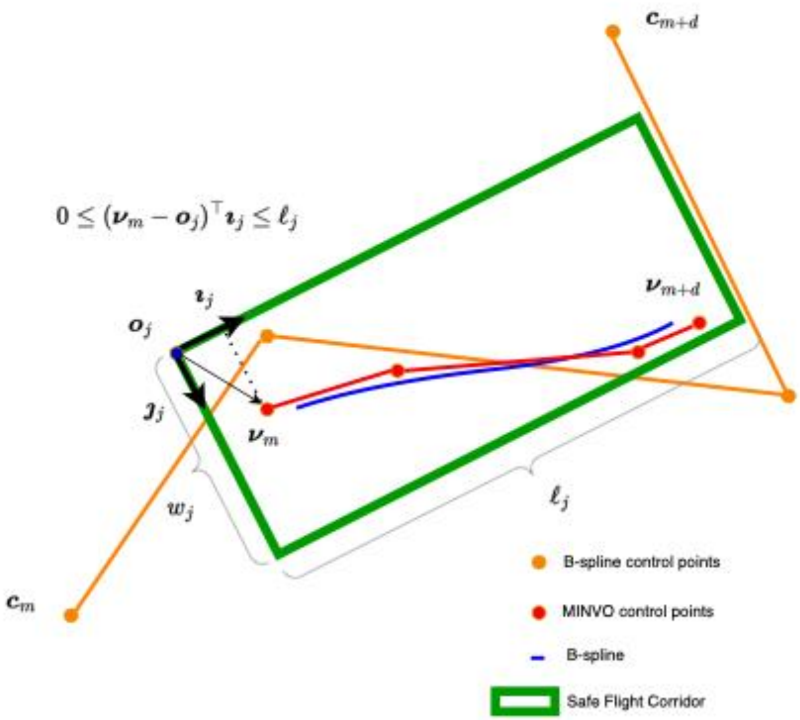
evaluation time: 3.32
path length: 26.27
num ctrl pts: 10



evaluation time: 3.64
path length: 30.13
num ctrl pts: 10

Two Obstacle

Applications to Planning: Safe Flight Corridor



- For the m^{th} spline segment, convert to MINVO control points:

$$\begin{pmatrix} \boldsymbol{\nu}_m & \cdots & \boldsymbol{\nu}_{m+d} \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}_m & \cdots & \boldsymbol{c}_{m+d} \end{pmatrix} F^{\text{Minvo}}_{\text{b-spline}}$$

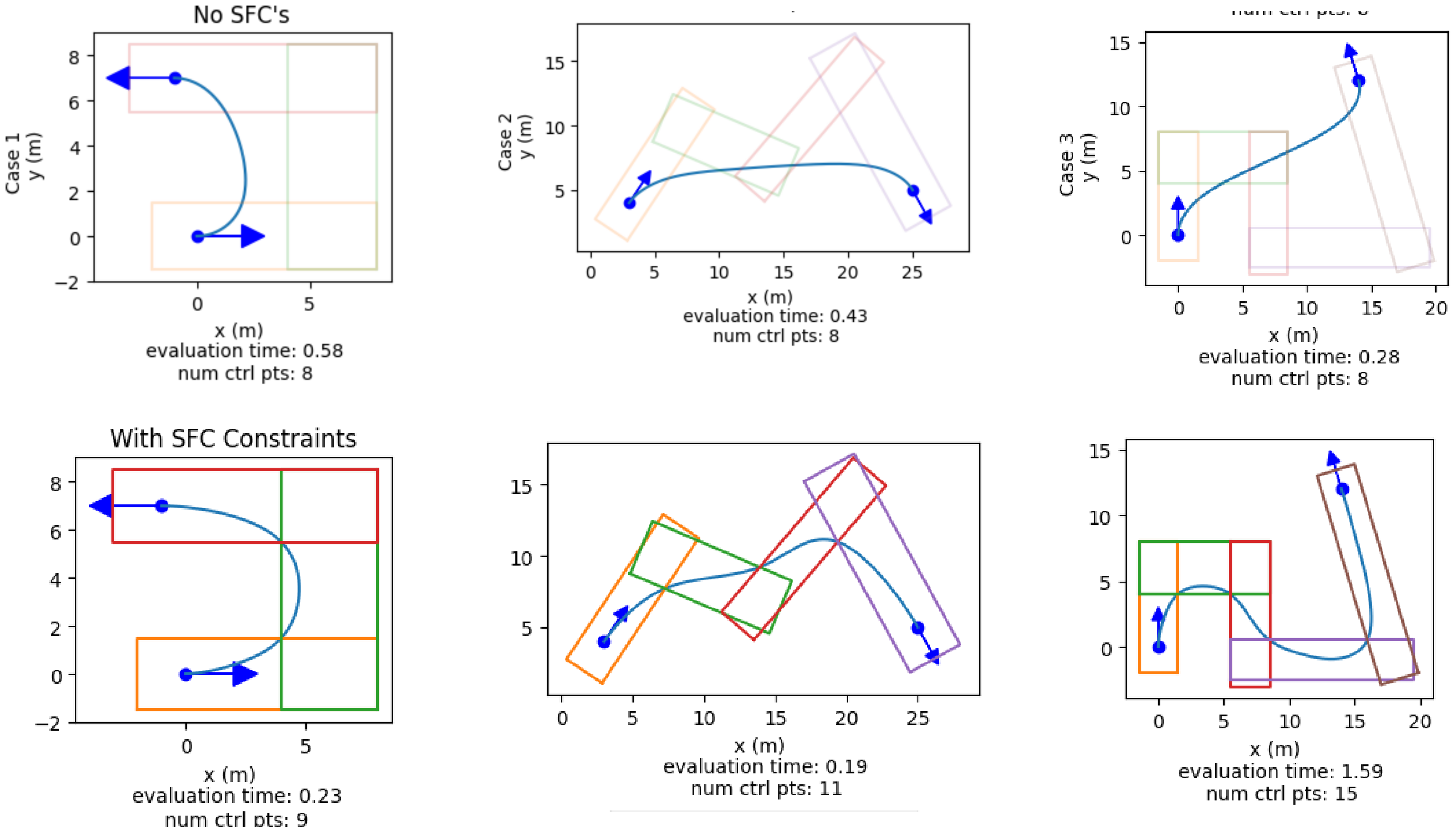
- The j^{th} flight corridor defined by corner $\boldsymbol{o}_j$, direction unit vectors $\boldsymbol{\imath}_j, \boldsymbol{\jmath}_j$, width w_j, and length ℓ_j.

- Corridor constraints

$$0 \leq (\boldsymbol{\nu}_m - \boldsymbol{o}_j)^\top \boldsymbol{\imath}_j \leq \ell_j$$
$$0 \leq (\boldsymbol{\nu}_m - \boldsymbol{o}_j)^\top \boldsymbol{\jmath}_j \leq w_j$$

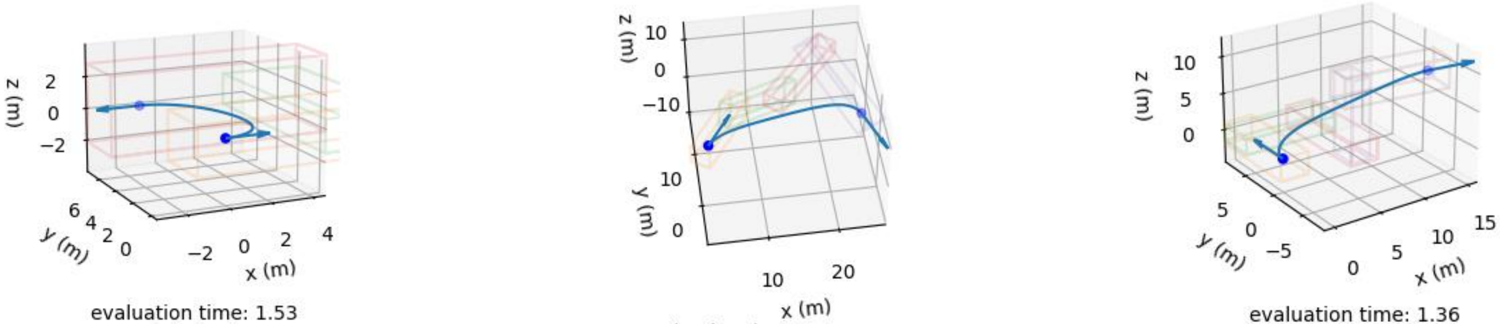
Convex constraint in control points $\boldsymbol{c}_m, \ldots, \boldsymbol{c}_{m+d}$
...and linear!

Applications to Planning: Safe Flight Corridor



Applications to Planning: Safe Flight Corridor

No SFC's

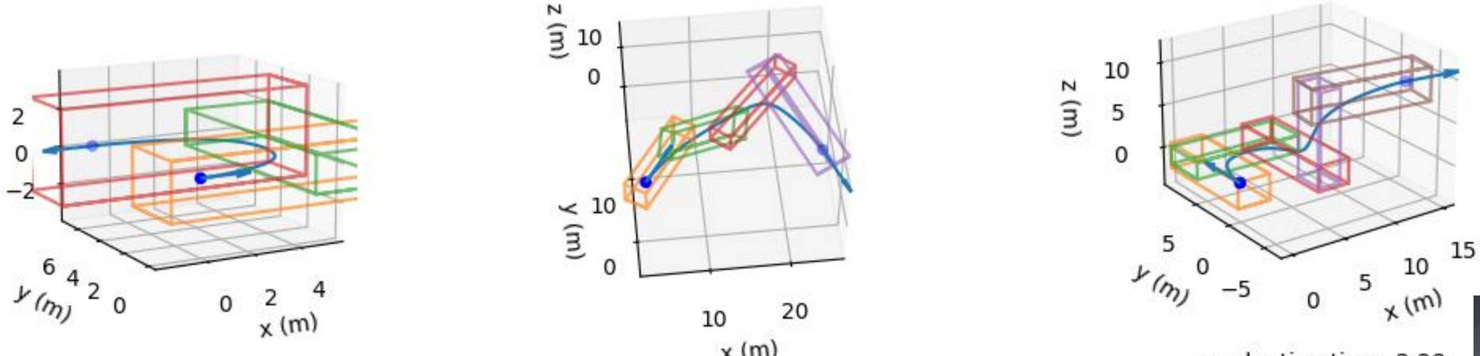


evaluation time: 1.53
num ctrl pts: 8

evaluation time: 0.8
num ctrl pts: 8

evaluation time: 1.36
num ctrl pts: 8

With SFC Constraints

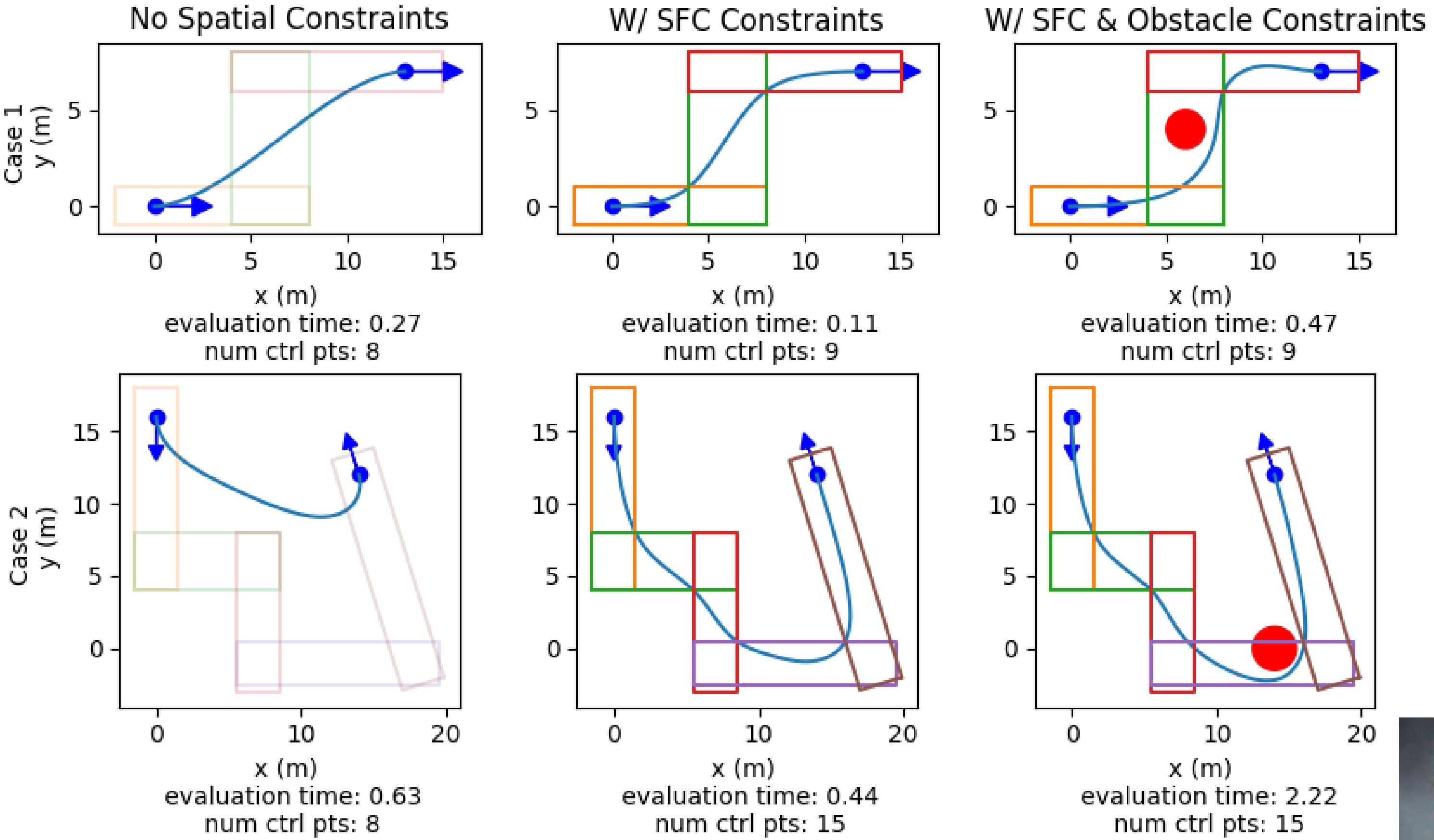


evaluation time: 0.61
num ctrl pts: 9

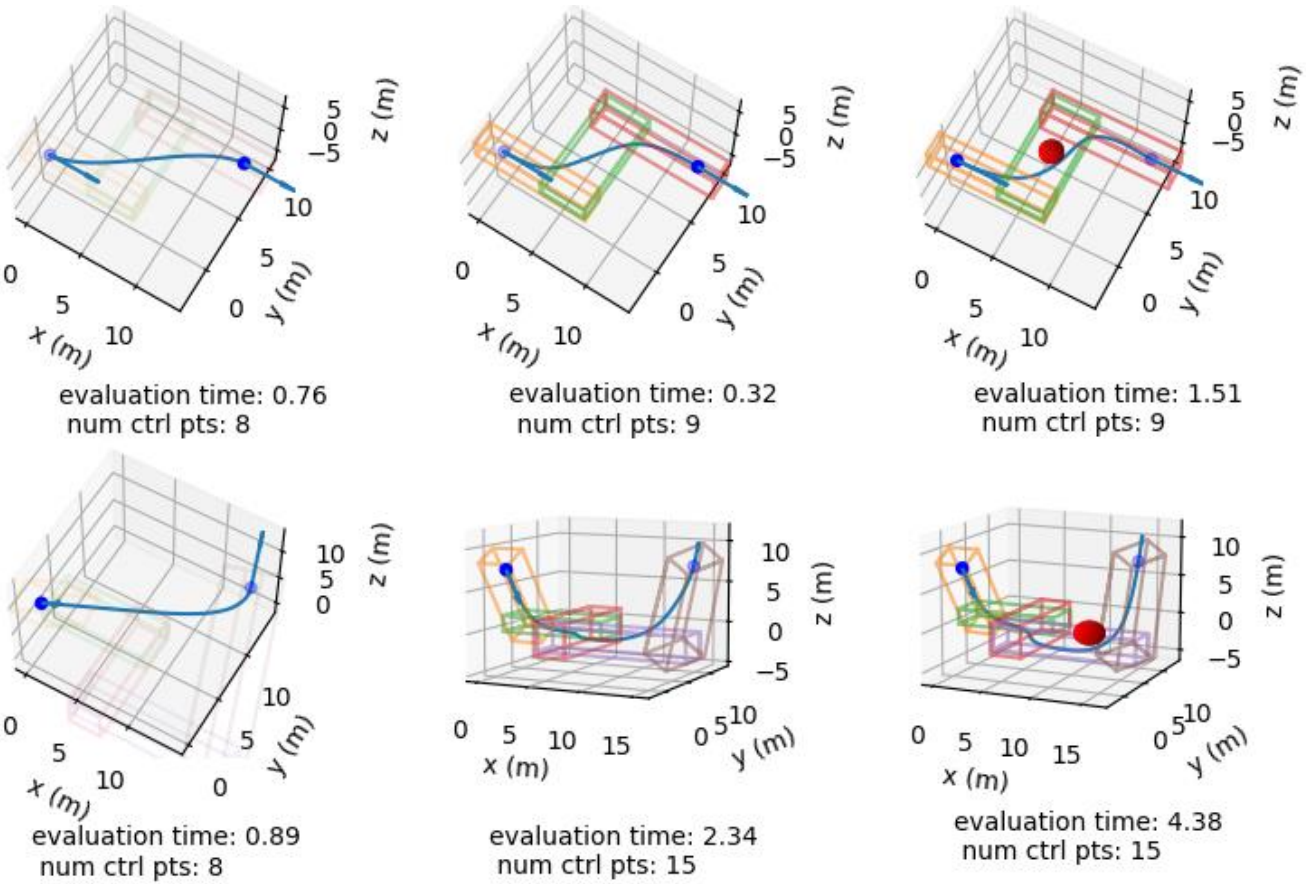
evaluation time: 0.93
num ctrl pts: 13

evaluation time: 2.28
num ctrl pts: 13

Obstacle Avoidance & Safe Flight Corridor



Obstacle Avoidance & Safe Flight Corridor



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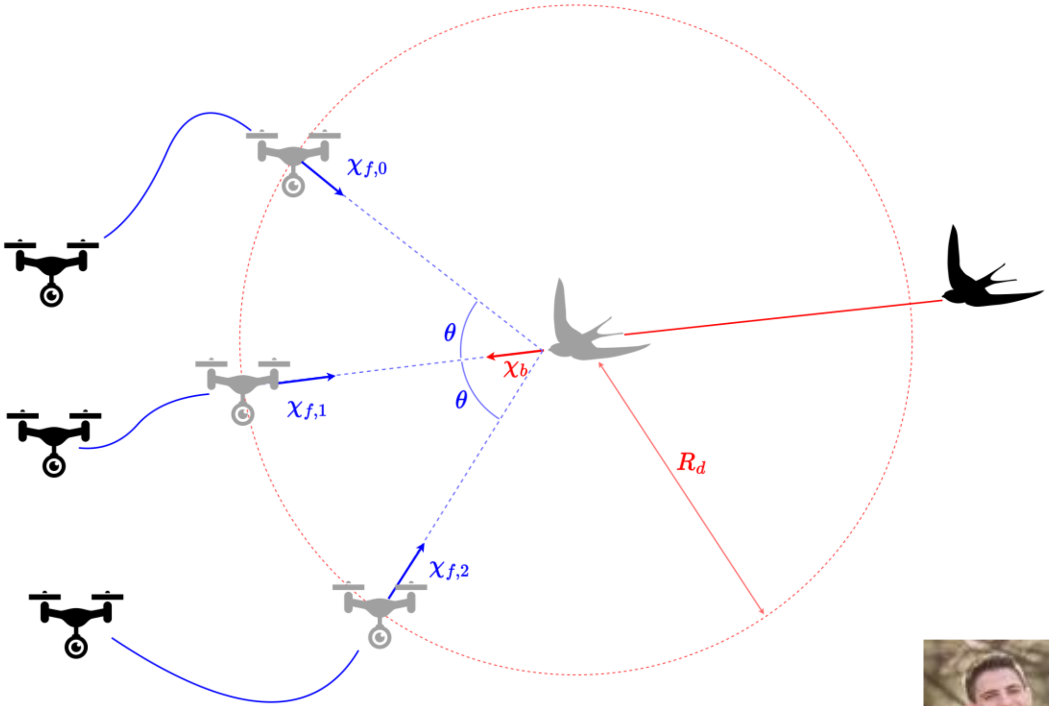
Applications to Planning: Cooperative Temporal-Spatial Sequencing

Wildlife cinematography:

- Capture video of bird at specified times (e.g. simultaneously) from different angles.
- [Key challenge:] Camera motion has pre-specified, but known, velocity profile (to enhance image quality)



Differentially Flat Systems

Definition: Differential Flatness
The dynamic system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

is *differentially flat with flat output* $\mathbf{z}$ if there exists function g^x and g^u such that

$$\mathbf{x}(t) = \mathbf{g}^x\left(\mathbf{z}(t), \dot{\mathbf{z}}(t), \ddot{\mathbf{z}}(t), \cdots\right)$$
$$\mathbf{u}(t) = \mathbf{g}^u\left(\mathbf{z}(t), \dot{\mathbf{z}}(t), \ddot{\mathbf{z}}(t), \cdots\right)$$

Examples include:
- Unicycle
- Quadrotor
- Fixed wing Aircraft
  (with reasonable assumptions)
- eVTOL
  (with reasonable assumptions)
- Mobile robot
  (with reasonable assumptions)
- Unmanned surface vehicle
  (with reasonable assumptions)
- …many others…




Informative Path Planning for Differentially Flat Systems

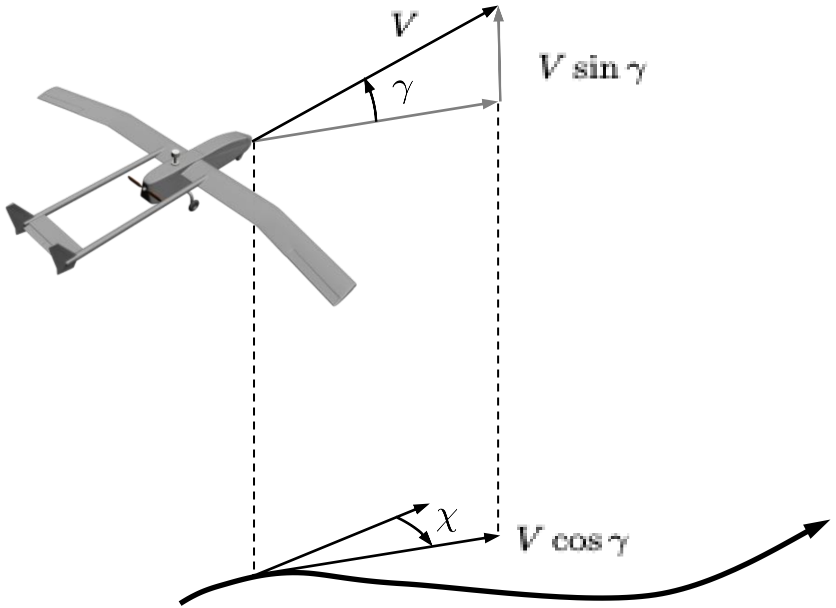
Dubins Airplane

$$\dot{r}_n = V_a \cos\psi$$
$$\dot{r}_e = V_a \sin\psi$$
$$\dot{r}_d = -V_a \sin\gamma$$
$$\dot{\psi} = \frac{g}{V_a}\tan\phi$$
$$\dot{V}_a = \underbrace{s(t)}_{\text{known and pre-defined}}$$

Subject to

$$|\phi| \leq \bar{\phi}$$
$$|\gamma| \leq \bar{\gamma}$$

Without the $\dot{V}_a$ equations, this system is differentially flat with flat output $z = (r_n, r_e, r_d)^\top$.



Flight path projected onto ground

Informative Path Planning for Differentially Flat Systems

Dubins Airplane

$$\dot{r}_n = V_a \cos\psi$$
$$\dot{r}_e = V_a \sin\psi$$
$$\dot{r}_d = -V_a \sin\gamma$$
$$\dot{\psi} = \frac{g}{V_a}\tan\phi$$

Subject to

$$|\phi| \leq \bar{\phi}$$
$$|\gamma| \leq \bar{\gamma}$$

Differentially flat with flat output $\mathbf{p} = (r_n, r_e, r_d)^\top$ since:

$$(r_n, r_e, r_d)^\top(\mathbf{p}) = \mathbf{p}$$
$$V(\dot{\mathbf{p}}) = \sqrt{\dot{p}_n^2 + \dot{p}_e^2 + \dot{p}_d^2}$$
$$\psi(\dot{\mathbf{p}}) = \frac{\dot{p}_e}{\dot{p}_n}$$
$$\gamma(\dot{\mathbf{p}}) = -\sin^{-1}\left(\frac{\dot{p}_d}{\sqrt{\dot{p}_n^2 + \dot{p}_e^2 + \dot{p}_d^2}}\right)$$
$$\phi(\dot{\mathbf{p}}, \ddot{\mathbf{p}}) = \tan^{-1}\left(\frac{(\dot{p}_n\ddot{p}_e - \dot{p}_e\ddot{p}_e)\sqrt{\dot{p}_n^2 + \dot{p}_e^2 + \dot{p}_d^2}}{g(\dot{p}_n^2 + \dot{p}_e^2)}\right)$$

Cooperative Temporal-Spatial Sequencing

B-Spline Approximation of Dubins Paths

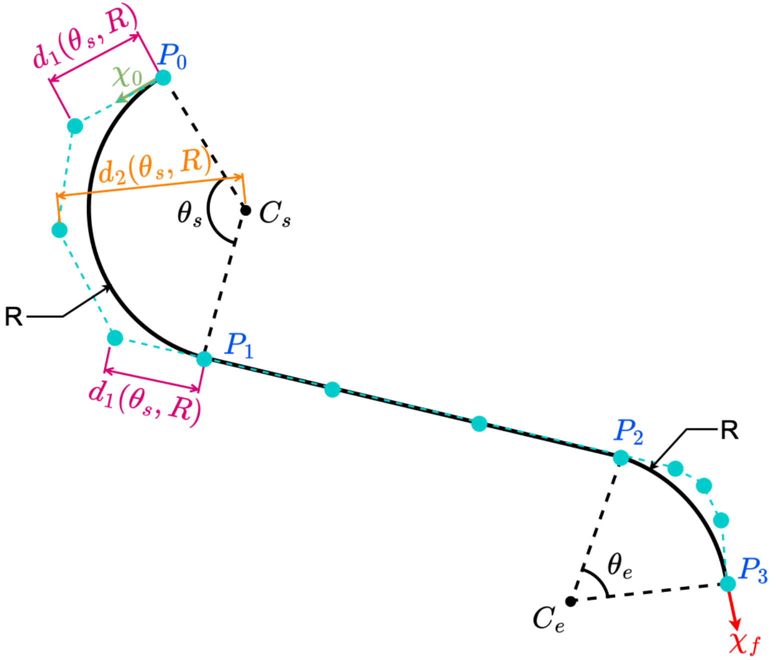
Given a Dubin's path:

- Use degree $d = 3$ (to ensure continuous acceleration), clamped, uniform, B-spline
- Select control points to correspond to P_0, P_1, P_2, P_3 shown in the figure
- Add control points to ensure appropriate specified heading at P_0, P_1, P_2, P_3
- Add addition waypoints on turns according to (empirical fit)

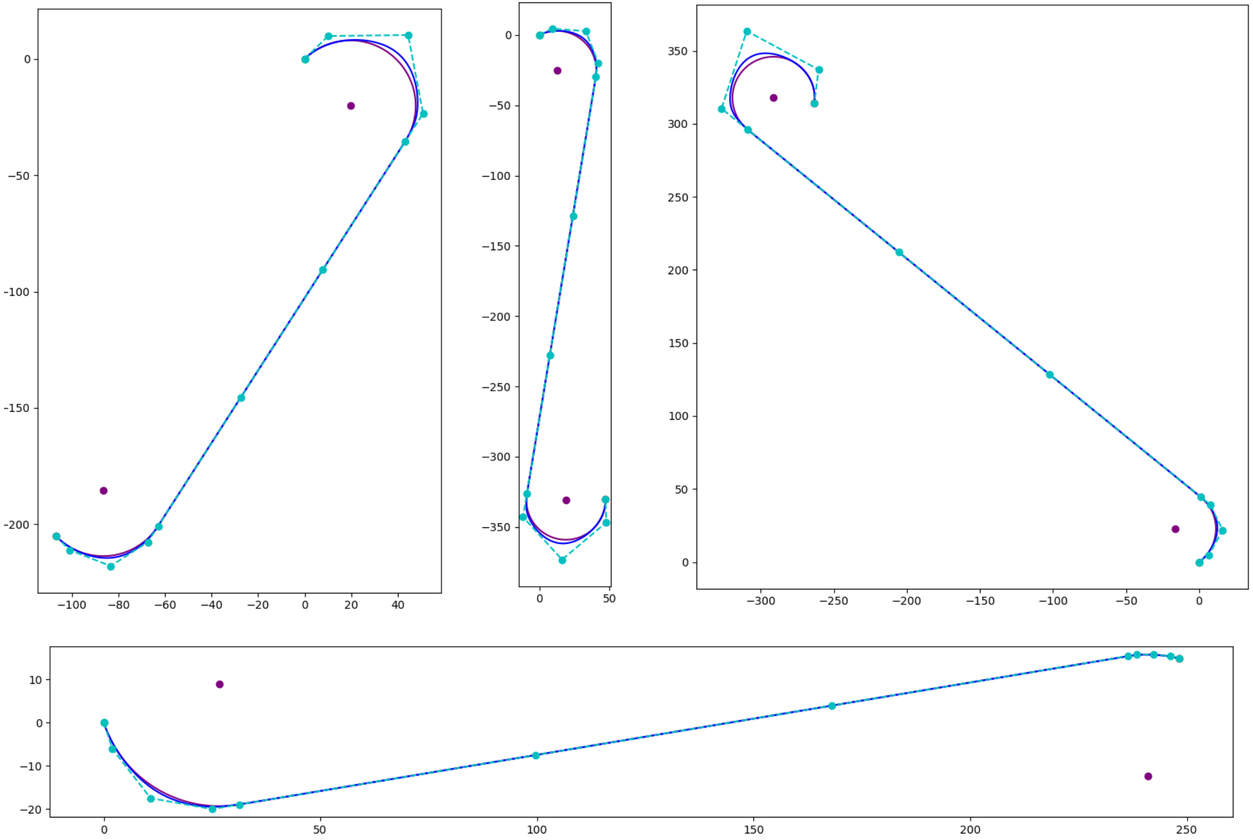
$$d_1 = R(0.004298534\theta_s^3 - 0.010226116\theta_s^2 + 0.163751234\theta_s + 0.00529229)$$

$$d_2 = R(0.002995584\theta_s^4 - 0.031004044\theta_s^3 + 0.142959764\theta_s^2 - 0.097444194\theta_s + 1.01900052)$$

- Add two addition control points along straight-line segment.



B-Spline Approximation of Dubins Paths



Cooperative Temporal-Spatial Sequencing

Path Parameterization for Known Speed Profile

Approximating the Dubin's path results in the B-spline:

$$\mathbf{p}(\sigma) = \sum_{m=0}^{M+d-1} \mathbf{c}_m b_m^d(\sigma, \mathbf{k}),$$

where $d = 3$, and

$$\mathbf{k} = \left[0, 0, 0, 0, \frac{1}{M}, \ldots, \frac{M-1}{M}, 1, 1, 1, 1\right]$$

where we use the path parameter σ instead of t.

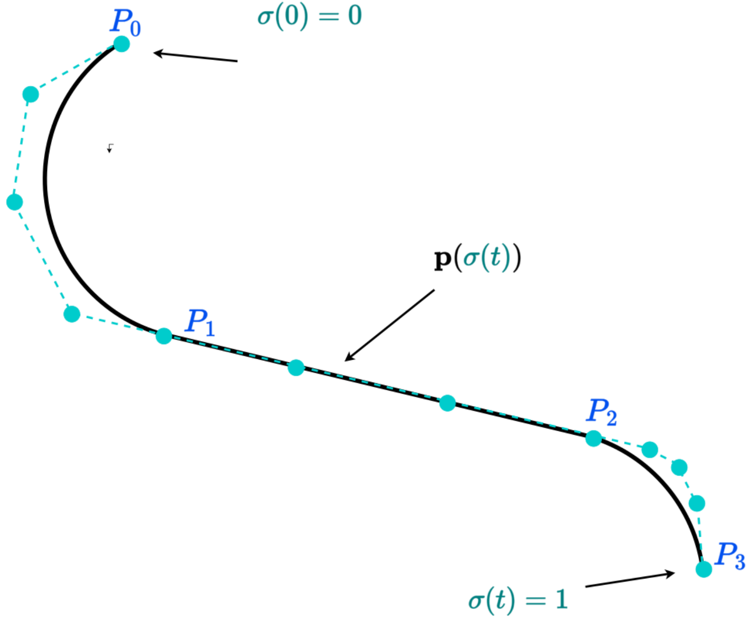
To achieve the desired speed profile, note that

$$s(t) = \|\frac{d}{dt}\mathbf{p}(\sigma(t))\| = \dot{\sigma}\|\frac{d\mathbf{p}}{d\sigma}\|$$

Therefore let

$$\dot{\sigma} = \frac{s(t)}{\|\frac{d\mathbf{p}}{d\sigma}(\sigma(t))\|}, \qquad \sigma(0) = 0$$

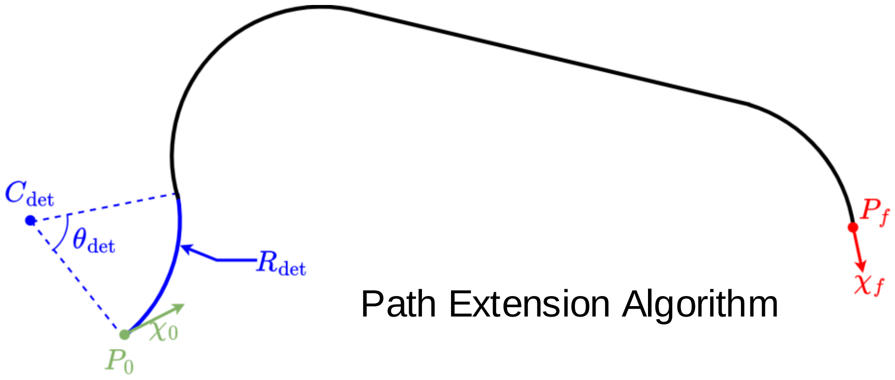
which is integrated until $\sigma(t) = 1$ corresponding to the end of the B-spline path.



Cooperative Temporal-Spatial Sequencing

Cooperation Auction Algorithm

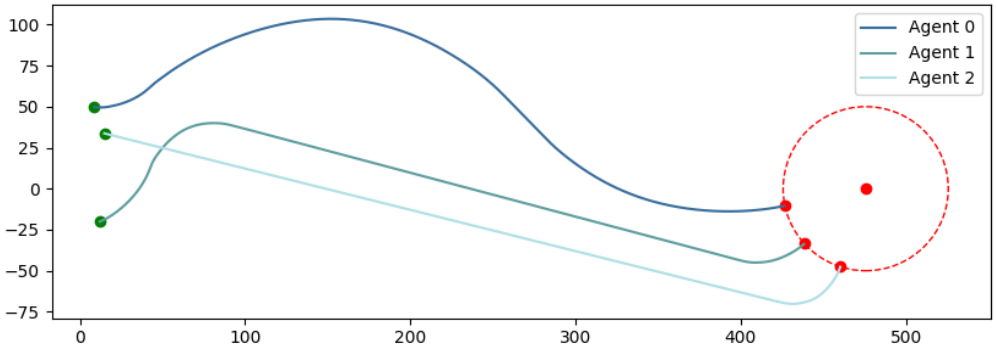
- Each vehicle plans its myopically best path, and submits bid: B-spline parameterization and arrival time
- The vehicle with the longest arrival time $t_{\max}$ is selected
- Remaining vehicles plan myopically best de-conflicted and elongated path matching arrival time $t_{\max}$ an
- Best bid selected, and repeat

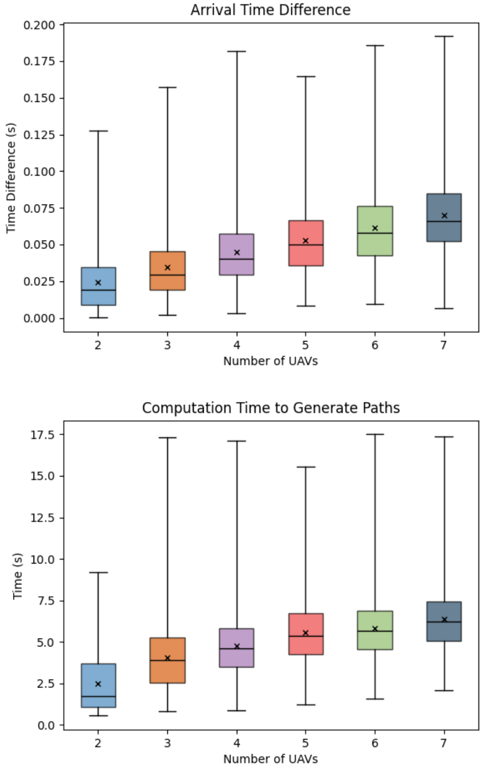


Path Extension Algorithm

Cooperative Temporal-Spatial Sequencing

Monte-Carlo runs:
Each whisker represents 1000 simulations





Outline

- Motivation
- B-Spline Tutorial
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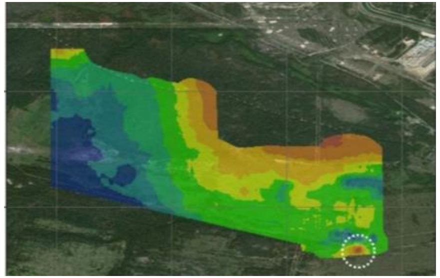
  - Collision Avoidance
  - Cooperative Temporal-Spatial Sequencing
  - Cooperative Informative Path Planning
- Applications to Estimation:
  - Moving horizon estimation exploiting Differential Flatness
  - Calibration

Applications to Planning: Informative Path Planning

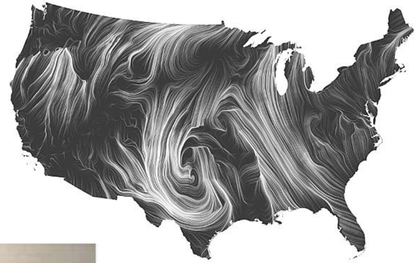
Problem Statement:

- Team of Mobile Agents
- Sensor to measure phenomenon:
- Objective is to create a map of the environment



Radiation

Wind



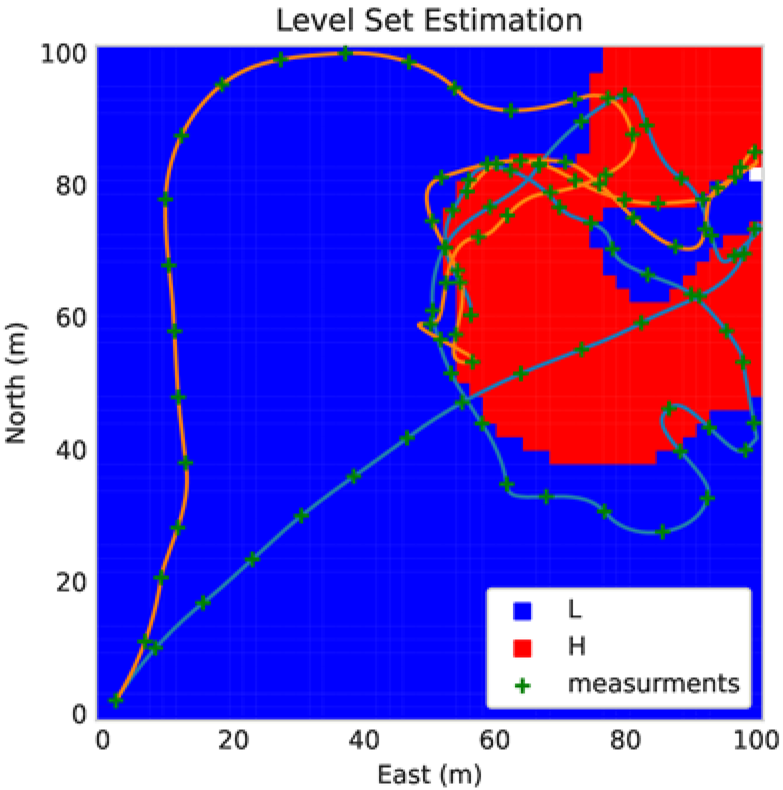
Chemical Spill

Informative Path Planning: Level Set Estimation

- D is the area of interest
- $f(x)$ is an unknown scaler field specifying concentration, where

$$H = \{\mathbf{x} \in D | f(\mathbf{x}) > h\}$$
$$L = \{\mathbf{x} \in D | f(\mathbf{x}) \leq h\}$$

- Sensor measures $z_i = f(\mathbf{x}_i) + \eta, \quad \eta \sim \mathcal{N}(0, \sigma_n^2)$.
- **Goal:** Plan cooperative paths to classify area into high and low sets using noisy measurements of the environment

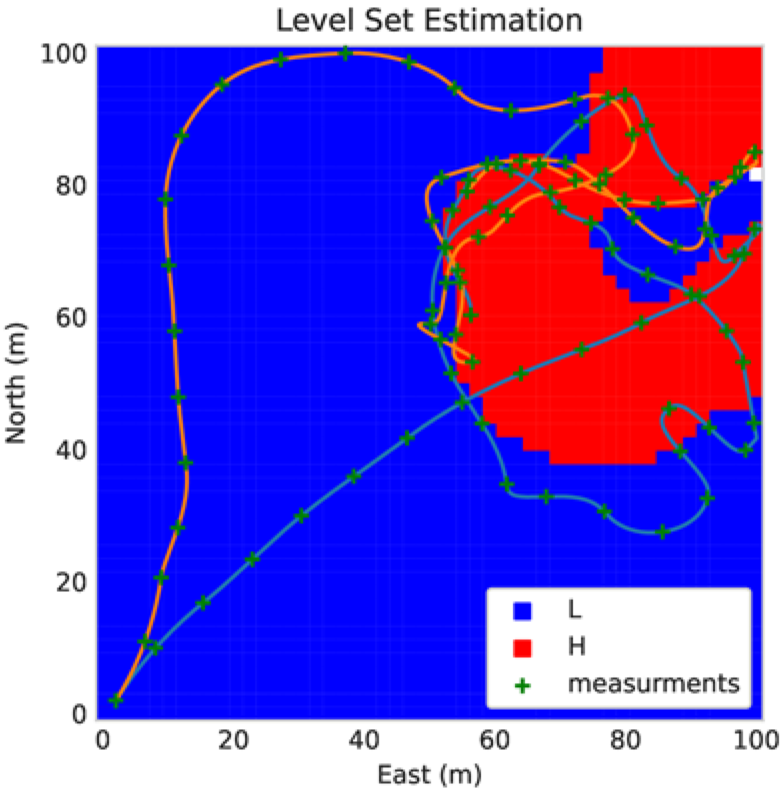


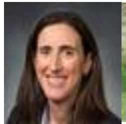
Informative Path Planning: Level Set Estimation

- As the aircraft moves and collects measurement, the scalar field is updated using Gaussian Process Regression:

$$p(f(\bar{X}_*)|z) \sim \mathcal{N}(\mu_p(\bar{X}_*, \mathrm{cov}_p(\bar{X}_*))$$
$$\mu_p(\bar{X}_*) = \mu(\bar{X}_*) + K_{*f}(K_{ff} + \sigma_n^2 I)^{-1}\,(z - \mu(\bar{X}))$$
$$\Sigma_p(\bar{X}_*) = K_{**} - K_{*f}(K_{ff} + \sigma_n^2 I)^{-1} K_{f*}$$

- Cost function for planning algorithm is to maximize the information gain over a receding horizon trajectory.





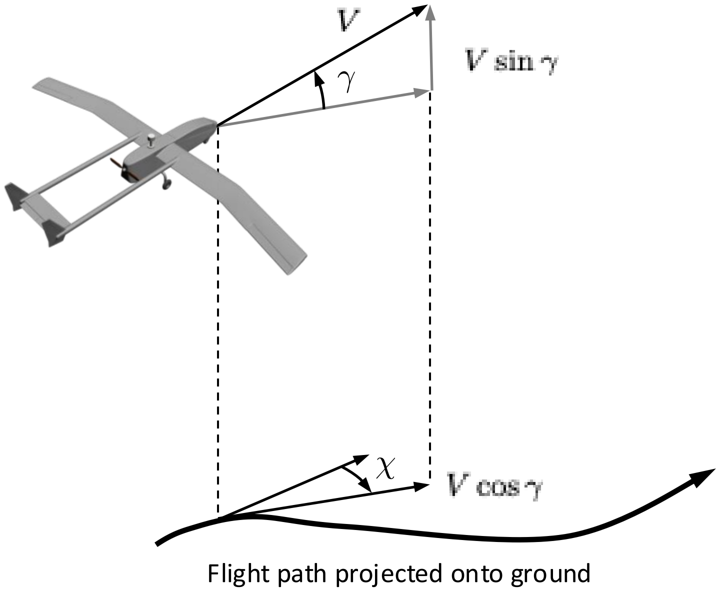
Informative Path Planning for Differentially Flat Systems

Dubins Airplane

$$\dot{r}_n = V_a \cos\psi$$
$$\dot{r}_e = V_a \sin\psi$$
$$\dot{r}_d = -V_a \sin\gamma$$
$$\dot{\psi} = \frac{g}{V_a}\tan\phi$$

Subject to

$$|\phi| \le \bar{\phi}$$
$$|\gamma| \le \bar{\gamma}$$

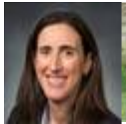


Informative Path Planning: Path Planning

- Differential flatness for feasibility constraints
  - Path specified as $\mathbf{p}$
  - Velocity contraint: $\|\dot{\mathbf{p}}\| \leq \bar{V}$
  - Curvature: $\frac{\|\dot{\mathbf{p}} \times \ddot{\mathbf{p}}\|}{\|\dot{\mathbf{p}}\|^2} \leq \bar{u}$
- Clamped and uniform B-splines to parameterize paths

$$\mathbf{p}(t) = \sum_{m=0}^{M+d-1} \mathbf{c}_m b_m^d(t, \mathbf{k})$$

- Level set estimation specific objective function
- Decentralization using block coordinate ascent




Informative Path Planning:
Path Planning – Optimization

The path optimization problem is therefore:

$$\mathbf{C}^* = \arg\max \sum_{i=1}^{N_c} \Gamma\left(\mathbf{p}(t_c + i\Delta_{tf}, \mathbf{k})\right)$$

subject to:

$$\mathbf{c}_1 = \mathbf{p}_{t_c}$$

$$\mathbf{c}_2 = \mathbf{c}_1 + \left(\frac{t_{k+2} - t_2}{k}\right) v_{t_c} \begin{pmatrix} \cos\theta_{t_c} \\ \sin\theta_{t_c} \end{pmatrix}$$

$$\mathbf{p}(t_i) \in D$$

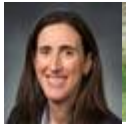
$$\|\dot{\mathbf{p}}(t_i)\| \leq \bar{v}$$

$$\underline{u} \leq \frac{\|\dot{\mathbf{p}}(t_i) \times \ddot{\mathbf{p}}(t_i)\|}{\|\dot{\mathbf{p}}(t_i)\|^2} \leq \bar{u}$$

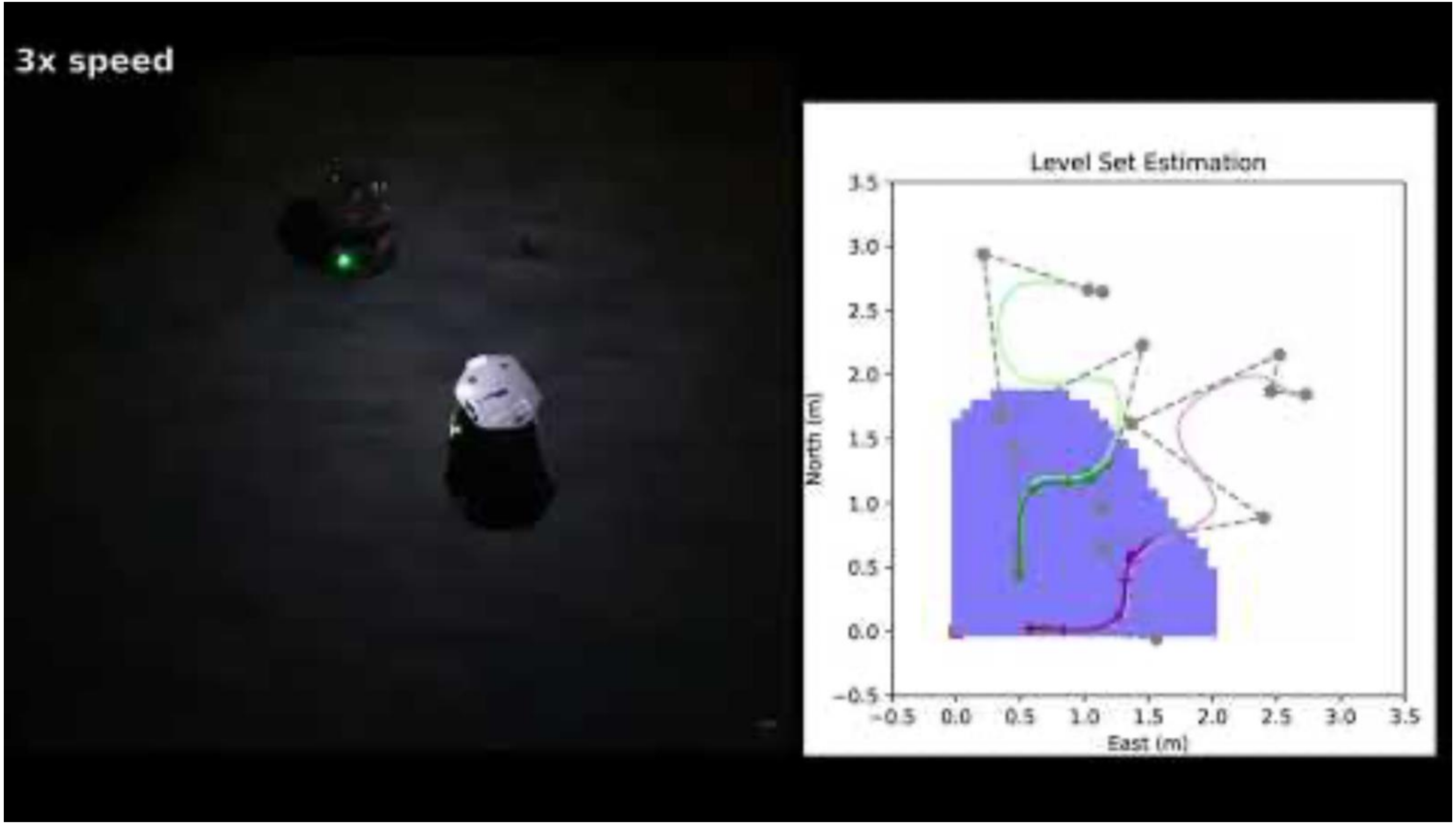
where

$$\Gamma(\mathbf{x}) = \alpha\|\Sigma_{t_c}(\mathbf{x})\| + (1-\alpha)(h - \mu_{t_c}(\mathbf{x}))^2$$

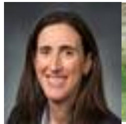
if the information gain at $\boldsymbol{x}$.



Informative Path Planning: Results



https://youtu.be/7jVc9QdYnJs?si=Y1jqu6_lnP3Hdizx&t=20

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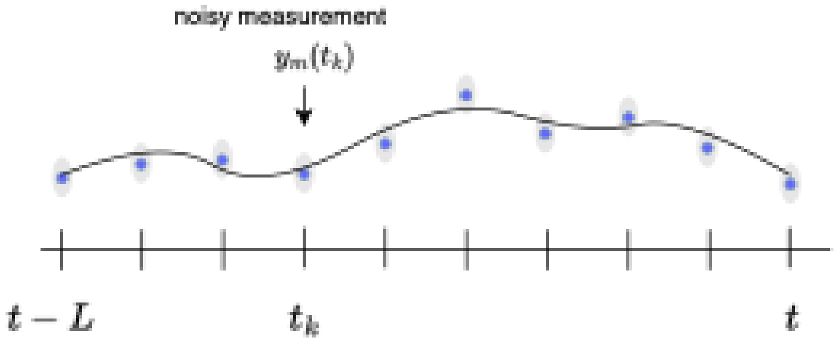
Moving Horizon Estimation (MHE) using B-splines

Given the nonlinear system

$$\dot{x} = f(x, u, w), \qquad x(t-L)$$
$$y = h(x, u) + \xi$$

where L is the window length, and the measurement noise satisfies $\xi \sim \mathcal{N}(0, \Sigma)$
The signal $w(t)$ can be thought of as an unknown noise and/or disturbance term, but with no Gaussian restrictions.



Represent $w(t) = \sum_{m=0}^{M+d-1} \mathbf{c}_m b_m^d(t, \mathbf{k}_M^d)$
and solve differential equation to obtain $x(t, \mathbf{C}, x(t-L))$.

Solve:

$$\mathbf{C}^*, x^*(t-L) = \arg\min \sum \| h(x(t_k, \mathbf{C}, x(t-L)), u(t_k)) - y_m(t_k) \|_{\Sigma^{-1}},$$

where $y_m(t_k)$ is a measurement at time $t - L \leq t_k \leq t$, and the sum is over all measurements in the window.

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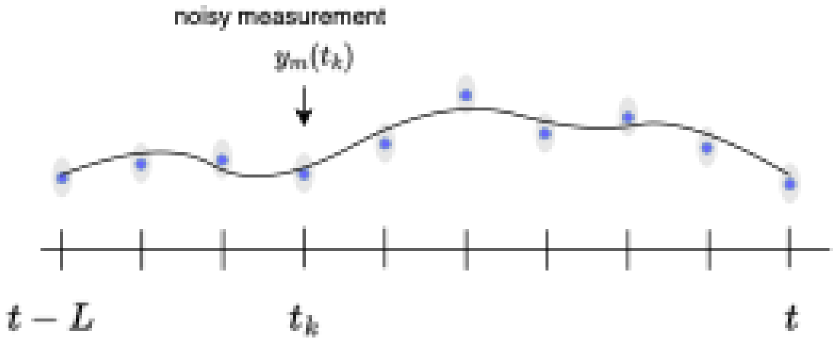

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where $y_m(t_k)$ is a measurement at time $t - L \leq t_k \leq t$, and the sum is over all measurements in the window.

Note optimization over spline coefficients $\mathbf{C}$ and initial condition $x(t-L)$, and that each iteration requires the solution of a differential equation.

Moving Horizon Estimation (MHE) using B-splines

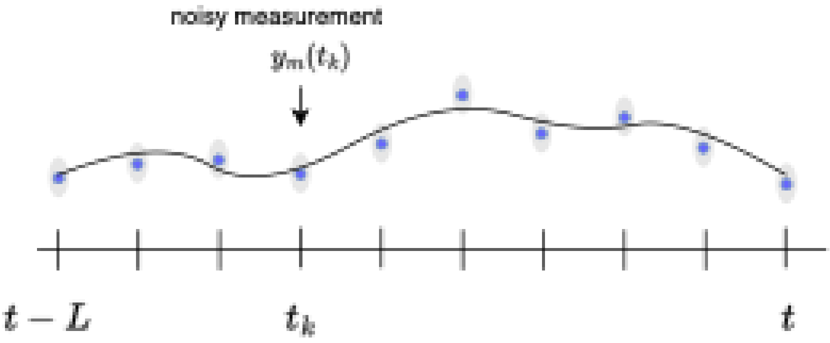
Given the nonlinear system

$$\dot{x} = f(x, u, w), \qquad x(t-L)$$
$$y = h(x, u) + \xi$$

where L is the window length, and the measurement noise satisfies $\xi \sim \mathcal{N}(0, \Sigma)$
The signal $w(t)$ can be thought of as an unknown noise and/or disturbance term, but with no Gaussian restrictions.

B-spline estimates the input disturbance:

Represent $w(t) = \sum_{m=0}^{M+d-1} \mathbf{c}_m b_m^d(t, \mathbf{k}_M^d)$
and solve differential equation to obtain $x(t, \mathbf{C}, x(t-L))$.



Solve:

$$\boxed{\mathbf{C}^*, x^*(t-L)} = \arg\min \sum \| h(x(t_k, \mathbf{C}, x(t-L)), u(t_k)) - y_m(t_k) \|_{\Sigma^{-1}},$$

where $y_m(t_k)$ is a measurement at time $t - L \leq t_k \leq t$, and the sum is over all measurements in the window.

Note optimization over spline coefficients $\mathbf{C}$ and initial condition $x(t-L)$, and that each iteration requires the solution of a differential equation.

Differentially Flat Systems

Definition: Differential Flatness
The dynamic system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

is *differentially flat with flat output* $\mathbf{z}$ if there exists function g^x and g^u such that

$$\mathbf{x}(t) = \mathbf{g}^x\left(\mathbf{z}(t), \dot{\mathbf{z}}(t), \ddot{\mathbf{z}}(t), \cdots\right)$$
$$\mathbf{u}(t) = \mathbf{g}^u\left(\mathbf{z}(t), \dot{\mathbf{z}}(t), \ddot{\mathbf{z}}(t), \cdots\right)$$

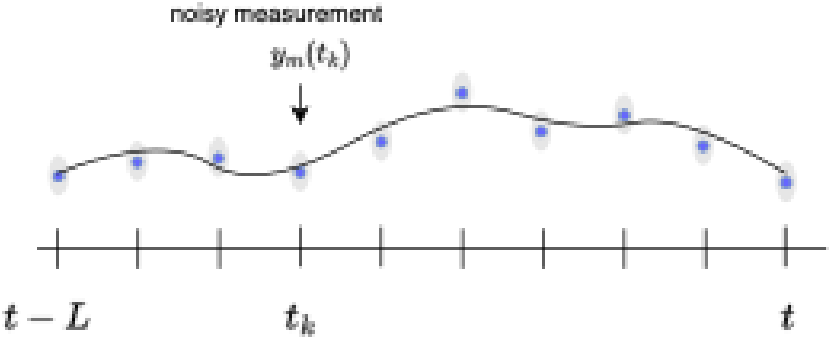
Examples include:
- Unicycle
- Quadrotor
- Fixed wing Aircraft
(with reasonable assumptions)
- eVTOL
(with reasonable assumptions)
- Mobile robot
(with reasonable assumptions)
- Unmanned surface vehicle
(with reasonable assumptions)
- …many others…




Moving Horizon Estimation (MHE) using DF + B-splines

Given the differentially flat nonlinear system with flat output $\mathbf{z}$:

$$\begin{aligned} \dot{x} &= f(x, u, w), \qquad x(t-L) \\ y &= h(x, u) + \xi \end{aligned}$$



Represent the flat output using $\hat{z}(t, \mathbf{C}) = \sum_{m=0}^{M+d-1} \mathbf{c}_m b_m^d(t, \mathbf{k}_M^d)$

giving $\hat{x}(t, \mathbf{C})$ and $\hat{w}(t, \mathbf{C})$

Solve:

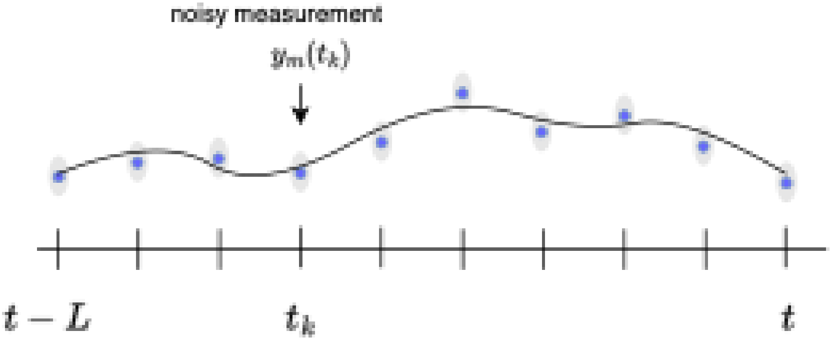
$$\mathbf{C}^* = \arg\min \sum \|h\left(\hat{x}(t_k, \mathbf{C}), u(t_k)\right) - y_m(t_k)\|_{\Sigma^{-1}},$$

where $y_m(t_k)$ is a measurement at time $t - L \leq t_k \leq t$, and the sum is over all measurements in the window.

Moving Horizon Estimation (MHE) using DF + B-splines

Given the differentially flat nonlinear system with flat output $\mathbf{z}$:

$$
\begin{aligned}
\dot{x} &= f(x, u, w), \qquad x(t-L) \\
y &= h(x, u) + \xi
\end{aligned}
$$



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Solve:

$$
\mathbf{C}^* = \arg\min \sum \| h\left(\hat{x}(t_k, \mathbf{C}), u(t_k)\right) - y_m(t_k) \|_{\Sigma^{-1}},
$$

where $y_m(t_k)$ is a measurement at time $t - L \leq t_k \leq t$, and the sum is over all measurements in the window.

Note: No initial condition, & do not solve differential equation

Moving Horizon Estimation (MHE) using DF + B-splines

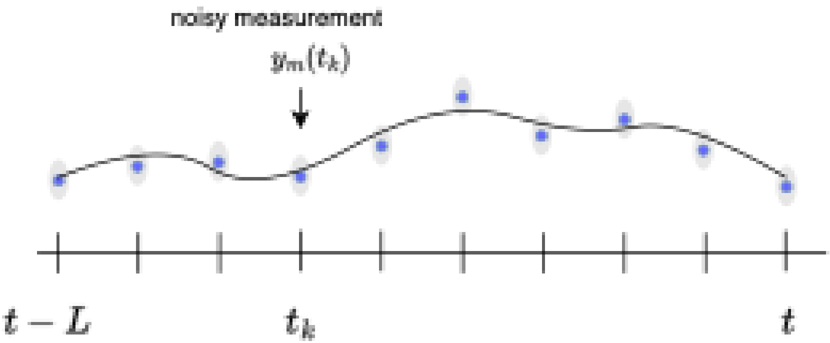
Given the differentially flat nonlinear system with flat output $\mathbf{z}$:

$$\dot{x} = f(x, u, w), \qquad x(t - L)$$
$$y = h(x, u) + \xi$$

Represent the flat output using $\hat{z}(t, \mathbf{C}) = \sum_{m=0}^{M+d-1} \mathbf{c}_m b_m^d(t, \mathbf{k}_M^d)$

giving $\hat{x}(t, \mathbf{C})$ and $\hat{w}(t, \mathbf{C})$

B-spline estimates output trajectory:



Solve:

$$\mathbf{C}^* = \arg\min \sum \left\| h\left(\hat{x}(t_k, \mathbf{C}), u(t_k)\right) - y_m(t_k) \right\|_{\Sigma^{-1}},$$

where $y_m(t_k)$ is a measurement at time $t - L \leq t_k \leq t$, and the sum is over all measurements in the window.

Note: No initial condition, & do not solve differential equation

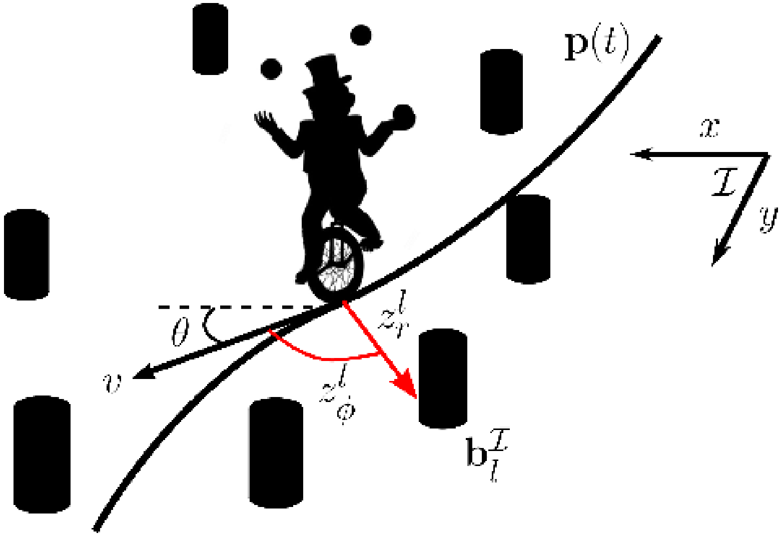
Continuous-time Trajectory Estimation for Differentially Flat Systems

Unicycle Dynamic Model

$$
\begin{aligned}
\dot{\mathbf{T}}^{\mathcal{I}}_{\mathcal{B}} &= \mathbf{T}^{\mathcal{I}}_{\mathcal{B}}\left(\begin{bmatrix} v & 0 & \omega \end{bmatrix}^\top\right)^\wedge \\
\dot{v} &= \frac{1}{m}(F - d_v v + \eta_F), \\
\dot{\omega} &= \frac{1}{J}(\tau - d_\omega \omega + \eta_\tau),
\end{aligned}
$$

where

$$
\mathbf{T}^{\mathcal{I}}_{\mathcal{B}} = \begin{bmatrix} \mathbf{R}^{\mathcal{I}}_{\mathcal{B}} & \mathbf{t}^{\mathcal{I}}_{\mathcal{B}/\mathcal{I}} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(2)
$$



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$$\dot{\mathbf{T}}^{\mathcal{I}}_{\mathcal{B}} = \mathbf{T}^{\mathcal{I}}_{\mathcal{B}} \left(\begin{bmatrix} v & 0 & \omega \end{bmatrix}^{\top} \right)^{\wedge}$$

$$\dot{v} = \frac{1}{m}(F - d_v v + \eta_F),$$

$$\dot{\omega} = \frac{1}{J}(\tau - d_\omega \omega + \eta_\tau),$$

where

$$\mathbf{T}^{\mathcal{I}}_{\mathcal{B}} = \begin{bmatrix} \mathbf{R}^{\mathcal{I}}_{\mathcal{B}} & \mathbf{t}^{\mathcal{I}}_{\mathcal{B}/\mathcal{I}} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(2)$$

Differentially flat with flat output $\mathbf{t}^{\mathcal{I}}_{\mathcal{B}/\mathcal{I}}$ since:

$$\mathbf{T}^{\mathcal{I}}_{\mathcal{B}}(\mathbf{p}, \dot{\mathbf{p}}) = \begin{bmatrix} \mathrm{Exp}\left(\tan^{-1}\left(\frac{\dot{p}_y}{\dot{p}_x}\right)\right) & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$v(\dot{\mathbf{p}}) = \|\dot{\mathbf{p}}\|,$$

$$\omega(\dot{\mathbf{p}}, \ddot{\mathbf{p}}) = -\frac{1}{\|\dot{\mathbf{p}}\|^2} \dot{\mathbf{p}}^{\top} 1^{\wedge} \ddot{\mathbf{p}}$$

$$\dot{v}(\dot{\mathbf{p}}, \ddot{\mathbf{p}}) = \frac{1}{\|\dot{\mathbf{p}}\|} \dot{\mathbf{p}}^{\top} \ddot{\mathbf{p}}$$

$$\dot{\omega}(\dot{\mathbf{p}}, \ddot{\mathbf{p}}\, \dddot{\mathbf{p}}) = \frac{2}{\|\dot{\mathbf{p}}\|^4} \left(\dot{\mathbf{p}}^{\top} \ddot{\mathbf{p}}\right) \left(\dot{\mathbf{p}}^{\top} 1^{\wedge} \ddot{\mathbf{p}}\right) - \frac{1}{\|\dot{\mathbf{p}}\|^2} \dot{\mathbf{p}}^{\top} 1^{\wedge} \dddot{\mathbf{p}}$$

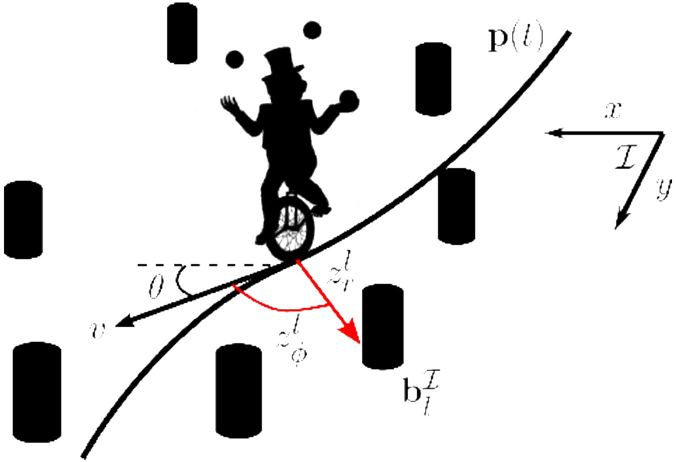
$$F(\dot{\mathbf{p}}, \ddot{\mathbf{p}}) = m\dot{v}(\dot{\mathbf{p}}, \ddot{\mathbf{p}}) + d_v v(\dot{\mathbf{p}})$$

$$\tau(\dot{\mathbf{p}}, \ddot{\mathbf{p}}, \dddot{\mathbf{p}}) = J\dot{\omega}(\dot{\mathbf{p}}, \ddot{\mathbf{p}}, \dddot{\mathbf{p}}) + d_\omega \omega(\dot{\mathbf{p}}, \ddot{\mathbf{p}}).$$

Continuous-time Trajectory Estimation for Differentially Flat Systems

Unicycle Measurement Model:

$$
\begin{aligned}
&\text{Gyroscope:} && z_g = \omega + \eta_g \\
&\text{Accelerometer:} && \mathbf{z}_a = \begin{pmatrix} \frac{1}{m}F - d_v v \\ 0 \end{pmatrix} + \boldsymbol{\eta}_a \\
&\text{Range:} && z_r^l = \|\mathbf{b}_l^B\| + \eta_r \\
&\text{Bearing:} && z_\phi^l = \tan^{-1}\left(\frac{b_{l_y}}{b_{l_x}}\right) + \eta_\phi \\
&\text{Known input force:} && z_F = F + \eta_F \\
&\text{Known input torque:} && z_\tau = \tau + \eta_\tau
\end{aligned}
$$



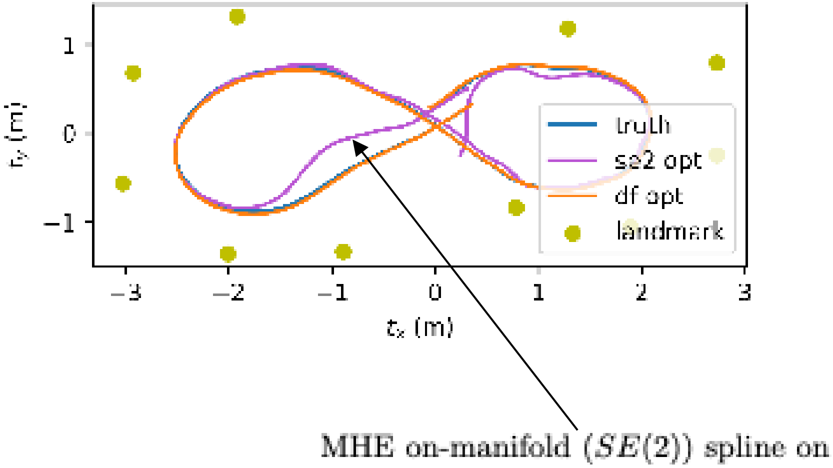
Continuous-time Trajectory Estimation for Differentially Flat Systems: Experimental Results

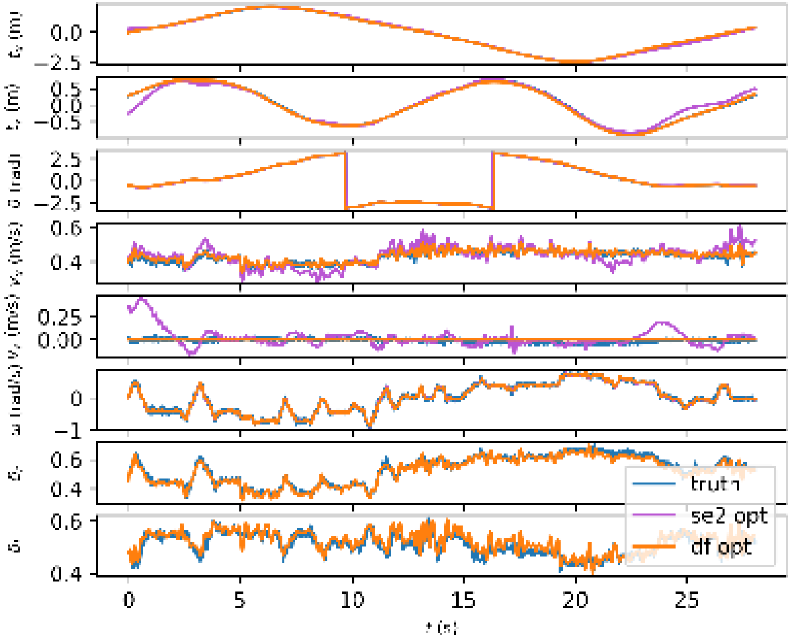
Turtlebot, with Intel RealSense D435i camera
(all measurements produced by RealSense)

MoCap room for truth data

Landmarks identified by ArUco markers

Continuous-time Trajectory Estimation for Differentially Flat Systems: Experimental Results





Spline Est. Type	Pose RMSE	Velocity RMSE	Input RMSE	Solve Time (s)	Iterations	Avg. Time per Iteration (s)
DF with IMU	0.07914	0.03440	0.11363	**0.03950**	10.20	**0.00418**
DF with input	0.07934	0.03292	0.11366	0.06195	14.90	0.00433
DF with IMU and input	**0.07801**	**0.02774**	**0.11283**	0.07557	13.44	0.00597
$SE(2)$ with IMU	0.07945	0.03852	NA	0.12425	**5.98**	0.02100
$SO(2) \times \mathbb{R}^2$ with IMU	0.08090	0.03839	NA	0.07891	8.06	0.01009

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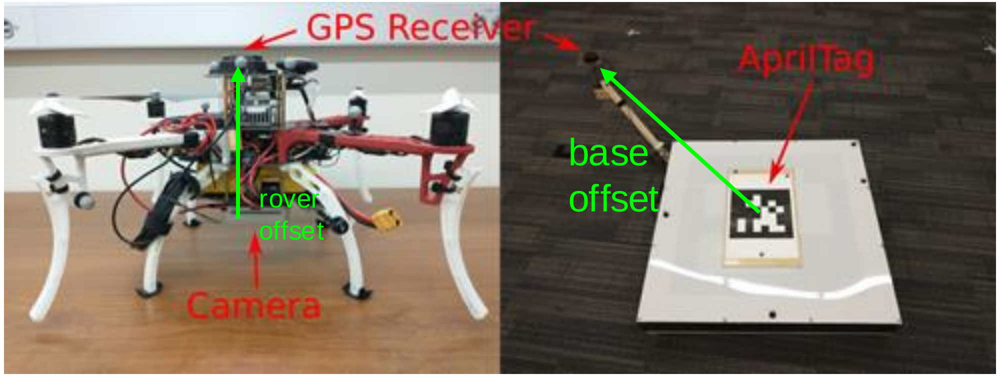
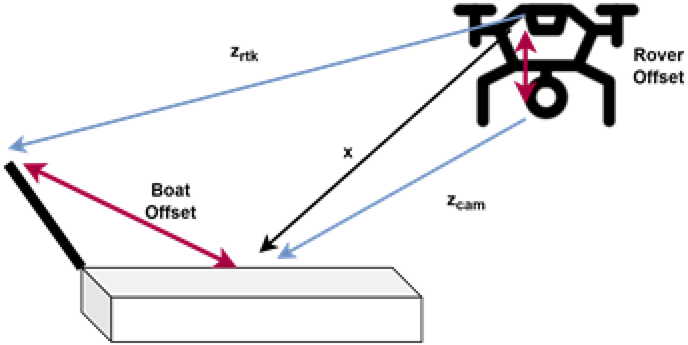
GNSS/Camera Extrinsic Calibration Using Splines on SE(3)

- State estimation by fusing GNSS and Camera
  - Precision Landing
  - Target geolocation
- Constant offsets needed for measurement models in online estimation
- Offsets not easily measured by hand

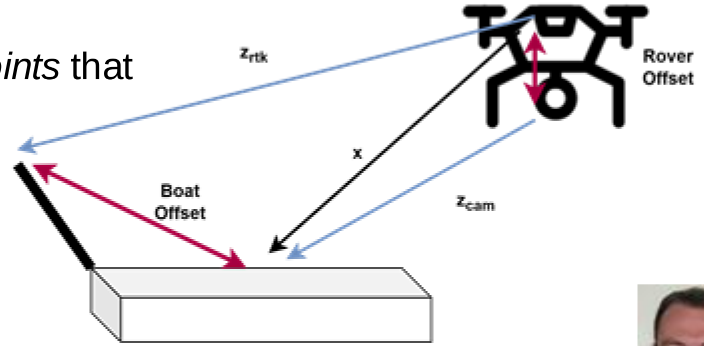




GNSS/Camera Extrinsic Calibration Using Splines on SE(3)

Overview of Method

- Offline full-batch calibration of sensor offsets
- Simultaneously estimate offsets and vehicle trajectory
    - Trajectory represented as a spline on SE(3)
- Result is a Maximum Likelihood estimation problem
    - Given a batch of measurements…
    - Choose the optimal *parameters* and *spline control points* that maximize the likelihood of the measurements.



B-Splines: (differential parameterization)

Splines on $\mathbb{R}^n$:
For any time $t \in [t_s, t_{s+1}]$, the spline

$$\mathbf{p}(t) = \sum_{m=0}^{M+d-1} \boldsymbol{c}_m b_m^d(t, \mathbf{k}_M^d) = \sum_{m=s}^{s+d} \boldsymbol{c}_m b_m^d(t, \mathbf{k}_M^d) = b_s^d \boldsymbol{c}_s + b_{s+1}^d \boldsymbol{c}_{s+1} + \cdots + b_{s+d-1}^d \boldsymbol{c}_{s+d-1} + b_{s+d}^d \boldsymbol{c}_{s+d}$$

because $\sum_{m=s}^{s+d} b_m^d(t) = 1$,

$$\begin{aligned}\mathbf{p}(t) &= \left(1 - (1 - \sum_{m=s}^{s} b_m^d)\right) \boldsymbol{c}_s + \left(1 - \sum_{m=s}^{s} b_m^d - (1 - \sum_{m=s}^{s+1} b_m^d)\right) \boldsymbol{c}_{s+1} + \cdots + \left(1 - \sum_{m=s}^{s+d-2} - (1 - \sum_{m=s}^{s+d-1} b_m^d)\right) \boldsymbol{c} \\ &= \boldsymbol{c}_s + \left(1 - \sum_{m=s}^{s} b_m^d\right)(\boldsymbol{c}_{s+1} - \boldsymbol{c}_s) + \cdots + \left(1 - \sum_{m=s}^{s+d-2} b_m^d\right)(\boldsymbol{c}_{s+d} - \boldsymbol{c}_{s+d-1})\end{aligned}$$

Letting $\beta_m^d(t, \mathbf{k}_M^d) \doteq 1 - \sum_{q=s}^{s+q} b_m^d(t, \mathbf{k}_M^d)$ gives

$$\mathbf{p}(t) = \boldsymbol{c}_s + \sum_{m=1}^{s+d} \beta_m^d(t, \mathbf{k}_M^d)(\boldsymbol{c}_{s+m} - \boldsymbol{c}_{s+m-1})$$

K. Qin, “General matrix representations for B-splines,” *The Visual Computer*, vol. 16, no. 3-4, pp. 177–186, 2000.

B-Splines on SE(3) (generalizes to other Lie Groups)

Splines on $\mathbb{R}^n$: For any time $t \in [t_s, t_{s+1}]$, the spline

$$\mathbf{p}(t) = \boldsymbol{c}_s + \sum_{m=1}^{s+d} \beta_m^d(t, \mathbf{k}_M^d)\,(\boldsymbol{c}_{s+m} - \boldsymbol{c}_{s+m-1})$$

Splines on $SE(3)$: For any time $t \in [t_s, t_{s+1}]$, the spline

$$\mathbf{T}(t) = \bar{\boldsymbol{C}}_s \Pi_{m=1}^{s+d} \mathrm{Exp}\left(\beta_m^d(t, \mathbf{k}_M^d) \mathrm{Log}\left(\bar{\boldsymbol{C}}_{s+m-1}^{-1} \bar{\boldsymbol{C}}_{s+m}\right)\right)$$

where

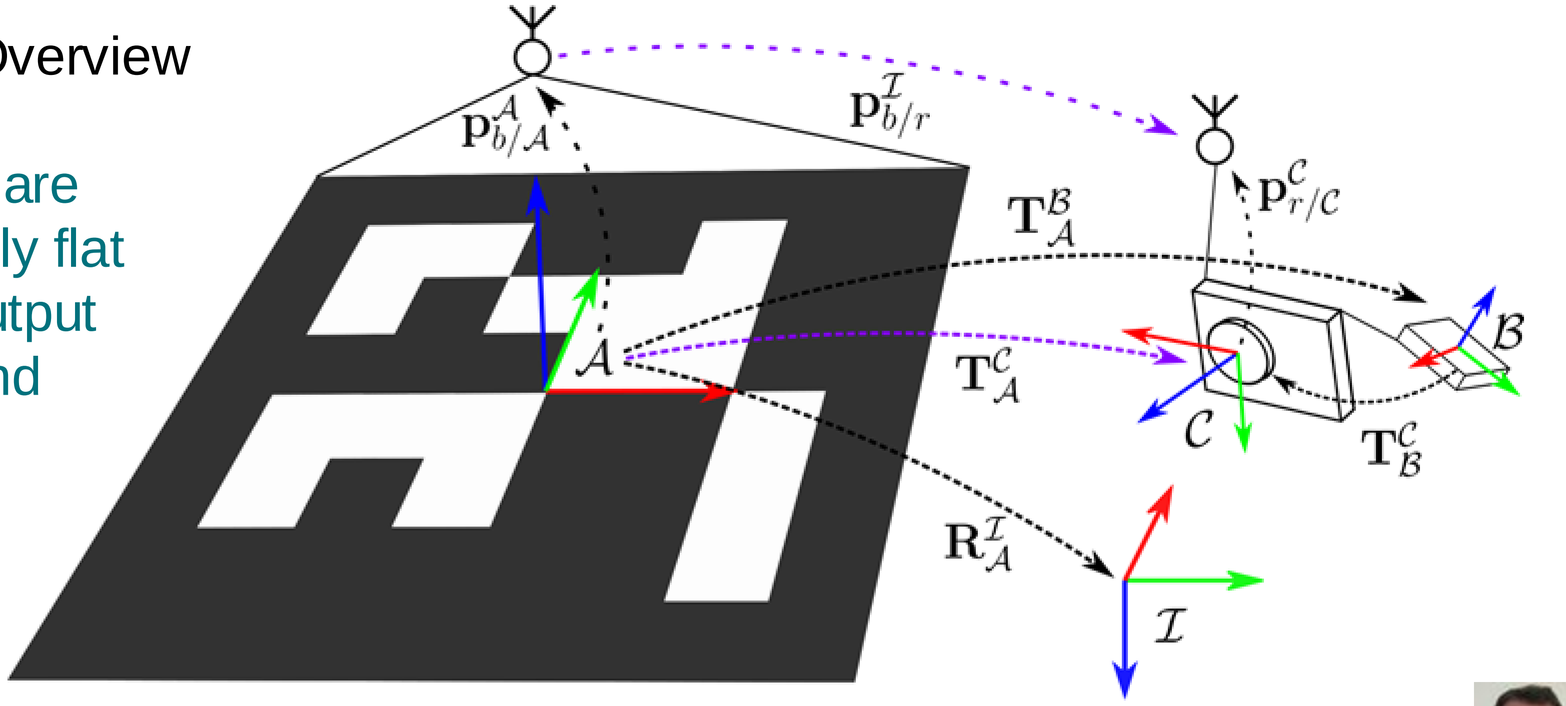
$$\begin{aligned} \bar{\boldsymbol{C}}_m &\in SE(3) && \text{are the control points} \\ \mathrm{Exp} = \exp \circ \wedge &: \mathbb{R}^6 \to SE(3) && \text{is the exponential map} \\ \mathrm{Log} = \vee \circ \log &: SE(3) \to \mathbb{R}^6 && \text{is the logarithmic map} \end{aligned}$$

A. Patron-Perez, S. Lovegrove, and G. Sibley, “A spline-based tra- jectory representation for sensor fusion and rolling shutter cameras,” *International Journal of Computer Vision*, vol. 113, no. 3, pp. 208– 219, 2015.

GNSS/Camera Extrinsic Calibration Using Splines on SE(3)

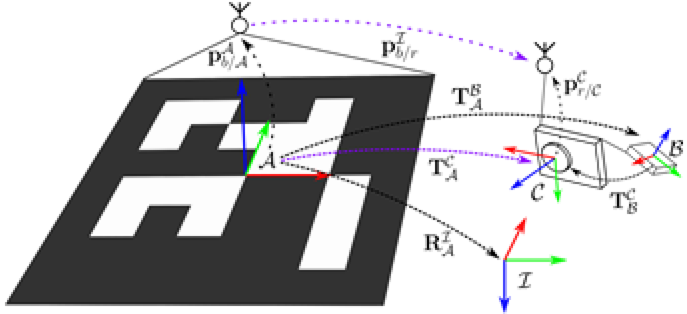
Problem Overview

Dynamics are differentially flat with flat output position and heading



δt : time delay between GPS and camera clocks

Calibration - Measurement Models



AprilTag: $z_{\text{cam}} = \text{Exp}(\eta_{\text{cam}} T_{\mathcal{B}}^{\mathcal{C}} T_{\mathcal{A}}^{\mathcal{B}}(t_{\text{cam}}))$

RTK: $z_{\text{rtk}} = R_{\mathcal{A}}^{\mathcal{I}} \left(-p_{b/\mathcal{A}}^{\mathcal{A}} + T_{\mathcal{B}}^{\mathcal{A}}(t_{\text{gnss}} + \delta t) T_{\mathcal{C}}^{\mathcal{B}} \mathbf{p}_{r/\mathcal{C}}^{\mathcal{C}} \right) + \eta_{\text{rtk}}$

GNSS Velocity: $z_{\text{vel}} = R_{\mathcal{A}}^{\mathcal{I}} R_{\mathcal{B}}^{\mathcal{A}} \left(v_{\mathcal{B}/\mathcal{A}}^{\mathcal{B}} + (\omega_{\mathcal{B}/\mathcal{A}}^{\mathcal{B}})^{\wedge} T_{\mathcal{C}}^{\mathcal{C}} \mathbf{p}_{r/\mathcal{C}}^{\mathcal{C}} \right) + \nu_{\text{vel}}$

Gyro: $z_{\text{gyro}} = \omega_{\mathcal{B}/\mathcal{I}}^{\mathcal{B}}(t_{\text{imu}}) + b_{\text{gyro}}(t_{\text{imu}}) + \eta_{\text{gyro}}$

Accelerometer: $z_{\text{accel}} = a_{\mathcal{B}/\mathcal{I}}^{\mathcal{B}}(t_{\text{imu}}) - g R_{\mathcal{A}}^{\mathcal{B}}(t_{\text{imu}}) R_{\mathcal{I}}^{\mathcal{A}} e_3 + b_{\text{acc}}(t_{\text{imu}}) + \eta_{\text{acc}}$

Calibration - Optimization

Quantities that need to be calibrated:

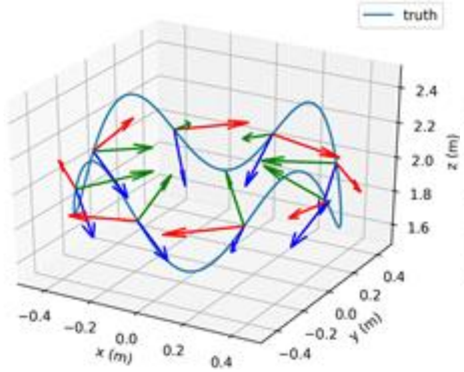
$\{\bar{T}_j\}_{j=0,\ldots,M}$	trajectory control points
$\boldsymbol{p}^{\mathcal{C}}_{r/\mathcal{C}}$	position of rover receiver in camera frame
$\boldsymbol{p}^{\mathcal{A}}_{b/\mathcal{A}}$	position of base receiver in fiducial frame
$R^{\mathcal{I}}_{\mathcal{A}}$	rotation from fiducial to NED frame
δt	time offset between camera and GNSS
$\{\bar{b}_{g_j}\}_{j=0,\ldots,M_{\text{gyro}}}$	gyro bias control points
$\{\bar{b}_{g_j}\}_{j=0,\ldots,M_{\text{gyro}}}$	gyro bias control points

Optimization Problem:

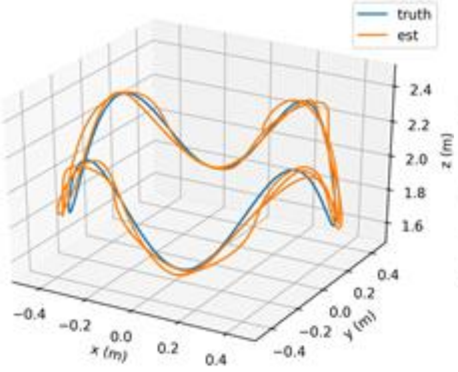
$$\begin{aligned}\arg\min_{\boldsymbol{\Gamma}} \Big\{ &\sum_{i_{\text{cam}}} \|\mathrm{Log}(\boldsymbol{h}_{\text{cam}}(\boldsymbol{\Gamma})\boldsymbol{z}^{-1}_{i_{\text{cam}}})\|^2_{\Sigma_{\text{cam}}^{-1}} \\ &+ \sum_{i_{\text{rtk}}} \|\boldsymbol{z}_{i_{\text{rtk}}} - \boldsymbol{h}_{\text{rtk}}(\boldsymbol{\Gamma})\|^2_{\Sigma^{-1}_{\text{rkt}}} \\ &+ \sum_{i_{\text{vel}}} \|\boldsymbol{z}_{i_{\text{vel}}} - \boldsymbol{h}_{\text{vel}}(\boldsymbol{\Gamma})\|^2_{\Sigma^{-1}_{\text{vel}}} \\ &+ \sum_{i_{\text{gyro}}} \|\boldsymbol{z}_{i_{\text{gyro}}} - \boldsymbol{h}_{\text{gyro}}(\boldsymbol{\Gamma})\|^2_{\Sigma^{-1}_{\text{gyro}}} \\ &+ \sum_{i_{\text{accel}}} \|\boldsymbol{z}_{i_{\text{accel}}} - \boldsymbol{h}_{\text{accel}}(\boldsymbol{\Gamma})\|^2_{\Sigma^{-1}_{\text{accel}}} \Big\}\end{aligned}$$

Optimization is solved iteratively using Levenberg-Marquardt

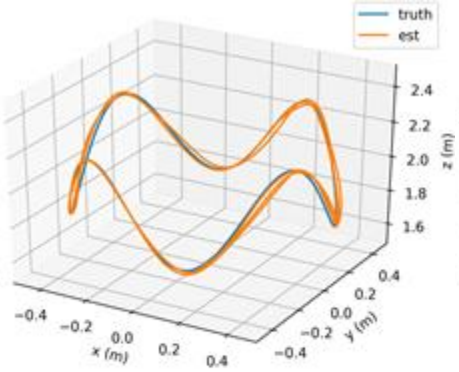
Simulation Results



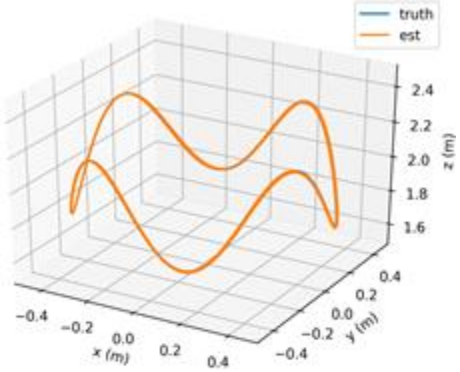
Trajector
y



Camera + GNSS



Camera + GNSS
+ Velocity



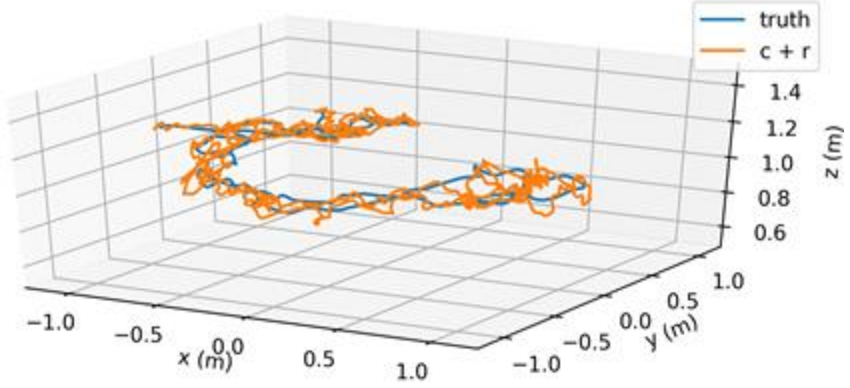
Camera + GNSS
+ Velocity + IMU

Motion Capture Results

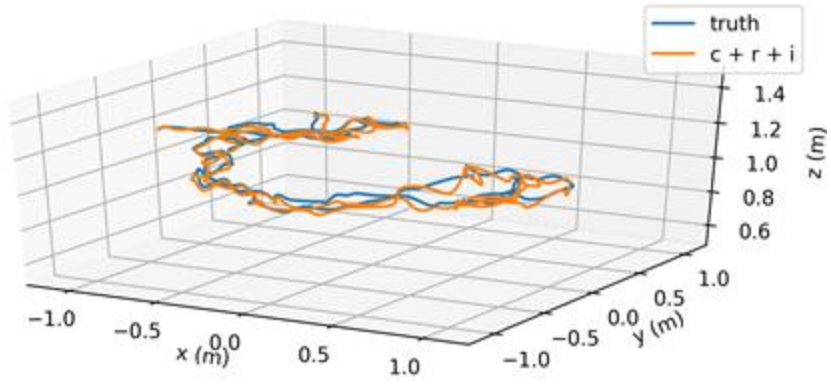
TABLE I

MOTION CAPTURE CALIBRATION RESULTS FOR C+R AND C+R+I FOR A 50 AND 100 s TRAJECTORY.

Type	$\mathbf{p}_{b/\mathcal{A}}^{\mathcal{A}}$ err. (m)	$\mathbf{R}_{\mathcal{A}}^{\mathcal{I}}$ err. (rad)	$\mathbf{p}_{r/\mathcal{C}}^{\mathcal{C}}$ (m)	δt (s)	Pose RMSE	Time (s)
c+r, 50	0.0136	0.0089	[0.100, 0.048, -0.196]	-0.0067	0.0560	2.90
c+r+i, 50	**0.0098**	**0.0067**	[0.099, 0.047, -0.198]	-0.0200	**0.0458**	24.88
c+r, 100	**0.0039**	0.0049	[0.100, 0.046, -0.201]	-0.0160	0.0460	9.90
c+r+i, 100	**0.0039**	**0.0047**	[0.100, 0.046, -0.202]	-0.0206	**0.0421**	35.58



Camera + GNSS



Camera + GNSS
+ IMU

Outdoor Results

TABLE II
OUTDOOR CALIBRATION RESULTS

	$\mathbf{p}^{\mathcal{A}}_{b/\mathcal{A}}$ (m)	$\mathbf{p}^{\mathcal{C}}_{r/\mathcal{C}}$ (m)	δt (s)
mean	[0.766, -0.650, 0.308]	[0.053, 0.005, -0.184]	-0.0570
std. dev.	[0.006, 0.005, 0.004]	[0.003, 0.005, 0.006]	0.0033

Hand Calibrated Values:

- Base offset: [0.78, -0.64, 0.33]
- Rover offset: [0.04, 0, -0.19]

Summary and Future work

- B-splines
    - continuous representation of trajectories and surfaces
    - Discrete parameterization not directly linked to time constants, sampling rates, noise levels, etc.
    - Sparse representation: control points have only local impact
    - Many properties can be guaranteed at every time by checking finite (and selectable) number of constraints

- Applications to Planning
    - Trajectories are in the convex hull of control points (locally)

- Applications to Estimation
    - Flexible parameterization with many advantages for Moving Horizion Estimation