The Empty Set: Much Ado About Nothing

ECEn 670: Stochastic Processes

The great thing about mathematics is that it allows one to be upset over nothing.

–Anonymous

Definitions

On page 15, the text reads [1]

> The *empty* or *null* set is by definition the set that contains no elements. This set will be denoted by $\{\emptyset\}$.

There are several issues with this two-sentence paragraph.

1. The terms *empty set* and *null set* used to be used interchangeably. The term *null set* now has a technical definition in measure theory [2] that sets it apart from *empty set*. So, for this class, the term *empty set* is used.

2. The definition "the empty set is the set that contains no elements" works for us.

3. Notation: "This set will be denoted by $\{\emptyset\}$" is not consistent with the widely adopted conventions in mathematical set theory. In fact, I would go as far as saying the notation is incorrect. To be consistent with the conventions in set theory, the sentence should read

   > This set will be denoted by $\emptyset$.

   That is, the symbol $\emptyset$ denotes the set with no elements. Curiously, this is the notation used in the lecture slides provided by U. Pillai on the publisher's website [3]. To put the nail in the coffin, we have from Malcolm Graham [4]

   > Note that the empty set is not designated by $\{\varnothing\}$; this notation would represent a set containing one element $\varnothing$, rather than the set with no elements. Similarly, the set $\{0\}$ contains the element zero and hence is not the empty set.

The third issue warrants a little more discussion. The cardinality of the set A, denoted $|A|$ is defined as the number of elements in the set. Some examples are

$$
\begin{aligned}
A &= \{a\} & |A| &= 1 \\
A &= \{a, b, c\} & |A| &= 3 \\
A &= \{\{a, b\}, c\} & |A| &= 2.
\end{aligned}
$$

Because $\emptyset$ contains no elements $|\emptyset| = 0$. But the notation $\{\emptyset\}$ means the set containing one element: the empty set. So we have $|\{\emptyset\}| = 1$.

Notation

As of this writing, the notation for the empty set most commonly encountered is either "{ }" or "$\varnothing$". With reference to the end of the previous section, the set containing the empty set is $\{\{\}\}$ or $\{\varnothing\}$, respectively.

The symbol $\varnothing$ for the empty set was introduced by Bourbaki in [5, pg. 4]:

> certaines propriétés ... ne sont vraies pour *aucun* élément de E ... la partie qu'elles définissent est appelée la *partie vide* de E, et designée par la notation $\varnothing$."

On the choice of the symbol $\varnothing$, André Weil (1906–1998), a student of Bourbaki, explains in [6, pg 114] (translation by Jennifer Gage via [7])

> Wisely, we had decided to publish an installment establishing the system of notation for set theory, rather than wait for the detailed treatment that was to follow: it was high time to fix these notations once and for all, and indeed the ones we proposed, which introduced a number of modifications to the notations previously in use, met with general approval. Much later, my own part in these discussions earned me the respect of my daughter Nicolette, when she learned the symbol $\varnothing$ for the empty set at school and I told her that I had been personally responsible for its adoption. The symbol came from the Norwegian alphabet, with which I alone among the Bourbaki group was familiar.

The text uses the symbol $\emptyset$ to represent the empty set. Comments:

1. The symbol $\emptyset$ looks like a zero with a slash through it. In LaTeX, the symbol is created using $\emptyset$.

2. Most authors (including your instructor) believe \emptyset produces the ugliest empty set symbol in the solar system. Evidently, the keepers of LaTeX also believe this and have provided the \varnothing command that produces the much better looking $\varnothing$. (\varnothing needs the amssymb package.)

3. In Unicode, the empty set symbol $\varnothing$ (∅) occupies code point U+2205. But many fonts in use today do not include this character and render it as a small rectangle.

4. The Greek letters ϕ (lowercase) and Φ (uppercase) are not equivalent to the symbol for the empty set, and should not be used. Never. Ever.

$$\phi \neq \varnothing \qquad \Phi \neq \varnothing.$$

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$\varnothing$ is *the* empty set, not *an* empty set. This is because the empty set is unique. Two sets are equal if they both contain the same elements. Consequently, there can only be one set with no elements. In other words, if there were more than one empty set, they would be equal.

Von Neumann's Definition of Ordinals

Ordinals are an extension to the natural numbers (non-negative integers). In the simplest terms, natural numbers count things (they answer the question "how many?") and ordinal numbers tell the position of something (its "order") in a list. For a finite number of things to be counted or ordered, there is little difference between natural numbers and the ordinal numbers. Where things get interesting to mathematicians is the case where an infinite number of things need to be ordered. The trick is to identify the definition of a number that satisfies all the required mathematical properties in the limit.

Von Neumann defined ordinals in terms of the cardinality of sets. If α is the set representing an ordinal, then the next ordinal is defined by the cardinality of the sucessor set $S_\alpha = \{\alpha \cup \{\alpha\}\}$.

The starting point is the empty set! The first few Von Neumann ordinals are

$$
\begin{aligned}
&0 \leftarrow \varnothing \\
&1 \leftarrow 0 \cup \{0\} = \varnothing \cup \{\varnothing\} = \{\varnothing\} \\
&2 \leftarrow 1 \cup \{1\} = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\} \\
&3 \leftarrow 2 \cup \{2\} = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\
&4 \leftarrow 3 \cup \{3\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}
\end{aligned}
$$

References

[1] A. Papoulis and U. Pillai, *Probability, Random Variables, and Stochastic Processes*, 4th ed. Boston: McGraw-Hill, 2002.

[2] Wikipedia, "Measure (mathematics)," 2017. [Online]. Available: https://en.wikipedia.org/wiki/Measure_(mathematics)

[3] McGraw-Hill Companies, "Probability, random variables, and stochastic processes: Instructor resources," 2001. [Online]. Available: http://www.mhhe.com/engcs/electrical/papoulis/ippt.mhtml

[4] M. Graham, *Modern Elementary Mathematics*, 4th ed. San Diego, CA: Harcourt College Publishers, 1984.

[5] N. Bourbaki, *Éléments de mathématique Fasc.1: Les structures fondamentales de l'analyse; Liv.1: Theorie de ensembles.* Paris: Hermann & C^{ie} Éditeurs, 1939.

[6] A. Weil, *The Apprenticeship of a Mathematician*. Basel-Boston-Berlin: Birkhaeuser Verlag, 1992.

[7] J. Miller, "Earliest uses of symbols of set theory and logic," 2017. [Online]. Available: http://jeff560.tripod.com/set.html