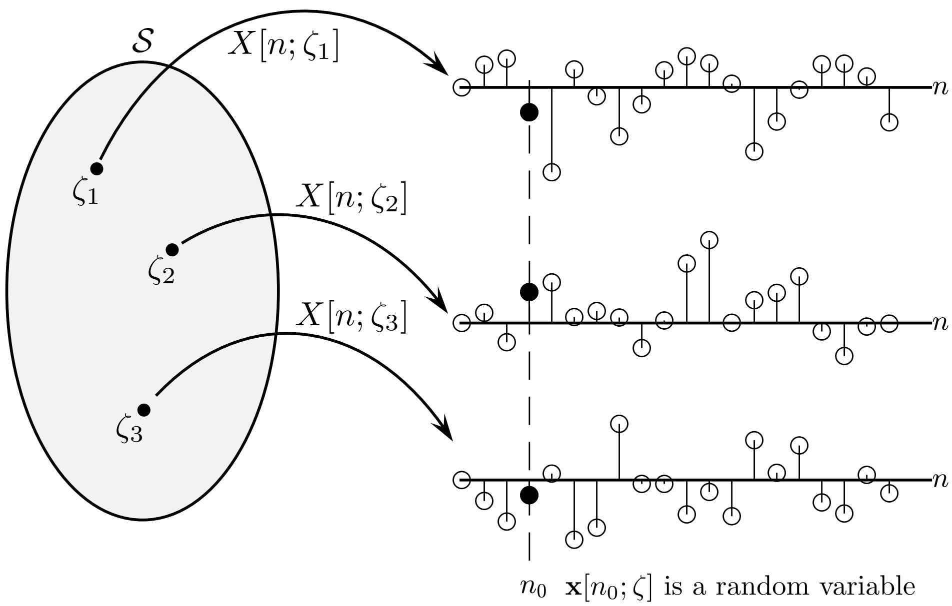
9-4 Discrete-Time Random Processes



Definition

A discrete-time *stochastic process* (also called *random process*) $\mathbf{x}[n;\zeta]$ is a rule for assigning to every $\zeta \in \mathcal{S}$ a discrete-time sequence.

Interpretations

1. If n and ζ are variables, the result is a family (or an *ensemble*) of sequences $\mathbf{x}[n,\zeta]$.
2. If n is a variable and ζ is fixed, the result is a single discrete-time sequence (or a *sample* of the stochastic process).
3. If n is fixed and ζ is variable, the result is a *random variable.*
4. If n and ζ are fixed, the result is a *number.*

Statistics of Discrete-Time Stochastic Processes

Definition

- The *k-th order distribution* of the real-valued process $\mathbf{x}[n]$ is the joint distribution of the real-valued random variables $\mathbf{x}[n_1], \ldots, \mathbf{x}[n_k]$:

$$F_{\mathbf{x}}(x_1, \ldots, x_k; n_1, \ldots, n_k) = P(\mathbf{x}[n_1] \leq x_1, \ldots, \mathbf{x}[n_k] \leq x_k)$$

  - If the random variables are jointly continuous, then the joint cdf is a continuous function.
  - If the random variables are jointly discrete, the the joint cdf is a k-dimensional stair-step function.
  - Do not confuse time and random variable type: a discete-*time* random process may be described by either continuous or discrete *random variables* at a fixed time index.

- The *k-th order density function* of the real-valued process $\mathbf{x}[n]$ is joint density of the real-valued random variables $\mathbf{x}[n_1], \ldots, \mathbf{x}[n_k]$:

$$f_{\mathbf{x}}(x_1, \ldots, x_k; n_1, \ldots, n_k) = \frac{\partial^k}{\partial x_1 \cdots \partial x_k} F_{\mathbf{x}}(x_1, \ldots, x_n; n_1, \ldots, n_k)$$

  - If the random variables are jointly continuous, then the joint pdf is smooth.
  - If the random variables are jointly discrete, then the joint pdf contains impulses (in the form of Dirac delta functions).
  - Alternatively, for jointly discrete random variables, the joint pmf may be used.
  - Do not confuse time and random variable type: a discrete-*time* random process may be described by either a continuous or discrete *random variable* at a fixed time index.

Special cases (real-valued random processes)

- First-order density:

  1. The *first-order* distribution/density is the special case $k = 1$:

  $$F_{\mathbf{x}}(x;n) = P(\mathbf{x}[n] \leq x)$$
  $$f_{\mathbf{x}}(x;n) = \frac{\partial F_{\mathbf{x}}(x;n)}{\partial x}$$

  2. The *mean* of the random process $\mathbf{x}[n]$ is the mean of the random variable $\mathbf{x}[n]$ for fixed n and is computed from the first-order pdf

  $$\mu_{\mathbf{x}}[n] = E\{\mathbf{x}[n]\} = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;n) dx$$

- Second-order density

  1. The *second-order* distribution/density is the special case $k = 2$:

  $$F_{\mathbf{x}}(x_1, x_2; n_1, n_2) = P(\mathbf{x}[n_1] \leq x_1, \mathbf{x}[n_2] \leq x_2)$$
  $$f_{\mathbf{x}}(x_1, x_2; n_1, n_2) = \frac{\partial^2 F_{\mathbf{x}}(x_1, x_2; n_1, n_2)}{\partial x_1 \partial x_2}$$

  2. The *autocorrelation function* is the expected value of the product $\mathbf{x}[n_1]\mathbf{x}[n_2]$ and is computed from the second order density:

  $$R_{\mathbf{xx}}[n_1, n_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{\mathbf{x}}(x_1, x_2; n_1, n_2)\, dx_1\, dx_2$$

  3. The *average power* of the random process $\mathbf{x}[n]$ is the value of $R_{\mathbf{xx}}[n_1, n_2]$ along the diagonal $n = n_1 = n_2$:

  $$\text{average power} = E\{\mathbf{x}^2[n]\} = R_{\mathbf{xx}}[n, n]$$

  4. The *autcovariance* of the random process $\mathbf{x}[n]$ is the covariance of the random variables $\mathbf{x}[n_1]$ and $\mathbf{x}[n_2]$ and is computed from the second order density

  $$\begin{aligned} C_{\mathbf{xx}}[n_1, n_2] &= E\{(\mathbf{x}[n_1] - \mu_{\mathbf{x}}[n_1])(\mathbf{x}[n_2] - \mu_{\mathbf{x}}[n_2])\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mu_{\mathbf{x}}[n_1])(x_2 - \mu_{\mathbf{x}}[n_2]) f_{\mathbf{x}}(x_1, x_2; n_1, n_2)\, dx_1\, dx_2 \end{aligned}$$

  5. The *variance* of the random process $\mathbf{x}[n]$ is the value of $C_{\mathbf{xx}}[n_1, n_2]$ along the diagonal $n = n_1 = n_2$:

  $$\text{variance} = E\{(\mathbf{x}[n] - \mu_{\mathbf{x}}[n])^2\} = C_{\mathbf{xx}}(n, n)$$

  6. The *correlation coefficient* is

  $$r_{\mathbf{xx}}[n_1, n_2] = \frac{C_{\mathbf{xx}}[n_1, n_2]}{\sqrt{C_{\mathbf{xx}}[n_1, n_1] C_{\mathbf{xx}}[n_2, n_2]}}$$

More Definitions (real-valued random processes)

- A *white* random process $\mathbf{x}[n]$ means

$$C_{\mathbf{xx}}[n_1, n_2] = q[n_1]\delta[n_1 - n_2]$$

It is *almost always* assumed that a white random process has zero mean:

$$\mu_{\mathbf{x}}[n] = 0$$

- A *normal random process* $\mathbf{x}[n]$ means the random variables $\mathbf{x}[n_1], \ldots, \mathbf{x}[n_k]$ are jointly normal for any k and any $n_1, \ldots, n_k$.

Two real-valued random processes

- Two real-valued random processes $\mathbf{x}[n]$ and $\mathbf{y}[n]$ are described by the joint distribution and density of the random variables

$$\mathbf{x}[n_1], \ldots, \mathbf{x}[n_k], \mathbf{y}[n_1'], \ldots, \mathbf{y}[n_m']$$

$$F_{\mathbf{xy}}(x_1, \ldots, x_k, y_1, \ldots, y_m; n_1, \ldots, n_k, n_1', \ldots, n_m') = \\ P(\mathbf{x}[n_1] \leq x_1, \ldots, \mathbf{x}[n_k] \leq x_k, \mathbf{y}[n_1'] \leq y_1, \ldots, \mathbf{y}[n_m'] \leq y_m)$$

$$f_{\mathbf{xy}}(x_1, \ldots, x_k, y_1, \ldots, y_m; n_1, \ldots, n_k, n_1', \ldots, n_m') = \\ \frac{\partial^{k+m} F_{\mathbf{xy}}(x_1, \ldots, x_k, y_1, \ldots, y_m; n_1, \ldots, n_k, n_1', \ldots, n_m')}{\partial x_1 \cdots \partial x_k \partial y_1 \cdots \partial y_m}$$

- The *cross-correlation function* is

$$R_{\mathbf{xy}}[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{y}[n_2]\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{\mathbf{xy}}(x, y; n_1, n_2)\, dx\, dy$$

- The *cross-covariance function* is

$$\begin{aligned} C_{\mathbf{xy}}[n_1, n_2] &= E\{(\mathbf{x}[n_1] - \mu_{\mathbf{x}}[n_1])(\mathbf{y}[n_2] - \mu_{\mathbf{y}}[n_2])\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_{\mathbf{x}}[n_1])(y - \mu_{\mathbf{y}}[n_2]) f_{\mathbf{xy}}(x, y; n_1, n_2)\, dx\, dy \end{aligned}$$

- Two processes $\mathbf{x}[n]$ and $\mathbf{y}[n]$ are *uncorrelated* if

$$C_{\mathbf{xy}}[n_1, n_2] = 0 \quad \text{for every } n_1 \text{ and } n_2$$

Comments on Complex-Valued Random Processes

- A *complex-valued* random process $\mathbf{z}[n,\zeta]$ maps each $\zeta \in \mathcal{S}$ to a complex-valued discrete-time sequence.

  1. If n and ζ are variable, the result is an *ensemble* of complex-valued discrete-time sequences $\mathbf{z}[n,\zeta]$.
  2. If n is variable and ζ is fixed, the result is a single complex-valued discrete-time sequence: a *sample* of the random process.
  3. If n is fixed and ζ is variable, the result is a complex-valued *random variable.*
  4. If n and ζ are fixed, the result is a *complex number.*

- The k-th order distribution and density of the complex-valued process $\mathbf{z}[n]$

  – Write

$$\begin{aligned} \mathbf{z}[n_1] &= \mathbf{x}[n_1] + j\mathbf{y}[n_1] & \qquad z_1 &= x_1 + jy_1 \\ &\vdots & &\vdots \\ \mathbf{z}[n_k] &= \mathbf{x}[n_k] + j\mathbf{y}[n_k] & \qquad z_k &= x_k + jy_k \end{aligned}$$

  – The k-th order distribution is the joint distribution of the complex-valued random variables $\mathbf{z}[n_1], \ldots, \mathbf{z}[n_k]$

$$\begin{aligned} F_{\mathbf{z}}(z_1,\ldots,z_k;n_1,\ldots,n_k) &= P(\mathbf{x}[n_1] \le x_1, \ldots, \mathbf{x}[n_k] \le x_k, \mathbf{y}[n_1] \le y_1, \ldots, \mathbf{y}[n_k] \le y_k) \\ &= F_{\mathbf{xy}}(x_1,\ldots,x_k,y_1\ldots,y_k;n_1,\ldots,n_k) \end{aligned}$$

  – The k-th order density of $\mathbf{z}[n]$ is expressed in terms of the (real-valued) real and imaginary components of $\mathbf{z}[n]$

$$\begin{aligned} f_{\mathbf{z}}(z_1,\ldots,z_k;n_1,\ldots,n_k) &= f_{\mathbf{xy}}(x_1,\ldots,x_k,y_1,\ldots,y_k;n_1,\ldots,n_k) \\ &= \frac{\partial^{2k}}{\partial x_1 \cdots \partial x_k \partial y_1 \cdots \partial y_k} F_{\mathbf{xy}}(x_1,\ldots,x_k,y_1\ldots,y_k;n_1,\ldots,n_k) \end{aligned}$$

- First two moments

  – mean

$$\mu_{\mathbf{z}}[n] = \int_{-\infty}^{\infty} z f_{\mathbf{z}}(z;n)dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+jy) f_{\mathbf{xy}}(x,y;n)\,dx\,dy = \mu_{\mathbf{x}}[n] + j\mu_{\mathbf{y}}[n]$$

  – Autocorrelation $R_{\mathbf{zz}}[n_1,n_2] = E\{\mathbf{z}[n_1]\mathbf{z}^*[n_2]\}$

$$R_{\mathbf{zz}}[n_1,n_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1+jy_1)(x_2-jy_2) f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;n_1,n_2)\,dx_1\,dx_2\,dy_1\,dy_2$$

  – Autocovariance: $C_{\mathbf{zz}}[n_1,n_2] = R_{\mathbf{zz}}[n_1,n_2] - \mu_{\mathbf{z}}[n_1]\mu_{\mathbf{z}}^*[n_2]$

Stationary Processes

Definitions

A stochastic process $\mathbf{x}[n]$ is called *strict-sense stationary* (abbreviated SSS) if its statistical properties are invariant to to a shift of the time origin.

- $\Rightarrow$ $\mathbf{x}[n]$ and $\mathbf{x}[n+c]$ have the same statistics.
- $\Rightarrow$ $f_{\mathbf{x}}(x_1,\ldots,x_k;n_1,\ldots,n_k) = f_{\mathbf{x}}(x_1,\ldots,x_k;n_1{+}c,\ldots,n_k{+}c)$ for any integer c and for all k.

Properties

1. First-order density:
   (a) $f_{\mathbf{x}}(x;n) = f_{\mathbf{x}}(x;n+c) \Rightarrow f_{\mathbf{x}}(x;n) = f_{\mathbf{x}}(x)$
   (b) $\mu_{\mathbf{x}}[n] = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;n)dx = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x)dx = \mu_{\mathbf{x}}$
2. Second-order density:
   (a)
   $$\begin{aligned} f_{\mathbf{x}}(x_1,x_2;n_1,n_2) &= f_{\mathbf{x}}(x_1,x_2;n_1+c,n_2+c) \\ \Rightarrow f_{\mathbf{x}}(x_1,x_2;n_1,n_2) &= f_{\mathbf{x}}(x_1,x_2;n_1-n_2,0) \\ &= f_{\mathbf{x}}(x_1,x_2;m,0), \quad m = n_1-n_2 \end{aligned}$$

   To paraphrase: "Thus the joint density of the random variables $\mathbf{x}[n+m]$ and $\mathbf{x}[n]$ is independent of [i.e., not a function of] n and it equals $f_{\mathbf{x}}(x_1,x_2;m)$."

   (b)
   $$\begin{aligned} R_{\mathbf{xx}}[n_1,n_2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{\mathbf{x}}(x_1,x_2;n_1n_2)\,dx_1\,dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{\mathbf{x}}(x_1,x_2;m,0)\,dx_1\,dx_2 \\ &= R_{\mathbf{xx}}[m,0] \end{aligned}$$

   (c) It is customary to express the autocorrelation function for a SSS random process by
   $$R_{\mathbf{xx}}[m] = E\{\mathbf{x}[n+m]\mathbf{x}[n]\} = E\{\mathbf{x}[n]\mathbf{x}[n-m]\}$$

   (d) $C_{\mathbf{xx}}[n_1,n_2] = C_{\mathbf{xx}}[m] = R_{\mathbf{xx}}[m] - \mu_{\mathbf{x}}^2$

Consequences

1. average power of SSS process $= R_{\mathbf{xx}}[0]$
2. variance of SSS process $= C_{\mathbf{xx}}[0]$
3. correlation coefficient of SSS process: $r_{\mathbf{xx}}[m] = \dfrac{C_{\mathbf{xx}}[m]}{C_{\mathbf{xx}}[0]}$

Definitions

A stochastic process $\mathbf{x}[n]$ is called *wide-sense stationary* (abbreviated WSS) if

$$f_{\mathbf{x}}(x;n) = f_{\mathbf{x}}(x;n+c)$$
$$f_{\mathbf{x}}(x_1,x_2;n_1,n_2) = f_{\mathbf{x}}(x_1,x_2;n_1+c,n_2+c)$$

Properties

1. $\mu_{\mathbf{x}}[n] = \mu_{\mathbf{x}}$
2. $R_{\mathbf{xx}}[n_1,n_2] = R_{\mathbf{xx}}[m]$

A stochastic process $\mathbf{x}[n]$ is *WSS white noise* means $C_{\mathbf{xx}}[m] = q\delta[m]$.

Comments on Complex-Valued WSS Random Processes

- The complex-valued WSS process $\mathbf{z}[n] = \mathbf{x}[n] + j\mathbf{y}[n]$ is described in terms of the joint statistics of the two real-valued processes $\mathbf{x}[n]$ and $\mathbf{y}[n]$.
- The first-order density property

$$f_{\mathbf{z}}(z;n) = f_{\mathbf{z}}(z;n+c) \Rightarrow f_{\mathbf{z}}(z;n) = f_{\mathbf{z}}(z)$$

becomes

$$f_{\mathbf{xy}}(x,y;n) = f_{\mathbf{xy}}(x,y;n+c) \Rightarrow f_{\mathbf{xy}}(x,y;n) = f_{\mathbf{xy}}(x,y)$$

- The complex-valued mean is a constant:

$$\mu_{\mathbf{z}}[n] = \mu_{\mathbf{z}} \Rightarrow \mu_{\mathbf{x}}[n] + j\mu_{\mathbf{y}}[n] = \mu_{\mathbf{x}} + j\mu_{\mathbf{y}}$$

- The second-order density property

$$f_{\mathbf{z}}(z_1,z_2;n_1,n_2) = f_{\mathbf{x}}(z_1,z_2;n_1+c,n_2+c)$$
$$\Rightarrow f_{\mathbf{z}}(z_1,z_2;n_1,n_2) = f_{\mathbf{x}}(z_1,z_2;n_1-n_2,0)$$

becomes

$$f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;n_1,n_2) = f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;n_1+c,n_2+c)$$
$$\Rightarrow f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;n_1,n_2) = f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;n_1-n_2,0)$$

- Autocorrelation function is

$$R_{\mathbf{zz}}[m] = E\{\mathbf{z}[n+m]z^*[n]\} = E\{\mathbf{z}[n]\mathbf{z}^*[n-m]\}$$

Properties of the auto- and cross-correlation functions

General Random Processes

1. $R_{\mathbf{xx}}[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{x}^*[n_2]\}$
2. $R_{\mathbf{xx}}[n_2, n_1] = R^*_{\mathbf{xx}}[n_1, n_2]$
3. $R_{\mathbf{xx}}[n, n] \geq 0$
4. $R_{\mathbf{xy}}[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{y}^*[n_2]\}$
5. $R_{\mathbf{yx}}[n_2, n_1] = R^*_{\mathbf{xy}}[n_1, n_2]$

WSS Random Processes

1. $R_{\mathbf{xx}}[m] = E\{\mathbf{x}[n+m]\mathbf{x}^*[n]\}$
2. $R_{\mathbf{xx}}[-m] = R^*_{\mathbf{xx}}[m]$
3. $R_{\mathbf{xx}}[0] \geq 0$
4. $R_{\mathbf{xy}}[m] = E\{\mathbf{x}[n+m]\mathbf{y}^*[n]\}$
5. $R_{\mathbf{yx}}[-m] = R^*_{\mathbf{xy}}[m]$
6. $R_{\mathbf{xx}}[m] \leq R_{\mathbf{xx}}[0]$

From Property 6 for WSS random processes

$R_{\mathbf{xx}}[m_1] = R_{\mathbf{xx}}[0]$ for some $m_1 \neq 0 \quad \Rightarrow \quad R_{\mathbf{xx}}[m+m_1] = R_{\mathbf{xx}}[m]$ for all m

$\Rightarrow \quad R_{\mathbf{xx}}[m]$ is *periodic* with period m_1

$R_{\mathbf{xx}}[1] = R_{\mathbf{xx}}[0] \quad \Rightarrow \quad R_{\mathbf{xx}}[m] = R_{\mathbf{xx}}[0]$ for all m.

Systems With Stochastic Inputs

Definitions

- A stochastic process $\mathbf{x}[n, \zeta]$ is a map from $\mathcal{S}$ to a real-valued discrete-time sequence.
  - For each $\zeta \in \mathcal{S}$, $\mathbf{x}[n, \zeta]$ is a *discrete-time signal.*
- Discrete-time system: input is a discrete-time signal $x[n]$. Output is another discrete-time signal $y[n]$.
  - If the input to a discrete-time system is a random process, then the input/relationship applies on a sample-by-sample basis.
  - For $\zeta_i \in \mathcal{S}$,

    $$\begin{aligned}\mathbf{x}[n, \zeta_i] &= \text{the input discrete-time signal} \\ \mathbf{y}[n, \zeta_i] &= \text{the output discrete-time signal}\end{aligned}$$

    $$\mathbf{x}[n, \zeta_i] \longrightarrow \boxed{\text{system}} \longrightarrow \mathbf{y}[n, \zeta_i]$$
- Kinds of Systems
  - The system is *deterministic* if it operates only the variable n, treating ζ as a parameter.
  - The system is called *stochastic* if it operates on both variables n and ζ.
  - If the system is specified in terms of physical elements or by an equation,
    * the system is deterministic if the elements or coefficients of the defining equations are deterministic
    * the system is stochastic of the elements or coefficients of the defining equations are random
  - Memoryless Systems: a system is called *memoryless* if its input/output relationship is given by $\mathbf{y}[n] = g(\mathbf{x}[n])$.
  - LTI Systems: a linear time-invariant system is described in the recorded lectures. The input/output relationship is given by the *discrete-time convolution* of the input signal with the *impulse response* $h[n]$ of the system:

    $$\mathbf{y}[n, \zeta] = \sum_{k=-\infty}^{\infty} \mathbf{x}[k, \zeta] h[n-k] = \sum_{k=-\infty}^{\infty} \mathbf{x}[n-k, \zeta] h[k]$$
  - In this class, all LTI systems will be described by linear constant-coefficient difference equations *with all zero initial conditions*

LTI System with WSS Input

$$\mathbf{x}[n] \longrightarrow \boxed{h[n]} \longrightarrow \mathbf{y}[n]$$

$$\mu_{\mathbf{y}} = \mu_{\mathbf{x}} \sum_{n=-\infty}^{\infty} h[n]$$

$$R_{\mathbf{xy}}[m] = \sum_{k=-\infty}^{\infty} R_{\mathbf{xx}}[m+k]h^*[k]$$

$$R_{\mathbf{yy}}[m] = \sum_{k=-\infty}^{\infty} R_{\mathbf{xy}}[m-k]h[k]$$

Special Case: $\mathbf{x}[n]$ is WSS white random process:

$$\mu_{\mathbf{x}} = 0 \qquad R_{\mathbf{xx}}[m] = q\delta[m]$$

$$\mu_{\mathbf{y}} = 0$$

$$R_{\mathbf{xy}}[m] = qh^*[-m]$$

$$R_{\mathbf{yy}}[m] = q \underbrace{\sum_{k=-\infty}^{\infty} h^*[k-m]h[k]}_{\rho[m]}$$

The Power Spectrum

Definitions

- The *power spectrum* (or *spectral density*) of a WSS discrete-time random process $\mathbf{x}[n]$, real or complex, is the DTFT of its autocorrelation function $R_{\mathbf{xx}}[m] = E\{\mathbf{x}[n+m]\mathbf{x}[n]\}$:

$$\mathsf{S}_{\mathbf{xx}}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R_{\mathbf{xx}}[m]e^{-j\omega m}$$

- From the DTFT inversion formula

$$R_{\mathbf{xx}}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathsf{S}_{\mathbf{xx}}(e^{j\omega})e^{j\omega m}\, d\omega$$

- The *cross power spectrum* of two random processes $\mathbf{x}[n]$ and $\mathbf{y}[n]$ is the Fourier transform of their cross correlation $R_{\mathbf{xy}}[m] = E\{\mathbf{x}[n+m]\mathbf{y}^*[n]\}$:

$$\mathsf{S}_{\mathbf{xy}}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R_{\mathbf{xy}}[m]e^{-j\omega m}$$

- From the DTFT inversion formula

$$R_{\mathbf{xy}}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathsf{S}_{\mathbf{xy}}(e^{j\omega})e^{j\omega m}\, d\omega$$

Properties

1. $\mathsf{S}_{\mathbf{xx}}(e^{j\omega})$ is a real-valued function of ω.
2. If $\mathbf{x}[n]$ is real, then $\mathsf{S}_{\mathbf{xx}}(e^{j\omega})$ is real and even.
3. $\mathsf{S}_{\mathbf{xx}}(e^{j\omega}) \geq 0$ for all ω.
4. $\mathsf{S}_{\mathbf{xy}}(e^{j\omega})$ is, in general, complex valued, even when both processes $\mathbf{x}[n]$ and $\mathbf{y}[n]$ are real-valued processes.
5. $\mathsf{S}_{\mathbf{xy}}(e^{j\omega}) = \mathsf{S}^*_{\mathbf{yx}}(e^{j\omega})$

The DTFT Version of TABLE 9-1

$R[m] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathsf{S}(e^{j\omega})e^{j\omega k}\,d\omega$	$\leftrightarrow$	$\mathsf{S}(e^{j\omega}) = \displaystyle\sum_{m=-\infty}^{\infty} R[m]e^{-j\omega m}$
$\delta[m]$	$\leftrightarrow$	1
1	$\leftrightarrow$	$2\pi \displaystyle\sum_{\ell=-\infty}^{\infty} \delta(\omega - 2\pi\ell)$
$e^{j\beta m}$	$\leftrightarrow$	$2\pi \displaystyle\sum_{\ell=-\infty}^{\infty} \delta(\omega - \beta - 2\pi\ell)$
$\cos(\beta m)$	$\leftrightarrow$	$\pi \displaystyle\sum_{\ell=-\infty}^{\infty} \delta(\omega - \beta - 2\pi\ell) + \delta(\omega + \beta - 2\pi\ell)$
$\rho^{\lvert m\rvert}$	$\leftrightarrow$	$\dfrac{\rho e^{j\omega}}{1-\rho e^{j\omega}} + \dfrac{1}{1-\rho e^{-j\omega}} = \dfrac{1}{1-\rho e^{j\omega}} + \dfrac{\rho e^{-j\omega}}{1-\rho e^{-j\omega}}$
$\rho^{\lvert m\rvert}\cos(\beta m)$	$\leftrightarrow$	$\dfrac{1}{2}\left[\dfrac{\rho e^{j(\omega-\beta)}}{1-\rho e^{j(\omega-\beta)}} + \dfrac{1}{1-\rho e^{-j(\omega-\beta)}} + \dfrac{\rho e^{j(\omega+\beta)}}{1-\rho e^{j(\omega+\beta)}} + \dfrac{1}{1-\rho e^{-j(\omega+\beta)}}\right] = \dfrac{1}{2}\left[\dfrac{1}{1-\rho e^{j(\omega-\beta)}} + \dfrac{\rho e^{-j(\omega-\beta)}}{1-\rho e^{-j(\omega-\beta)}} + \dfrac{1}{1-\rho e^{j(\omega+\beta)}} + \dfrac{\rho e^{-j(\omega+\beta)}}{1-\rho e^{-j(\omega+\beta)}}\right]$
$\begin{cases} 1 & -M \le m \le M \\ 0 & \text{otherwise} \end{cases}$	$\leftrightarrow$	$\dfrac{\sin\left(\omega\left(\dfrac{2M+1}{2}\right)\right)}{\sin\left(\dfrac{\omega}{2}\right)}$

$$\rho^{|m|}\sin(\beta|m|) \leftrightarrow \frac{1}{j2}\left[\frac{1}{1-\rho e^{-j(\omega-\beta)}} - \frac{\rho e^{j(\omega-\beta)}}{1-\rho e^{j(\omega-\beta)}} + \right.$$

$$\left. + \frac{\rho e^{j(\omega+\beta)}}{1-\rho e^{j(\omega+\beta)}} - \frac{1}{1-\rho e^{-j(\omega+\beta)}}\right]$$

$$= \frac{1}{j2}\left[\frac{\rho e^{-j(\omega-\beta)}}{1-\rho e^{-j(\omega-\beta)}} - \frac{1}{1-\rho e^{j(\omega-\beta)}} + \right.$$

$$\left. + \frac{1}{1-\rho e^{j(\omega+\beta)}} - \frac{\rho e^{-j(\omega-\beta)}}{1-\rho e^{-j(\omega+\beta)}}\right]$$

$$|m|\rho^{|m|} \leftrightarrow \frac{\rho e^{j\omega}}{(1-\rho e^{j\omega})^2} + \frac{\rho e^{-j\omega}}{(1-\rho e^{-j\omega})^2}$$

These are valid DTFT pairs, but the left-hand sides by themselves are not valid auto-correlation functions. (See Property 6.)

LTI System with WSS Input

$$\mathbf{x}[n] \longrightarrow \boxed{h[n]} \longrightarrow \mathbf{y}[n]$$

$$\mu_{\mathbf{y}} = \mu_{\mathbf{x}} \sum_{n-\infty}^{\infty} h[n]$$

$$\mu_{\mathbf{y}} = \mu_{\mathbf{x}} \mathbf{H}(1)$$

$$R_{\mathbf{xy}}[m] = \sum_{n=-\infty}^{\infty} R_{\mathbf{xx}}[n+m]h^*[m]$$

$$\mathbf{S}_{\mathbf{xy}}(e^{j\omega}) = \mathbf{S}_{\mathbf{xx}}(e^{j\omega})\mathbf{H}^*(e^{j\omega})$$

$$R_{\mathbf{yy}}[m] = \sum_{n-\infty}^{\infty} R_{\mathbf{xy}}[m-n]h[n]$$

$$\mathbf{S}_{\mathbf{yy}}(e^{j\omega}) = \mathbf{S}_{\mathbf{xy}}(e^{j\omega})\mathbf{H}(e^{j\omega}) = \mathbf{S}_{\mathbf{xx}}(e^{j\omega})|\mathbf{H}(e^{j\omega})|^2$$

Example 9-27 Revisited

$$\mathbf{x}[n] \longrightarrow \boxed{h[n]} \longrightarrow \mathbf{y}[n]$$

$$\text{(a)} \quad \mathbf{y}[n] - b\mathbf{y}[n-1] = \mathbf{x}[n] \quad \text{all } n \quad \mu_{\mathbf{x}} = 0 \quad R_{\mathbf{xx}}[m] = q\delta[m]$$

$$\mathbf{H}(z) = \frac{1}{1 - bz^{-1}}$$

$$\mathbf{H}(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}}$$

$$|\mathbf{H}(e^{j\omega})|^2 = \frac{1}{(1 - be^{-j\omega})(1 - be^{j\omega})}$$

$$= \frac{1}{1 - b^2}\left[\frac{be^{-j\omega}}{1 - be^{-j\omega}} + \frac{1}{1 - be^{j\omega}}\right]$$

$$\mathbf{S}_{\mathbf{yy}}(e^{j\omega}) = \mathbf{S}_{\mathbf{xx}}(e^{j\omega})|\mathbf{H}(e^{j\omega})|^2$$

$$= \frac{q}{1 - b^2}\left[\frac{be^{-j\omega}}{1 - be^{-j\omega}} + \frac{1}{1 - be^{j\omega}}\right]$$

$$R_{\mathbf{yy}}[m] = \frac{q}{1 - b^2} b^{|m|}$$

$$\mathbf{H}(z) = \frac{1}{1 - bz^{-1}}$$

$$h[n] = b^n \, U[n]$$

$$R_{\mathbf{xy}}[m] = \begin{cases} qb^m & m \le 0 \\ 0 & m > 0 \end{cases}$$

$$R_{\mathbf{yy}}[m] = \begin{cases} \dfrac{q}{1 - b^2} b^{-m} & m < 0 \\ \dfrac{q}{1 - b^2} b^m & m \ge 0 \end{cases}$$

$$= \frac{q}{1 - b^2} b^{|m|}$$

$$\rho[m] = \begin{cases} \displaystyle\sum_{n=0}^{\infty} b^{n-m} b^n & m < 0 \\ \displaystyle\sum_{n=m}^{\infty} b^{n-m} b^n & m \ge 0 \end{cases} = \begin{cases} \dfrac{b^{-m}}{1 - b^2} & m < 0 \\ \dfrac{b^m}{1 - b^2} & m \ge 0 \end{cases} = \frac{b^{|m|}}{1 - b^2}$$

$$R_{\mathbf{yy}}[m] = q\rho[m] = \frac{q}{1 - b^2} b^{|m|}$$

$$E\{\mathbf{y}^2[n]\} = R_{\mathbf{yy}}[0] = \frac{q}{1 - b^2}$$

Example 9-27 Revisited

$$\mathbf{x}[n] \longrightarrow \boxed{h[n]} \longrightarrow \mathbf{y}[n]$$

(b) $\quad \mathbf{y}[n] - b\mathbf{y}[n-1] + c\mathbf{y}[n-2] = \mathbf{x}[n] \quad \text{all } n \quad \mu_{\mathbf{x}} = 0 \quad R_{\mathbf{xx}}[m] = q\delta[m]$

$$\mathbf{H}(z) = \frac{1}{1 - bz^{-1} + cz^{-2}}$$

$$\mathbf{H}(e^{j\omega}) = \frac{1}{1 - be^{-j\omega} + ce^{-j2\omega}}$$

$$|\mathbf{H}(e^{j\omega})|^2 = \frac{1}{(1 - be^{-j\omega} + ce^{-j2\omega})(1 - be^{j\omega} + ce^{j2\omega})}$$

$$\mathbf{S}_{\mathbf{yy}}(e^{j\omega}) = \mathbf{S}_{\mathbf{xx}}(e^{j\omega})|\mathbf{H}(e^{j\omega})|^2 = \frac{q}{(1 - be^{-j\omega} + ce^{-j2\omega})(1 - be^{j\omega} + ce^{j2\omega})}$$

$\underline{b^2 < 4c}$

$$R_{\mathbf{yy}}[m] = \frac{q}{(1+c)^2 - b^2}\, \rho^{|m|} \left[\frac{1+c}{1-c} \cos(\beta m) + \frac{b}{2\gamma} \sin(\beta |m|) \right]$$

$$\alpha = \frac{b}{2}, \quad \gamma^2 + \alpha^2 = c, \quad \rho^2 = \alpha^2 + \gamma^2 = c, \quad \beta = \operatorname{atan}\left(\frac{\gamma}{\alpha}\right)$$

$\underline{b^2 = 4c}$

$$R_{\mathbf{yy}}[m] = \frac{q}{(1-c)^2} \left[|m| + \frac{1+c}{1-c} \right] \alpha^{|m|} \qquad \alpha = \frac{b}{2}$$

$\underline{b^2 > 4c}$

$$R_{\mathbf{yy}}[m] = \frac{q}{2\gamma(1-c)} \left[\frac{\alpha+\gamma}{1-(\alpha+\gamma)^2} (\alpha+\gamma)^{|m|} - \frac{\alpha-\gamma}{1-(\alpha-\gamma)^2} (\alpha-\gamma)^{|m|} \right]$$

$$\alpha = \frac{b}{2}, \qquad \alpha^2 - \gamma^2 = c$$

In all cases, $E\{\mathbf{y}^2[n]\} = \dfrac{q(1+c)}{(1-c)[(1+c)^2 - b^2]}$.