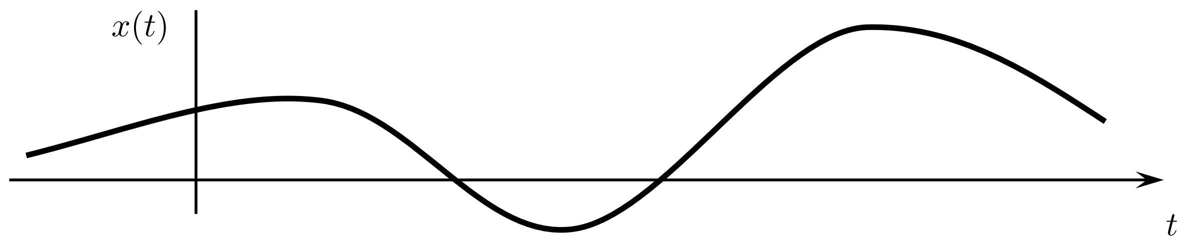
A Review of Discrete Time Linear Systems

- Description of Discrete Time Signals
- A Description of Discrete Time Systems
- Frequency Domain Analysis

Discrete Time Signals



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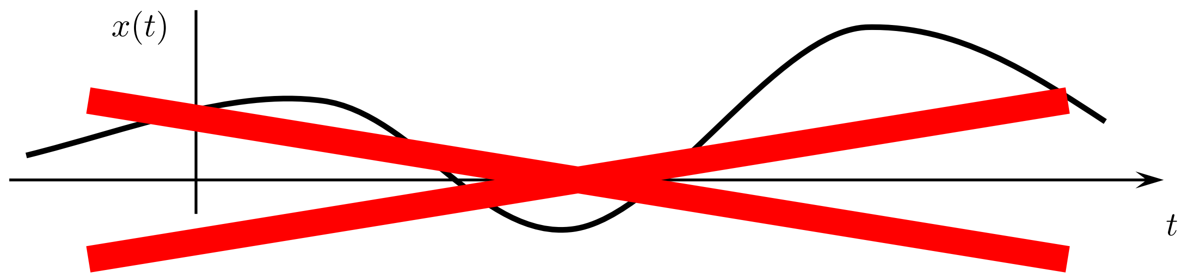
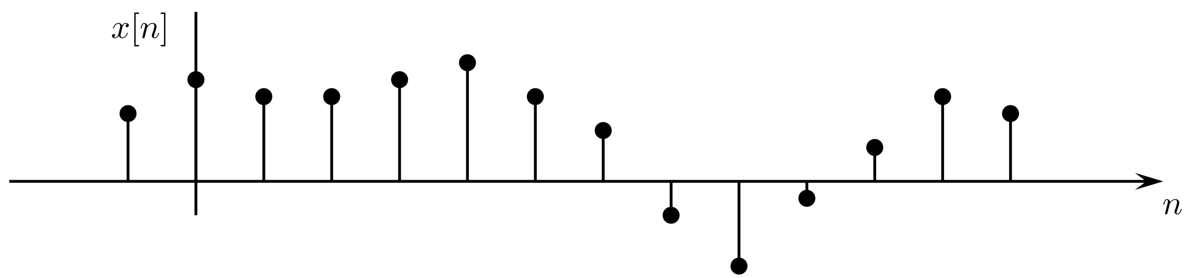
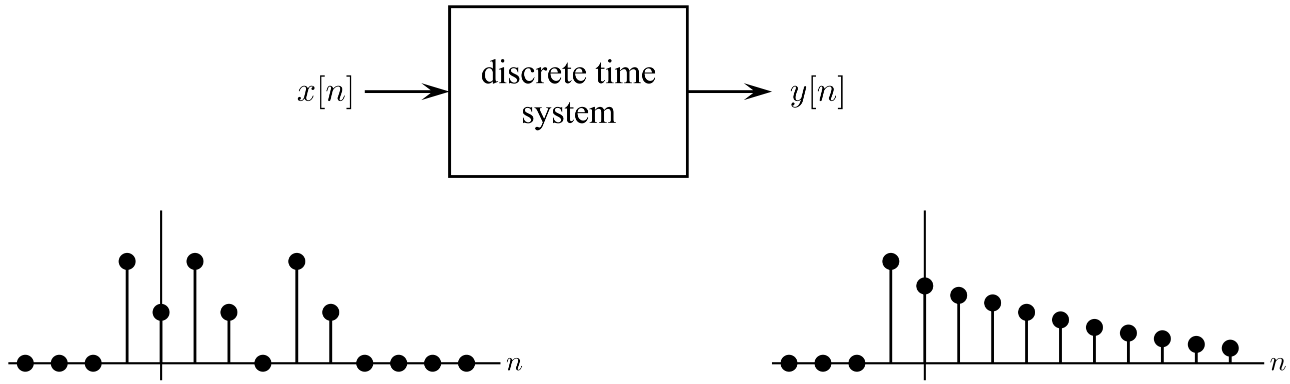


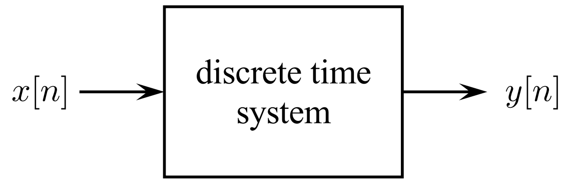


Discrete Time System



A discrete time system produces an output discrete time signal from an input discrete time signal.

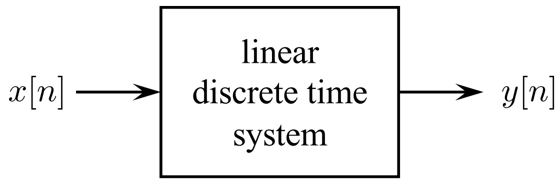
Linear Discrete Time System



$$y[n] + a_1 y[n-1] = b_0 x[n] + b_1 x[n-1]$$

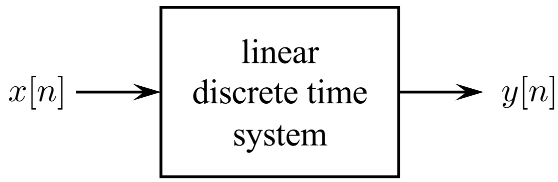
Linear **C**onstant **C**oefficient **D**ifference **E**quation
LCCDE

Recursions in the form of an LCCDE are the most popular for linear discrete time systems because only multiplications by fixed constants and additions are required.



The linear systems of interest to us have an input/output relationship defined by LCCDE

$$\text{e.g., } y[n] - ay[n-1] = x[n]$$



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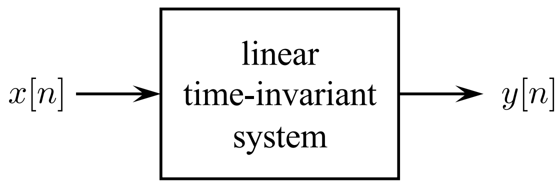
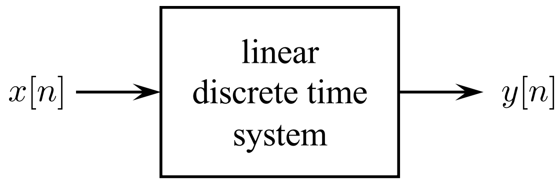


Systems defined by an LCCDE are also time-invariant



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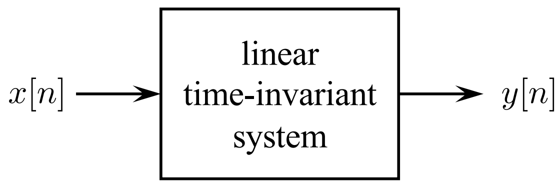

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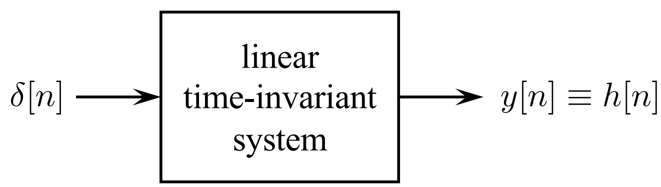
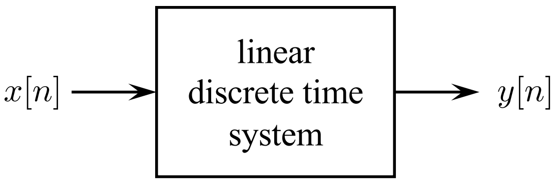


The input/output relationship of a linear time-invariant system may also be described using the impulse response and convolution.



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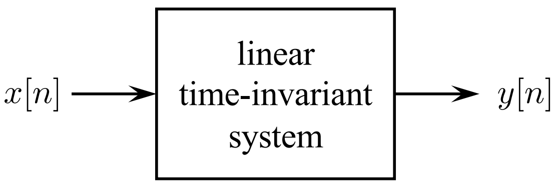

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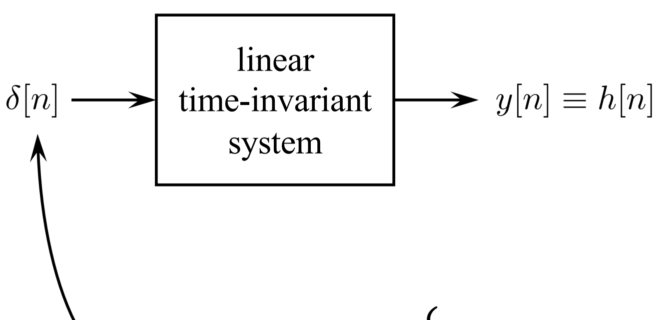

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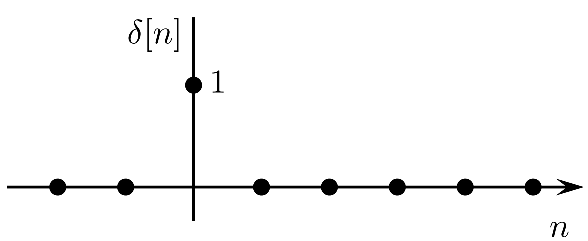
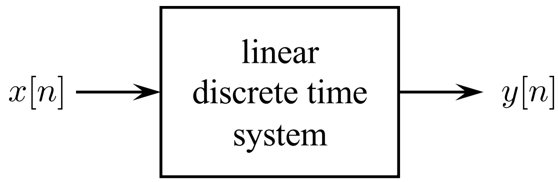
$$\text{Kronecker “delta”: } \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$





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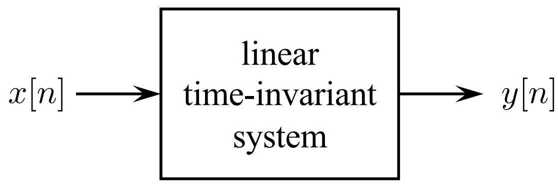

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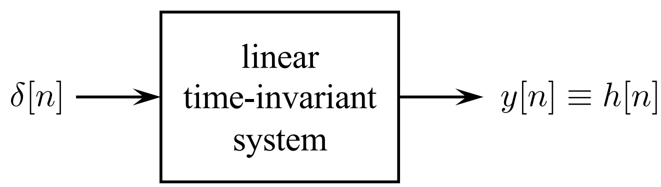

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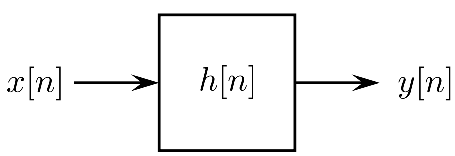

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Convolution Example

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$x[n] = U[n] \qquad h[n] = a^n U[n]$$

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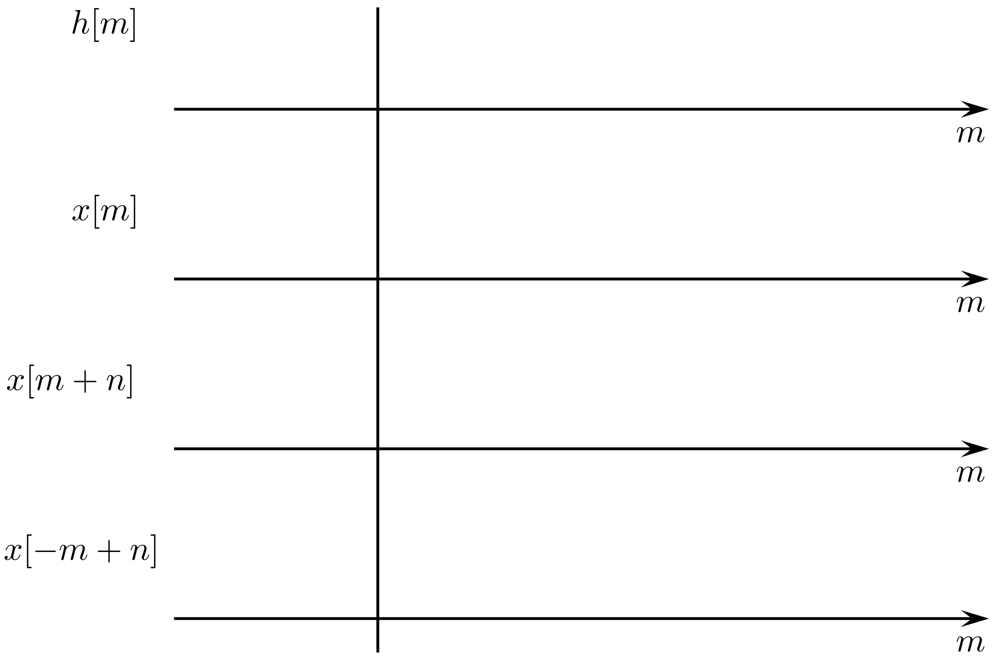

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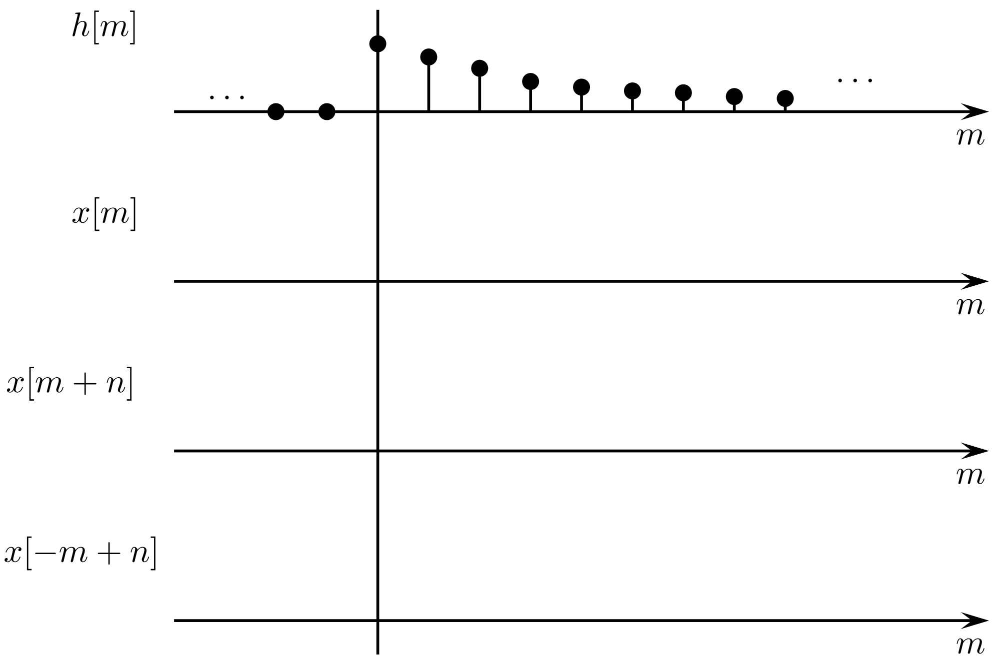

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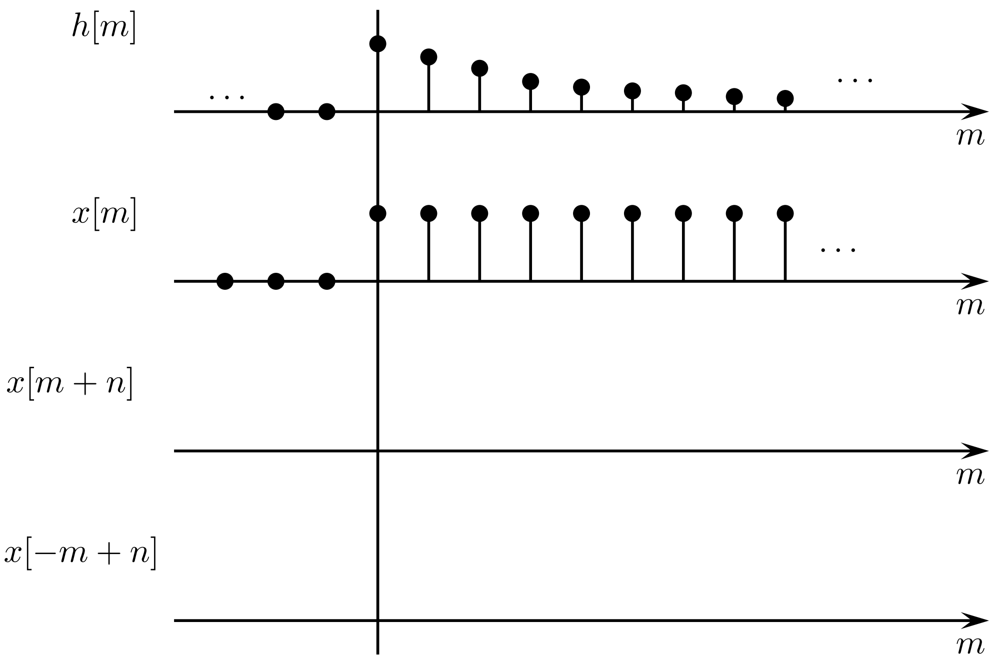

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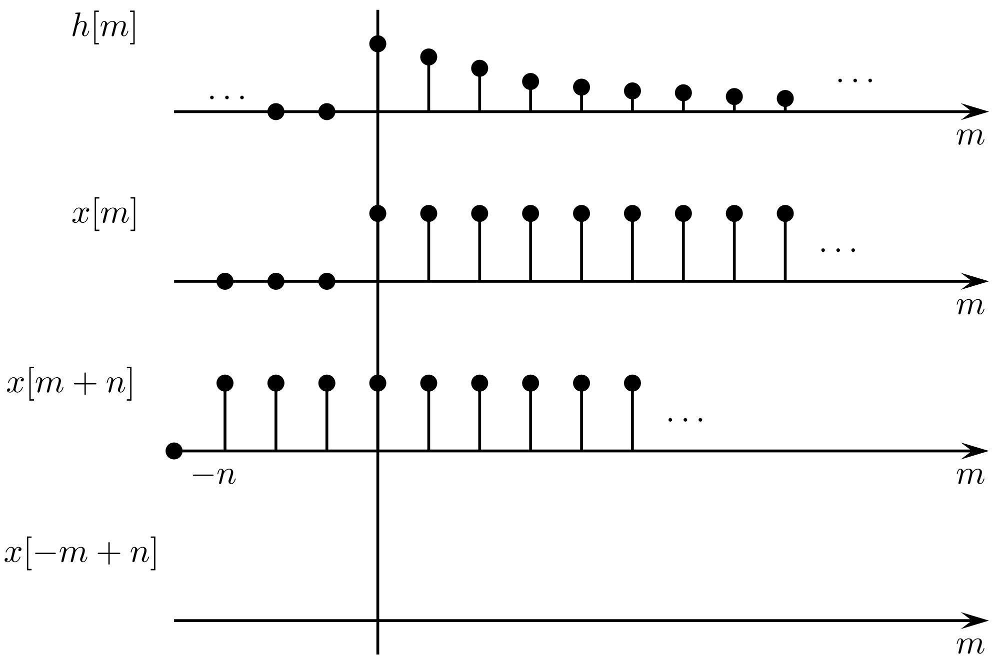

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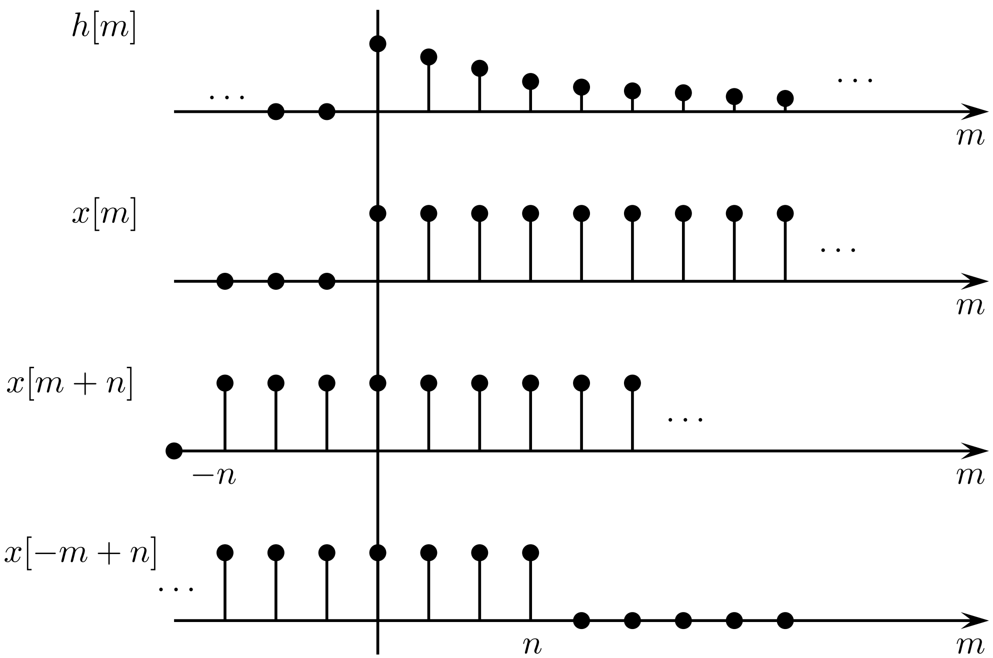

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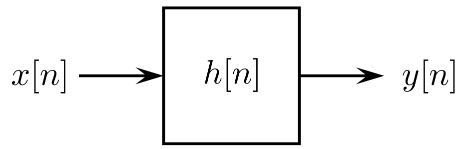

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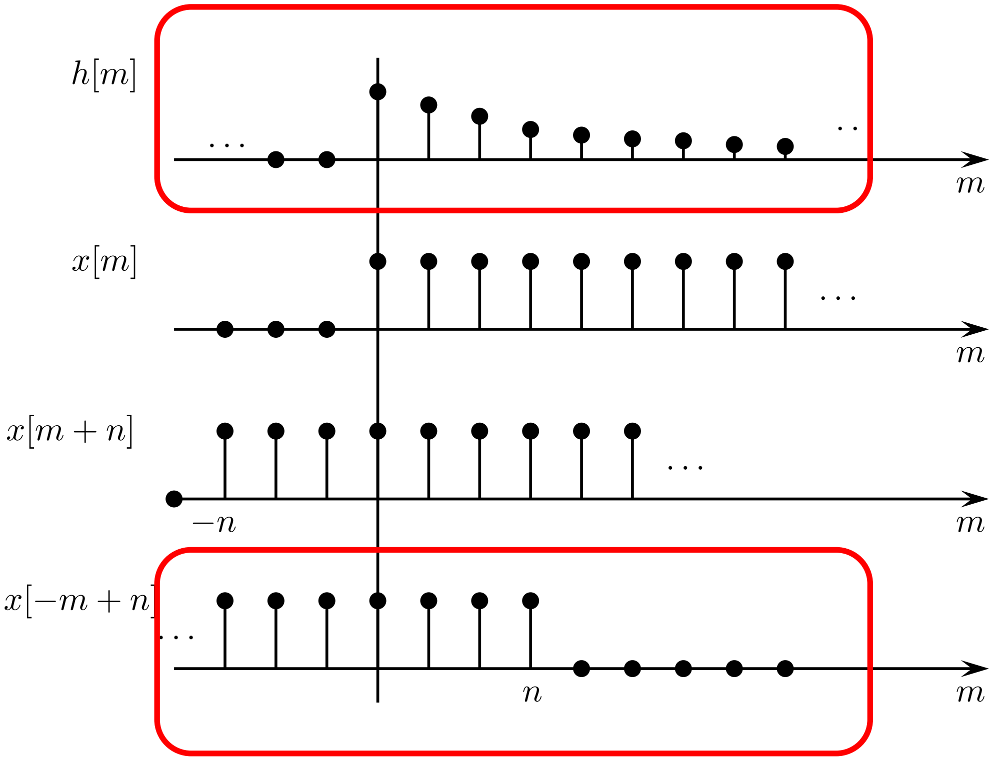

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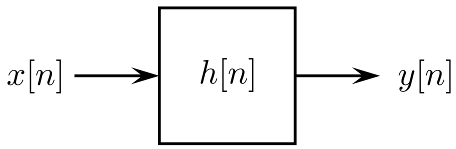

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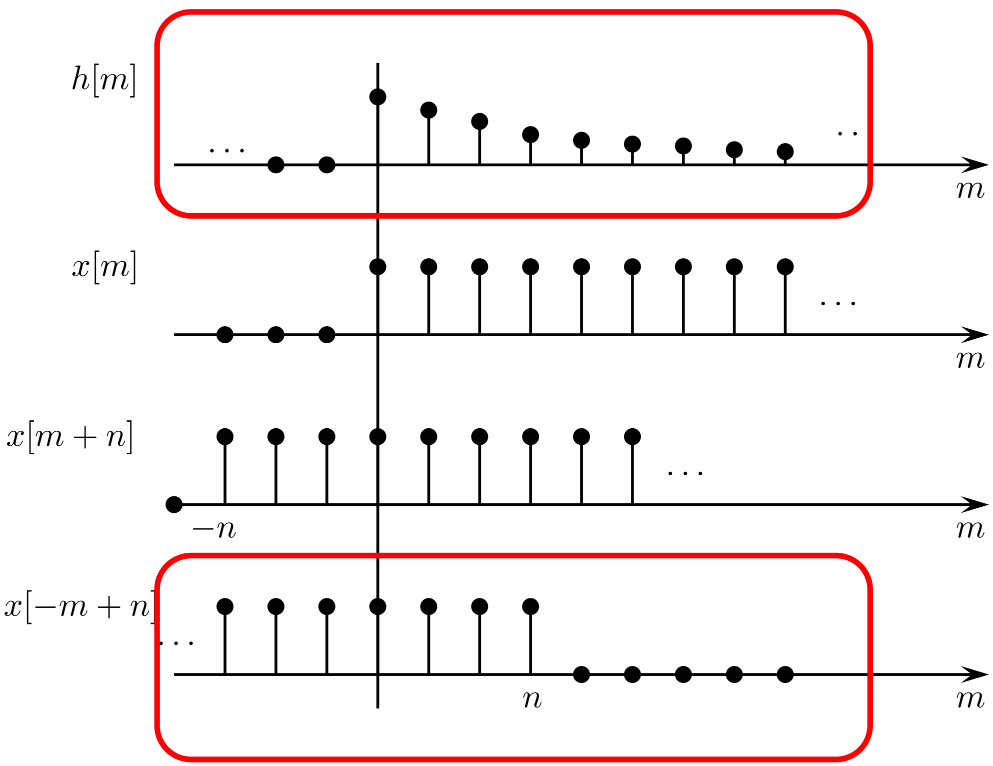

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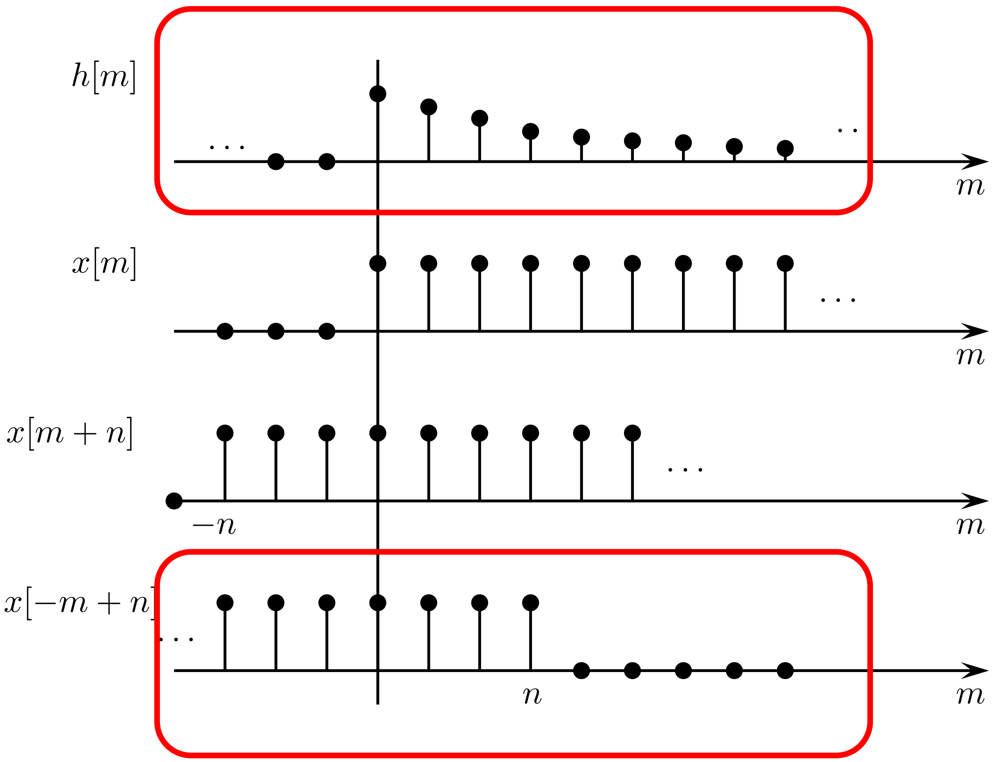

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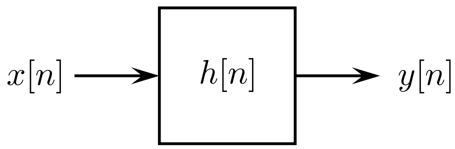

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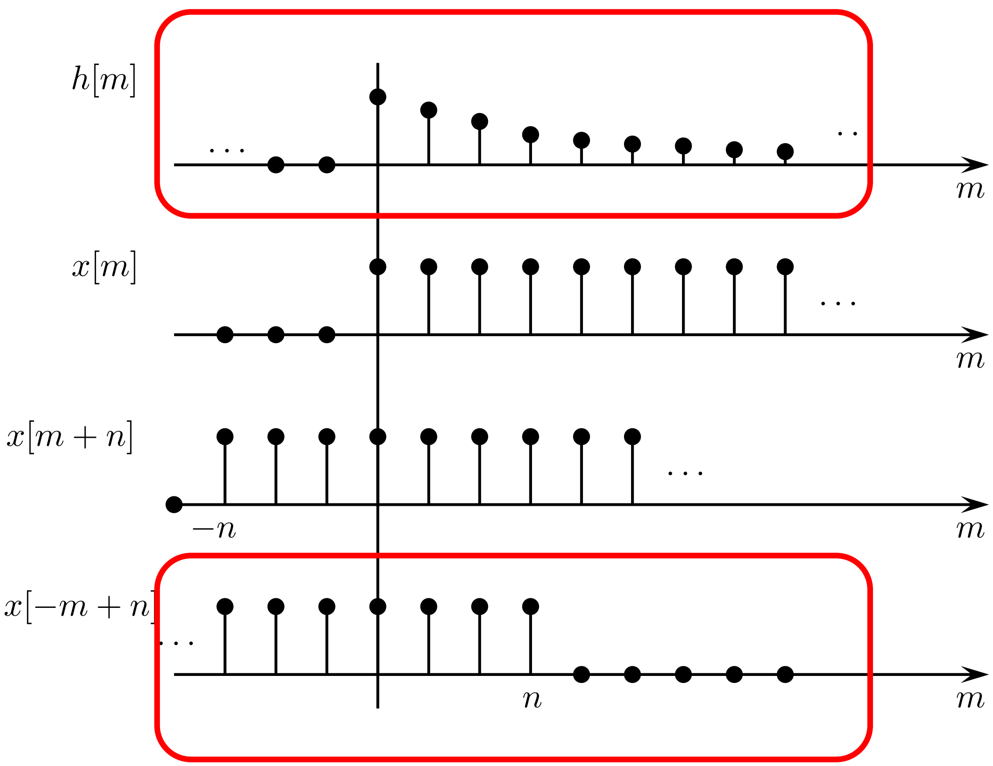

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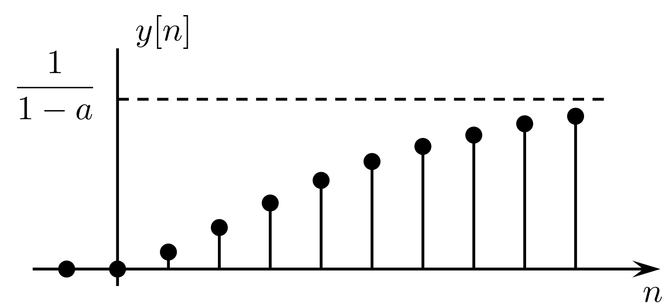

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$$\left.\begin{aligned}
\sum_{n=0}^{\infty} r^n &= \frac{1}{1-r} \\
\sum_{n=0}^{N-1} r^n &= \frac{1-r^N}{1-r} \\
\sum_{n=1}^{\infty} r^n &= \frac{r}{1-r} \\
\sum_{n=0}^{\infty} r^{n-1} &= \frac{1}{r(1-r)} \\
\sum_{n=0}^{\infty} nr^{n-1} &= \frac{1}{(1-r)^2} \\
\sum_{n=0}^{\infty} nr^n &= \frac{r}{(1-r)^2}
\end{aligned}\right\}$$

geometric sums and their variations are helpful in this discrete time business

$|r| < 1$

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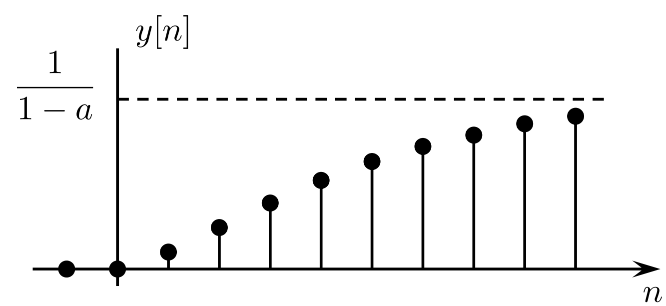

Frequency Domain Analysis

z Transform

$$\mathbf{X}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{j2\pi} \oint \mathbf{X}(z) z^{n-1} dz$$

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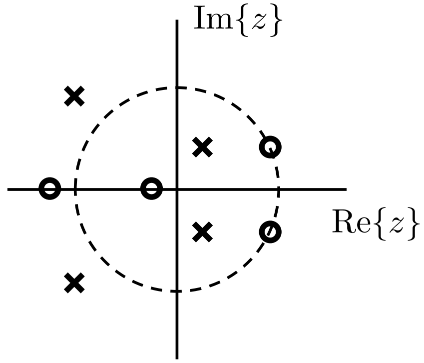

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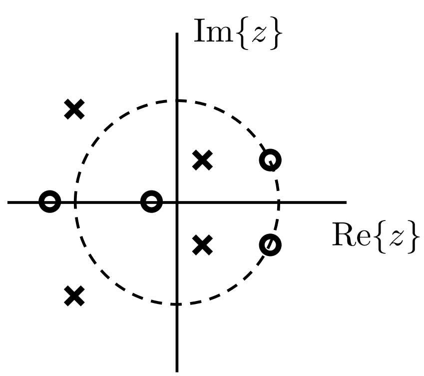

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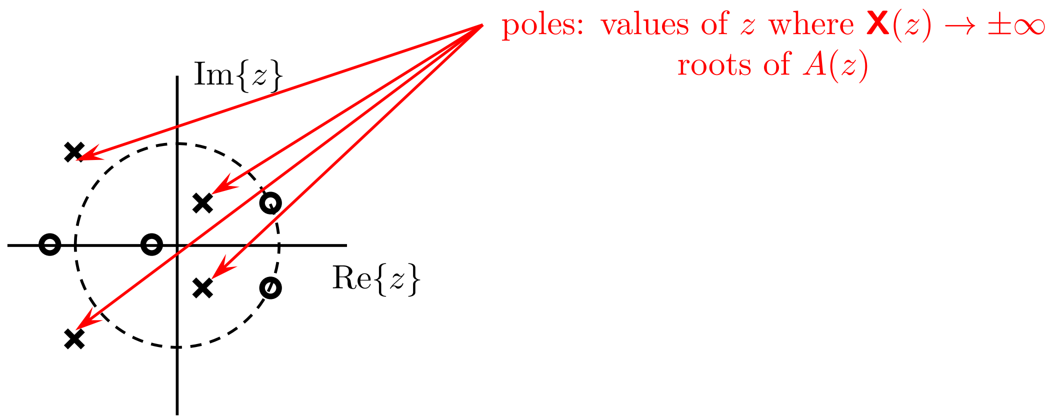

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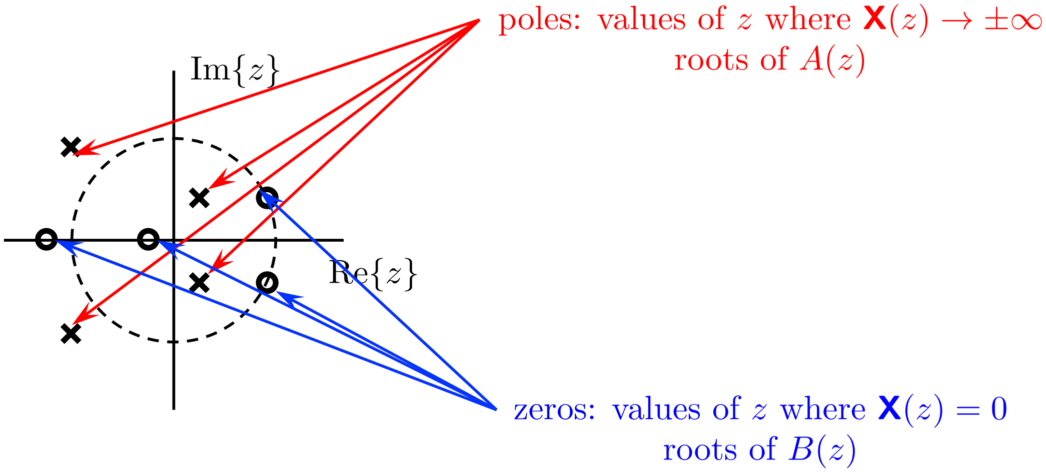

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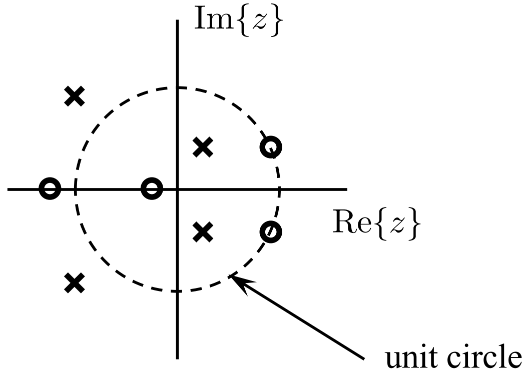

The z transform sum

$$\mathbf{X}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

only converges for certain values of z. These values are called *the region of convergence* (ROC).

- The ROC cannot contain any poles.
- The ROC for the z transform of a stable (bounded) time-domain signal contains the countour $e^{j\omega}$. This contour is called the *unit circle*.

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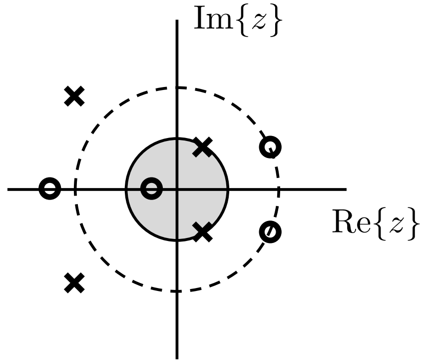
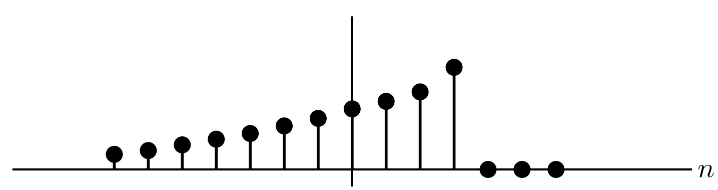


ROC for left-sided signals



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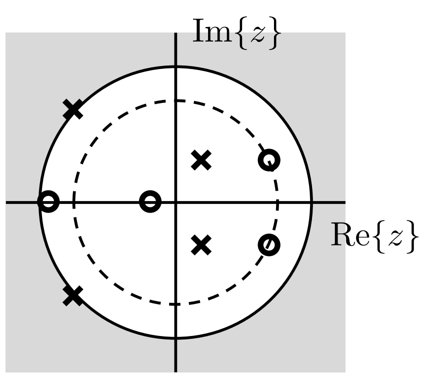
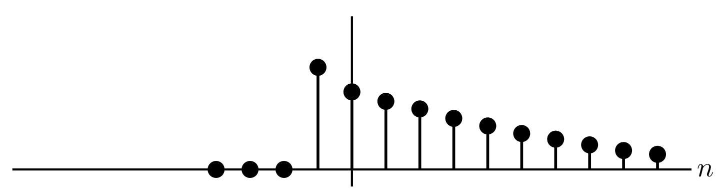


ROC for right-sided signals



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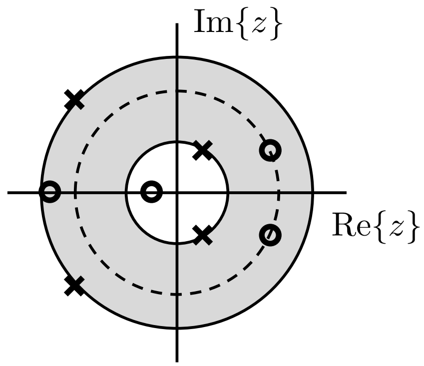
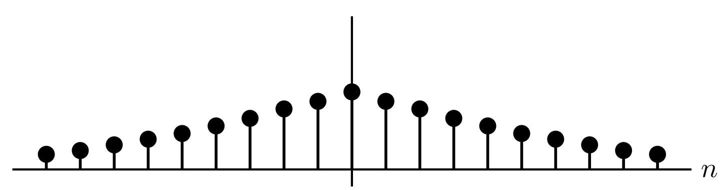


ROC for two-sided signals



z Transform

$$\mathbf{X}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{j2\pi} \oint \mathbf{X}(z) z^{n-1} dz$$

Comments:

- z is a complex variable and $\mathbf{X}(z)$ is a complex-valued function of the complex variable z.
- LCCDEs always produce a ratio of polynomials in z^{-1}:

$$\mathbf{X}(z) = \frac{B(z)}{A(z)} = \frac{b_q z^{-q} + \cdots + b_1 z^{-1} + b_0}{a_p z^{-p} + \cdots + a_1 z^{-1} + a_0}$$

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Signal	Transform	ROC
$\delta[n]$	1	all z
$U[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
$a^n U[n]$	$\frac{1}{1 - a z^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$n a^n U[n]$	$\frac{a z^{-1}}{(1 - a z^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$\delta[n-m]$	z^{-m}	all z[a]
$\cos(\omega_0 n)\, U[n]$	$\frac{1 - \cos(\omega_0) z^{-1}}{1 - [2\cos(\omega_0)] z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)\, U[n]$	$\frac{\sin(\omega_0) z^{-1}}{1 - [2\cos(\omega_0)] z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)\, U[n]$	$\frac{1 - r\cos(\omega_0) z^{-1}}{1 - [2r\cos(\omega_0)] z^{-1} + z^{-2}}$	$\lvert z\rvert > r$
$r^n \sin(\omega_0 n)\, U[n]$	$\frac{r\sin(\omega_0) z^{-1}}{1 - [2r\cos(\omega_0)] z^{-1} + z^{-2}}$	$\lvert z\rvert > r$

[a] except 0 if $m > 0$ or ∞ if $m > 0$

Property	Signal	z-transform	ROC
	$x[n]$	$\mathbf{X}(z)$	R_x
	$y[n]$	$\mathbf{Y}(z)$	R_y
Linearity	$ax[n] + by[n]$	$a\mathbf{X}(z) + b\mathbf{Y}(z)$	at least $R_x \cap R_y$
Time Shifting	$x[n - n_0]$	$z^{-n_0}\mathbf{X}(z)$	R_x
Time Scaling (Upsampling)	$\begin{cases} x(n/K) & n \text{ is a multiple of K} \\ 0 & \text{otherwise} \end{cases}$	$\mathbf{X}(z^K)$	$R_x^{1/K}$
Conjugation	$x^*(n)$	$\mathbf{X}^*(z^*)$	R_x
Convolution	$x[n] * y[n]$	$\mathbf{X}(z)\mathbf{Y}(z)$	at least $R_x \cap R_y$
First Difference	$x[n] - x[n-1]$	$(1 - z^{-1})\mathbf{X}(z)$	at least $R_x \cap \left\{\lvert z\rvert > 0\right\}$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}\mathbf{X}(z)$	at least $R_x \cap \left\{\lvert z\rvert > 1\right\}$
Initial Value Theorem	$x[0]$	$\lim_{z\to\infty} \mathbf{X}(z)$	
Final Value Theorem	$\lim_{N\to\infty} x[n]$	$\lim_{z\to 1}(1 - z^{-1})\mathbf{X}(z)$	poles of $(1 - z^{-1})\mathbf{X}(z)$ inside unit circle.

Frequency Domain Analysis

z Transform

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Discrete Time Fourier Transform (DTFT)

$$\mathbf{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{X}(e^{j\omega}) e^{j\omega} d\omega$$

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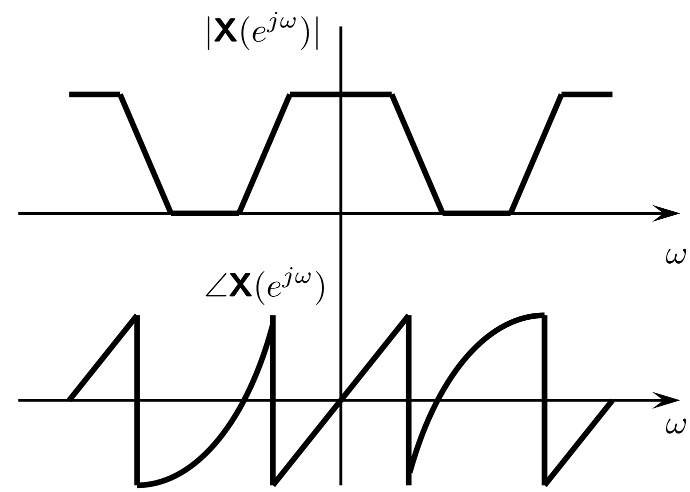

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Comments:

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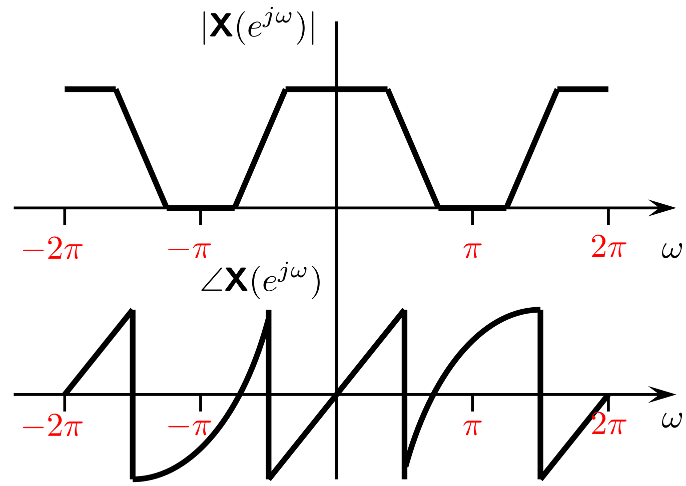
- $\mathbf{X}(e^{j\omega})$ is *periodic* in ω with period 2π.



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$$\mathbf{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{X}(e^{j\omega}) e^{j\omega} d\omega$$

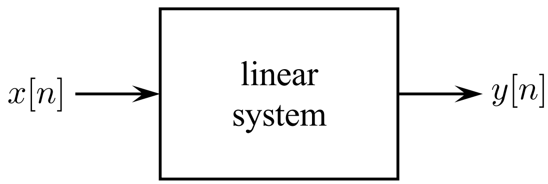
Comments:

- w is a real variable and $\mathbf{X}(e^{j\omega})$ is a complex-valued function of the real variable ω.
- $\mathbf{X}(e^{j\omega})$ is *periodic* in ω with period 2π.
- Because ω is a real variable, the integral that defines the inverse transform is the familiar integral introduced in your first and second calculus courses.
  - Integral wizards compute the inverse DTFT transform from the definition.
  - Everyone else uses uses tables.

Signal	DTFT
$x[n]$	$\mathbf{X}\left(e^{j\omega}\right)$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$U[n]$	$\dfrac{1}{1-e^{-j\omega}}+\pi\sum_{\ell=-\infty}^{\infty}\delta\left(\omega-2\pi\ell\right)$
$a^nU[n], \lvert a\rvert<1$	$\dfrac{1}{1-ae^{-j\omega}}$
$\sum_{k=k_0}^{k_0+N-1}c_ke^{jk(2\pi/N)n}$	$2\pi\sum_{k=-\infty}^{\infty}c_k\delta\left(\omega-\dfrac{2\pi k}{N}\right)$
$\sum_{k=-\infty}^{\infty}\delta[n-kN]$	$\dfrac{2\pi}{N}\sum_{\ell=-\infty}^{\infty}\delta\left(\omega-\dfrac{2\pi\ell}{N}\right)$
1	$2\pi\sum_{\ell=-\infty}^{\infty}\delta\left(\omega-2\pi\ell\right)$
$\begin{cases}1 & \lvert n\rvert\le N\\ 0 & \lvert n\rvert>N\end{cases}$	$\dfrac{\sin\left(\omega\left[N+\frac{1}{2}\right]\right)}{\sin\left(\frac{\omega}{2}\right)}$
$e^{j\omega_0 n}$	$2\pi\sum_{\ell=-\infty}^{\infty}\delta\left(\omega-\omega_0-2\pi\ell\right)$
$\cos\left(\omega_0 n\right)$	$\pi\sum_{\ell=-\infty}^{\infty}\left\{\delta(\omega-\omega_0-2\pi\ell)+\delta(\omega+\omega_0-2\pi\ell)\right\}$
$\sin\left(\omega_0 n\right)$	$\dfrac{\pi}{j}\sum_{\ell=-\infty}^{\infty}\left\{\delta(\omega-\omega_0-2\pi\ell)-\delta(\omega+\omega_0-2\pi\ell)\right\}$

Property	Signal	DTFT
	$x[n]$	$\mathbf{X}\left(e^{j\omega}\right)$
	$y[n]$	$\mathbf{Y}\left(e^{j\omega}\right)$
Linearity	$ax[n]+by[n]$	$a\mathbf{X}\left(e^{j\omega}\right)+b\mathbf{Y}\left(e^{j\omega}\right)$
Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}\mathbf{X}\left(e^{j\omega}\right)$
Time Scaling (Upsampling)	$\begin{cases} x[n/K] & n \text{ is a multiple of K} \\ 0 & \text{otherwise} \end{cases}$	$\mathbf{X}\left(e^{jK\omega}\right)$
Conjugation	$x^*[n]$	$\mathbf{X}^*\left(e^{-j\omega}\right)$
Convolution	$x[n]*y[n]$	$\mathbf{X}\left(e^{j\omega}\right)\mathbf{Y}\left(e^{j\omega}\right)$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\mathbf{X}\left(e^{j\omega}\right)*\mathbf{Y}\left(e^{j\omega}\right)$
First Difference	$x[n]-x[n-1]$	$\left(1-e^{-j\omega}\right)\mathbf{X}\left(e^{j\omega}\right)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}\mathbf{X}\left(e^{j\omega}\right)$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi}\left\lvert\mathbf{X}\left(e^{j\omega}\right)\right\rvert^2 d\omega$	

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



The linear systems of interest to us have an input/output relationship defined by LCCDE

e.g., $y[n] - ay[n-1] = x[n]$

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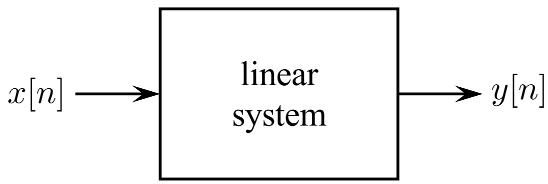

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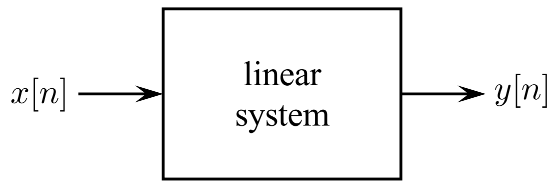

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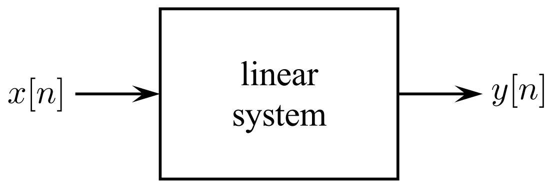

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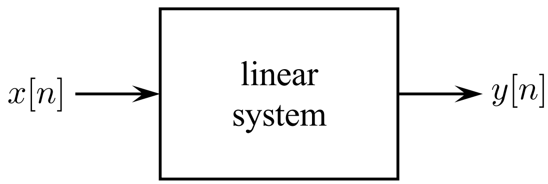

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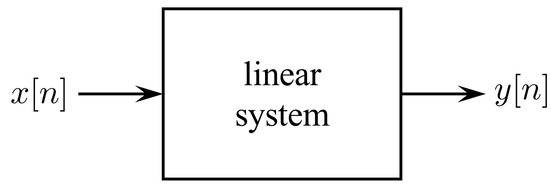

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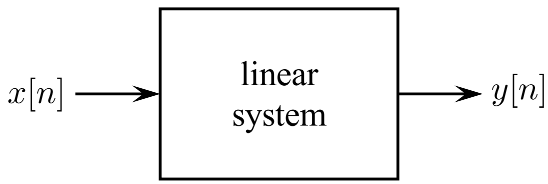

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Transfer function $\mathbf{H}(z)$

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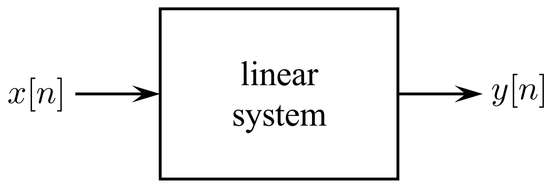

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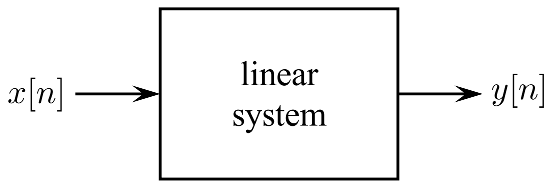
$$\mathbf{Y}(z) = \mathbf{H}(z)\mathbf{X}(z) \quad \Rightarrow \quad y[n] = \sum_{m=\infty}^{\infty} h[m]x[n-m]$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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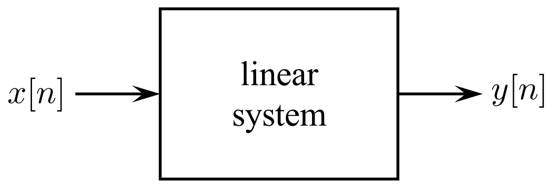

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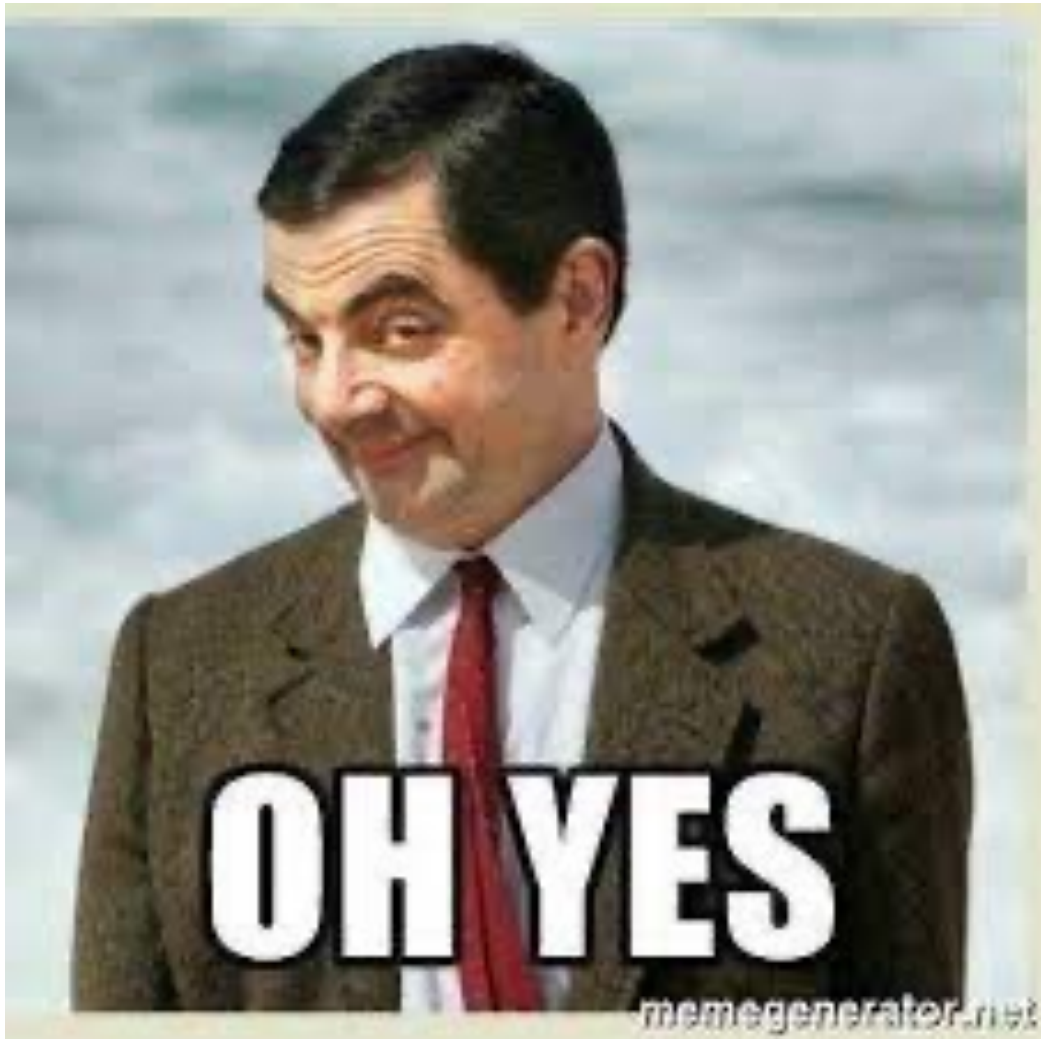
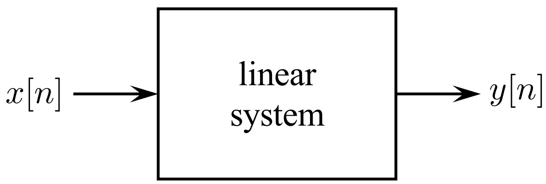
Is this really the impulse response?


OH YES
memegenerator.net

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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$$(1 - az^{-1})\mathbf{Y}(z) = \mathbf{X}(z)$$

$$\mathbf{Y}(z) = \frac{1}{1 - az^{-1}}\mathbf{X}(z)$$

$$x[n] = \delta[n] \quad \Rightarrow \quad \mathbf{X}(z) = 1$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain

$$x[n] \longrightarrow \boxed{\text{linear system}} \longrightarrow y[n]$$

The linear systems of interest to us have an input/output relationship defined by LCCDE

$$\text{e.g., } y[n] - ay[n-1] = x[n]$$

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$$\mathbf{Y}(z) - a\left(z^{-1}\mathbf{Y}(z) + \cancelto{0}{y[-1]}\right) = \mathbf{X}(z)$$

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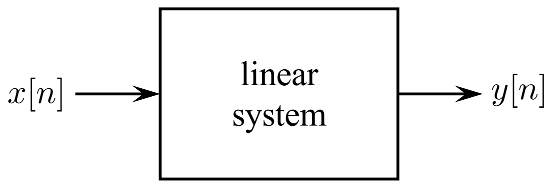

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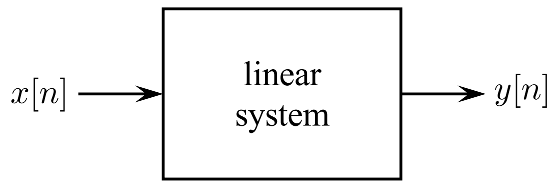

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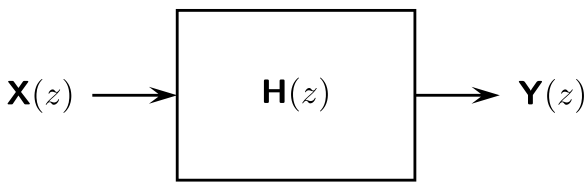
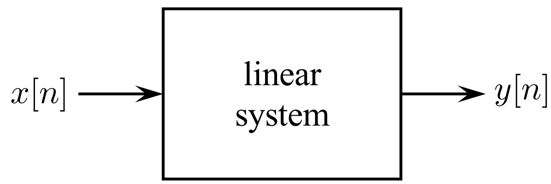


z domain transfer function

$$\mathbf{Y}(z) = \mathbf{H}(z)\mathbf{X}(z)$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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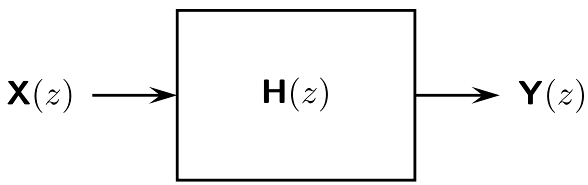

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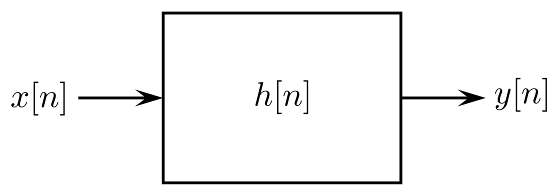

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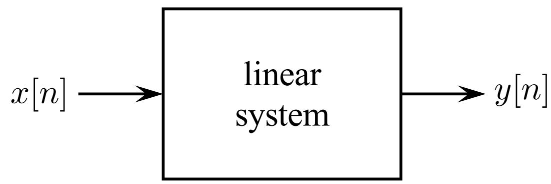
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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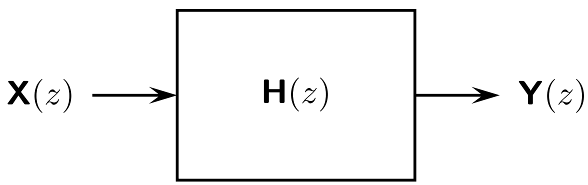

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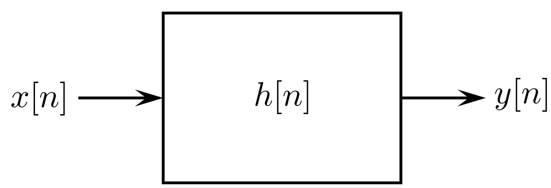

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To find the impulse response of an LTI system described by an LCCDE

1. Solve the LCCDE using the z transform with all-zeros initial conditions.
2. Write the z domain solution as $\mathbf{Y}(z) = \mathbf{H}(z)\mathbf{X}(z)$.
3. Identify $\mathbf{H}(z)$ in the solution.
4. The impulse response $h[n]$ is the inverse Laplace transform of $\mathbf{H}(z)$.

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain + DTFT

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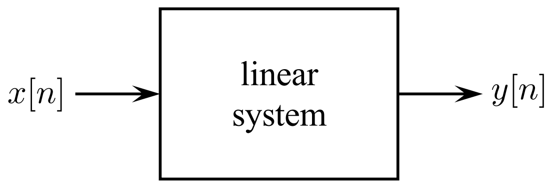

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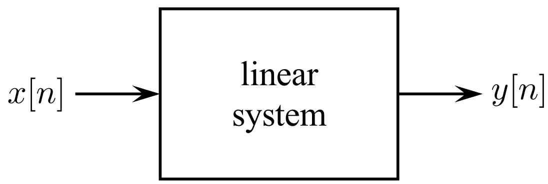

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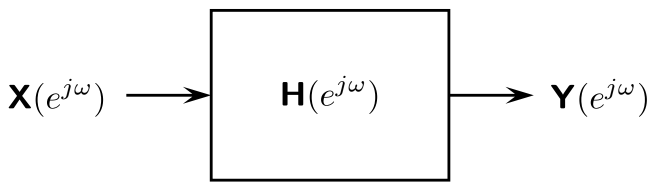

DTFT domain frequency response

$$\mathbf{Y}(e^{j\omega}) = \mathbf{H}(e^{j\omega})\mathbf{X}(e^{j\omega})$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain + DTFT

$$x[n] \longrightarrow \boxed{\begin{array}{c}\text{linear}\\ \text{system}\end{array}} \longrightarrow y[n]$$

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Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain + DTFT

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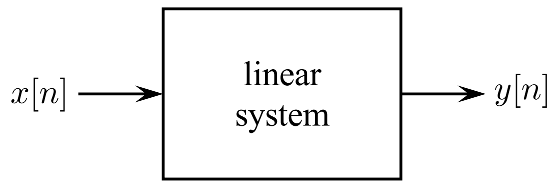
- After identifying the z domain transfer function $\mathbf{H}(z)$, the DTFT domain frequency response is

$$\mathbf{H}(e^{j\omega}) = \mathbf{H}(z)\Big|_{z=e^{j\omega}}$$

This works only as long as the ROC of $\mathbf{H}(z)$ contains the unit circle ($z = e^{j\omega}$).

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain + DTFT



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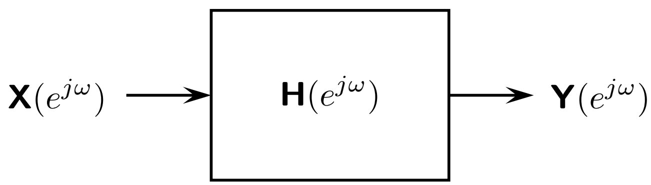

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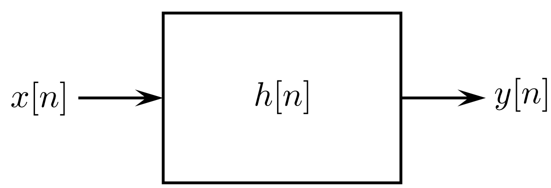

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- For stable systems, this is always true.