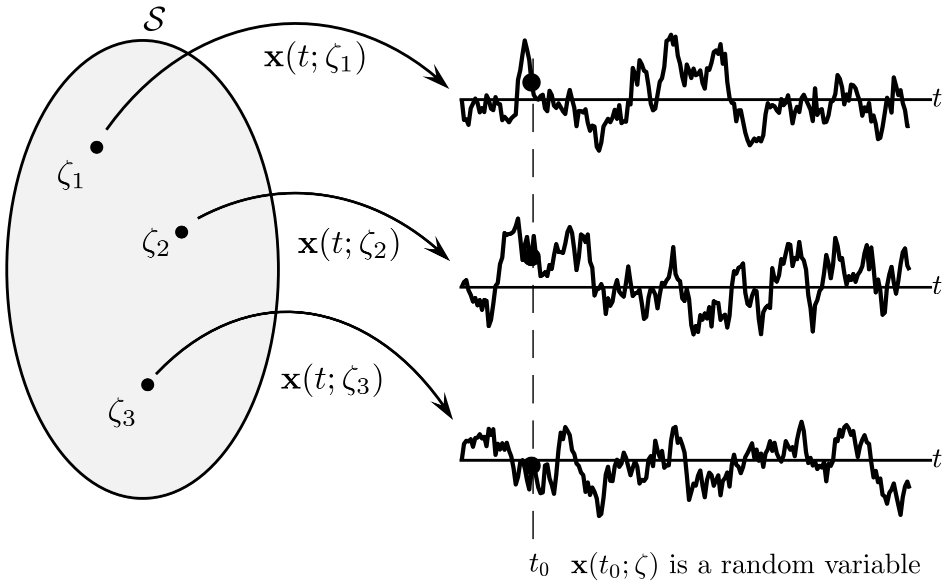
9-1 Definitions



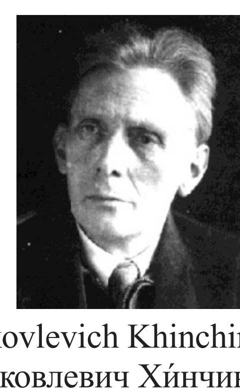
Definition

- A *stochastic process* (also called *random process*) $\mathbf{x}(t;\zeta)$ is a rule for assigning to every $\zeta \in \mathcal{S}$ a function of time.
  - A stochastic process is a *family of time functions* depending on the the parameter $\zeta \in \mathcal{S}$.
  - A stochastic process is a *function* of t and ζ.
- The functions of time that comprise the stochastic process may be either *continuous time* functions or *discrete time* functions.

Interpretations

1. If t and ζ are variables, the result is a family (or an *ensemble*) of waveforms $\mathbf{x}(t,\zeta)$.
2. If t is a variable and ζ is fixed, the result is a single function of time (or a *sample* of the stochastic process).
3. If t is fixed and ζ is variable, the result is a *random variable*.
4. If t and ζ are fixed, the result is a *number*.

στόχος, στόχου, ὁ: target, guess, conjecture



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1894-1959

Statistics of Stochastic Processes

Definition

- The *n-th order distribution* of the real-valued process $\mathbf{x}(t)$ is the joint distribution of the real-valued random variables $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)$:

$$F_{\mathbf{x}}(x_1, \ldots, x_n; t_1, \ldots, t_n) = P(\mathbf{x}(t_1) \leq x_1, \ldots, \mathbf{x}(t_n) \leq x_n)$$

  - If the random variables are jointly continuous, then the joint cdf is a continuous function.
  - If the random variables are jointly discrete, the the joint cdf is an n-dimensional stair-step function.
  - Do not confuse time and random variable type: a continuous-*time* random process may be described by either continuous or discrete *random variables* at a fixed time instant.

- The *n-th order density function* of the real-valued process $\mathbf{x}(t)$ is joint density of the real-valued random variables $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)$:

$$f_{\mathbf{x}}(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{\mathbf{x}}(x_1, \ldots, x_n; t_1, \ldots, t_n)$$

  - If the random variables are jointly continuous, then the joint pdf is smooth.
  - If the random variables are jointly discrete, then the joint pdf contains impulses (in the form of Dirac delta functions).
  - Alternatively, for jointly discrete random variables, the joint pmf may be used.
  - Do not confuse time and random variable type: a continuous-*time* random process may be described by either a continuous or discrete *random variable* at a fixed time instant.

Special cases (real-valued random processes)

- First-order density:
  1. The *first-order* distribution/density is the special case $n = 1$:
  $$F_{\mathbf{x}}(x;t) = P(\mathbf{x}(t) \leq x)$$
  $$f_{\mathbf{x}}(x;t) = \frac{\partial F_{\mathbf{x}}(x;t)}{\partial x}$$
  2. The *mean* of the random process $\mathbf{x}(t)$ is the mean of the random variable $\mathbf{x}(t)$ for fixed t and is computed from the first-order pdf
  $$\mu_{\mathbf{x}}(t) = E\{\mathbf{x}(t)\} = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;t) dx$$
- Second-order density
  1. The *second-order* distribution/density is the special case $n = 2$:
  $$F_{\mathbf{x}}(x_1, x_2; t_1, t_2) = P(\mathbf{x}(t_1) \leq x_1, \mathbf{x}(t_2) \leq x_2)$$
  $$f_{\mathbf{x}}(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_{\mathbf{x}}(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$
  2. The *autocorrelation function* is the expected value of the product $\mathbf{x}(t_1)\mathbf{x}(t_2)$ and is computed from the second order density:
  $$R_{\mathbf{xx}}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{\mathbf{x}}(x_1, x_2; t_2, t_2)\, dx_1\, dx_2$$
  3. The *average power* of the random process $\mathbf{x}(t)$ is the value of $R_{\mathbf{xx}}(t_1, t_2)$ along the diagonal $t = t_1 = t_2$:
  $$\text{average power} = E\{\mathbf{x}^2(t)\} = R_{\mathbf{xx}}(t, t)$$
  4. The *autcovariance* of the random process $\mathbf{x}(t)$ is the covariance of the random variables $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ and is computed from the second order density
  $$\begin{aligned} C_{\mathbf{xx}}(t_1, t_2) &= E\{(\mathbf{x}(t_1) - \mu_{\mathbf{x}}(t_1))(\mathbf{x}(t_2) - \mu_{\mathbf{x}}(t_2))\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mu_{\mathbf{x}}(t_1))(x_2 - \mu_{\mathbf{x}}(t_2)) f_{\mathbf{x}}(x_1, x_2; t_1, t_2)\, dx_1\, dx_2 \end{aligned}$$
  5. The *variance* of the random process $\mathbf{x}(t)$ is the value of $C_{\mathbf{xx}}(t_1, t_2)$ along the diagonal $t = t_1 = t_2$:
  $$\text{variance} = E\{(\mathbf{x}(t) - \mu_{\mathbf{x}}(t))^2\} = C_{\mathbf{xx}}(t, t)$$
  6. The *correlation coefficient* is
  $$r_{\mathbf{xx}}(t_1, t_2) = \frac{C_{\mathbf{xx}}(t_1, t_2)}{\sqrt{C_{\mathbf{xx}}(t_1, t_1) C_{\mathbf{xx}}(t_2, t_2)}}$$

More Definitions (real-valued random processes)

- A *white* random process $\mathbf{x}(t)$ means

$$C_{\mathbf{xx}}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$$

  It is *almost always* assumed that a white random process has zero mean:

$$\mu_{\mathbf{x}}(t) = 0$$

- A *normal random process* $\mathbf{x}(t)$ means the random variables $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)$ are jointly normal for any n and any $t_1, \ldots, t_n$.

Two real-valued random processes

- Two real-valued random processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are described by the joint distribution and density of the random variables

$$\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n), \mathbf{y}(t'_1), \ldots, \mathbf{y}(t'_m)$$

$$\begin{aligned} F_{\mathbf{xy}}(x_1, \ldots, x_n, y_1, \ldots, y_m; t_1, \ldots, t_n, t'_1, \ldots, t'_m) &= \\ &P(\mathbf{x}(t_1) \leq x_1, \ldots, \mathbf{x}(t_n) \leq x_n, \mathbf{y}(t'_1) \leq y_1, \ldots, \mathbf{y}(t'_m) \leq y_m) \end{aligned}$$

$$\begin{aligned} f_{\mathbf{xy}}(x_1, \ldots, x_n, y_1, \ldots, y_m; t_1, \ldots, t_n, t'_1, \ldots, t'_m) &= \\ &\frac{\partial^{n+m} F_{\mathbf{xy}}(x_1, \ldots, x_n, y_1, \ldots, y_m; t_1, \ldots, t_n, t'_1, \ldots, t'_m)}{\partial x_1 \cdots \partial x_n \partial y_1 \cdots \partial y_m} \end{aligned}$$

- The *cross-correlation function* is

$$R_{\mathbf{xy}}(t_1, t_2) = E\{\mathbf{x}(t_1)\mathbf{y}(t_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{\mathbf{xy}}(x, y; t_1, t_2)\, dx\, dy$$

- The *cross-covariance function* is

$$\begin{aligned} C_{\mathbf{xy}}(t_1, t_2) &= E\{(\mathbf{x}(t_1) - \mu_{\mathbf{x}}(t_1))(\mathbf{y}(t_2) - \mu_{\mathbf{y}}(t_2))\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_{\mathbf{x}}(t_1))(y - \mu_{\mathbf{y}}(t_2)) f_{\mathbf{xy}}(x, y; t_1, t_2)\, dx\, dy \end{aligned}$$

- Two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are *uncorrelated* if

$$C_{\mathbf{xy}}(t_1, t_2) = 0 \quad \text{for every } t_1 \text{ and } t_2$$

Comments on Complex-Valued Random Processes

- A *complex-valued* random process $\mathbf{z}(t,\zeta)$ maps each $\zeta \in \mathcal{S}$ to a complex-valued waveform.

  1. If t and ζ are variables, the result is an *ensemble* of complex-valued waveforms $\mathbf{z}(t,\zeta)$.
  2. If t is variable and ζ is fixed, the result is a single complex-valued function of time: a *sample* of the random process.
  3. If t is fixed and ζ is variable, the result is a complex-valued *random variable.*
  4. If t and ζ are fixed, the result is a *complex number.*

- The n-th order distribution and density of the complex-valued process $\mathbf{z}(t)$

  – Write

$$\begin{aligned} \mathbf{z}(t_1) &= \mathbf{x}(t_1) + j\mathbf{y}(t_1) \qquad & z_1 &= x_1 + jy_1 \\ &\vdots & &\vdots \\ \mathbf{z}(t_n) &= \mathbf{x}(t_n) + j\mathbf{y}(t_n) \qquad & z_n &= x_n + jy_n \end{aligned}$$

  – The n-th order distribution is the joint distribution of the complex-valued random variables $\mathbf{z}(t_1), \ldots, \mathbf{z}(t_n)$

$$\begin{aligned} F_{\mathbf{z}}(z_1,\ldots,z_n;t_1,\ldots,t_n) &= P(\mathbf{x}(t_1) \le x_1, \ldots, \mathbf{x}(t_n) \le x_n, \mathbf{y}(t_1) \le y_1, \ldots, \mathbf{y}(t_n) \le y_n) \\ &= F_{\mathbf{xy}}(x_1,\ldots,x_n,y_1\ldots,y_n;t_1,\ldots,t_n) \end{aligned}$$

  – The n-th order density of $\mathbf{z}(t)$ is expressed in terms of the (real-valued) real and imaginary components of $\mathbf{z}(t)$

$$\begin{aligned} f_{\mathbf{z}}(z_1,\ldots,z_n;t_1,\ldots,t_n) &= f_{\mathbf{xy}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1,\ldots,t_n) \\ &= \frac{\partial^{2n}}{\partial x_1 \cdots \partial x_n \partial y_1 \cdots \partial y_n} F_{\mathbf{xy}}(x_1,\ldots,x_n,y_1\ldots,y_n;t_1,\ldots,t_n) \end{aligned}$$

- First two moments

  – mean

$$\mu_{\mathbf{z}}(t) = \int_{-\infty}^{\infty} z f_{\mathbf{z}}(z;t) dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+jy) f_{\mathbf{xy}}(x,y;t)\, dx\, dy = \mu_{\mathbf{x}}(t) + j\mu_{\mathbf{y}}(t)$$

  – Autocorrelation $R_{\mathbf{zz}}(t_1,t_2) = E\{\mathbf{z}(t_1)\mathbf{z}^*(t_2)\}$

$$R_{\mathbf{zz}}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1+jy_1)(x_2-jy_2) f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;t_1,t_2)\, dx_1\, dx_2\, dy_1\, dy_2$$

  – Autocovariance: $C_{\mathbf{zz}}(t_1,t_2) = R_{\mathbf{zz}}(t_1,t_2) - \mu_{\mathbf{z}}(t_1)\mu_{\mathbf{z}}^*(t_2)$

Stationary Processes

Definitions

A stochastic process $\mathbf{x}(t)$ is called *strict-sense stationary* (abbreviated SSS) if its statistical properties are invariant to to a shift of the time origin.

$\Rightarrow$ $\mathbf{x}(t)$ and $\mathbf{x}(t+c)$ have the same statistics.

$\Rightarrow$ $f_{\mathbf{x}}(x_1,\ldots,x_n;t_1,\ldots,t_n) = f_{\mathbf{x}}(x_1,\ldots,x_n;t_1+c,\ldots,t_n+c)$ for any c and for all n.

Properties

1. First-order density:
   (a) $f_{\mathbf{x}}(x;t) = f_{\mathbf{x}}(x;t+c) \Rightarrow f_{\mathbf{x}}(x;t) = f_{\mathbf{x}}(x)$
   (b) $\mu_{\mathbf{x}}(t) = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;t)dx = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x)dx = \mu_{\mathbf{x}}$
2. Second-order density:
   (a)
$$\begin{aligned} f_{\mathbf{x}}(x_1,x_2;t_1,t_2) &= f_{\mathbf{x}}(x_1,x_2;t_1+c,t_2+c) \\ \Rightarrow f_{\mathbf{x}}(x_1,x_2;t_1,t_2) &= f_{\mathbf{x}}(x_1,x_2;t_1-t_2,0) \\ &= f_{\mathbf{x}}(x_1,x_2;\tau,0), \quad \tau = t_1-t_2 \end{aligned}$$

   "Thus the joint density of the random variables $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$ is independent of [i.e., not a function of] t and it equals $f(x_1,x_2;\tau)$."

   (b)
$$\begin{aligned} R_{\mathbf{xx}}(t_1,t_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{\mathbf{x}}(x_1,x_2;t_1t_2)\,dx_1\,dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{\mathbf{x}}(x_1,x_2;\tau,0)\,dx_1\,dx_2 \\ &= R_{\mathbf{xx}}(\tau,0) \end{aligned}$$

   (c) It is customary to express the autocorrelation function for a WSS random process by

$$R_{\mathbf{xx}}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}(t)\} = E\{\mathbf{x}(t)\mathbf{x}(t-\tau)\}$$

   (d) $C_{\mathbf{xx}}(t_1,t_2) = C_{\mathbf{xx}}(\tau) = R_{\mathbf{xx}}(\tau) - \mu_{\mathbf{x}}^2$

Consequences

1. average power of SSS process $= R_{\mathbf{xx}}(0)$
2. variance of SSS process $= C_{\mathbf{xx}}(0)$
3. correlation coefficient of SSS process: $r_{\mathbf{xx}}(\tau) = \dfrac{C_{\mathbf{xx}}(\tau)}{C_{\mathbf{xx}}(0)}$

Definitions

A stochastic process $\mathbf{x}(t)$ is called *wide-sense stationary* (abbreviated WSS) if

$$f_{\mathbf{x}}(x;t) = f_{\mathbf{x}}(x;t+c)$$
$$f_{\mathbf{x}}(x_1,x_2;t_1,t_2) = f_{\mathbf{x}}(x_1,x_2;t_1+c,t_2+c)$$

Properties

1. $\mu_{\mathbf{x}}(t) = \mu_{\mathbf{x}}$
2. $R_{\mathbf{xx}}(t_1,t_2) = R_{\mathbf{xx}}(\tau)$

A stochastic process $\mathbf{x}(t)$ is *WSS white noise* means $C_{\mathbf{xx}}(\tau) = q\delta(\tau)$.

Comments on Complex-Valued WSS Random Processes

- The complex-valued WSS process $\mathbf{z}(t) = \mathbf{x}(t) + j\mathbf{y}(t)$ is described in terms of the joint statistics of the two real-valued processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$.
- The first-order density property

$$f_{\mathbf{z}}(z;t) = f_{\mathbf{z}}(z;t+c) \Rightarrow f_{\mathbf{z}}(z;t) = f_{\mathbf{z}}(z)$$

  becomes

$$f_{\mathbf{xy}}(x,y;t) = f_{\mathbf{xy}}(x,y;t+c) \Rightarrow f_{\mathbf{xy}}(x,y;t) = f_{\mathbf{xy}}(x,y)$$

- The complex-valued mean is a constant:

$$\mu_{\mathbf{z}}(t) = \mu_{\mathbf{z}} \Rightarrow \mu_{\mathbf{x}}(t) + j\mu_{\mathbf{y}}(t) = \mu_{\mathbf{x}} + j\mu_{\mathbf{y}}$$

- The second-order density property

$$\begin{aligned} f_{\mathbf{z}}(z_1,z_2;t_1,t_2) &= f_{\mathbf{x}}(z_1,z_2;t_1+c,t_2+c) \\ &\Rightarrow f_{\mathbf{z}}(z_1,z_2;t_1,t_2) = f_{\mathbf{x}}(z_1,z_2;t_1-t_2,0) \end{aligned}$$

  becomes

$$\begin{aligned} f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;t_1,t_2) &= f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;t_1+c,t_2+c) \\ &\Rightarrow f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;t_1,t_2) = f_{\mathbf{xy}}(x_1,x_2,y_1,y_2;t_1-t_2,0) \end{aligned}$$

- Autocorrelation function is

$$R_{\mathbf{zz}}(\tau) = E\{\mathbf{z}(t+\tau)z^*(t)\} = E\{\mathbf{z}(t)\mathbf{z}^*(t-\tau)\}$$

Properties of the auto- and cross-correlation functions

General Random Processes

1. $R_{\mathbf{xx}}(t_1, t_2) = E\{\mathbf{x}(t_1)\mathbf{x}^*(t_2)\}$
2. $R_{\mathbf{xx}}(t_2, t_1) = R^*_{\mathbf{xx}}(t_1, t_2)$
3. $R_{\mathbf{xx}}(t, t) \geq 0$
4. $R_{\mathbf{xy}}(t_1, t_2) = E\{\mathbf{x}(t_1)\mathbf{y}^*(t_2)\}$
5. $R_{\mathbf{yx}}(t_2, t_1) = R^*_{\mathbf{xy}}(t_1, t_2)$

WSS Random Processes

1. $R_{\mathbf{xx}}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}^*(t)\}$
2. $R_{\mathbf{xx}}(-\tau) = R^*_{\mathbf{xx}}(\tau)$
3. $R_{\mathbf{xx}}(0) \geq 0$
4. $R_{\mathbf{xy}}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{y}^*(t)\}$
5. $R_{\mathbf{yx}}(-\tau) = R^*_{\mathbf{xy}}(\tau)$
6. $R_{\mathbf{xx}}(\tau) \leq R_{\mathbf{xx}}(0)$

From Property 6 for WSS random processes

$R_{\mathbf{xx}}(\tau_1) = R_{\mathbf{xx}}(0)$ for some $\tau_1 \neq 0 \quad \Rightarrow \quad R_{\mathbf{xx}}(\tau + \tau_1) = R_{\mathbf{xx}}(\tau)$ for all τ

$\Rightarrow \quad R_{\mathbf{xx}}(\tau)$ is *periodic* with period τ_1

$R_{\mathbf{xx}}(\tau_1) = R_{\mathbf{xx}}(\tau_2) = R_{\mathbf{xx}}(0)$ for τ_1, τ_2 noncommensurate $\quad \Rightarrow \quad R_{\mathbf{xx}}(\tau) = \text{constant}.$