BYU

Nonlinear Systems Theory

Jan 9, 2024

Overview

- About Dr. Usevitch
- About You
- Introduction to the Course
  - Why Nonlinear Systems?
  - State Space Representations
    - Linear Systems
    - Nonlinear Systems
  - Nonlinear Phenomena
  - Linearization
- Class Logistics

About Me

- BS from BYU
- MS, PhD from UMich
- 2 years technical staff at MIT Lincoln Laboratory

My Research:

Robust

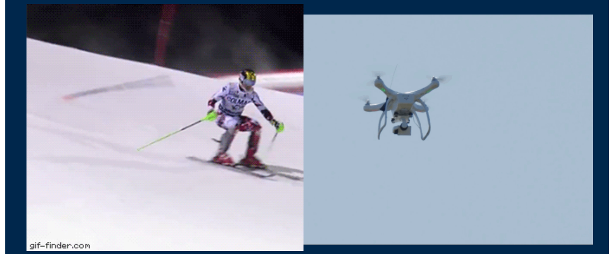
Resilient

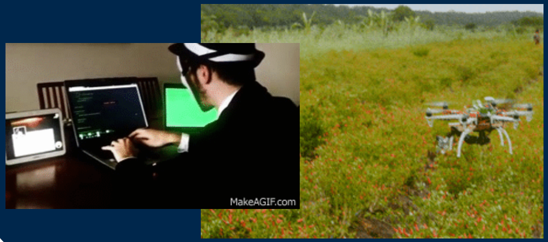
Autonomous Systems

Safe

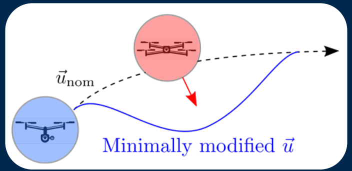
Intelligent

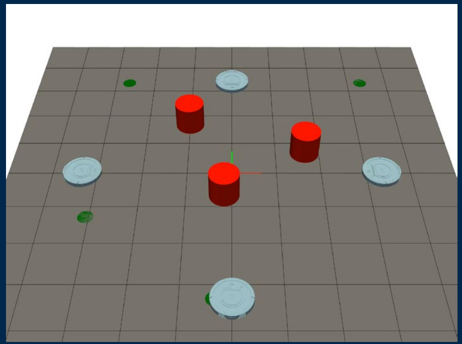
Robustness and Resilience



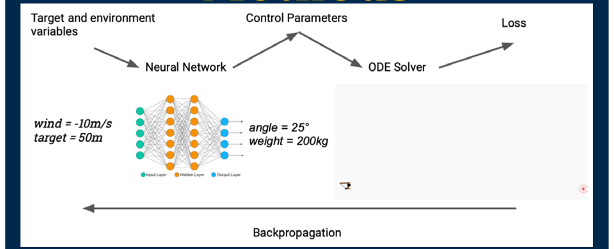


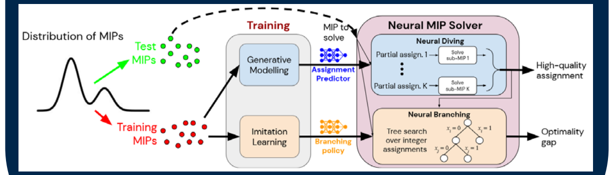
Algorithmic Control Theory





Uniting Classical and Learning Methods





Very Important Announcement

Michigan Wolverines football

1st in B10 East

GAMES NEWS STANDINGS PLAYERS

NCAA football · Yesterday

Final

13 - 34

2 Washington Huskies (14 - 1)

1 Michigan Wolverines (15 - 0)

Final

Team	1	2	3	4	T
Washington Huskies	3	7	3	0	13
Michigan Wolverines	14	3	3	14	34

About You

- Where are you from?
- What year are you?
- Best thing that happened over Christmas Break?
- What are future plans?
- Have you ever…
  - Taken a course on linear systems or control?
  - Programmed in Python / JAX?
  - Worked with mathematical theorems and proofs?

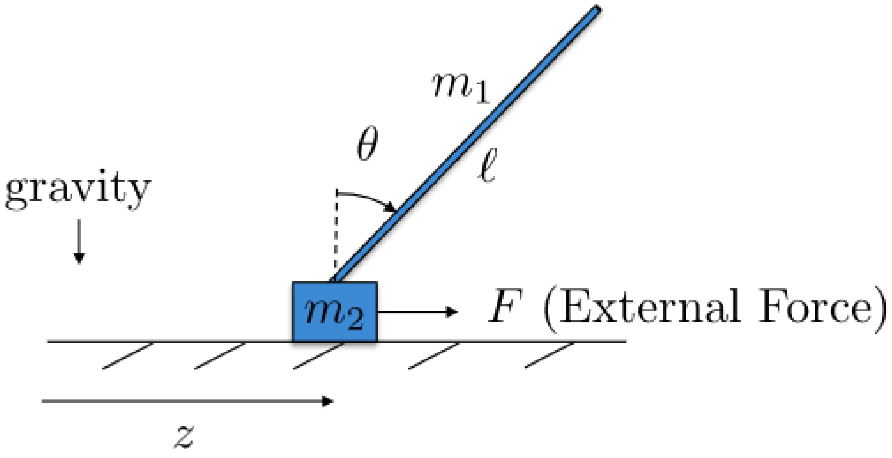
Class Logistics

- Syllabus
  - Midterm / Final Exams
  - Homework
  - AI Policy
  - Feedback!

What is a (Nonlinear) System?

State Space Representations

- **State: Minimum amount of information required to predict past, present, and future behavior of a system.**



State Variables: z, θ

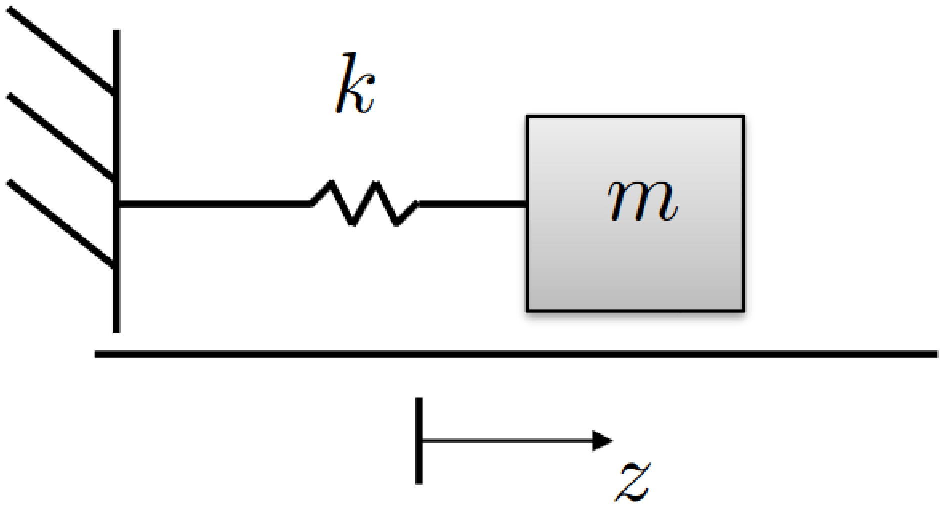
$$
\begin{aligned}
p_n &= \text{the inertial (north) position of the quadrotor along } \hat{i}^i \text{ in } \mathcal{F}^i, \\
p_e &= \text{the inertial (east) position of the quadrotor along } \hat{j}^i \text{ in } \mathcal{F}^i, \\
h &= \text{the altitude of the aircraft measured along } -\hat{k}^i \text{ in } \mathcal{F}^i, \\
u &= \text{the body frame velocity measured along } \hat{i}^b \text{ in } \mathcal{F}^b, \\
v &= \text{the body frame velocity measured along } \hat{j}^b \text{ in } \mathcal{F}^b, \\
w &= \text{the body frame velocity measured along } \hat{k}^b \text{ in } \mathcal{F}^b, \\
\phi &= \text{the roll angle defined with respect to } \mathcal{F}^{v2}, \\
\theta &= \text{the pitch angle defined with respect to } \mathcal{F}^{v1}, \\
\psi &= \text{the yaw angle defined with respect to } \mathcal{F}^{v}, \\
p &= \text{the roll rate measured along } \hat{i}^b \text{ in } \mathcal{F}^b, \\
q &= \text{the pitch rate measured along } \hat{j}^b \text{ in } \mathcal{F}^b, \\
r &= \text{the yaw rate measured along } \hat{k}^b \text{ in } \mathcal{F}^b.
\end{aligned}
$$

Given a System with States…

- Can we reconstruct past states?
- Can we predict future states?
- Can we control how the states evolve?
- Typically study state as a function of time $x(t)$
- We also may study…
  - Velocity $\dot{x}(t)$
  - Acceleration $\ddot{x}(t)$
  - Jerk, snap, crackle, pop, etc.

How do states evolve / change with time?

- **Damped Mass / Spring Example**
  - **State:** z
  - **From Newton:** $F = ma = m\ddot{x}(t)$
  - **What are the forces?**
    - **Spring:** $F_s = -kx$
    - **Damping:** $F_d = -b\dot{x}$
    - **External force input:** $F_{ex} = u(t)$
- **Sum of forces:**
  - $m\ddot{x} + b\dot{x} + kx - u = 0$
- **How do we predict where $x(t)$ will be at time t?**



Beard, R., et al. "Introduction to Feedback Control using Design Studies." (2016).

Ordinary Differential Equations (ODEs)

- **Equations that relate one unknown function and its derivatives**
  - **Allow us to solve for $x(t)$ – make predictions!**
  - **Order: highest order of derivative in the unknown function**
- **General form (for this class):**

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_m)
\end{aligned}
$$

More compact form!

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ \vdots \\ f_n(t, x, u) \end{bmatrix}
$$

$$
\dot{x} = f(t, x, u)
$$

https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/Lect_1.pdf

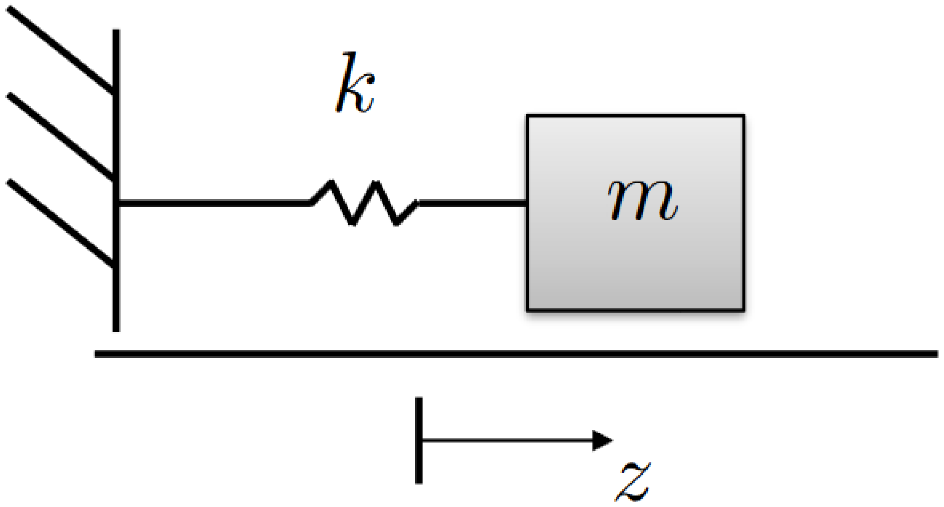
"But the Mass / Spring has a *second* derivative!"

- **We can cheat.**
- **Define the state** $z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$
- **We have:**

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ 1/m(u - b\dot{x} - kx) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

$$\dot{z} = f(z, u)$$

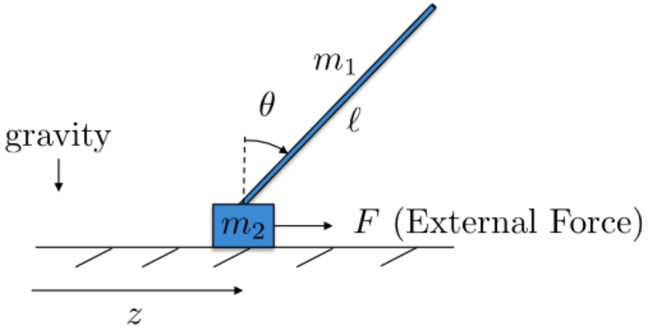


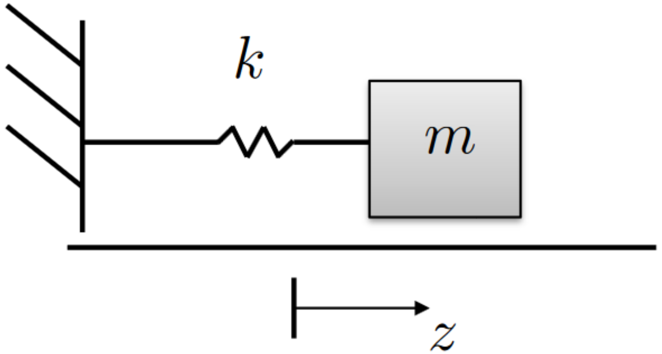
$$m\ddot{x} + b\dot{x} + kx - u = 0$$

Beard, R., et al. "Introduction to Feedback Control using Design Studies." (2016).

In Summary...

- The state equation: $\dot{x} = f(t, x, u)$
- Output equation: $y = h(t, x, u)$
- These equations describe how our system changes over time!





Beard, R., et al. "Introduction to Feedback Control using Design Studies." (2016).

Special Forms

- **Linear systems:**
$$\begin{aligned}\dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u\end{aligned}$$

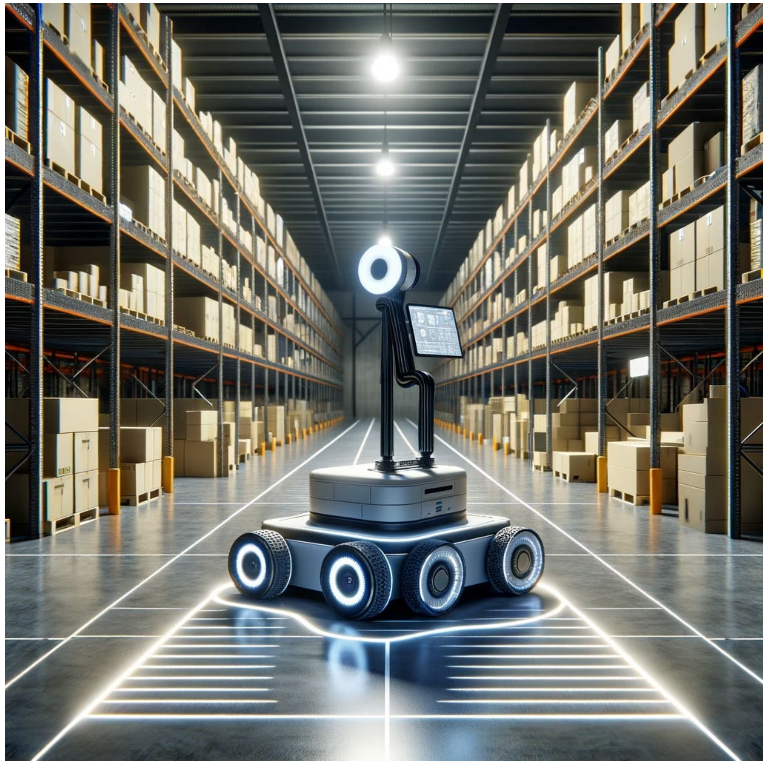
- **Unforced:** $\dot{x} = f(t, x)$
  - **No external input**

- **Autonomous System:** $\dot{x} = f(x)$
  - **Doesn't depend on time or an external input**

- **Time Invariant:**
$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u)\end{aligned}$$
  - **Shifting the start time from t_0 to $t_0 + a$ has no effect!**

"Great…so how do I *actually* predict the value of the state $x(t)$ from the state equation??"

- A *solution* of $\dot{x} = f(t, x)$:
  - A function $\phi(t)$ such that $\dot{\phi}(t) = f(t, \phi(t))$
- Example: $\dot{x} = ax$
  - Solution: $\phi(t) = ce^{at}$
  - $\frac{d}{dt}ce^{at} = a(ce^{at})$
- If we have the solution $\phi(t)$, we can predict the value of the state at any time given:
  - Initial state x_0
  - Initial time t_0



Equilibrium Points

- **In many cases there are points where the state "doesn't move"**

A point $x = x^*$ in the state space is said to be an equilibrium point of $\dot{x} = f(t, x)$ if

$$x(t_0) = x^* \quad \Rightarrow \quad x(t) \equiv x^*, \quad \forall\, t \geq t_0$$

For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real solutions of the equation

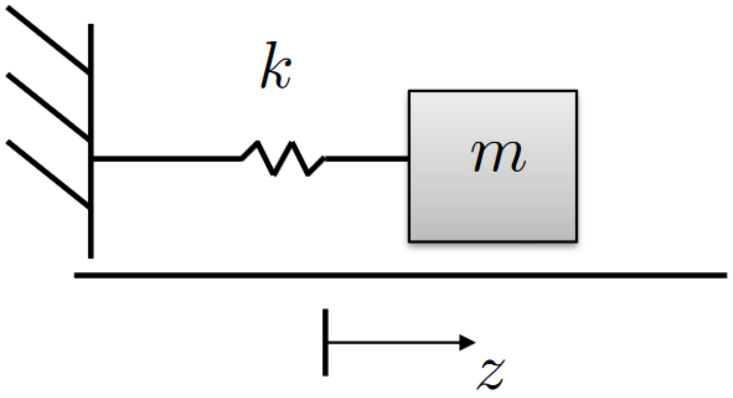
$$f(x) = 0$$

- **System behavior is studied around equilibrium points**

https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/Lect_1.pdf

Linear Systems are "Easy"…

- **Given a linear system model…**
  - **We have a closed-form solution for how it evolves!**
    - $x(t) = e^{At}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$
  - **We can analyze whether and where we can control it!**
  - **We can compute control commands in real time!**
  - **Everything is awesome!**



$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

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…but the Real World is Nonlinear.

- "Classification of mathematical problems as linear and nonlinear is like classification of the Universe as bananas and non-bananas"*
- Almost everything in the real world is nonlinear.
- Nonlinear is harder:
  - In general, no closed-form solution
  - Harder to predict future
  - Harder to reconstruct past
  - Control inputs may be NP-hard to compute

*Mesbahi and Egerstedt, "Graph Theoretic Methods in Multi-Agent Networks"

Why do we study Linear Systems?

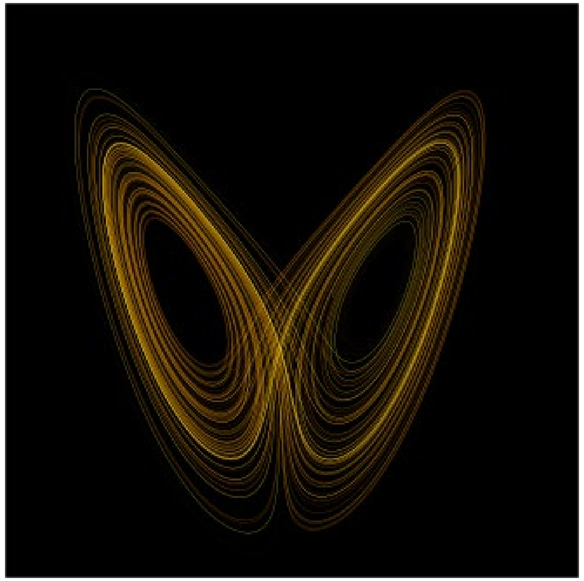
- We can approximate nonlinear systems with linear ones.
  - It works in the real world.
  - It is easy to compute
- How do we do this? Linearization!
  - Typically around equilibrium points

Example: Simple Pendulum

Second Example: With Control Inputs!

"How else are Nonlinear Systems Different?"

- Nonlinear systems exhibit several phenomena that Linear systems don't have:
  - Finite Escape Time
  - Multiple Isolated Equilibria
  - Limit Cycles
  - Subharmonic, harmonic, or almost-periodic oscillations
  - Chaos
  - Multiple modes of behavior



https://en.wikipedia.org/wiki/Chaos_theory

Finite Escape Time

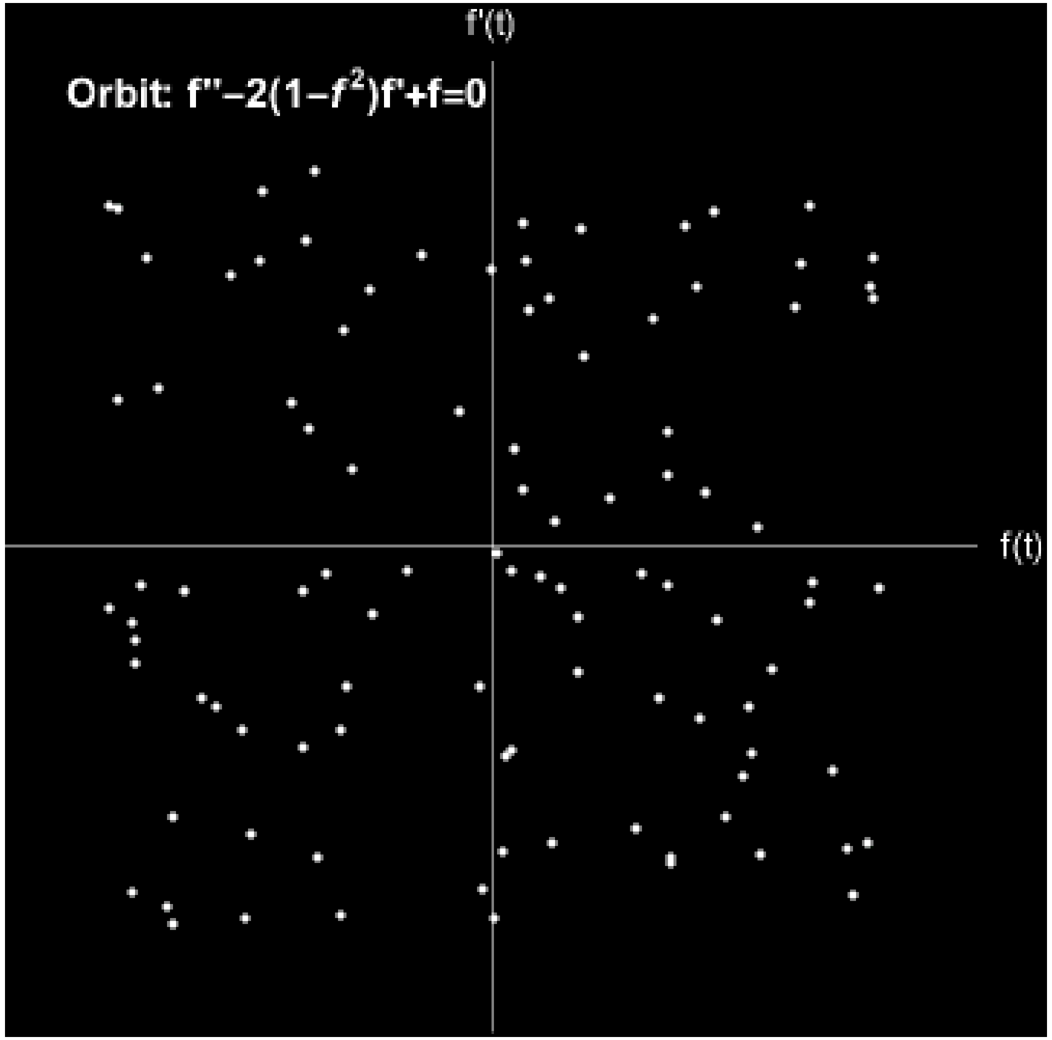
- The state of the system “blows up” to infinity in a finite amount of time
- Linear systems require infinite time to escape to infinity

Equilibria

- Linear Systems: One of the following:
  - One isolated equilibrium
  - A subspace of equilibria
- Nonlinear systems
  - Multiple isolated equilibria

Limit Cycles

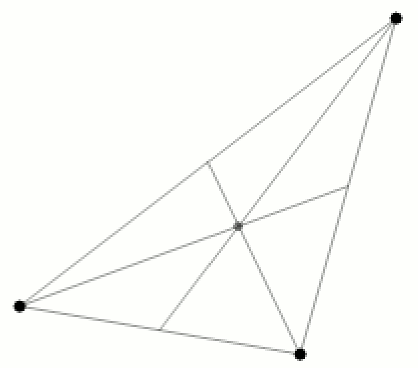
- **Linear Systems: Oscillation possible, but amplitude depends on initial state**
- **Nonlinear Systems:**
  - **Oscillation does not depend on initial state**
  - **Oscillation is *stable***

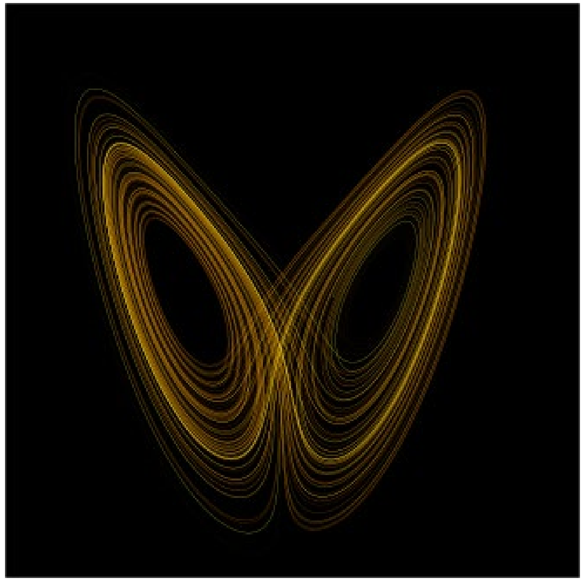


https://en.wikipedia.org/wiki/Van_der_Pol_oscillator

Chaos

- System may demonstrate "random" motions, even though the system itself is deterministic
- Small changes in initial condition lead to massive changes in future states





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Course Outline (Selected Topics)

- Modeling and Simulation
- Math Review
- Lyapunov Stability and LaSalle Invariance Theorem
- Boundedness and Ultimate Boundedness
- Input – Output Stability
- Feedback Control and Linearization
- Sliding Mode Control
- Control Barrier Functions
- Adaptive Control

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Nonlinear Systems Theory

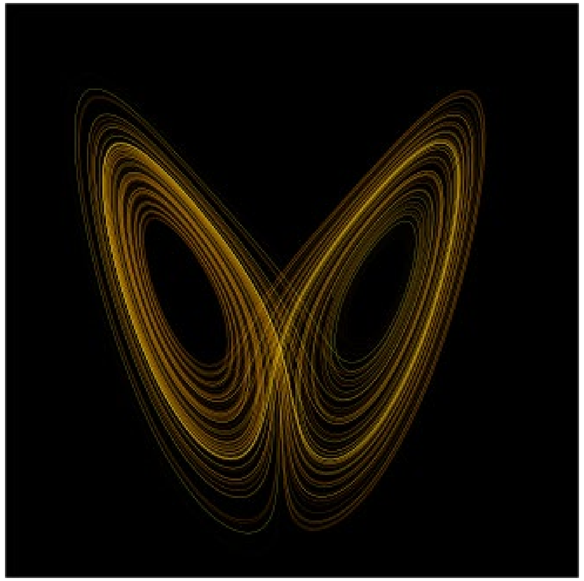
Jan 11, 2024

Overview

- Review
- Nonlinear Phenomena
- Simulating Nonlinear Systems
  - ODE solvers
  - Introduction to JAX
  - Hands-on experimentation
- Modeling Nonlinear Systems

"How else are Nonlinear Systems Different?"

- Nonlinear systems exhibit several phenomena that Linear systems don't have:
  - Finite Escape Time
  - Multiple Isolated Equilibria
  - Limit Cycles
  - Subharmonic, harmonic, or almost-periodic oscillations
  - Chaos
  - Multiple modes of behavior



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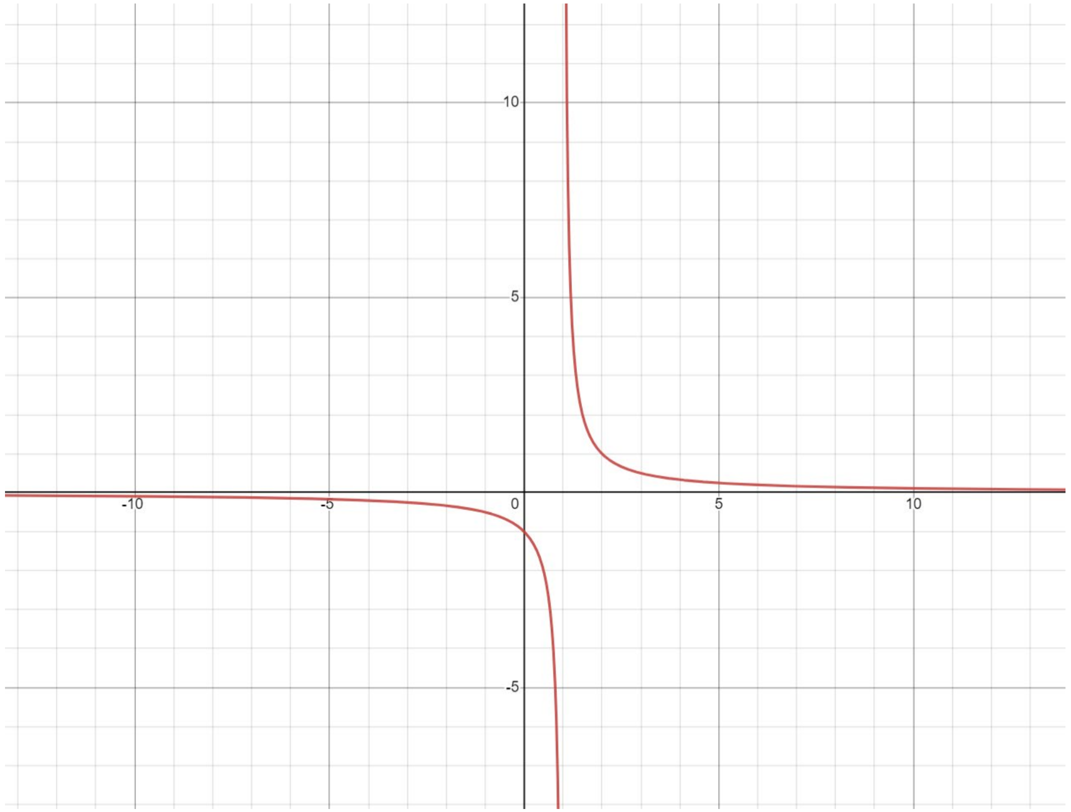
Finite Escape Time

- The state of the system "blows up" to infinity in a finite amount of time
  - Linear systems require infinite time to escape to infinity
- Example: $\dot{x} = -x^2$
  - Solution: $x(t) = \frac{1}{t-1}$
  - As t --> 1, we divide by zero and the universe explodes.

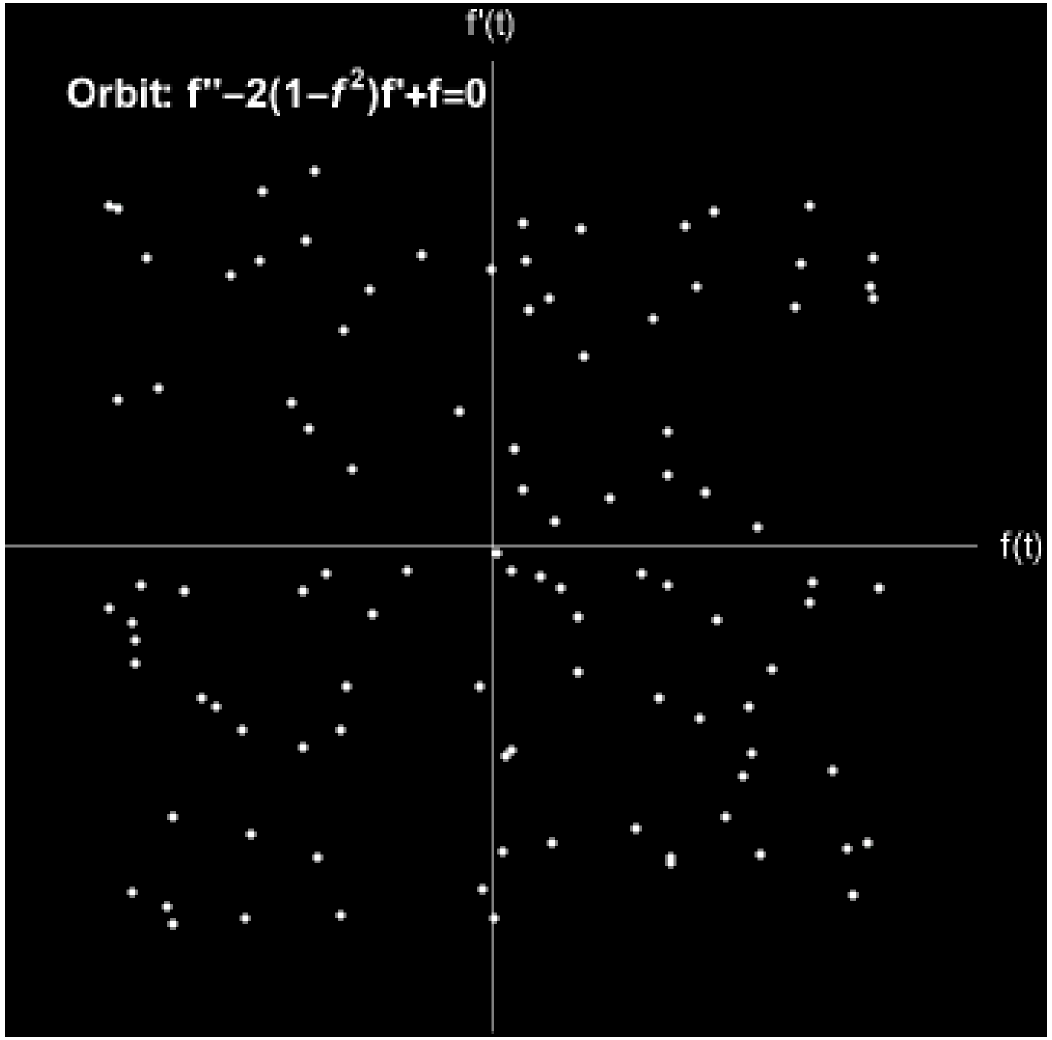


Equilibria

- An equilibrium point x^* is isolated if $\forall\epsilon > 0$, the set $\{z: ||z - x|| < \epsilon\}$ contains no other equilibria (besides x^*itself)
- Linear Systems: One of the following:
  - One isolated equilibrium
  - A subspace of equilibria
- Why?
  - $\dot{x} = Ax$, consider the null space of A
- Nonlinear systems
  - Multiple isolated equilibria

Limit Cycles

- **Linear Systems: Oscillation possible, but amplitude depends on initial state**
- **Nonlinear Systems:**
  - **Oscillation does not depend on initial state**
  - **Oscillation is *stable***



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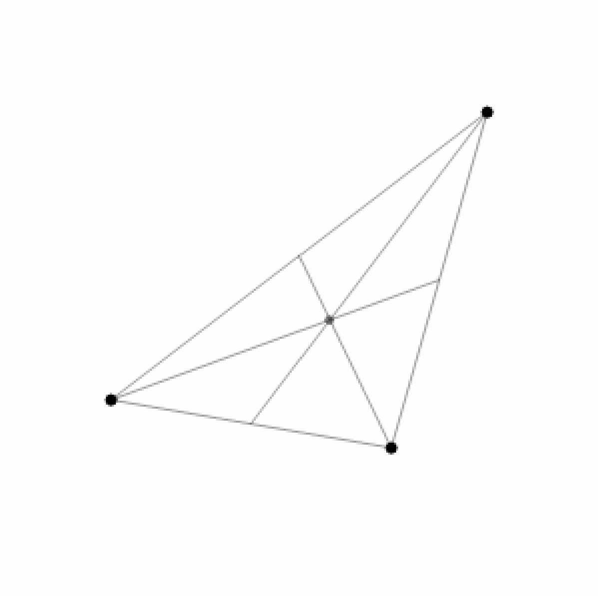
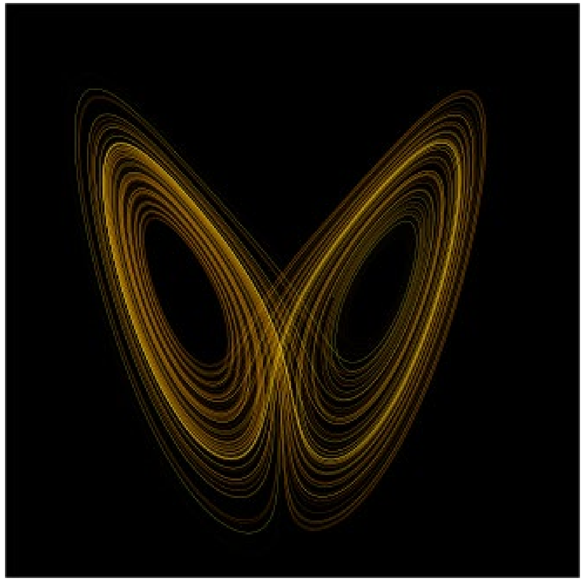
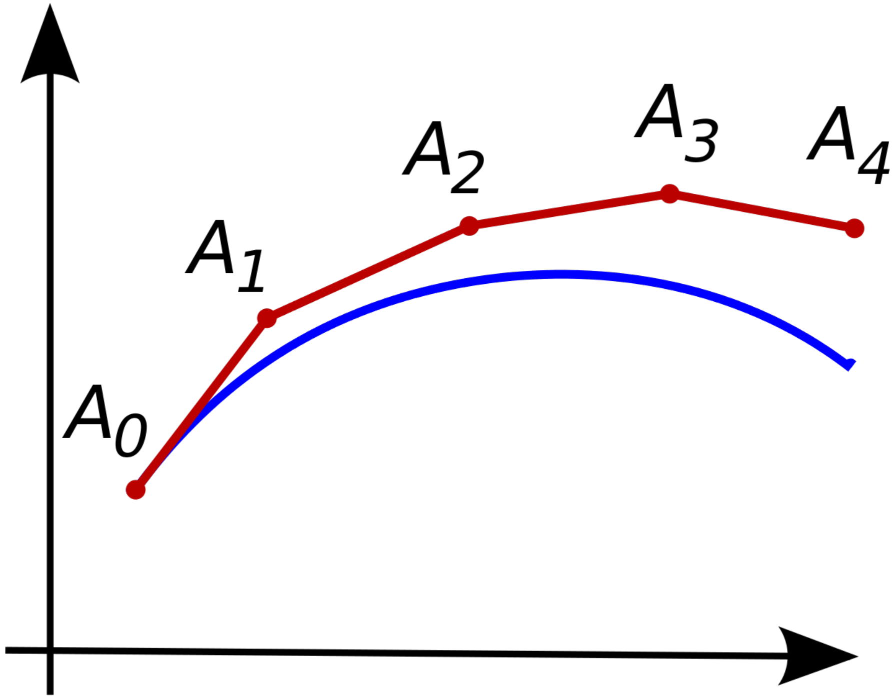




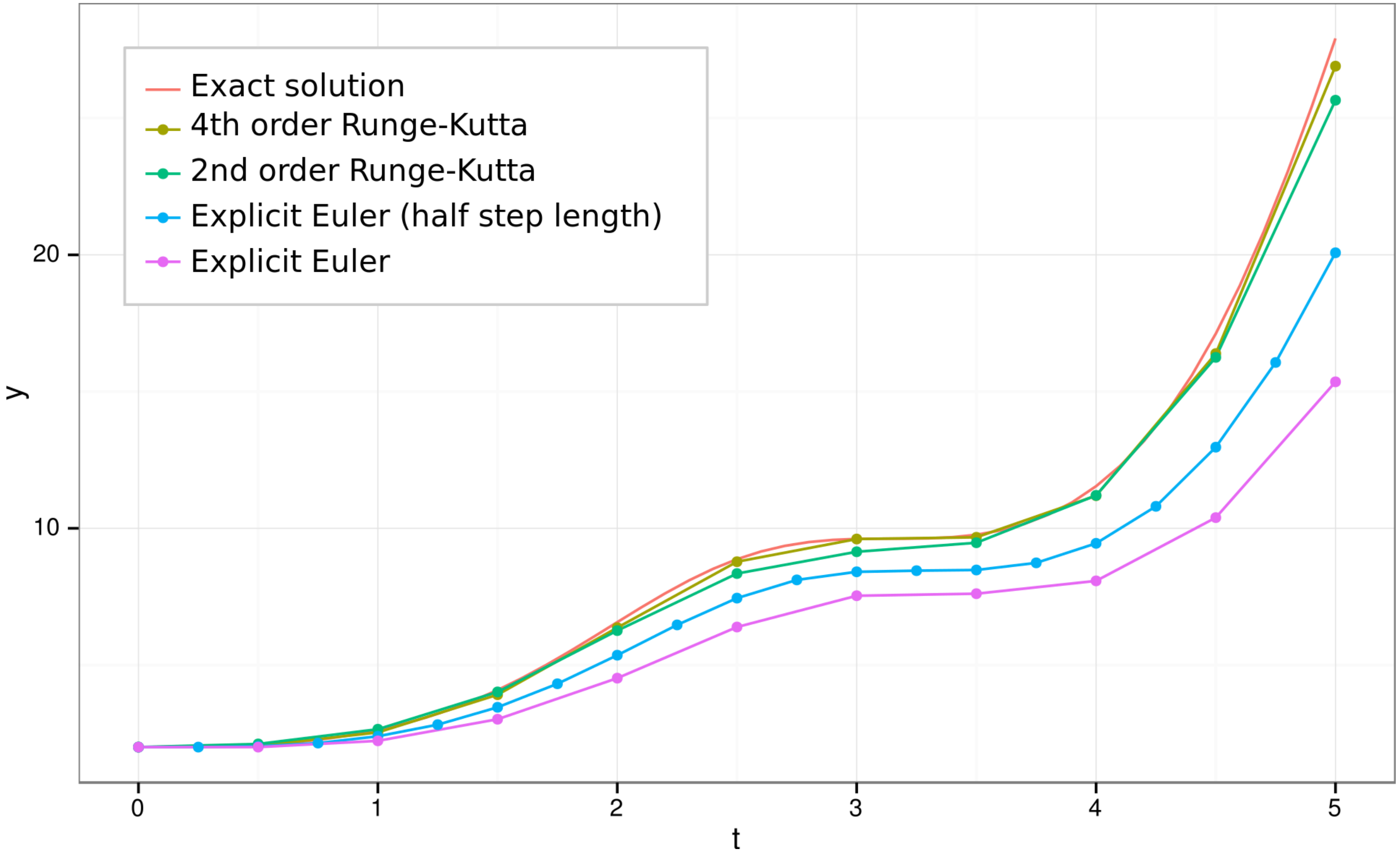
https://en.wikipedia.org/wiki/Chaos_theory

Simulating Nonlinear Systems

- **Our models are coupled ordinary differential equations**
- **For nonlinear systems, closed-form solutions are rare**
- **Instead, we use *numerical methods***
- **Basic: Euler's method!**
  - $x_{k+1} = x_k + f(t, x_k, u_k)\Delta t$
  - **Easy, but inaccurate**
- **Less basic: Runge-Kutta methods!**
- **A plethora of other options**



Comparison of Methods



Solver Options

- MATLAB: at least 10 options
- Python: Scipy, JAX
- Julia: DifferentialEquations.jl
- C++: Sundials
- ...And many more!

JAX

- Domain-specific Language built on Python
- "Numpy with three superpowers":
  - Automatic Differentiation
  - Just-in-time (JIT) compilation
  - GPU / TPU vectorization and acceleration
- One Kryptonite: Functional programming
  - Different style than you're used to!

Why JAX?

- Solid ODE solver suite (Diffrax)
- State-of-the-art machine learning platforms (Equinox, Flax)
- Excellent option for Scientific Machine Learning
  - Other state-of-the-art option is Julia
- Free and open source
  - You can keep your scripts after grad school 😃

Demonstration: Simple Pendulum

Practice: Pendulum on a Cart

Demonstration of AutoDiff: Kalman Filter

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Nonlinear Systems Theory

Jan 16, 2024

Overview

- Nonlinear Systems Demo (continued)
- Math Review

Practice: Pendulum on a Cart

https://colab.research.google.com/drive/1P4eofZ-8b-1kYji7LVeqzETdo18KPwZ8?usp=sharing

Demonstration of AutoDiff: Kalman Filter

- https://colab.research.google.com/drive/1FxPA1CuYKoC7ZbQx0xFxe92C6d02wvdA?usp=sharing

- (Created by Patrick Kidger, author of Diffrax / Equinox)

Math Review

- **Sources:**
  - **Cunningham, Daniel. *A logical introduction to proof*. Springer Science & Business Media, 2012.**
  - **Ross, Kenneth A. *Elementary Analysis the Theory of calculus*. springer publication, 2013.**

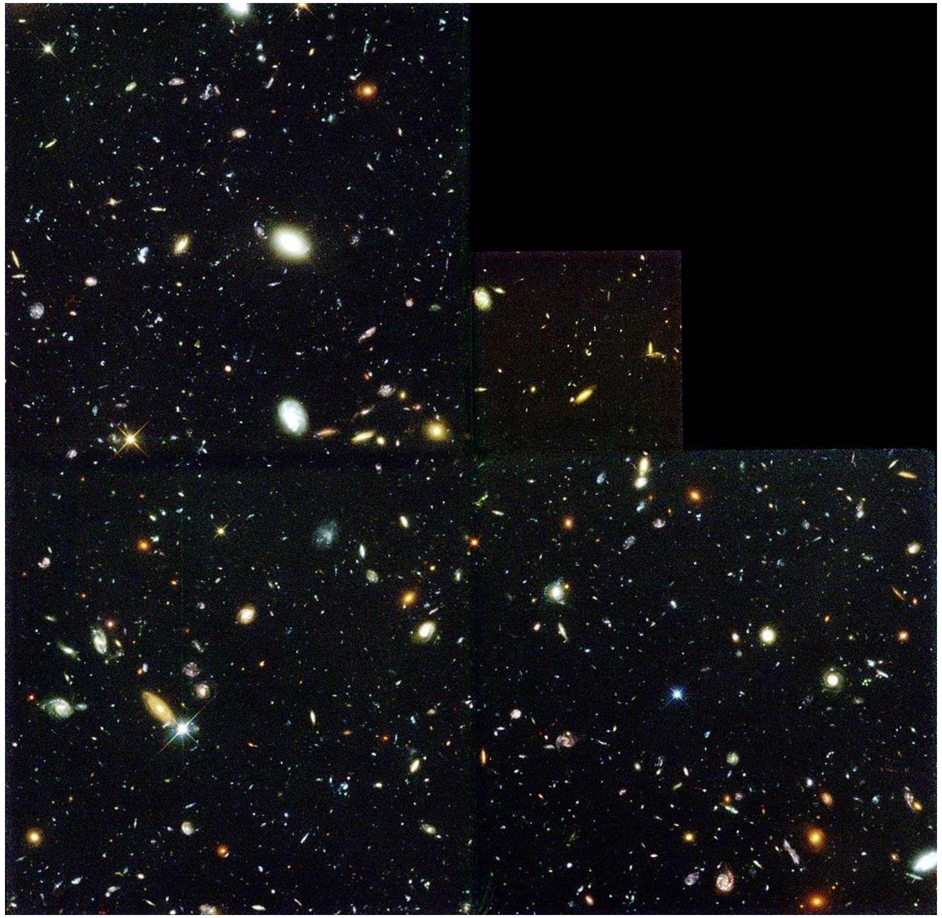
Why learn theoretical math?

- Math is a *language of truth and precision*

All kingdoms have a law given;

And unto every kingdom is given a law; and unto every law there are certain bounds also and conditions.

Doctrine and Covenants 88:36, 38

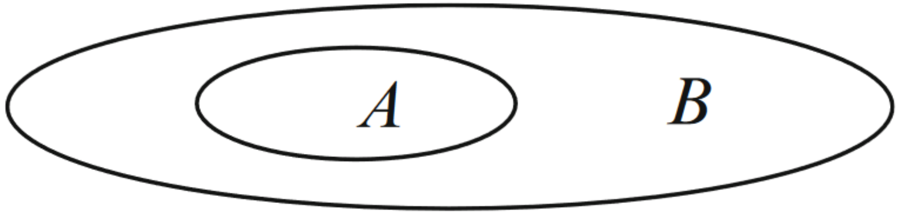


https://youtu.be/bBT4c5jWTms?si=Fg76RxDahPJplDov&t=428

President Russell M. Nelson: Divine Law

Sets

- Collections of objects
  - Usually defined by some rules / properties
- Examples:
  - $\mathbb{R}$: Set of all real numbers
  - $\mathbb{Z}$: Set of integers
  - $\mathbb{R}^n$: Set of all n-dimensional vectors with real entries
- Subsets: $A \subset B$ iff $x \in A \Longrightarrow x \in B$
- Empty set: $\emptyset$
- Cartesian product: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$



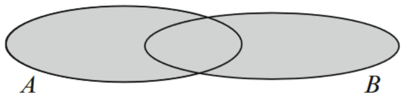


Fig. 5.2a Venn diagram of $A \cup B$

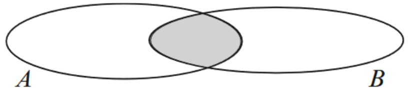


Fig. 5.2b Venn diagram of $A \cap B$

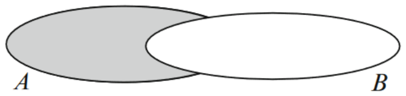


Fig. 5.2c Venn diagram of $A \setminus B$

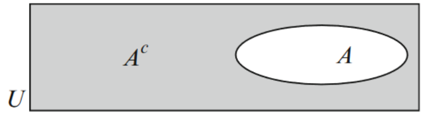


Fig. 5.2d Venn diagram of A^c

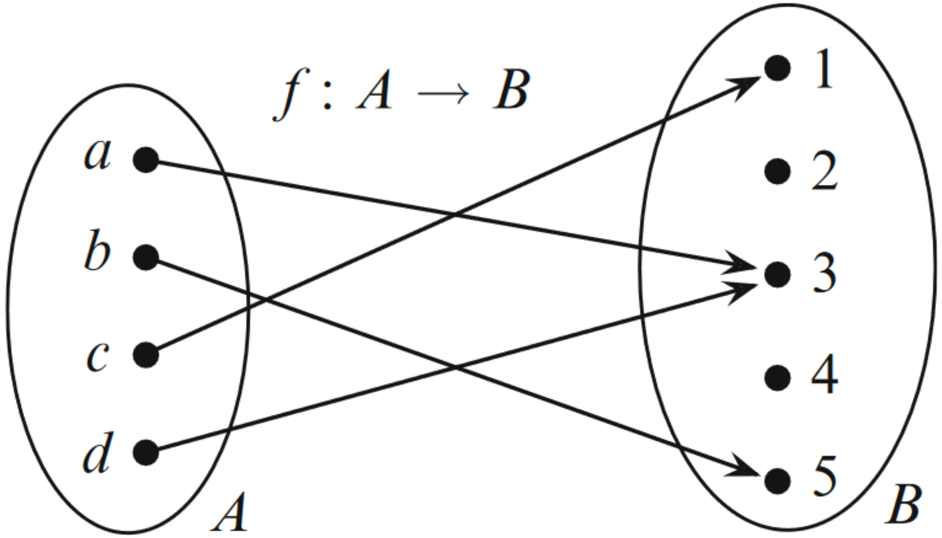
Properties of Sets

- **ϵ-Neighborhood: $\mathrm{N}(\mathrm{x}, \epsilon) = \{z \in \mathbb{R}^n : ||z - x|| < \epsilon\}$**
- **A set S is *open* if, for every $x \in S$, one can find an $\epsilon > 0$ such that $N(x, \epsilon) \subset S$**
- **A set S is *closed* iff its complement is open**
- **A set is *bounded* iff there exists $r > 0$ such that $||x|| \leq r$ for all $x \in S$**
- **A set is *compact* if it is both closed and bounded.**

Functions

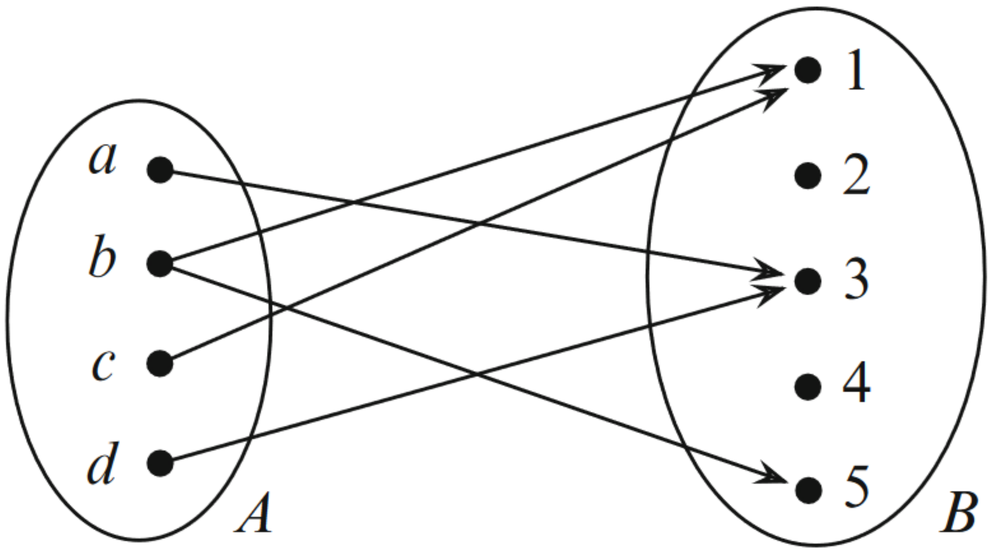
- A function from A to B is a subset $f \subseteq A \times B$ such that:
  - For each $x \in A$ there exists $y \in B$ such that $(x, y) \in f$.
  - (Single-valued function): If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$
- A is called the *domain* of f
- B is called the *codomain* of f
- We typically write $f: A \to B$
- *Range* of function:

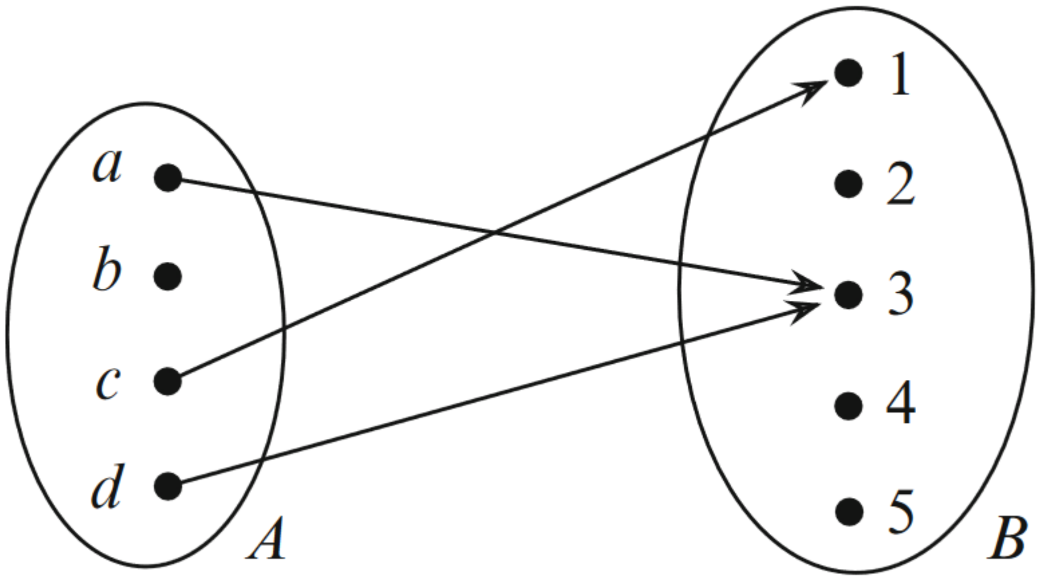
$$\operatorname{ran}(f) = \{f(a) : a \in A\} = \{b \in B : b = f(a) \text{ for some } a \in A\}.$$



Quick Quiz

- **Why are these not functions?**





Injective, Surjective, Bijective Functions

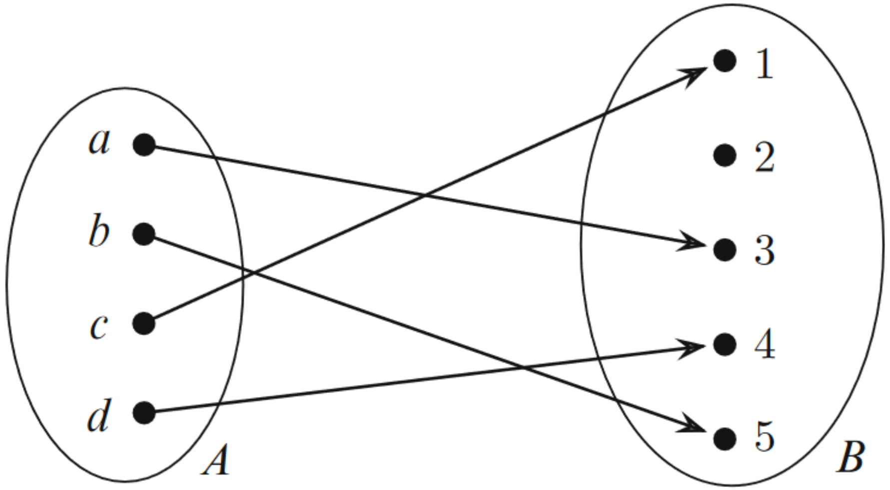
- **A function $f \subset A$ is…**
  - **Injective (aka "one-to-one") if distinct elements in A are mapped to distinct elements in B:**

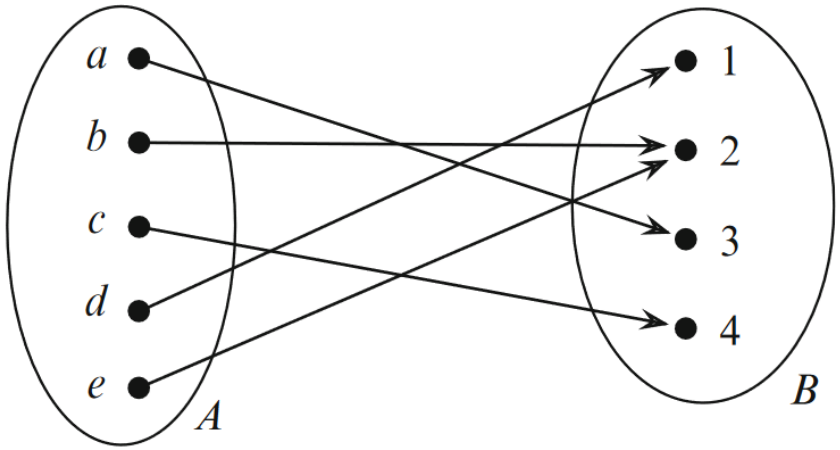
$$(\forall x \in A)(\forall y \in A)[f(x) = f(y) \to x = y].$$

  - **Surjective (aka "onto") if every element y in B has an associated element x in A such that $f(x) = y$:**

$$(\forall y \in B)(\exists x \in A)[f(x) = y].$$

  - **Bijective if is both injective and surjective simultaneously**





Why do we care? Inverses.

- An inverse function "reverses" the effects of the original function.

Theorem 6.2.12. *Suppose that $f : A \to B$ is one-to-one and onto. Then there is a function $f^{-1} : B \to A$ that satisfies*

$$f^{-1}(b) = a \text{ iff } f(a) = b \tag{6.12}$$

for all $b \in B$ and $a \in A$.

- Only bijections have an inverse!
- ...But sometimes we can "cheat" with surjective functions
  - Example: Moore Penrose pseudoinverse

Fields

- Set of elements on which addition, subtraction, multiplication, and division are defined
- Examples: $\mathbb{R}$, $\mathbb{C}$
- Two binary operations: addition and multiplication
- Field axioms:
  - Associativity
  - Commutativity
  - Additive and multiplicative identities
  - Additive inverse
  - Multiplicative inverse
  - Distributivity

Vector Space over a Field

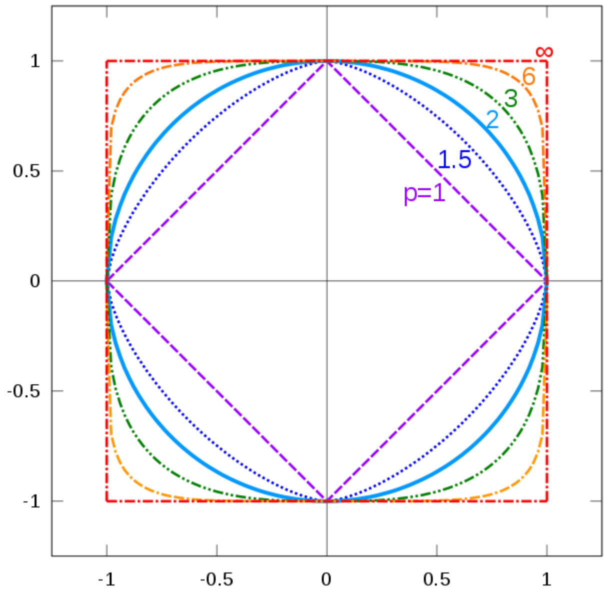
- Set of objects (vectors) with
  - Vector addition
  - Scalar multiplication
- Examples:
  - $\mathbb{R}^n, \mathbb{C}^n$
- Axioms:
  - Associativity
  - Commutativity
  - Identity
  - Inverse
  - Scalar associativity
  - Scalar identity
  - Scalar distributivity
  - Vector distributivity

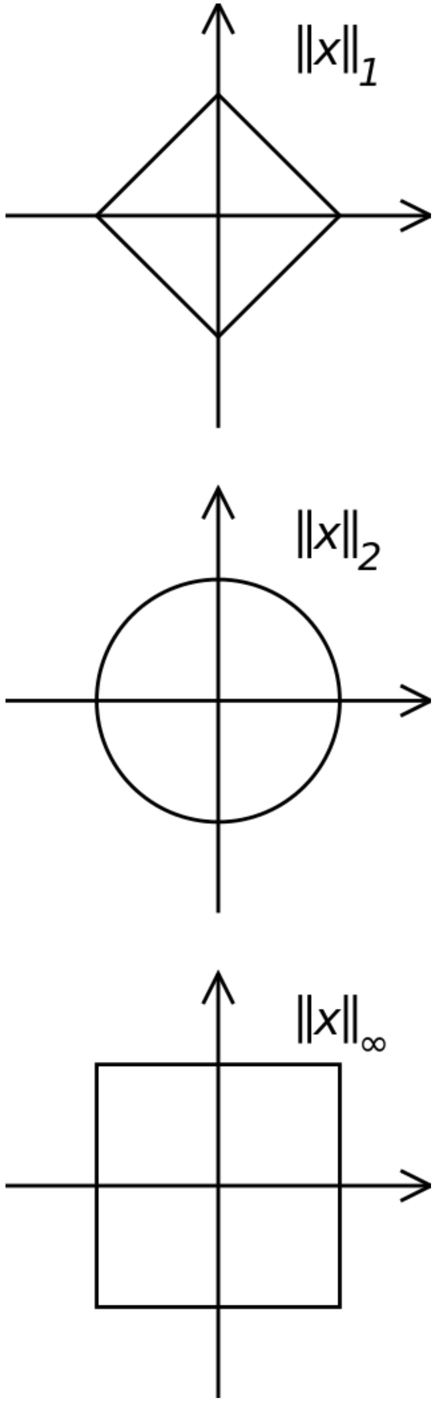
Norms

- **A *norm* is a function mapping a vector space to real scalars $f: V \to \mathbb{R}$ that satisfies the following properties:**
  - $||x|| \geq 0$ **for all** $x \in V$, **and** $||x|| = 0 \Leftrightarrow x = 0$.
  - $||x + y|| \leq ||x|| + ||y||$ **for all** $x, y \in V$ **(Triangle inequality)**
  - $||\alpha x|| = |\alpha| \; ||x||$ **for all** $\alpha \in \mathbb{R}$ **and** $x \in V$
- **Norms quantify *distance* between vectors.**

Types of Vector Norms

- **Euclidean:** $||x||_2 = \left(\sum_i x_i^2\right)^{\frac{1}{2}}$
- **1-norm:** $||x||_1 = \sum_i |x_i|$
- **∞-norm:** $||x||_\infty = \max(|x_1|, |x_2|, \ldots)$
- ***p*-norm:** $||x||_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$





Equivalence of p-Norms

- **All p-norms are equivalent in the following sense:**

If $||\cdot||_a$ and $||\cdot||_b$ are two different p-norms, there exist positive constants c_1, c_2 such that

$$c_1||x||_a \leq ||x||_b \leq c_2||x||_a$$

Matrix Norms

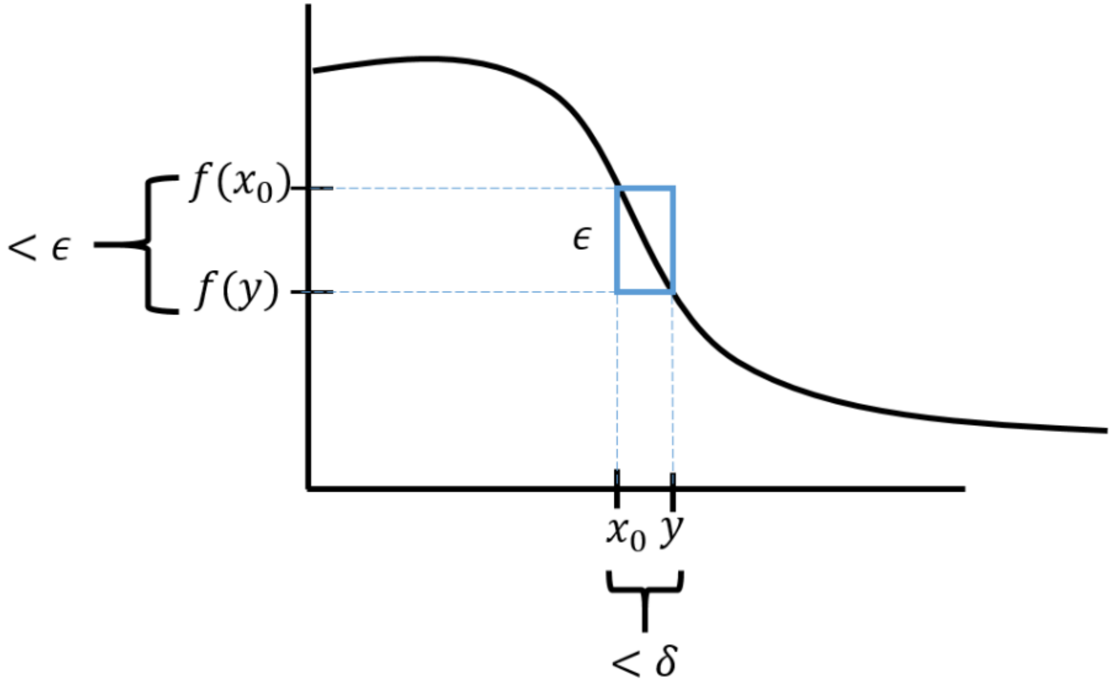
- **Matrices can represent two things:**
  - ***Linear functions*** $A: \mathbb{R}^n \to \mathbb{R}^n$
  - **Multidimensional arrays**
- **Matrix norms induced by vector *p*-norms:** $\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$
  - **Matrix 2-norm:** $\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)} = \sigma_{\max}(A).$
  - **Matrix 1-norm:** $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|,$
  - **Matrix ∞-norm:** $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|,$
- **There are entry-wise norms (Frobenius, $L_{p,q}$)**

https://en.wikipedia.org/wiki/Matrix_norm

Continuity

- **A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a point x if for all $\epsilon > 0$ there exists $\delta > 0$ such that**

$$||x - y|| < \delta \Longrightarrow ||f(x) - f(y)|| < \epsilon$$



Continuity (cont.)

- **A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous on a set S if it is continuous at every point $x \in S$**

- **A function is *uniformly continuous* on S if, given $\epsilon > 0$ there is $\delta > 0$ (dependent only on ϵ) such that**

  $||x - y|| < \delta \Longrightarrow ||f(x) - f(y)|| < \epsilon$ holds for all $x, y \in S$

Differentiability

- **A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at x if the following limit exists:**

$$f'(x) = \lim_{h \rightarrow 0} (f(x+h) - f(x))/h$$

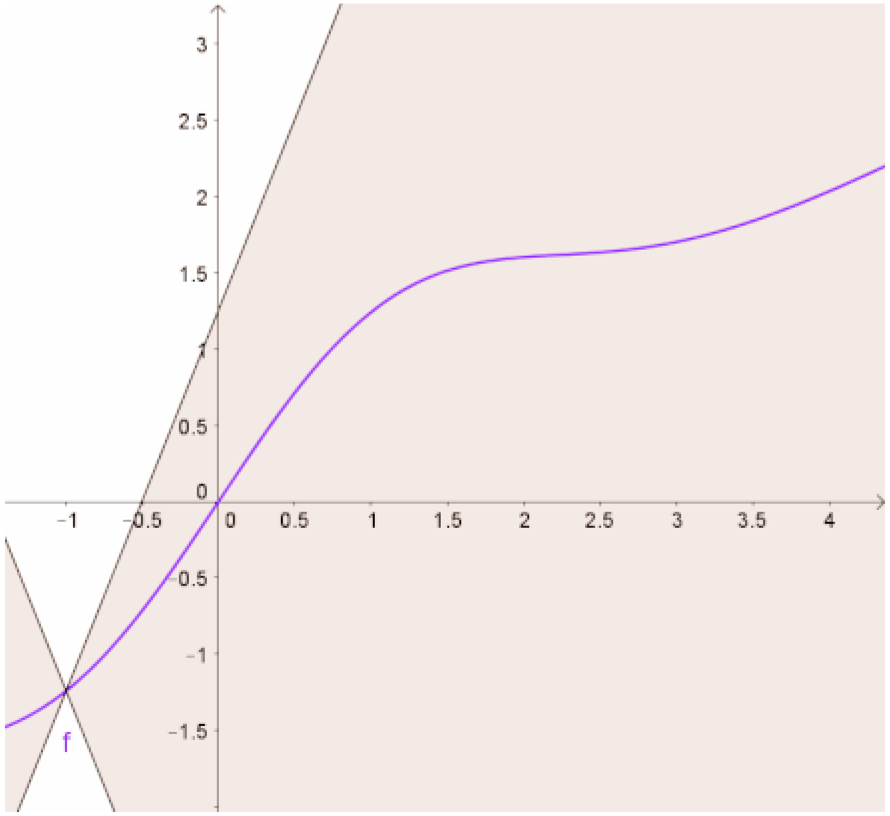
- **A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *continuously differentiable* at a point x_0 if the partial derivatives $\left(\frac{\partial f_i}{\partial x_j}\right)$ exist and are continuous at x_0 for all i, j.**

Lipschitz Continuity

- A function $f(t, x)$ is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0, r)$ and a positive constant L such that

$$||f(t, x) - f(t, y)|| \leq L||x - y||$$

for all t and for all $x, y \in N(x_0, r)$



https://en.wikipedia.org/wiki/Lipschitz_continuity

Lipschitz Continuity (cont.)

$$||f(t,x) - f(t,y)|| \leq L||x - y||$$

- A function $f(t,x)$ is locally Lipschitz in x on an open and connected set $D \subset \mathbb{R}^n$ if it is locally Lipschitz at every point $x_0 \in D$

- A function is Lipschitz in x on a set W if the condition holds for all $x \in W$ with the same Lipschitz constant L

Lipschitz Continuity (cont.)

$$||f(t,x) - f(t,y)|| \leq L||x - y||$$

- **A function is globally Lipschitz if it is Lipschitz on $\mathbb{R}^n$.**

What’s the difference??

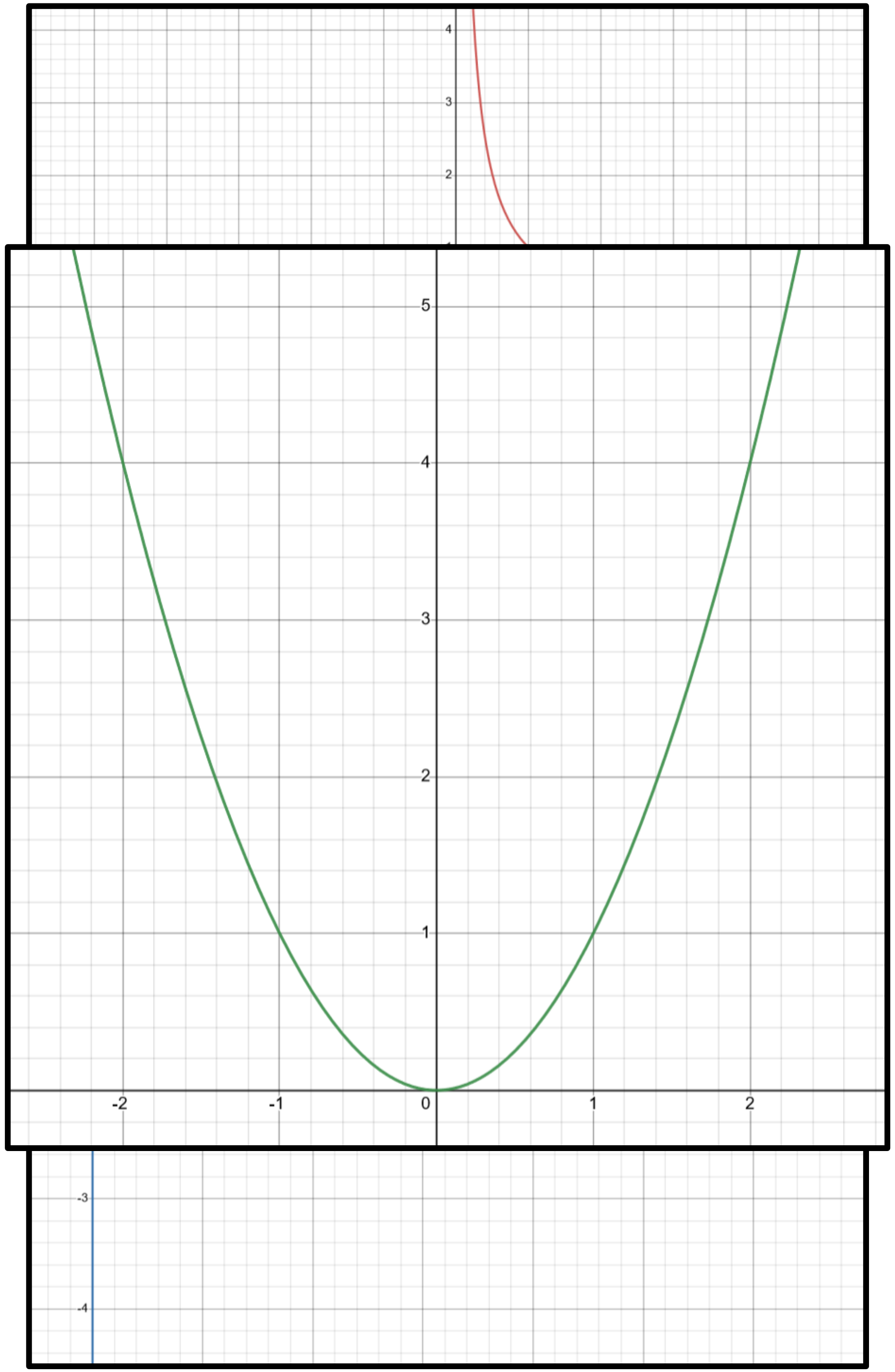
Globally Lipschitz $\xrightarrow{\text{Implies}}$ Lipschitz on $W \subset \mathbb{R}^n$ $\xrightarrow{\text{Implies}}$ Locally Lipschitz on $W \subset \mathbb{R}^n$ $\xrightarrow{\text{Implies}}$ Locally Lipschitz at $x_0 \in W$

- A locally Lipschitz function f on a domain D is *not* necessarily Lipschitz on D.
  - There may not exist an L that holds for all points in an open domain

- BUT, f is Lipschitz on every *compact subset* $S \subset D$
  - If we look at a closed, bounded set, we can simply set $L = max_x(L(x))$

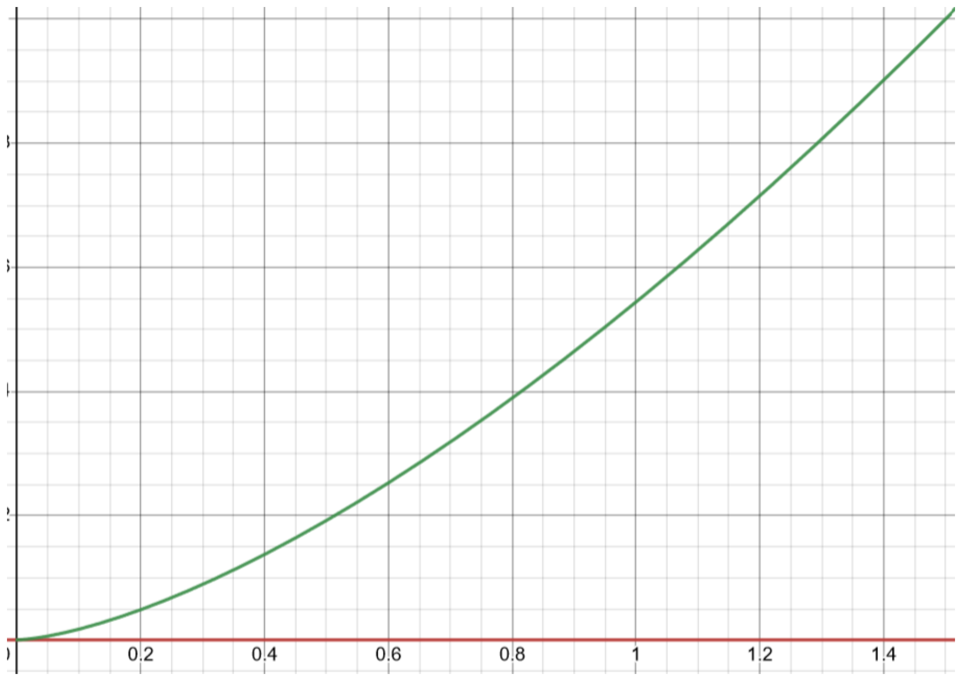
Examples

- $f(x) = \frac{1}{x}$
  - On $\mathbb{R}$?
  - On $(0, \infty)$?
  - On $[1, 2]$?
- $f(x) = \log(x)$
  - On $(0, \infty)$?
  - On $(0, 1)$?
  - On $(1, 2)$?
- $f(x) = x^2$
  - On $\mathbb{R}$?
  - On $(-1, 1)$?



Why do we care?

- Existence and uniqueness of solutions.
- Given an equation $\dot{x} = f(t, x)$…
  - How do we tell if a solution exists?
  - If it exists, is it the *only* solution?
- Example:
  - $\dot{x} = x^{\frac{1}{3}}$
  - Two solutions:
    - $x = 0$
    - $x = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$
- How do we avoid this ambiguity?



Existence and Uniqueness Theorems

Lemma 1.1

Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$

https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/Lect_1.pdf

Example

Example 1.3

$$\dot{x} = -x^2$$

$f(x) = -x^2$ is locally Lipschitz for all x

$$x(0) = -1 \quad \Rightarrow \quad x(t) = \frac{1}{(t-1)}$$

$$x(t) \to -\infty \quad \text{as} \quad t \to 1$$

The solution has a *finite escape time* at $t = 1$

https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/Lect_1.pdf

Existence and Uniqueness Theorems (cont.)

Lemma 1.2

Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$

https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/Lect_1.pdf

Existence and Uniqueness Theorems (cont.)

Lemma 1.3

Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, and suppose that every solution of

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

with $x_0 \in W$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$

https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/Lect_1.pdf

Example

Example 1.4

$$\dot{x} = -x^3 = f(x)$$

$f(x)$ is locally Lipschitz on R, but not globally Lipschitz because $f'(x) = -3x^2$ is not globally bounded

If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive

Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in R \mid |x| \leq |a|\}$

Thus, the equation has a unique solution for all $t \geq 0$

https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/Lect_1.pdf

Homeomorphism

- **Suppose we have a bijective function f.**
  - **This implies that there exists an inverse function f^{-1}**

- **A function $f: X \to Y$ is a homeomorphism if:**
  - **f is a bijection**
  - **f is continuous**
  - **The inverse f^{-1} is also continuous**

Diffeomorphisms

- **Homeomorphisms, but with derivatives!**

- **The function f is a diffeomorphism if both f and f^{-1} are continuously differentiable.**
- **$f: D \to C$ is a local diffeomorphism at x_0 if there exists a neighborhood N of x_0 such that $f: N \to C$ is a diffeomorphism**
- **f is a global diffeomorphism if it is a diffeomorphism on $\mathbb{R}$^n and $f(\mathbb{R}^n) = \mathbb{R}^n$.**

Why do we care???

- Sometimes we want to change the state variables.

$$z = T(x)$$

Linear: $z = Px$

  - This can make analysis easier.
  - This can make control easier.

- If we change variables…
  - We need to be able to change back (bijection + inverse)
  - The derivatives of the new variables z should be continuous (continuously differentiable)

Easier way to check for local diffeomorphisms

Lemma 1.4

The continuously differentiable map $z = T(x)$ is a local diffeomorphism at x_0 if the Jacobian matrix $[\partial T/\partial x]$ is nonsingular at x_0. It is a global diffeomorphism if and only if $[\partial T/\partial x]$ is nonsingular for all $x \in R^n$ and T is proper; that is, $\lim_{\|x\| \to \infty} \|T(x)\| = \infty$

Example

- Let $z = T(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$

- Is $T(\cdot)$ a global diffeomorphism?
- Is it proper?

BYU

Nonlinear Systems Theory

Jan 18, 2024

Overview

- Two-Dimensional Systems
  - Why we care
  - Phase portraits
  - Qualitative Behavior

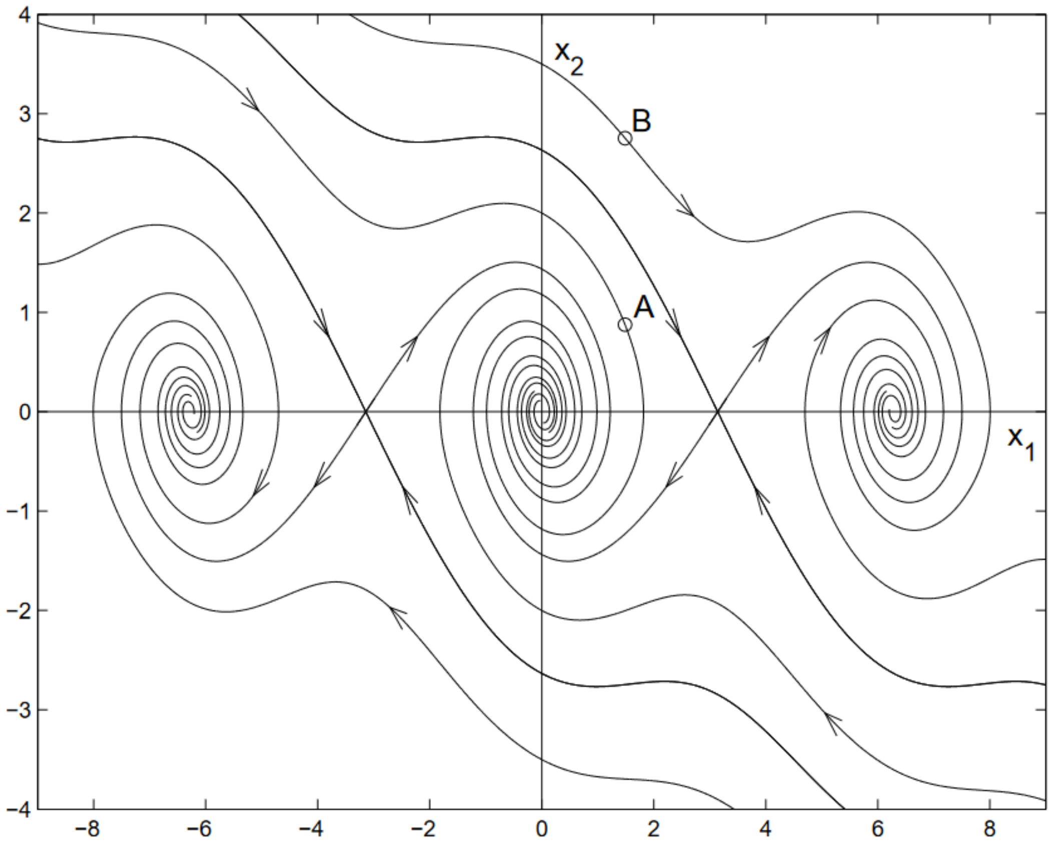
Credit Where Credit is Due

- Sources:
  - Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html

Why do we care about 2D systems?

- **How do you visualize *n*-dimensional space?**
  - **You don't.**

- **BUT, 2D systems can be easily visualized**
- **Studying properties of 2D systems gives intuition for higher dimensions**



2D System Model

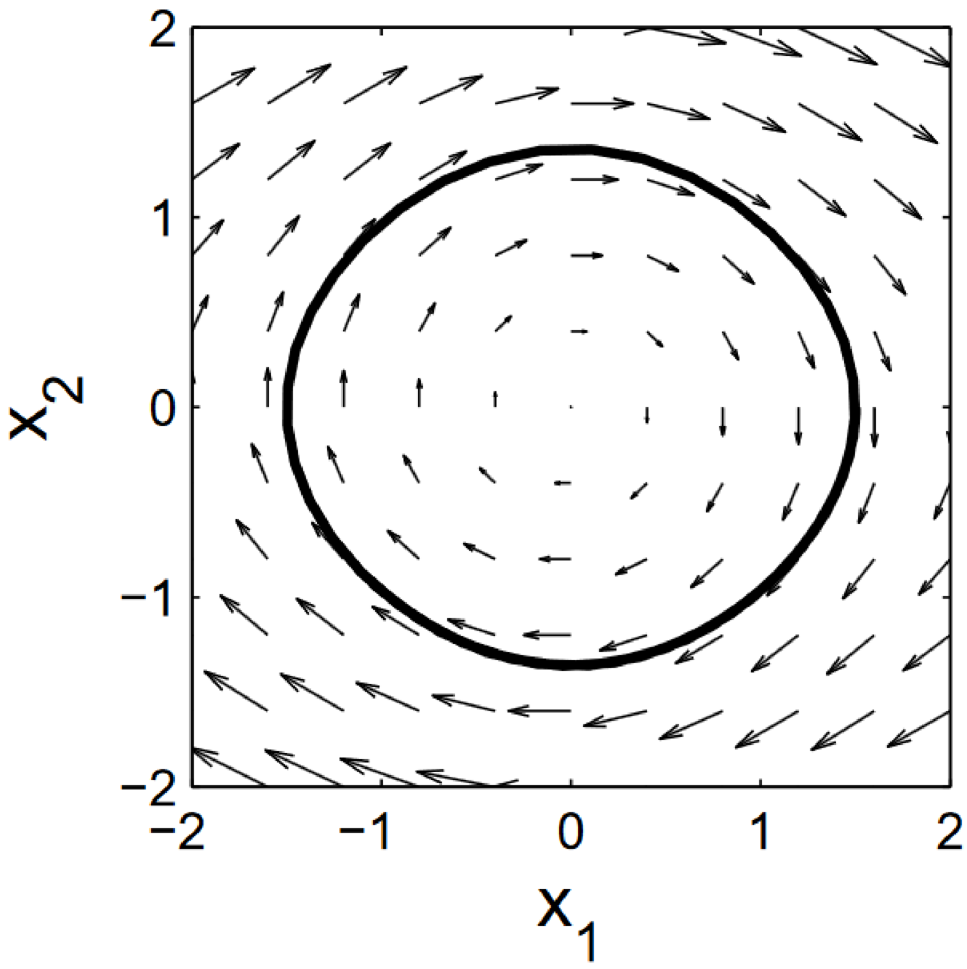
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

- Vector notation: $\dot{x} = f(x)$
- Assumptions:
  - f is locally Lipschitz on the domain of interest
- Solution: $x(t) = [x_1(t), x_2(t)]^T$
- Phase plane: The x_1, x_2 plane

Vector Fields

- **Vector field on phase plane: Assignment of vector $f(x)$ to every point x**

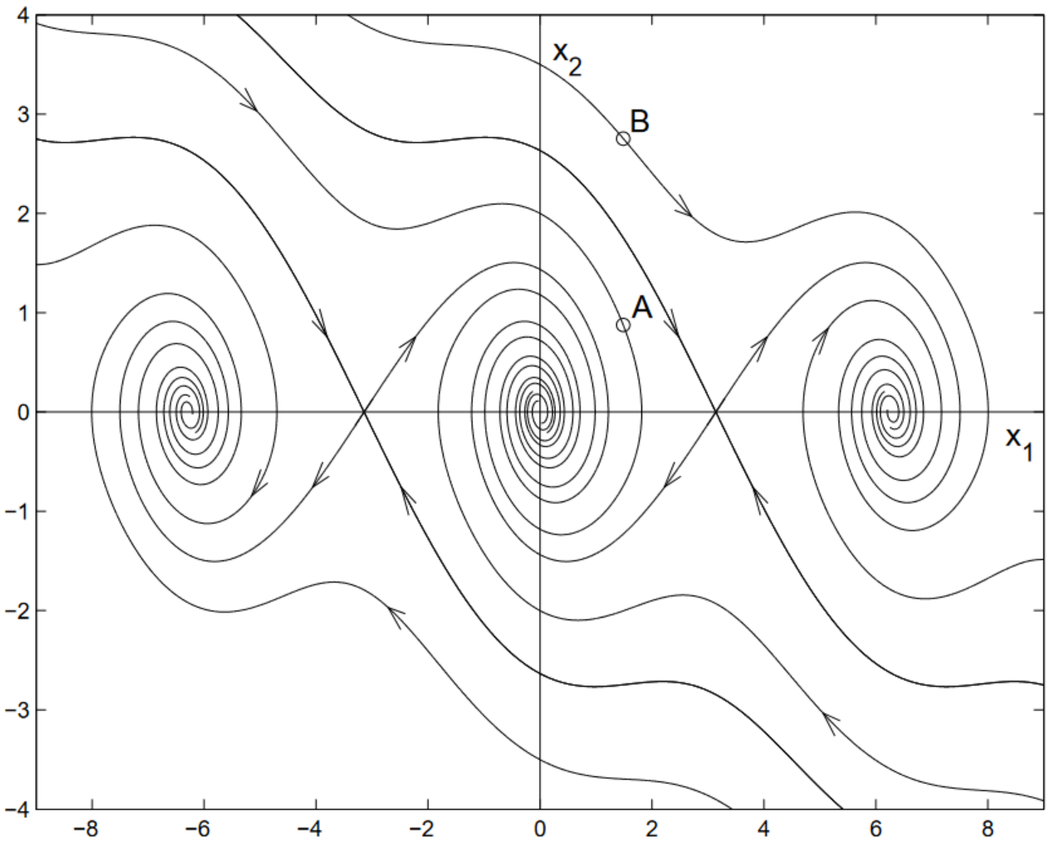
- **Vector field diagram: Plot of the vector field at every point in grid**



$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1$$

Phase Portrait

- **Plot of the family of all trajectories for a system**
  - **Typically approximated by plotting a large number of trajectories starting from different initial conditions**



Linear Systems: Qualitative Behavior

- **Consider $\dot{x} = Ax$**
- **Solutions:**
  - $x(t) = e^{At}x_0$
  - **Change of basis:** $x(t) = Ve^{(\Lambda t)}\,V^{-1}x_0$
- **We focus on when eigenvalues of A are *distinct***
- **Jordan form has two possibilities:**

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

Eigenvalues are All You Need

- **We have two separate systems:**

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2$$

$$z_1(t) = z_{10} e^{\lambda_1 t}, \qquad z_2(t) = z_{20} e^{\lambda_2 t}$$

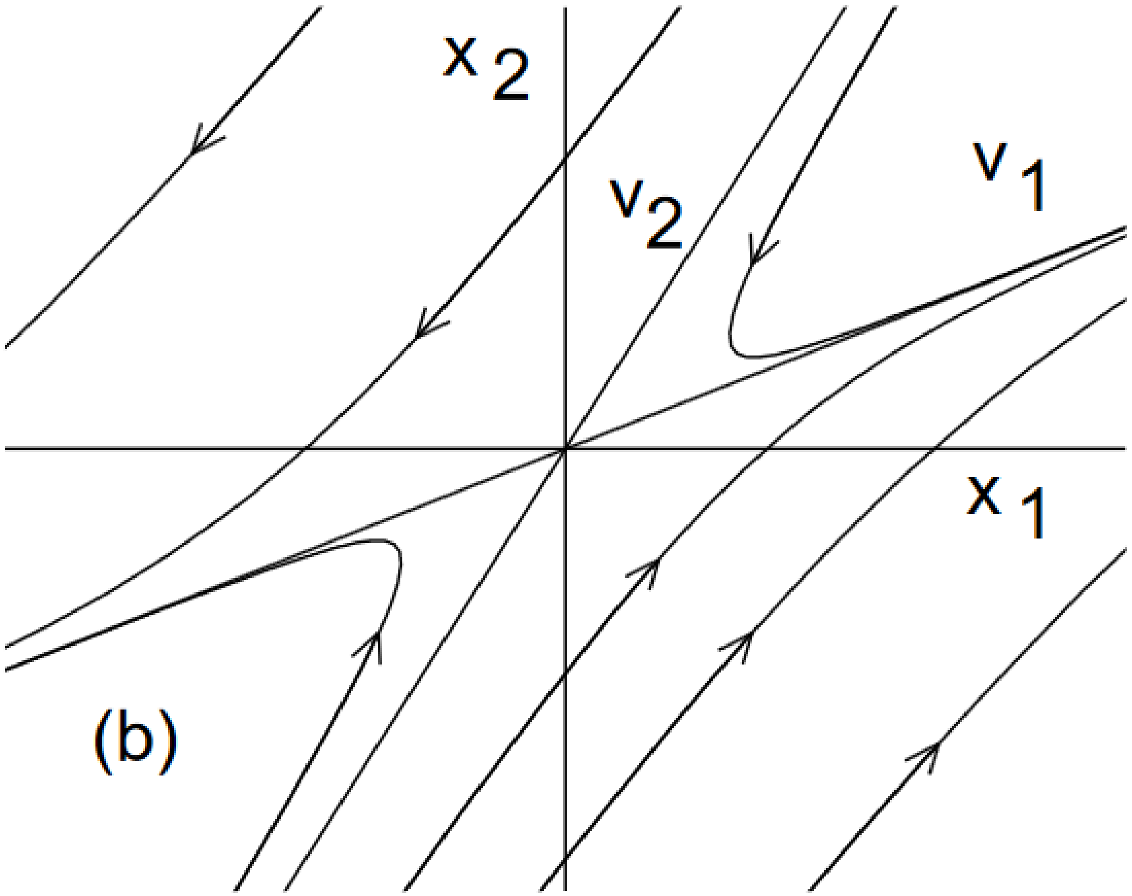
$$\boxed{z_2 = c z_1^{\lambda_2/\lambda_1}, \qquad c = z_{20}/(z_{10})^{\lambda_2/\lambda_1}}$$

Rewriting z_2 as a function of z_1

- **The shape of the phase portrait is determined by the eigenvalues λ_1, λ_2**

Case 1: Both Eigenvalues are Real

- Both eigenvalues are **negative**: equilibrium is **stable node**
- Both eigenvalues are **positive**: equilibrium is **unstable node**
- **One positive, one negative** eigenvalue: equilibrium is **saddle point**



Case 2: Both Eigenvalues are Complex

- **In this case, $\lambda_k = \alpha_k \pm i\beta_k$ for $k = \{1, 2\}$**
- **Our system:** $\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$
- **Polar coordinates!**

$$r = \sqrt{z_1^2 + z_2^2}, \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

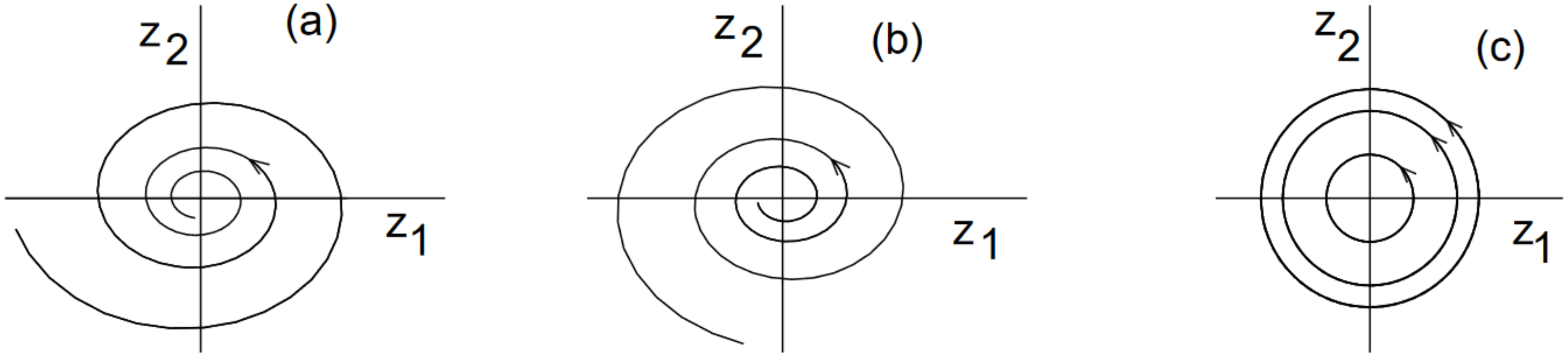
$$r(t) = r_0 e^{\alpha t} \quad \text{and} \quad \theta(t) = \theta_0 + \beta t$$

- **Behavior depends on value of α:** $\alpha < 0 \quad \Rightarrow \quad r(t) \to 0$ as $t \to \infty$

$$\alpha > 0 \quad \Rightarrow \quad r(t) \to \infty \text{ as } t \to \infty$$

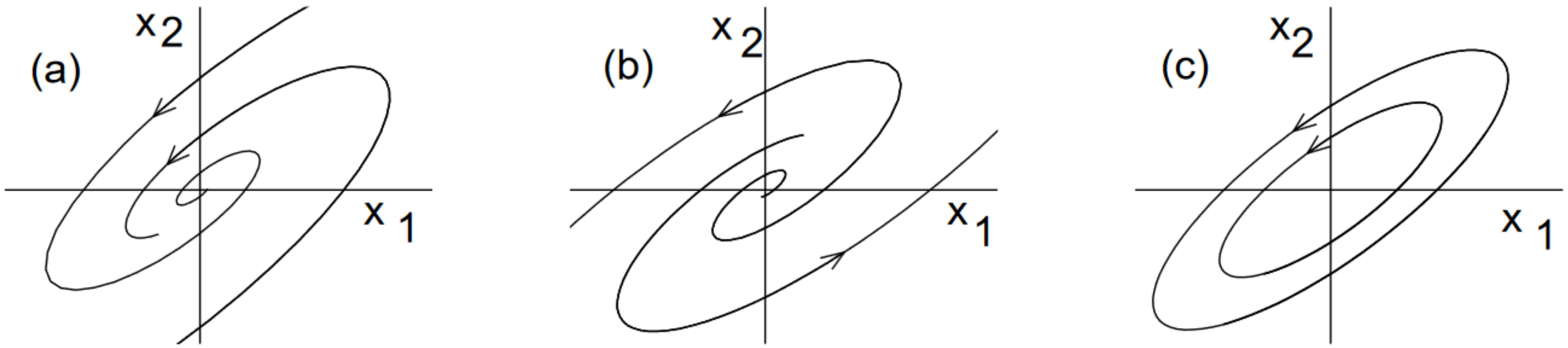
$$\alpha = 0 \quad \Rightarrow \quad r(t) \equiv r_0 \ \forall\ t$$

Stable / Unstable Foci, Centers



$\alpha < 0$ | $\alpha > 0$ | $\alpha = 0$

Stable Focus | Unstable Focus | Center



What if our Matrix is wrong?

- We may not have exact knowledge of A
- Perturbations: $A + \Delta A$
- Fact: The eigenvalues of a matrix depend continuously on its parameters
- Any eigenvalue in open right hand plane (RHP) will remain in RHP for sufficiently small perturbations ΔA
- Similarly for open LHP
- What about on the imaginary axis?
  - ¯_(ツ)_/¯
  - Perturbations may push it either way
  - Example: $\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$
  - Equilibrium is stable focus when $\mu < 0$, unstable focus when $\mu > 0$, center when $\mu = 0$.

What about Nonlinear Systems?

- We can study the local behavior around an equilibrium point
- How? By using linearization
- Why does this work?
  - Let $[p_1, p_2]^T$ be an equilibrium point of 2D system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$
- Expand the Taylor series:

$$\begin{aligned}\dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.} \\ \dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.}\end{aligned}$$

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$a_{21} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{22} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

Behavior Around Equilibria

$$\begin{aligned}\dot{x}_1 &= f_1(p_1,p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.}\\ \dot{x}_2 &= f_2(p_1,p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.}\end{aligned}$$

$$a_{11} = \left.\frac{\partial f_1(x_1,x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{12} = \left.\frac{\partial f_1(x_1,x_2)}{\partial x_2}\right|_{x=p}$$

$$a_{21} = \left.\frac{\partial f_2(x_1,x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{22} = \left.\frac{\partial f_2(x_1,x_2)}{\partial x_2}\right|_{x=p}$$

- **Remember that for equilibrium points,** $f_1(p_1,p_2) = f_2(p_1,p_2) = 0$
- **Define** $y_1 = x_1 - p_1 \qquad y_2 = x_2 - p_2$
- **Then we have:** $$\begin{aligned}\dot{y}_1 &= \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + \text{H.O.T.}\\ \dot{y}_2 &= \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + \text{H.O.T.}\end{aligned}$$

Behavior Around Equilibria (cont.)

- **Linearized system:**

$$\dot{y} \approx Ay$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x=p} = \frac{\partial f}{\partial x}\Bigg|_{x=p}$$

- **We can determine the local behavior of nonlinear system around $[p_1, p_2]^T$ by inspecting the eigenvalues of the linearized system matrix A!**

Inspecting Nonlinear Behavior

- **Let $\dot{x} = f(x)$, and let $\dot{y} \approx Ay$ be the linearized model around $[p_1, p_2]^T$.**
  - **If the origin of $\dot{y} \approx Ay$ is a stable (unstable) node with distinct eigenvalues, the trajectories of the nonlinear system will behave like a stable (unstable) node in a neighborhood around $[p_1, p_2]^T$.**
  - **Similarly, if the origin of $\dot{y} \approx Ay$ is a stable (unstable) focus with distinct eigenvalues, the trajectories of the nonlinear system will behave like a stable (unstable) focus**
  - **Similarly, if the origin of $\dot{y} \approx Ay$ is a saddle, the trajectories of the nonlinear system will behave like a saddle**

What about eigenvalues on the imaginary axis?

- **If the Jacobian is a center, then we cannot draw any conclusions about the (qualitative) system behavior.**

- **The behavior of the linearized system may be quite different than the behavior of the nonlinear system.**

Backup Slides

BYU

NONLINEAR SYSTEMS THEORY

Jan 23, 2024

Overview

- Limit Cycles / Oscillation
- Introduction to Lyapunov Stability

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Oscillation

- Systems oscillate when they have nontrivial periodic solutions

$$x(t+T) = x(t), \quad \forall\, t \geq 0$$

- Example: Linear (Harmonic) Oscillator

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$$

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \qquad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

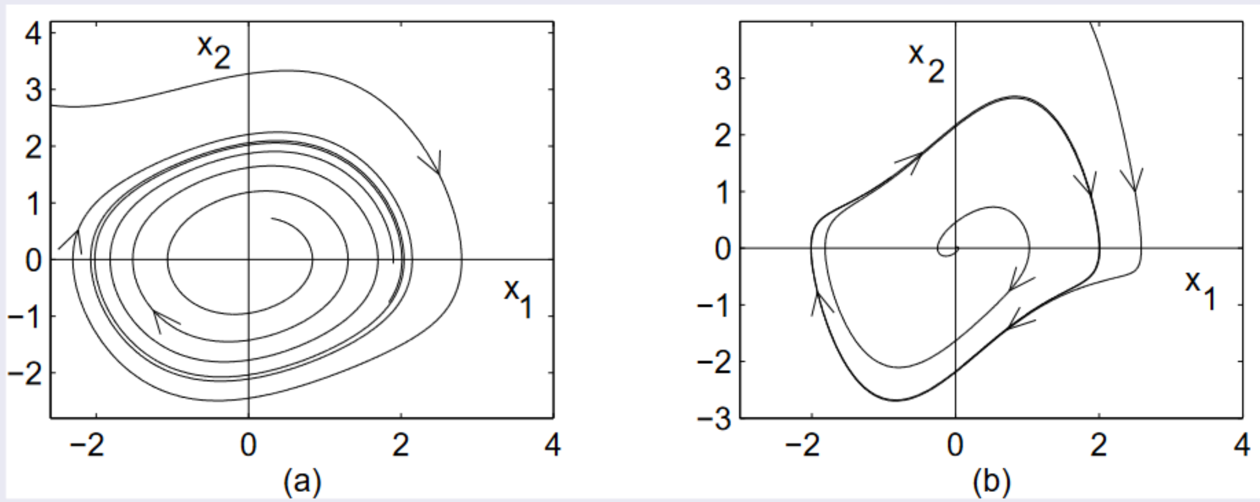
$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \qquad \theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right]$$

Do Linear Oscillators Occur in the Wild?

- No, not really.
- Why?
  - Infinitely small perturbations may change the equilibrium behavior to stable focus or unstable focus
  - Amplitude of oscillation depends on initial conditions

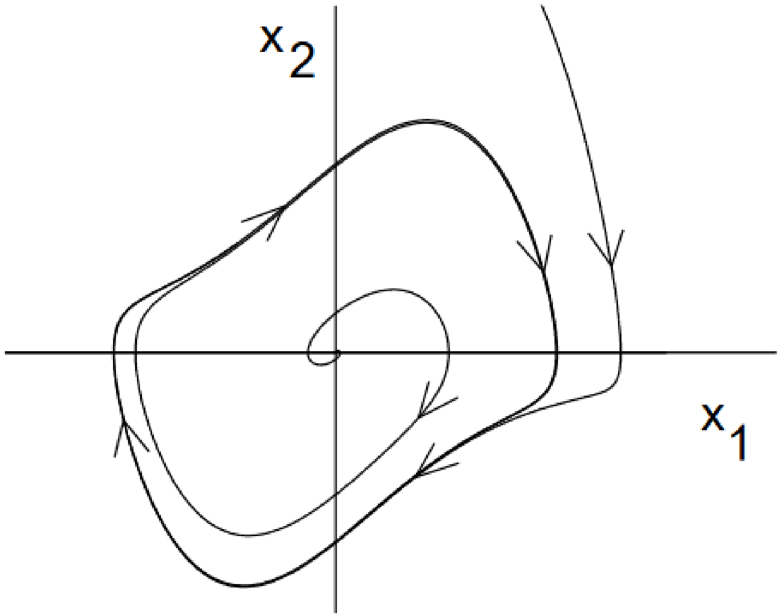
Nonlinear Oscillators

- Structurally stable (perturbations don't change qualitative behavior)
- Amplitude independent of initial conditions
- Example: Van der Pol Oscillator

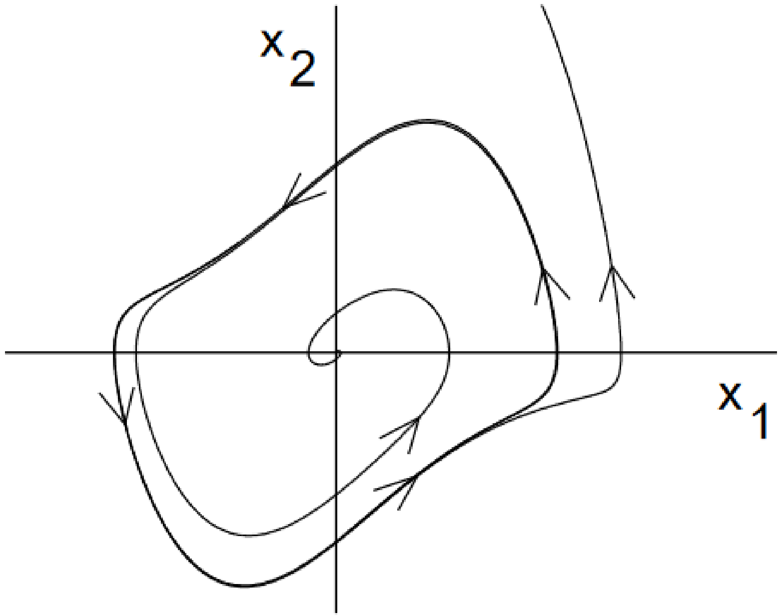


$\varepsilon = 0.2$ $\varepsilon = 1$

Stable vs. Unstable Limit Cycle



(a)



(b)

Stable Limit Cycle

Unstable Limit Cycle

Stability in Nonlinear Systems

- Fundamental concept when studying nonlinear systems!
- Multiple types of stability
  - Stability of Equilibrium points
  - Input-to-State stability
  - Input-Output stability

- Heads up!
  - You need to know stability theory backwards and forwards when doing competency exams

Basic Concepts

- **Start with time-invariant systems:** $\dot{x} = f(x)$
  - f **is locally Lipschitz over a domain** $D \subset \mathbb{R}^n$
- **We consider Equilibrium point** $\bar{x} = 0$
  - $f(\bar{x}) = 0$
- **What if we want to study different equilibrium point?**
  - **Change of variables:** $y = x - \bar{x}$
  - $\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) = g(y)$

Definition of Stability

Definition 3.1

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- stable if for each $\varepsilon > 0$ there is $\delta > 0$ (dependent on ε) such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq 0$$

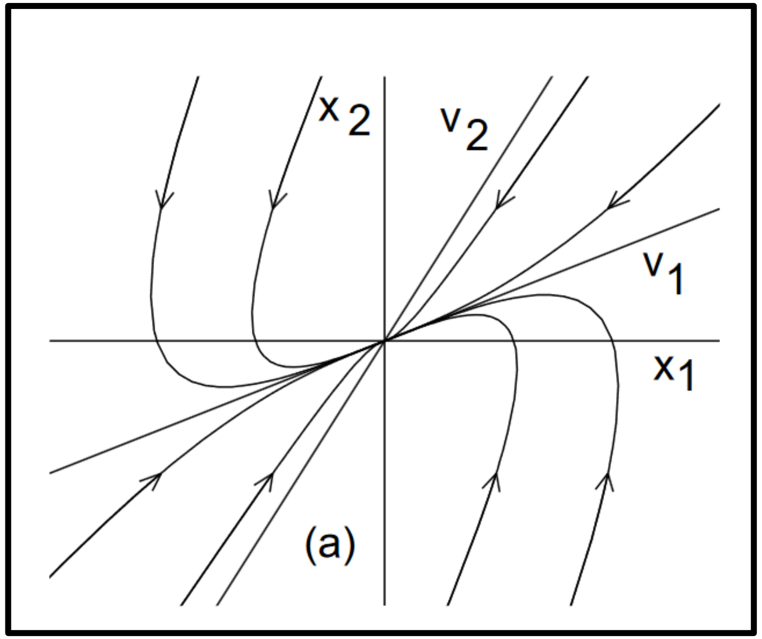
- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

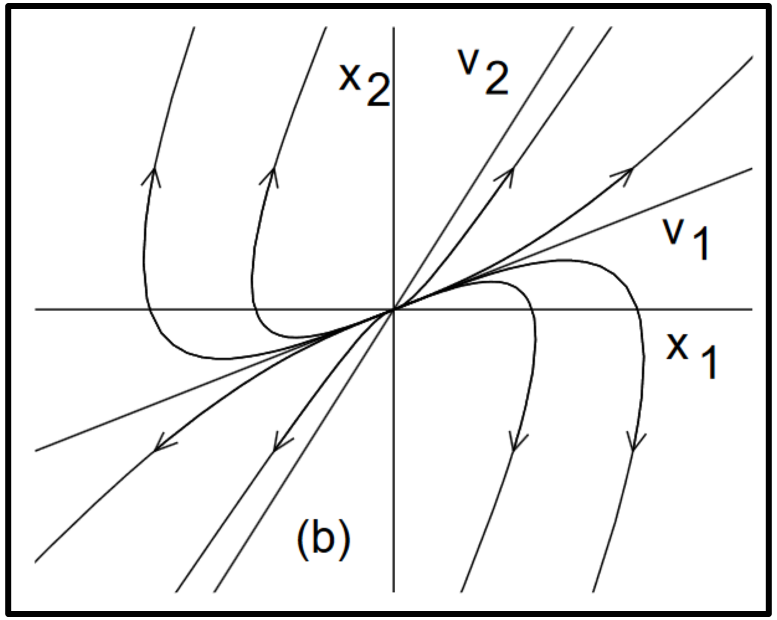
$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

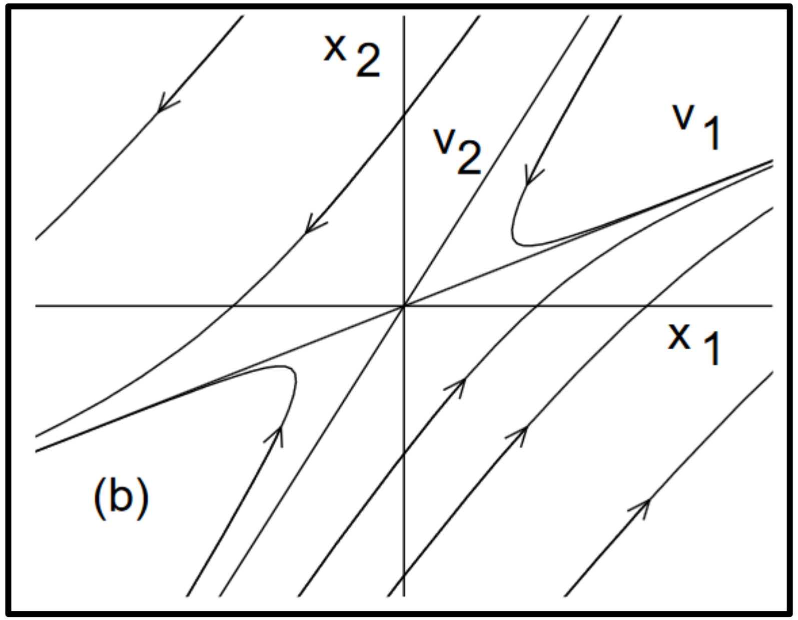
Examples: 2D Systems

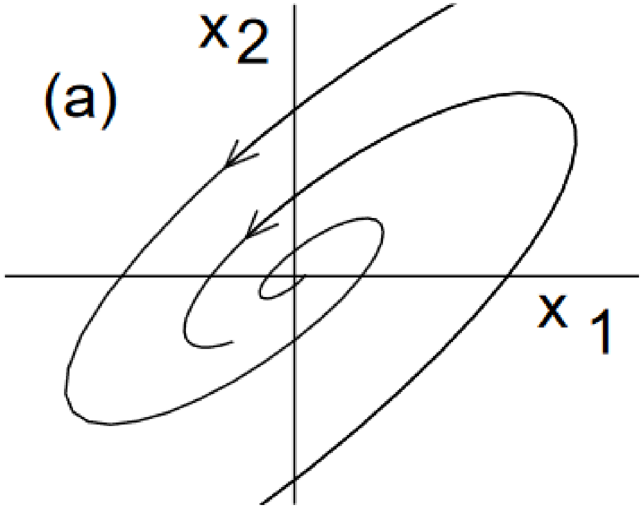
- **What kind of stability does each qualitative behavior exhibit?**

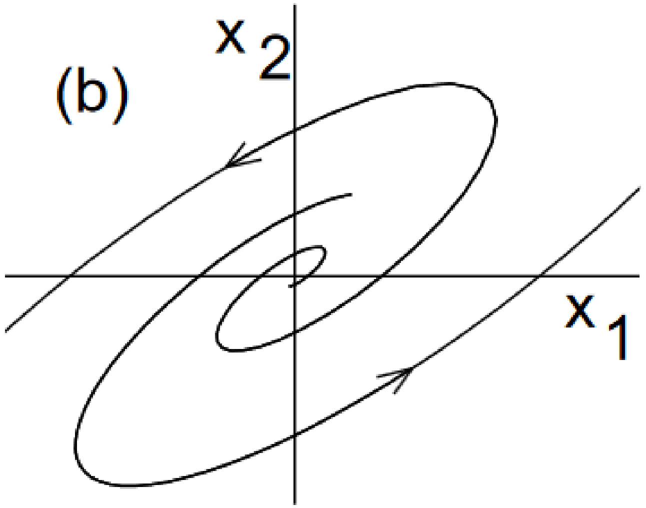
Type of equilibrium point	Stability Property
Center	
Stable Node	
Stable Focus	
Unstable Node	
Unstable Focus	
Saddle	

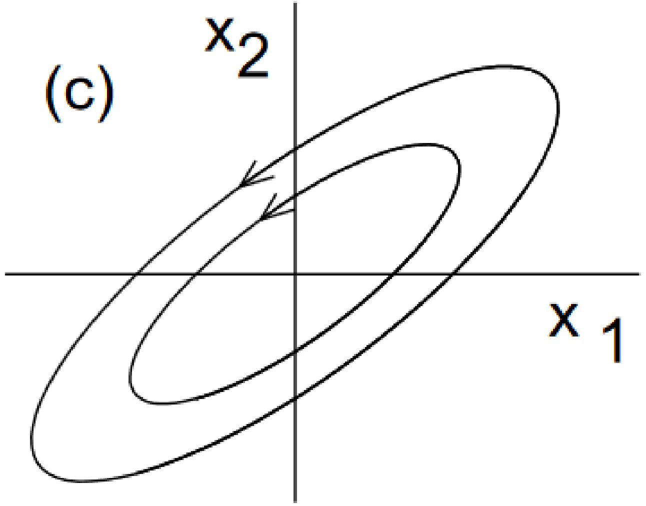

x_2
v_2
v_1
x_1
(a)


x_2
v_2
v_1
x_1
(b)


x_2
v_2
v_1
x_1
(b)


x_2
(a)
x_1


x_2
(b)
x_1


x_2
(c)
x_1

Regions of Attraction

- System: $\dot{x} = f(x)$
- Origin: asymptotically stable equilibrium point
- f is Locally Lipschitz on $D \subset \mathbb{R}^n$, $0 \in D$

- Region of attraction of the origin: set of all points $x_0 \in D$ such that $x(t), x(0) = 0$ is defined for all t and converges to origin as $t \to \infty$

- Origin is globally asymptotically stable if region of attraction is $\mathbb{R}^n$

Backup Slides

Nonlinear Systems Theory

Jan 25, 2024

Overview

- Logistics
  - HW Rubric
  - Quizzes
- Stability (cont.)
  - 1D Systems
  - Linear Time-Invariant Systems
  - Exponential Stability
  - Lyapunov’s Indirect Method (Linearization)
  - Lyapunov’s Direct Method

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Logistics

- HW Rubric is on Learning Suite, *Homework Solutions* Tab
- Quiz 3 begins today
  - Learning Suite error, so everyone gets full credit for Quizzes 1-2.

Definition of Stability

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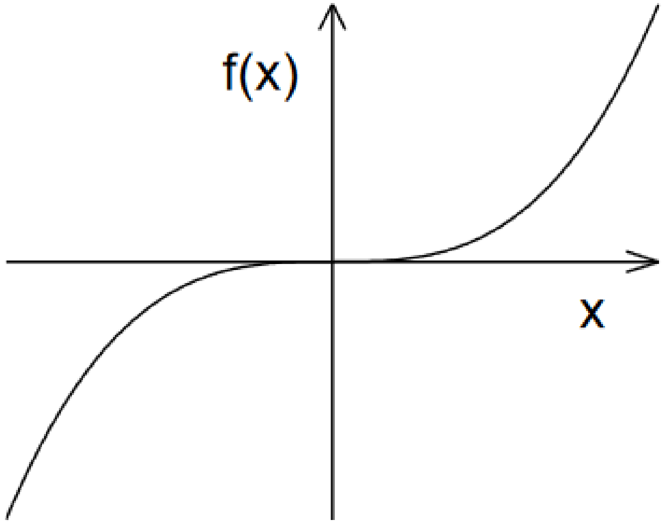
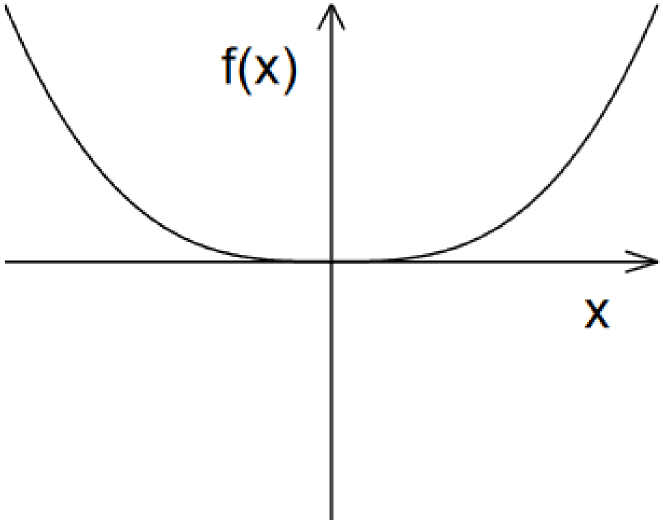
- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

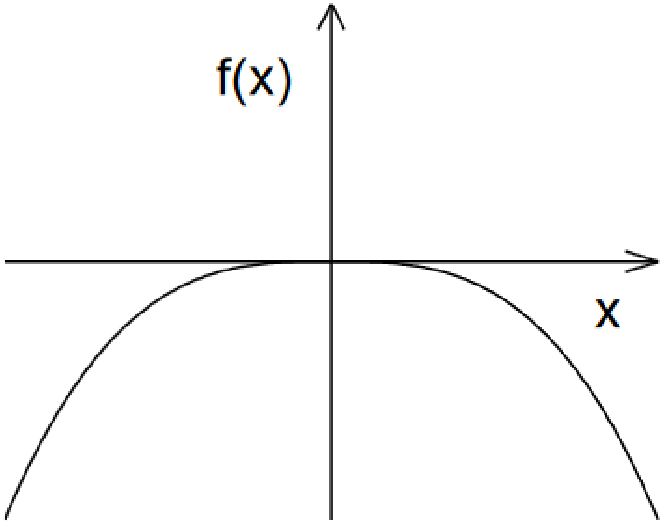
$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

1D Systems

- $\dot{x} = f(x), x \in \mathbb{R}$
  - Equilibrium point at $x = 0$
- We can determine stability by looking at sign of $xf(x)$ in a neighborhood of the origin.
- $xf(x) > 0$: Origin is **unstable**.

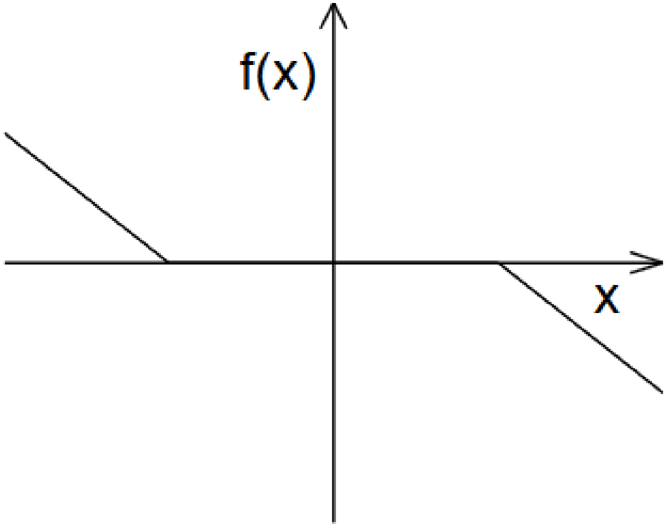


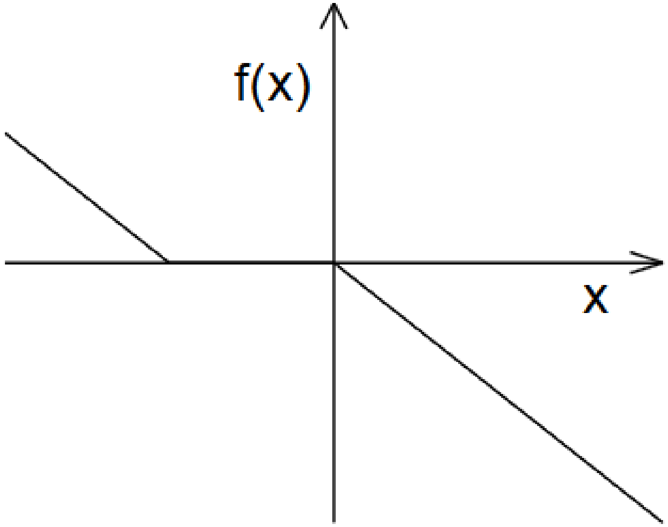


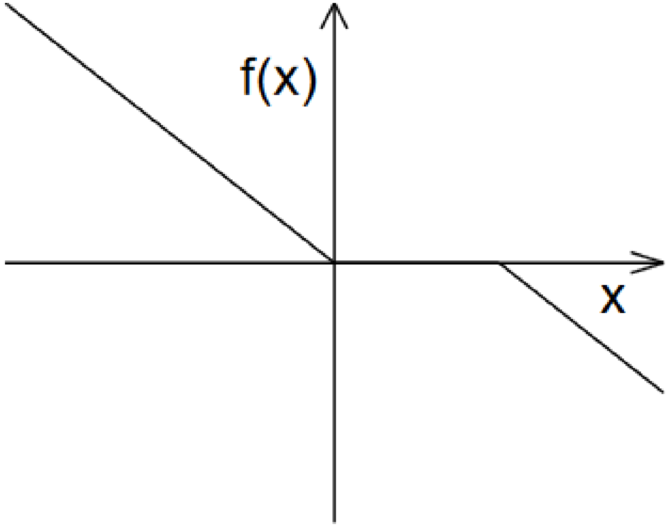


1D Systems (cont.)

- $\dot{x} = f(x), x \in \mathbb{R}$
  - **Equilibrium point at $x = 0$**
- $xf(x) \leq 0$**: Origin is stable.**

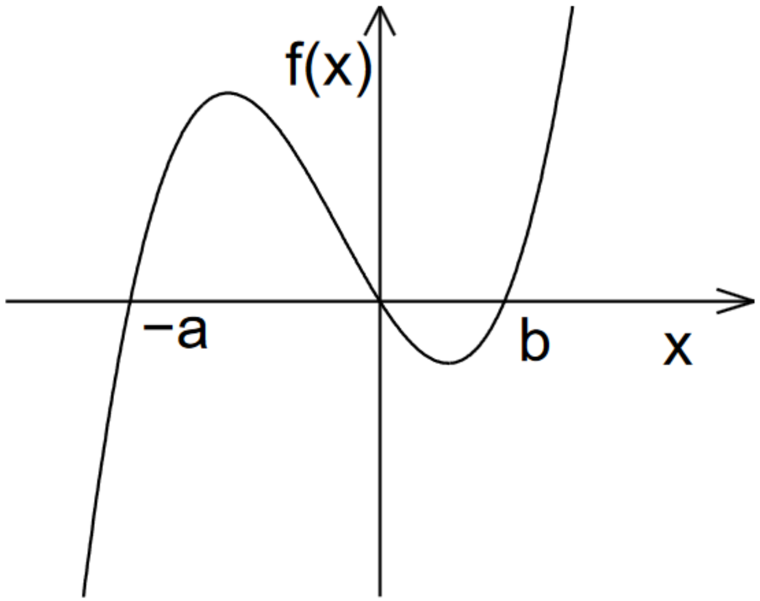


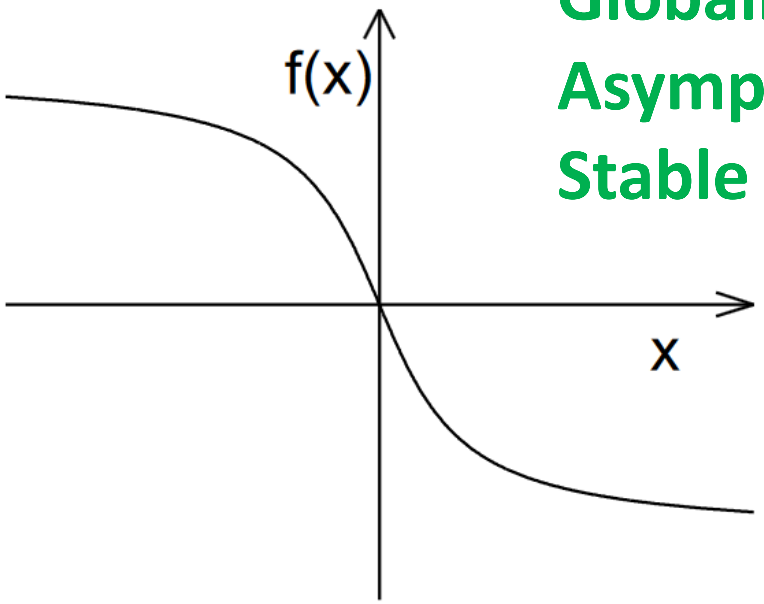




1D Systems (cont.)

- $\dot{x} = f(x), x \in \mathbb{R}$
  - **Equilibrium point at $x = 0$**
- $xf(x) < 0$: **Origin is asymptotically stable.**





Globally Asymptotically Stable

Try it out!

- **Find the equilibrium, determine the stability**
  - $\dot{x} = x \sin(x)$
  - $\dot{x} = -x \cos(x)$
  - $\dot{x} = x \left(\cos\left(\frac{1}{x}\right) - \epsilon\right)$ **(for all possible ϵ values)**
  - $\dot{x} = -|x^5|$
  - $\dot{x} = -xe^x$
  - $\dot{x} = -\log(x + 1)$

Regions of Attraction

- **System:** $\dot{x} = f(x)$
- **Origin: asymptotically stable equilibrium point**
- f **is Locally Lipschitz on** $D \subset \mathbb{R}^n$, $0 \in D$

- **Region of attraction of the origin: set of all points** $x_0 \in D$ **such that** $x(t), x(0) = 0$ **is defined for all** t **and converges to origin as** $t \to \infty$

- **Origin is globally asymptotically stable if region of attraction is** $\mathbb{R}^n$

Can there be more than 1 globally asymptotically stable points?

- **No.**
- **Can you prove this?**
  - **Suppose $x_1 \neq x_2$ are two globally asymptotically stable equilibrium points.**
    - **This implies $||x_1 - x_2|| > 0$.**
  - **By definition of GAS, we have that $\forall x(t_0)$,**
    - $\lim_{t\to\infty} ||x(t) - x_1|| = 0$ **and**
    - $\lim_{t\to\infty} ||x(t) - x_2|| = 0$
  - **However, this is a contradiction since**
    $$0 < ||x_1 - x_2|| = ||x_1 - x_2 + \left(x(t) - x(t)\right)|| \leq ||x_1 - x(t)|| + ||x_2 - x(t)||$$
  - **Therefore, any globally asymptotically stable point is unique.**

Linear Time-Invariant Systems

- $\dot{x} = Ax$
- Stability is (unsurprisingly!) determined by the eigenvalues
- Throwback Thursday:
  - Solutions: $x(t) = e^{At}x(0)$
  - For any matrix A there exists a nonsingular matrix P that transforms A into Jordan form:

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

Linear Time-Invariant Systems

- **Using P we can change the basis for e^{At}!**

$$\exp(At) = P\exp(Jt)P^{-1} \qquad P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

$$e^{J_i} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2 e^{\lambda_i t}}{2!} & \cdots \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \cdots \\ 0 & 0 & e^{\lambda_i t} & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}_{m \times m}$$

- **m_i is the order of the Jordan block**

LTI Stability

Theorem 3.1

The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \leq 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$

Hurwitz Matrices

- **If all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$, A is called a Hurwitz matrix.**
- **The origin of $\dot{x} = Ax$ is asymptotically stable if and only if (iff) A is Hurwitz**

Examples: Classify stability!

- $A_1 = \begin{bmatrix} -3 & 0 \\ 10 & -4.5 \end{bmatrix}$

- $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- $A_3 = \begin{bmatrix} \epsilon & -14 \\ 0 & -1.6 \end{bmatrix}$

- **Bonus: Which matrices are Hurwitz?**

Exponential Stability

Exponential Stability$\Longrightarrow$Asymptotic Stability!

- **When the origin of a linear system is asymptotically stable, its solutions satisfy**

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0, \quad k \geq 1, \ \lambda > 0$$

Definition 3.3

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is exponentially stable if

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0$$

$k \geq 1, \ \lambda > 0$, for all $\|x(0)\| < c$

It is globally exponentially stable if the inequality is satisfied for any initial state $x(0)$

Lyapunov’s Indirect Method

- Stability of linear system determined by eigenvalues
- We’ve seen that we can predict *qualitative* behavior of *nonlinear* systems around equilibria by looking at eigenvalues of linearization
- Can we predict *stability* of nonlinear systems using linearization?
- Yes! This is called Lyapunov’s Indirect Method.

Theorem: Lyapunov's Indirect Method

- **Let $x^* = 0$ be an equilibrium point of nonlinear system $\dot{x} = f(x)$.**
- **Let $A = \frac{\partial f}{\partial x}\Big|_{x^*=0}$**

- **The equilibrium point x^* is exponentially stable if and only if $\mathrm{Re}[\lambda_i] < 0$ for all eigenvalues of A**

- **The equilibrium point x^* is unstable if $\mathrm{Re}[\lambda_i] > 0$ for some i.**

Be careful!

- **What if** $\mathrm{Re}[\lambda_i] \le 0$**?**
  - **In other words, what if we have at least one eigenvalue such that** $\mathrm{Re}[\lambda_i] = 0$ **?**
- **Linearization fails to determine stability in this case.**
  - **We can't say whether the equilibrium point is stable or unstable *using linearization*.**
  - **(We can try other methods though!)**

- **The most linearization can conclude in this case is that the origin is at most asymptotically stable**

Example

- $\dot{x} = ax^3$
- Try using linearization!

- $\frac{\partial f}{\partial x}|_0 = 3ax^2|_0 = 0$
- One eigenvalue at zero.
- $a > 0$: origin is unstable (remember $xf(x)$!)
- $a < 0$: origin is stable

…So how do we determine stability?

- We need some sort of *certificate* showing that an equilibrium is stable.
- Example: Energy of Pendulum
  - $\dot{x}_1 = x_2, \;\; \dot{x}_2 = -\sin(x_1) - bx_2$
- $E(x) = \left(1 - \cos(x_1)\right) + \frac{1}{2}x_2^2$
- What is the time rate of change $\frac{d}{dt}E(x)$?
- (Try this out!)

Energy of Pendulum

- **Use the chain rule!**
- $\frac{d}{dt}E(x) = \frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2$

$$= \sin(x_1)x_2 + x_2(-\sin(x_1) - bx_2)$$
$$= \sin(x_1)x_2 - \sin(x_1)x_2 - bx_2^2$$
$$= -bx_2^2$$

^^ For $b > 0$, this is a non-positive value

- **This means that the energy constantly decreases.**
- **The value of $\frac{d}{dt}E(x)$ is a certificate that origin is asymptotically stable.**

Lyapunov's Direct Method

- **Lyapunov's Direct Method generalizes the Energy approach**
- **Given $\dot{x} = f(x)$ with equilibrium $x^* = 0$…**
- **…we're looking for a similar certificate proving stability**
  - **Works for *all* initial conditions in a neighborhood**
  - **Stronger than numerical simulation / testing**

- **These certificates are called Lyapunov candidate functions**

Lyapunov Candidate Functions

- **Let $V: D \to \mathbb{R}$ be a continuously differentiable function**
  - **Defined in some domain $D \subseteq \mathbb{R}^n$**
- **Treat $V(x)$ like our energy!**
- **What's the time derivative?**

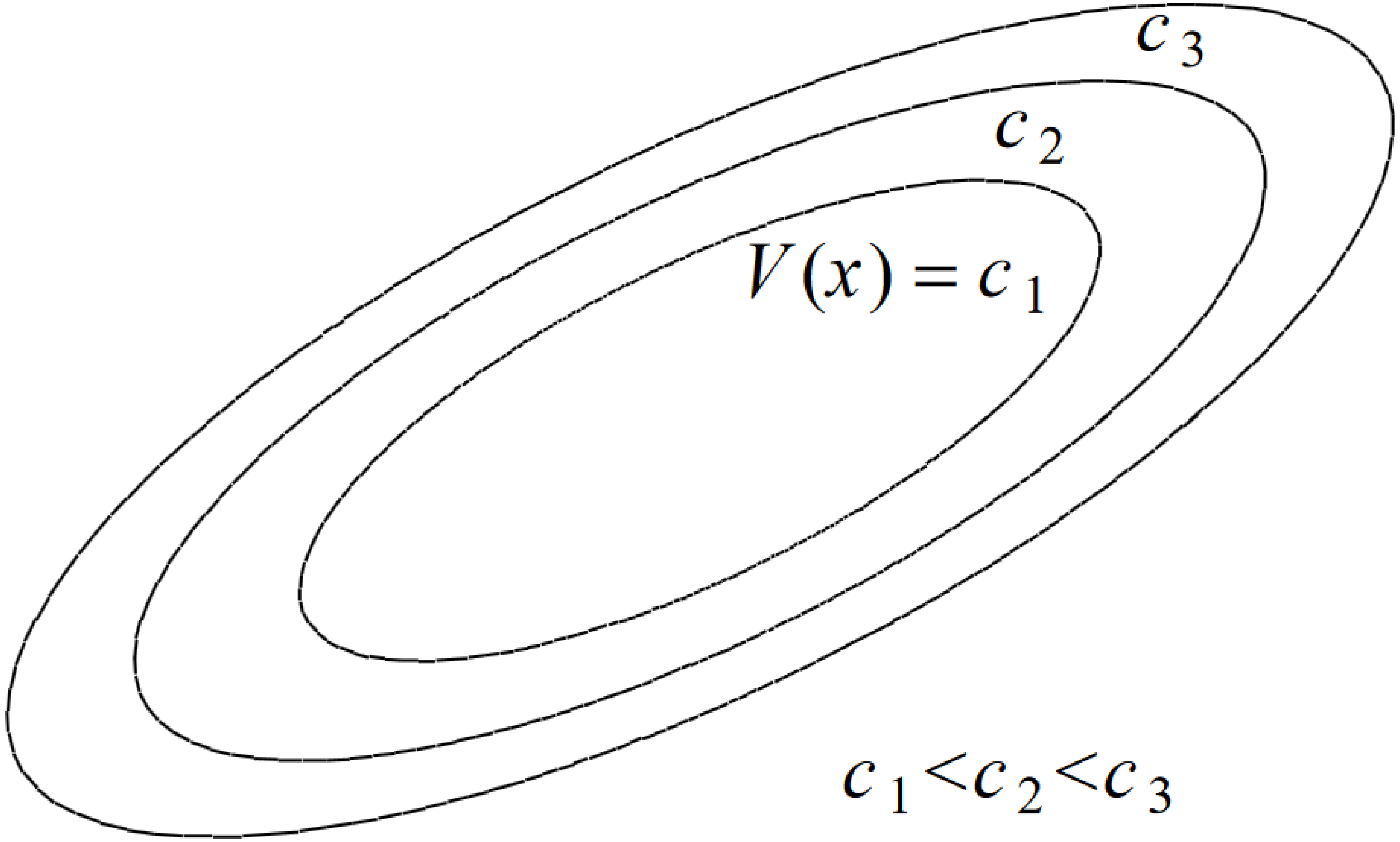
$$
\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\
&= \left[\frac{\partial V}{\partial x_1}, \quad \frac{\partial V}{\partial x_2}, \quad \ldots, \quad \frac{\partial V}{\partial x_n} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \\
&= \frac{\partial V}{\partial x} f(x)
\end{aligned}
$$

Some Terminology

$V(0) = 0,\ V(x) \geq 0$ for $x \neq 0$	Positive semidefinite
$V(0) = 0,\ V(x) > 0$ for $x \neq 0$	Positive definite
$V(0) = 0,\ V(x) \leq 0$ for $x \neq 0$	Negative semidefinite
$V(0) = 0,\ V(x) < 0$ for $x \neq 0$	Negative definite
$\|x\| \to \infty \ \Rightarrow\ V(x) \to \infty$	Radially unbounded

What does this look like?

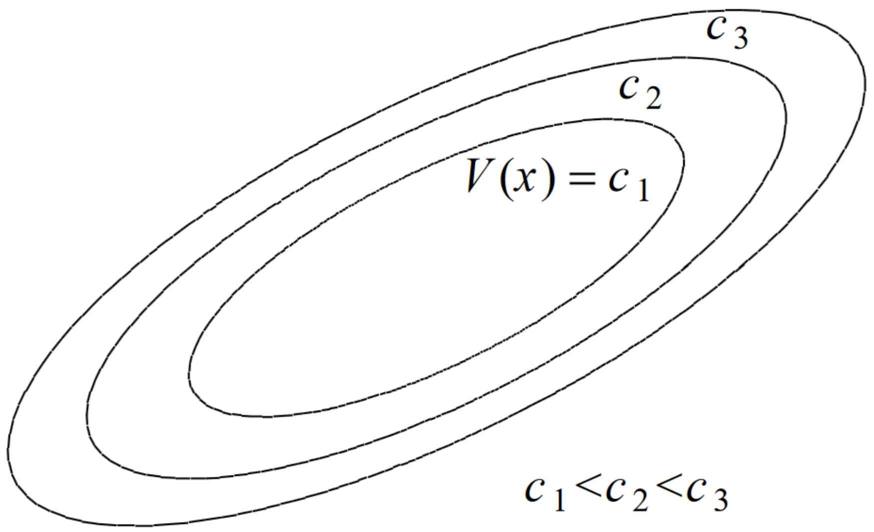
- **The surface $V(x) = c$ for some $c > 0$ is called a *level surface* or *level set*.**



Lyapunov Candidate Functions (cont.)

- **If $\phi(t, ; x)$ is the solution of $\dot{x} = f(x)$ that starts at x_0 at time $t = 0$, then**

$$\dot{V}(x) = \frac{d}{dt} V(\phi(t; x))\Big|_{t=0}$$



If $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$

If $\dot{V}(x)$ is positive, V will increase along the solution of $\dot{x} = f(x)$

Theorem: Lyapunov's Direct Method

Lyapunov's Theorem (3.3)

- If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \in D \text{ with } x \neq 0$$

$$\dot{V}(x) \leq 0, \quad \forall\, x \in D$$

then the origin is a stable

- Moreover, if

$$\dot{V}(x) < 0, \quad \forall\, x \in D \text{ with } x \neq 0$$

then the origin is asymptotically stable

(…But wait, there's more!)

Theorem: Lyapunov's Direct Method

- Furthermore, if $V(x) > 0$, $\forall\ x \neq 0$,

$$\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty$$

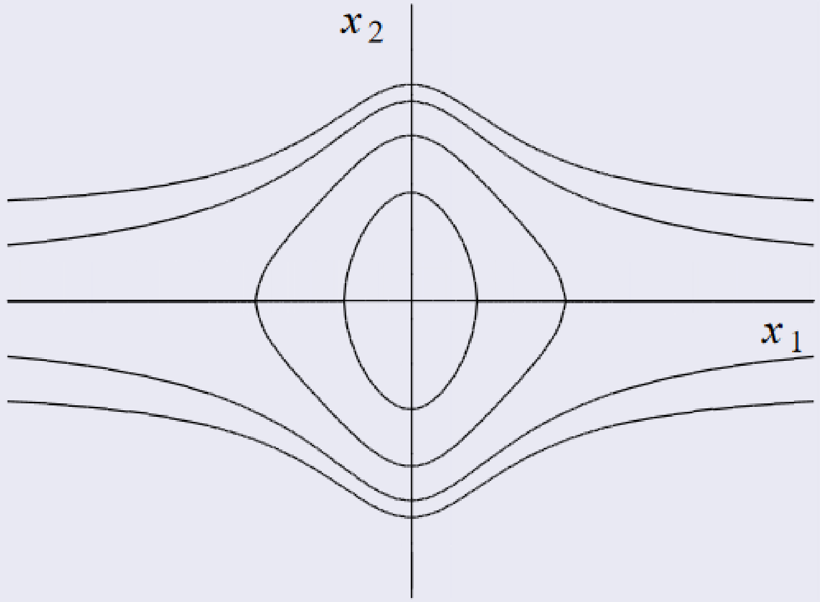
and $\dot{V}(x) < 0$, $\forall\ x \neq 0$, then the origin is globally asymptotically stable

- **If $V(x)$ satisfies this condition, it is called radially unbounded.**

What does *Not* Radially Unbounded look like?

Example

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$



Important!

- **Lyapunov's direct theorem is a *sufficient condition only!***
- **If it doesn't work, *we cannot conclude anything!***

Backup Slides

BYU

Nonlinear Systems Theory

Jan 30, 2024

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- Logistics
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  - Linear Time-Invariant Systems
  - Exponential Stability
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  - Lyapunov’s Direct Method

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- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

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1D Systems

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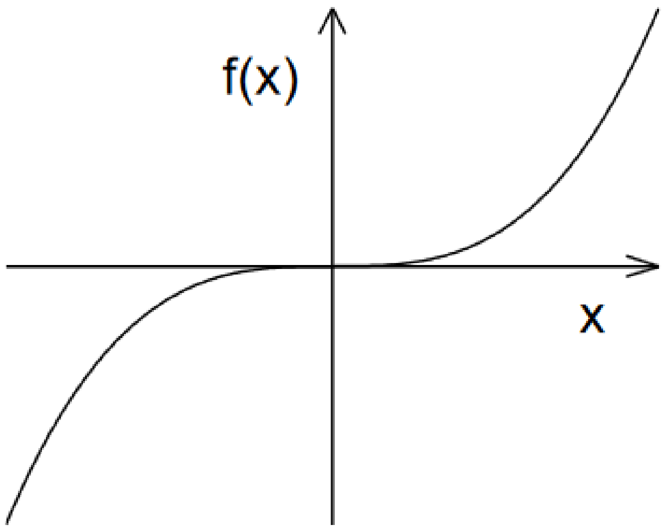
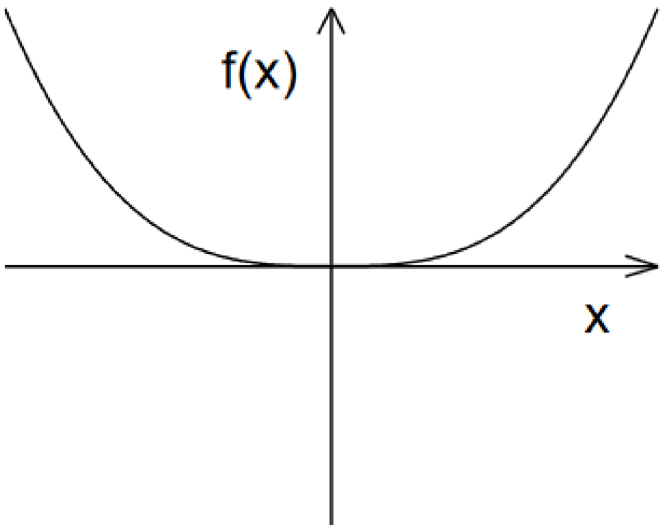
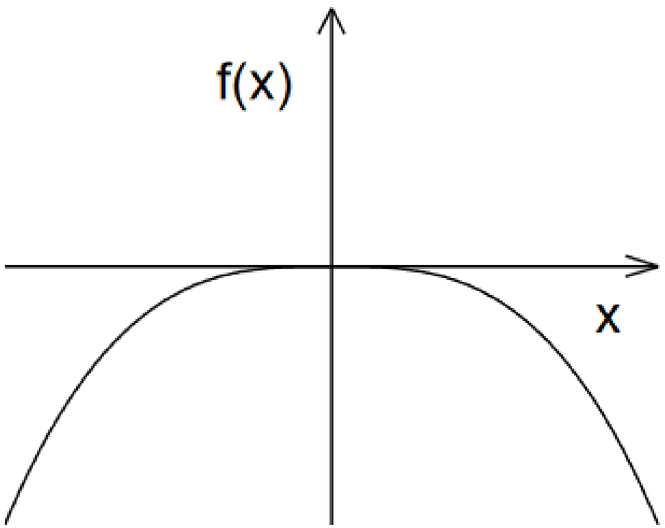
  - Equilibrium point at $x = 0$
- We can determine stability by looking at sign of $xf(x)$ in a neighborhood of the origin.
- $xf(x) > 0$: Origin is **unstable**.

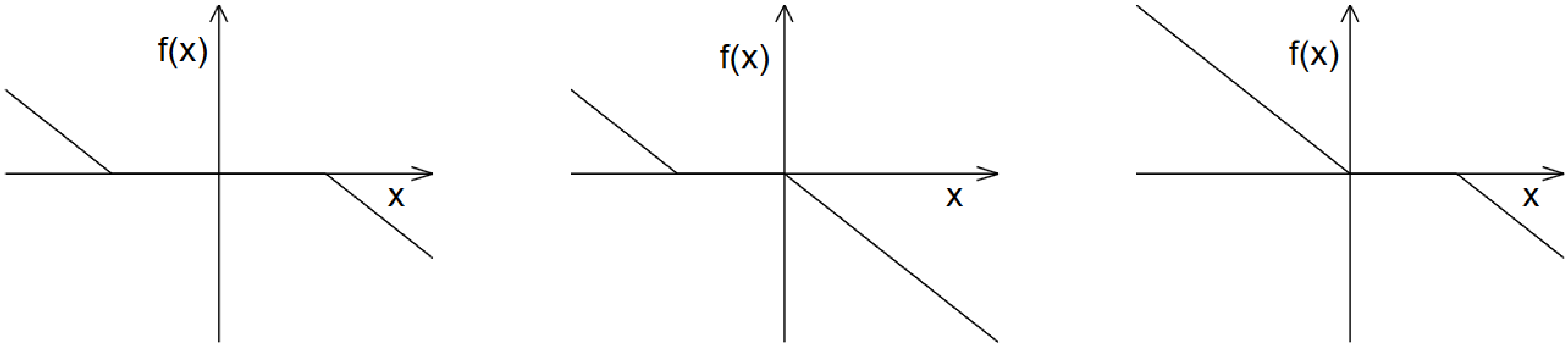






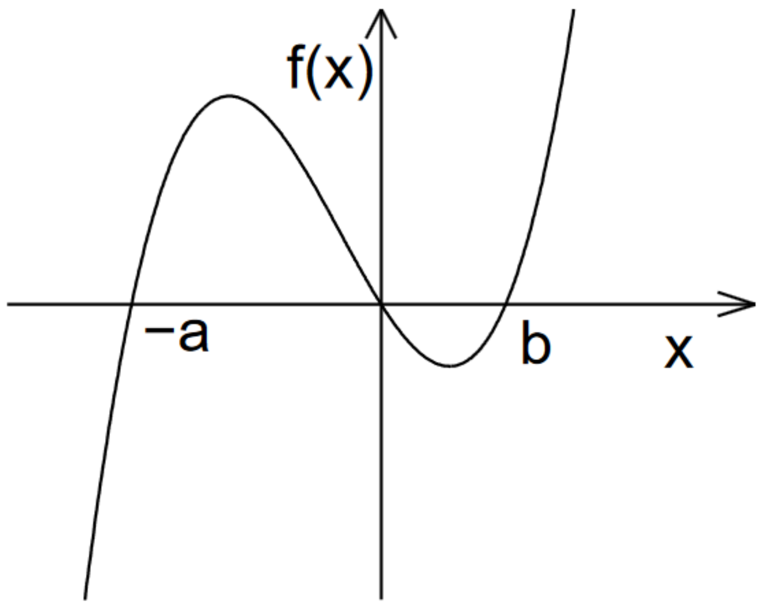
1D Systems (cont.)

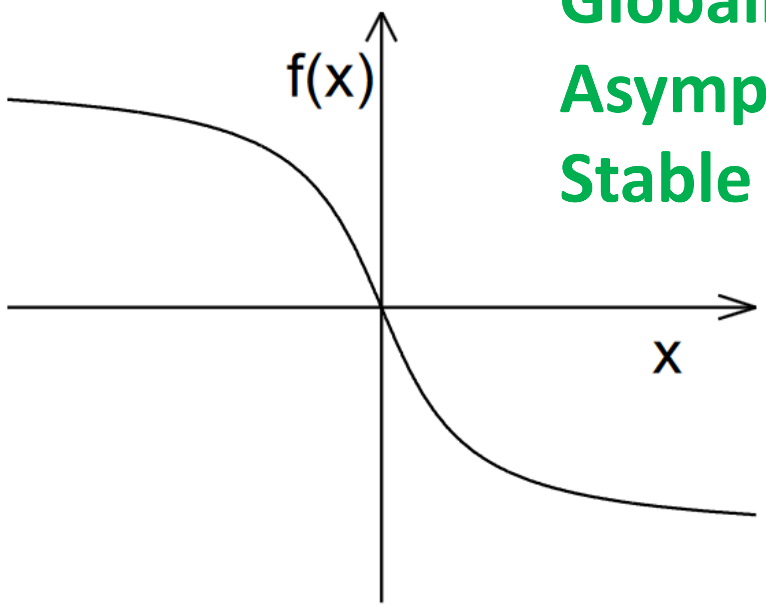
- $\dot{x} = f(x), x \in \mathbb{R}$
  - Equilibrium point at $x = 0$
- $xf(x) \leq 0$: Origin is stable.



1D Systems (cont.)

- $\dot{x} = f(x), x \in \mathbb{R}$
  - Equilibrium point at $x = 0$
- $xf(x) < 0$: Origin is asymptotically stable.





Globally Asymptotically Stable

Try it out!

- **Find the equilibrium, determine the stability**
  - $\dot{x} = x\sin(x)$
  - $\dot{x} = -x\cos(x)$
  - $\dot{x} = x\left(\cos\left(\frac{1}{x}\right) - \epsilon\right)$ **(for all possible ϵ values)**
  - $\dot{x} = -|x^5|$
  - $\dot{x} = -xe^x$
  - $\dot{x} = -\log(x+1)$

Regions of Attraction

- System: $\dot{x} = f(x)$
- Origin: asymptotically stable equilibrium point
- f is Locally Lipschitz on $D \subset \mathbb{R}^n$, $0 \in D$

- Region of attraction of the origin: set of all points $x_0 \in D$ such that $x(t)$, $x(0) = 0$ is defined for all t and converges to origin as $t \to \infty$

- Origin is globally asymptotically stable if region of attraction is $\mathbb{R}^n$

Can there be more than 1 globally asymptotically stable points?

- No.
- Can you prove this?
  - Suppose $x_1 \neq x_2$ are two globally asymptotically stable equilibrium points.
    - This implies $||x_1 - x_2|| > 0$.
  - By definition of GAS, we have that $\forall x(t_0)$,
    - $\lim_{t\to\infty} ||x(t) - x_1|| = 0$ and
    - $\lim_{t\to\infty} ||x(t) - x_2|| = 0$
  - However, this is a contradiction since
    $$0 < ||x_1 - x_2|| = ||x_1 - x_2 + \big(x(t) - x(t)\big)|| \leq ||x_1 - x(t)|| + ||x_2 - x(t)||$$
  - Therefore, any globally asymptotically stable point is unique.

Linear Time-Invariant Systems

- $\dot{x} = Ax$
- **Stability is (unsurprisingly!) determined by the eigenvalues**
- **Throwback Thursday:**
  - **Solutions:** $x(t) = e^{At}x(0)$
  - **For any matrix** A **there exists a nonsingular matrix** P **that transforms** A **into Jordan form:**

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

Linear Time-Invariant Systems

- **Using P we can change the basis for e^{At}!**

$$\exp(At) = P\exp(Jt)P^{-1} \qquad P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

$$e^{J_i} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \dfrac{t^2 e^{\lambda_i t}}{2!} & \cdots \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \cdots \\ 0 & 0 & e^{\lambda_i t} & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}_{m \times m}$$

- **m_i is the order of the Jordan block**

LTI Stability

Theorem 3.1

The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \leq 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$

Hurwitz Matrices

- **If all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$, A is called a Hurwitz matrix.**

- **The origin of $\dot{x} = Ax$ is asymptotically stable if and only if (iff) A is Hurwitz**

Examples: Classify stability!

- $A_1 = \begin{bmatrix} -3 & 0 \\ 10 & -4.5 \end{bmatrix}$

- $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- $A_3 = \begin{bmatrix} \epsilon & -14 \\ 0 & -1.6 \end{bmatrix}$

- **Bonus: Which matrices are Hurwitz?**

Exponential Stability

Exponential Stability ⟹ Asymptotic Stability!

- **When the origin of a linear system is asymptotically stable, its solutions satisfy**

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0, \quad k \geq 1, \ \lambda > 0$$

Definition 3.3

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is exponentially stable if

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0$$

$k \geq 1, \ \lambda > 0$, for all $\|x(0)\| < c$

It is globally exponentially stable if the inequality is satisfied for any initial state $x(0)$

Lyapunov's Indirect Method

- **Stability of linear system determined by eigenvalues**

- **We've seen that we can predict *qualitative* behavior of *nonlinear* systems around equilibria by looking at eigenvalues of linearization**

- **Can we predict *stability* of nonlinear systems using linearization?**

- **Yes! This is called Lyapunov's Indirect Method.**

Theorem: Lyapunov's Indirect Method

- **Let $x^* = 0$ be an equilibrium point of nonlinear system $\dot{x} = f(x)$.**
- **Let $A = \frac{\partial f}{\partial x}|_{x^*=0}$**
- **The equilibrium point x^* is exponentially stable if and only if $\mathrm{Re}[\lambda_i] < 0$ for all eigenvalues of A**
- **The equilibrium point x^* is unstable if $\mathrm{Re}[\lambda_i] > 0$ for some i.**

Be careful!

- **What if** $\mathrm{Re}[\lambda_i] \leq 0$**?**
  - **In other words, what if we have at least one eigenvalue such that** $\mathrm{Re}[\lambda_i] = 0$ **?**
- **Linearization fails to determine stability in this case.**
  - **We can't say whether the equilibrium point is stable or unstable *using linearization*.**
  - **(We can try other methods though!)**

- **The most linearization can conclude in this case is that the origin is at most asymptotically stable**

Example

- $\dot{x} = ax^3$
- **Try using linearization!**

- $\frac{\partial f}{\partial x}|_0 = 3ax^2|_0 = 0$
- **One eigenvalue at zero.**
- $a > 0$**: origin is unstable (remember** $xf(x)$**!)**
- $a < 0$**: origin is stable**

…So how do we determine stability?

- We need some sort of *certificate* showing that an equilibrium is stable.
- Example: Energy of Pendulum
  - $\dot{x}_1 = x_2, \;\; \dot{x}_2 = -\sin(x_1) - bx_2$
- $E(x) = \left(1 - \cos(x_1)\right) + \frac{1}{2}x_2^2$
- What is the time rate of change $\frac{d}{dt}E(x)$?
- (Try this out!)

Energy of Pendulum

- **Use the chain rule!**
- $\frac{d}{dt}E(x) = \frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2$

$$= \sin(x_1)x_2 + x_2(-\sin(x_1) - bx_2)$$

$$= \sin(x_1)x_2 - \sin(x_1)x_2 - bx_2^2$$

$$= -bx_2^2$$

^^ For $b > 0$, this is a non-positive value

- **This means that the energy constantly decreases for non-zero values of x_2^2**

Lyapunov's Direct Method

- Lyapunov's Direct Method generalizes the Energy approach
- Given $\dot{x} = f(x)$ with equilibrium $x^* = 0$…
- …we're looking for a similar certificate proving stability
  - Works for *all* initial conditions in a neighborhood
  - Stronger than numerical simulation / testing

- These certificates are called Lyapunov candidate functions

Lyapunov Candidate Functions

- Let $V: D \to \mathbb{R}$ be a continuously differentiable function
  - Defined in some domain $D \subseteq \mathbb{R}^n$
- Treat $V(x)$ like our energy!
- What's the time derivative?

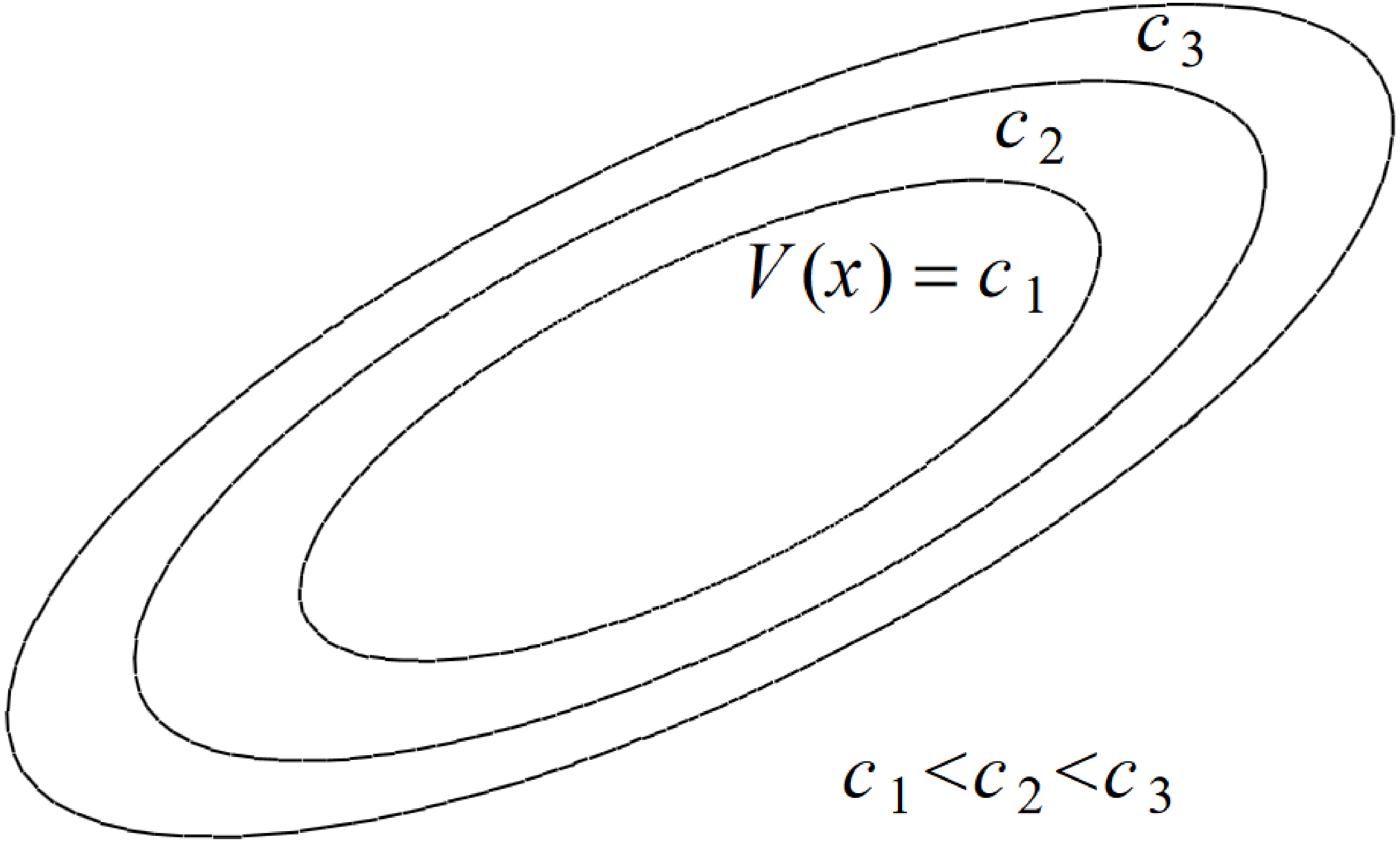
$$\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\
&= \begin{bmatrix} \frac{\partial V}{\partial x_1}, & \frac{\partial V}{\partial x_2}, & \cdots, & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \\
&= \frac{\partial V}{\partial x} f(x)
\end{aligned}$$

Some Terminology

$V(0) = 0,\ V(x) \geq 0$ for $x \neq 0$	Positive semidefinite
$V(0) = 0,\ V(x) > 0$ for $x \neq 0$	Positive definite
$V(0) = 0,\ V(x) \leq 0$ for $x \neq 0$	Negative semidefinite
$V(0) = 0,\ V(x) < 0$ for $x \neq 0$	Negative definite
$\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty$	Radially unbounded

What does this look like?

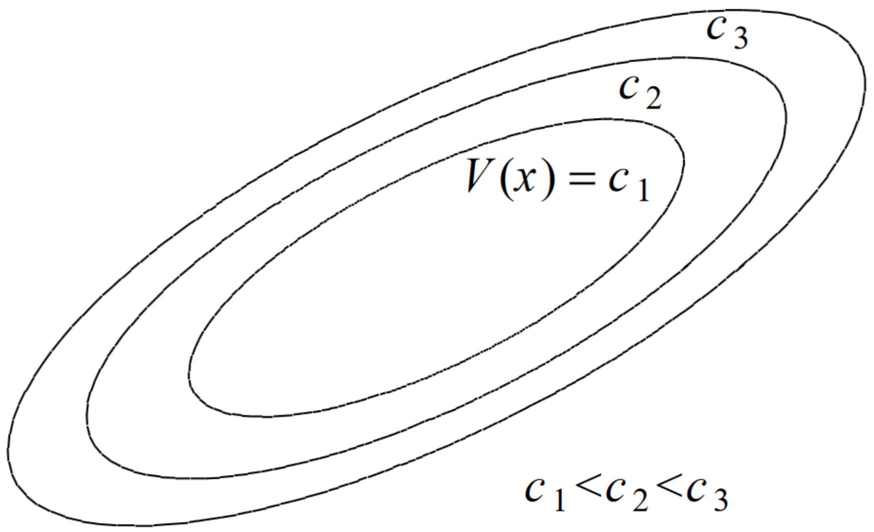
- The surface $V(x) = c$ for some $c > 0$ is called a ***level surface*** or ***level set***.



Lyapunov Candidate Functions (cont.)

- **If $\boldsymbol{\phi(t, ; x)}$ is the solution of $\boldsymbol{\dot{x} = f(x)}$ that starts at $\boldsymbol{x_0}$ at time $\boldsymbol{t = 0}$, then**

$$\dot{V}(x) = \frac{d}{dt} V(\phi(t; x))\Big|_{t=0}$$



If $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$

If $\dot{V}(x)$ is positive, V will increase along the solution of $\dot{x} = f(x)$

Theorem: Lyapunov's Direct Method

Lyapunov's Theorem (3.3)

- If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \in D \text{ with } x \neq 0$$

$$\dot{V}(x) \leq 0, \quad \forall\, x \in D$$

then the origin is a stable

- Moreover, if

$$\dot{V}(x) < 0, \quad \forall\, x \in D \text{ with } x \neq 0$$

then the origin is asymptotically stable

(…But wait, there's more!)

Theorem: Lyapunov's Direct Method

- Furthermore, if $V(x) > 0,\ \forall\ x \neq 0$,

$$\|x\| \to \infty \ \Rightarrow\ V(x) \to \infty$$

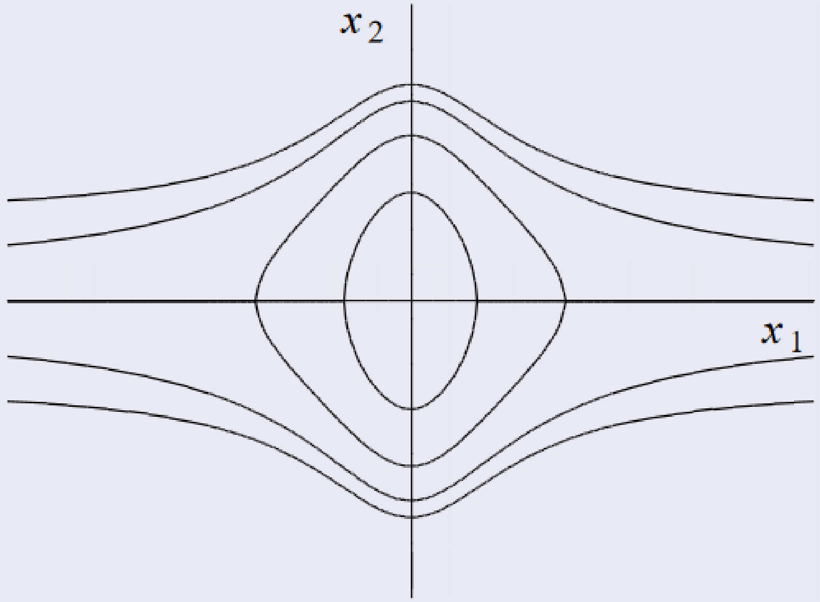
and $\dot{V}(x) < 0,\ \forall\ x \neq 0$, then the origin is globally asymptotically stable

- **If $V(x)$ satisfies this condition, it is called radially unbounded.**

What does *Not* Radially Unbounded look like?

Example

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$



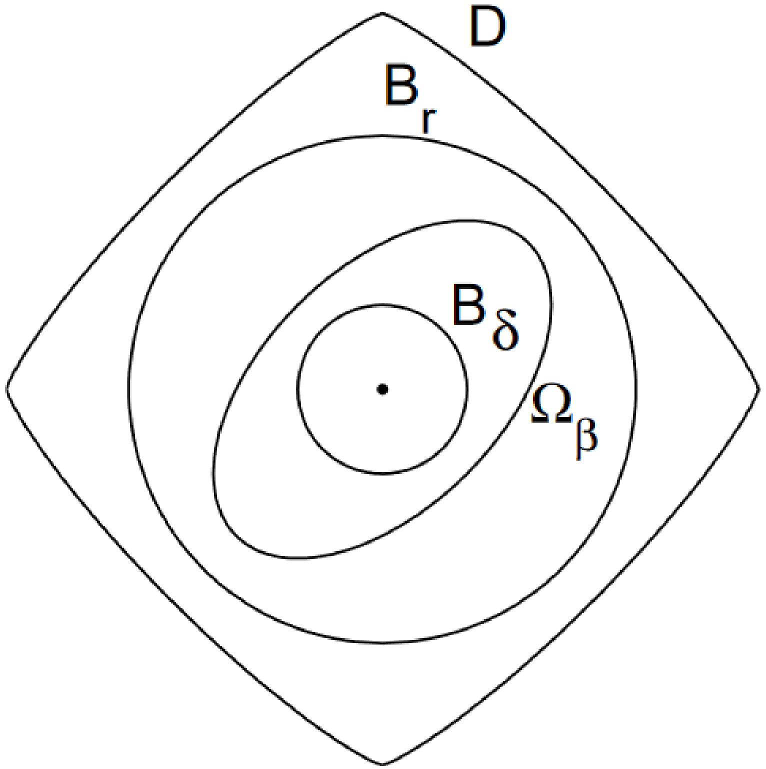
Important!

- **Lyapunov's direct theorem is a *sufficient condition only!***

- **If it doesn't work, *we cannot conclude anything!***

Proof of Lyapunov's Theorem

Proof



$$0 < r \le \varepsilon, \quad B_r = \{\|x\| \le r\}$$

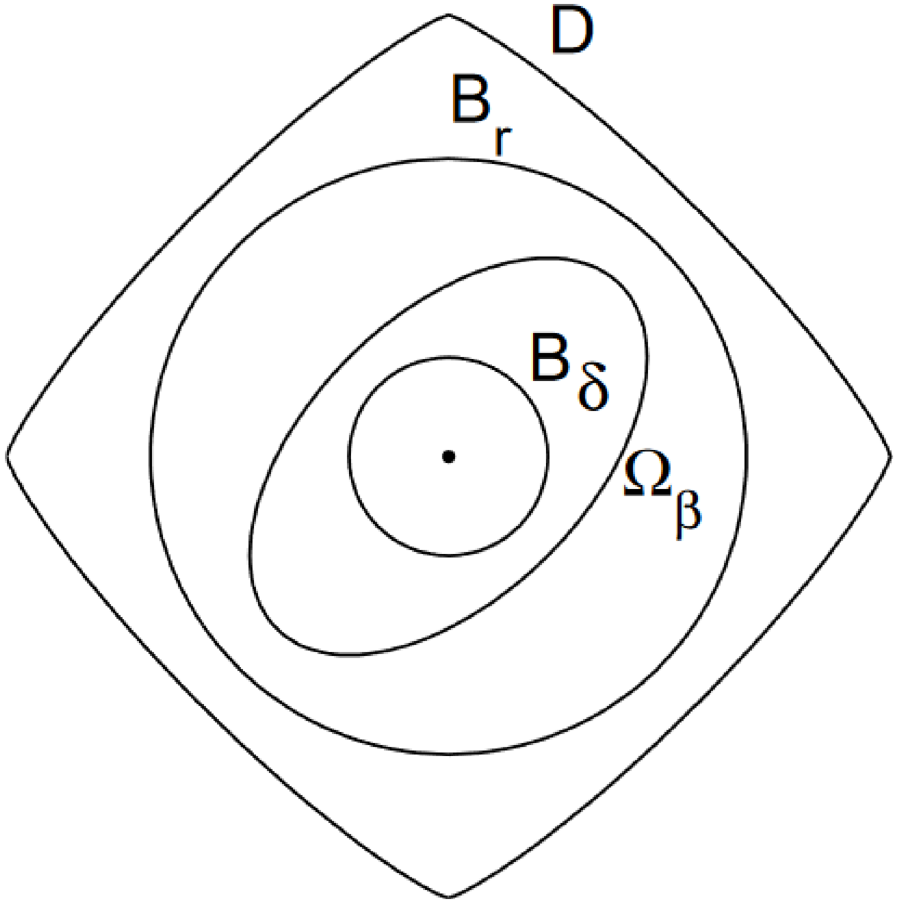
$$\alpha = \min_{\|x\|=r} V(x) > 0$$

$$0 < \beta < \alpha$$

$$\Omega_\beta = \{x \in B_r \mid V(x) \le \beta\}$$

$$\|x\| \le \delta \;\Rightarrow\; V(x) < \beta$$

Proof of Lyapunov's Theorem



Solutions starting in Ω_β stay in Ω_β because $\dot{V}(x) \le 0$ in Ω_β

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \le \varepsilon, \quad \forall\, t \ge 0$$

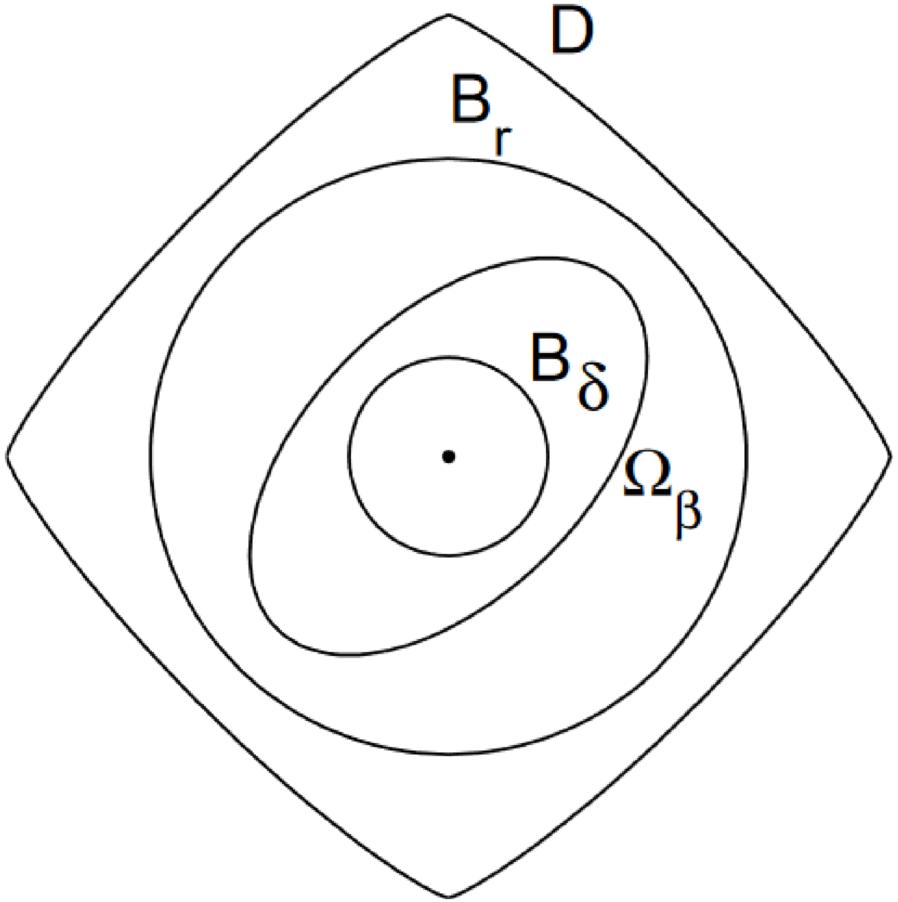
$\Rightarrow$ The origin is stable

Now suppose $\dot{V}(x) < 0 \; \forall \; x \in D, \;\; x \neq 0$. $V(x(t))$ is monotonically decreasing and $V(x(t)) \ge 0$

$$\lim_{t \to \infty} V(x(t)) = c \ge 0 \qquad \text{Show that } c = 0$$

Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. Then, $x(t)$ lies outside B_d for all $t \ge 0$

Proof of Lyapunov's Theorem



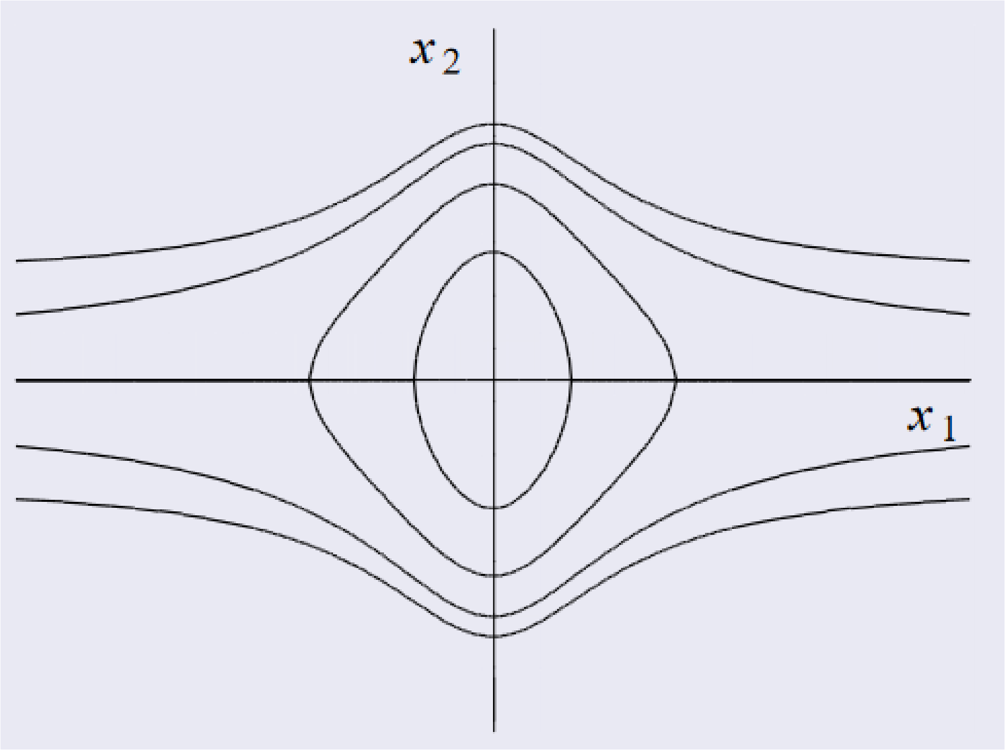
$$\gamma = -\max_{d \le \|x\| \le r} \dot{V}(x)$$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \le V(x(0)) - \gamma t$$

This inequality contradicts the assumption $c > 0$

$\Rightarrow$ The origin is asymptotically stable

Global Asymptotic Stability



The condition $\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty$ implies that the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is compact for every $c > 0$. This is so because for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$. All solutions starting Ω_c will converge to the origin. For any point $p \in R^n$, choosing $c = V(p)$ ensures that $p \in \Omega_c$

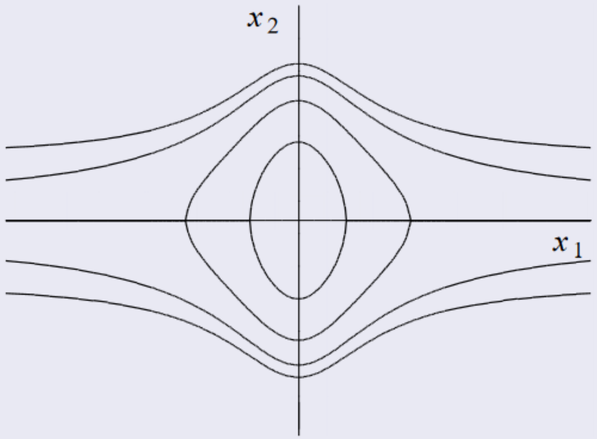
$\Rightarrow$ The origin is globally asymptotically stable

Need for Radially Unbounded Functions

- The proof assumes that $\Omega_c = \{V(x) \leq c\}$ is bounded for every $c > 0$.
  - This guarantees existence and uniqueness of solutions

Example

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$



In Other Words…

- **Lyapunov's Theorem using positive / negative (semi)definiteness**

Lyapunov' Theorem

The origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite. It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded

How do I Choose Lyapunov Functions?

- **This is an open question.**
- **A few techniques:**
  - **System Energy**
  - **Quadratic Form**
  - **Variable Gradient method**
  - **Sum-of-Squares optimization**

Quadratic Form

- Remember the pendulum example?
  - Derivative was negative semidefinite, so we can only conclude it is stable.
  - But we know it converges to zero!
  - Let's try again.

Try

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T P x + a(1 - \cos x_1) \\ &= \tfrac{1}{2}[x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1) \end{aligned}$$

$$p_{11} > 0, \qquad p_{11}p_{22} - p_{12}^2 > 0$$

Quadratic Form

$$\begin{aligned}\dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a\sin x_1)\,x_2 \\ &\quad + (p_{12}x_1 + p_{22}x_2)\,(-a\sin x_1 - bx_2) \\ &= a(1-p_{22})x_2\sin x_1 - ap_{12}x_1\sin x_1 \\ &\quad + (p_{11} - p_{12}b)\,x_1x_2 + (p_{12} - p_{22}b)\,x_2^2\end{aligned}$$

$p_{22} = 1, \quad p_{11} = bp_{12} \;\Rightarrow\; 0 < p_{12} < b,$ Take $p_{12} = b/2$

$$\dot{V}(x) = -\,\tfrac{1}{2}abx_1\sin x_1 - \tfrac{1}{2}bx_2^2$$

$$D = \{|x_1| < \pi\}$$

$V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D.
The origin is asymptotically stable

Backup Slides

BYU

Nonlinear Systems Theory

Feb 1, 2024

Overview

- Methods for Finding Lyapunov Candidates
  - Quadratic candidates
  - Variable gradient method
  - Sum-of-squares polynomials
- Discrete Time Lyapunov Theory

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Logistics

- **Next week: Out of Town**
- **Readings:**
  - **R. E. Kalman, "Control System Analysis and Design via the "Second Method" of Lyapunov"**
    - **Part I: https://asmedigitalcollection.asme.org/fluidsengineering/article/82/2/371/442524/Control-System-Analysis-and-Design-Via-the-Second**
    - **Part II: https://asmedigitalcollection.asme.org/fluidsengineering/article/82/2/394/442467/Control-System-Analysis-and-Design-Via-the-Second**

Quadratic Lyapunov Candidates

- **Basic idea: set $V(x) = x^T P x$**
  - **P is positive (semi-)definite matrix**
  - **$V(x)$ is positive definite if P positive definite**
  - **$V(x)$ is positive semidefinite if P positive semidefinite**

- **Form: $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$**

Example: Linear Systems

- $\dot{x} = Ax$
- $V(x) = x^T P x$
- $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x$

  $= x^T \left(A^T P + P A \right) x$
- If $(A^T P + PA)$ is negative definite, we've shown global asymptotic stability!
- Set $\left(A^T P + P A \right) = -Q$
  - Rearranging: $A^T P + PA + Q = 0$

Example: Linear Systems

- $A^T P + PA + Q = 0$
- How to solve?
- Special case of Sylvester Equation

- Can be solved by:
  - Bartel-Stewart Algorithm
  - Hessenberg-Schur Algorithm
- (Use a software toolbox)

Variable Gradient Method

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

$$g(x) = \nabla V = (\partial V / \partial x)^T$$

Choose $g(x)$ as the gradient of a positive definite function $V(x)$ that would make $\dot{V}(x)$ negative definite

$g(x)$ is the gradient of a scalar function if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, \ldots, n$$

Choose $g(x)$ such that $g^T(x) f(x)$ is negative definite

Computing $V(x)$ from $g(x)$

Compute the integral

$$V(x) = \int_0^x g^T(y)\ dy = \int_0^x \sum_{i=1}^{n} g_i(y)\ dy_i$$

over any path joining the origin to x; for example

$$\begin{aligned} V(x) \ &= \ \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\ dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\ dy_2 \\ &\quad + \ \cdots \ + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\ dy_n \end{aligned}$$

Leave some parameters of $g(x)$ undetermined and choose them to make $V(x)$ positive definite

Example of Variable Gradient Method

Example 3.7

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2$$

$a > 0$, $h(\cdot)$ is locally Lipschitz,

$$h(0) = 0; \quad yh(y) > 0 \; \forall \; y \neq 0, \; y \in (-b, c), \quad b > 0, \; c > 0$$

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + ax_2] < 0, \quad \text{for } x \neq 0$$

$$V(x) = \int_0^x g^T(y)\, dy > 0, \quad \text{for } x \neq 0$$

Example cont.

Try $\qquad g(x) = \begin{bmatrix} \phi_1(x_1) + \psi_1(x_2) \\ \phi_2(x_1) + \psi_2(x_2) \end{bmatrix}$

To satisfy the symmetry requirement, we must have

$$\frac{\partial \psi_1}{\partial x_2} = \frac{\partial \phi_2}{\partial x_1}$$

$$\psi_1(x_2) = \gamma x_2 \quad \text{and} \quad \phi_2(x_1) = \gamma x_1$$

$$\begin{aligned} \dot{V}(x) &= -\gamma x_1 h(x_1) - a x_2 \psi_2(x_2) + \gamma x_2^2 \\ &\quad + x_2 \phi_1(x_1) - a\gamma x_1 x_2 - \psi_2(x_2) h(x_1) \end{aligned}$$

Example cont.

To cancel the cross-product terms, take

$$\psi_2(x_2) = \delta x_2 \quad \text{and} \quad \phi_1(x_1) = a\gamma x_1 + \delta h(x_1)$$

$$g(x) = \left[\begin{array}{c} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{array} \right]$$

$$\begin{aligned} V(x) &= \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\ dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\ dy_2 \\ &= \tfrac{1}{2} a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\ dy + \gamma x_1 x_2 + \tfrac{1}{2}\delta x_2^2 \\ &= \tfrac{1}{2} x^T P x + \delta \int_0^{x_1} h(y)\ dy, \quad P = \left[\begin{array}{cc} a\gamma & \gamma \\ \gamma & \delta \end{array} \right] \end{aligned}$$

Example cont.

$$V(x) = \tfrac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)\, dy, \quad P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$$

$$\dot{V}(x) = -\gamma x_1 h(x_1) - (a\delta - \gamma)x_2^2$$

Choose $\delta > 0$ and $0 < \gamma < a\delta$

If $yh(y) > 0$ holds for all $y \neq 0$, the conditions of Lyapunov's theorem hold globally and $V(x)$ is radially unbounded

Sum of Squares (SOS) Lyapunov Functions

- Can computers be used to find Lyapunov candidates?
- In some cases, yes!

- A multi-variate polynomial $p(x)$ is a **sum-of-squares polynomial** if there exist polynomials $f_i(x)$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2(x)$$

- $p(x)$ being SOS implies that $p(x) \geq 0$ for all x

Sum of Squares Polynomials

- **Equivalent definition:**

- **A polynomial $p(x)$ of degree $2d$ is an SOS polynomial if and only if**
  - **There exists a positive definite matrix Q and**
  - **There exists a vector of monomials $Z(x)$ (with degree $\leq d$) such that**

$$p(x) = Z(x)^T Q Z(x)$$

Proving that a Polynomial is SOS

- Need to find a Q such that $p(x) = Z(x)^T Q Z(x)$ holds
    - How to find this?
- Optimization
    - Can be cast as a semidefinite programming problem (SDP)
- Finding a Q is a certificate that polynomial is SOS

What's the point of all this?!?

- **Nonlinear system: $\dot{x} = f(x)$**
  - **We want to find a Lyapunov candidate:**
  - **$V(x) > 0$ for all $x \neq 0$**
  - **$\dot{V}(x) \leq 0$, or equivalently $-\dot{V}(x) = -\frac{(\partial V)}{\partial x} f(x) \geq 0$**

- **Suppose that $f(x)$ is a polynomial.**
- **Suppose we're looking for a polynomial Lyapunov candidate $V(x)$**
- **The above stability conditions are SOS conditions!**

...But we need a *positive definite* $V(x)$

- **Proposition: Given a polynomial $V(x)$ of degree $2d$...**
- **...let $\varphi(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \varepsilon_{ij} x_i^{2j}$ such that $\sum_{j=1}^{m} \varepsilon_{ij} > \gamma > 0$.**

- **Then if $V(x) - \varphi(x)$ is an SOS, $V(x)$ is positive definite**

Wrapping it all together:

- Suppose for $\dot{x} = f(x)$ with $f(x)$ polynomial and an equilibrium at the origin…
- …there exists polynomial $V(x)$ such that
  - $V(x) = 0$
  - $V(x) - \varphi(x)$ is SOS
  - $-\frac{\partial V}{\partial x} f(x)$ is SOS

- Then the origin equilibrium is globally stable in the sense of Lyapunov.

Discrete Time Lyapunov Theory

- $x_{t+1} = f(x_t)$
  - f is locally Lipschitz in a domain D
  - Equilibrium at $x = 0$
- Suppose there exists continuous positive definite function $V(x)$ such that

$$V\big(f(x_t)\big) - V(x_t) \leq 0 \; \forall x \in D$$

Then the origin is stable. If we have

$$V\big(f(x_t)\big) - V(x_t) < 0 \; \forall x \in D - \{0\}$$

then the origin is asymptotically stable.

Backup Slides

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Nonlinear Systems Theory

Feb 8, 2024

Overview

- Clarifications
- Lyapunov Analysis Examples

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Clarifications

- Quadratic Lyapunov candidate: $V(x) = x^T P x$
  - What is the time derivative for $\dot{x} = f(x)$?

- For Linear systems ONLY:
  - $f(x) = Ax$
  - $\dot{V}(x) = x^T \left(A^T P + P A \right) x$
  - You shouldn't ever use this for nonlinear systems!

Example 1: Quadratic Lyapunov Candidates

- $\dot{x}_1 = -x_1 - x_2$
- $\dot{x}_2 = 2x_1 - x_2^3$
- **Determine stability of origin.**
- **Strategy:**
  - **Try** $V(x) = \frac{1}{2}x^T x$
  - **Try** $V(x) = \frac{1}{2}x^T P x$ **where** $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$
    - **Remember we need p.s.d.** P **matrix!**
    - $p_{11} > 0, p_{11}p_{22} - (p_{12})^2 > 0$

Example 2: Quadratic Lyapunov Candidates

- $\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2)$
- $\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$
- Determine stability of origin.
- Strategy:
  - Try $V(x) = \frac{1}{2}x^T x$
  - Can we write the resulting expression in terms of V itself?
  - What do we know about V?

Example 3: Quadratic Lyapunov Candidates

- $\dot{x}_1 = -x_1 + x_1 x_2$
- $\dot{x}_2 = -x_2$
- **Determine stability of origin.**
- **Strategy:**
  - **Try** $V(x) = \frac{1}{2} x^T x$
  - **Try** $V(x) = \frac{1}{2} |x|^T |x|$

Example 4: Quadratic Lyapunov Candidates

- $\dot{x}_1 = x_2(1 - x_1^2)$
- $\dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$
- Determine stability of origin.
- Strategy:
  - Where are the equilibria?
  - Try $V(x) = \frac{1}{2}x^T x$
  - Try $V(x) = \frac{1}{2}x^T P x$ where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$
    - Remember we need p.s.d. P matrix!
    - $p_{11} > 0, p_{11}p_{22} - (p_{12})^2 > 0$

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Nonlinear Systems Theory

Feb 13, 2024

Overview

- LaSalle’s Invariance Principle
- Stabilizing Feedback Control Laws

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Back to the Pendulum

- $\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin(x_1) - bx_2$
- Remember that we attempted to use the energy as a Lyapunov candidate
- $E(x) = \big(1 - \cos(x_1)\big) + \frac{1}{2}x_2^2$
- This gave us $\frac{d}{dt}E(x) = -bx_2^2$
  - Time derivative is positive *semi*-definite
  - By Lyapunov's direct method, we can only conclude stability
- But intuitively, we know that the pendulum stops!
  - How can we prove this mathematically?

Analyzing the Pendulum Equation

- $\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin(x_1) - bx_2$
- $\frac{d}{dt}E(x) = -bx_2^2$
- For $x \neq 0$, notice that:
  - $\dot{E}(x) = 0 \Leftrightarrow x_2 = 0$
  - But the system can't stay on that line!
    - $\dot{x}_2 = -sin(x_1)$ is nonzero for $-\pi \leq x_1 \leq \pi$
- Since the condition $\dot{E}(x) = 0$ can only be maintained at the origin, the system energy must decrease over time!

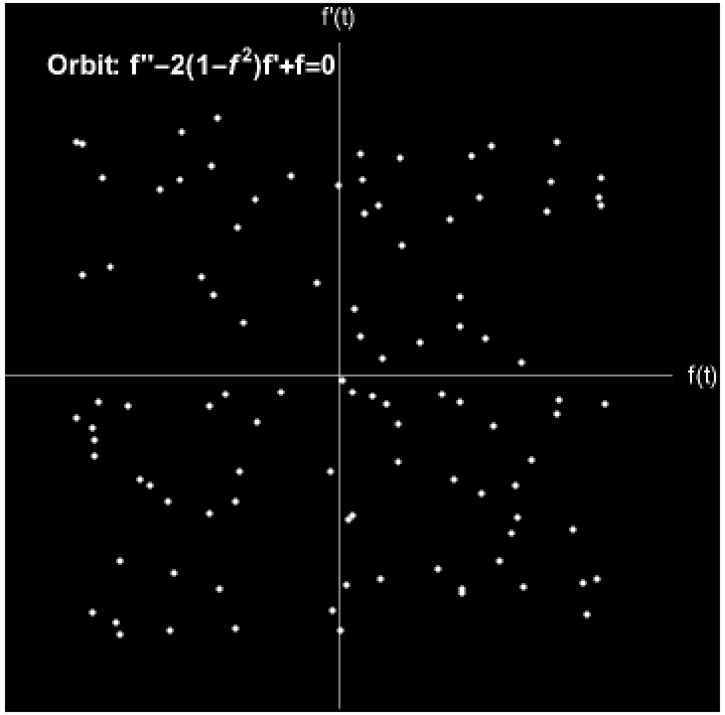
General Principles

- Summing up what we just saw:
  - If we find a Lyapunov candidate $V(x)$…
  - If the derivative $\dot{V}(x) \leq 0$…
  - If no trajectory can stay in the set $\{x : \dot{V}(x) = 0\}$ except at the equilibrium point…
  - Then the equilibrium point must be asymptotically stable!

- This is a rough outline of LaSalle's Invariance Principle.

- …But we need to be mathematically rigorous!

Positive Limit Sets



- System: $\dot{x} = f(x)$
  - Solution: $x(t)$
- Sequence: a function with the domain $\{n \in \mathbb{Z}: n \geq 0\}$
  - Example: $s_n = \frac{1}{n^2}$
- A point is a positive limit point of $x(t)$ if:
  - there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that
  - $x(t_n) \to p$ as $n \to \infty$
- Positive limit set: the set of all positive limit points of $x(t)$
- Examples:
  - Asymptotically stable equilibrium
  - Periodic orbits
  - Limit cycle (e.g. Van der Pol oscillator)

Invariant Sets and Distances

- **System:** $\dot{x} = f(x)$
  - **Solution:** $x(t)$
- **A set M is an invariant set w.r.t. the system if**
$$x(0) \in M \Longrightarrow x(t) \in M \quad \forall t \in \mathbb{R}$$
  - **State belongs to M for all *past and future* time.**
- **A set M is a positively invariant set w.r.t. the system if**
$$x(0) \in M \Longrightarrow x(t) \in M, \qquad \forall t \geq t_0$$
  - **State belongs to M for all *future* time.**
- **Distance from a set:**
$$\operatorname{dist}(p, M) = \inf_{x \in M} ||p - x||$$
- **The point $x(t)$ approaches a set M as $t \to \infty$ if for all $\epsilon > 0$ there exists $T > 0$ such that**
$$\operatorname{dist}(x(t), M) < \epsilon, \qquad \forall t > T.$$

Examples of Invariant Sets

- **Invariant sets:**
  - **Equilibrium points**
  - **Limit cycles**
  - **Solution remains in the set for all past and future time**
- **Positively invariant sets:**
  - **Lyapunov level sets for stable equilibria**
    - $\{x \in \mathbb{R}^n : V(x) \leq c\}$ **with** $\dot{V}(x) \leq 0$

Properties of Limit Sets

Lemma 3.1

If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$

LaSalle's Invariance Theorem

LaSalle's Theorem (3.4)

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$

Proof

$$\dot V(x) \le 0 \text{ in } \Omega \quad \Rightarrow \quad V(x(t)) \text{ is a decreasing}$$

$$V(x) \text{ is continuous in } \Omega \quad \Rightarrow \quad V(x) \ge b = \min_{x\in\Omega} V(x)$$

$$\Rightarrow \quad \lim_{t\to\infty} V(x(t)) = a$$

$$x(t) \in \Omega \quad \Rightarrow \quad x(t) \text{ is bounded} \quad \Rightarrow \quad L^+ \text{ exists}$$

Moreover, $L^+ \subset \Omega$ and $x(t)$ approaches L^+ as $t \to \infty$

For any $p \in L^+$, there is $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $x(t_n) \to p$ as $n \to \infty$

$$V(x) \text{ is continuous} \quad \Rightarrow \quad V(p) = \lim_{n\to\infty} V(x(t_n)) = a$$

$$V(x) = a \text{ on } L^+ \text{ and } L^+ \text{ invariant} \quad \Rightarrow \quad \dot V(x) = 0,\ \forall\, x \in L^+$$

$$L^+ \subset M \subset E \subset \Omega$$

$$x(t) \text{ approaches } L^+ \ \Rightarrow \ x(t) \text{ approaches } M \ (\text{as } t \to \infty)$$

A Few Things to Note

- **LaSalle's Theorem does not require $V(x)$ to be positive definite**
  - **But it *does* require you to prove that the set Ω is positively invariant**
  - **Usually a Lyapunov candidate is used to show this.**

- **How to show asymptotic stability?**
  - **Need to show that the largest invariant set E is the origin $x = 0$ (assuming the origin is the equilibrium point)**
  - **We do this by showing that no solution $x(t)$ can stay in E other than the origin.**

Corollary

Theorem 3.5

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable
- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction
- Furthermore, if $D = R^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example: Pendulum

- $\dot{x}_1 = x_2, \;\; \dot{x}_2 = -\sin(x_1) - bx_2$
- Rename $E(x)$ to $V(x)$
  - $V(x) = \left(1 - \cos(x_1)\right) + \frac{1}{2}x_2^2$
  - $\dot{V}(x) = -bx_2^2$
- Use LaSalle's Invariance Principle
  - $V(x)$ is positive definite on $x \in [-\pi, \pi]$
  - $\dot{V}(x) \leq 0$
  - Set of points where $\dot{V}(x) = 0$: $S = \{x : x_2 = 0\}$
  - The only solution that can stay in S is $x = 0$
- Therefore the origin is asymptotically stable
- Is it *globally* asymptotically stable?

Try It Out!

- **Consider the following mass-spring system:**

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

Hints:

- **Find the equilibrium point**
- **Use a change of variables to make the equilibrium point at the origin**
- **Try $V(x) = ax_1^2 + bx_2^2$ with $a, b > 0$.**

Stabilizing Feedback Control Laws

- Suppose we have a system with a control input:
- $\dot{x} = f(x, u)$

- Typically u is a function of t and x: $u(t, x)$
- We as the designers can choose $u(t, x)$

- Choose $u(t, x)$ such that $\dot{V}(x) < 0$!

(Very Simple) Example

- $\dot{x}_1 = x_1 x_2 + u_1$
- $\dot{x}_2 = x_1^2 + u_2$

- Let $\dot{V}(x) = \frac{1}{2}(x_1^2 + x_2^2)$
- What u_1, u_2 to pick to make the origin globally asymptotically stable?

Answer

- Choose
  - $u_1 = -x_1 - x_1 x_2$
  - $u_2 = -x_2 - x_1^2$

- (You can see stability by inspection, but this makes the Lyapunov candidate have a negative definite derivative)

Example

- Consider robots with the following equations of motion:
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$
- $\dot{M} - 2C$ is skew-symmetric for all $q, \dot{q}$
- $g(q) = \frac{\partial P(q)}{\partial q}^T$ where $P(q)$ is potential energy
- Assume $P(q)$ is positive definite, $g(q)$ has isolated root at $q = 0$
- With $u = 0$, use $V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov candidate to show that the origin is stable.

Example (cont.)

- With $u = -K_d \dot{q}$ where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

- With $u = g(q) - K_p(q - q^*) - K_d(\dot{q})$ where K_p, K_d are positive diagonal matrices and q^* is a desired position, show that the point $q = q^*, \dot{q} = 0$ is an asymptotically stable equilibrium point.

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Nonlinear Systems Theory

Feb 15, 2024

Overview

- **Stabilizing Feedback Control Laws**
- **Exponential Stability**
- **Region of Attraction**
- **Nonautonomous Systems**

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Stabilizing Feedback Control Laws

- **Suppose we have a system with a control input:**
- $\dot{x} = f(x, u)$

- **Typically u is a function of t and x: $u(t, x)$**
- **We as the designers can choose $u(t, x)$**

- **Choose $u(t, x)$ such that $\dot{V}(x) < 0$!**

Another Simple Example

- $\dot{x}_1 = u$
- $\dot{x}_2 = g(x_1, x_2)x_1$
- $x \neq 0 \implies g(x_1, x_2) \neq 0$
- Let $\mathrm{V}(x) = \frac{1}{2}(x_1^2 + x_2^2)$

- What u to pick to make the origin globally asymptotically stable?

Answer

- **Choose**
  - $\boldsymbol{u = -x_2 g(x_1, x_2) - k x_1, k > 0}$

More Complex Example

- Consider robots with the following equations of motion:
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$
- $\dot{M} - 2C$ is skew-symmetric for all $q, \dot{q}$
- $g(q) = \frac{\partial P(q)}{\partial q}^T$ where $P(q)$ is potential energy
- Assume $P(q)$ is positive definite, $g(q)$ has isolated root at $q = 0$
- With $u = 0$, use $V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov candidate to show that the origin is stable.

Example (cont.)

- With $u = -K_d \dot{q}$ where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

- With $u = g(q) - K_p(q - q^*) - K_d(\dot{q})$ where K_p, K_d are positive diagonal matrices and q^* is a desired position, show that the point $q = q^*, \dot{q} = 0$ is an asymptotically stable equilibrium point.

Exponential Stability: From Local to Global

The origin of $\dot{x} = f(x)$ is exponentially stable if and only if the linearization of $f(x)$ at the origin is Hurwitz

Theorem 3.6

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable function such that

$$k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a, \qquad \dot{V}(x) \leq -k_3\|x\|^a$$

for all $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is an exponentially stable equilibrium point of $\dot{x} = f(x)$. If the assumptions hold globally, the origin will be globally exponentially stable

Example 3.10

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - x_2$$

$$c_1 y^2 \le y h(y) \le c_2 y^2, \quad \forall\ y,\ c_1 > 0,\ c_2 > 0$$

$$V(x) = \tfrac{1}{2}\ x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2 \int_0^{x_1} h(y)\ dy$$

$$c_1 x_1^2 \le 2 \int_0^{x_1} h(y)\ dy \le c_2 x_1^2$$

$$\begin{aligned} \dot{V} &= [x_1 + x_2 + 2h(x_1)]x_2 + [x_1 + 2x_2][-h(x_1) - x_2] \\ &= -x_1 h(x_1) - x_2^2 \ \le\ -c_1 x_1^2 - x_2^2 \end{aligned}$$

The origin is globally exponentially stable

Region of Attraction

- Sometimes it's not enough to know that a point is asymptotically stable
- Sometimes we need to estimate the region in which the system converges to the equilibrium.

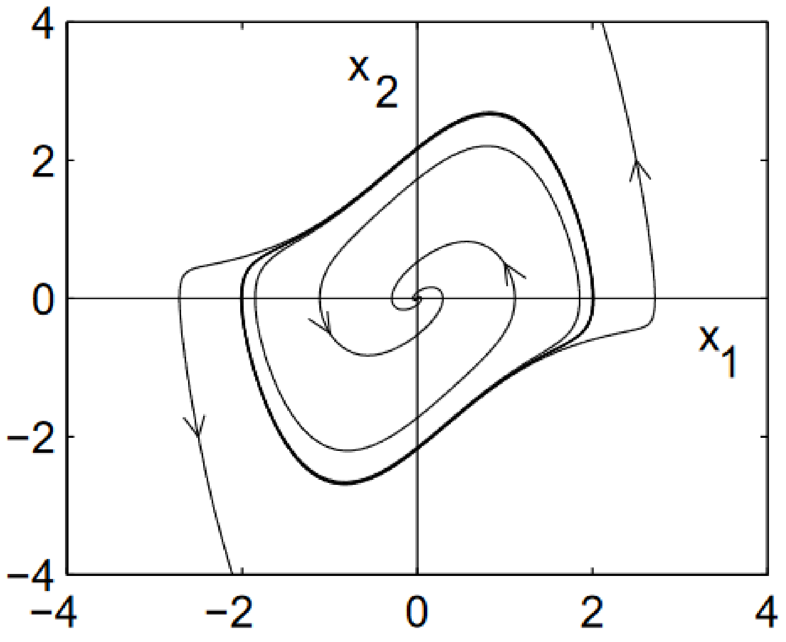
Lemma 3.2

The region of attraction of an asymptotically stable equilibrium point is an open, connected, invariant set, and its boundary is formed by trajectories

Example: Van der Pol in Reverse Time

Example 3.11

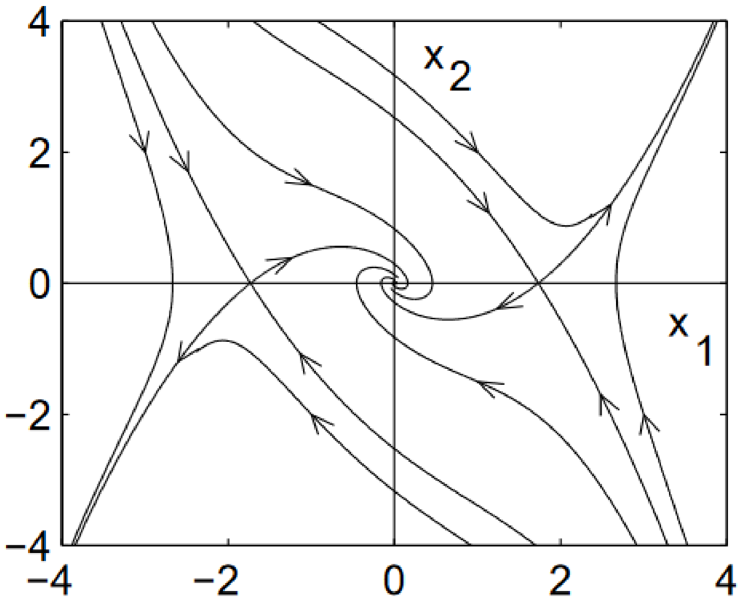
$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$



Another Example

Example 3.12

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \tfrac{1}{3}x_1^3 - x_2$$



Estimating the Region of Attraction

By Theorem 3.5, if D is a domain that contains the origin such that $\dot{V}(x) \leq 0$ in D, then the region of attraction can be estimated by a compact positively invariant set $\Gamma \in D$ if

- $\dot{V}(x) < 0$ for all $x \in \Gamma$, $x \neq 0$, or
- No solution can stay identically in $\{x \in D \mid \dot{V}(x) = 0\}$ other than the zero solution.

The simplest such estimate is the set $\Omega_c = \{V(x) \leq c\}$ when Ω_c is bounded and contained in D

Two Remarks

Remark 3.1

If $\Omega_1, \Omega_2, \ldots, \Omega_m$ are positively invariant subsets of the region of attraction, then their union $\cup_{i=1}^{m} \Omega_i$ is also a positively invariant subset of the region of attraction. Therefore, if we have multiple Lyapunov functions for the same system and each function is used to estimate the region of attraction, we can enlarge the estimate by taking the union of all the estimates

Remark 3.2

we can work with any compact set $\Gamma \subset D$ provided we can show that Γ is positively invariant. This typically requires investigating the vector field at the boundary of Γ to ensure that trajectories starting in Γ cannot leave it

Time-Varying Systems

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D$, $(0 \in D)$. The origin is an equilibrium point at $t = 0$ if

$$f(t, 0) = 0, \quad \forall\, t \geq 0$$

While the solution of the time-invariant system

$$\dot{x} = f(x), \qquad x(t_0) = x_0$$

depends only on $(t - t_0)$, the solution of

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

may depend on both t and t_0

Comparison Functions

Comparison Functions

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$, belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$
- A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$, belongs to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$

Examples: Class K Functions

Example 4.1

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1+r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$
- $\alpha(r) = r^c$, $c > 0$, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$
- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$. It is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $\mathcal{K}$ function

Examples: Class KL Functions

- $\beta(r,s) = r/(ksr+1)$, for any positive constant k, is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$$

$\beta(r,s) \to 0$ as $s \to \infty$. It belongs to class $\mathcal{KL}$

- $\beta(r,s) = r^c e^{-as}$, with positive constants a and c, belongs to class $\mathcal{KL}$

Properties of Comparison Functions

Lemma 4.1

Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a_1)$ and $[0, a_2)$, respectively, with $a_1 \geq \lim_{r \to a_2} \alpha_2(r)$, and β be a class $\mathcal{KL}$ function defined on $[0, \lim_{r \to a_2} \alpha_2(r)) \times [0, \infty)$ with $a_1 \geq \lim_{r \to a_2} \beta(\alpha_2(r), 0)$. Let α_3 and α_4 be class $\mathcal{K}_\infty$ functions. Denote the inverse of α_i by α_i^{-1}. Then,

- α_1^{-1} is defined on $[0, \lim_{r \to a_1} \alpha_1(r))$ and belongs to class $\mathcal{K}$
- α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\alpha_1 \circ \alpha_2$ is defined on $[0, a_2)$ and belongs to class $\mathcal{K}$
- $\alpha_3 \circ \alpha_4$ is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is defined on $[0, a_2) \times [0, \infty)$ and belongs to class $\mathcal{KL}$

Class-*K* Functions and Lyapunov Candidates

Lemma 4.2

Let $V : D \to R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = R^n$ and $V(x)$ is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such that the foregoing inequality holds for all $x \in R^n$

Stability Definitions for Time-Varying Systems

Definition 4.2

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \alpha(\|x(t_0)\|), \ \forall\ t \ge t_0 \ge 0, \ \forall\ \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \ \forall\ t \ge t_0 \ge 0, \ \forall\ \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

Stability Definitions for Time-Varying Systems

- exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \ \forall \ \|x(t_0)\| < c$$

- globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

Lyapunov's Direct Method for Time-Varying Systems

Theorem 4.1

Let the origin $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$ and $D \subset R^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, the origin is uniformly stable

Lyapunov's Direct Method for Time-Varying Systems

Theorem 4.2

Suppose the assumptions of the previous theorem are satisfied with

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \geq t_0 \geq 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable

Lyapunov's Direct Method for Time-Varying Systems

Theorem 4.3

Suppose the assumptions of the previous theorem are satisfied with

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a$$

for all $t \geq 0$ and $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable

Notions of Definiteness for Time-varying Functions

Terminology: A function $V(t,x)$ is said to be

- positive semidefinite if $V(t,x) \geq 0$
- positive definite if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$
- radially unbounded if $V(t,x) \geq W_1(x)$ and $W_1(x)$ is radially unbounded
- decrescent if $V(t,x) \leq W_2(x)$
- negative definite (semidefinite) if $-V(t,x)$ is positive definite (semidefinite)

English Explanation of Stability Conditions

Theorems 4.1 and 4.2 say that *the origin is uniformly stable if there is a continuously differentiable, positive definite, decrescent function* $V(t,x)$, *whose derivative along the trajectories of the system is negative semidefinite. It is uniformly asymptotically stable if the derivative is negative definite, and globally uniformly asymptotically stable if the conditions for uniform asymptotic stability hold globally with a radially unbounded* $V(t,x)$

Examples

Example 4.2

$$\dot{x} = -[1 + g(t)]x^3, \qquad g(t) \geq 0, \quad \forall \;\; t \geq 0$$

$$V(x) = \tfrac{1}{2}x^2$$

$$\dot{V}(t, x) = -[1 + g(t)]x^4 \leq -x^4, \quad \forall \; x \in R, \; \forall \; t \geq 0$$

The origin is globally uniformly asymptotically stable

Examples

Example 4.3

$$\dot{x}_1 = -x_1 - g(t)x_2, \qquad \dot{x}_2 = x_1 - x_2$$

$$0 \le g(t) \le k \ \text{ and } \ \dot{g}(t) \le g(t), \quad \forall\, t \ge 0$$

$$V(t,x) = x_1^2 + [1 + g(t)]x_2^2$$

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2, \quad \forall\, x \in R^2$$

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

$$2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$$

$$\dot{V}(t,x) \le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\, x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x$$

The origin is globally exponentially stable

Backup Slides

HW 3, Problem 1

- For a scalar non-autonomous differential equation in the form $\dot{x} = a(t)x$, define sufficient conditions on $a(t)$ such that the equilibrium of the scalar dynamics is
  - Stable, and
  - Asymptotically stable

The Wrong Answer

The Wrong Answer

- $\dot{x} = f(x), x \in \mathbb{R}$
  - Equilibrium point at $x = 0$
- We can determine stability by looking at sign of $xf(x)$ in a neighborhood of the origin.
- $xf(x) > 0$: Origin is unstable.
- $xf(x) \leq 0$: Origin is stable.
- $xf(x) < 0$: Origin is asymptotically stable.
- Why is this wrong???
  - $\dot{x} = f(t, x)$
  - This is a non-autonomous system

The Right Answer™

- Let $x(t) = \left(e^{\int_{t_0}^{t} -a(\tau)d\tau}\right)x(0)$
- Is this a solution to $\dot{x} = -a(t)x$?

- Is this solution unique?
  - $f(x) = -a(t)x$ is globally Lipschitz in x for all $t \in [t_0, t]$ for any finite t
  - By Lemma 1.2 (N.C.), the solution is unique

Stability

- Define $z_{max}(t_0) = \sup_{t \geq t_0} e^{\int_{t_0}^{t} -a(\tau)d\tau}$
- We have $|x(t)| \leq |z_{max}(t_0)x_0| = z_{max}(t_0)|x_0|$ for all $t \geq t_0$.
- Stability:
  - Choose any ϵ
  - Set $\epsilon = |z_{max}(t_0)x_0| = z_{max}(t_0)|x_0|$
  - Set $\delta = \frac{\epsilon}{z_{max}(t_0)}$
  - Then $|x_0| \leq \delta \Longrightarrow |x(t)| \leq \epsilon$

Asymptotic Stability

- **If $-a(t) < 0$ for all time, then $\dot{x} = -a(t)x$ is asymptotically stable, right?**
- **...Right?**

Counterexample #1

- Let $a(t) = e^{-t}$
  - $a(t) > 0 \; \forall t \geq t_0$
- $\lim_{t\to\infty} \int_{t_0}^{t} e^{-t} = \frac{1}{e^{t_0}}$.
- Therefore $\lim_{\mathrm{t}\to\infty} x(t) = \lim_{\mathrm{t}\to\infty} \left(e^{\int_{t_0}^{t} -a(\tau)d\tau}\right) x(0) = e^{\left(-e^{-t_0}\right)} x(0) \neq 0$.
- The system never reaches zero!
- We need $\lim_{t\to 0} \int_{t_0}^{t} a(\tau)d\tau = \infty$ in order for asymptotic stability to hold

Counterexample #2

- Suppose $t_0 = 0$.
- Let $a(t) = \frac{1}{x^2+1}$
  - $a(t) > 0 \; \forall t \geq 0$
- $\int_0^\infty \frac{1}{x^2+1} = \frac{\pi}{2}$

The Comparison Lemma

- Used to compute bounds on solution to $x(t)$ using an "easier" differential equation
- Consider $\dot{u} = f(t, u), u(t_0) = u_0$
  - $f(t, u)$ is continuous in t, locally L.C. in u for all $t \geq 0$ and all $u \in J \subset \mathbb{R}$
- Let $[t_0, T)$ be the maximal interval of existence of the solution $u(t)$
  - Suppose $u(t) \in J$ for all $t \in [t_0, T)$
- Let $v(t)$ be a continuous, differentiable function such that $\dot{v}(t) \leq f\big(t, v(t)\big), v(t_0) \leq u_0$
  - Suppose $v(t) \in J$ for all $t \in [t_0, T)$
- Then $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Using the Comparison Lemma

- **Define** $z_{max}(t_0) = \sup_{t \geq t_0} e^{\int_{t_0}^{t} a(\tau)d\tau}$
- **Define** $g(t, z) = 0$.
- **Then:**
  - $\dot{x} = f(t, x) = -a(t$

BYU

Nonlinear Systems Theory

Feb 27, 2024

Overview

- Project Discussion
- Lyapunov Stability for Nonautonomous Systems
- Boundedness and Ultimate Boundedness
- Input-to-State Stability

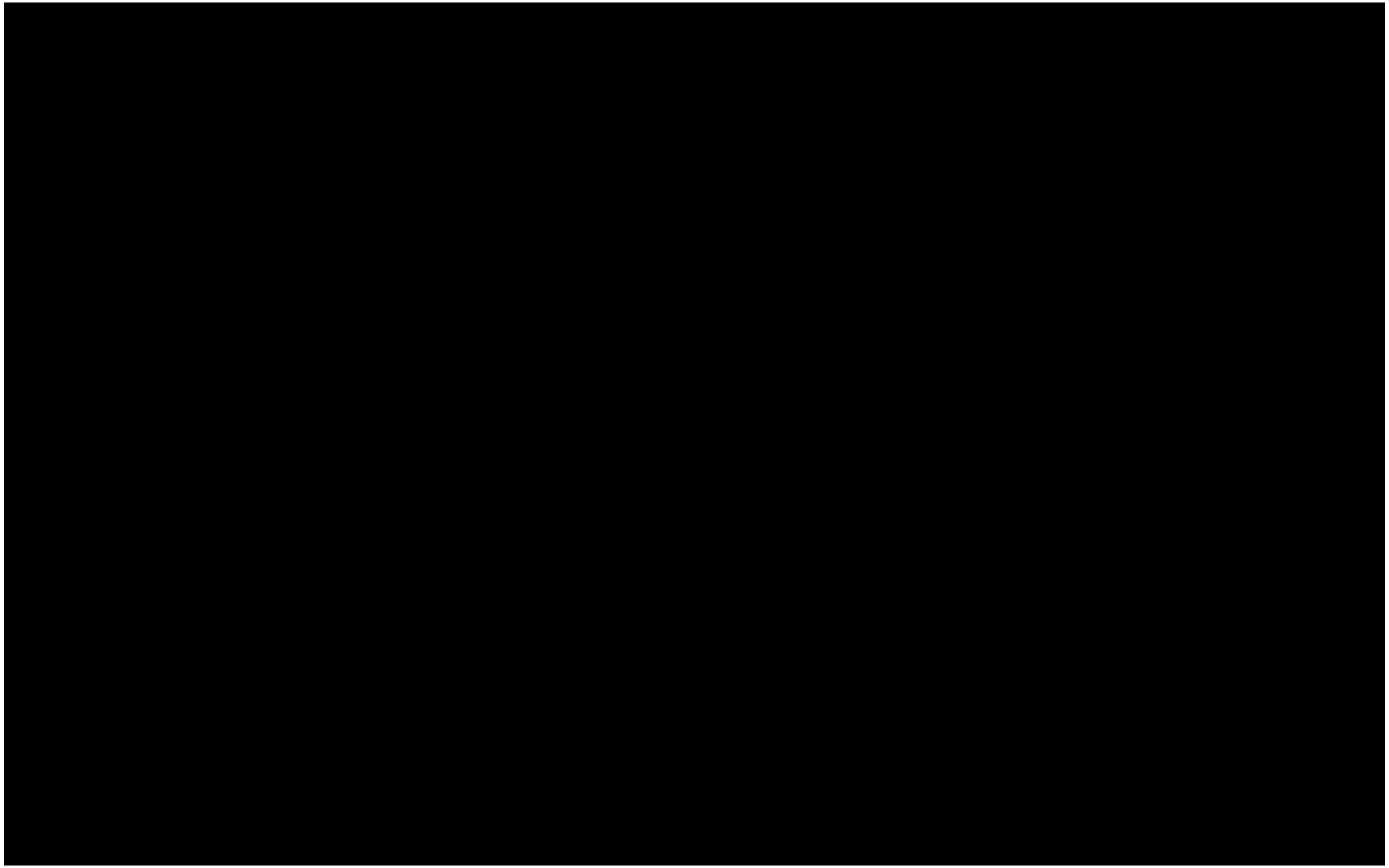
Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Project Discussion

- One of two options
- Application
  - Simple demonstration / simulation
  - Replicate a small result in a paper
- Advanced Topic
  - Pick a topic on nonlinear systems that we will not cover in class
  - Dive deeper into a topic that we cover in class
  - Focus is on *teaching the rest of the class*
    - Why should we care?
    - Tutorial on fundamental principles
- It is OK to change topics after this initial proposal
  - Just be sure to have a completed project at the end of the semester 🥴

Project Example



Seeing the Forest Despite the Trees

- We’ve been doing a lot of stability / Lyapunov analysis.
- How does this relate to control?

Fundamental Strategy of Control Theory

- **Define a *reference***

$$x_r(t)$$
$$\dot{x}_r = f(t, x_r, u_r)$$

- **Define an *error***

$$e_r = x_r - x$$

- **Make the error *converge to zero***

$$V(e_r) \geq 0$$
$$\dot{V}(e_r) < 0 \ \forall e_r \neq 0$$

This may not seem glamorous.

- Basketball: We are drilling layups
- Wrestling: We are drilling shots and takedowns.
- Soccer: Give-and-go.

- Most other "cool" control concepts are proven using Lyapunov stability theory.
- Hang in there!

Lyapunov Stability for Non-Autonomous Systems

- **(From Day 12 slides)**

Ultimate Boundedness: Why do we care?

- When analyzing some systems, there may be some terms we can't cancel out.
  - Noise
  - External signals / inputs

- We may not be able to show asymptotic convergence.
  - We may not even be able to show an equilibrium exists!

- Boundedness / Ultimate boundedness is another tool to quantify where the system "converges" to.
  - If we can't show an equilibrium, we may at least show the system remains in a certain region.

Boundedness and Ultimate Boundedness

- Sometimes, time-varying systems never converge to an equilibrium.
- Example: $\dot{x} = -x + \delta \sin(t), x(t_0) = a, \; a > \delta > 0$
  - No points where $\dot{x} = 0$
- Solutions: $x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin(\tau) d\tau$
  - Transient: $e^{-(t-t_0)}$
  - Non-transient: $\delta \int_{t_0}^{t} e^{-(t-\tau)} \sin(\tau) d\tau$
- (See Desmos Plot)

Nasty, but we can bound it.

- **For all $t \geq t_0$:**

$$\begin{aligned} |x(t)| &\leq e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)}\, d\tau \\ &= e^{-(t-t_0)}a + \delta\left(1 - e^{-(t-t_0)}\right) \\ &= e^{-(t-t_0)}(a-\delta) + \delta \leq a \end{aligned}$$

- **Solution is bounded *uniformly* in t_0**
  - **I.e. bound is independent of t_0**

- **But this bound doesn't consider exponential decay**
- **Can we do better?**

A Better Bound

- **Observe that for any b with $\delta < b < a$:**

$$|x(t)| \leq b \ \ \forall t \geq t_0 + ln\left(\frac{a-\delta}{b-\delta}\right)$$

- **So b is a tighter bound as time progresses**

- **Solution is said to be uniformly ultimately bounded**
- **The constant b is called an ultimate bound.**

Definitions

Definition 4.3

The solutions of $\dot{x} = f(t,x)$ are

- uniformly bounded if there exists $c > 0$, independent of t_0, and for every $a \in (0,c)$, there is $\beta > 0$, dependent on a but independent of t_0, such that

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq \beta, \quad \forall\, t \geq t_0$$

- uniformly ultimately bounded with ultimate bound b if there exists a positive constant c, independent of t_0, and for every $a \in (0,c)$, there is $T \geq 0$, dependent on a and b but independent of t_0, such that

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq b, \quad \forall\, t \geq t_0 + T$$

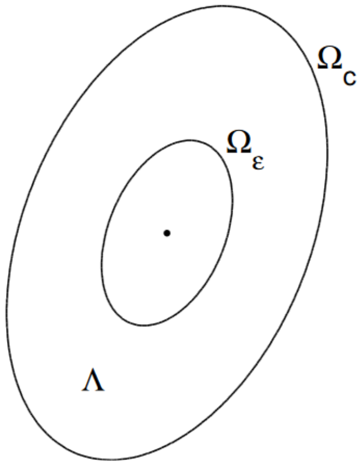
- Add "Globally" if a can be arbitrarily large
- Drop "uniformly" if $\dot{x} = f(x)$

Using Lyapunov Methods to determine Ultimate Boundedness

Lyapunov Analysis: Let $V(x)$ be a cont. diff. positive definite function and suppose the sets

$$\Omega_c = \{V(x) \le c\},\ \Omega_\varepsilon = \{V(x) \le \varepsilon\},\ \Lambda = \{\varepsilon \le V(x) \le c\}$$

are compact for some $c > \varepsilon > 0$



Suppose

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall\, x \in \Lambda,\ \forall\, t \ge 0$$

$W_3(x)$ is continuous and positive definite

Ω_c and Ω_ε are positively invariant

$$k = \min_{x \in \Lambda} W_3(x) > 0$$

$$\dot{V}(t,x) \le -k, \quad \forall\, x \in \Lambda,\ \forall\, t \ge t_0 \ge 0$$

$$V(x(t)) \le V(x(t_0)) - k(t - t_0) \le c - k(t - t_0)$$

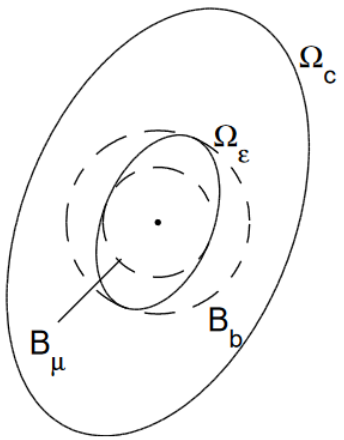
$x(t)$ enters the set Ω_ε within the interval $[t_0, t_0 + (c - \varepsilon)/k]$

Using Lyapunov Methods to determine Ultimate Boundedness

Suppose

$$\dot{V}(t,x) \le -W_3(x), \quad \forall\, x \in D \text{ with } \|x\| \ge \mu, \ \forall\, t \ge 0$$

Choose c and ε such that $\Lambda \subset D \cap \{\|x\| \ge \mu\}$



Let α_1 and α_2 be class $\mathcal{K}$ functions such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

$$V(x) \le c \ \Rightarrow\ \alpha_1(\|x\|) \le c \ \Leftrightarrow\ \|x\| \le \alpha_1^{-1}(c)$$

$$\text{If } B_r \subset D, \quad c = \alpha_1(r) \quad \Rightarrow \quad \Omega_c \subset B_r \subset D$$

$$\|x\| \le \mu \ \Rightarrow\ V(x) \le \alpha_2(\mu)$$

$$\varepsilon = \alpha_2(\mu) \quad \Rightarrow \quad B_\mu \subset \Omega_\varepsilon$$

What is the ultimate bound?

$$V(x) \le \varepsilon \Rightarrow \alpha_1(\|x\|) \le \varepsilon \Leftrightarrow \|x\| \le \alpha_1^{-1}(\varepsilon) = \alpha_1^{-1}(\alpha_2(\mu))$$

Theorem for Determining U.U.B

Theorem 4.4

Suppose $B_\mu \subset D \subset R^n$ and

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \quad \forall\ x \in D \text{ with } \|x\| \geq \mu, \ \forall\ t \geq 0$$

where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Choose $c > 0$ such that $\Omega_c = \{V(x) \leq c\}$ is compact and contained in D and suppose $\mu < \alpha_2^{-1}(c)$. Then, Ω_c is positively invariant and there exists a class $\mathcal{KL}$ function β such that for every $x(t_0) \in \Omega_c$,

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0), \alpha_1^{-1}(\alpha_2(\mu))\right\}, \quad \forall\ t\ \geq t_0$$

If $D = R^n$ and $\alpha_1 \in \mathcal{K}_\infty$, the inequality holds $\forall x(t_0)$, $\forall \mu$

Remarks

- The ultimate bound is independent of the initial state
- The ultimate bound is a class $\mathcal{K}$ function of μ; hence, the smaller the value of μ, the smaller the ultimate bound. As $\mu \to 0$, the ultimate bound approaches zero

Another Theorem for determining U.U.B.

Theorem 4.5

Suppose

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(t,x) \le -c_3\|x\|^2, \quad \forall\ x \in D \text{ with } \|x\| \ge \mu,\ \forall\ t \ge 0$$

for some positive constants c_1 to c_3, and $\mu < \sqrt{c/c_2}$. Then, $\Omega_c = \{V(x) \le c\}$ is positively invariant and $\forall\ x(t_0) \in \Omega_c$

$$V(x(t)) \le \max\left\{V(x(t_0))e^{-(c_3/c_2)(t-t_0)},\ c_2\mu^2\right\}, \quad \forall\ t \ge t_0$$

$$\|x(t)\| \le \sqrt{c_2/c_1}\max\left\{\|x(t_0)\|e^{-(c_3/c_2)(t-t_0)/2},\ \mu\right\}, \ \forall\ t \ge t_0$$

If $D = R^n$, the inequalities hold $\forall x(t_0)$, $\forall \mu$

Example: Mass-Spring System with Periodic Force

- **Dynamics:** $m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos(\omega t)$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(1+x_1^2)x_1 - x_2 + M\cos\omega t, \qquad M \ge 0$$

With $M = 0$, $\quad \dot{x}_2 = -(1+x_1^2)x_1 - x_2 = -h(x_1) - x_2$

$$V(x) = x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2\int_0^{x_1} (y + y^3)\, dy \quad \text{(Example 3.7)}$$

$$V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \tfrac{1}{2}x_1^4 \stackrel{\text{def}}{=} x^T P x + \tfrac{1}{2}x_1^4$$

Example: Mass-Spring System with Periodic Force

$$\lambda_{min}(P)\|x\|^2 \le V(x) \le \lambda_{max}(P)\|x\|^2 + \tfrac{1}{2}\|x\|^4$$

$$\alpha_1(r) = \lambda_{min}(P)r^2, \qquad \alpha_2(r) = \lambda_{max}(P)r^2 + \tfrac{1}{2}r^4$$

$$\begin{array}{rcl} \dot{V} & = & -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t \\ & \le & -\|x\|^2 - x_1^4 + M\sqrt{5}\|x\| \\ & = & -(1-\theta)\|x\|^2 - x_1^4 - \theta\|x\|^2 + M\sqrt{5}\|x\| \\ & & \hfill (0 < \theta < 1) \\ & \le & -(1-\theta)\|x\|^2 - x_1^4, \quad \forall\ \|x\| \ge M\sqrt{5}/\theta \stackrel{\text{def}}{=} \mu \end{array}$$

The solutions are GUUB by

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{max}(P)\mu^2 + \mu^4/2}{\lambda_{min}(P)}}$$

Input-to-State Stability: Why do we care?

- **Consider the system $\dot{x} = f(x, u)$**
  - **$u(t)$ is piecewise continuous, bounded function of t for all $t \geq t_0$**

- **If $\dot{x} = f(x, 0)$ has a globally asymptotic stable equilibrium point at $x = 0$…**
  - **What can we say about solution $x(t)$ for non-zero, bounded $u(t)$?**

Example: Linear System

- Consider $\dot{x} = Ax + Bu$ with Hurwitz A matrix
  - Remember: Hurwitz means all eigenvalues have negative real part, and the origin is globally asymptotically stable!
- For Hurwitz A, we can write:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

- Using the bound $||e^{(t-t_0)A}|| \leq ke^{-\lambda(t-t_0)}$:

$$||x(t)|| \leq ke^{-\lambda(t-t_0)}||x(t_0)|| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)}||B||||u(\tau)||d\tau$$

$$\leq ke^{-\lambda(t-t_0)}||x(t_0)|| + \frac{k||B||}{\lambda} \sup_{t_0 \leq \tau \leq t} ||u(\tau)||$$

Example: Linear System

- **Bound:** $||\mathrm{x(t)}|| \leq ke^{-\lambda(t-t_0)}||x(t_0)|| + \frac{k||B||}{\lambda} \sup_{t_0 \leq \tau \leq t} ||u(\tau)||$
- **Two parts:**
  - **Transient:** $ke^{-\lambda(t-t_0)}||x(t_0)||$
  - **Non-transient:** $\frac{k||B||}{\lambda} \sup_{t_0 \leq \tau \leq t} ||u(\tau)||$
- **The best bound we get is a function of the max norm of $u(t)$!**

How to extend this to Nonlinear Systems?

- These properties may or may not immediately hold for nonlinear systems
- Example: $\dot{x} = -3x + (1 + 2x^2)u$
  - $u(t) = 0$: Origin is globally asymptotically stable
  - $u(t) = 1$: Solution is $x(t) = \frac{3-e^t}{3-2e^t}$.
    - Unbounded!
    - Finite escape time!

- So, as usual, we need to add a bit more rigor to our analysis.

General Strategy

- **Use the notions of Boundedness and Ultimate Boundedness!**
- **View the system $\dot{x} = f(x, u)$ as a *perturbation* of the system $\dot{x} = f(x, 0)$**
- **Suppose we have a Lyapunov candidate $V(x)$ for $\dot{x} = f(x, 0)$, and $\dot{V}(x) < 0$ when $u = 0$.**
- **Since u is bounded…**
  - **In some cases, we expect $\dot{V} < 0$ outside a ball of radius μ**
  - **μ is a function of $\sup ||u||$**
  - **We'd expect this if $f(x, u)$ is Lipschitz in u:**

$$||f(x, u) - f(x, 0)|| \leq L||u||$$

  - **If we can show $\dot{V}(x) < 0$ outside of μ ball, then we can use Thm. 4.4:**

$$||x(t)|| \leq \max\{\beta(||x(t_0)||, t - t_0), \alpha_1^{-1}(\alpha_2(\mu))\}$$

Definitions

Definition 4.4

The system $\dot{x} = f(x, u)$ is input-to-state stable if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}$$

for all $t \geq t_0$

ISS of $\dot{x} = f(x, u)$ implies

- BIBS stability
- $x(t)$ is ultimately bounded by $\gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$
- $\lim_{t \to \infty} u(t) = 0 \quad \Rightarrow \lim_{t \to \infty} x(t) = 0$
- The origin of $\dot{x} = f(x, 0)$ is GAS

Sufficient Condition for I.S.S.

Theorem 4.6

Let $V(x)$ be a continuously differentiable function

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x,u) \le -W_3(x), \quad \forall \; \|x\| \ge \rho(\|u\|) > 0$$

$\forall \; x \in R^n, \; u \in R^m$, where $\alpha_1, \; \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, and $W_3(x)$ is a continuous positive definite function. Then, the system $\dot{x} = f(x,u)$ is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Leveraging Global Exponential Stability

Lemma 4.5

Suppose $f(x, u)$ is continuously differentiable and globally Lipschitz in (x, u). If $\dot{x} = f(x, 0)$ has a globally exponentially stable equilibrium point at the origin, then the system $\dot{x} = f(x, u)$ is input-to-state stable

Examples

Example 4.12

$$\dot{x} = -x^3 + u$$

The origin of $\dot{x} = -x^3$ is globally asymptotically stable

$$V = \tfrac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu \\ &\le -(1-\theta)x^4, \quad \forall\ |x| \ge \left(\frac{|u|}{\theta}\right)^{1/3} \end{aligned}$$

$$0 < \theta < 1$$

The system is ISS with $\gamma(r) = (r/\theta)^{1/3}$

Examples

Example 4.13

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2$$

The origin of $\dot{x} = -x - 2x^3$ is globally exponentially stable

$$V = \tfrac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^2 - 2x^4 + x(1 + x^2)u^2 \\ &= -x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \\ &\le -x^4, \quad \forall \ |x| \ge u^2 \end{aligned}$$

The system is ISS with $\gamma(r) = r^2$

Cascading Systems

Lemma 4.6

If the systems $\dot{\eta} = f_1(\eta, \xi)$ and $\dot{\xi} = f_2(\xi, u)$ are input-to-state stable, then the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi, u)$$

is input-to-state stable. Consequently, If $\dot{\eta} = f_1(\eta, \xi)$ is input-to-state stable and the origin of $\dot{\xi} = f_2(\xi)$ is globally asymptotically stable, then the origin of the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi)$$

is globally asymptotically stable

Global vs. Regional vs. Local I.S.S.

- **Important: the previous definition of Input-to-State stability is a *Global* definition!**
- **What if we want to define *local* I.S.S.?**
  - **Global: Everywhere**
  - **Regional: On subsets**
  - **Local: There exists a subset around the origin (eq. point) such that system is I.S.S. on that subset**

Regional and Local I.S.S.

Definition 4.5

Let $\mathcal{X} \subset R^n$ and $\mathcal{U} \subset R^m$ be bounded sets containing their respective origins as interior points. The system $\dot{x} = f(x, u)$ is regionally input-to-state stable with respect to $\mathcal{X} \times \mathcal{U}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0) \in \mathcal{X}$ and any input u with $u(t) \in \mathcal{U}$ for all $t \geq t_0$, the solution $x(t)$ belongs to $\mathcal{X}$ for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}$$

The system $\dot{x} = f(x, u)$ is locally input-to-state stable if it is regionally input-to-state stable with respect to some neighborhood of the origin $(x = 0, u = 0)$

Lyapunov Analysis of Regional I.S.S.

Theorem 4.7

Suppose $f(x,u)$ is locally Lipschitz in (x,u) for all $x \in B_r$ and $u \in B_\lambda$. Let $V(x)$ be a continuously differentiable function that satisfies

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -W_3(x), \quad \forall \ \|x\| \geq \rho(\|u\|) > 0$$

for all $x \in B_r$ and $u \in B_\lambda$, where α_1, α_2, $\rho \in \mathcal{K}$ and $W_3(x)$ is a continuous positive definite function. Suppose $\alpha_1(r) > \alpha_2(\rho(\lambda))$ and let $\Omega = \{V(x) \leq \alpha_1(r)\}$. Then, the system $\dot{x} = f(x,u)$ is regionally input-to-state stable with respect to $\Omega \times B_\lambda$ and $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Equivalence of I.S.S. and Asymptotic Stability

Local input-to-state stability of $\dot{x} = f(x,u)$ is equivalent to asymptotic stability of the origin of $\dot{x} = f(x,0)$

Lemma 4.7

Suppose $f(x,u)$ is locally Lipschitz in (x,u) in some neighborhood of $(x=0, u=0)$. Then, the system $\dot{x} = f(x,u)$ is locally input-to-state stable if and only if the unforced system $\dot{x} = f(x,0)$ has an asymptotically stable equilibrium point at the origin

Backup Slides

BYU

NONLINEAR SYSTEMS THEORY

Feb 29, 2024

Overview

- Input-to-State Stability
- Input-Output Stability
- Feedback Control

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

Input-Output Stability

- Most of Khalil's book addresses state space models
- Alternative approach: *input-output*
  - Relationship of inputs to outputs with no knowledge of internal structure
  - System is "black box"
- IO approach can handle systems that can't be represented by state models
  - E.g. systems with time delay

System Model and Function Norms

Input-Output Models: $y = Hu$

$u(t)$ is a piecewise continuous function of t and belongs to a linear space of signals

- The space of bounded functions: $\sup_{t \geq 0} \|u(t)\| < \infty$
- The space of square-integrable functions: $\int_0^\infty u^T(t)u(t)\, dt < \infty$

Norm of a signal $\|u\|$:

- $\|u\| \geq 0$ and $\|u\| = 0 \quad \Leftrightarrow \quad u = 0$
- $\|au\| = a\|u\|$ for any $a > 0$
- Triangle Inequality: $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$

$\mathcal{L}_p^m$ Spaces: Quantifying the "Size" of Functions

$\mathcal{L}_p$ spaces:

$$\mathcal{L}_\infty : \quad \|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$$

$$\mathcal{L}_2 : \quad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\ dt} < \infty$$

$$\mathcal{L}_p : \quad \|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p\ dt\right)^{1/p} < \infty, \quad 1 \leq p < \infty$$

Notation $\mathcal{L}_p^m$: p is the type of p-norm used to define the space and m is the dimension of u

Extended Spaces

- We think of $u \in \mathcal{L}^m$ as a "well-behaved" input.
- Some systems $y = Hu$ have the property that $u \in \mathcal{L}^m$ but $y \notin \mathcal{L}^p$
  - E.g. $y = \frac{1}{u-e^{-1}}, u = e^{-(2-t)}$
- To handle this, we use extended spaces

Extended Space: $\mathcal{L}_e = \{u \mid u_\tau \in \mathcal{L}, \forall\ \tau \in [0,\infty)\}$

u_τ is a truncation of u: $u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$

$\mathcal{L}_e$ is a linear space and $\mathcal{L} \subset \mathcal{L}_e$

Example

$$u(t) = t, \qquad u_\tau(t) = \begin{cases} t, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

$u \notin \mathcal{L}_\infty$ but $u_\tau \in \mathcal{L}_{\infty e}$

Causality and Gain Functions

Causality: A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the input up to time t

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Definition 6.1

A scalar continuous function $g(r)$, defined for $r \in [0, a)$, is a gain function if it is nondecreasing and $g(0) = 0$

A class $\mathcal{K}$ function is a gain function but not the other way around. By not requiring the gain function to be strictly increasing we can have $g = 0$ or $g(r) = \mathrm{sat}(r)$

$\mathcal{L}$ Stability

Definition 6.2

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a gain function g, defined on $[0, \infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq g\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma\|u_\tau\|_{\mathcal{L}} + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

In this case, we say that the system has $\mathcal{L}$ gain $\leq \gamma$. The bias term β is included in the definition to allow for systems where Hu does not vanish at $u = 0$.

$\mathcal{L}$ Stability and Input-Output Stability

- **For any causal, $\mathcal{L}$ stable system $y = Hu$:**
  - $u \in \mathcal{L}^m \Longrightarrow Hu \in \mathcal{L}^q$
  - **A bounded input results in a bounded output (where bound refers to the function norm)**

- **When we use $\mathcal{L}_\infty$, we get the traditional notion of BIBO (bounded-input-bounded-output) stability!**

Example

Example 6.1: Memoryless function $y = h(u)$

Suppose $\quad |h(u)| \leq a + b|u|, \quad \forall\ u \in R$

Finite-gain $\mathcal{L}_\infty$ stable with $\beta = a$ and $\gamma = b$

If $a = 0$, then for each $p \in [1, \infty)$

$$\int_0^\infty |h(u(t))|^p \ dt \leq (b)^p \int_0^\infty |u(t)|^p \ dt$$

Finite-gain $\mathcal{L}_p$ stable with $\beta = 0$ and $\gamma = b$

For $h(u) = u^2$, H is $\mathcal{L}_\infty$ stable with zero bias and $g(r) = r^2$. It is not finite-gain $\mathcal{L}_\infty$ stable because $|h(u)| = u^2$ cannot be bounded $\gamma|u| + \beta$ for all $u \in R$

The Local Case

- **What if the conditions of $\mathcal{L}$ stability only hold locally?**

$$y = \tan u$$

The output $y(t)$ is defined only when the input signal is restricted to $|u(t)| < \pi/2$ for all $t \geq 0$

$$u(t) \in \{|u| \leq r < \pi/2\} \quad \Rightarrow \quad |y| \leq \left(\frac{\tan r}{r}\right)|u|$$

$$\|y\|_{\mathcal{L}_p} \leq \left(\frac{\tan r}{r}\right)\|u\|_{\mathcal{L}_p}, \quad p \in [1, \infty]$$

Small Signal $\mathcal{L}$ Stability

Definition 6.3

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is small-signal $\mathcal{L}$ stable (respectively, small-signal finite-gain $\mathcal{L}$ stable) if there is a positive constant r such that the condition for $\mathcal{L}$ stability (respectively, finite-gain $\mathcal{L}$ stability) is satisfied for all $u \in \mathcal{L}_e^m$ with $\sup_{0 \le t \le \tau} \|u(t)\| \le r$

- **Notes:**
  - $||u(t)||$ **refers to a vector norm, not a function norm**

Applying $\mathcal{L}$ Stability to State-Space Models

$$\dot{x} = f(x,u), \quad y = h(x,u), \qquad 0 = f(0,0), \; 0 = h(0,0)$$

Case 1: The origin of $\dot{x} = f(x,0)$ is exponentially stable

Theorem 6.1

Suppose, $\forall \; \|x\| \le r$, $\forall \; \|u\| \le r_u$,

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x,0) \le -c_3\|x\|^2, \qquad \left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

$$\|f(x,u) - f(x,0)\| \le L\|u\|, \qquad \|h(x,u)\| \le \eta_1\|x\| + \eta_2\|u\|$$

Then, for each x_0 with $\|x_0\| \le r\sqrt{c_1/c_2}$, the system is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1,\infty]$. It is finite-gain $\mathcal{L}_p$ stable $\forall \; x_0 \in R^n$ if the assumptions hold globally [see the textbook for β and γ]

Applying $\mathcal{L}$ Stability to State-Space Models

Case 2: The origin of $\dot{x} = f(x, 0)$ is asymptotically stable

Theorem 6.2

Suppose that, for all (x, u), f is locally Lipschitz and h is continuous and satisfies

$$\|h(x, u)\| \le g_1(\|x\|) + g_2(\|u\|) + \eta, \quad \eta \ge 0$$

for some gain functions g_1, g_2. If $\dot{x} = f(x, u)$ is ISS, then, for each $x(0) \in R^n$, the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is $\mathcal{L}_\infty$ stable

Small-Signal Stability for State-Space Systems

Theorem 6.3

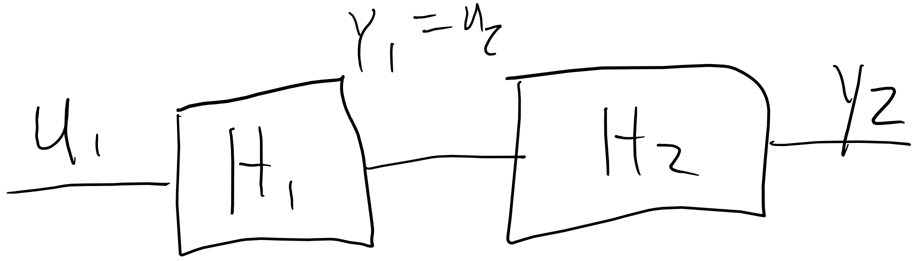
Suppose f is locally Lipschitz and h is continuous in some neighborhood of $(x = 0,\ u = 0)$. If the origin of $\dot{x} = f(x, 0)$ is asymptotically stable, then there is a constant $k_1 > 0$ such that for each $x(0)$ with $\|x(0)\| < k_1$, the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

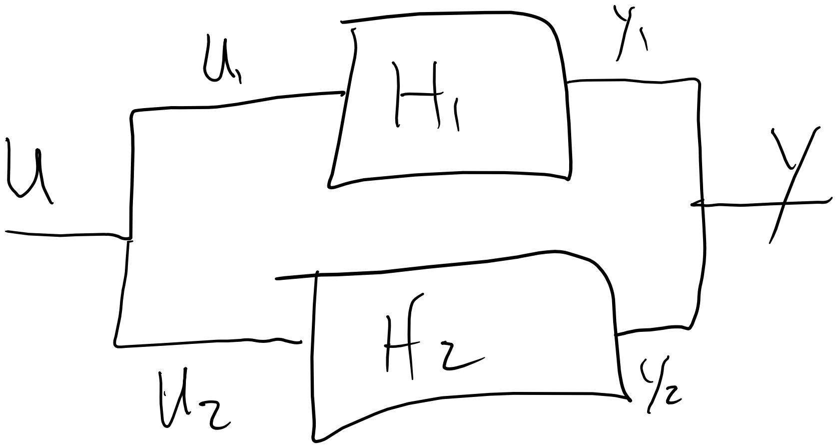
is small-signal $\mathcal{L}_\infty$ stable

Chaining Systems Together

- $\boldsymbol{y_1 = H_1 u_1}$
- $\boldsymbol{y_2 = H_2 u_2}$
- **Series:**



- **Parallel:**



Feedback Control

- Getting into the Fun Stuff™
- Here's what's ahead:
  - Basic Feedback Control Concepts
  - Normal Forms of Nonlinear Dynamics
  - Lie Derivatives
  - Feedback / Full State Linearization
  - Sliding Mode Control
  - Backstepping
  - Control Barrier Functions (CBFs)
  - Model Reference Adaptive Control (MRAC; if time permits)

Feedback Control: Stabilizing Equilibria

We want to stabilize the system

$$\dot{x} = f(x, u)$$

at the equilibrium point $x = x_{\text{ss}}$

Steady-State Problem: Find steady-state control u_{ss} s.t.

$$0 = f(x_{\text{ss}}, u_{\text{ss}})$$

$$x_\delta = x - x_{\text{ss}}, \qquad u_\delta = u - u_{\text{ss}}$$

$$\dot{x}_\delta = f(x_{\text{ss}} + x_\delta, u_{\text{ss}} + u_\delta) \stackrel{\text{def}}{=} f_\delta(x_\delta, u_\delta)$$

$$f_\delta(0, 0) = 0$$

$$u_\delta = \phi(x_\delta) \quad \Rightarrow \quad u = u_{\text{ss}} + \phi(x - x_{\text{ss}})$$

State Feedback Stabilization

State Feedback Stabilization: Given

$$\dot{x} = f(x, u) \qquad\qquad [f(0,0) = 0]$$

find

$$u = \phi(x) \qquad\qquad [\phi(0) = 0]$$

s.t. the origin is an asymptotically stable equilibrium point of

$$\dot{x} = f(x, \phi(x))$$

f and ϕ are locally Lipschitz functions

Notions of Stabilization

$$\dot{x} = f(x, u), \qquad u = \phi(x)$$

Local Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and a given region G is a subset of the region of attraction (for all $x(0) \in G$, $\lim_{t\to\infty} x(t) = 0$) (e.g., $G \subset \Omega_c = \{V(x) \le c\}$ where Ω_c is an estimate of the region of attraction)

Global Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is globally asymptotically stable

Notions of Stabilization

Semiglobal Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and $\phi(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \phi_p(x)$ is dependent on a parameter p such that for any compact set G, p can be chosen to ensure that G is a subset of the region of attraction)

What is the difference between global stabilization and semiglobal stabilization?

Example

$$\dot{x} = x^2 + u$$

Linearization:

$$\dot{x} = u, \qquad u = -kx, \quad k > 0$$

Closed-loop system:

$$\dot{x} = -kx + x^2$$

Linearization of the closed-loop system yields $\dot{x} = -kx$. Thus, $u = -kx$ achieves local stabilization

The region of attraction is $\{x < k\}$. Thus, for any set $\{-a \le x \le b\}$ with $b < k$, the control $u = -kx$ achieves regional stabilization

Example

The control $u = -kx$ does not achieve global stabilization

But it achieves semiglobal stabilization because any compact set $\{|x| \leq r\}$ can be included in the region of attraction by choosing $k > r$

The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable

Special Case: Linear Systems

Linear Systems

$$\dot{x} = Ax + Bu$$

(A, B) is stabilizable (controllable or every uncontrollable eigenvalue has a negative real part)

Find K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Typical methods:

- Eigenvalue Placement
- Eigenvalue-Eigenvector Placement
- LQR

Linearization of Nonlinear Systems

Linearization

$$\dot{x} = f(x, u)$$

$f(0, 0) = 0$ and f is continuously differentiable in a domain $D_x \times D_u$ that contains the origin $(x = 0,\ u = 0)$ $(D_x \subset R^n$, $D_u \subset R^m)$

$$\dot{x} = Ax + Bu$$

$$A = \frac{\partial f}{\partial x}(x, u)\bigg|_{x=0,u=0} \ ; \qquad B = \frac{\partial f}{\partial u}(x, u)\bigg|_{x=0,u=0}$$

Assume (A, B) is stabilizable. Design a matrix K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Linearization of Nonlinear Systems

Closed-loop system:

$$\dot{x} = f(x, -Kx)$$

Linearization:

$$\begin{aligned}\dot{x} &= \left[\frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx)\ (-K)\right]_{x=0} x \\ &= (A - BK)x\end{aligned}$$

Since $(A - BK)$ is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system

Example

Example 9.2 (Pendulum Equation)

$$\ddot{\theta} = -\sin\theta - b\dot{\theta} + cu$$

Stabilize the pendulum at $\theta = \delta_1$

$$0 = -\sin\delta_1 + cu_{\text{ss}}$$

$$x_1 = \theta - \delta_1, \qquad x_2 = \dot{\theta}, \qquad u_\delta = u - u_{\text{ss}}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -[\sin(x_1 + \delta_1) - \sin\delta_1] - bx_2 + cu_\delta \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\cos(x_1 + \delta_1) & -b \end{bmatrix}_{x_1 = 0} = \begin{bmatrix} 0 & 1 \\ -\cos\delta_1 & -b \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ -\cos\delta_1 & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -(\cos\delta_1 + ck_1) & -(b + ck_2) \end{bmatrix}$$

$$k_1 > -\frac{\cos\delta_1}{c}, \quad k_2 > -\frac{b}{c}$$

$$u = \frac{\sin\delta_1}{c} - Kx = \frac{\sin\delta_1}{c} - k_1(\theta - \delta_1) - k_2\dot{\theta}$$

Backup Slides

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NONLINEAR SYSTEMS THEORY

Mar 5, 2024

Overview

- **Midterm Review**
- **Input-Output Stability**
- **Feedback Control**

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

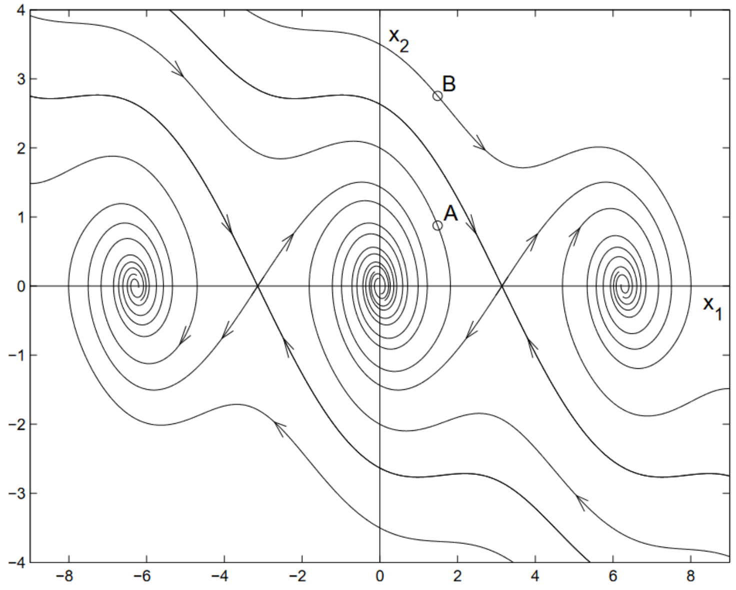
Equilibrium points

- **Autonomous Systems:**
  - **All x where $f(x) = 0$**

- **Time-Varying Systems:**
  - **x is eq. point at $t = 0$ if $f(t, x) = 0$ for all $t \geq 0$**

Phase Portraits

- **Linearize!**
- **Let $\dot{x} = f(x)$, and let $\dot{y} \approx Ay$ be the linearized model around $[p_1, p_2]^T$.**
  - **If the origin of $\dot{y} \approx Ay$ is a stable (unstable) node with distinct eigenvalues, the trajectories of the nonlinear system will behave like a stable (unstable) node in a neighborhood around $[p_1, p_2]^T$.**
  - **Similarly, if the origin of $\dot{y} \approx Ay$ is a stable (unstable) focus with distinct eigenvalues, the trajectories of the nonlinear system will behave like a stable (unstable) focus**
  - **Similarly, if the origin of $\dot{y} \approx Ay$ is a saddle, the trajectories of the nonlinear system will behave like a saddle**



Stability of Equilibria: Autonomous Systems

- $\dot{x} = f(x)$
- **Lyapunov's Indirect method: Linearize!**
  - $\dot{x} \approx Ax = \frac{\partial f}{\partial x}\Big|_{x=0}$
  - **Origin is asymptotically stable if A Hurwitz**
  - **Origin is unstable if there is an eigenvalue with Re$\lambda_i > 0$**

- **Lyapunov's Direct Method: Find a p.d. $V(x)$ such that $\dot{V}(x) \leq 0$**
  - **Quadratic: $V(x) = x^T P x$**
  - **Higher terms: $V(x) = ax^4 + bx^2$**
  - **Put $\dot{V}(x)$ in terms of $V(x)$**

LaSalle

- **Only for autonomous systems!**
- **Find continuously differentiable $V(x)$**
- **Find a positively invariant Ω**
- **Find $E = \{x : \dot{V}(x) = 0\}$**
- **Find largest invariant set M**
- **Everything converges to M!**

Time-Varying Systems

- **Lyapunov Stability**
- **Does LaSalle's Invariance principle (as given by Khalil) work for Time-Varying Systems?**
  - **NO…**
  - ***(…Unless* you use an upper / lower bound to get rid of time-varying terms and convert system to autonomous)**

Notions of Stability

- Autonomous
  - Unstable
  - Stable (in the sense of Lyapunov)
  - Asymptotically Stable
  - Exponentially Stable
  - Globally {Asymp, Exponent.} Stable
- Time-Varying
  - Uniformly Stable
  - Uniformly Asymptotically Stable
  - Exponentially Stable
  - Globally {Uniformly Asymptotically, Exponentially} Stable

Odds and Ends

- Diffeomorphisms
    - Continuously differentiable with continuously differentiable inverse
- Input to state Stability
    - What is the relationship between:
        - Global Asymptotic Stability
        - Global Exponential Stability
        - Input-to-state Stability
- Lipschitz continuity
- Radially Unbounded $V(x)$
- Descrescent (See Day 12 slides)

Backup Slides

BYU

NONLINEAR SYSTEMS THEORY

Mar 7, 2024

Overview

- Input-Output Stability
- Feedback Control
- Lie Derivatives
- Lie Brackets

Credit Where Credit is Due

- **Sources:**
  - **Hassan Khalil, Nonlinear Systems Lecture slides, https://www.egr.msu.edu/~khalil/NonlinearControl/Slides-Full/index.html**

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$$\mathcal{L}_p : \quad \|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p\ dt\right)^{1/p} < \infty, \quad 1 \leq p < \infty$$

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Example

$$u(t) = t, \qquad u_\tau(t) = \begin{cases} t, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

$u \notin \mathcal{L}_\infty$ but $u_\tau \in \mathcal{L}_{\infty e}$

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A class $\mathcal{K}$ function is a gain function but not the other way around. By not requiring the gain function to be strictly increasing we can have $g = 0$ or $g(r) = \mathrm{sat}(r)$

$\mathcal{L}$ Stability

Definition 6.2

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a gain function g, defined on $[0, \infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq g\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma\|u_\tau\|_{\mathcal{L}} + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

In this case, we say that the system has $\mathcal{L}$ gain $\leq \gamma$. The bias term β is included in the definition to allow for systems where Hu does not vanish at $u = 0$.

$\mathcal{L}$ Stability and Input-Output Stability

- **For any causal, $\mathcal{L}$ stable system $y = Hu$:**
  - $u \in \mathcal{L}^m \Longrightarrow Hu \in \mathcal{L}^q$
  - **A bounded input results in a bounded output (where bound refers to the function norm)**

- **When we use $\mathcal{L}_\infty$, we get the traditional notion of BIBO (bounded-input-bounded-output) stability!**

Example

Example 6.1: Memoryless function $y = h(u)$

Suppose $\quad |h(u)| \le a + b|u|, \quad \forall\ u \in R$

Finite-gain $\mathcal{L}_\infty$ stable with $\beta = a$ and $\gamma = b$

If $a = 0$, then for each $p \in [1, \infty)$

$$\int_0^\infty |h(u(t))|^p \, dt \le (b)^p \int_0^\infty |u(t)|^p \, dt$$

Finite-gain $\mathcal{L}_p$ stable with $\beta = 0$ and $\gamma = b$

For $h(u) = u^2$, H is $\mathcal{L}_\infty$ stable with zero bias and $g(r) = r^2$. It is not finite-gain $\mathcal{L}_\infty$ stable because $|h(u)| = u^2$ cannot be bounded $\gamma|u| + \beta$ for all $u \in R$

The Local Case

- **What if the conditions of $\mathcal{L}$ stability only hold locally?**

$$y = \tan u$$

The output $y(t)$ is defined only when the input signal is restricted to $|u(t)| < \pi/2$ for all $t \geq 0$

$$u(t) \in \{|u| \leq r < \pi/2\} \quad \Rightarrow \quad |y| \leq \left(\frac{\tan r}{r}\right)|u|$$

$$\|y\|_{\mathcal{L}_p} \leq \left(\frac{\tan r}{r}\right)\|u\|_{\mathcal{L}_p}, \quad p \in [1, \infty]$$

Small Signal $\mathcal{L}$ Stability

Definition 6.3

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is small-signal $\mathcal{L}$ stable (respectively, small-signal finite-gain $\mathcal{L}$ stable) if there is a positive constant r such that the condition for $\mathcal{L}$ stability (respectively, finite-gain $\mathcal{L}$ stability) is satisfied for all $u \in \mathcal{L}_e^m$ with $\sup_{0 \le t \le \tau} \|u(t)\| \le r$

- **Notes:**
  - **$||u(t)||$ refers to a vector norm, not a function norm**

Applying $\mathcal{L}$ Stability to State-Space Models

$$\dot{x} = f(x,u), \quad y = h(x,u), \qquad 0 = f(0,0),\ 0 = h(0,0)$$

Case 1: The origin of $\dot{x} = f(x,0)$ is exponentially stable

Theorem 6.1

Suppose, $\forall\ \|x\| \le r$, $\forall\ \|u\| \le r_u$,

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x,0) \le -c_3\|x\|^2, \qquad \left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

$$\|f(x,u) - f(x,0)\| \le L\|u\|, \qquad \|h(x,u)\| \le \eta_1\|x\| + \eta_2\|u\|$$

Then, for each x_0 with $\|x_0\| \le r\sqrt{c_1/c_2}$, the system is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1,\infty]$. It is finite-gain $\mathcal{L}_p$ stable $\forall\ x_0 \in R^n$ if the assumptions hold globally [see the textbook for β and γ]

Applying $\mathcal{L}$ Stability to State-Space Models

Case 2: The origin of $\dot{x} = f(x, 0)$ is asymptotically stable

Theorem 6.2

Suppose that, for all (x, u), f is locally Lipschitz and h is continuous and satisfies

$$\|h(x, u)\| \le g_1(\|x\|) + g_2(\|u\|) + \eta, \quad \eta \ge 0$$

for some gain functions g_1, g_2. If $\dot{x} = f(x, u)$ is ISS, then, for each $x(0) \in R^n$, the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is $\mathcal{L}_\infty$ stable

Small-Signal Stability for State-Space Systems

Theorem 6.3

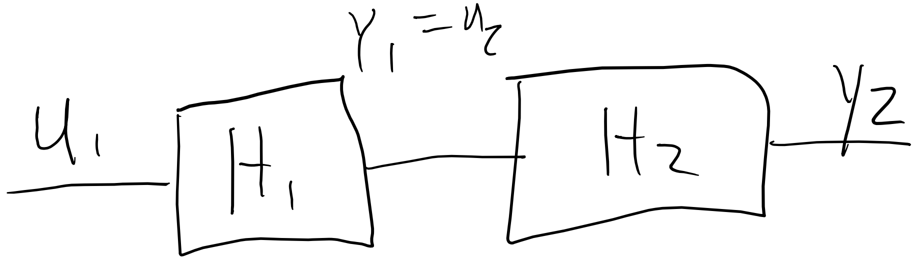
Suppose f is locally Lipschitz and h is continuous in some neighborhood of $(x = 0,\ u = 0)$. If the origin of $\dot{x} = f(x, 0)$ is asymptotically stable, then there is a constant $k_1 > 0$ such that for each $x(0)$ with $\|x(0)\| < k_1$, the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

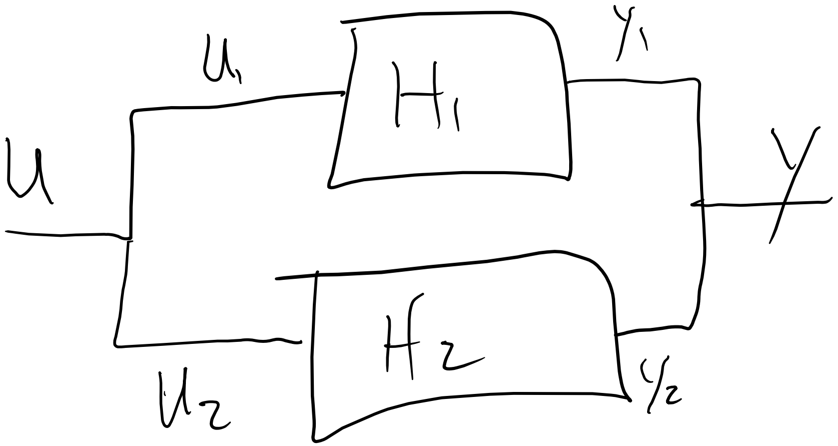
is small-signal $\mathcal{L}_\infty$ stable

Chaining Systems Together

- $\boldsymbol{y_1 = H_1 u_1}$
- $\boldsymbol{y_2 = H_2 u_2}$
- **Series:**



- **Parallel:**



Feedback Control: Stabilizing Equilibria

We want to stabilize the system

$$\dot{x} = f(x, u)$$

at the equilibrium point $x = x_{\mathrm{ss}}$

Steady-State Problem: Find steady-state control u_{ss} s.t.

$$0 = f(x_{\mathrm{ss}}, u_{\mathrm{ss}})$$

$$x_\delta = x - x_{\mathrm{ss}}, \qquad u_\delta = u - u_{\mathrm{ss}}$$

$$\dot{x}_\delta = f(x_{\mathrm{ss}} + x_\delta, u_{\mathrm{ss}} + u_\delta) \stackrel{\mathrm{def}}{=} f_\delta(x_\delta, u_\delta)$$
$$f_\delta(0, 0) = 0$$

$$u_\delta = \phi(x_\delta) \quad \Rightarrow \quad u = u_{\mathrm{ss}} + \phi(x - x_{\mathrm{ss}})$$

State Feedback Stabilization

State Feedback Stabilization: Given

$$\dot{x} = f(x, u) \qquad\qquad [f(0,0) = 0]$$

find

$$u = \phi(x) \qquad\qquad [\phi(0) = 0]$$

s.t. the origin is an asymptotically stable equilibrium point of

$$\dot{x} = f(x, \phi(x))$$

f and ϕ are locally Lipschitz functions

Notions of Stabilization

$$\dot{x} = f(x, u), \qquad u = \phi(x)$$

Local Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and a given region G is a subset of the region of attraction (for all $x(0) \in G$, $\lim_{t\to\infty} x(t) = 0$) (e.g., $G \subset \Omega_c = \{V(x) \le c\}$ where Ω_c is an estimate of the region of attraction)

Global Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is globally asymptotically stable

Notions of Stabilization

Semiglobal Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and $\phi(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \phi_p(x)$ is dependent on a parameter p such that for any compact set G, p can be chosen to ensure that G is a subset of the region of attraction)

What is the difference between global stabilization and semiglobal stabilization?

Example

$$\dot{x} = x^2 + u$$

Linearization:

$$\dot{x} = u, \qquad u = -kx, \quad k > 0$$

Closed-loop system:

$$\dot{x} = -kx + x^2$$

Linearization of the closed-loop system yields $\dot{x} = -kx$. Thus, $u = -kx$ achieves local stabilization

The region of attraction is $\{x < k\}$. Thus, for any set $\{-a \le x \le b\}$ with $b < k$, the control $u = -kx$ achieves regional stabilization

Example

The control $u = -kx$ does not achieve global stabilization

But it achieves semiglobal stabilization because any compact set $\{|x| \leq r\}$ can be included in the region of attraction by choosing $k > r$

The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable

Special Case: Linear Systems

Linear Systems

$$\dot{x} = Ax + Bu$$

(A, B) is stabilizable (controllable or every uncontrollable eigenvalue has a negative real part)

Find K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Typical methods:

- Eigenvalue Placement
- Eigenvalue-Eigenvector Placement
- LQR

Linearization of Nonlinear Systems

Linearization

$$\dot{x} = f(x, u)$$

$f(0,0) = 0$ and f is continuously differentiable in a domain $D_x \times D_u$ that contains the origin $(x = 0,\ u = 0)$ $(D_x \subset R^n$, $D_u \subset R^m)$

$$\dot{x} = Ax + Bu$$

$$A = \frac{\partial f}{\partial x}(x, u)\bigg|_{x=0,u=0} \; ; \qquad B = \frac{\partial f}{\partial u}(x, u)\bigg|_{x=0,u=0}$$

Assume (A, B) is stabilizable. Design a matrix K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Linearization of Nonlinear Systems

Closed-loop system:

$$\dot{x} = f(x, -Kx)$$

Linearization:

$$\begin{aligned} \dot{x} &= \left[\frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx)\ (-K) \right]_{x=0} x \\ &= (A - BK)x \end{aligned}$$

Since $(A - BK)$ is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system

Example

Example 9.2 (Pendulum Equation)

$$\ddot{\theta} = -\sin\theta - b\dot{\theta} + cu$$

Stabilize the pendulum at $\theta = \delta_1$

$$0 = -\sin\delta_1 + cu_{\text{ss}}$$

$$x_1 = \theta - \delta_1, \qquad x_2 = \dot{\theta}, \qquad u_\delta = u - u_{\text{ss}}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -[\sin(x_1 + \delta_1) - \sin\delta_1] - bx_2 + cu_\delta \end{aligned}$$

$$A = \left[\begin{array}{cc} 0 & 1 \\ -\cos(x_1 + \delta_1) & -b \end{array}\right]_{x_1=0} = \left[\begin{array}{cc} 0 & 1 \\ -\cos\delta_1 & -b \end{array}\right]$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ -\cos\delta_1 & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -(\cos\delta_1 + ck_1) & -(b + ck_2) \end{bmatrix}$$

$$k_1 > -\frac{\cos\delta_1}{c}, \quad k_2 > -\frac{b}{c}$$

$$u = \frac{\sin\delta_1}{c} - Kx = \frac{\sin\delta_1}{c} - k_1(\theta - \delta_1) - k_2\dot{\theta}$$

BYU

Nonlinear Systems Theory

Mar 12, 2024

Overview

- Big Picture: Controllability
- Flow Operator
- Lie Derivatives
- Lie Groups and Lie Algebras
  - The Exponential Map
- Lie Brackets
- Controllability of Nonlinear Systems

Controllability (and Reachability)

- Say we have a system $\dot{x} = f(x, u)$.
- We have:
  - An initial state $x_0 = x(t_0)$
  - A target state x_r
- Does there exist a control input $u(t)$ (or $u(x)$) that takes us from x_0 to x_r?

$$x_r = x(t) = \int_{t_0}^{t} f\big(x(t), u(t)\big)dt$$

- Controllability:
  - $x_0 \in \mathbb{R}^n$
  - $x_r = 0$ (origin)
- Reachability:
  - $x_0 = 0$ (origin)
  - $x_r \in \mathbb{R}^n$

Very simple example of a system that is *not* reachable:

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = u$$

Can we ever reach the state $[1, 0]$?

Linear Time-Invariant Systems

- **Discrete time:** $x_{k+1} = Ax_k + Bu_k$
- **Expanding terms:**
  - $x_1 = Ax_0 + Bu_0$
  - $x_2 = Ax_1 + Bu_1 = A^2x_0 + ABu_0 + Bu_1$
  - $x_3 = Ax_2 + Bu_2 = A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2$
  - …
  - $x_r = x_n = A^n x_0 + \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ u_{n-3} \\ \vdots \\ u_0 \end{bmatrix}$
  - $x_n - A^n x_0 = \mathcal{C}(A, B)\vec{u}$
    - $\mathcal{C}(A, B)$ is called the controllability matrix.

Linear Time-Invariant Systems

- $x_n - A^n x_0 = \mathcal{C}(A,B)\vec{u}$
- If $\mathcal{C}(A,B)$ is full rank, then we can solve for our control inputs $\vec{u}$!
- $\vec{u} = \mathcal{C}^+(A,B)(x_r - A^n x_0)$
  - (where $\mathcal{C}^+$ is the pseudoinverse)
- If the matrix $\mathcal{C}(A,B)$ has full rank, the pair (A,B) is said to be *controllable*.

- Similar result holds for continuous-time systems:
  - $x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$
  - Let $u(\tau) = B^T\left(e^{A(t-\tau)}\right)^T v$
  - $x(t) - e^{A(t-t_0)}x_0 = \left(\int_{t_0}^{t} e^{A(t-\tau)}BB^T\left(e^{A(t-\tau)}\right)^T d\tau\right) v$
  - $v = \left(\int_{t_0}^{t} e^{A(t-\tau)}BB^T\left(e^{A(t-\tau)}\right)^T d\tau\right)^+ \left(x(t) - e^{A(t-t_0)}x_0\right)$

Example: 3D Double Integrator (Discrete-Time)

- $A = \begin{bmatrix} 0 & I_{3\times3} \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_{3\times3} \end{bmatrix}$
- **Try computing $\mathcal{C}(A, B)$!**

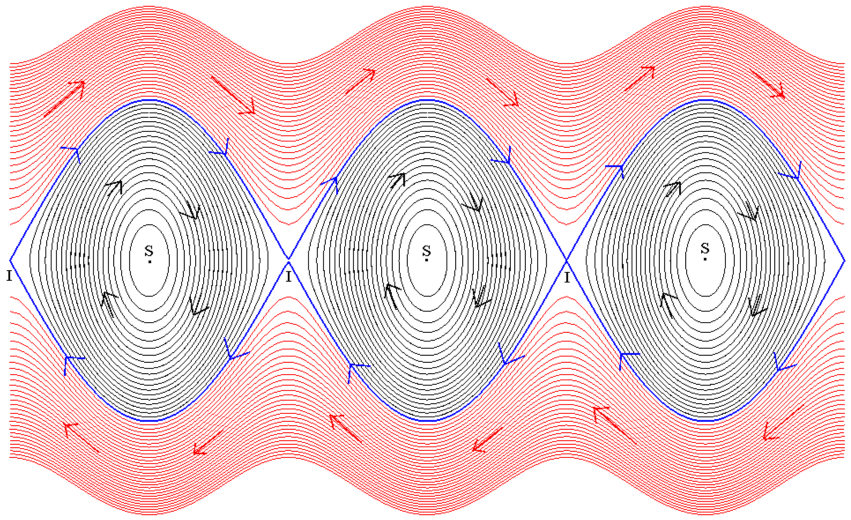
- $\mathcal{C}(A, B) = \begin{bmatrix} 0 & I & 0 & \cdots \\ I & 0 & 0 & \cdots \end{bmatrix}$

- **Full rank → Controllable**

Nonlinear Systems

- **As you can guess, the concept of controllability is harder in this case.**
- **We need some additional tools.**
- **Let's talk about *flows*.**

Go with the Flow



- Let $\dot{x} = f(x)$ be an autonomous system
  - Vector field f
- A *flow* is a function $\phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined as:
  - $\phi(x, 0) = x$
  - $\phi(x, t) = \int_0^t f(x) dt$
- Flows compose (as everyone knows)
  - $\phi(\phi(x, t), s) = \phi(x, s + t)$
- But this is a pain to write. Let's use this notation:
  - $\phi^t(x) = \phi(x, t)$
  - $\phi^t \circ \phi^s(x) = \phi^t\big(\phi^s(x)\big) = \phi^{s+t}(x)$
- This is very similar to the exponential function! Let's use the following notation:
  - $e^{tf} x = \phi^t(x)$ for $\dot{x} = f(x)$
  - $e^{tg} x = \phi^t(x)$ for $\dot{x} = g(x)$

Flows form *Groups*

- Group: Set of elements G with binary operation $\circ$ that satisfies:
  - Closure: $(e^{sf} e^{tf}) \in G$ for all $e^{sf}, e^{tf} \in G$
  - Associativity: $\left(e^{rf} e^{sf}\right) e^{tf} = e^{rf}\left(e^{sf} e^{tf}\right)$
  - Identity: There exists identity e^{0f} such that $e^{0f} e^{sf} = e^{sf}$ for all $e^{sf} \in G$
  - Inverse: For all $e^{sf} \in G$ there exists $e^{-sf} \in G$ such that $e^{sf} e^{-sf} = e^{0f}$.

- A special class of groups are *Lie Groups*:
  - Group binary operation is smooth
  - Inversion operation is smooth

Lie Derivatives

- **Let's say we have the following nonlinear system:**

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

- **Assume single-input, single-output SISO system for now:**
  - $x \in \mathbb{R}^n, u \in \mathbb{R}$
  - $f: \mathbb{R}^n \to \mathbb{R}^n$
  - $g: \mathbb{R}^n \to \mathbb{R}^{n \times 1}$
  - $h: \mathbb{R}^n \to \mathbb{R}$
- **Question: What is the rate of change of the output y?**

Lie Derivatives

- Using the chain rule:
  - $\dot{y} = \dot{h}(x)$
  - $\quad = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}f(x) + \frac{\partial h}{\partial x}g(x)u$
  - $\quad = \qquad L_f h(x) + L_g h(x)u$
- The term $L_f h(x)$ is the Lie derivative of h with respect to f (or along f)
- Very similar to Lyapunov candidates!
  - For $\dot{x} = f(x) + g(x)u$
  - $\dot{V}(x) = L_f V(x) + L_g V(x)u$
  - For non-affine $\dot{x} = f(t, x, u)$: $\dot{V}(t, x, u) = L_f V(t, x, u)$
- Lie derivatives measure *rate of change of h(x)* in the direction of a vector field

Examples

- **Dynamics:**
  - $f(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$
  - $g(x) = \begin{bmatrix} x_1^2 \\ x_1^3 x_2^2 \end{bmatrix}$
  - $h(x) = x_1$
- **Try finding:**
  - $L_f h(x)$
  - $L_g h(x)$
  - $L_f g(x)$
- **Answers:**
  - $L_f h(x) = -x_2$
  - $L_g h(x) = x_1^2$
  - $L_f g(x) = \begin{bmatrix} -x_1^3 x_2^2 \\ x_1^2 \end{bmatrix}$

Example: Euler Lagrange

- **Dynamics:** $M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$
- **States:** $z = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$
- **Revised dynamics:**
- $\dot{z} = f(z) + g(z)u$
- $\quad = \begin{bmatrix} z_2 \\ M^{-1}(z)\left(\tau_g(z) - C(z)z_2\right) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u$
- **Given** $h(z) = R - \frac{1}{2} z^T z$**, what are:**
  - $L_f h(x)$
  - $L_g h(x)$

Answers

- $L_f h(x) = -z_1^T z_2 - z_2^T M^{-1}(z)\left(\tau_g(z) - C(z) z_2\right)$
- $L_g h(x) = -z_2^T B$

Lie Brackets

- Consider two smooth vector fields f and g on a manifold,
  - $f(x) \in \mathbb{R}^n$
  - $g(x) \in \mathbb{R}^n$
- The *Lie Bracket* $[\cdot,\cdot]$ maps two vector fields into a new vector field

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

Lie Bracket Properties

- **Bilinearity:**
  - $[af + bg, h] = a[f, h] + b[g, h]$
  - $[h, af + bg] = a[h, f] + b[h, g]$
- **Alternating:** $[f, f] = 0$

- **Skew Symmetric:** $[f, g] = -[g, f]$

- **Jacobi identity:**

$$\big[[f, g], h\big] + \big[[g, h], f\big] + \big[[h, f], g\big] = 0$$

Example

- **Find the Lie bracket of**
  - $f(x) = Ax$
  - $g(x) = Bx$

- **Answer:**
  - $[f, g](x) = (BA - AB)x$

Example

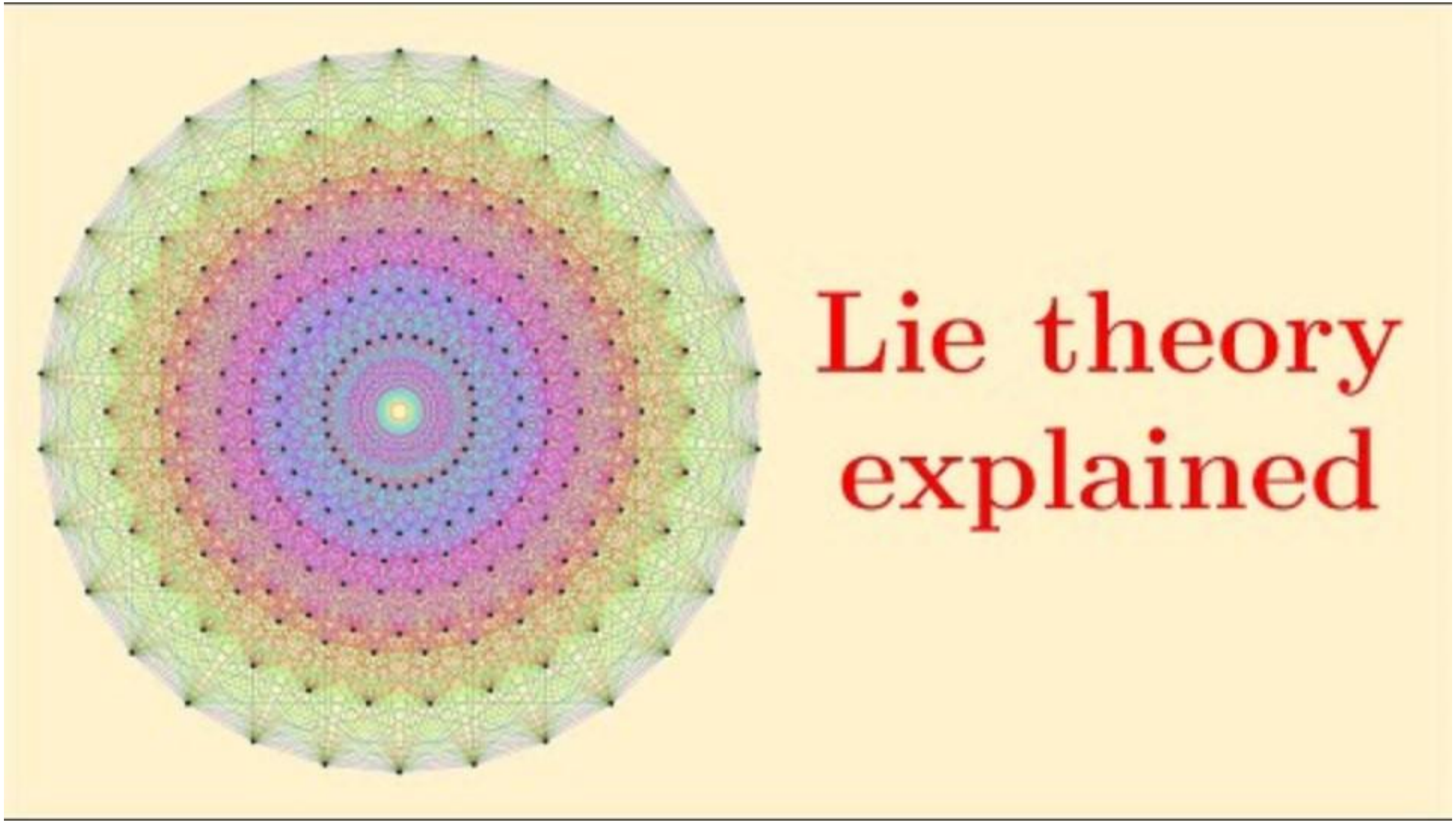
- **Find the Lie Bracket of**
  - $f(x) = \begin{bmatrix} x_2^3 \\ -x_1^2 \\ 0 \end{bmatrix}$
  - $g(x) = \begin{bmatrix} 0 \\ x_3^2 \\ -x_2 \end{bmatrix}$
- **Answer:**
  - $[f, g](x) = \begin{bmatrix} 0 \\ 0 \\ x_1^2 \end{bmatrix} - \begin{bmatrix} 3x_2^2 x_3^2 \\ 0 \\ 0 \end{bmatrix}$

Lie Algebras (and Lie Groups)

- A *Lie Algebra* is a vector space over a field equipped with a Lie bracket.
- ...But what does it represent??

- Lie Algebras are a way to relate two objects:
  - Lie Groups (what we care about, but hard to work with!)
  - Tangent spaces (Easier to work with!)
- Analogously:
  - Flows $\phi(x_0, t)$
  - Vector (velocity) fields $f(x)$

A Video Is Worth a Thousand Slides



The Exponential Map

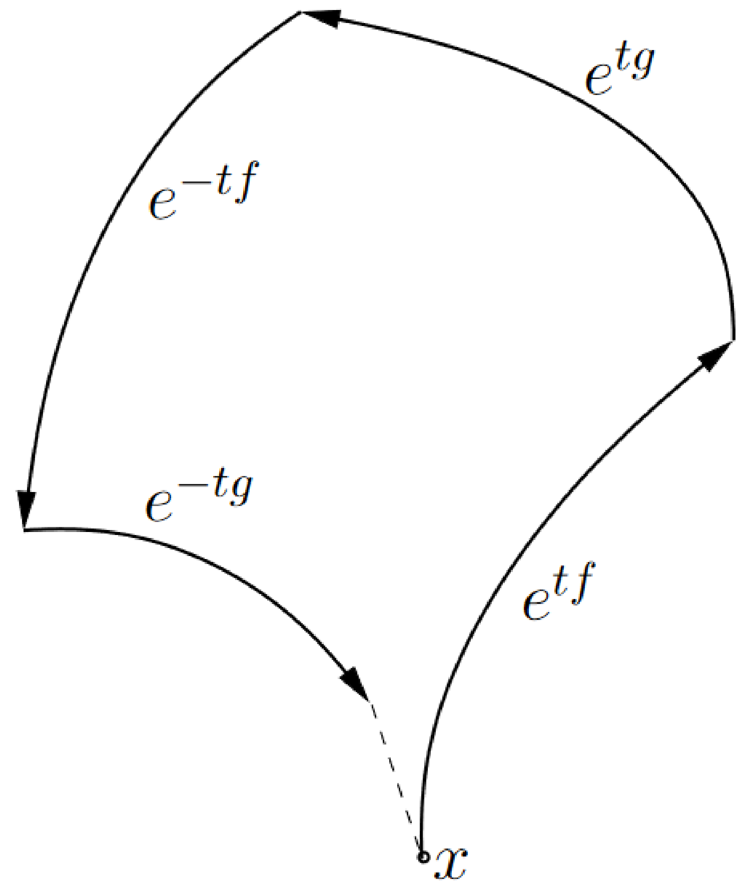
- The exponential map transforms an element of the Lie *Algebra* into an associated element of the Lie *Group*
- Example: e^{ix} for unit circle
  - Lie Algebra / Tangent space: Line $1 + ix$
  - Lie Group: 2D rotations
- Example: e^{At} for flow of $\dot{x} = Ax$.
  - Lie Algebra / Tangent Space: Vector field Ax
    - Tangent to the flow
  - Lie Group: Flow $\phi(x_0, t) = e^{At}x_0$

So What Exactly Does a Lie Bracket Do?

- A Lie Bracket quantifies how well two flows from two vector fields commute
    - Flow 1: e^{tf}, vector field $\dot{x} = f(x)$
    - Flow 2: e^{tg}, vector field $\dot{x} = g(x)$

- Does $e^{gt}e^{ft}x_0 = e^{ft}e^{gt}x_0$?

- In English, is following flow f and flow g the same as following flow g first and then flow f?



Baker-Campbell-Hausdorff Formula

- **Suppose X, Y are in the Lie algebra of a Lie group**
- **Solve the equation $e^X e^Y = e^Z$:**

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots$$

- **If $[X, Y] = 0$, then $e^X e^Y = e^{X+Y} = e^{Y+X} = e^Y e^X$**
- **If $[X, Y] \neq 0$, then $e^X e^Y \neq e^Y e^X$**

We Need More Tools

- **Given a set of smooth vector fields $\boldsymbol{f}_1, \boldsymbol{f}_2, \dots, \boldsymbol{f}_k$, define**
$$\boldsymbol{\Delta}(\boldsymbol{x}) = \text{span}\{\boldsymbol{f}_1(\boldsymbol{x}), \boldsymbol{f}_2(\boldsymbol{x}), \dots, \boldsymbol{f}_k(\boldsymbol{x})\}$$
- **This is the subspace of $\mathbb{R}^n$ spanned by the vectors $\boldsymbol{f}_i(\boldsymbol{x})$**
- **The distribution is defined as**
$$\boldsymbol{\Delta} = \bigcup_{\mathbf{x} \in \boldsymbol{D}} \boldsymbol{\Delta}(\boldsymbol{x}) = \text{span}\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_k\}$$
- **The dimension of $\boldsymbol{\Delta}(\boldsymbol{x})$ is defined as**
$$\dim\big(\boldsymbol{\Delta}(\boldsymbol{x})\big) = \text{rank}[\boldsymbol{f}_1(\boldsymbol{x}), \boldsymbol{f}_2(\boldsymbol{x}), \dots, \boldsymbol{f}_k(\boldsymbol{x})]$$

Examples

- $\boldsymbol{f}_1(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{x}_1 \\ \mathbf{0} \end{bmatrix}$
- $\boldsymbol{f}_2(\boldsymbol{x}) = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x}_2 \end{bmatrix}$
- $\boldsymbol{\Delta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$
- $\dim\left(\boldsymbol{\Delta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)\right) = \backslash\text{rank}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}\right) = \mathbf{1}$
- $\boldsymbol{\Delta} = \backslash\text{span}\left\{\begin{bmatrix} \boldsymbol{x}_1 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x}_2 \end{bmatrix}\right\} = \mathbb{R}^{\mathbf{2}}$

Involutive Distributions

- **A distribution $\Delta = \text{span}\{f_1, \dots, f_k\}$ is nonsingular on $D \subseteq \mathbb{R}^n$ if $\dim(\Delta(x)) = k$ for all $x \in D$.**

- **A distribution is involutive if**

$$f_1 \in \Delta \text{ and } f_2 \in \Delta \Longrightarrow [f_1, f_2] \in \Delta$$

  - **In other words, the Lie Bracket is also in the distribution**

- **Non-singular distributions are involutive if and only if $[f_i, f_j] \in \Delta$ for all i, j.**

Why do we care?

- **Being completely integrable is the generalization of Lipschitz continuity for ODEs on multi-dimensional manifolds.**

- **Equivalent to saying solutions exist and are unique on 1D manifolds.**

- **Frobenius theorem: A nonsingular distribution Δ is completely integrable if and only if it is involutive.**

What Does This Have To Do With Controllability?

- Remember the controllability matrix $\mathcal{C}(A, B)$?
- In essence, the Lie bracket produces the columns of the controllability matrix.
  - Tells us what directions a nonlinear system can go it.

- The Lie bracket will also help us determine if we can translate a nonlinear system into a linear one

Drift-Free Systems

- **Special case: Drift-free control affine system**

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

- **A drift-free system is controllable if:**

$$\text{span}\left\{g_1, g_2, \dots, g_m, [g_i, g_j], \dots \left[g_i, [g_j, g_k]\right], \dots\right\} = \mathbb{R}^n$$

- **Involves checking all possible Lie bracket combinations in the distribution!**

Example: Unicycle Model

BYU

Nonlinear Systems Theory

Mar 19, 2024

Overview

- Feedback Linearization
- Input-Output Linearization

The Big Idea

- *Controllability* and *Reachability* of nonlinear systems:
  - Binary “Yes / No” answer
  - “Can I drive this system to any desired reference state?”
- Determining a *good* control law is much harder.
- Much easier to design control law for *linear systems*
- Given the following model:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Can we transform our system into an *equivalent* linear system?

- (In other words, can we cheat?)

The Big Idea

- **Given the following model:**

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

- **We're looking for:**
  - **A state feedback control $u(x) = \alpha(x) + \beta(x)v$**
  - **A change of variables $z = T(x)$**
- **We want to end up with a new linear system equivalent to the nonlinear system:**

$$\dot{z} = Az + Bv$$

- **Two methods to do this:**
  - **Feedback Linearization**
  - **Input-Output Linearization**

Example: Pendulum on Cart

- **Dynamics:**

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu$$

- **In this case, we can cancel out the nonlinearities:**
  - $u = \frac{a}{c}[\sin(x_1 + \delta)] + \frac{v}{c}$
  - **Plugging in gives:**

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -bx_2 + v$$

  - **Equivalently:** $\dot{x} = Ax + Bv$ **where** $A = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - **Is this system controllable?**
  - **We can let** $v = -k_1x_1 - k_2x_2 = \mathrm{Kx}$
- **Final control law:** $u = \frac{a}{c}[\sin(x_1 + \delta)] - \frac{1}{c}(k_1x_1 + k_2x_2)$

Can We Generalize This?

- **When can we expect to be able to cancel nonlinearities?**
- **General form:**

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$$

  - $A \in \mathbb{R}^{n \times n}$
  - $B \in \mathbb{R}^{n \times p}$
  - (A, B) are controllable
  - $\gamma: \mathbb{R}^n \to \mathbb{R}^{p \times p}$ **(Matrix!) – Assumed to be nonsingular in our domain**
  - $\alpha: \mathbb{R}^n \to \mathbb{R}^p$
- **If our system has this form, we can use the feedback law**

$$u = \alpha(x) + \gamma^{-1}(x)v$$

- **Resulting system:** $\dot{x} = Ax + Bv$
- **We can then choose** $v = -Kx$ **using LQR, pole placement, etc.**
- **Final control law:** $u = \alpha(x) - \gamma^{-1}(x)Kx$

Definition of Feedback Linearizable Systems

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

$$f(0) = 0, \quad x \in R^n, \quad u \in R^m$$

Suppose there is a change of variables $z = T(x)$, defined for all $x \in D \subset R^n$, that transforms the system into the controller form

$$\dot{z} = Az + B[\psi(x) + \gamma(x)u]$$

where (A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \gamma^{-1}(x)[-\psi(x) + v] \quad \Rightarrow \quad \dot{z} = Az + Bv$$

The system is said to be feedback linearizable

Notes on the Change of Variables

- **The function $T(\cdot)$ must be a diffeomorphism.**
- **Why?**
  - **It must be invertible: $z = T(x)$ and $x = T^{-1}(z)$**
  - **The derivatives $\dot{z}, \dot{x}$ must be continuous, therefore $T(\cdot), T^{-1}(\cdot)$ must both be continuously differentiable.**
- **These are the conditions for a diffeomorphism!**
- **Choosing $T(\cdot)$ may be tricky; no closed form general solution.**

Example

- **System:**

$$\dot{x}_1 = a\sin(x_2)$$
$$\dot{x}_2 = -x_1^2 + u$$

- **We can't cancel the nonlinearity with u…**
- **Do we give up?**
  - **Never.** **Unless we can't find a Lyapunov candidate. Then giving up is understandable.**
- **Use the following change of variables:**
  - $z_1 = x_1$
  - $z_2 = a\sin(x_2) = \dot{x}_1$
- **Try computing the new equations for $\dot{z}_1, \dot{z}_2$!**

Example

- **New system:**
  - $\dot{z}_1 = z_2$
  - $\dot{z}_2 = a\cos(x_2)\left(-x_1^2 + u\right)$
- **Now we can cancel nonlinearities! Give it a shot.**
- **Answer:** $u = x_1^2 + \dfrac{1}{a\cos x_2} v$
  - **Well-defined for** $-\frac{\pi}{2} < z_2 < \frac{\pi}{2}$
- **Mapping from** $z \rightarrow x$**:**

$$x_1 = z_1$$
$$x_2 = \sin^{-1}\left(\frac{z_2}{a}\right)$$

Example

- **Plugging in for x values:**

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a\cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)\left(-z_1^2 + u\right)$$

- **Control law:**

$$u(z) = z_1^2 + \frac{1}{a\cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)}v$$

- **Final system after plugging in control law:**

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v$$

Input-Output Linearization

- **Feedback linearization involves directly controlling the state x**
- **What if we instead want to control our system *output*?**
  - **Output $h(x)$ may be a nonlinear function of x**
- **System:**

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

- **Objective: Control the output y**
  - **In particular, our system becomes $\dot{y} = F(x, u)$**

Special Case: Single-Input-Single-Output (SISO)

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

- $x \in \mathbb{R}, \ \ y \in \mathbb{R}$
- **Let's look at $\dot{y}$:**

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = L_f h(x) + L_g h(x)u$$

- **Question: What happens if $L_g h(x)u = 0$?**
- **Example: Double Integrator that only observes position**

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- **What is $L_g h(x)$?**
- **Answer:** $\frac{\partial h}{\partial x} g(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

Relative Degree

- If $L_g h(x) = 0$, then the control input doesn't appear in $\dot{y}$.
- So how do we control our output?

Relative Degree

- **Let's take the second derivative of y!**

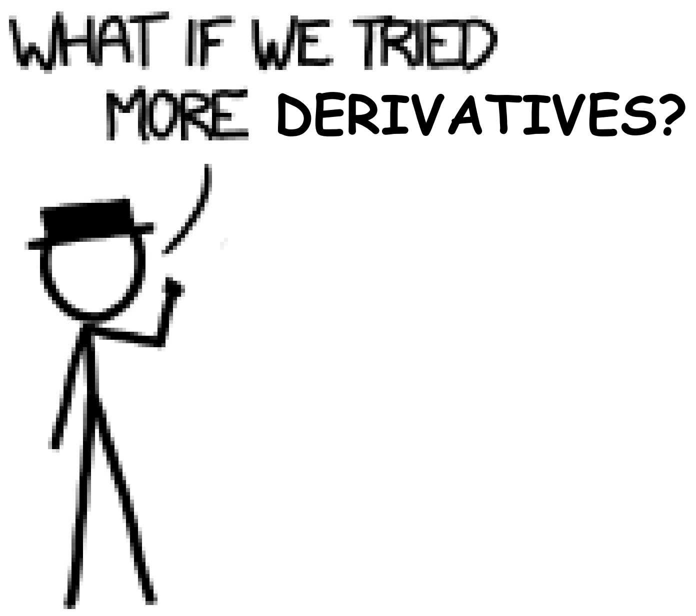
$$y^{(1)} = L_f h(x)$$

$$y^{(2)} = \frac{\partial\left(L_f h(x)\right)}{\partial x}[f(x) + g(x)u]$$

$$= L_f L_f h(x) + L_g L_f h(x) u$$

$$= L_f^2 h(x) + L_g L_f h(x) u$$

- **...But in some cases we may have $L_g L_f h(x) u = 0$**



Relative Degree

- **We can keep taking derivatives until u appears:**

$$
\begin{gathered}
y^{(3)} = L_f^3 h(x) \\
y^{(4)} = L_f^4 h(x) \\
\vdots \\
y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u
\end{gathered}
$$

- **We stop when $L_g L_f^{\rho-1} h(x) \neq 0$ for some $\rho \in \mathbb{Z}_{\geq 0}$**
- **Informally, the minimum value of ρ for which u appears is called the *relative degree* of the nonlinear system.**

Relative Degree

Definition 8.1

The system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ, $1 \leq \rho \leq n$, in $\mathcal{R} \subset D$ if $\forall\ x \in \mathcal{R}$

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \qquad L_g L_f^{\rho-1} h(x) \neq 0$$

Example: Controlled Van der Pol Equation

Controlled van der Pol equation

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2/\varepsilon, \quad \ddot{y} = \dot{x}_2/\varepsilon = -x_1 + x_2 - \tfrac{1}{3}x_2^3 + u$$

Relative degree two over R^2

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_2$$

$\dot{y} = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u]$, Relative degree one over R^2

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = \tfrac{1}{2}(\varepsilon^2 x_1^2 + x_2^2)$$

$$\dot{y} = \varepsilon^2 x_1 \dot{x}_1 + x_2 \dot{x}_2 = \varepsilon x_2^2 - (\varepsilon/3)x_2^4 + \varepsilon x_2 u$$

Relative degree one in $\{x_2 \neq 0\}$

- **The relative degree depends on *both*:**
  - **The dynamics f, g**
  - **The output function h**
- **Different output functions can result in different relative degrees**

Example: Field-Controlled DC Motor

- System:

$$\dot{x}_1 = -ax_1 + u$$
$$\dot{x}_2 = -bx_2 + k - cx_1x_3$$
$$\dot{x}_3 = \theta x_1 x_2$$

- Where x_1, x_2, x_3 are field, armature current, angular velocity
- Speed control: $y = x_3$
- Output derivatives:

$$y^{(1)} = \dot{x}_3 = \theta x_1 x_2$$
$$y^{(2)} = \theta(x_1\dot{x}_2 + \dot{x}_1 x_2) = (\cdots) + \theta x_2 u$$

- System has relative degree two in the region

$$D = \{x \in \mathbb{R}^3 : x_2 \neq 0\}$$

Example

- **System:**

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2 + u$$
$$y = x_1$$

- **What is the relative degree?**
- **Trick question:** $y^{(1)} = \dot{x}_1 = x_1 = y$
  - **The output is independent of the input** u

Try it!

- **Modified Double Integrator Unicycle**

$$\begin{aligned}\dot{x}_1 &= x_4 \cos(x_3)\\ \dot{x}_2 &= x_4 \sin(x_3)\\ \dot{x}_3 &= x_5\\ \dot{x}_4 &= -cu\\ \dot{x}_5 &= u\end{aligned}$$

- **What is relative degree for $h(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$?**
- $L_g h(x) = 0$
- $L_f h(x) = (x_1 c\theta + x_2 s\theta) x_4$
- $L_g L_f h(x) = -c(x_1 c\theta + x_2 s\theta)$ **which is nonzero for $x_1, x_2 \neq 0$**
- **System has relative degree 2 on $D = \{x: x_1, x_2 \neq 0\}$**

Controlling SISO Systems with Relative Degree

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

- For SISO system with relative degree ρ, we have
$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u$$
- Control law:
$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left[-L_f^{\rho} h(x) + v \right]$$
- Plugging in control law yields $y^{\rho} = v$
  - We have a linear system! Define $z = \left[y, y^{(1)}, y^{(2)}, \cdots, y^{(\rho-1)} \right]$
  - Then $\dot{z} = Az + Bv$
  - (Chain of integrators)

Wrench in the Gears

- Consider the following system:

$$\dot{x}_1 = x_1 u$$
$$\dot{x}_2 = -x_1^2 + u$$
$$y = x_2$$

- Choose a control input u to make $\dot{y}$ a linear system.
- Answer: $u = x_1^2 + v$
- ...But what happens to x_1?
    - x_1 is unobservable – it doesn't show up in y
    - The control law that linearizes y can make x_1 unstable
- This is an example of internal dynamics

Linear Transfer Functions and Internal Dynamics

- **Let's take a closer look at internal dynamics.**
- **We'll start with a linear transfer function**

- **What's the relative degree?**

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & & & \dots & 0 \\ 0 & 0 & 1 & \dots & & & \dots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ & & & \ddots & & & & \\ & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & & & & & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_m & \dots & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \end{bmatrix}$$

Relative Degree of Linear Transfer Function

- We have…
  - $y^{(1)} = CAx + CBu$
  - If $m = n - 1$, then $CB = b_{n-1} \neq 0$ and we have relative degree 1
- Otherwise…
  - $CA^{i-1}B = 0$ for $i = 1, \dots, n - m - 1$
  - $CA^{n-m-1}B = b_m \neq 0$
- Basically, u appears first in the expression for $y^{(n-m)}$
- The relative degree is equivalent to relative degree between polynomials

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & & & \dots & 0 \\ 0 & 0 & 1 & \dots & & & \dots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ & & & \ddots & & & & \\ & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & & & & & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_m & \dots & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \end{bmatrix}$$

Splitting into Observable / Internal Dynamics

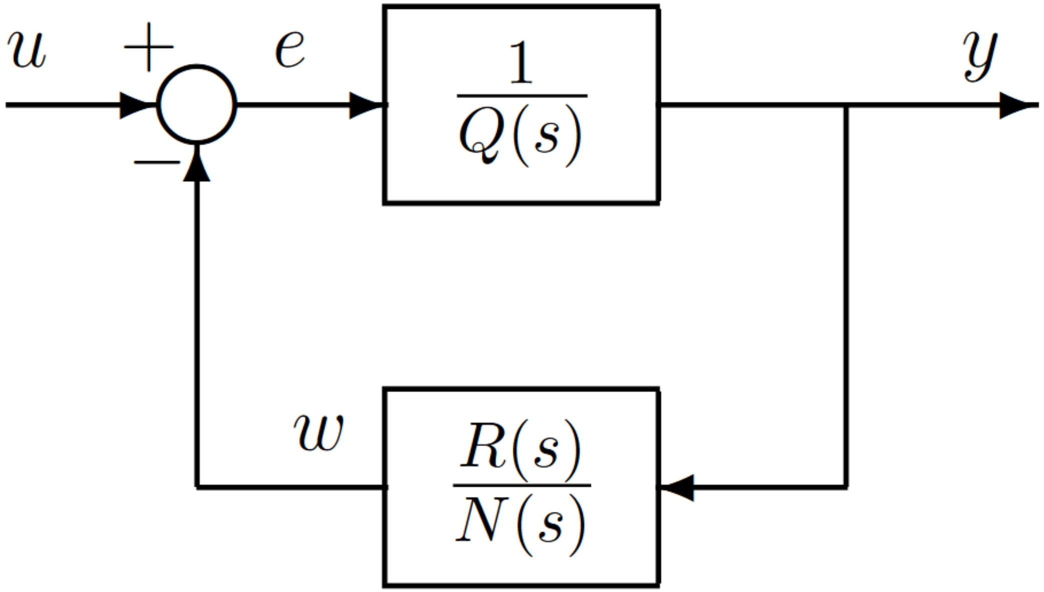
- **Let's try to split this linear system into observable dynamics and internal dynamics**
- **First, we can split $H(s) = \frac{N(s)}{D(s)}$ as follows:**
  - **The denominator polynomial satisfies**
    $$D(s) = Q(s)N(s) + R(s)$$
    - $Q(s)$**: Quotient polynomial**
    - $R(s)$**: Remainder polynomial**
  - **The degrees are:**
    - $deg(Q) = n - m = \rho$ **(relative degree)**
    - $deg(R) < m$

Splitting into Observable / Internal Dynamics

- **Using this, we can write $H(s)$ as:**

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{Q(s)N(s) + R(s)} = \frac{\frac{1}{Q(s)}}{1 + \frac{1}{Q(s)} \frac{R(s)}{N(s)}}$$

- **Block diagram:**
  - **Negative feedback!**
  - **Forward path:** $\frac{1}{Q(s)}$
  - **Feedback path:** $\frac{R(s)}{N(s)}$



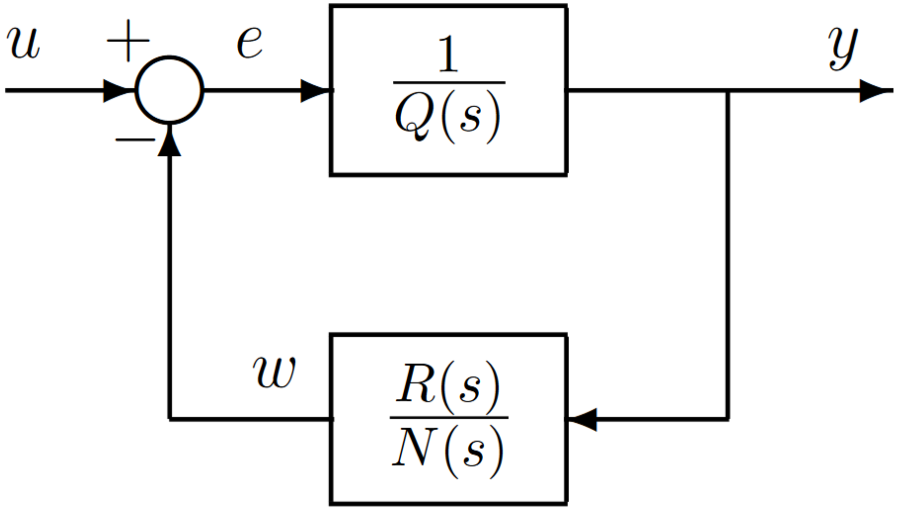
Splitting into Observable / Internal Dynamics

- **Since the numerator is 1, the transfer function $\frac{1}{Q(s)}$ has no zeros and has an equivalent integrator state space form:**

State model of $1/Q(s)$: $\xi = \mathrm{col}\left(y, \dot{y}, \ldots, y^{(\rho-1)}\right)$

$$\dot{\xi} = (A_c + B_c\lambda^T)\xi + B_c b_m e, \qquad y = C_c\xi$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; C_c = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

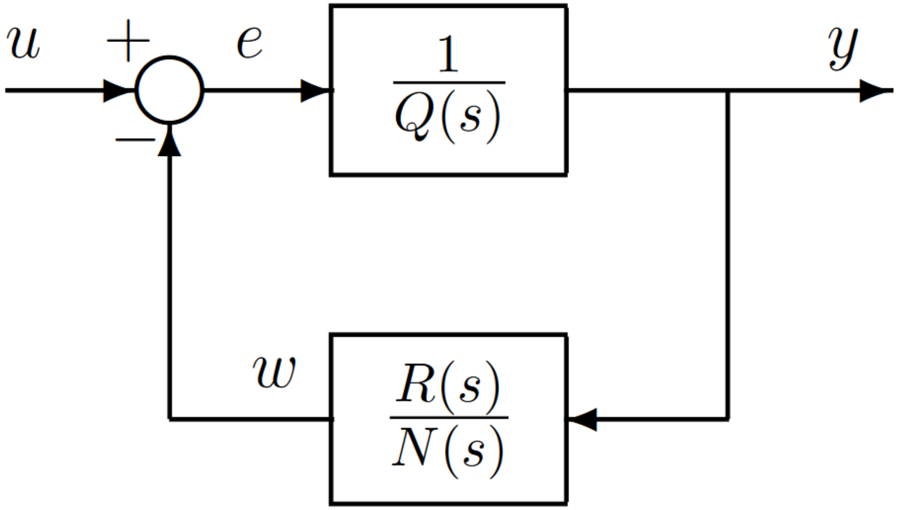


Splitting into Observable / Internal Dynamics

- **The feedback transfer function $\frac{R(s)}{N(s)}$ also has a state space equivalent form:**

State model of $R(s)/N(s)$

$$\dot{\eta} = A_0\eta + B_0 y, \qquad w = C_0\eta$$



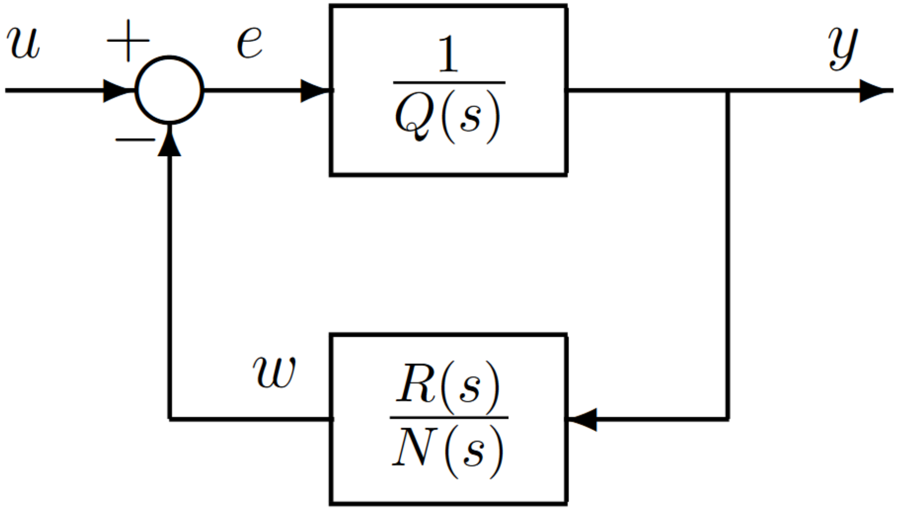
Splitting into Observable / Internal Dynamics

- **With their powers combined…**
- **We have a state space model for $H(s)$**

State model of $H(s)$

$$\begin{aligned}\dot{\eta} &= A_0\eta + B_0 C_c \xi \\ \dot{\xi} &= A_c\xi + B_c(\lambda^T\xi - b_m C_0 \eta + b_m u) \\ y &= C_c\xi\end{aligned}$$

The eigenvalues of A_0 are the zeros of $H(s)$



Controlling the Observable Dynamics

- **You can verify that $y^{(\rho)} = \lambda^T \xi - b_m C_0 \eta + b_m u$**
- **How do we cancel the difficult terms?**
  - $u = \frac{1}{b_m}[-\lambda^T \xi + b_m C_0 \eta + v]$ **cancels out the nonlinearities!**
- **We then have:**
  - $\dot{\eta} = A_0 \eta + B_0 C_c \xi$ **(Internal Dynamics)**
  - $\dot{\xi} = A_c \xi + B_c v$ **(Observable Dynamics)**
  - $y = C_c \xi$
- **Our observable dynamics are a linear chain of integrators!**

Controlling the Observable Dynamics

- Suppose we have a reference r
- We track it by setting $\xi^* = [r, 0, \dots, 0]^T$
- Define error $e_r = (\xi - \xi^*)$
- Define $v = -K(\xi - \xi^*)$ such that $(A_c - B_c K)$ is Hurwitz
- Then we have:

$$u = \frac{1}{b_m}[-\lambda^T \xi + b_m C_0 \eta - K(\xi - \xi^*)]$$

- Final closed-loop system:

$$\dot{\eta} = A_0 + B_0 C_c(\xi^* + e_r)$$
$$\dot{e}_r = (A_c - B_c K) e_r$$

What did we just do??

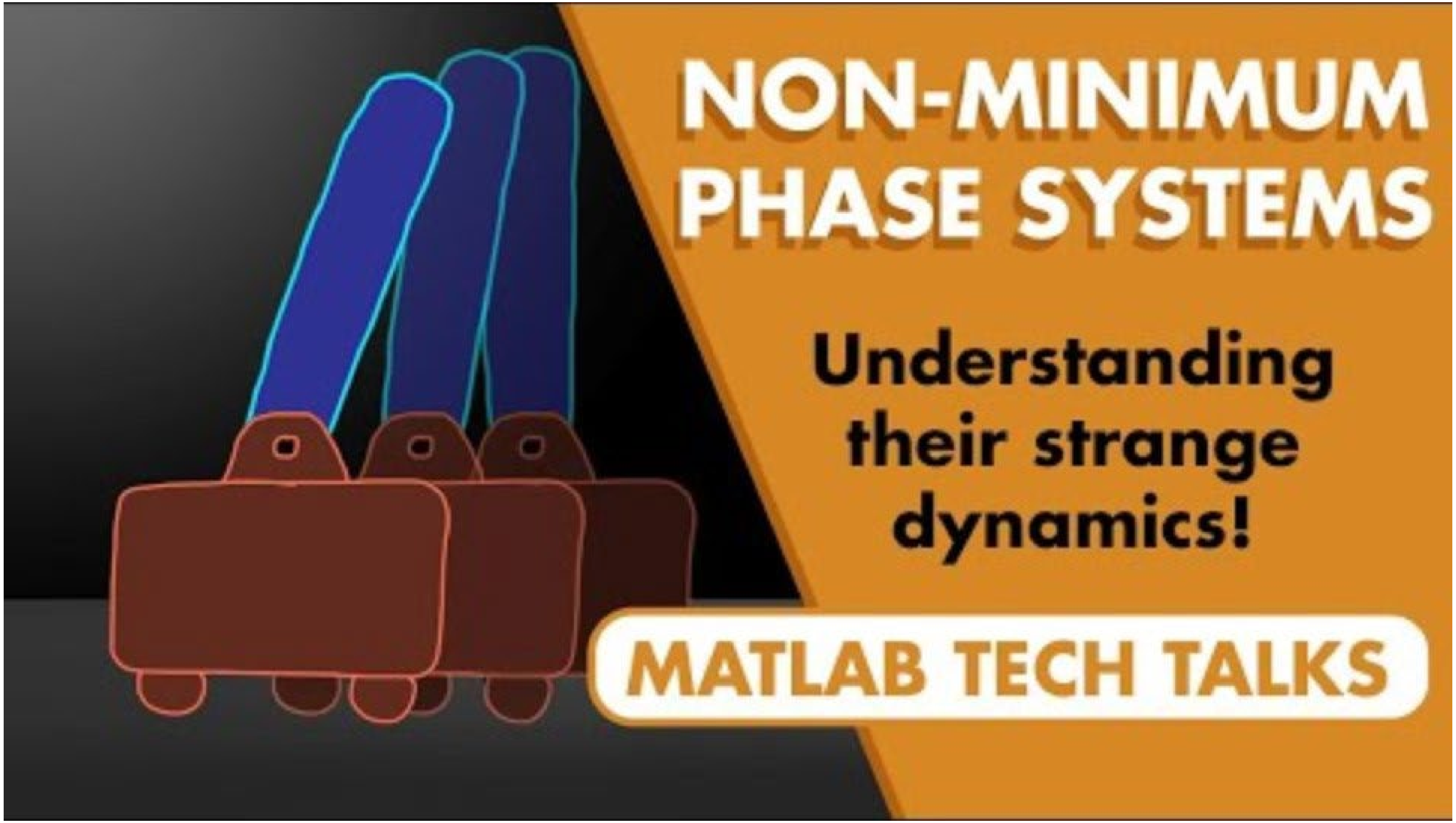
- We took a linear transfer function
- We divided it into two subsystems:
  - Observable dynamics
  - Internal dynamics
- We transformed the observable dynamics into a chain of integrators
- We derived a control law that controls the output y
  - This can be used to stabilize the observable dynamics

But Wait!

- **What about $\dot{\eta} = A_0 + B_0 C_c(\xi^* + e_r)$? (Internal dynamics)**
  - **Does it have a (asymptotically) stable equilibrium?**
  - **Is $\eta(t)$ at least bounded?**
  - **Remember, u is a function of η!**
- **To ensure $\eta(t)$ is bounded:**
  - **A_0 must be Hurwitz**
  - **Equivalently, the zeros of $H(s)$ must be in the open left half plane**
- **A transfer function with all zeros in the open left-half plane is called minimum phase.**

https://youtu.be/jGEkmDRsq_M?si=UFW6-PBsyNdhQDxY

Non-minimum Phase Systems



What about Nonlinear Systems?

- We divided the linear system into:
  - Observable states ξ
  - Internal states η
- Just like earlier, it would be nice if we had a diffeomorphism to transform our system

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix}$$

ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on a domain $D_x \subset \mathcal{R}$

$$\text{When } \rho = n, \quad z = T(x) = \psi(x) = \xi$$

Conditions on the Diffeomorphism

$$
\begin{aligned}
\dot{\eta} &= \frac{\partial \phi}{\partial x}[f(x) + g(x)u] \;=\; f_0(\eta, \xi) + g_0(\eta, \xi)u \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\; u \\
y &= \xi_1
\end{aligned}
$$

Choose $\phi(x)$ such that $T(x)$ is a diffeomorphism and

$$
\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \le i \le n - \rho, \;\forall\; x \in D_x
$$

Always possible (at least locally)

$$
\dot{\eta} = f_0(\eta, \xi)
$$

Conditions for this Diffeomorphism to Exist

Theorem 8.1

Suppose the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ $(\leq n)$ in $\mathcal{R}$. If $\rho = n$, then for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 exists such that the map $T(x) = \psi(x)$, restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 and smooth functions $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ exist such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n - \rho$$

is satisfied for all $x \in N$ and the map $T(x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}$, restricted to N, is a diffeomorphism on N

The Normal Form

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c \xi + B_c \left[L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x)\ u \right] \\
y &= C_c \xi
\end{aligned}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

How to Control the Normal Form

- **Can you determine the control input that linearizes this system?**

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c \xi + B_c \left[L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\ u \right] \\
y &= C_c \xi
\end{aligned}
$$

- **Answer:**
  - $\boldsymbol{u(x) = \dfrac{v - L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}}$
- **If x^* is an open loop equilibrium point of the original dynamics $\dot{x} = f(x) + g(x)u$…**
- **…Then (η^*, ξ^*) is an equilibrium point of the normal form system:**

$$
\begin{gathered}
\eta^* = \phi(x^*) \\
\xi^* = [h(x), 0, \cdots, 0]
\end{gathered}
$$

Zero Dynamics

- **What happens when we reach the equilibrium?**

$$y(t) \equiv 0 \;\Rightarrow\; \xi(t) \equiv 0 \;\Rightarrow\; u(t) \equiv -\frac{L_f^\rho h(x(t))}{L_g L_f^{\rho-1} h(x(t))}$$

$$\Rightarrow\; \dot{\eta} = f_0(\eta, 0)$$

Definition

The equation $\dot{\eta} = f_0(\eta, 0)$ is called the *zero dynamics* of the system. The system is said to be *minimum phase* if the zero dynamics have an asymptotically stable equilibrium point in the domain of interest (at the origin if $T(0) = 0$)

Zero Dynamics Behavior at Zero Output

- **If the output $y(t) = 0$, our state is restricted to the following set:**

$$Z^* = \{x \in \mathcal{R} \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0\}$$

$$y(t) \equiv 0 \;\Rightarrow\; x(t) \in Z^*$$

- **The control input is then:**

$$u = u^*(x) \stackrel{\text{def}}{=} - \left. \frac{L_f^{\rho} h(x)}{L_g L_f^{\rho-1} h(x)} \right|_{x \in Z^*}$$

- **Restricted system motion:**

$$\dot{x} = f^*(x) \stackrel{\text{def}}{=} \left[f(x) - g(x) \frac{L_f^{\rho} h(x)}{L_g L_f^{\rho-1} h(x)} \right]_{x \in Z^*}$$

Special Case

- **If our relative degree is equal to the dimension of the state space $\rho = n$, then we don't have zero dynamics!**
- **Normal form in this case:**

$$\dot{z} = A_c z + B_c \gamma(x)[u - \alpha(x)]$$
$$y = C_c z$$

- **Here, $z = \left[h(x), \cdots, L_f^{n-1} h(x)\right]^T$ and the η variable doesn't exist!**
- **This special case system:**
  - **Has no zero dynamics**
  - **Is said to be minimum phase by default**

Full State Linearization

- You may have noticed that the special case looks suspiciously familiar to the formula for Feedback Linearization
- System: $(x \in \mathbb{R}^n, u \in \mathbb{R}^m)$
$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$
- This system is feedback linearizable if and only if a function $h(x)$ exists such that the system has relative degree n
- ...But when does such an $h(x)$ exist?

Your Life is a Lie (Bracket)

- **Remember Lie Brackets?**

$$[f, g](x) = L_f g(x) - L_g f(x)$$

- **Let's define some more notation:**

$$ad_f^0 g(x) = g(x)$$
$$ad_f^1 g(x) = [f, g](x)$$
$$ad_f^k g(x) = \left[f, ad_f^{\{k-1\}} g\right](x), k \geq 1$$

- **Also, remember involutive distributions?**

Involutive Distributions

- **A distribution $\Delta = \mathrm{span}\{f_1, \dots, f_k\}$ is nonsingular on $D \subseteq \mathbb{R}^n$ if $\dim(\Delta(x)) = k$ for all $x \in D$.**

- **A distribution is involutive if**

$$f_1 \in \Delta \text{ and } f_2 \in \Delta \Longrightarrow [f_1, f_2] \in \Delta$$

  - **In other words, the Lie Bracket is also in the distribution**

- **Non-singular distributions are involutive if and only if $[f_i, f_j] \in \Delta$ for all i, j.**

Feedback Linearization Theorem

Theorem 8.2

The n-dimensional single-input system

$$\dot{x} = f(x) + g(x)u$$

is feedback linearizable in a neighborhood of $x_0 \in D$ if and only if there is a domain $D_x \subset D$, with $x_0 \in D_x$, such that

1. the matrix $\mathcal{G}(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ has rank n for all $x \in D_x$;
2. the distribution $\mathcal{D} = \mathrm{span}\ \{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in D_x.

What It All Means

- We can treat certain nonlinear systems as linear systems
- We can divide nonlinear systems into:
  - Observable states
  - Internal states
- If we can pick an observation function $h(x)$ so the system has relative degree $\rho = n$,
  - Our system is feedback linearizable
  - There are no internal states!
- We can find such an $h(x)$ if and only if the conditions in the previous theorem are satisfied
  - In general there is no closed-form solution
  - We only know it exists

Backup Slides

BYU

Nonlinear Systems Theory

Mar 28, 2024

Overview

- How to Train Your Diffeomorphism
- Examples of {Feedback, Full State} Linearization
- Sliding Mode Control

From Last Time:

Definition of Feedback Linearizable Systems

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

$$f(0) = 0, \quad x \in R^n, \quad u \in R^m$$

Suppose there is a change of variables $z = T(x)$, defined for all $x \in D \subset R^n$, that transforms the system into the controller form

$$\dot{z} = Az + B[\psi(x) + \gamma(x)u]$$

where (A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \gamma^{-1}(x)[-\psi(x) + v] \quad \Rightarrow \quad \dot{z} = Az + Bv$$

The system is said to be feedback linearizable

Notes on the Change of Variables

- **The function $T(\cdot)$ must be a diffeomorphism.**
- **Why?**
  - **It must be invertible: $z = T(x)$ and $x = T^{-1}(z)$**
  - **The derivatives $\dot{z}, \dot{x}$ must be continuous, therefore $T(\cdot), T^{-1}(\cdot)$ must both be continuously differentiable.**

- **These are the conditions for a diffeomorphism!**

- **Choosing $T(\cdot)$ may be tricky; no closed form general solution.**

Feedback Linearization Theorem

Theorem 8.2

The n-dimensional single-input system

$$\dot{x} = f(x) + g(x)u$$

is feedback linearizable in a neighborhood of $x_0 \in D$ if and only if there is a domain $D_x \subset D$, with $x_0 \in D_x$, such that

1. the matrix $\mathcal{G}(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ has rank n for all $x \in D_x$;
2. the distribution $\mathcal{D} = \text{span}\,\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in D_x.

“How do I choose a Diffeomorphism?”

- **The previous theorem says that a diffeomorphism *exists*…**
- **…But doesn’t tell us how to find it.**
  - **Classic mathematical attitude.**

- **Finding $T(x)$ is hard in general, but a few tips are given in the Nonlinear *Systems* book (Section 13.3).**

Partial Differential Equalities

- **Start with our system models:**

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

- **Diffeomorphism:** $z = T(x)$
- **Time derivative of diffeomorphism:**

$$\dot{z} = \frac{\partial T}{\partial x}\dot{x} = \frac{\partial T}{\partial x}(f(x) + g(x)u)$$

- **Feedback Linearized system:**

$$\dot{z} = A_c z + B_c \gamma(x)[u - \alpha(x)]$$

- **Let's set the two expressions for $\dot{z}$ equal and see what happens**

Finding $T(x)$

- We have:

$$A_c z + B_c \gamma(x)[u - \alpha(x)] = \frac{\partial T}{\partial x}(f(x) + g(x)u)$$

- Let's set $u = 0$ and see what happens:

$$\frac{\partial T}{\partial x} f(x) = A_c T(x) - B_c \alpha(x)\gamma(x)$$

$$\frac{\partial T}{\partial x} g(x) = B_c \gamma(x)$$

- Remember the structure of A_c and B_c?
  - Chains of integrators
  - We can leverage this

Finding $T(x)$

- The equation $\frac{\partial T}{\partial x} f(x) = A_c T(x) - B_c \alpha(x)\gamma(x)$ is equivalent to:

$$\frac{\partial T_1}{\partial x} f(x) = T_2(x)$$
$$\frac{\partial T_2}{\partial x} f(x) = T_3(x)$$
$$\vdots$$
$$\frac{\partial T_{n-1}}{\partial x} f(x) = T_n(x)$$
$$\frac{\partial T_n}{\partial x} f(x) = -\alpha(x)\gamma(x)$$

Finding $T(x)$

- **The equation** $\frac{\partial T}{\partial x} g(x) = B_c \gamma(x)$ **is equivalent to:**

$$\frac{\partial T_1}{\partial x} g(x) = 0$$

$$\frac{\partial T_2}{\partial x} g(x) = 0$$

$$\vdots$$

$$\frac{\partial T_{n-1}}{\partial x} g(x) = 0$$

$$\frac{\partial T_n}{\partial x} g(x) = \gamma(x) \neq 0$$

Relating $T(x)$ to $h(x)$

- **Let's set $h(x) = T_1(x)$!**
- **We get:**
  - $T_{i+1}(x) = L_f T_i(x) = L_f^i(x)$ **for** $i = 1, 2, \dots, n-1$
- $h(x)$ **satisfies the condition** $L_g L_f^{i-1} h(x) = 0, i = 1, \dots, n-1$ **subject to:**
  - $L_g L_f^{n-1} h(x) \neq 0$
- **The functions $\alpha(x)$ and $\gamma(x)$ are given by:**
  - $\gamma(x) = L_g L_f^{n-1} h(x)$
  - $\alpha(x) = -\frac{L_f^n h(x)}{L_g L_f^{n-1} h(x)}$

Example

- **Show that the following system is feedback linearizable (in a local neighborhood) on the domain $D_0 = \{x \in \mathbb{R}^2 : cos(x_2) \neq 0\}$:**

$$\dot{x} = \begin{bmatrix} a\,sin(x_2) \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- **Hint: Check the two conditions:**

1. the matrix $\mathcal{G}(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ has rank n for all $x \in D_x$;
2. the distribution $\mathcal{D} = \text{span}\,\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in D_x.

Example

- **Condition 1:** $n = 2$ and $n - 1 = 1$, so need to check rank of $[g, ad_f g(x)]$

$$[g, ad_f g(x)] = \begin{bmatrix} 0 & -acos(x_2) \\ 1 & 0 \end{bmatrix}$$

- This is full rank in the domain $D_0 = \{x \in \mathbb{R}^2 : cos(x_2) \neq 0\}$
- **Condition 2:** $n - 2 = 0$, so we need to check whether $\Delta = \mathrm{span}\{g\} = \mathrm{span}\{ad_f^{n-2} g\}$ is involutive
  - But $[g, g] = 0$ by definition of the Lie bracket!
  - And $0 = [g, g] \in \mathrm{span}\{g\} = \Delta$
- Both conditions satisfied, so our system is feedback linearizable

But what about $T(x)$?

- **Let's use the method on the previous slides to reconstruct a possible $T(x)$**
  - **Multiple may exist**
- **First, let $h(x) = T_1(x)$**
- **Then we should have:**

$$\frac{\partial h}{\partial x} g = 0$$

$$\frac{\partial (L_f h)}{\partial x} g \neq 0$$

- **We also want $h(0) = 0$**

Example: Finding $T(x)$

- Since $g = [0, 1]^T$, we have $\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0$
- Therefore $h(x)$ is independent of x_2.
- What can we say about $L_f h(x)$ from this?

$$L_f h(x) = \frac{\partial h}{\partial x_1} a \sin(x_2)$$

- We can plug this into our second condition $\frac{\partial(L_f h)}{\partial x} g \neq 0$:

$$\frac{\partial(L_f h)}{\partial x} g = \frac{\partial(L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos(x_2) \neq 0$$

- The condition $\frac{\partial h}{\partial x_1} a \cos(x_2) \neq 0$ is satisfied in our domain for any choice of $h(x)$ for which $\frac{\partial h}{\partial x_1} \neq 0$

Example: Finding $T(x)$

- **To summarize our "clues" so far:**
  - $h(x)$ **is independent of** x_2
  - $\frac{\partial h}{\partial x_1} \neq 0$
- **Let's try** $h(x) = x_1$
  - **Start simple**
  - **We could also try** $h(x) = x_1 + x_1^3$
- **This implies:**
  - $T_1(x) = h(x) = x_1$
  - $T_2(x) = \frac{\partial T_1}{\partial x} f(x) = a\backslash\sin(x_2)$
- **Therefore** $z = T(x) = \begin{bmatrix} x_1 \\ a\sin(x_2) \end{bmatrix}$
- **Try checking if** $\dot{z}$ **is in feedback linearizable form!**

Example: Checking $T(x)$

- **We have:**

$$\dot{z} = \begin{bmatrix} \dot{x}_1 \\ a\cos(x_2)\dot{x}_2 \end{bmatrix} = \begin{bmatrix} a\sin(x_2) \\ a\cos(x_2)(-x_1^2 + u) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a\cos(x_2)[u - x_1^2]$$

$$\dot{z} = A_c z + B_c \gamma(x)[u - \alpha(x)]$$

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Nonlinear Systems Theory

Apr 2, 2024

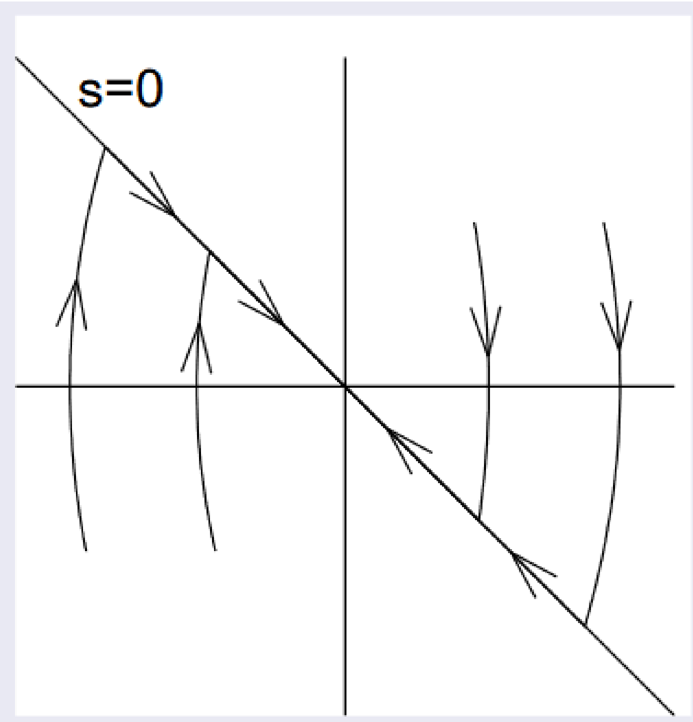
Overview

- Logistics
- Sliding Mode Control

Final Project

- Due next Tuesday, April 9th
- Submit a 10-15 minute video presentation to Learning Suite
- Tips:
  - Follow the rubric!
  - Please be organized!
    - Powerpoint is recommended
    - Screensharing Brian Douglas / 3B1B style is also OK

Sliding Mode Control



- Second order System:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = h(x) + g(x)u$$

- Here, $h(x)$ is NOT the observation!
  - Just a notational change
- Goal: Design a state feedback law that drives state to origin
- Key ideas:
  - Find a manifold where the dynamics are “well-behaved” (e.g. asymptotically stable)
  - Drive the system to the manifold

Example

Example 10.1

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) + g(x)u, \qquad g(x) \geq g_0 > 0$$

Sliding Manifold (Surface):

$$s = ax_1 + x_2 = 0$$

$$s(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_1 = -ax_1$$

$$a > 0 \quad \Rightarrow \quad \lim_{t \to \infty} x_1(t) = 0$$

How can we bring the trajectory to the manifold $s = 0$?

How can we maintain it there?

Example

$$\dot{s} = a\dot{x}_1 + \dot{x}_2 = ax_2 + h(x) + g(x)u$$

Suppose

$$\left|\frac{ax_2 + h(x)}{g(x)}\right| \le \varrho(x)$$

$$V = \tfrac{1}{2}s^2$$

$$\dot{V} = s\dot{s} = s[ax_2 + h(x)] + g(x)su \le g(x)|s|\varrho(x) + g(x)su$$

$$\beta(x) \ge \varrho(x) + \beta_0, \qquad \beta_0 > 0$$

$$s > 0, \qquad u = -\beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)\beta(x)|s| \le -g(x)\beta_0|s|$$

Example

$$s < 0, \qquad u = \beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) + g(x)su = g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$\operatorname{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

$$u = -\beta(x)\ \operatorname{sgn}(s) \ \Rightarrow\ \dot{V} \le -g(x)\beta_0|s| \le -g_0\beta_0|s|$$

$$\dot{V} \le -g_0\beta_0\sqrt{2V}$$

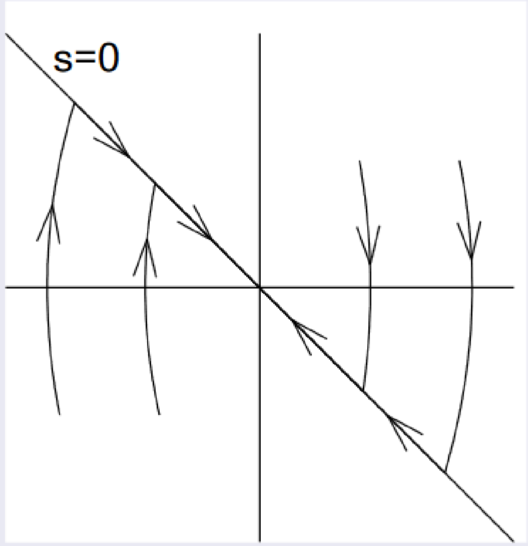
$$\frac{dV}{\sqrt{V}} \le -g_0\beta_0\sqrt{2}\ dt \ \Rightarrow\ \sqrt{V(s(t))} \le \sqrt{V(s(0))} - g_0\beta_0\frac{1}{\sqrt{2}}\ t$$

Example

$$|s(t)| \leq |s(0)| - g_0\beta_0\ t$$

$s(t)$ reaches zero in finite time

The trajectory cannot leave the surface $s = 0$



What Just Happened?

- We found a sliding manifold where the state converges to the origin “naturally”
  - $s = ax_1 + x_2$
  - $s(t) = 0$ implies:
    - $x_2 = -ax_1$
    - $\dot{x}_1 = x_2 = -ax_1 \Longrightarrow x \to 0$
- We designed a control law that drives our state to the sliding manifold in finite time
  - $u = -\beta(x)sgn(s)$
- Once on the manifold, our system converges to the origin
  - Reaching phase: system converges to manifold
  - Sliding phase: system “slides” along manifold

What are the Advantages?

- **Sliding mode control is robust to model error**
- **Don't need to know $h(x)$, $g(x)$ exactly:**
  - $\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq \varrho(x)$
  - $\varrho(x) + \beta_0 \leq \beta(x)$ **where** $\beta_0 > 0$
- **Don't need to know $s(x)$ exactly**
  - **Just need to know $sgn(s)$**
- **We can use sliding mode control in combination with feedback linearization**
  - **More on this next time!**

Special Case

- **We can simplify if, in some domain, we have (for $k_1 \geq 0$):**

$$\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq k_1$$

- **In this case our control law is**

$$u = -k\,\mathrm{sgn}(x), \qquad k > k_1$$

- **If we do this however, we will usually have a finite region of attraction**

Region of Attraction

Estimate the region of attraction when

$$\left|\frac{ax_2 + h(x)}{g(x)}\right| \le \varrho(x), \quad \forall x \in D \subset R^2$$

$$\dot{x}_1 = -ax_1 + s \quad \dot{s} = ax_2 + h(x) - g(x)\beta(x)\mathrm{sgn}(s)$$

$$s\dot{s} \le -g_0\beta_0|s| \;\Rightarrow\; \{|s| \le c\} \text{ is positively invariant}$$

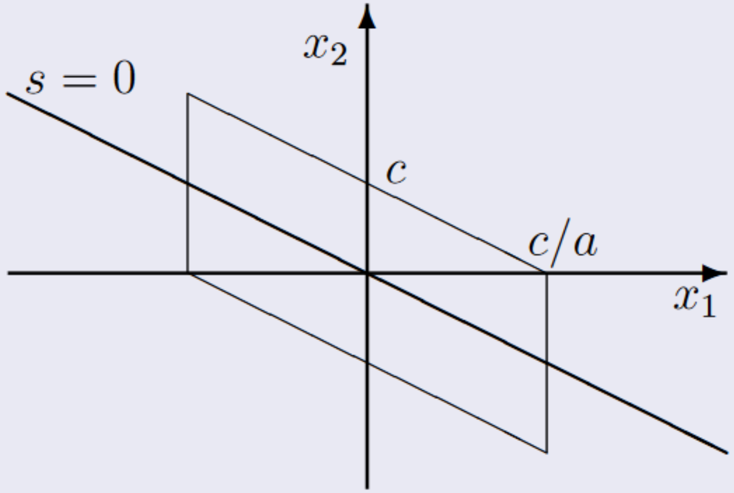
$$V_0 = \tfrac{1}{2}x_1^2 \;\Rightarrow\; \dot{V}_0 = x_1\dot{x}_1 = -ax_1^2 + x_1 s \le -ax_1^2 + |x_1|c$$

$$\dot{V}_0 \le 0, \quad \forall\ |s| \le c \text{ and } |x_1| \ge \frac{c}{a}$$

$$\Omega = \left\{|x_1| \le \frac{c}{a},\ |s| \le c\right\} \subset D \text{ is positively invariant}$$

Region of Attraction

$$\Omega = \left\{ |x_1| \le \frac{c}{a}, \; |s| \le c \right\}$$

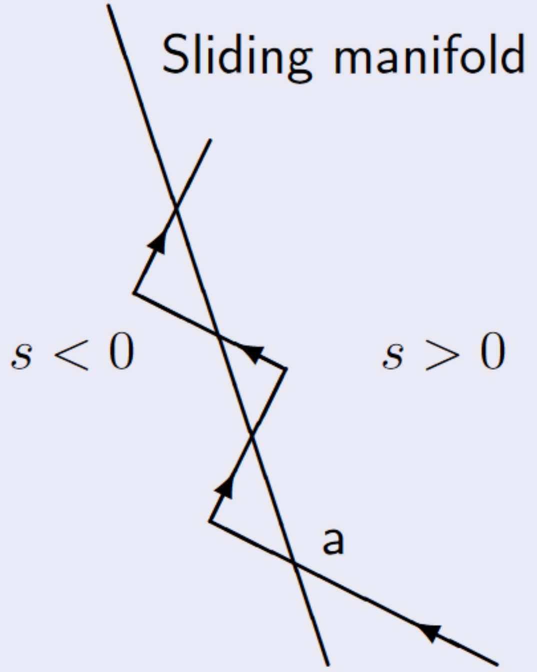


$$\left| \frac{a x_2 + h(x)}{g(x)} \right| \le k_1 < k, \quad \forall \; x \in \Omega$$

$$u = -k \; \mathrm{sgn}(s)$$

What Could Possibly Go Wrong?

Chattering



How can we reduce or eliminate chattering?

Dealing with Chattering: Reducing Amplitude

Reduce the amplitude of the signum function

$$\dot{s} = ax_2 + h(x) + g(x)u$$

$$u = -\frac{[ax_2 + \hat{h}(x)]}{\hat{g}(x)} + v$$

$$\dot{s} = \delta(x) + g(x)v$$

$$\delta(x) = a\left[1 - \frac{g(x)}{\hat{g}(x)}\right]x_2 + h(x) - \frac{g(x)}{\hat{g}(x)}\hat{h}(x)$$

$$\left|\frac{\delta(x)}{g(x)}\right| \le \varrho(x), \qquad \beta(x) \ge \varrho(x) + \beta_0$$

$$v = -\beta(x)\,\mathrm{sgn}(s)$$

Example: Pendulum Equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\cos x_1 - bx_2 + cu, \qquad \hat{b} = 0, \hat{c}$$

$$u = \frac{-x_2 + \cos x_1}{\hat{c}} + v \;\Rightarrow\; \dot{s} = \delta + cv$$

$$\frac{\delta}{c} = \left(\frac{1-b}{c} - \frac{1}{\hat{c}}\right) x_2 - \left(\frac{1}{c} - \frac{1}{\hat{c}}\right) \cos x_1$$

Take $\hat{c} = 1/1.2$ to minimize $|(1-b)/c - 1/\hat{c}|$

$$\left|\frac{\delta}{c}\right| \le 0.8|x_2| + 0.8$$

$$u = 1.2 \cos x_1 - 1.2x_2 - (1 + 0.8|x_2|)\,\mathrm{sgn}(s)$$

Example: Pendulum Equation

Simulation with unmodeled actuator dynamics

$$\frac{1}{(0.01s+1)^2}$$

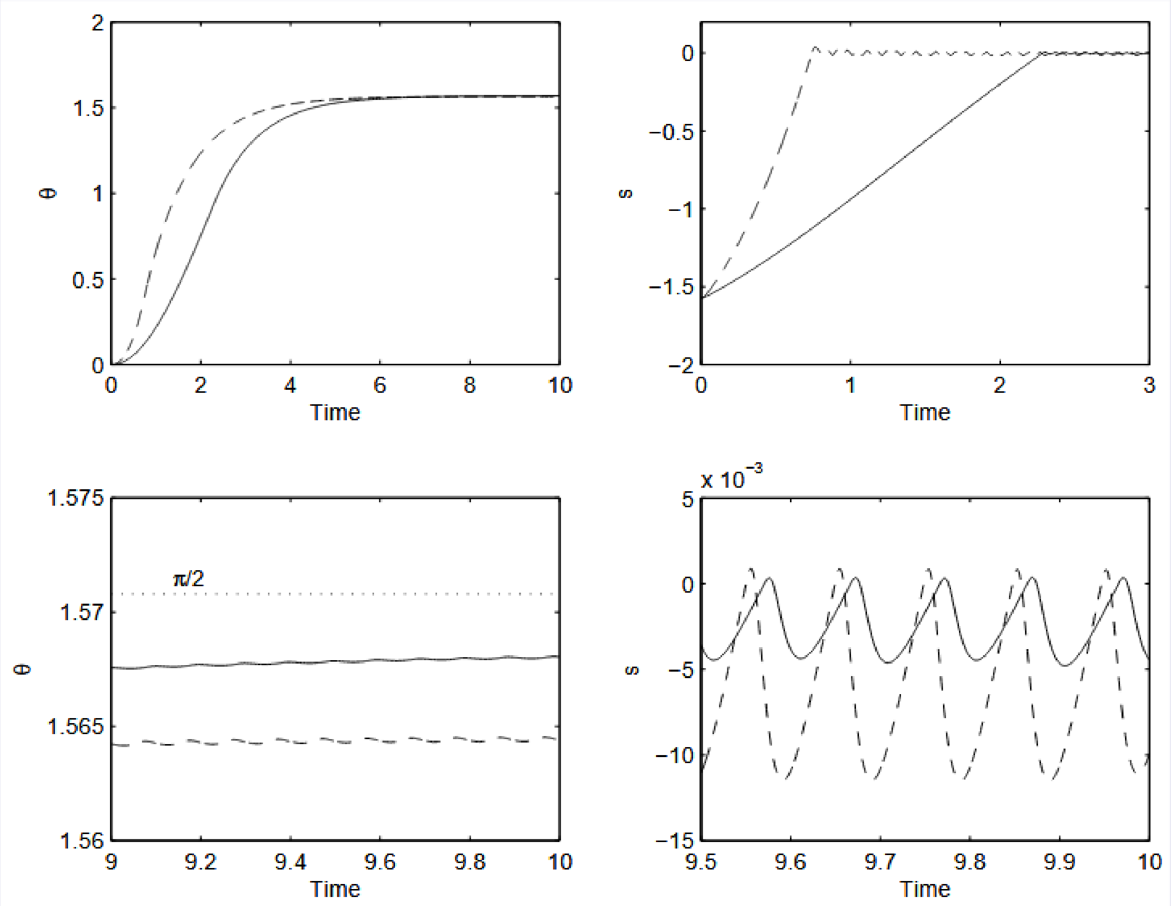
Dashed lines:

$$u = -(2.5 + 2|x_2|)\ \mathrm{sgn}(s)$$

Solid lines:

$$u = 1.2\cos x_1 - 1.2x_2 - (1 + 0.8|x_2|)\ \mathrm{sgn}(s)$$

Example: Pendulum Equation



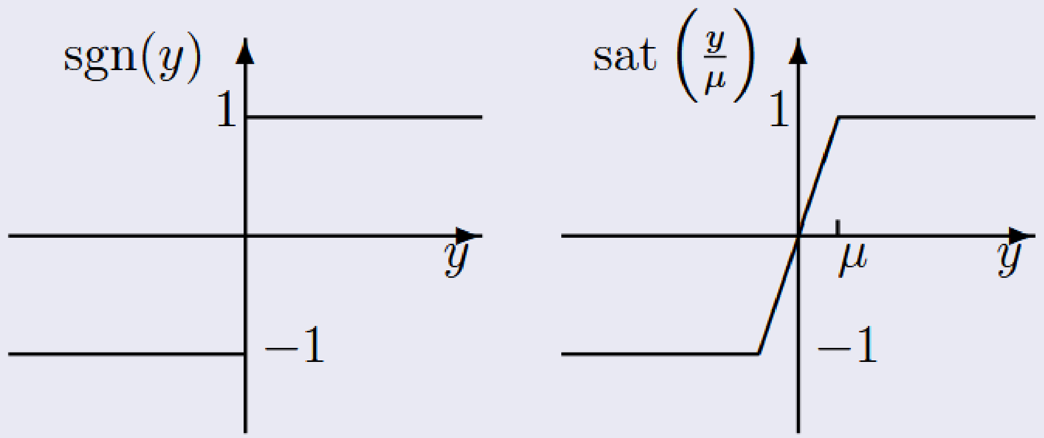
$$b = 0.01,\ c = 0.5,\ \theta(0) = \dot{\theta}(0) = 0$$

Dealing with Chattering: Saturation Function

Replace the signum function by a high-slope saturation function

$$u = -\beta(x)\ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

$$\mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \mathrm{sgn}(y), & \text{if } |y| > 1 \end{cases}$$



Analyzing System Behavior: Saturation Function

$$\text{For } |s| \geq \mu, \quad u = -\beta(x)\,\mathrm{sgn}(s)$$

With $c \geq \mu$

- $\Omega = \{|x_1| \leq \frac{c}{a},\ |s| \leq c\}$ is positively invariant
- The trajectory reaches the boundary layer $\{|s| \leq \mu\}$in finite time
- The boundary layer is positively invariant

Analyzing System Behavior: Saturation Function

Inside the boundary layer:

$$\dot{x}_1 = -ax_1 + s \qquad \dot{s} = ax_2 + h(x) - g(x)\beta(x)\frac{s}{\mu}$$

$$x_1\dot{x}_1 \leq -ax_1^2 + |x_1|\mu$$

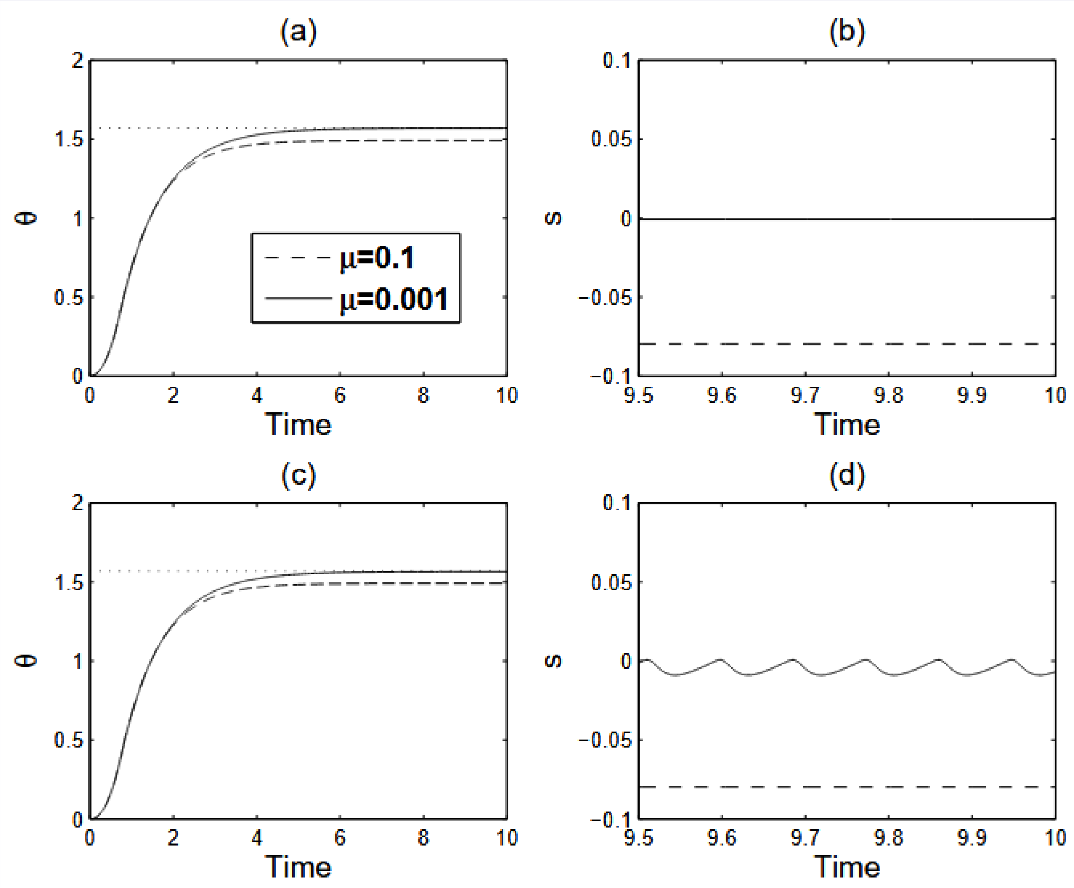
$$x_1\dot{x}_1 \leq -(1-\theta_1)ax_1^2, \ \ \forall\ |x_1| \geq \frac{\mu}{\theta_1 a}, \quad 0 < \theta_1 < 1$$

The trajectories reach the positively invariant set

$$\Omega_\mu = \{|x_1| \leq \frac{\mu}{\theta_1 a},\ |s| \leq \mu\}$$

in finite time

Analyzing System Behavior: Saturation Function



(a) & (b) without and (c) & (d) with actuator dynamics

Finite Time Stability

- **Sliding mode control deals with finite-time stability**
- **System:** $\dot{x} = f(x)$
- **The origin is said to be finite time stable if there exists an open neighborhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin and a function $T: \mathcal{D}\backslash\{0\} \to (0, \infty)$, called the settling time function, such that the following statements hold:**
  - **(Stability) The origin is stable in the sense of Lyapunov**
  - **(Finite-time convergence) For all $x(0) \in \mathcal{D}\backslash\{0\}$,**
    - $x(\cdot)$ **is defined on** $\left[0, T\left(x(0)\right)\right)$
    - $x(t) \in D\backslash\{0\}$ **for all** $t \in \left[0, T\left(x(0)\right)\right)$,
    - $x(t) \to 0$ **for** $t \to T\left(x(0)\right)$
- **Origin is globally finite time stable if it is finite-time stable with $\mathcal{D} = \mathbb{R}^n$**

Lyapunov Theorem for Finite-Time Stability

- Let $\dot{x} = f(x)$
- Suppose there exists continuous $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is continuously differentiable on $\mathbb{R}^n \backslash \{0\}$.
- Suppose $\alpha_1, \alpha_2 \in K_\infty$ and $\kappa > 0$ such that

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||),$$
$$\dot{V}(x) \leq -\kappa\sqrt{V(x)}$$

for all $x \neq 0$. Then the origin is globally finite time stable.

- Moreover, the settling time $T(x)$ is upper bounded by

$$T(x) \leq \frac{2}{\kappa}\sqrt{\alpha_2(||x||)}$$

Backup Slides

NONLINEAR SYSTEMS THEORY

Apr 4, 2024

Overview

- **Logistics**
- **Sliding Mode Control**

Final Project

- Due next Tuesday, April 9th
- Submit a 10-15 minute video presentation to Learning Suite
- Tips:
  - Follow the rubric!
  - Please be organized!
    - Powerpoint is recommended
    - Screensharing Brian Douglas / 3B1B style is also OK

Finite Time Stability

- **Sliding mode control deals with finite-time stability**
- **System:** $\dot{x} = f(x)$
- **The origin is said to be finite time stable if there exists an open neighborhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin and a function $T: \mathcal{D} \backslash \{0\} \to (0, \infty)$, called the settling time function, such that the following statements hold:**
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    - $x(t) \to 0$ **for** $t \to T\big(x(0)\big)$
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Lyapunov Theorem for Finite-Time Stability

- Let $\dot{x} = f(x)$
- Suppose there exists continuous $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$.
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for all $x \neq 0$. Then the origin is globally finite time stable.

- Moreover, the settling time $T(x)$ is upper bounded by

$$T(x) \leq \frac{2}{\kappa}\sqrt{\alpha_2(||x||)}$$

- (Note: Other finite-time conditions besides this one exist. This is just one result that applies easily to our analysis.)

More General Approach

- **Consider the system**

$$\dot{x} = f_1(x,z)$$
$$\dot{z} = f_2(x,z) + g(x,z)\big(u + \delta(t,x,z)\big)$$

- $x \in \mathbb{R}^n$, $z \in \mathbb{R}$
- **Assume that** $f_1(0,0) = 0$ **(origin is equilibrium point)**
- **Assume that** $||\delta(t,x,z)|| \leq L_\delta$ **for all** t, x, z
  - $\delta: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ **is an unknown disturbance function**
  - **"Matched disturbance": Only appears in** z **dynamics as an additive term with** u**.**
- **Assume that** $g(x,z) \neq 0$ **and** $||g(x,z)|| \leq L_g$ **for all** $(x,z) \in \mathbb{R}^{n+1}$
- **Assume we know a control law** $z = k(x)$ **that asymptotically stabilizes the origin of** $\dot{x} = f_1(x,z)$

More General Approach

- **Design an error variable:**

$$\sigma = z - k(x)$$

- **Time derivative:**

$$\dot{\sigma} = f_2(x,z) + g(x,z)\big(u + \delta(t,x,z)\big) - \frac{\partial k}{\partial x}(x) f_1(x,z)$$

- **Define the Lyapunov candidate** $V(\sigma) = \frac{1}{2}\sigma^2$
- **Then**

$$\dot{V}(\sigma) = \sigma\dot{\sigma} = \sigma\left(f_2(x,z) + g(x,z)\big(u + \delta(t,x,z)\big) - \frac{\partial k}{\partial x}(x) f_1(x,z) \right)$$

More General Approach

$$\dot{V}(\sigma) = \sigma\dot{\sigma} = \sigma \cdot \left(f_2(x,z) + g(x,z)\big(u + \delta(t,x,z)\big) - \frac{\partial k}{\partial x}(x) f_1(x,z) \right)$$

- Notice that there are terms we can cancel, and terms we can't cancel.
  - Can't cancel $\delta(t,x,z)$ because it is an unknown disturbance
- What control input cancels the known terms?

$$u = \frac{1}{g(x,z)}\left(-f_2(x,z) + \frac{\partial k}{\partial x} f_1(x,z) + v\right)$$

- The v term is our new linear input
- Remember that we assumed $g(x,z) \neq 0$ for all $(x,z) \in \mathbb{R}^{n+1}$
- Substituting in this u:

$$\dot{V}(\sigma) = \sigma \cdot \big(v + g(x,z)\delta(t,x,z)\big)$$

More General Approach

$$\dot{V}(\sigma) = \sigma \cdot \big(v + g(x, z)\delta(t, x, z)\big)$$

- Notice that this expression is independent of both dynamics functions $f_1(x, z)$ and $f_2(x, z)$!
- To get finite time stability to the manifold, choose v as follows:

$$v = -\left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right)\operatorname{sgn}(\sigma)$$

where $\operatorname{sgn}(\sigma)$ is the sign / signum function

  - (FYI, "signum" is Latin for "sign". I say "signum" it because it's harder to confuse with "sin" 🙂)

- Assuming that $||g(x, z)|| \leq L_g$ and $||\delta(t, x, z)|| \leq L_\delta$, we get:

$$\dot{V}(\sigma) \leq \sigma v + |\sigma| L_g L_\delta$$

- Plugging in v from above:

$$\dot{V}(\sigma) \leq -\frac{\kappa}{\sqrt{2}}|\sigma| = -\kappa\sqrt{V(\sigma)}$$

- This proves that, under our controls u, v, our state converges to the sliding manifold in finite time!

Theorem

- Consider the dynamics on slide 6 with $f_1(0,0) = 0$
- Assume that $g(x,z) \neq 0$
- Assume there exists constant $L_g > 0$ such that $||g(x,z)|| \leq L_g$ for all $(x,z) \in \mathbb{R}^{n+1}$
- Assume that $k(x)$ is defined such that the origin of $\dot{x} = f_1\big(x, k(x)\big)$ is asymptotically stable, and $k(0) = 0$
- Then for all disturbances $\delta(\cdot)$ satisfying $||\delta(t,x,z)|| \leq L_\delta\ \forall (t,x,z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+1}$ for some $L_\delta > 0$, the feedback law

$$u = \frac{1}{g(x,z)}\left(-f_2(x,z) + \frac{\partial k}{\partial x} f_1(x,z)\right) - \frac{\left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right) \operatorname{sgn}\big(z - k(x)\big)}{g(x,z)}$$

Asymptotically stabilizes the origin of the system for all $\kappa > 0$.

- Additionally, the sliding surface $\sigma = z - k(x) = 0$ is reached no later than

$$T\big(\sigma(0)\big) = T\Big(z(0) - k\big(x(0)\big)\Big) = \frac{1}{\sqrt{2\kappa}}\left|z(0) - k\big(x(0)\big)\right|$$

Example

- **System Model:**

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a\left(\cos(x_1)\right)^3 + bx_1^3 + cu$$

- **Sliding Manifold:** $s = \gamma x_1 + x_2$
- **Uncertainties:**
  - $0 < a < 3$
  - $1 < b < 2$
  - $0.5 < c < 10$
- **What can we choose for our controller?**

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a\left(\cos(x_1)\right)^3 + bx_1^3 + cu$$

- $s = \gamma x_1 + x_2$
- **Uncertainties:**
  - $0 < a < 3$
  - $1 < b < 2$
  - $0.5 < c < 10$
- **Take** $u = \beta(x)\,\mathrm{sgn}(s)$
- **We need to satisfy:**
  - $\beta(x) \geq \varrho(x) + \beta_0$
  - $\beta_0 > 0$
  - $\varrho(x) \geq \left|\frac{a_1 x_2 + h(x)}{g(x)}\right|$ **for all** $x \in \mathbb{R}^2$

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a\left(\cos(x_1)\right)^3 + bx_1^3 + cu$$

- $s = \gamma x_1 + x_2$
- **Uncertainties:**
  - $0 < a < 3$
  - $1 < b < 2$
  - $0.5 < c < 10$
- $\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq \frac{1}{0.5}\left(\gamma|x_2| + \left(3 + 2|x_1|^3\right)\right)$
- **For any** β_0**, we can take** $\beta(x) = 2\gamma|x_2| + 6 + 4|x_1|^3 + \beta_0$

Backup Slides

BYU

NONLINEAR SYSTEMS THEORY

Apr 9, 2024

Overview

- **Logistics**
- **Control Lyapunov Functions**
- **Integrated Backstepping**

Final Project Proposition

- **Move due date to Thursday, April 11th at 10 pm**

- **15 points extra credit to those who submit tonight (Tues, April 9th) by 8 pm**

- **Binge watch videos before Quiz 17 on Wednesday**

Thought from Conference

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
ARTIFICIAL INTELLIGENCE LABORATORY

MEMO No. 299 SEPTEMBER 1973

PROPOSAL TO ARPA
FOR
RESEARCH ON INTELLIGENT AUTOMATA
AND
MICRO-AUTOMATION
1974 - 1976

Work reported herein was conducted at the Artificial Intelligence Laboratory, a Massachusetts Institute of Technology research program supported in part by the Advanced Research Projects Agency of the Department of Defense and monitored by the Office of Naval Research under Contracts N00014-70-A-0362-0003 and N00014-70-A-0362-0005.

Extended residencies:

Bledsoe (Chairman, Mathematics, Univ of Texas)
Cocke (IBM research)
Voyat (Genevà) Prof. CUNY Graduate Center
McConkie (Cornell)
Marr (Cambridge)
Paterson (Cambridge)
Nevins (current)
Rabin (Hebrew Univ.)
Forte (Chairman, Yale Univ. Music Dept.)
Slawson (Chairman, Music, Univ. Pit.)
Binford (vision research, Stanford)
Samuel (IBM Laboratories)

Our Laboratory has very close ties to the AI groups at
Bolt, Beranek and Newman
Stanford
Carnegie-Mellon
Stanford Research Institute
Edinburgh

Russell M. Nelson

- “Consider how your life would be different if priesthood keys had not been restored to the earth.”

- The temple is the gateway to the greatest blessings God has for each of His children.
- The temple is the only place on earth where individuals may receive all of the blessings promised to Abraham.

Control Lyapunov Functions (CLFs)

- Consider (yet again) our typical nonlinear system with equilibrium at the origin:

$$\dot{x} = f(x, u)$$

- Say we have a continuously differentiable, positive definite Lyapunov candidate $V(x)$.
- We all (should) know the equation for $\dot{V}(x)$ by now:

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x, u)$$

Definition of Control Lyapunov Function

- **Consider the nonlinear system $\dot{x} = f(x, u)$**
- **A continuously differentiable, positive semidefinite function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a control Lyapunov function if:**
  - **There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that**
  $$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||),$$
  - **For all $x \in \mathbb{R}^n \backslash \{0\}$ there exists $u \in \mathbb{R}^m$ such that:**
  $$\frac{\partial V}{\partial x} f(x, u) < 0.$$

- **For control affine functions, the last inequality becomes**
$$L_f V + L_g V u < 0$$

Say we have a CLF…

- …What do we do next?
- In general, we want to choose a control input $u(x)$ such that $\frac{\partial V}{\partial x} f(x, u) < 0.$
  - Sufficient condition for asymptotic stability
- How can we choose such a $u(x)$?
- Two methods:
  - Optimization algorithms
  - Sontag's Universal Formula

Choosing Control Inputs: Optimization Methods

- Case 1: We can use optimization!

$$\boldsymbol{u}^*(\boldsymbol{x}) = \text{minimize}_{\boldsymbol{u}} \ ||\boldsymbol{u}||_2^2$$
$$\text{s.t.} \quad \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\boldsymbol{\alpha}\big(\boldsymbol{V}(\boldsymbol{x})\big)$$

  - $\alpha(\cdot)$ is an extended class-$\mathcal{K}_\infty$ function
  - Quadratic objective function
  - Nonconvex constraint
- Special case: control affine dynamics

$$\boldsymbol{u}^*(\boldsymbol{x}) = \text{minimize}_{\boldsymbol{u}} \ ||\boldsymbol{u}||_2^2$$
$$\text{s.t.} \quad \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \leq -\boldsymbol{\alpha}\big(\boldsymbol{V}(\boldsymbol{x})\big)$$

  - Quadratic objective function
  - Linear constraint! (Linear in optimization variable u)
- Special case is called a convex quadratic program.
  - Typically can be solved in real time
  - Control input $u^*(x)$ is repeatedly updated
- Solvers: OSQP, CVXPy, JaxOpt, Gurobi, and more

Choosing Control Inputs: Sontag's Universal Formula

- Case 2: Closed form expression for control affine systems
- Let $u = k(x)$ and define $k(x)$ as:

$$k(x) = \begin{cases} -\left(\kappa + \dfrac{L_fV(x) + \sqrt{L_fV(x)^2 + L_gV(x)^4}}{L_gV(x)^2}\right) L_gV(x), & \text{if } L_gV(x) \neq 0, \\ 0, & \text{if } L_gV(x) = 0 \end{cases}$$

- What happens when we plug $u = k(x)$ into the expression for $\dot{V}(x)$?

$$\dot{V}(x) = L_fV(x) + L_gV(x)u = L_fV(x) - L_gV(x)\left(\kappa + \frac{L_fV(x) + \sqrt{L_fV(x)^2 + L_gV(x)^4}}{L_gV(x)^2}\right) L_gV(x)$$

$$= L_fV(x) - L_fV(x) - \sqrt{L_fV(x)^2 + L_gV(x)^4} - \kappa(L_gV(x))^2$$

$$= -\sqrt{L_fV(x)^2 + L_gV(x)^4} - \kappa(L_gV(x))^2 < 0 \ \forall x \neq 0$$

Backstepping

- **Finding CLFs may be trickier for higher order systems**
- **How can we recursively construct CLFs for such systems?**
- **Consider systems in strict feedback form:**

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}, x_n) \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u)
\end{aligned}$$

Kellett, Christopher M., and Philipp Braun. *Introduction to nonlinear control: Stability, control design, and estimation*. Princeton University Press, 2023.

Backstepping: Simple Example

- **System:**

$$\dot{x} = x^3 + x\xi$$
$$\dot{\xi} = u$$

- **We can't cancel out the nonlinearities…**
- **Four Steps to a More Stable System™:**
    1. **Define a Virtual Control**
    2. **Define an Error Variable**
    3. **Construct a Control Lyapunov Function**
    4. **Construct a Feedback Stabilizer**

Example: Define Virtual Control (Step 1)

$$\dot{x} = x^3 + x\xi$$
$$\dot{\xi} = u$$

- If you squint and tilt your head, ξ looks like a control input for $\dot{x} = x^3 + x\xi$
- Let's choose a "control law"!
  - $\xi = -2x^2 \implies \dot{x} = x^3 - 2x^3 = -x^3 \implies$ Origin is asymptotically stable
  - Lyapunov candidate: $V(x) = \frac{1}{2}x^2$
  - Define $k(x) = -2x^2$
- Add $k(x)$ into the original dynamics by "adding zero":

$$\dot{x} = x^3 + x\xi + \big(xk(x) - xk(x)\big) = -x^3 + x\big(\xi + 2x^2\big)$$

Example: Define an Error Variable (Step 2)

$$\dot{x} = x^3 + x\xi$$
$$\dot{\xi} = u$$

- **Problem: ξ is not really a control input.**
- **Can we "control" it to behave like our desired control input $k(x)$?**
- **Define an error: $z = \xi - k(x) = \xi + 2x^2$**
  - **Equivalently, $\xi = z - 2x^2$**
- **Computing $\dot{z}$:**

$$\dot{z} = \dot{\xi} - \frac{d}{dt}k(x) = u - \frac{\partial}{\partial x}k(x)\dot{x} = u + 4x\left(x^3 + x\xi\right) = u + 4x\left(-x^3 + xz\right)$$
$$= u - 4x^4 + 4x^2 z$$

- **Rewrite our dynamics in terms of (x, z):**

$$\dot{x} = -x^3 + xz$$
$$\dot{z} = u - 4x^4 + 4x^2 z$$

Example: Construct a CLF (Step 3)

$$\dot{x} = -x^3 + xz$$
$$\dot{z} = u - 4x^4 + 4x^2 z$$

- Remember we had $V(x) = \frac{1}{2}x^2$ as a Lyapunov candidate for the original x dynamics $\dot{x} = x^3 + x\xi$
- We *augment* this candidate to create a new control Lyapunov function (CLF)
- In this case, choose $V_a(x, z) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}(x^2 + z^2)$
- Sanity check: Is this really a CLF?
  - $\dot{V}_a(x, z) = x\dot{x} + z\dot{z} = -x^4 + x^2 z + z\left(u - 4x^4 + 4x^2 z\right) = f(x, z) + zu$
- Notice that
  - $z = 0 \Longrightarrow \dot{V}_a(x, z) = -x^4 < 0.$
  - $z \neq 0 \Longrightarrow$ We can choose u to cancel nonlinearities and set $\dot{V}_a(x, z) < 0.$

Example: Construct a Feedback Stabilizer (Step 4)

$$\dot{x} = -x^3 + xz$$
$$\dot{z} = u - 4x^4 + 4x^2 z$$
$$V_a(x,z) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}(x^2 + z^2)$$
$$\dot{V}_a(x,z) = -x^4 + x^2 z + z\left(u - 4x^4 + 4x^2 z\right)$$

- Choose a u to make $\dot{V}_a(x,z) < 0$
- In this case, we can do it by inspection:

$$u = k_1(x,z) = -x^2 + 4x^4 - 4x^2 z - z$$

- Plugging this into our expression for $\dot{V}_a$ gives:

$$\dot{V}_a(x,z) = -x^4 - z^2 < 0$$

- We could have used other methods to choose u
  - E.g. Sonntag's Universal Formula

Asymptotic Stabilizability

- Question: When do CLFs exist for a given system?
    - Are we guaranteed to be able to find a CLF?
- First, we need to know whether a control input $u(\cdot)$ exists that renders the origin of the closed-loop system asymptotically stable
- Asymptotic stabilizability:
    - Consider $\dot{x} = f(x, u)$
    - The origin is (locally) asymptotically stabilizable if there exist $\epsilon > 0$ and $\beta \in \mathcal{KL}$ such that:
    - For all $x_0 = x(0) \in \mathcal{B}_\epsilon$, there exists a bounded and continuous function $u_{x_0}: \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ such that
$$||x(t, x_0, u_{x_0})|| \leq \beta(||x_0||, t) \qquad \forall t \geq 0$$
    - If ϵ can be chosen arbitrarily large, then the origin is called *globally asymptotically stabilizable.*

Lipschitz Continuous Feedback Stabilizability

- **Consider $\dot{x} = f(x, u)$**
- **The origin is called (locally) Lipschitz continuous feedback stabilizable if there exist $\epsilon > 0$, $\beta \in \mathcal{KL}$, and a Lipschitz continuous feedback law $k\colon \mathbb{R}^n \to \mathbb{R}^m$ such that:**

$$||x\big(t, x_0, k(x)\big)|| \leq \beta(||x_0||, t)$$

- **If $\epsilon > 0$ can be chosen arbitrarily large, then the origin is called *globally Lipschitz continuous feedback stabilizable***
- **Lipschitz cont. feedback stabilizable $\Longrightarrow$ asymptotic stabilizable**

How to tell if a system is L.C.F.S.

- **We can check the linearization:**

$$A = \frac{\partial f}{\partial x}\bigg|_{x^*,u^*}$$

$$B = \frac{\partial f}{\partial u}\bigg|_{x^*,u^*}$$

- **Theorem: Consider $\dot{x} = f(x, u)$, the equilibrium point x^*, and the above linearization.**
- **If (A, B) is stabilizable, then the equilibrium point x^* is Lipschitz continuous feedback stabilizable.**
  - **(Stabilizable: All eigenvalues with non-negative real parts can be moved to the open left hand plane with feedback).**

What does this have to do with the existence of CLFs?

- **Theorem: Consider the system $\dot{x} = f(x, u)$**
- **The origin is asymptotically stabilizable if and only if there exists a (possibly nonsmooth) Lipschitz continuous CLF.**

- **But what is a nonsmooth Lipschitz continuous CLF?**

Dini Derivatives

- **Generalization of the directional derivative to cases where a function is not continuously differentiable**

- **Consider a Lipschitz continuous function $V: \mathbb{R}^n \to \mathbb{R}$**
- **The (upper right) Dini derivative of V at $x \in \mathbb{R}^n$ in the direction $w \in \mathbb{R}^n$ is:**

$$DV(x, w) = \limsup_{h \downarrow 0} \frac{1}{h}\big(V(x + hw) - V(x)\big)$$

- **If V is continuously differentiable, then**

$$DV(x, w) = \frac{\partial V}{\partial x} w$$

Nonsmooth Control Lyapunov Function

- Consider $\dot{x} = f(x, u)$
- Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$
- A Lipschitz continuous function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a control Lyapunov function for $\dot{x} = f(x, u)$ if

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$$

and for all $x \in \mathbb{R}^n \setminus \{0\}$ there exists $u \in \mathbb{R}^m$ such that

$$DV\big(x, f(x, u)\big) < 0$$

Summary

- **Control Lyapunov Functions allow us to find feedback laws $u(x)$ that render equilibrium points asymptotically stable**
  - **We can use optimization**
  - **We can use Sonntag's Universal Formula**

- **Integrated Backstepping allows us to control systems in strict feedback form**

- **CLFs exist for an equilibrium point if and only if the equilibrium point is asymptotically stabilizable**

Backup Slides

BYU

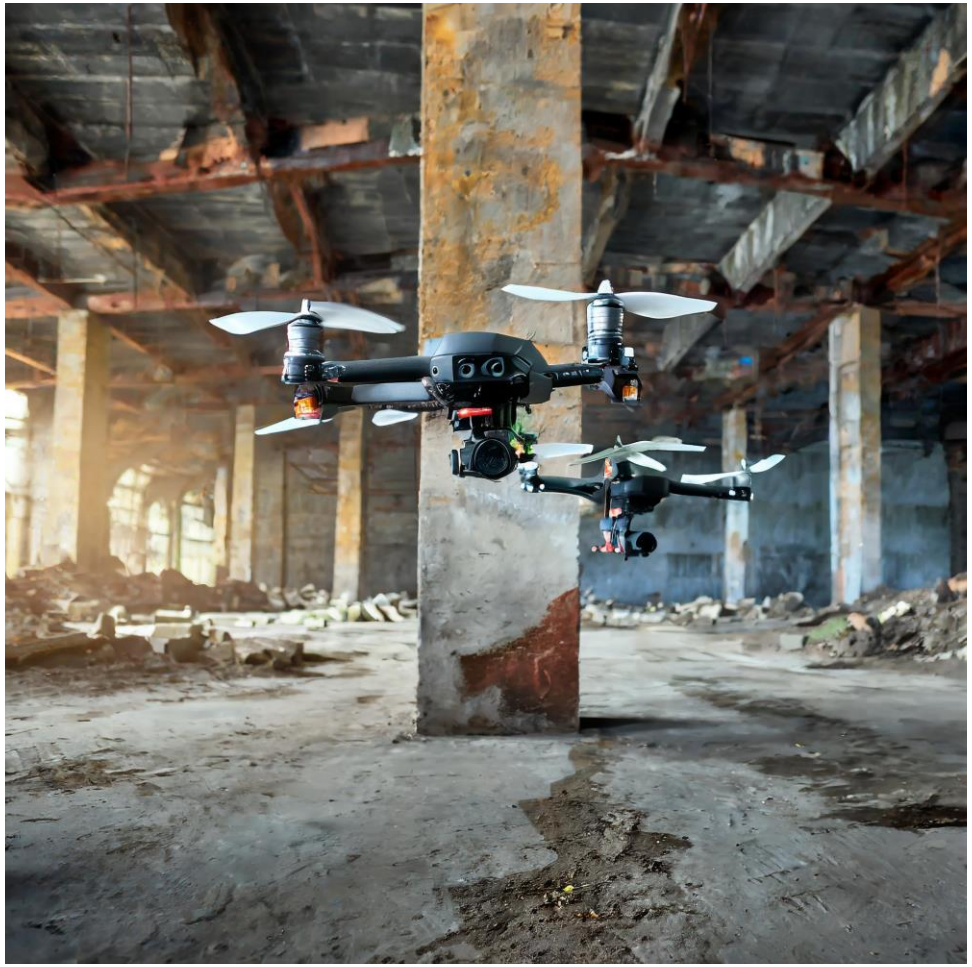
NONLINEAR SYSTEMS THEORY

Apr 16, 2024

Overview

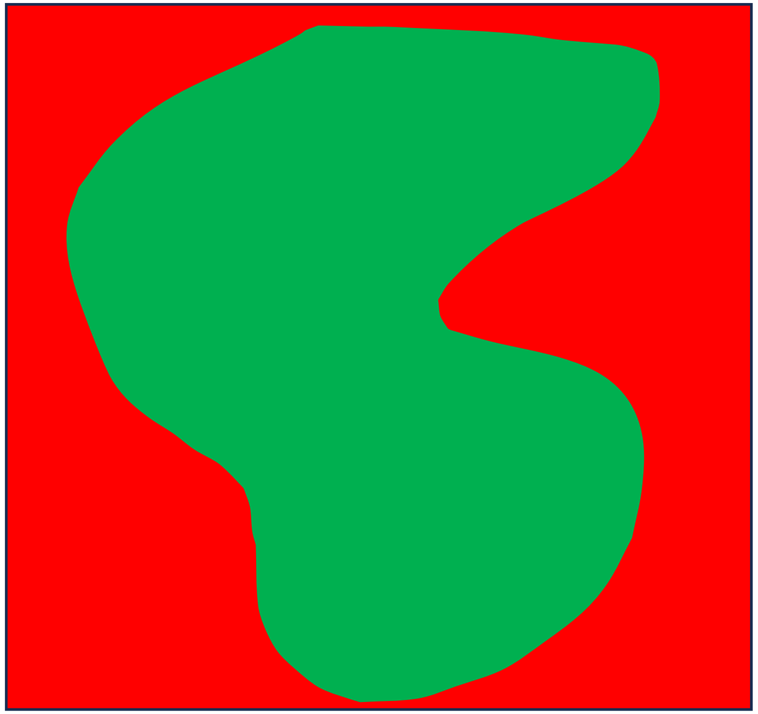
- Logistics
- Control Barrier Functions
- Higher Order Control Barrier Functions

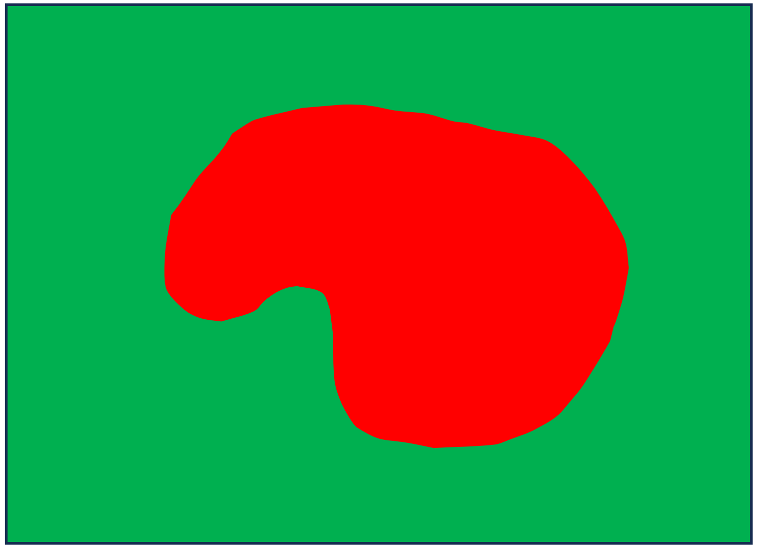
Safety in Nonlinear Systems



Quantifying Safety

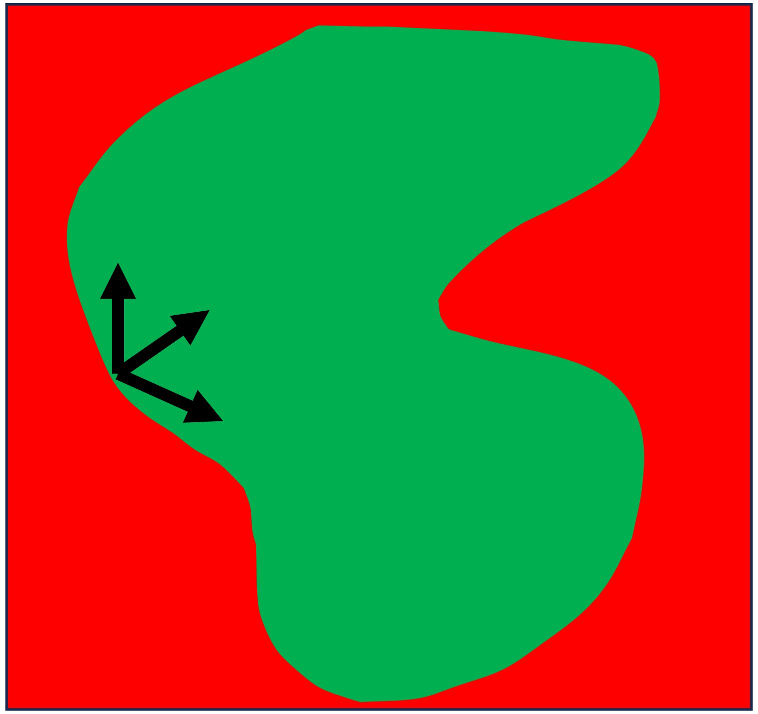
- System: $\dot{x} = f(x) + g(x)u$
  - $x \in \mathbb{R}^n$
  - $u \in \mathbb{R}^m$
  - f, g both locally Lipschitz
- Divide the state space into two sets:
  - Safe set: $S \subseteq \mathbb{R}^n$
  - Unsafe set: $\overline{S} = \mathbb{R}^n \backslash S$
- We define the safe set S using a function $h: \mathbb{R}^n \to \mathbb{R}$ as follows:
  - $S = \{x \in \mathbb{R}^n: h(x) \geq 0\}$
  - $\partial S = \{x \in \mathbb{R}^n: h(x) = 0\}$
  - $\mathrm{Int}(S) = \{x \in \mathbb{R}^n: h(x) < 0\}$
  - $\overline{S} = \{x \in \mathbb{R}^n: h(x) > 0\}$
- NOTE: Others typically define safety as $h(x) \geq 0$
  - It doesn't matter. The math works out the same.





Invariance of the Safe Set

- We want our state to remain in the safe set for all forward time
- The set S is forward invariant if for all $x_0 \in S$, $x(t) \in S$ for $x(0) = x_0$ and all $t \in I(x_0) = [0, \tau_{max})$ where $I(x_0)$ is the maximum interval of existence.
- How do we determine safety if we can't compute $x(t)$ explicitly?
- Nagumo's Theorem:
  - Consider $\dot{x} = f(x)$
  - Let $S = \{x: h(x) \leq 0\}$
  - Assume that $\frac{\partial h}{\partial x}(x) \neq 0$ for all $x \in \{x: h(x) = 0\}$
  - Then S is invariant if and only if $\dot{h}(x) \leq 0$ for all $x \in \partial S$
- This is similar to invariant sets in Lyapunov analysis
- What about the condition $\frac{\partial h}{\partial x}(x) \neq 0$ for all $x \in \{x: h(x) = 0\}$?
  - This can be relaxed using a comparison lemma
  - See Konda 2020, *Characterizing Safety: Minimal Control Barrier Functions from Scalar Comparison Systems*



Preliminaries

- **Extended class-$\mathcal{K}_\infty$ functions:**
  - $\boldsymbol{\alpha}: \mathbb{R} \rightarrow \mathbb{R}$
  - $\boldsymbol{\alpha(0) = 0}$
  - $\boldsymbol{\alpha(x)}$ **is strictly increasing**
- **Similar to class-$\mathcal{K}_\infty$ functions, but defined on the entire number line**

Control Barrier Functions

- **Similar to Control Lyapunov Functions!**

- **Consider a safe set $S \subseteq D \subseteq \mathbb{R}^n$ defined by the sublevel sets of a continuously differentiable function $h: D \to \mathbb{R}$**
- **Then h is a Control Barrier Function (CBF) if there exists a class-$\mathcal{K}_\infty$ function α such that**

$$\sup_{u \in U} \left(L_f h(x) + L_g h(x) u \right) \geq -\alpha\big(h(x)\big)$$

for all $x \in D$.

Forward Invariance with CBFs

- **Intuitively, with CBFs we can always choose a u that keeps our state in the safe set**
- **Set of safe control inputs:**

$$K_{\mathrm{cbf}}(x) = \{u \in U : L_f h(x) + L_g h(x)u + \alpha(h(x)) \geq 0\}.$$

- **Theorem (the safe set is denoted as C):**

Theorem 2. *Let $\mathcal{C} \subset \mathbb{R}^n$ be a set defined as the superlevel set of a continuously differentiable function $h : D \subset \mathbb{R}^n \to \mathbb{R}$. If h is a control barrier function on D and $\frac{\partial h}{\partial x}(x) \neq 0$ for all $x \in \partial\mathcal{C}$, then any Lipschitz continuous controller $u(x) \in K_{\mathrm{cbf}}(x)$ for the system (1) renders the set $\mathcal{C}$ safe. Additionally, the set $\mathcal{C}$ is asymptotically stable in D.*

Examples of Safe Set Functions

- **Spherical obstacle:**
  - **Location x_{obs}**
  - **We want $||x - x_{obs}||_2 \leq R_{safe} \Leftrightarrow R_{safe}^2 - ||x - x_{obs}||_2^2 \leq 0$**
  - **Define $h(x) = R_{safe}^2 - ||x - x_{obs}||_2^2$**
  - **Then $h(x) \leq 0 \Leftrightarrow x$ is collision free**

- **Hyperplane**
  - **We want $a^T x \leq b$**
  - **Define $h(x) = a^T x - b$**

- **Nonlinear (and possibly non-convex) function**
  - **We want $f(x) \leq g(x)$**
  - **Define $h(x) = f(x) - g(x)$**

Safety-Critical Control

- **Suppose we have a nominal feedback control $u_{nom} = k(x)$**
- **We want to guarantee safety ($x(t) \in S$)**
- **We can minimally modify our u_{nom} to preserve safety!**

$$\begin{aligned} u(x) = \underset{u \in \mathbb{R}^m}{\operatorname{argmin}} \quad & \frac{1}{2}\|u - k(x)\|^2 \qquad \text{(CBF-QP)} \\ \text{s.t.} \quad & L_f h(x) + L_g h(x) u \geq -\alpha(h(x)) \end{aligned}$$

- **This guarantees that safety is preserved**
  - **It does NOT guarantee that we converge to the equilibrium point**

Simultaneous Safety and Convergence

- **CBFs can be used simultaneously with CLFs:**

$$
\begin{aligned}
u(x) = \underset{(u,\delta)\in\mathbb{R}^{m+1}}{\operatorname{argmin}} \quad & \frac{1}{2}u^T H(x) u + p\delta^2 \qquad \text{(CLF-CBF QP)} \\
\text{s.t.} \quad & L_f V(x) + L_g V(x) u \leq -\gamma(V(x)) + \delta \\
& L_f h(x) + L_g h(x) u \geq -\alpha(h(x))
\end{aligned}
$$

- **Two constraints:**
  - **The first attempts to enforce convergence to the equilibrium point**
    - **The $+\delta$ term makes this a soft constraint**
  - **The second enforces safety**
    - **Hard constraint**
- **Convergence will be sacrificed to preserve safety**

Composition of Multiple CBFs

- **Suppose there are multiple obstacles.**
- **How can we guarantee safety for all of them simultaneously?**
- **Safe set functions:**

$$\boldsymbol{h_1(x), h_2(x), \ldots h_p(x)}$$

- **We can add separate CBF constraints to the QP:**

$$\begin{aligned} &\min_{u \in \mathrm{U}} ||\boldsymbol{u_{nom}} - \boldsymbol{u}|| \\ \boldsymbol{s.t.}\quad &\boldsymbol{L_f h_1(x) + L_g h_1(x) u \leq \alpha_1\big(h_1(x)\big)} \\ &\vdots \\ &\boldsymbol{L_f h_p(x) + L_g h_p(x) u \leq \alpha_p\left(h_p(x)\right)} \end{aligned}$$

Time-Varying Obstacles

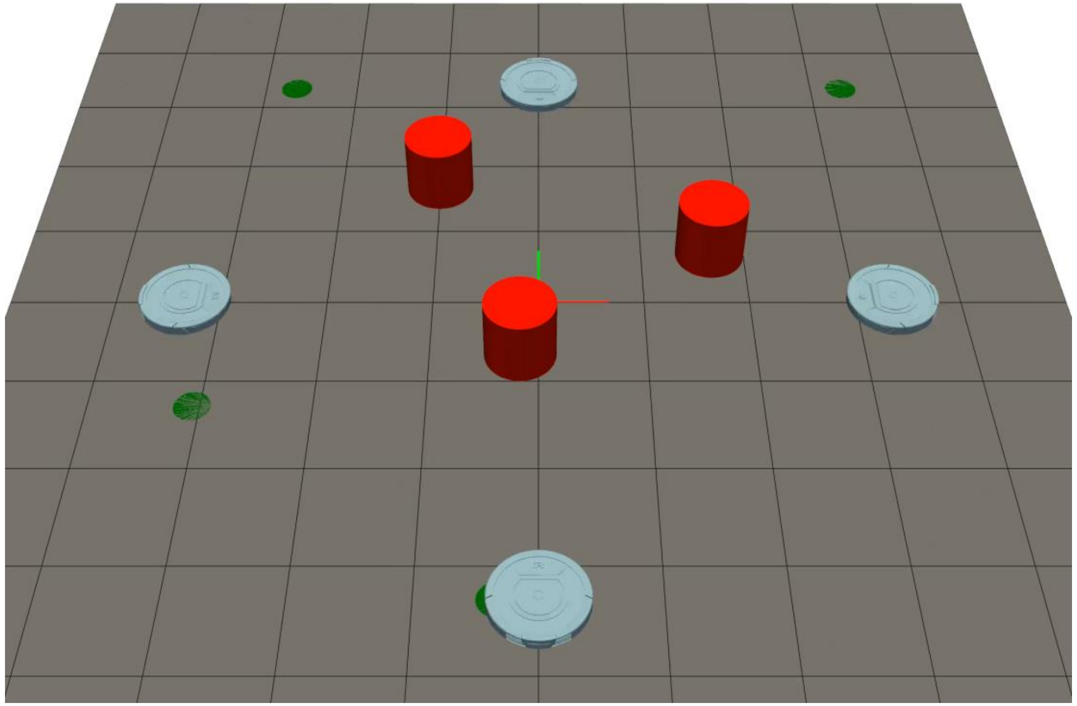
- What if we have a time-varying obstacle?
  - E.g. two mobile robots trying to avoid collisions with each other
  - $\dot{x}_i = f_i(x_i) + g_i(x_i)$
- First define safe set function in terms of both states: $h(x_1, x_2)$
- Let's go back to Nagumo's condition: $\dot{h}(x_1, x_2) \leq -\alpha\big(h(x_1, x_2,)\big)$
- By the chain rule:
  - $\dot{h}(x_1, x_2) = \frac{\partial h}{\partial x_1}(f_1(x_1) + g_1(x_1)u_1) + \frac{\partial h}{\partial x_2}(f_2(x_2) + g_2(x_2)u_2) \leq -\alpha\big(h(x_1, x_2)\big)$
- This is the new CBF condition
  - Two optimization variables: u_1, u_2
  - If agents are cooperative, they can collaborate
  - If agents are noncooperative or adversarial, more advanced methods are needed (e.g. Adversarially Resilient CBFs)

Example: Single Integrator in the Plane

- $\dot{x} = u, x \in \mathbb{R}^2, u \in \mathbb{R}^2$
- $h(x) = R_{safe}^2 - ||x - x_{obs}||_2^2$
- **What is the CBF QP constraint?**

Example: Unicycle Agent

- Trick: Use input / output linearization
    - Use CBFs on the resulting linear system



Example: 2D Double Integrator

- **Spherical safe set function:**

$$h(x) = R_{safe}^2 - ||C(x - x_{obs})||$$
$$C = [I, 0]$$

- **Try computing the CBF QP constraint!**
- **...What happened?**
- **The system has relative degree 2 – no control inputs?**
- **What do we do in this case?**

Higher Order CBFs (HOCBF)

- Original safe set function: $h(x)$
- Define the following safe set functions:

$$\phi_0(x) = h(x)$$
$$\phi_1(x) = \dot{\phi}_0(x) + \alpha_0(x)$$
$$\phi_2(x) = \dot{\phi}_1(x) + \alpha_1(x)$$
$$\vdots$$
$$\phi_r(x, u) = \dot{\phi}_{r-1}(x) + \alpha_{r-1}(x)$$

- Each has an associated safe set $S_i = \{x: \phi_i(x) \leq 0\}$
- r is the minimum integer such that u appears
- The function $h(x)$ is a HOCBF if

$$\sup_{u \in U} \phi_r(x, u) \leq 0$$

Equivalently,

$$\sup_{u \in U} L_f^r h(x) + L_g L_f^{r-1} h(x) \leq -\alpha_{r-1}\big(\phi_{r-1}(x)\big)$$

“What if my system isn’t control affine?”

- **You can add integrators.**
- **Let** $\dot{x} = f(x, u)$
- **Define** $z = [x, u]^T$
- **Let** $\dot{u} = v$
- **Then** $\dot{z} = f(z) + Bv$
- **Use HOCBF methods to control the system!**

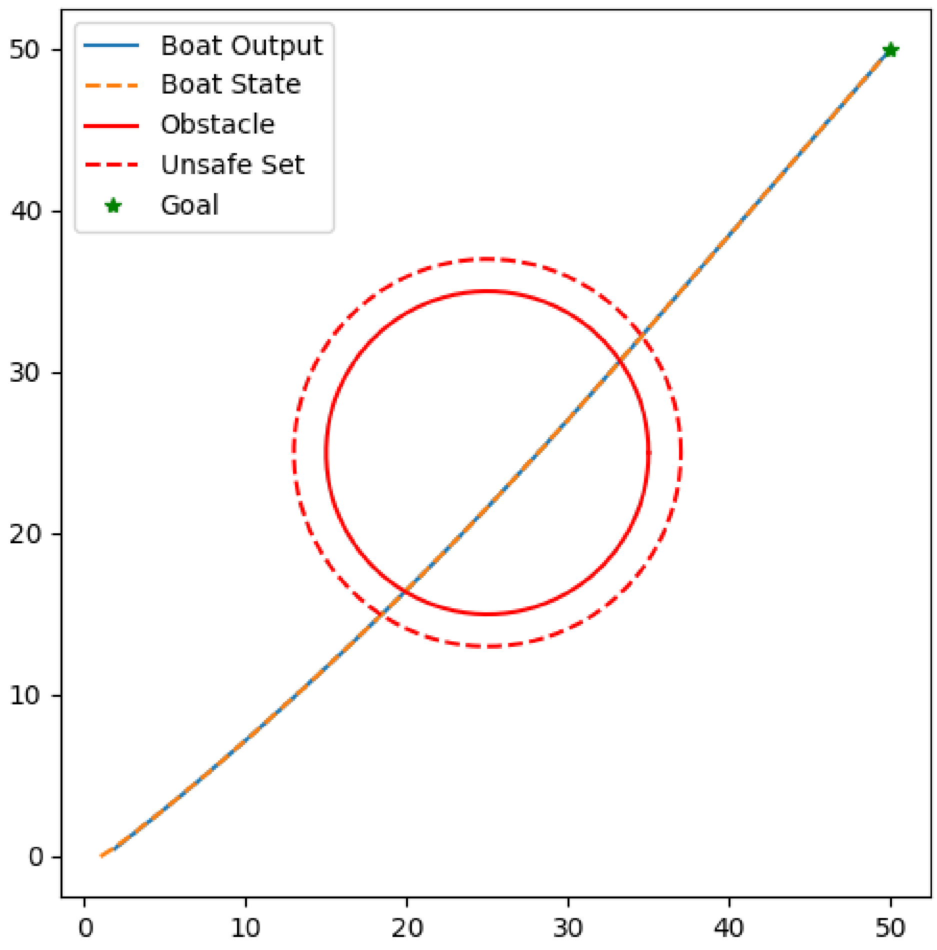
Into the Abyss

- Consider the following system:

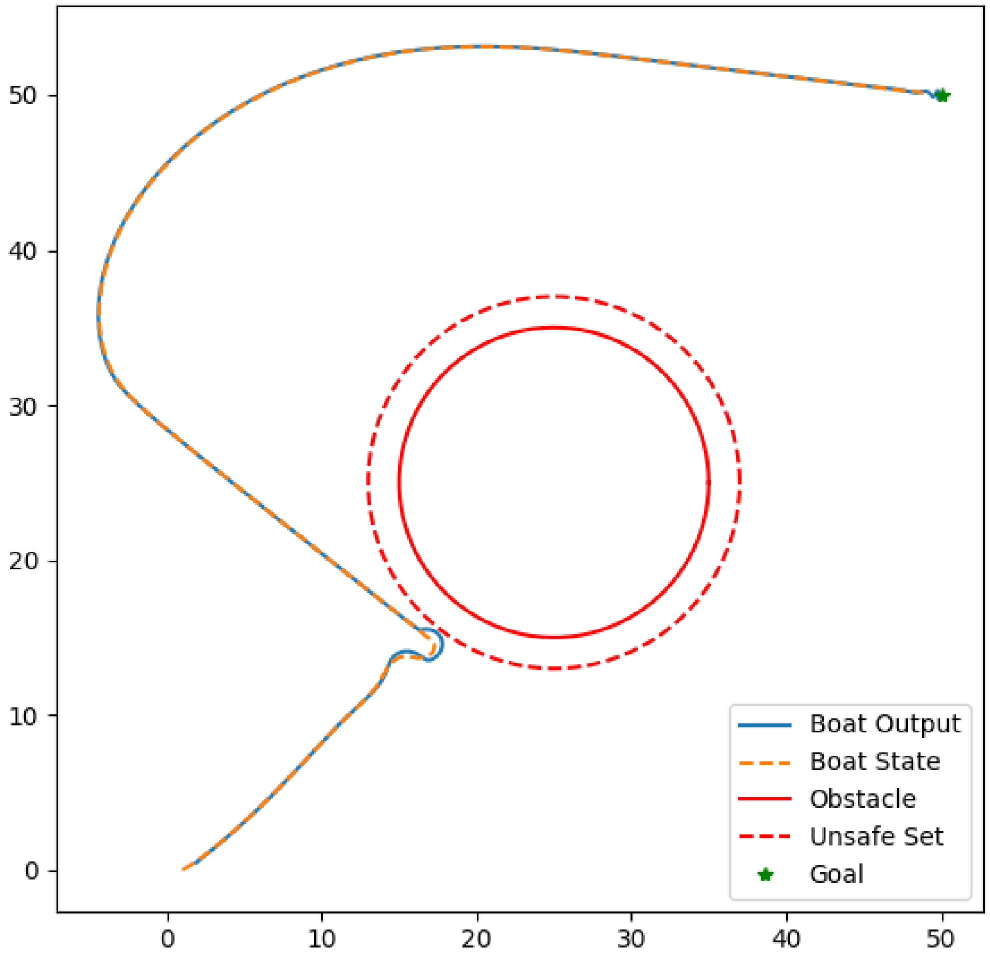
$$\vec{x} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} V \cos\delta \cos\theta \\ V \cos\delta \sin\theta \\ -V \sin\delta \end{bmatrix}$$

- Boat with an outboard motor
- Is it control affine?
- How to tackle this?

The Results



Boat Output
Boat State
Obstacle
Unsafe Set
Goal



Boat Output
Boat State
Obstacle
Unsafe Set
Goal

(PLACEHOLDER) Thoughts on Safety & Trust

Backup Slides