2-1 Set Theory

Definitions

- A *set* is a collection of objects called *elements*. Example

$$A = \{\text{car}, \text{apple}, \text{pencil}\}$$

- The *cardinality* of the set A is the number of elements in the set. The cardinality is often denoted $|A|$. For the previous example, $|A| = 3$.
- The notation that represents "ζ is a member of A" is $\zeta \in A$.
- The notation that represents "ζ is not a member of A" is $\zeta \notin A$.
- The *empty set* is the set that contains no elements. The empty set is denoted $\varnothing$.

$$|\varnothing| = 0.$$

- A *subset* B of a set A is anther set whose elements are also elements of A. This is denoted $B \subset A$ or $B \subseteq A$. The formal mathematical definition is

$$B \subseteq A \text{ means } \zeta \in B \Rightarrow \zeta \in A.$$

  If $|A| = n$, then there are 2^n subsets of A.
- In probability theory, all sets are subsets of a set $\mathcal{S}$ called a *space*. The *space* is the set of all possible experimental outcomes.
- For any set A,

$$\varnothing \subseteq A \subseteq \mathcal{S}$$

Set Properties and Operations

- *Transitivity*: if $C \subseteq B$ and $B \subseteq A$ then $C \subseteq A$.
- *Equality*: $A = B$ if and only if (iff) $B \subseteq A$ and $A \subseteq B$.
- The *union* of two sets A and B is a set whose elements are all elements of A or of B or of both.

$$A \cup B$$

  - *The union operation is commutative*: $A \cup B = B \cup A$.
  - *The union operation is associative*: $(A \cup B) \cup C = A \cup (B \cup C)$.

- The *intersection* of two sets A and B is the set comprising all the elements that are common to the sets A and B.

$$A \cap B$$

  - *The intersection operation is commutative*: $A \cap B = B \cap A$.
  - *The intersection operation is associative*: $(A \cap B) \cap C = A \cap (B \cap C)$.
  - *Intersection distributes over union*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

- Two sets are *mutually exclusive* or *disjoint* if they have no common elements.

$$A \cap B = \varnothing.$$

  Several sets $A_1, A_2, \ldots$ are mutually exclusive if

$$A_i \cap A_j = \varnothing \quad \text{for all } i \text{ and for every } j \neq i.$$

- A *partition* of the set $\mathcal{S}$ is a set of mutually exclusive subsets $A_1, \ldots, A_n$ whose union equals $\mathcal{S}$.

$$A_1 \cup \cdots \cup A_n = \mathcal{S} \qquad A_i \cap A_j = \varnothing \text{ for } i \neq j.$$

$$\bigcup_{i=1}^{n} A_i = \mathcal{S} \qquad A_i \cap A_j = \varnothing \text{ for } i \neq j.$$

- The *complement* of a set A, denoted $\overline{A}$ is the set comprising all elemets of $\mathcal{S}$ that are not in A.
- *De Morgan's Law*:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Some Results

$$B \subseteq A \Rightarrow A \cup B = A \qquad \begin{aligned} A \cup A &= A \\ A \cup \varnothing &= A \\ \mathcal{S} \cup A &= \mathcal{S} \end{aligned}$$

$$B \subseteq A \Rightarrow A \cap B = B \qquad \begin{aligned} \varnothing \cap A &= \varnothing \\ A \cap \mathcal{S} &= A \end{aligned}$$

$$\left.\begin{aligned} A \cup \overline{A} &= \mathcal{S} \\ A \cap \overline{A} &= \varnothing \end{aligned}\right\} \; A \text{ and } \overline{A} \text{ form a \textit{partition} of } \mathcal{S}$$

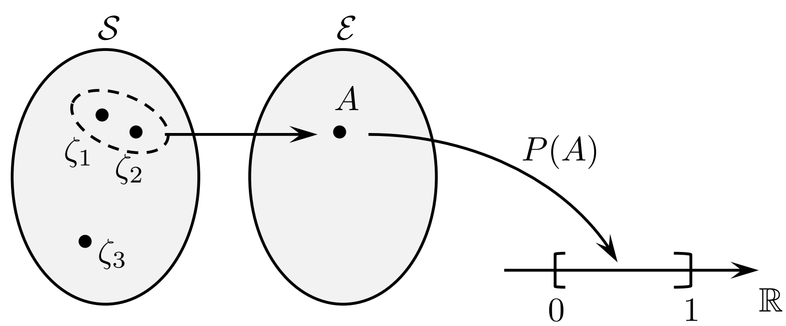
$$\overline{\overline{A}} = A$$

$$\overline{\mathcal{S}} = \varnothing$$

$$\overline{\varnothing} = \mathcal{S}$$

2-2 Probability Space

- Probability theory (in this course) is conceptualized as an experiment with possible *outcomes* $\zeta_1, \zeta_2, \ldots, \zeta_n$ where $\zeta_i \in \mathcal{S}$.
- An *event* is a subset of $\mathcal{S}$.
- The experimental outcome is *uncertain*.
- *Probability* is the measure of uncertainty.



$$
\begin{aligned}
\mathcal{S} &= \text{the set of experimental outcomes} \\
\mathcal{E} &= \text{the set of all events} \\
&= \text{the set of all subsets of } \mathcal{S} \\
P(A) &= \text{a probability measure of the event } A \\
&= P(\cdot) \text{ is a map } \mathcal{E} \to [0,1] \text{ in } \mathbb{R}
\end{aligned}
$$

Axioms of Probability for the event $A \subseteq \mathcal{S}$

1. $P(A) \geq 0$
2. $P(\mathcal{S}) = 1$
3. $A \cap B = \varnothing \quad \Rightarrow \quad P(A \cup B) = P(A) + P(B).$

Prove these: $P(\varnothing) = 0$

$$P(\overline{A}) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$B \subseteq A \Rightarrow P(A) \geq P(B)$$