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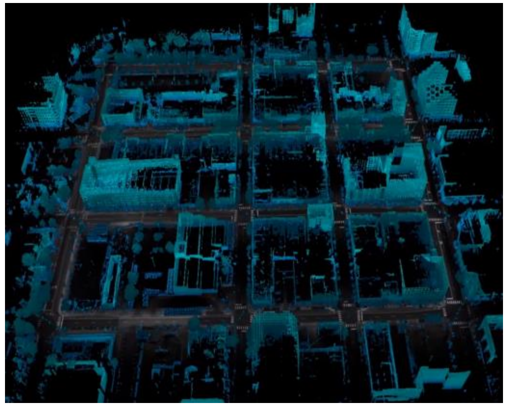
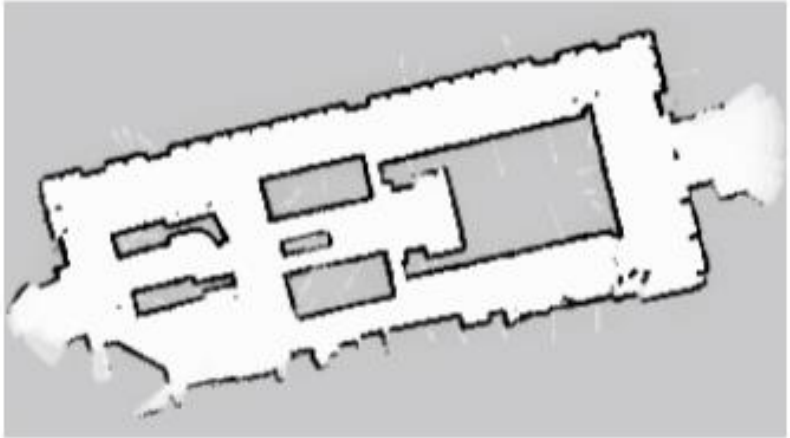
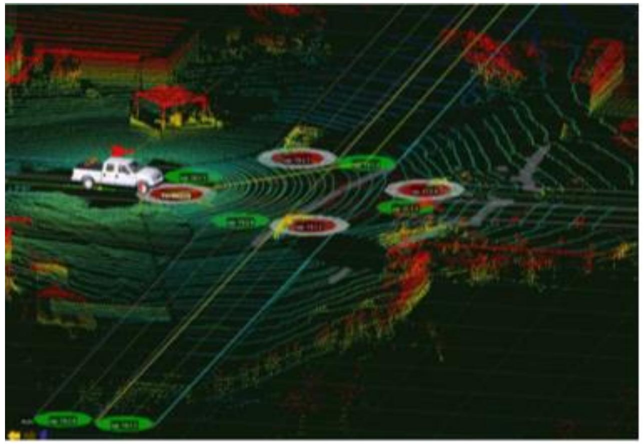


INTRO TO KALMAN FILTERING

ECEN 633: Robotic Localization and Mapping

Some slides courtesy of Ryan Eustice.

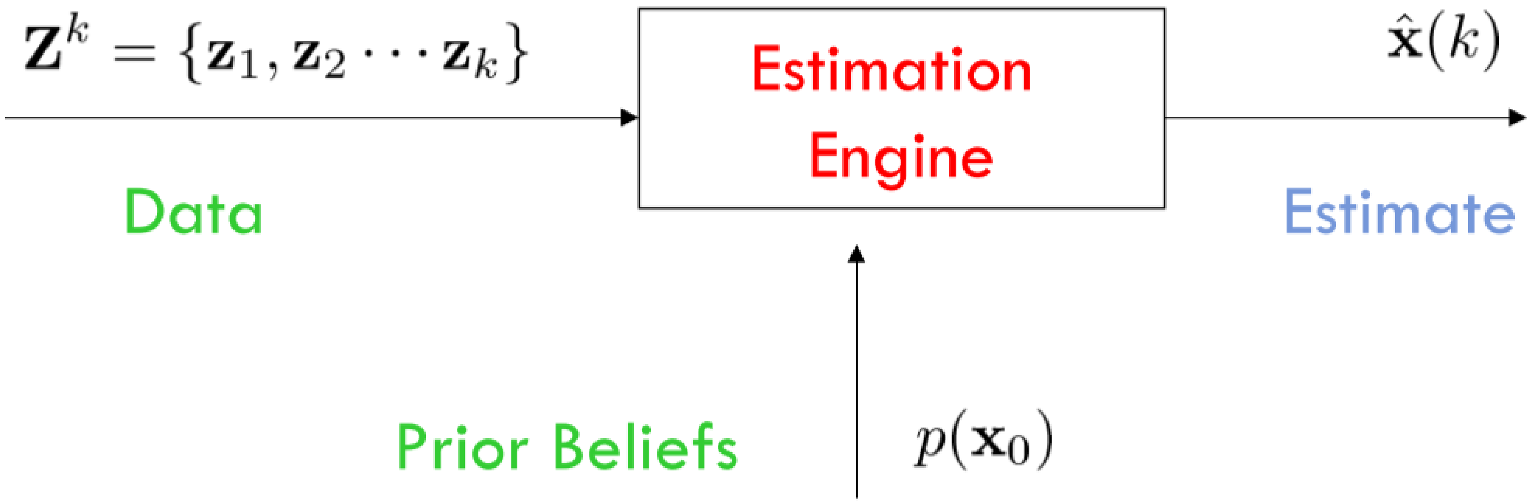
Agenda

- Minimum Mean Square Error
- Linear Kalman Filter
  - Gaussian Systems
    - Optimal Unbiased Estimator
  - Non-Gaussian Systems
    - Best Linear Unbiased Estimator

- KF Falling Body Demo

Estimation is …

"Estimation is the process by which we infer the value of a quantity of interest, $\mathbf{x}$*, by processing data that is in some way dependent on* $\mathbf{x}$ *."*



From P. Newman, Oxford

What does it mean for something to be the "best" or "optimal" filter?

- Minimum Mean Squared Error Estimation

$$\hat{\mathbf{x}}_{mmse} = \arg\min_{\hat{\mathbf{x}}} \mathcal{E}\{\underbrace{(\hat{\mathbf{x}} - \mathbf{x})^T(\hat{\mathbf{x}} - \mathbf{x})|\mathbf{Z^k}}_{\text{Cost Function}}\}$$

Choose $\hat{\mathbf{x}}$ so argument is minimised

Expectation operator ("average")

$\hat{\mathbf{x}}$ is estimate
$\mathbf{x}$ is truth

From P. Newman, Oxford

Evaluating

$$\mathcal{E}\{g(x)|y\} = \int_{-\infty}^{\infty} g(\mathbf{x})p(\mathbf{x}|\mathbf{y})dx$$

From probability theory

$$J(\hat{\mathbf{x}}, \mathbf{x}) = \mathcal{E}\{(\hat{\mathbf{x}} - \mathbf{x})^T(\hat{\mathbf{x}} - \mathbf{x})|\mathbf{Z^k}\} = \int_{-\infty}^{\infty} (\hat{\mathbf{x}} - \mathbf{x})^T(\hat{\mathbf{x}} - \mathbf{x})p(\mathbf{x}|\mathbf{Z^k})d\mathbf{x}$$

$$\frac{\partial J(\hat{\mathbf{x}}, \mathbf{x})}{\partial \hat{\mathbf{x}}} = 2\int_{-\infty}^{\infty} (\hat{\mathbf{x}} - \mathbf{x})p(\mathbf{x}|\mathbf{Z^k})d\mathbf{x} = 0$$

Splitting apart the integral, noting that $\hat{\mathbf{x}}$ is a constant:

$$\int_{-\infty}^{\infty} \hat{\mathbf{x}}p(\mathbf{x}|\mathbf{Z^k})d\mathbf{x} = \int_{-\infty}^{\infty} \mathbf{x}p(\mathbf{x}|\mathbf{Z^k})d\mathbf{x}$$

$$\hat{\mathbf{x}}\int_{-\infty}^{\infty} p(\mathbf{x}|\mathbf{Z^k})d\mathbf{x} = \int_{-\infty}^{\infty} \mathbf{x}p(\mathbf{x}|\mathbf{Z^k})d\mathbf{x}$$

$$\hat{\mathbf{x}} = \int_{-\infty}^{\infty} \mathbf{x}p(\mathbf{x}|\mathbf{Z^k})d\mathbf{x}$$

Very Important Thing → $\hat{\mathbf{x}}_{mmse} = \mathcal{E}\{\mathbf{x}|\mathbf{Z^k}\}$

From P. Newman, Oxford

Recursive Bayesian Estimation

Key idea: "one mans posterior is another's prior" ;-)

$\mathbf{Z}^k = \{\mathbf{z}_1, \mathbf{z}_2 \cdots \mathbf{z}_k\}$ Sequence of data (measurements)

We want conditional mean (mmse) of x given Z^k

Can we iteratively calculate this – i.e. every time a new measurement comes in, update our estimate?

$$p(\mathbf{x}|\mathbf{Z^k}) = f(p(\mathbf{x}|\mathbf{Z}^{k-1}), p(\mathbf{z}_k|\mathbf{x}))$$

From P. Newman, Oxford

Yes…

Implicitly dropped dependence on Z^{k-1}

$$p(\mathbf{x}|\mathbf{Z^k}) = \frac{p(\mathbf{z}_k|\mathbf{x})p(\mathbf{x}|\mathbf{Z}^{k-1})}{p(\mathbf{z}_k|\mathbf{Z}^{k-1})}$$

$$\underbrace{p(\mathbf{x}|\mathbf{Z^k})}_{\text{Estimate}} \propto \underbrace{p(\mathbf{z}_k|\mathbf{x})}_{\text{Likelihood}} \; \underbrace{p(\mathbf{x}|\mathbf{Z}^{k-1})}_{\text{Last Estimate}}$$

At time k

Explains data at time k
as function of x at time k

At time k-1

And if these distributions are Gaussian turning the handle leads to the Kalman filter……

From P. Newman, Oxford

Bayes Filter Reminder

- Prior $bel(\mathbf{x}_0)$
- Prediction

$$\overline{bel}(\mathbf{x}_t) = \int p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1})\, bel(\mathbf{x}_{t-1})\, d\mathbf{x}_{t-1}$$

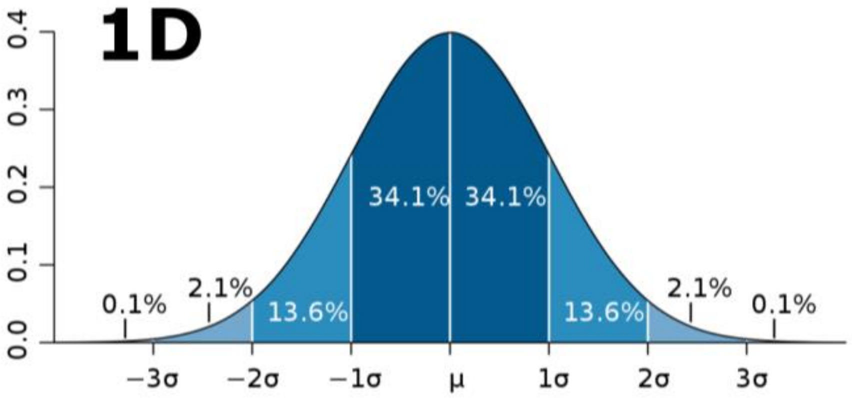
- Correction

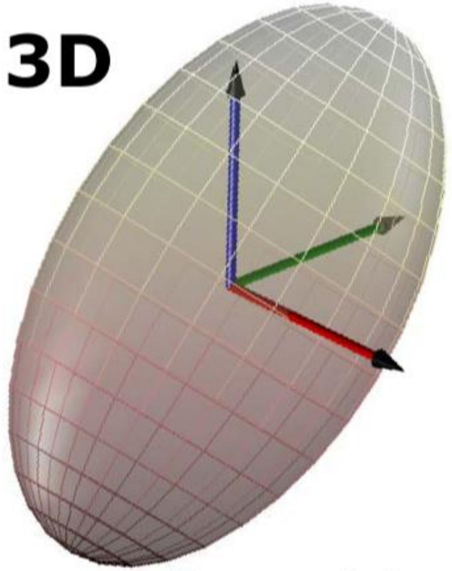
$$bel(\mathbf{x}_t) = \eta\, p(\mathbf{z}_t \mid \mathbf{x}_t)\, \overline{bel}(\mathbf{x}_t)$$

Kalman Filter Distribution

- Everything is Gaussian

$$p(\mathbf{x}) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$





Courtesy: K. Arras

Properties of Gaussians

- Univariate

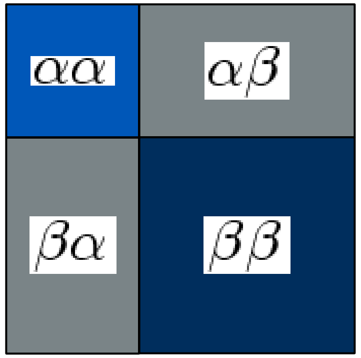
$$\left.\begin{array}{l} x \sim N(\mu, \sigma^2) \\ y = ax + b \end{array}\right\} \Rightarrow \quad y \sim N(a\mu + b, a^2\sigma^2)$$

- Multivariate

$$\left.\begin{array}{l} \mathbf{x} \sim N(\mu, \Sigma) \\ \mathbf{y} = A\mathbf{x} + \mathbf{b} \end{array}\right\} \Rightarrow \quad \mathbf{y} \sim N(A\mu + \mathbf{b}, A\Sigma A^T)$$

- We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations.

Gaussian Covariance & Information Parameterizations:



	Covariance Form	Information Form
Marginalization $p(\boldsymbol{\alpha}) = \int p(\boldsymbol{\alpha}, \boldsymbol{\beta}) d\boldsymbol{\beta}$	$\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$ $\Sigma = \Sigma_{\alpha\alpha}$ (sub-block)	$\boldsymbol{\eta} = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\boldsymbol{\eta}_\beta$ $\Lambda = \Lambda_{\alpha\alpha} - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha}$ (Schur complement)
Conditioning $p(\boldsymbol{\alpha}\|\boldsymbol{\beta}) = \frac{p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p(\boldsymbol{\beta})}$	$\boldsymbol{\mu}' = \boldsymbol{\mu}_\alpha + \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta)$ $\Sigma' = \Sigma_{\alpha\alpha} - \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\Sigma_{\beta\alpha}$ (Schur complement)	$\boldsymbol{\eta}' = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\boldsymbol{\beta}$ $\Lambda' = \Lambda_{\alpha\alpha}$ (sub-block)

Discrete Kalman Filter

Estimates the (n x 1) state $\mathbf{x}_t$ of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$\mathbf{x}_t = A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t + \varepsilon_t$$

Observed through (k x 1) measurements $\mathbf{z}_t$

$$\mathbf{z}_t = C_t \mathbf{x}_t + \delta_t$$

Components of a Kalman Filter

A_t — Matrix (n x n) that describes how the state evolves from t-1 to t without controls or noise.

B_t — Matrix (n x m) that describes how the control u_t changes the state from t-1 to t .

C_t — Matrix (k x n) that describes a projection of state x_t to an observation z_t .

ε_t, δ_t — Random variables representing the process and measurement noise that are assumed to be zero mean, independent, and normally distributed with covariance R_t and Q_t, respectively.

Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$bel(\mathbf{x}_0) = N\left(\mathbf{x}_0; \mu_0, \Sigma_0\right)$$

Linear Gaussian Systems Dynamics

- Dynamics are linear function of state and control plus additive noise:

$$\mathbf{x}_t = A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t + \varepsilon_t$$

$$p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) = N\left(\mathbf{x}_t; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t\right)$$

$$\overline{bel}(\mathbf{x}_t) = \int p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) \qquad bel(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1}$$

$$\Downarrow \qquad \qquad \Downarrow$$

$$\sim N\left(\mathbf{x}_t; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t\right) \qquad \sim N\left(\mathbf{x}_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

Linear Gaussian Systems: Dynamics

$$\overline{bel}(\mathbf{x}_t) = \int p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) \qquad bel(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\sim N\left(\mathbf{x}_t; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t\right) \sim N\left(\mathbf{x}_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

$$\Downarrow$$

$$\overline{bel}(\mathbf{x}_t) = \eta \int \exp\left\{-\frac{1}{2}(\mathbf{x}_t - A_t \mathbf{x}_{t-1} - B_t \mathbf{u}_t)^T R_t^{-1} (\mathbf{x}_t - A_t \mathbf{x}_{t-1} - B_t \mathbf{u}_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(\mathbf{x}_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (\mathbf{x}_{t-1} - \mu_{t-1})\right\} d\mathbf{x}_{t-1}$$

$$\overline{bel}(\mathbf{x}_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Linear Gaussian Systems: Observations

- Observations are linear function of state plus additive noise:

$$\boxed{\mathbf{z}_t = C_t \mathbf{x}_t + \delta_t}$$

$$\boxed{p(\mathbf{z}_t \mid \mathbf{x}_t) = N\left(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t\right)}$$

$$\begin{array}{ll} bel(\mathbf{x}_t) = \eta\, p(\mathbf{z}_t \mid \mathbf{x}_t) & \overline{bel}(\mathbf{x}_t) \\ \qquad\qquad\qquad\Downarrow & \quad\Downarrow \\ \qquad \sim N\left(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t\right) & \sim N\left(\mathbf{x}_t; \bar{\mu}_t, \bar{\Sigma}_t\right) \end{array}$$

Linear Gaussian Systems: Observations

$$bel(\mathbf{x}_t) = \eta \quad p(\mathbf{z}_t \mid \mathbf{x}_t) \qquad \overline{bel}(\mathbf{x}_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(\mathbf{z}_t ; C_t \mathbf{x}_t , Q_t\right) \qquad \sim N\left(\mathbf{x}_t ; \bar{\mu}_t , \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$bel(\mathbf{x}_t) = \eta \exp\left\{-\frac{1}{2}(\mathbf{z}_t - C_t\mathbf{x}_t)^T Q_t^{-1}(\mathbf{z}_t - C_t\mathbf{x}_t)\right\} \exp\left\{-\frac{1}{2}(\mathbf{x}_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(\mathbf{x}_t - \bar{\mu}_t)\right\}$$

Kalman Gain

$$bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t \boxed{(\mathbf{z}_t - C_t \bar{\mu}_t)} \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad \boxed{K_t} = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Innovation

Kalman Filter Algorithm

1: **Kalman_filter**$(\boldsymbol{\mu}_{t-1}, \Sigma_{t-1}, \mathbf{u}_t, \mathbf{z}_t)$**:**

2: $\bar{\boldsymbol{\mu}}_t = A_t\, \boldsymbol{\mu}_{t-1} + B_t\, \mathbf{u}_t$

3: $\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^\top + R_t$

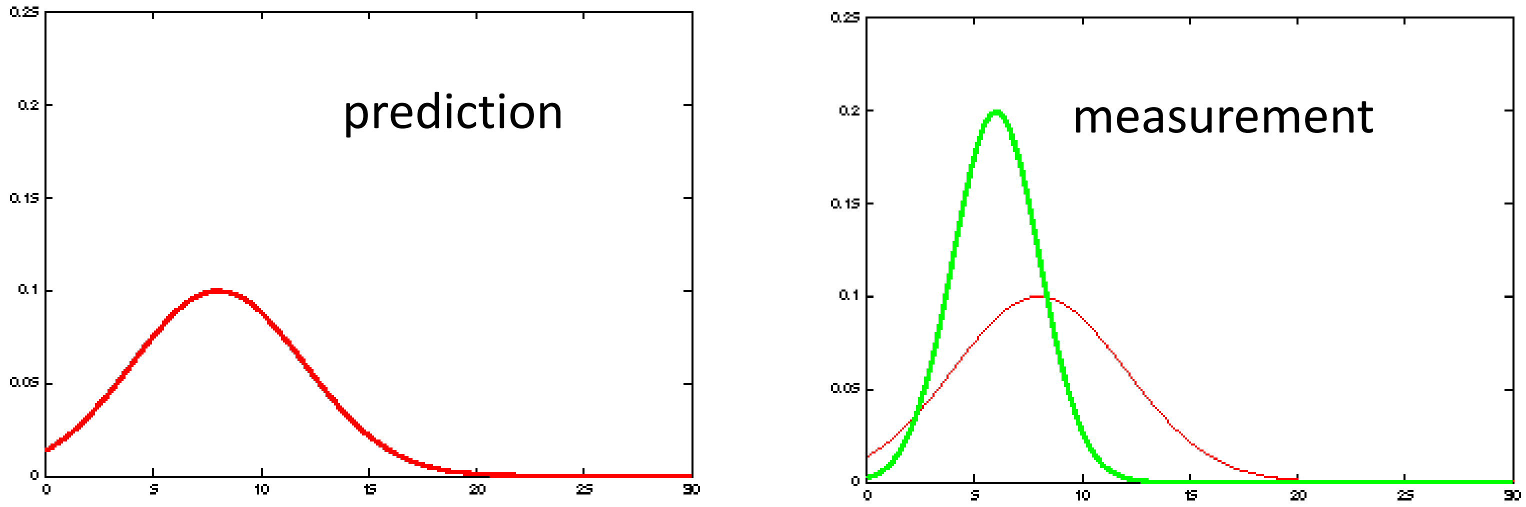
4: $K_t = \bar{\Sigma}_t\, C_t^\top (C_t\, \bar{\Sigma}_t\, C_t^\top + Q_t)^{-1}$

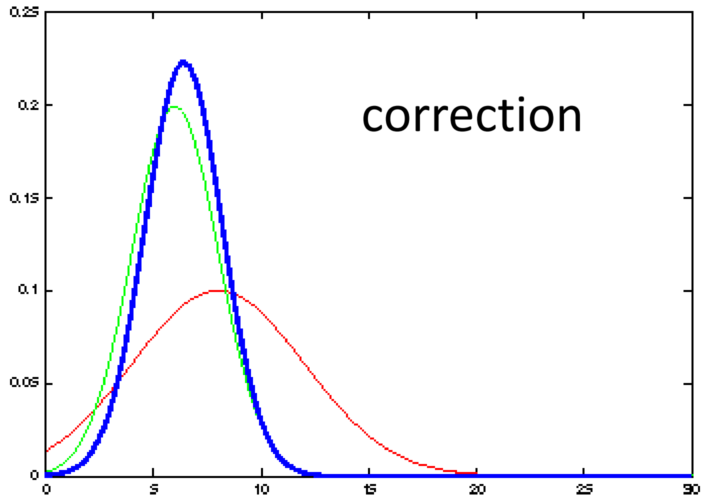
5: $\boldsymbol{\mu}_t = \bar{\boldsymbol{\mu}}_t + K_t(\mathbf{z}_t - C_t\, \bar{\boldsymbol{\mu}}_t)$

6: $\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$

7: *return* $\boldsymbol{\mu}_t, \Sigma_t$

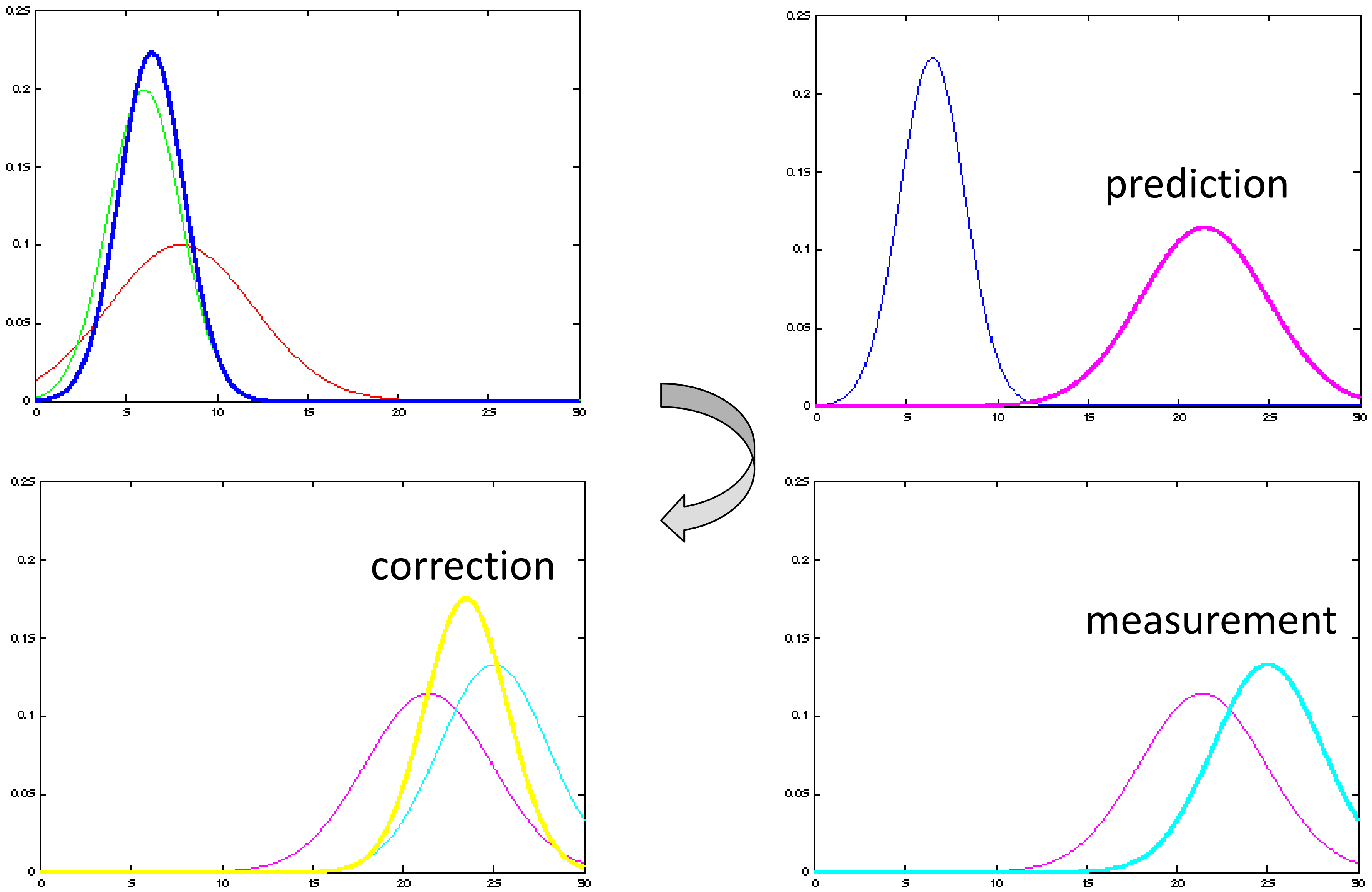
1D Kalman Filter Example (1)





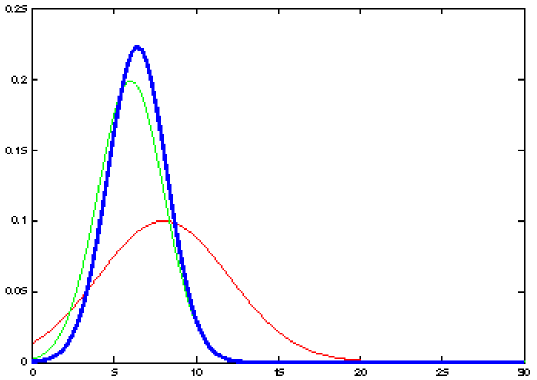
It's a weighted mean!

1D Kalman Filter Example (2)



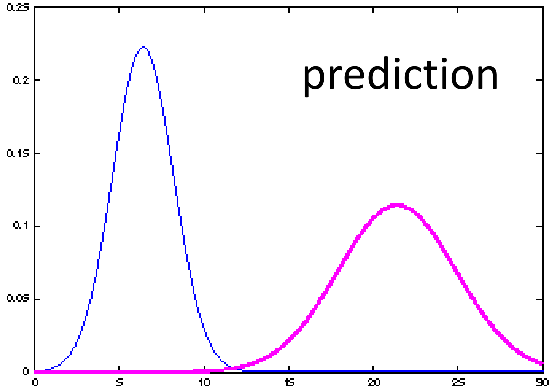
The Prediction-Correction Cycle

Prediction

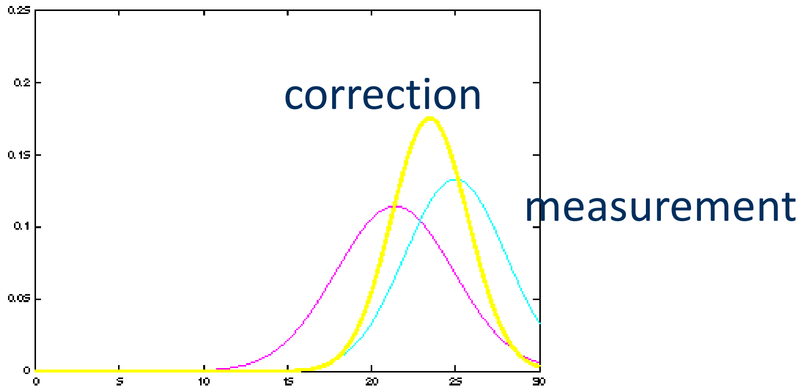


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{\varepsilon_t}^2 \end{cases}$$

$$\overline{bel}(\mathbf{x}_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

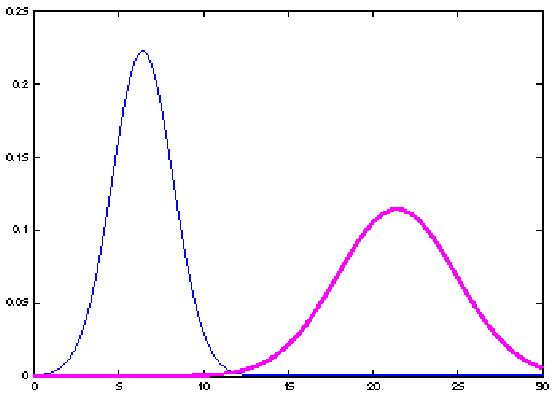


The Prediction-Correction Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + k_t(z_t - c_t\bar{\mu}_t) \\ \sigma_t^2 = (1 - k_t c_t)\bar{\sigma}_t^2 \end{cases}, k_t = \frac{c_t\bar{\sigma}_t^2}{c_t^2\bar{\sigma}_t^2 + \sigma_{\delta_t}^2}$$

$$bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(\mathbf{z}_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$



Correction

The Prediction-Correction Cycle

Prediction

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{\varepsilon_t}^2 \end{cases}$$

$$\overline{bel}(\mathbf{x}_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Correction

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + k_t (z_t - c_t \bar{\mu}_t) \\ \sigma_t^2 = (1 - k_t c_t) \bar{\sigma}_t^2 \end{cases}, \quad k_t = \frac{c_t \bar{\sigma}_t^2}{c_t^2 \bar{\sigma}_t^2 + \sigma_{\delta_t}^2}$$

$$bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (\mathbf{z}_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Alternative (Equivalent) Covariance Update Expressions

- Defining innovation/observation covariance as

$$S_t = C_t \bar{\Sigma}_t C_t^\top + Q_t$$

- Alternative Update Expressions (see Bar-Shalom Chap 5)

1. $$\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$$

2. $$\Sigma_t = \bar{\Sigma}_t - K_t S_t K_t^\top$$

3. $$\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t (I - K_t C_t)^\top + K_t Q_t K_t^\top$$

"Joseph form"

What if the statistics are **not** Gaussian?

- Structure of KF corresponds to the Best Linear Unbiased Estimator (BLUE)
  - i.e., if we restrict our estimator to the class of linear estimators, then the KF is the best *linear* MMSE estimator*

$$\hat{\mathbf{x}} = A\mathbf{z} + \mathbf{b} \longleftarrow \text{Affine function of } \mathbf{z}$$

- Estimator $\hat{\mathbf{x}} = \boldsymbol{\mu}_x + \Sigma_{\mathbf{xz}}\Sigma_{\mathbf{zz}}^{-1}(\mathbf{z} - \boldsymbol{\mu}_z)$

- Matrix MSE $E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^\top] = \Sigma_{\mathbf{xx}} - \Sigma_{\mathbf{xz}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{zx}}$

- Remarks
  - The *best estimator* (in the MMSE sense) for *Gaussian Random variables* is identical to
    - The *best linear estimator* for *arbitrarily distributed* random variables with the *same first- and second-order moments.*

*Note: a nonlinear estimator could do *better!*

Falling Body Example

- Governing Equations

$$\ddot{y}(t) = -g$$

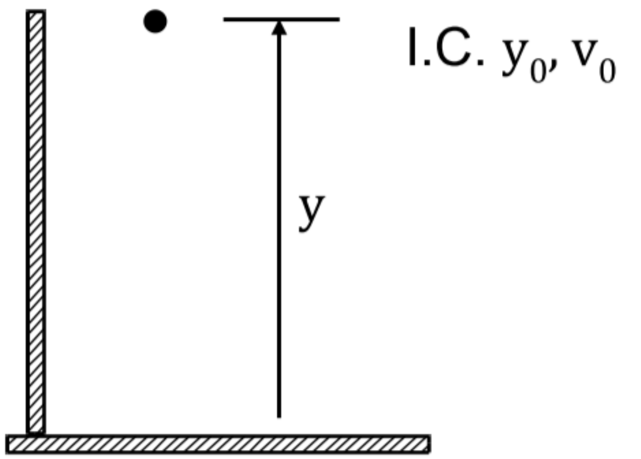
$$\dot{y}(t) = -gt + v_o$$

$$y(t) = -\frac{1}{2}t^2 + v_o t + y_o$$

- CT State-Space Description

$$\mathbf{x}(t) = [y(t), \dot{y}(t)]^\top$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} g$$



- DT State-Space Description

$$\mathbf{x}[k] = [y[k],\ \dot{y}[k]]^\top$$

$$\mathbf{x}[k] = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}[k-1] + \begin{bmatrix} -0.5\Delta t^2 \\ -\Delta t \end{bmatrix} g$$

$$z[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}[k] + \delta[k]$$

ProbRob Notation

- Initial State

$$\mathbf{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_o,\ \Sigma_o) = \mathcal{N}\left(\begin{bmatrix}\mu_y \\ \mu_{\dot{y}}\end{bmatrix}, \begin{bmatrix}\sigma_y^2 & 0 \\ 0 & \sigma_{\dot{y}}^2\end{bmatrix}\right)$$

- Process Model

$$\mathbf{x}_t = \underbrace{\begin{bmatrix}1 & \Delta t \\ 0 & 1\end{bmatrix}}_{A_t} \mathbf{x}_{t-1} + \underbrace{\begin{bmatrix}-0.5\Delta t^2 \\ -\Delta t\end{bmatrix}}_{B_t} \underbrace{g}_{\mathbf{u}_t} + \underbrace{\varepsilon_t}_{\mathbf{0}}$$

- Observation Model

$$z_t = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C_t} \mathbf{x}_t + \delta_t$$

Falling Body: Prediction

$$\bar{\boldsymbol{\mu}}_1 = A_1\boldsymbol{\mu}_0 + B_1\mathbf{u}_1 = \begin{bmatrix} \mu_y + \mu_{\dot{y}}\Delta t - 0.5g\Delta t^2 \\ \mu_{\dot{y}} - g\Delta t \end{bmatrix}$$

$$\bar{\Sigma}_1 = A_1\Sigma_0 A_1^\top + R_1 = \begin{bmatrix} \sigma_y^2 + \sigma_{\dot{y}}^2\Delta t^2 & \sigma_{\dot{y}}^2\Delta t \\ \sigma_{\dot{y}}^2\Delta t & \sigma_{\dot{y}}^2 \end{bmatrix}$$

Process model builds correlation beetween position and velocity

Falling body position uncertainty *increases* during open-loop prediction due to uncertainty in velocity

Falling Body: Correction

$$K_1 = \bar{\Sigma}_1 C_1^\top (C_1 \bar{\Sigma}_1 C_1^\top + Q_1)^{-1} = \begin{bmatrix} \bar{\sigma}_y^2 \\ \bar{\sigma}_{\dot{y}y} \end{bmatrix} (\bar{\sigma}_y^2 + \sigma_q^2)^{-1}$$

$$\mu_1 = \bar{\mu}_1 + K_1(z_1 - C_1\bar{\mu}_1) = \begin{bmatrix} \bar{\mu}_y + \frac{\bar{\sigma}_y^2}{(\bar{\sigma}_y^2 + \sigma_q^2)} \overset{0}{\cancel{(z_1 - \bar{\mu}_y)}} \\ \bar{\mu}_{\dot{y}} + \frac{\bar{\sigma}_{\dot{y}y}}{(\bar{\sigma}_y^2 + \sigma_q^2)} (z_1 - \bar{\mu}_y) \end{bmatrix}$$

$$\Sigma_1 = (I - K_1 C_1)\bar{\Sigma}_1 = \begin{bmatrix} \bar{\sigma}_y^2 \left(1 - \frac{\bar{\sigma}_y^2}{\bar{\sigma}_y^2 + \sigma_q^2}\right) & \bar{\sigma}_{y\dot{y}} \left(1 - \frac{\bar{\sigma}_y^2}{\bar{\sigma}_y^2 + \sigma_q^2}\right) \\ \bar{\sigma}_{\dot{y}y} \left(1 - \frac{\bar{\sigma}_y^2}{\bar{\sigma}_y^2 + \sigma_q^2}\right) & \bar{\sigma}_{\dot{y}}^2 - \frac{\sigma_{\dot{y}y}^2}{\bar{\sigma}_y^2 + \sigma_q^2} \end{bmatrix}$$

Zero state uncertainty case: $\bar{\sigma}_y = 0$

Falling Body: Correction

$$K_1 = \bar{\Sigma}_1 C_1^\top (C_1 \bar{\Sigma}_1 C_1^\top + Q_1)^{-1} = \begin{bmatrix} \bar{\sigma}_y^2 \\ \bar{\sigma}_{\dot{y}y} \end{bmatrix} (\bar{\sigma}_y^2 + \sigma_q^2)^{-1}$$

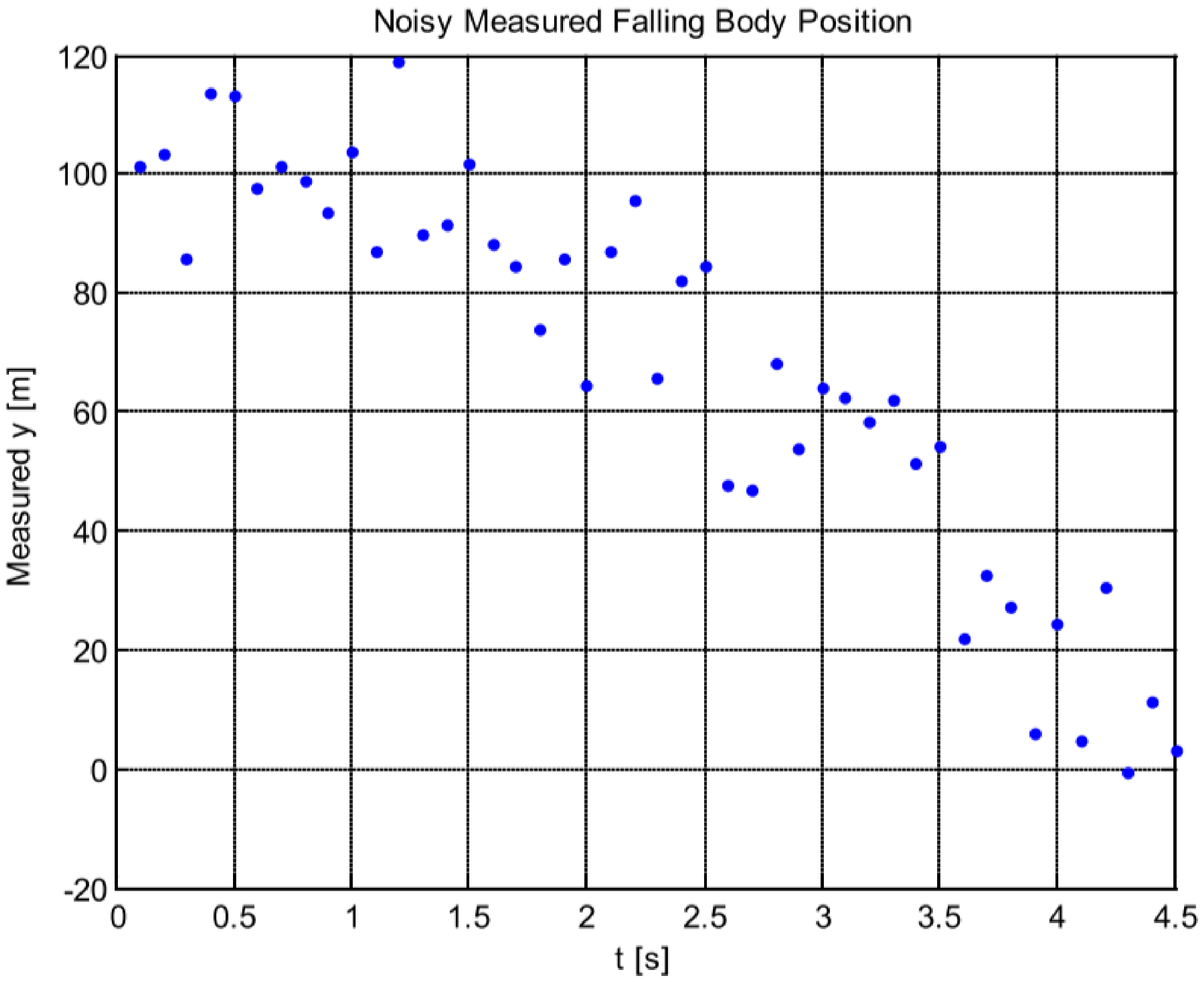
1

$$\mu_1 = \bar{\mu}_1 + K_1(z_1 - C_1\bar{\mu}_1) = \begin{bmatrix} \cancel{\bar{\mu}_y} + \frac{\bar{\sigma}_y^2}{(\bar{\sigma}_y^2 + \sigma_q^2)}(z_1 - \cancel{\bar{\mu}_y}) \\ \bar{\mu}_{\dot{y}} + \frac{\bar{\sigma}_{\dot{y}y}}{(\bar{\sigma}_y^2 + \sigma_q^2)}(z_1 - \bar{\mu}_y) \end{bmatrix}$$

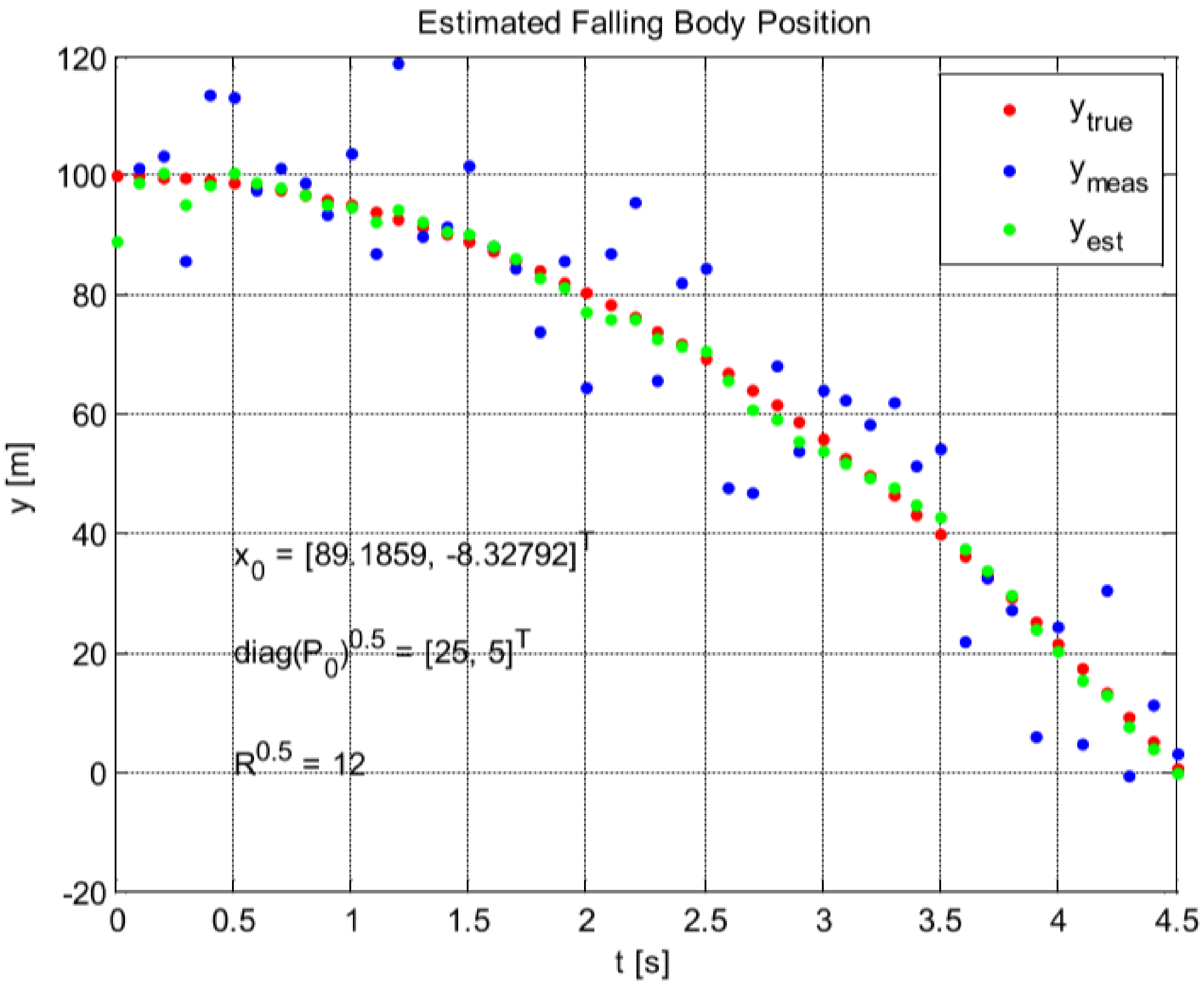
$$\Sigma_1 = (I - K_1 C_1)\bar{\Sigma}_1 = \begin{bmatrix} \bar{\sigma}_y^2 (1 - \frac{\bar{\sigma}_y^2}{\bar{\sigma}_y^2 + \sigma_q^2}) & \bar{\sigma}_{y\dot{y}} (1 - \frac{\bar{\sigma}_y^2}{\bar{\sigma}_y^2 + \sigma_q^2}) \\ \bar{\sigma}_{\dot{y}y} (1 - \frac{\bar{\sigma}_y^2}{\bar{\sigma}_y^2 + \sigma_q^2}) & \bar{\sigma}_{\dot{y}}^2 - \frac{\sigma_{\dot{y}y}^2}{\bar{\sigma}_y^2 + \sigma_q^2} \end{bmatrix}$$

Infinite state uncertainty case: $\bar{\sigma}_y = \infty$

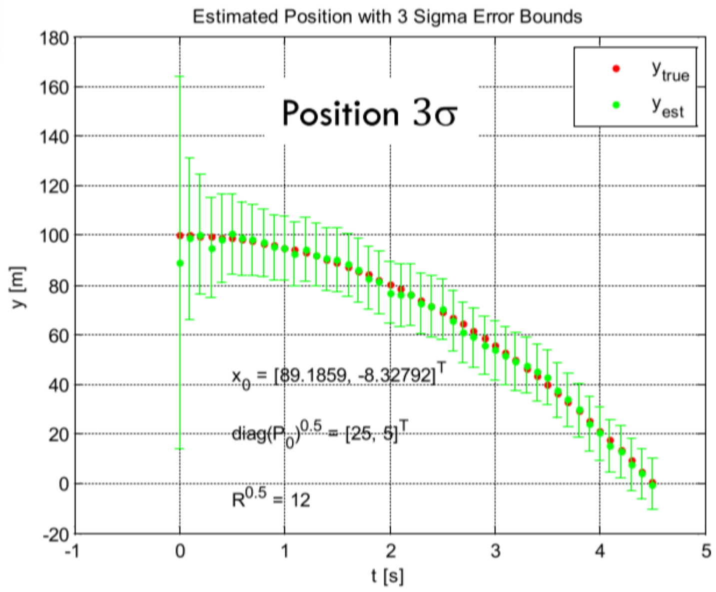
Noisy Range Measurements (σ = 12 m)



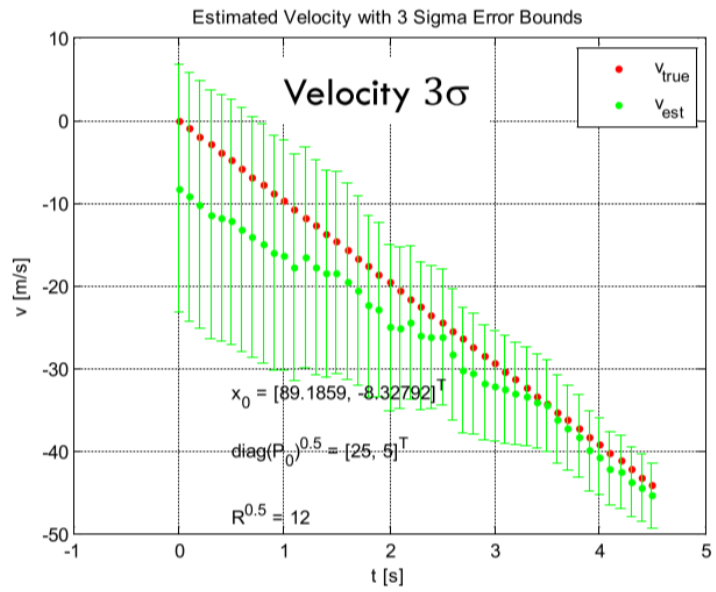
KF Results



Estimated Falling Body Position
y [m]
t [s]
y_{true}
y_{meas}
y_{est}
x_0 = [89.1859, -8.32792]^T
diag(P_0)$^{0.5}$ = [25, 5]^T
$R^{0.5}$ = 12



Estimated Position with 3 Sigma Error Bounds
Position 3σ
y [m]
t [s]
y_{true}
y_{est}
x_0 = [89.1859, -8.32792]^T
diag(P_0)$^{0.5}$ = [25, 5]^T
$R^{0.5}$ = 12



Estimated Velocity with 3 Sigma Error Bounds
Velocity 3σ
v [m/s]
t [s]
v_{true}
v_{est}
x_0 = [89.1859, -8.32792]^T
diag(P_0)$^{0.5}$ = [25, 5]^T
$R^{0.5}$ = 12

Note:

- Can process multiple measurements at once if necessary!

A. **Update** via Process model (Dynamics) with time or control inputs

B. **Correct** via measurements whenever available

Kalman Filter Summary

- **Highly efficient:** Polynomial in measurement dimensionality k and state dimensionality n:

$$O(k^{2.376} + n^2)$$

- **Kalman Gain**
  - Weights measurement update by:
    - State Uncertainty
    - Measurement Uncertainty

- **Optimal for linear Gaussian systems**!
  - No other estimator can do better

- Most robotics systems are **nonlinear**!