

# Introduction to Feedback Control

USING DESIGN STUDIES

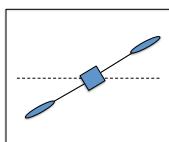
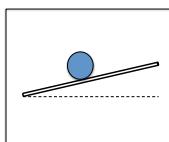
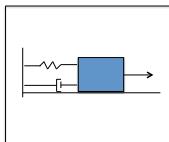
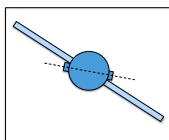
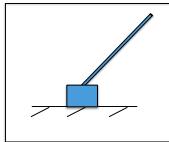
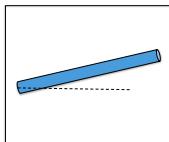
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## Preface

Does the world really need another introductory textbook on feedback control? For many years we have felt that the many quality textbooks that currently exist are more than sufficient. However, several years ago the first author (RWB) had an experience that changed my mind, and the minds of my colleagues. I had just finished teaching the introductory feedback control course at BYU. I had a need for a new master's student in my research group and hired one of the top students in the class. We had designed a gimbal system for a small unmanned air vehicle, and we needed a control system implemented on a microcontroller for the gimbal. I tasked this new master's student to model the gimbal and design the control system. I was surprised by how much the student struggled with this task, especially understanding where to begin and how to model the gimbal. If I had given him a transfer function and asked him to design a PID controller, or if I had given him a state space model and asked him to design an observer and controller for the system, he would not have had any difficulty. But he did not know how to do an end-to-end design that required developing models for the system, including physical constraints. It was this experience that convinced me that my current approach to teaching feedback control was inadequate.

The next time that I taught the course, rather than focusing on the theory of feedback design, I focused the lectures on the end-to-end design process. Accordingly, I spent significantly more time talking about the modeling process, adding several lectures on the Euler-Lagrange method. I also added several lectures on linearization about set points, and on developing linear models. When I taught feedback design methods, I focused on design specifications and the need to account for saturation, sensor noise, model uncertainty, and external disturbances on the system. Over the next several years, we developed several complete design studies and then used these design studies throughout the course to illustrate the material. That course reorganization and the realization that existing textbooks on

feedback control did not fit the pedagogy were the genesis of this textbook.

Therefore, this textbook is unique in several ways. Most importantly, it is focused, in its organization, in its examples, and in its homework problems, on a particular, but fairly general, end-to-end design strategy and methodology. Topics were included in the book only if they aided in the design process. For example, the book does not include a chapter on the root locus (although it is discussed in an appendix). The second unique feature is that the examples and homework problems follow a small set of design studies throughout the book. The text provides complete worked examples for three design studies: a single link robot arm, a pendulum on a cart, and a satellite attitude control problem. The homework problems parallel the examples with three additional design studies: a mass-spring-damper, a ball on beam, and a planar vertical take-off and landing aircraft, similar to a quadrotor. A third unique feature of the book is that complete solutions to the three example design studies are provided in Matlab/Simulink and Python at <http://controlbook.byu.edu>.

The design methodology culminates in computer code that implements the controllers. In the examples provided in the book, the computer code interacts with a simulation model of the plant composed of coupled nonlinear differential equations, rather than the physical plant. However, the example computer code is written in a way that can be directly implemented on a microcontroller, or other computing devices. We have used Matlab/Simulink because many students are already familiar with these tools. But we also provide Python examples because Python is free of charge and therefore accessible to everyone, and because for many applications, Python can be used to implement the controller in hardware.

One criticism of the book might be that all of the design problems are mechanical systems, and that they are fairly similar to each other. The student may come away with the impression that feedback control and the design methods discussed in the book are only applicable to mechanical systems. Of course this is not true. The tools taught in the book are widely used in many applications including electrical power systems, disk drives, aerospace systems, chemical processing, biological systems, queueing systems, and many more. Although this book is focused on mechanical systems, we believe that the student who masters the material will be able to easily transition to other application domains. Mechanical systems, like those covered in this book, have a compelling pedagogical advantage because the control objectives are intuitive, and the behavior of the system can be easily visualized. Therefore, we feel that they offer the best setting for developing intuition behind feedback design.

We had two other motivations for writing this textbook. The first is the exorbitant prices currently being charged for textbooks. Accordingly, we are providing an electronic version of this book to students free of charge at <http://controlbook.byu.edu>. The book has required significant time and energy to develop, and so we hope that “free” does not equate to “low quality.” An Amazon version of the book is also available for those desiring a hard copy. Given the amount of high quality open source software, students may get the wrong impression about monetary compensation for technical work. Good work should be

compensated. However, since our day jobs provide adequate financial remuneration, we ask to be compensated by your feedback about your educational experience with the book, including typos and other errors. This will allow us to update and improve the book. We also hope that you will understand that we will probably not be very responsive to email asking for help with the homework problems. Another motivation for writing the book is our observation that this generation of students seems to prefer to learn using electronic resources. The book has therefore been written with the assumption that it will be studied electronically. Therefore, hyperlinks and other electronic aids are embedded throughout the text. We hope that these enhance the educational experience.

RWB



# 1

## Introduction

### Overview

The objective of this book is to prepare the reader to design feedback control systems for dynamic systems and to lay the groundwork for further studies in the area. The design philosophy that we follow throughout the book is illustrated schematically in [Fig. 1-1](#). The objective is to design a control system for the “Physical System” shown in [Fig. 1-1](#). The physical system includes actuators like motors and propellers, and sensors like accelerometers, gyros, GPS, cameras, and pressure sensors. The first step in the design process is to model the physical system using nonlinear differential equations. While approximations and simplifications will be necessary at this step, the hope is to capture, in mathematics, all of the important characteristics of the physical system. While there are many different methods for developing the equations of motion of the physical systems, in this book we will introduce one specific method, the Euler-Lagrange method, that is applicable to a wide variety of mechanical systems. We will, for the most part, abstract the actuators to applied forces and torques on the system. Similarly, the sensors will be abstracted to general measurements of certain physical quantities like position and velocity. The resulting model is called the Simulation Model as shown in [Fig. 1-1](#). In the design process, the simulation model is used for the high fidelity computer simulation of the physical system. Every control design will be tested thoroughly on the simulation model. However, the simulation model is only a mathematical approximation of the physical system, and simply because a design is effective on the simulation model, we should not assume that it will function properly on the physical system. Part I of the book introduces the Euler-Lagrange method and demonstrates how to derive simulation models for mechanical systems using this method.

The simulation model is typically nonlinear and high order and is too mathematically complex to be useful for control design. Therefore, to facilitate design,

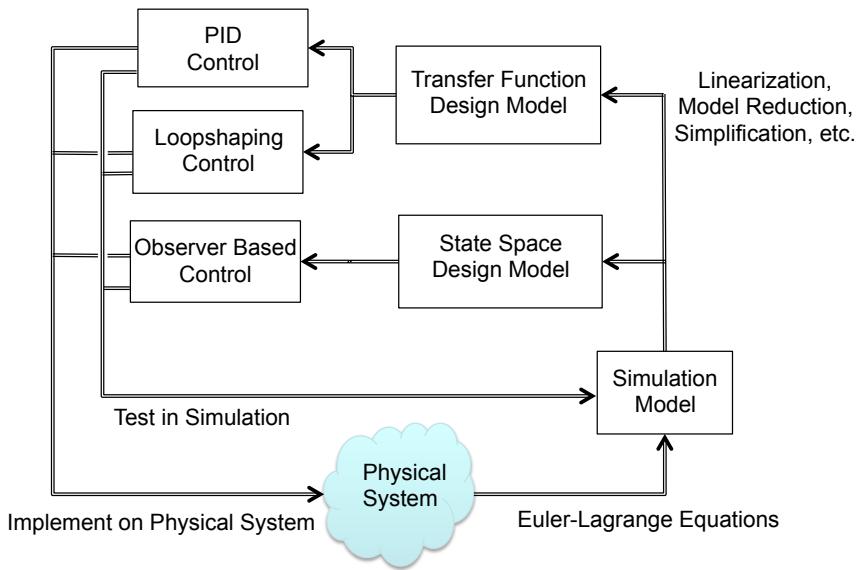


Figure 1-1: The design process. Using principles of physics, the physical system is modeled mathematically, resulting in the simulation model. The simulation model is simplified to create design models that are used for the control design. The control design is then tested and debugged in simulation and finally implemented on the physical system.

the simulation model is simplified and usually linearized to create lower-order design models. For any physical system, there may be multiple design models that capture certain aspects of the system that are important for a particular design. In this book we will introduce the two most common design models used for control systems analysis and design: transfer-function models and state-space models. In both cases we will linearize the system about a particular operating condition. The advantage of transfer function models is that they capture the frequency dependence of the input-output relationship in dynamic systems. The advantage of state space models is that they are more intuitive and are better adapted to numerical implementation. Part II of the book shows how to linearize the equations of motion and how to create transfer function and state space design models.

The next three parts, Part III–V, introduce three different control design methods. Part III shows how to design Proportional-Integral-Derivative (PID) controllers for second order systems. The PID method is the most common control technique used in industry. It can be understood from both a time-domain and frequency-domain perspective. The advantage of PID is its simplicity in design and implementation and the fact that it can be used to achieve high performance for a surprising variety of systems. However, the disadvantage of PID control is that stability and performance can typically only be guaranteed for second order systems. Therefore, Part III is focused exclusively on second order systems.

In Part IV we introduce the observer-based control method. Observer based methods can be thought of as a natural extension of PID controllers to high order systems. However, observer based methods are much more general than PID and can be used to obtain higher performance. In addition, most observer based techniques extend to nonlinear and time-varying systems. The disadvantages of observer-based methods are that they require a higher level of mathematical sophistication to understand and use and they are primarily time-domain methods.

The final design technique, which is covered in Part V, is the loopshaping control design method. The loopshaping method is a frequency domain technique that explicitly addresses robustness of the system. The loopshaping method is also a natural extension of PID control to higher order systems. One of the advantages of loopshaping techniques is that they can be used when only a frequency response model of the system is available.

The book is organized around several design studies. For each new concept that is introduced, we will apply that concept to three specific design problems. The design problems are introduced in the following three sections and include a single link robot arm, a pendulum on a cart, and a simple satellite attitude control problem. The homework problems, which are provided in Part VI, will follow the same format with problems that are identical to those worked in the book, but they will be applied to different physical systems. Our hope is to illustrate the control design process with specific, easy-to-understand problems. We note, however, that the concepts introduced in this book are applicable to a much wider variety of problems than those illustrated in the book. We focus on mechanical systems, but a similar process (with the exception of the modeling section) can be followed for any system that can be modeled using ordinary differential equations.

Examples include aerodynamic systems, electrical systems, chemical processes, large structures like buildings and bridges, biological feedback systems, economic systems, queuing systems, and many more.

## 1.1 Design Studies

### A Design Study: Single Link Robot Arm

Figure 1-2 shows a single link robot arm system. The robot arm has mass  $m$  and length  $\ell$ . The angle of the robot is given by  $\theta$  as measured from horizontal to the ground. The angular speed of the arm is  $\dot{\theta}$ . There is an applied torque  $\tau$  at the joint. There is also a damping torque that opposes the rotation of the joint of magnitude, which can be modeled as  $-b\dot{\theta}$ .

The actual physical parameters of the system are  $m = 0.5 \text{ kg}$ ,  $\ell = 0.3 \text{ m}$ ,  $g = 9.8 \text{ m/s}^2$ ,  $b = 0.01 \text{ Nms}$ . The torque is limited by  $|\tau| \leq 1 \text{ Nm}$ .

#### Examples (with solutions):

**A.2** Kinetic energy.

**A.3** Equations of motion.

**A.4** Linearize equations of motion.

**A.5** Transfer function model.

**A.6** State space model.

**A.7** Pole placement using PD.

**A.8** Second order design.

**A.9** Integrators and system type.

**A.P.6** Root locus.

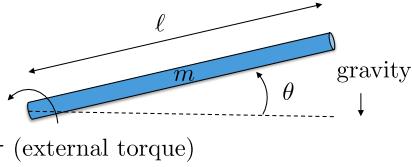


Figure 1-2: Single Link Robot Arm

**A.10** Digital PID.

**A.11** Full state feedback.

**A.12** Full state with integrator.

**A.13** Observer based control.

**A.14** Disturbance observer.

**A.15** Frequency Response.

**A.16** Loop gain.

**A.17** Stability margins.

**A.18** Loopshaping design.

Note: The notation identifies the design configuration and the chapter in which it is used: **Example A.4** refers to the application of approximations in Chapter 4 to linearize design configuration A. Similarly, **Example A.P.6** refers to the root locus assignment associated with Appendix A.P.6.

## B Design Study: Pendulum on a Cart

Figure 1-3 shows the pendulum on a cart system. The position of the cart measured from the origin is  $z$  and the linear speed of the cart is  $\dot{z}$ . The angle of the pendulum from straight up is given by  $\theta$ , and the angular velocity is  $\dot{\theta}$ . The rod is of length  $\ell$ , and has mass  $m_1$ , and is approximated as being infinitely thin. The cart has mass  $m_2$ . Gravity acts in the down direction. The only applied force is  $F$ , which acts in the direction of  $z$ . The cart slides on a frictionless surface, but air friction produces a damping force equal to  $-b\dot{z}$ .

The physical constants are  $m_1 = 0.25 \text{ kg}$ ,  $m_2 = 1.0 \text{ kg}$ ,  $\ell = 1.0 \text{ m}$ ,  $g = 9.8 \text{ m/s}^2$ ,  $b = 0.05 \text{ Ns}$ . The force is limited by  $|F| \leq 5 \text{ N}$ .

### Examples (with solutions):

- B.2** Kinetic energy.
- B.3** Equations of Motion.
- B.4** Linearize equations of motion.
- B.5** Transfer function model.
- B.6** State space model.
- B.8** Successive loop closure.
- B.9** Integrators and system type.
- B.P.6** Root locus.
- B.10** Digital PID.
- B.11** Full state feedback.
- B.12** Full state with integrator.
- B.13** Observer based control.
- B.14** Disturbance observer.
- B.15** Frequency Response.
- B.16** Loop gain.
- B.17** Stability margins.
- B.18** Loopshaping design.

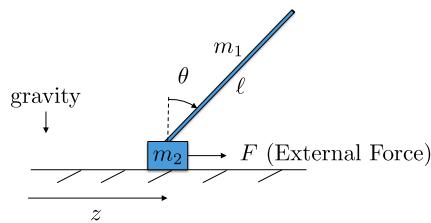


Figure 1-3: Pendulum on a cart.

## C Design Study: Satellite Attitude Control

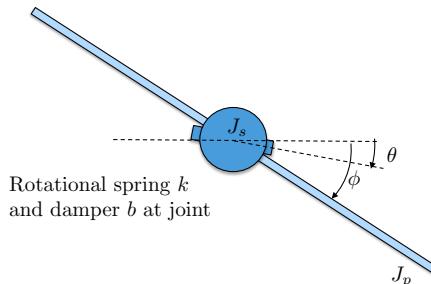


Figure 1-4: Satellite with flexible solar panels.

Figure 1-4 shows a simplified version of a satellite with flexible solar panels. We will model the flexible panels using a rigid sheet with moment of inertia  $J_p$  connected to the main satellite body by a torsional compliant element with spring constant  $k$  and damping constant  $b$ . The moment of inertia of the satellite is  $J_s$ . The angle of the satellite body from the inertial reference is  $\theta$  and the angle of the panel from the inertial reference is denoted as  $\phi$ . Thrusters are used to command an external torque of  $\tau$  about the axis of the satellite body.

The physical constants are  $J_s = 5 \text{ kg m}^2$ ,  $J_p = 1 \text{ kg m}^2$ ,  $k = 0.15 \text{ N m}$ ,  $b = 0.05 \text{ Nms}$ . The torque is limited by  $|\tau| \leq 5 \text{ Nm}$ .

### Examples (with solutions):

[C.2](#) Kinetic energy.

[C.2](#) Animate the satellite.

[C.3](#) Equations of motion.

[C.4](#) Equilibria.

[C.5](#) Transfer function model.

[C.6](#) State space model.

[C.8](#) Successive loop closure.

[C.9](#) Integrators and system type.

[C.P.6](#) Root locus.

[C.10](#) Digital PID.

[C.11](#) Full state feedback.

[C.12](#) Full state with integrator.

[C.13](#) Observer based control.

[C.14](#) Disturbance observer.

[C.15](#) Frequency Response.

[C.16](#) Loop gain.

[C.17](#) Stability margins.

[C.18](#) Loopshaping design.

# Part I

## Simulation Models

To design efficient feedback control systems, we use mathematical models to understand the system, and then we design controllers that achieve the desired specifications. Developing suitable mathematical models is problem specific and can be very complicated requiring specialized knowledge. However, there are some well known techniques for certain classes of systems. This book only covers one general technique that is useful for a large class of rigid body mechanical systems. The technique is known as the Euler-Lagrange method and requires a mathematical description of the kinetic and potential energy of the system, given its (generalized) position and velocity. The theory behind Euler-Lagrange method is deep and well beyond the scope of this book. However, we will give a brief introduction that will enable the reader to apply Euler-Lagrange methods to a surprisingly large class of problems.

As motivation for the Euler-Lagrange equations, consider a particle of mass  $m$  in a gravity field, as shown in [Fig. 1-5](#), where  $g$  is the force of gravity at sea level,  $f_a$  is an externally applied force, and the friction due to air resistance is modeled by a gain  $b$  times the velocity  $\dot{y}$ . Since all of the forces are in the vertical direction, the motion of the vehicle can be completely described by the vertical position of particle  $y(t)$ . From Newton's law, the equations of motion are

$$m\ddot{y} = f_a - mg - b\dot{y}, \quad (1.1)$$

where dots over the variable will be used throughout the book to denote differentiation with respect to time, i.e.,  $\dot{y} \triangleq \frac{dy}{dt}$  and  $\ddot{y} \triangleq \frac{d^2y}{dt^2}$ . Rearranging (1.1) gives

$$m\ddot{y} + mg = f_a - b\dot{y}. \quad (1.2)$$

The kinetic energy for the particle is given by  $K(\dot{y}) = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2$ . The potential energy, which is induced by gravitational pull proportional to the distance from the center of the earth, is given by  $P(y) = P_0 + mgy$ , where  $P_0$  is the potential energy of the particle when  $y = 0$ . Note that if the velocity  $\dot{y}$  is thought of

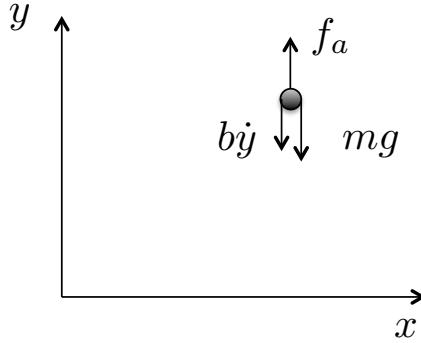


Figure 1-5: Point mass in a gravity field.

as independent of  $y$ , then

$$\frac{\partial K(\dot{y})}{\partial \dot{y}} = m\dot{y}$$

and

$$\frac{d}{dt} \left( \frac{\partial K(\dot{y})}{\partial \dot{y}} \right) = m\ddot{y}.$$

Also note that

$$\frac{\partial P(y)}{\partial y} = mg.$$

Therefore, Equation (1.2) can be written as

$$\frac{d}{dt} \left( \frac{\partial K(\dot{y})}{\partial \dot{y}} \right) + \frac{\partial P(y)}{\partial y} = f_a - b\dot{y}. \quad (1.3)$$

Equation (1.3) is a particular form of the so-called Euler-Lagrange equation. The generalization to a more complicated system takes the form

$$\frac{d}{dt} \left( \frac{\partial K(y, \dot{y})}{\partial \dot{y}} - \frac{\partial P(y)}{\partial \dot{y}} \right) - \left( \frac{\partial K(y, \dot{y})}{\partial y} - \frac{\partial P(y)}{\partial y} \right) = f_a - b\dot{y}, \quad (1.4)$$

which reduces to Equation (1.3) for our particular case since  $K$  is only a function of  $\dot{y}$ . To simplify the notation, we define the so-called *Lagrangian* as

$$L(y, \dot{y}) \triangleq K(y, \dot{y}) - P(y),$$

and write Equation (1.4) as

$$\frac{d}{dt} \left( \frac{\partial L(y, \dot{y})}{\partial \dot{y}} \right) - \left( \frac{\partial L(y, \dot{y})}{\partial y} \right) = f_a - b\dot{y}. \quad (1.5)$$

The position of the system can be generalized to both positions and angles, and the velocity can be generalized to both linear and angular velocities. Throughout

the book we will use the notation  $\mathbf{q} = (q_1, q_2, \dots, q_n)^\top$  to denote the general configuration of the mechanical system that we are modeling. The vector  $\mathbf{q}$  is called the generalized coordinates. The kinetic energy of the system is generally a function of both the generalized coordinates  $\mathbf{q}$  and the generalized velocity  $\dot{\mathbf{q}}$ . Therefore, we denote the kinetic energy as  $K(\mathbf{q}, \dot{\mathbf{q}})$ . On the other hand, in this book we will only consider potential fields that are conservative, and that only depend on the configuration variable  $\mathbf{q}$ . Under these assumptions, the general form of the Euler-Lagrange equations is given by

$$\begin{aligned}\frac{d}{dt} \left( \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_1} \right) - \frac{\partial L(\mathbf{q})}{\partial q_1} &= \tau_1 - b_{11}\dot{q}_1 - \dots - b_{1n}\dot{q}_n \\ \frac{d}{dt} \left( \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_2} \right) - \frac{\partial L(\mathbf{q})}{\partial q_2} &= \tau_2 - b_{21}\dot{q}_1 - \dots - b_{2n}\dot{q}_n \quad (1.6)\end{aligned}$$

$$\vdots \quad (1.7)$$

$$\frac{d}{dt} \left( \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_n} \right) - \frac{\partial L(\mathbf{q})}{\partial q_n} = \tau_n - b_{n1}\dot{q}_1 - \dots - b_{nn}\dot{q}_n,$$

where  $\tau_i$  are the generalized forces and torques that are applied to the system. Or in vector form we can write

$$\frac{d}{dt} \left( \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L(\mathbf{q})}{\partial \mathbf{q}} = \boldsymbol{\tau} - B\dot{\mathbf{q}}, \quad (1.8)$$

where  $B$  is a matrix with positive elements and  $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)^\top$  are the generalized forces. The Euler-Lagrange equation will be discussed in more detail in Chapter 3. However, before showing how to use Equation (1.8) to derive the equations of motion for practical systems, we need to show how to compute the kinetic and potential energy for general (holonomic) rigid body systems.

### Important Concepts:

- The equations of motion for many mechanical systems can be derived using the Euler-Lagrange method.
- The Euler-Lagrange method utilizes the scalar quantities of kinetic and potential energy.
- The Lagrangian is the difference between the system's kinetic and potential energy.
- The Euler-Lagrange method produces equations of motion that are equivalent to those derived using Newton's 2nd law of motion.

# 2

## Kinetic Energy of Mechanical Systems

### Learning Objectives:

- Compute the kinetic energy for a translating and rotating point mass.
- Calculate the inertia matrix for rigid bodies.
- Derive the kinetic energy for rotating and translating rigid bodies.

### 2.1 Theory

Consider a point mass with mass  $m$ , traveling at velocity  $\mathbf{v}$ , as shown in [Fig. 2-1](#). If the velocity is resolved with respect to a coordinate axis, then  $\mathbf{v} = (v_x, v_y, v_z)^\top$ . The kinetic energy of the point mass is given by

$$\begin{aligned} K &= \frac{1}{2}m\|\mathbf{v}\|^2 \\ &= \frac{1}{2}m\mathbf{v}^\top\mathbf{v} \\ &= \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2). \end{aligned}$$

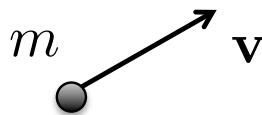


Figure 2-1: Point mass moving at velocity  $\mathbf{v}$ .

Now consider two point masses as shown in [Fig. 2-2](#). The total kinetic energy

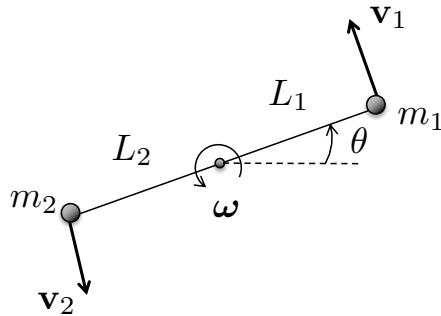


Figure 2-2: Two constrained point masses spinning about a point on an adjoining axis.

of the system is given by summing the kinetic energy of each of the point masses as

$$K = \frac{1}{2}m_1\mathbf{v}_1^\top \mathbf{v}_1 + \frac{1}{2}m_2\mathbf{v}_2^\top \mathbf{v}_2. \quad (2.1)$$

If the two masses are connected by a massless rod and the particles are spinning about a point on the rod, as shown in Fig. 2-2, where the angular velocity vector  $\omega$  points out of the page, and the magnitude is given by  $\dot{\theta}$ , i.e.,  $\omega = (0, 0, \dot{\theta})^\top$ , then from elementary physics, the velocity of point  $i$  is given by

$$\mathbf{v}_i = \omega \times \mathbf{p}_i,$$

where in our case  $\mathbf{p}_1 = (L_1 \cos \theta, L_1 \sin \theta, 0)^\top$ , and  $\mathbf{p}_2 = (-L_2 \cos \theta, -L_2 \sin \theta, 0)^\top$ . Therefore

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \times \begin{pmatrix} L_1 \cos \theta \\ L_1 \sin \theta \\ 0 \end{pmatrix} = \begin{pmatrix} -L_1 \dot{\theta} \sin \theta \\ L_1 \dot{\theta} \cos \theta \\ 0 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \times \begin{pmatrix} -L_2 \cos \theta \\ -L_2 \sin \theta \\ 0 \end{pmatrix} = \begin{pmatrix} L_2 \dot{\theta} \sin \theta \\ -L_2 \dot{\theta} \cos \theta \\ 0 \end{pmatrix}. \end{aligned}$$

An alternative method for computing the velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is to differentiate

$\mathbf{p}_1$  and  $\mathbf{p}_2$  with respect to time to obtain

$$\begin{aligned}\mathbf{v}_1 &\stackrel{\triangle}{=} \dot{\mathbf{p}}_1 = \frac{d\mathbf{p}_1}{dt} = \frac{d}{dt}(L_1 \cos \theta, L_1 \sin \theta, 0)^\top \\ &= (-L_1 \left(\frac{d\theta}{dt}\right) \sin \theta, \quad L_1 \left(\frac{d\theta}{dt}\right) \cos \theta, \quad 0)^\top \\ &= (-L_1 \dot{\theta} \sin \theta, \quad L_1 \dot{\theta} \cos \theta, \quad 0)^\top \\ \mathbf{v}_2 &\stackrel{\triangle}{=} \dot{\mathbf{p}}_2 = \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(-L_2 \cos \theta, -L_2 \sin \theta, 0)^\top \\ &= (L_2 \left(\frac{d\theta}{dt}\right) \sin \theta, \quad -L_2 \left(\frac{d\theta}{dt}\right) \cos \theta, \quad 0)^\top \\ &= (L_2 \dot{\theta} \sin \theta, \quad -L_2 \dot{\theta} \cos \theta, \quad 0)^\top.\end{aligned}$$

Therefore, using Equation (2.1) the kinetic energy is

$$\begin{aligned}K &= \frac{1}{2}m_1 \mathbf{v}_1^\top \mathbf{v}_1 + \frac{1}{2}m_2 \mathbf{v}_2^\top \mathbf{v}_2 \\ &= \frac{1}{2}m_1 \left[ (-L_1 \dot{\theta} \sin \theta)^2 + (L_1 \dot{\theta} \cos \theta)^2 \right] \\ &\quad + \frac{1}{2}m_2 \left[ (L_2 \dot{\theta} \sin \theta)^2 + (-L_2 \dot{\theta} \cos \theta)^2 \right] \\ &= \frac{1}{2}m_1 \left[ L_1^2 \dot{\theta}^2 \sin^2 \theta + L_1^2 \dot{\theta}^2 \cos^2 \theta \right] + \frac{1}{2}m_2 \left[ L_2^2 \dot{\theta}^2 \sin^2 \theta + L_2^2 \dot{\theta}^2 \cos^2 \theta \right] \\ &= \frac{1}{2}(m_1 L_1^2 + m_2 L_2^2) \dot{\theta}^2.\end{aligned}$$

Now consider the case where there are  $N$  point masses spinning about a general angular velocity vector  $\boldsymbol{\omega}$ , where the position of the  $i^{th}$  point mass,  $i = 1, \dots, N$  in a specified coordinate system is given by  $\mathbf{p}_i$ , as shown in **Fig. 2-3**. We will see in the discussion below that it is most convenient to pick the coordi-

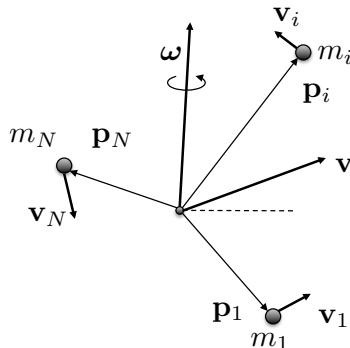


Figure 2-3: Many masses in a rigid body spinning about a common axis and moving at velocity  $\mathbf{v}$ .

nate system to be at the center of mass. The kinetic energy of the  $i^{th}$  particle is

$K_i = \frac{1}{2}m_i\mathbf{v}_i^\top \mathbf{v}_i$ . Therefore, the kinetic energy of the total system is given by

$$K = \sum_{i=1}^N \frac{1}{2}m_i\mathbf{v}_i^\top \mathbf{v}_i.$$

Using the fact that  $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{p}_i$  gives

$$K = \sum_{i=1}^N \frac{1}{2}m_i (\boldsymbol{\omega} \times \mathbf{p}_i)^\top (\boldsymbol{\omega} \times \mathbf{p}_i).$$

Recalling the cross product rule

$$(\mathbf{a} \times \mathbf{b})^\top (\mathbf{c} \times \mathbf{d}) = (\mathbf{a}^\top \mathbf{c})(\mathbf{b}^\top \mathbf{d}) - (\mathbf{a}^\top \mathbf{d})(\mathbf{b}^\top \mathbf{c})$$

gives

$$\begin{aligned} K &= \frac{1}{2} \sum_{i=1}^N m_i [(\boldsymbol{\omega}^\top \boldsymbol{\omega})(\mathbf{p}_i^\top \mathbf{p}_i) - (\boldsymbol{\omega}^\top \mathbf{p}_i)(\mathbf{p}_i^\top \boldsymbol{\omega})] \\ &= \frac{1}{2} \sum_{i=1}^N m_i \boldsymbol{\omega}^\top [(\mathbf{p}_i^\top \mathbf{p}_i)I_3 - \mathbf{p}_i \mathbf{p}_i^\top] \boldsymbol{\omega} \\ &= \frac{1}{2} \boldsymbol{\omega}^\top \left[ \sum_{i=1}^N m_i ((\mathbf{p}_i^\top \mathbf{p}_i)I_3 - \mathbf{p}_i \mathbf{p}_i^\top) \right] \boldsymbol{\omega} \\ &\triangleq \frac{1}{2} \boldsymbol{\omega}^\top J \boldsymbol{\omega}. \end{aligned}$$

where  $I_3$  is the  $3 \times 3$  identity matrix. The matrix

$$J = \sum_{i=1}^N m_i ((\mathbf{p}_i^\top \mathbf{p}_i)I_3 - \mathbf{p}_i \mathbf{p}_i^\top)$$

is called the inertia matrix of the mass system. If the mass system is solid then there is an infinite number of particles, and the sum becomes an integral resulting in

$$J = \int_{\mathbf{p} \in \text{body}} ((\mathbf{p}^\top \mathbf{p})I_3 - \mathbf{p}\mathbf{p}^\top) dm(\mathbf{p}),$$

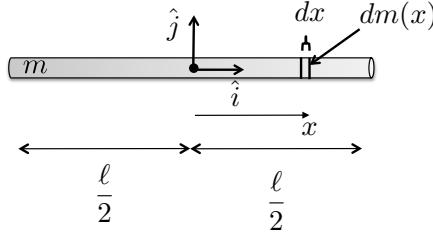
where  $dm(\mathbf{p})$  is the elemental mass located at position  $\mathbf{p}$  and the integral is over all elemental masses in the system.

Suppose that  $\mathbf{p} = (x, y, z)^\top$  is expressed with respect to a coordinate system

fixed in a rigid body. Then

$$\begin{aligned}
 J &= \int_{(x,y,z) \in \text{body}} \left[ (x \ y \ z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} (x \ y \ z) \right] dm(x, y, z) \\
 &= \int_{(x,y,z) \in \text{body}} \left[ (x^2 + y^2 + z^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix} \right] dm(x, y, z) \\
 &= \int_{(x,y,z) \in \text{body}} \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{pmatrix} dm(x, y, z).
 \end{aligned}$$

As an example, to compute the inertia matrix of the thin rod of mass  $m$  about its center of mass, as shown in [Fig. 2-4](#), the first step is to define a coordinate system at the center of mass, as shown in [Fig. 2-4](#), where  $\hat{i}$  and  $\hat{j}$  are orthogonal unit vectors pointing parallel and perpendicular to the rod, respectively. The unit vector  $\hat{k}$  forms a right-handed coordinate system and points out of the page. Assuming that the rod is infinitely thin and that the mass is evenly distributed



[Figure 2-4](#): Computing the inertia of a thin rod.

throughout the rod, then the elemental mass is given by

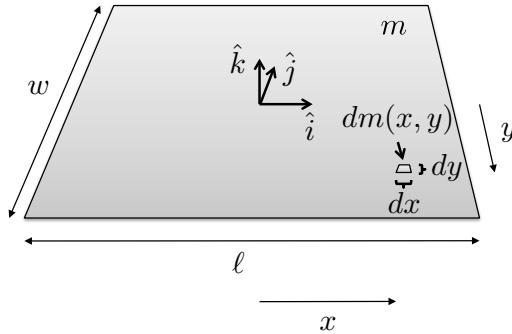
$$dm(x) = \frac{m}{\ell} dx.$$

Since the rod is infinitely thin in the  $\hat{j}$  and  $\hat{k}$  directions, the inertia matrix is given by

$$\begin{aligned}
 J &= \int_{x=-\ell/2}^{\ell/2} \int_{y=0}^0 \int_{z=0}^0 \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{pmatrix} dz dy \frac{m}{\ell} dx \\
 &= \frac{m}{\ell} \int_{x=-\ell/2}^{\ell/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & x^2 & 0 \\ 0 & 0 & x^2 \end{pmatrix} dx \\
 &= \frac{m}{\ell} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{x^3}{3} \Big|_{x=-\frac{\ell}{2}}^{\frac{\ell}{2}} & 0 \\ 0 & 0 & \frac{x^3}{3} \Big|_{x=-\frac{\ell}{2}}^{\frac{\ell}{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m\ell^2}{12} \end{pmatrix}. \tag{2.2}
 \end{aligned}$$

Note that the infinitely thin rod assumption implies that there is only inertia, or rotational mass, about the  $\hat{i}$  and  $\hat{k}$  axes.

As another example, to compute the inertia matrix of the thin plate of mass  $m$  about its center of mass, as shown in [Fig. 2-5](#), the first step is to define a coordinate system at the center of mass, as shown in [Fig. 2-5](#), where  $\hat{i}$  and  $\hat{j}$  are orthogonal unit vectors pointing along the length and width of the plate, respectively, and  $\hat{k}$  forms a right-handed coordinate system and points perpendicular to the plate. Assuming that the plate is infinitely thin, and that the mass is evenly distributed



[Figure 2-5](#): Computing the inertia of a thin plate.

throughout the plate, then the elemental mass is given by

$$dm(x, y) = \frac{m}{\ell w} dx dy.$$

Since the plate is infinitely thin in the  $\hat{k}$  direction, the inertia matrix is given by

$$\begin{aligned} J &= \int_{x=-\ell/2}^{\ell/2} \int_{y=-w/2}^{w/2} \int_{z=0}^0 \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{pmatrix} dz dy \frac{m}{\ell w} dx \\ &= \frac{m}{\ell w} \int_{x=-\ell/2}^{\ell/2} \int_{y=-w/2}^{w/2} \begin{pmatrix} y^2 & -xy & 0 \\ -yx & x^2 & 0 \\ 0 & 0 & x^2 + y^2 \end{pmatrix} dy dx \\ &= \frac{m}{\ell w} \int_{x=-\ell/2}^{\ell/2} \begin{pmatrix} \frac{w^3}{12} & 0 & 0 \\ 0 & x^2 w & 0 \\ 0 & 0 & x^2 w + \frac{w^3}{12} \end{pmatrix} dx \\ &= \frac{m}{\ell w} \begin{pmatrix} \frac{w^3}{12} \ell & 0 & 0 \\ 0 & \frac{\ell^3}{12} w & 0 \\ 0 & 0 & \frac{\ell^3}{12} w + \frac{w^3}{12} \ell \end{pmatrix} = \begin{pmatrix} \frac{mw^2}{12} & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m\ell^2}{12} + \frac{mw^2}{12} \end{pmatrix}. \end{aligned}$$

The diagonal elements of  $J$  are called the moments of inertia and represent the rotational mass about the  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  axes respectively.

We have shown that if the motion of the system of  $N$  masses is caused purely by spinning about the angular velocity vector  $\omega$ , then the kinetic energy is given by  $K = \frac{1}{2}\omega^\top J\omega$ . If the system of masses is also translating with velocity  $\mathbf{v}$ , then the velocity of the  $i^{th}$  particle is given by  $\mathbf{v}_i = \mathbf{v} + \omega \times \mathbf{p}_i$ . In that case, the kinetic energy is given by

$$\begin{aligned} K &= \frac{1}{2} \sum_{i=1}^N m_i \mathbf{v}_i^\top \mathbf{v}_i = \frac{1}{2} \sum_{i=1}^N m_i (\mathbf{v} + \omega \times \mathbf{p}_i)^\top (\mathbf{v} + \omega \times \mathbf{p}_i) \\ &= \frac{1}{2} \sum_{i=1}^N m_i \left( \mathbf{v}^\top \mathbf{v} + \mathbf{v}^\top (\omega \times \mathbf{p}_i) + (\omega \times \mathbf{p}_i)^\top \mathbf{v} + (\omega \times \mathbf{p}_i)^\top (\omega \times \mathbf{p}_i) \right) \\ &= \frac{1}{2} \left( \sum_{i=1}^N m_i \right) \mathbf{v}^\top \mathbf{v} + \frac{1}{2} \sum_{i=1}^N m_i (\omega \times \mathbf{p}_i)^\top (\omega \times \mathbf{p}_i) \\ &\quad + \frac{1}{2} \sum_{i=1}^N m_i [\mathbf{v}^\top (\omega \times \mathbf{p}_i) + (\omega \times \mathbf{p}_i)^\top \mathbf{v}]. \end{aligned}$$

Using the cross product property

$$\mathbf{a}^\top (\mathbf{b} \times \mathbf{c}) = \mathbf{b}^\top (\mathbf{c} \times \mathbf{a}) = \mathbf{c}^\top (\mathbf{a} \times \mathbf{b})$$

we get that

$$\begin{aligned} \sum_{i=1}^N m_i [\mathbf{v}^\top (\omega \times \mathbf{p}_i) + (\omega \times \mathbf{p}_i)^\top \mathbf{v}] &= \\ \left( \sum_{i=1}^N m_i \mathbf{p}_i \right)^\top (\mathbf{v} \times \omega) + \left( \sum_{i=1}^N m_i \mathbf{p}_i \right)^\top (\mathbf{v} \times \omega). \end{aligned}$$

If we choose the center of the coordinate system to be the center of mass, then  $\sum_{i=1}^N m_i \mathbf{p}_i = 0$ . Defining the total mass as  $m = \sum_{i=1}^N m_i$ , the kinetic energy is given by

$$\begin{aligned} K &= \frac{1}{2} \left( \sum_{i=1}^N m_i \right) \mathbf{v}_{cm}^\top \mathbf{v}_{cm} + \frac{1}{2} \sum_{i=1}^N m_i (\omega \times \mathbf{p}_i)^\top (\omega \times \mathbf{p}_i) \\ &= \frac{1}{2} m \mathbf{v}_{cm}^\top \mathbf{v}_{cm} + \frac{1}{2} \omega^\top J_{cm} \omega. \end{aligned}$$

Therefore, the kinetic energy of a rigid body moving with velocity  $\mathbf{v}_{cm}$  and spinning at angular velocity  $\omega$  is therefore

$$K = \frac{1}{2} m \mathbf{v}_{cm}^\top \mathbf{v}_{cm} + \frac{1}{2} \omega^\top J_{cm} \omega. \quad (2.3)$$

If the system consists of  $n$  rigid bodies, then the total kinetic energy is computed by summing the kinetic energy for each body as

$$K = \sum_{j=1}^n \left[ \frac{1}{2} m_j \mathbf{v}_{cm,j}^\top \mathbf{v}_{cm,j} + \frac{1}{2} \omega_j^\top J_{cm,j} \omega_j \right].$$

### 2.1.1 Example: Mass Spring System

As an example, consider the mass spring system shown in [Fig. 2-6](#) where two masses are moving along a surface and are connected by springs with constants  $k_1$  and  $k_2$  and dampers with constants  $b_1$  and  $b_2$ . The kinetic energy of the system is determined by the velocity of the two masses and is therefore given by

$$K = \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2.$$

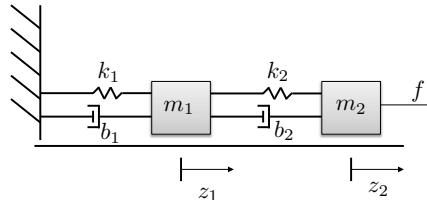


Figure 2-6: Mass spring system with two masses connected by springs and dampers.

### 2.1.2 Example: Spinning Dumbbell

As another simple example, consider the spinning dumbbell system shown in [Fig. 2-7](#), where two masses are spinning about a point on the adjoining line between the masses. The angular velocity vector is pointing out of the page with magnitude  $\dot{\theta}$  and the dumbbell system is moving to the right at velocity  $\dot{z}$ .

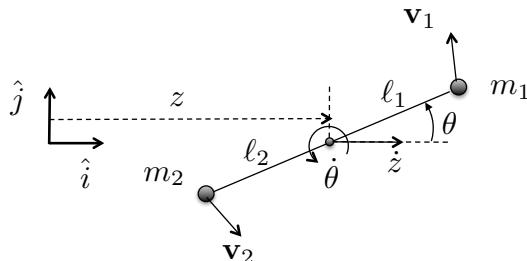


Figure 2-7: Spinning dumbbell. The system is spinning about the axis pointing out of the page and moving to the right at velocity  $\dot{z}$ .

The first step in computing the kinetic energy is to define an inertial coordinate system as shown in [Fig. 2-7](#) where  $\hat{i}$  is a unit vector along the direction of travel,  $\hat{j}$  is perpendicular to the direction of travel, and where  $\hat{k}$  forms a right-handed coordinate system and points out of the page. In this coordinate system, the position of the point of rotation is given by  $\mathbf{p}(t) = (z(t), 0, 0)^\top$ . Accordingly, the positions

of  $m_1$  and  $m_2$  are given by

$$\begin{aligned}\mathbf{p}_1(t) &= (z(t) + \ell_1 \cos \theta(t), \ell_1 \sin \theta(t), 0)^\top, \\ \mathbf{p}_2(t) &= (z(t) - \ell_2 \cos \theta(t), -\ell_2 \sin \theta(t), 0)^\top.\end{aligned}$$

Since  $z$  and  $\theta$  are functions of time, using the chain rule

$$\frac{d}{dt} f(g(t)) = \frac{\partial f}{\partial x} \frac{dg}{dt}$$

and differentiating with respect to time, we get the velocities of the masses as

$$\begin{aligned}\mathbf{v}_1 &= (\dot{z} - \ell_1 \dot{\theta} \sin \theta, \ell_1 \dot{\theta} \cos \theta, 0)^\top, \\ \mathbf{v}_2 &= (\dot{z} + \ell_2 \dot{\theta} \sin \theta, -\ell_2 \dot{\theta} \cos \theta, 0)^\top.\end{aligned}$$

The kinetic energy is therefore given by

$$\begin{aligned}K &= \frac{1}{2} m_1 \mathbf{v}_1^\top \mathbf{v}_1 + \frac{1}{2} m_2 \mathbf{v}_2^\top \mathbf{v}_2 \\ &= \frac{1}{2} m_1 \left[ (\dot{z} - \ell_1 \dot{\theta} \sin \theta)^2 + (\ell_1 \dot{\theta} \cos \theta)^2 \right] \\ &\quad + \frac{1}{2} m_2 \left[ (\dot{z} + \ell_2 \dot{\theta} \sin \theta)^2 + (-\ell_2 \dot{\theta} \cos \theta)^2 \right] \\ &= \frac{1}{2} m_1 \left[ \dot{z}^2 - 2\ell_1 \dot{z} \dot{\theta} \sin \theta + \ell_1^2 \dot{\theta}^2 \sin^2 \theta + \ell_1^2 \dot{\theta}^2 \cos^2 \theta \right] \\ &\quad + \frac{1}{2} m_2 \left[ \dot{z}^2 + 2\ell_2 \dot{z} \dot{\theta} \sin \theta + \ell_2^2 \dot{\theta}^2 \sin^2 \theta + \ell_2^2 \dot{\theta}^2 \cos^2 \theta \right] \\ &= \frac{1}{2} (m_1 + m_2) \dot{z}^2 + \frac{1}{2} (m_1 \ell_1^2 + m_2 \ell_2^2) \dot{\theta}^2 + (m_2 \ell_2 - m_1 \ell_1) \dot{z} \dot{\theta} \sin \theta.\end{aligned}$$

Note that if the pivot point is located at the center of mass, then  $m_1 \ell_1 = m_2 \ell_2$ , and the kinetic energy is simply given by

$$K = \frac{1}{2} (m_1 + m_2) \dot{z}^2 + \frac{1}{2} (m_1 \ell_1^2 + m_2 \ell_2^2) \dot{\theta}^2,$$

which is in the form

$$K = \frac{1}{2} m \mathbf{v}_{cm}^\top \mathbf{v}_{cm} + \frac{1}{2} \boldsymbol{\omega}^\top J \boldsymbol{\omega},$$

where the total mass is  $m = m_1 + m_2$ , the velocity of the center of mass is  $\mathbf{v}_{cm} = (\dot{z}, 0, 0)^\top$ , the angular velocity vector is  $\boldsymbol{\omega} = (0, 0, \dot{\theta})^\top$  and the (3, 3) element of  $J$  is given by  $J_z = m_1 \ell_1^2 + m_2 \ell_2^2$ .

## 2.2 Design Study A: Single Link Robot Arm

A definition of the single link robot arm is given in Design Study A.



### Example Problem A.2

- (a) Using the configuration variable  $\theta$ , write an expression for the kinetic energy of the system.
- (b) Referring to Appendices P.1, P.2, and P.3, write a Python or Matlab class, or a Matlab function that creates an animation of the single link robot arm. Simulate the animation to display a sinusoidal variation on the configuration variable  $q = \theta$ .

### Solution

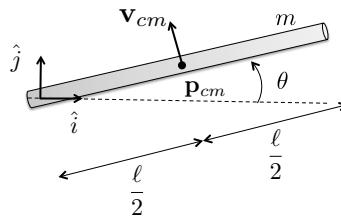


Figure 2-8: Single link robot arm. Compute the velocity of the center of mass and the angular velocity about the center of mass.

The first step is to define an inertial coordinate system as shown in Figure 2-8. If the pivot point is the origin of the coordinate system as shown in Figure 2-8, then the position of the center of mass is

$$\mathbf{p}_{cm}(t) = \begin{pmatrix} \frac{\ell}{2} \cos \theta(t) \\ \frac{\ell}{2} \sin \theta(t) \\ 0 \end{pmatrix}.$$

Differentiating to obtain the velocity we get

$$\mathbf{v}_{cm} = \frac{\ell}{2} \dot{\theta} \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}.$$

The angular velocity about the center of mass is given by

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}.$$

Observe that

$$\boldsymbol{\omega}^\top J \boldsymbol{\omega} = (0 \ 0 \ \dot{\theta}) \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} = J_z \dot{\theta}^2,$$

therefore we only need the moment of inertia about the  $\hat{k}$  axis  $J_z$ . From the inertia matrix for a thin rod given in Equation (2.2), we have that  $J_z = \frac{m\ell^2}{12}$ .

Therefore, the kinetic energy of the single link robot arm is

$$\begin{aligned} K &= \frac{1}{2}mv_{cm}^\top v_{cm} + \frac{1}{2}\omega^\top J_{cm}\omega \\ &= \frac{1}{2}m\frac{\ell^2}{4}\dot{\theta}^2(\sin^2\theta + \cos^2\theta) + \frac{1}{2}\dot{\theta}^2\frac{m\ell^2}{12} \\ &= \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2. \end{aligned}$$

Throughout the book, we will demonstrate how to simulate the single link robot arm using Python object-oriented code. The Python class that animates the single link robot arm is shown below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import matplotlib.patches as mpatches
5 import numpy as np
6 import armParam as P
7
8
9 class armAnimation:
10     """
11         Create arm animation
12     """
13     def __init__(self):
14         # Used to indicate initialization
15         self.flagInit = True
16
17         # Initializes a figure and axes object
18         self.fig, self.ax = plt.subplots()
19
20         # Initializes a list object that will be used to
21         # contain handles to the patches and line objects.
22         self.handle = []
23
24         self.length=P.length
25         self.width=P.width
26
27         # Change the x,y axis limits
28         plt.axis([-2.0*P.length, 2.0*P.length, -2.0*P.length,
29                   2.0*P.length])
30
31         # Draw a base line
32         plt.plot([0, P.length], [0, 0],'k--')
33
34
35         # Draw pendulum is the main function that will call the
36         # functions:
37         # drawCart, drawCircle, and drawRod to create the animation.
38
39     def update(self, u):
```

```

40     # Process inputs to function
41     theta = u[0]    # angle of arm, rads
42
43     X = [0, self.length*np.cos(theta)]  # X data points
44     Y = [0, self.length*np.sin(theta)]  # Y data points
45
46     # When the class is initialized, a line object will be
47     # created and added to the axes. After initialization, the
48     # line object will only be updated.
49     if self.flagInit == True:
50         # Create the line object and append its handle
51         # to the handle list.
52         line, =self.ax.plot(X, Y, lw=5, c='blue')
53         self.handle.append(line)
54         self.flagInit=False
55     else:
56         self.handle[0].set_xdata(X)      # Update the line
57         self.handle[0].set_ydata(Y)
58
59
60     plt.draw()

```

Listing 2.1: armAnimation.py

The Python code that implements the animation is given below.

```

1 import matplotlib.pyplot as plt
2 import numpy as np
3 import sys
4 sys.path.append('..')  # add parent directory
5 import armParam as P
6 from signalGenerator import signalGenerator
7 from armAnimation import armAnimation
8 from dataPlotter import dataPlotter
9
10 # instantiate reference input classes
11 reference = signalGenerator(amplitude=0.5, frequency=0.1)
12 thetaRef = signalGenerator(amplitude=2.0*np.pi, frequency=0.1)
13 tauRef = signalGenerator(amplitude=5, frequency=.5)
14
15 # instantiate the simulation plots and animation
16 dataPlot = dataPlotter()
17 animation = armAnimation()
18
19 t = P.t_start  # time starts at t_start
20 while t < P.t_end:  # main simulation loop
21     # set variables
22     r = reference.square(t)
23     theta = thetaRef.sin(t)
24     tau = tauRef.sawtooth(t)
25     # update animation
26
27     state = np.array([[theta], [0.0]])
28     animation.update(state)
29     dataPlot.update(t, r, state, tau)
30
31     #plt.show()

```

```

32 t = t + P.t_plot # advance time by t_plot
33 plt.pause(0.1)
34
35 # Keeps the program from closing until the user presses a button.
36 print('Press key to close')
37 plt.waitforbuttonpress()
38 plt.close()

```

Listing 2.2: armSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 2.3 Design Study B. Pendulum on Cart

A definition of the inverted pendulum is given in Design Study B.



### Example Problem B.2

- (a) Using the configuration variables  $z$  and  $\theta$ , write an expression for the kinetic energy of the system.
- (b) Referring to Appendices P.1, P.2, and P.3, write a Python or Matlab class, or a Matlab function that creates an animation of the inverted pendulum. Simulate the animation to display sinusoidal variations on the configuration variables  $q = (z, \theta)^\top$ .

### Solution

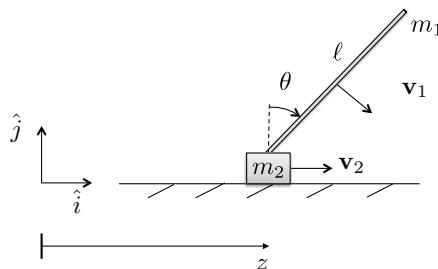


Figure 2-9: Pendulum on a cart. Compute the velocity of the pendulum and of the cart.

Define the inertial coordinate frame as in Figure 2-9, with  $\hat{k}$  out of the page.

The position of the center of mass of the rod with mass  $m_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} z(t) + \frac{\ell}{2} \sin \theta(t) \\ \frac{\ell}{2} \cos \theta(t) \\ 0 \end{pmatrix},$$

and the horizontal position of the cart with mass  $m_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} z(t) \\ 0 \\ 0 \end{pmatrix}.$$

Differentiating to obtain the velocities of  $m_1$  and  $m_2$  we obtain

$$\mathbf{v}_1 = \begin{pmatrix} \dot{z} + \frac{\ell}{2} \dot{\theta} \cos \theta \\ -\frac{\ell}{2} \dot{\theta} \sin \theta \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \dot{z} \\ 0 \\ 0 \end{pmatrix}.$$

We are modeling the cart as a point mass and the rod as being infinitely thin. Therefore the kinetic energy of the rod is the summation of its translational and rotational kinetic energy about the center of mass. The total kinetic energy of the system is given by

$$\begin{aligned} K &= \frac{1}{2} m_1 \mathbf{v}_1^\top \mathbf{v}_1 + \frac{1}{2} \boldsymbol{\omega}^\top J_{rod,cm} \boldsymbol{\omega} + \frac{1}{2} m_2 \mathbf{v}_2^\top \mathbf{v}_2 \\ &= \frac{1}{2} m_1 \left[ (\dot{z} + \frac{\ell}{2} \dot{\theta} \cos \theta)^2 + (-\frac{\ell}{2} \dot{\theta} \sin \theta)^2 \right] + \frac{1}{2} \dot{\theta}^2 \frac{m_1 \ell^2}{12} + \frac{1}{2} m_2 \dot{z}^2 \\ &= \frac{1}{2} m_1 \left[ \dot{z}^2 + 2 \frac{\ell}{2} \dot{z} \dot{\theta} \cos \theta + \frac{\ell^2}{4} \dot{\theta}^2 \cos^2 \theta + \frac{\ell^2}{4} \dot{\theta}^2 \sin^2 \theta \right] \\ &\quad + \frac{1}{2} \dot{\theta}^2 \frac{m_1 \ell^2}{12} + \frac{1}{2} m_2 \dot{z}^2 \\ &= \frac{1}{2} (m_1 + m_2) \dot{z}^2 + \frac{1}{2} m_1 \frac{\ell^2}{3} \dot{\theta}^2 + m_1 \frac{\ell}{2} \dot{z} \dot{\theta} \cos \theta. \end{aligned} \tag{2.4}$$

Throughout the book, we will demonstrate how to simulate the pendulum on a cart using Python code. The Python class that animates the system is shown below.

```

1 import matplotlib.pyplot as plt
2 import matplotlib.patches as mpatches
3 import numpy as np
4 import pendulumParam as P
5
6 class pendulumAnimation:
7     def __init__(self):
8         self.flag_init = True # Used to indicate initialization
9         # Initialize a figure and axes object
10        self.fig, self.ax = plt.subplots()
11        # Initializes a list of objects (patches and lines)

```

```

12     self.handle = []
13     # Specify the x,y axis limits
14     plt.axis([-3*P.ell, 3*P.ell, -0.1, 3*P.ell])
15     # Draw line for the ground
16     plt.plot([-2*P.ell, 2*P.ell], [0, 0], 'b--')
17     # label axes
18     plt.xlabel('z')
19
20 def update(self, state):
21     z = state.item(0) # Horizontal position of cart, m
22     theta = state.item(1) # Angle of pendulum, rads
23     # draw plot elements: cart, bob, rod
24     self.draw_cart(z)
25     self.draw_bob(z, theta)
26     self.draw_rod(z, theta)
27     self.ax.axis('equal')
28     # Set initialization flag to False after first call
29     if self.flag_init == True:
30         self.flag_init = False
31
32 def draw_cart(self, z):
33     # specify bottom left corner of rectangle
34     x = z-P.w/2.0
35     y = P.gap
36     corner = (x, y)
37     # create rectangle on first call, update on subsequent calls
38     if self.flag_init is True:
39         # Create the Rectangle patch and append its handle
40         # to the handle list
41         self.handle.append(
42             mpatches.Rectangle(corner, P.w, P.h,
43                                 fc='blue', ec='black'))
44         # Add the patch to the axes
45         self.ax.add_patch(self.handle[0])
46     else:
47         self.handle[0].set_xy(corner) # Update patch
48
49 def draw_bob(self, z, theta):
50     # specify center of circle
51     x = z+(P.ell+P.radius)*np.sin(theta)
52     y = P.gap+P.h+(P.ell+P.radius)*np.cos(theta)
53     center = (x, y)
54     # create circle on first call, update on subsequent calls
55     if self.flag_init is True:
56         # Create the CirclePolygon patch and append its handle
57         # to the handle list
58         self.handle.append(
59             mpatches.CirclePolygon(center, radius=P.radius,
60                                   resolution=15, fc='limegreen', ec='black'))
61         # Add the patch to the axes
62         self.ax.add_patch(self.handle[1])
63     else:
64         self.handle[1].xy = center
65
66 def draw_rod(self, z, theta):
67     # specify x-y points of the rod
68     X = [z, z+P.ell*np.sin(theta)]

```

```

69     Y = [P.gap+P.h, P.gap+P.h+P.ell*np.cos(theta)]
70     # create rod on first call, update on subsequent calls
71     if self.flag_init is True:
72         # Create the line object and append its handle
73         # to the handle list.
74         line, = self.ax.plot(X, Y, lw=1, c='black')
75         self.handle.append(line)
76     else:
77         self.handle[2].set_xdata(X)
78         self.handle[2].set_ydata(Y)

```

Listing 2.3: pendulumAnimation.py

The Python code that implements the animation functionality is given below.

```

1 import matplotlib.pyplot as plt
2 import numpy as np
3 import sys
4 sys.path.append('..') # add parent directory
5 import pendulumParam as P
6 from signalGenerator import signalGenerator
7 from pendulumAnimation import pendulumAnimation
8 from dataPlotter import dataPlotter
9
10
11 # instantiate reference input classes
12 reference = signalGenerator(amplitude=0.5, frequency=0.1)
13 zRef = signalGenerator(amplitude=0.5, frequency=0.1)
14 thetaRef = signalGenerator(amplitude=.25*np.pi, frequency=.5)
15 fRef = signalGenerator(amplitude=5, frequency=.5)
16
17 # instantiate the simulation plots and animation
18 dataPlot = dataPlotter()
19 animation = pendulumAnimation()
20
21 t = P.t_start # time starts at t_start
22 while t < P.t_end: # main simulation loop
23     # set variables
24     r = reference.square(t)
25     z = zRef.sin(t)
26     theta = thetaRef.square(t)
27     f = fRef.sawtooth(t)
28     # update animation
29     state = np.array([[z], [theta], [0.0], [0.0]])
30     animation.update(state)
31     dataPlot.update(t, r, state, f)
32
33     t = t + P.t_plot # advance time by t_plot
34     plt.pause(0.05)
35
36 # Keeps the program from closing until the user presses a button.
37 print('Press key to close')
38 plt.waitforbuttonpress()
39 plt.close()

```

Listing 2.4: pendulumSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 2.4 Design Study C. Satellite Attitude Control

A definition of the satellite attitude control problem is given in Design Study C.



### Example Problem C.2

- (a) Using the configuration variables  $\theta$  and  $\phi$ , write an expression for the kinetic energy of the system.
- (b) Referring to Appendices P.1, P.2, and P.3, write a Python or Matlab class, or a Matlab function that creates an animation of the satellite. Simulate the animation to display sinusoidal variations on the configuration variables  $q = (\theta, \phi)^\top$ .

### Solution

Since the satellite consists of two rotational masses, the kinetic energy is the sum of the rotational kinetic energy of each mass. Therefore the kinetic energy is given by

$$K = \frac{1}{2} J_s \dot{\theta}^2 + \frac{1}{2} J_p \dot{\phi}^2. \quad (2.5)$$

Throughout the book, we will demonstrate how to simulate the satellite using Python code. The code that animates the satellite and implements the animation is given below.

```

1 import matplotlib.pyplot as plt
2 import matplotlib.patches as mpatches
3 import numpy as np
4 import satelliteParam as P
5
6
7 class satelliteAnimation:
8     """
9         Create satellite animation
10    """
11    def __init__(self):
12        # Used to indicate initialization
13        self.flagInit = True
14
15        # Initializes a figure and axes object
16        self.fig, self.ax = plt.subplots()
17
18        # Initializes a list object that will be used to contain
19        # handles to the patches and line objects.
20        self.handle = []

```

```

21     plt.axis([-2.0*P.length, 2.0*P.length, -2.0*P.length,
22                 2.0*P.length])
23     plt.plot([-2.0*P.length, 2.0*P.length], [0, 0], 'b--')
24     self.length = P.length
25     self.width = P.width
26
27
28 def update(self, u):
29     # Process inputs to function
30     theta = u.item(0)      # Angle of base, rad
31     phi = u.item(1)        # angle of panel, rad
32
33     self.drawBase(theta)
34     self.drawPanel(phi)
35
36     # This will cause the image to not distort
37     # self.ax.axis('equal')
38
39     # After each function has been called, initialization is
40     # over.
41     if self.flagInit == True:
42         self.flagInit = False
43
44 def drawBase(self, theta):
45     # points that define the base
46     pts = np.matrix([
47         [self.width/2.0, -self.width/2.0],
48         [self.width/2.0, -self.width/6.0],
49         [self.width/2.0 + self.width/6.0, -self.width/6.0],
50         [self.width/2.0 + self.width/6.0, self.width/6.0],
51         [self.width/2.0, self.width/6.0],
52         [self.width/2.0, self.width/2.0],
53         [-self.width/2.0, self.width/2.0],
54         [-self.width/2.0, self.width/6.0],
55         [-self.width/2.0, -self.width/6.0],
56         [-self.width/2.0 - self.width/6.0, self.width/6.0],
57         [-self.width/2.0 - self.width/6.0, -self.width/6.0],
58         [-self.width/2.0, -self.width/6.0],
59         [-self.width/2.0, -self.width/2.0]]).T
60     R = np.matrix([[np.cos(theta), np.sin(theta)],
61                   [-np.sin(theta), np.cos(theta)]])
62     pts = R*pts
63     xy = np.array(pts.T)
64
65     # When the class is initialized, a polygon patch object
66     # will be created and added to the axes. After
67     # initialization, the polygon patch object will only be
68     # updated.
69     if self.flagInit == True:
70
71         # Create the Rectangle patch and append its handle
72         # to the handle list
73         self.handle.append(mpatches.Polygon(xy,
74                                         facecolor='blue',
75                                         edgecolor='black'))
76
77         # Add the patch to the axes
78         self.ax.add_patch(self.handle[0])

```

```

78     else:
79
80         # Update polygon
81         self.handle[0].set_xy(xy)
82
83     def drawPanel(self, phi):
84         # points that define the base
85         pts = np.matrix([
86             [-self.length, -self.width/6.0],
87             [self.length, -self.width/6.0],
88             [self.length, self.width/6.0],
89             [-self.length, self.width/6.0]]).T
90         R = np.matrix([[np.cos(phi), np.sin(phi)],
91                       [-np.sin(phi), np.cos(phi)]])
92         pts = R * pts
93         xy = np.array(pts.T)
94
95         # When the class is initialized, a polygon patch
96         # object will be created and added to the
97         # axes. After initialization, the polygon patch
98         # object will only be updated.
99         if self.flagInit == True:
100
101             # Create the Rectangle patch and append its
102             # handle to the handle list
103             self.handle.append(mpatches.Polygon(xy,
104                                              facecolor='green',
105                                              edgecolor='black'))
106
107             # Add the patch to the axes
108             self.ax.add_patch(self.handle[1])
109
110     else:
111
112         # Update polygon
113         self.handle[1].set_xy(xy)

```

Listing 2.5: satelliteAnimation.py

The Python code that implements the animation is given below.

```

1 import matplotlib.pyplot as plt
2 import numpy as np
3 import sys
4 sys.path.append('..') # add parent directory
5 import satelliteParam as P
6 from signalGenerator import signalGenerator
7 from satelliteAnimation import satelliteAnimation
8 from dataPlotter import dataPlotter
9
10
11 # instantiate reference input classes
12 reference = signalGenerator(amplitude=0.5, frequency=0.1)
13 thetaRef = signalGenerator(amplitude=2.0*np.pi, frequency=0.1)
14 phiRef = signalGenerator(amplitude=0.5, frequency=0.1)
15 tauRef = signalGenerator(amplitude=5, frequency=.5)
16

```

```

17 # instantiate the simulation plots and animation
18 dataPlot = dataPlotter()
19 animation = satelliteAnimation()
20
21 t = P.t_start # time starts at t_start
22 while t < P.t_end: # main simulation loop
23
24     # set variables
25     r = reference.square(t)
26     theta = thetaRef.sin(t)
27     phi = phiRef.sin(t)
28     tau = tauRef.sawtooth(t)
29
30     # update animation
31     state = np.array([[theta], [phi], [0.0], [0.0]])
32     animation.update(state)
33     dataPlot.update(t, r, state, tau)
34
35     # advance time by t_plot
36     t = t + P.t_plot
37     plt.pause(0.1)
38
39 # Keeps the program from closing until the user presses a button.
40 print('Press key to close')
41 plt.waitforbuttonpress()
42 plt.close()

```

Listing 2.6: satelliteSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### Important Concepts:

- A rigid body's total kinetic energy is equal to the summation of (a) the translational velocity of its center of mass and (b) the rotational kinetic energy spinning about its center of mass.
- The kinetic energy of a system is comprised of the summation of each component's kinetic energy.
- The inertia matrix is a mass-like value that plays a similar role in rotational kinetic energy as mass does in translational kinetic energy.
- The inertia matrix for a rigid body is an integral with respect to the elemental mass of the object.
- The diagonal elements of the inertia matrix represent the rotational mass about the  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  axes.

## Notes and References

The inertia matrix in Equation (2.3) is computed about the center of mass of the object. When an object is composed of several rigid bodies connected to each

other, where the inertia matrix may be known for each rigid body, to find the total inertia about the center of mass, it is convenient to use the parallel axis theorem which states that if  $J_{cm}$  is the inertia matrix about the center of mass, then the inertia matrix about a point that is located at vector  $\mathbf{r}$  from the center of mass is

$$J(\mathbf{r}) = J_{cm} + m (\mathbf{r}^\top \mathbf{r} I_3 - \mathbf{r} \mathbf{r}^\top).$$

For the moment of inertia about a single axis, where the axis is shifted from the center of mass by distance  $d$ , the parallel axis theorem implies that

$$I(d) = I_{cm} + md^2.$$



# 3

## The Euler-Lagrange Equations

### Learning Objectives:

- Compute the potential energy of a system due to gravity and springs (rotational or translational).
- Identify a set of generalized coordinates and forces of a system.
- Use the potential and kinetic energy to calculate the Lagrangian of a system.
- Derive the equations of motion for a system in terms of its generalized coordinates.

## 3.1 Theory

In this chapter we show how to define the potential energy for simple mechanical systems and how to define the generalized forces. The Euler-Lagrange equations given in Equation (1.8) can then be used to derive the equations of motion for the system.

### 3.1.1 Potential Energy

In this book we will be concerned with two sources of potential energy: potential energy due to gravity and potential energy due to springs. Consider a point mass  $m$  in a gravity field that is at position  $y$  as shown in Fig. 3-1. The potential energy is given by

$$P = mgy + P_0,$$

where  $P_0$  is the potential energy when  $y = 0$ . Typically we desire an expression for potential energy that is zero when the configuration variables are zero.

The second source of potential energy is from translational and rotational springs. The spring shown in Fig. 3-2 is stretched from rest by a distance of  $z$ .

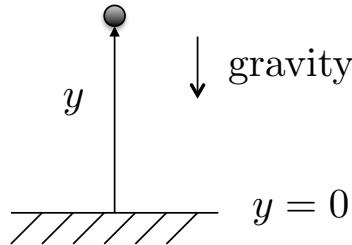


Figure 3-1: The potential energy of a unit mass  $m$  in gravity is given by  $P = mgy$ .

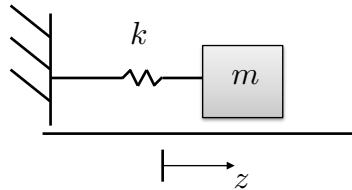


Figure 3-2: The potential energy of a spring stretched by distance  $z$  is  $P = \frac{1}{2}kz^2$ .

The potential energy of a linear stretched/compressed spring is given by

$$P = \frac{1}{2}kz^2,$$

where  $k$  is the spring constant. Similarly, for a rotational spring, the potential energy is given by  $P = \frac{1}{2}k\theta^2$ , where  $\theta$  is the rotational angle and  $\theta = 0$  is the angle at which the rotational spring is at rest.

### 3.1.2 Generalized Coordinates and Generalized Forces

The *generalized coordinates* are the minimum set of configuration variables. For example, for the mass-spring-damper system shown in [Fig. 2-6](#), the generalized coordinates are  $\mathbf{q} = (z_1, z_2)^\top$ . For the single link robot arm shown in [Fig. 2-8](#), the generalized coordinate is  $\mathbf{q} = \theta$ . For the spinning dumbbell system shown in [Fig. 2-7](#), and the pendulum on a cart system shown in [Fig. 2-9](#), the generalized coordinates are  $\mathbf{q} = (z, \theta)^\top$ . Finally, for the satellite system shown in [Fig. 1-4](#), the generalized coordinates are  $\mathbf{q} = (\theta, \phi)^\top$ .

The *generalized forces* are the nonconservative forces and torques acting along each generalized coordinate. In other words, they are the applied forces and torques on the system. For example, for the mass-spring-damper system shown in [Fig. 2-6](#), the generalized force acting along  $z_1$  is zero and the generalized force acting along  $z_2$  is the applied force  $F$ . Therefore the generalized force is  $\boldsymbol{\tau} = (0, F)^\top$ .

### 3.1.3 Damping Forces and Torques

Friction and applied dampers also exert forces on the system. Damping forces are externally applied forces that are proportional to the velocity or angular velocity of the system. For example, in the mass-spring-damper system shown in Fig. 2-6, the damping force acting along  $z_1$  is due to the first damper and the damper between the two masses and is given by  $-b_1\dot{z}_1 - b_2(\dot{z}_1 - \dot{z}_2)$ . The damping acting along  $z_2$  is given by  $-b_2(\dot{z}_2 - \dot{z}_1)$ . Therefore the damping force on the system is given by

$$-B\dot{\mathbf{q}} = -\begin{pmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 \end{pmatrix} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -b_1\dot{z}_1 - b_2(\dot{z}_1 - \dot{z}_2) \\ -b_2(\dot{z}_2 - \dot{z}_1) \end{pmatrix}.$$

### 3.1.4 Euler-Lagrange Equations

The equations of motion are given by the Euler-Lagrange equations. Let the Lagrangian be given by

$$L(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) - P(\mathbf{q}),$$

where  $K(\mathbf{q}, \dot{\mathbf{q}})$  is the kinetic energy of the system and is, in general, a function of the generalized coordinates and the generalized velocities, and  $P(\mathbf{q})$  is the potential energy which is, in general, a function of the generalized coordinates. As shown in Equation (1.8) the Euler-Lagrange equations are given in vector form as

$$\frac{d}{dt} \left( \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L(\mathbf{q})}{\partial \mathbf{q}} = \boldsymbol{\tau} - B\dot{\mathbf{q}}.$$

## 3.2 Design Study A. Single Link Robot Arm



### Example Problem A.3

- (a) Find the potential energy for the system.
- (b) Define the generalized coordinates.
- (c) Find the generalized forces and damping forces.
- (d) Derive the equations of motion using the Euler-Lagrange equations.
- (e) Referring to Appendices P.1, P.2, and P.3, write a class or s-function that implements the equations of motion. Simulate the system using a variable torque input. The output should connect to the animation function developed in homework A.2.

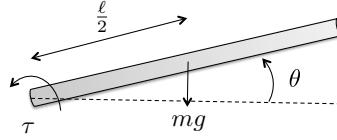


Figure 3-3: Energy calculation for the single link robot arm

### Solution

Let  $P_0$  be the potential when  $\theta = 0$ . The height of the center of mass is  $\ell/2 \sin \theta$ , which is zero when  $\theta = 0$ . Accordingly, the potential energy is given by

$$P = P_0 + mg \frac{\ell}{2} \sin \theta.$$

The generalized coordinate is

$$q_1 = \theta.$$

The generalized force is

$$\tau_1 = \tau,$$

and the generalized damping is

$$-B\dot{q} = -b\dot{\theta}.$$

From Homework A.1 we found that the kinetic energy is given by

$$K = \frac{1}{2} \left( \frac{m\ell^2}{3} \right) \dot{\theta}^2.$$

Therefore the Lagrangian is

$$L = K - P = \frac{1}{2} \left( \frac{m\ell^2}{3} \right) \dot{\theta}^2 - P_0 - mg \frac{\ell}{2} \sin \theta.$$

For this case, the Euler-Lagrange equation is given by

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \tau_1 - b\dot{\theta}$$

where

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) &= \frac{d}{dt} \left( \frac{m\ell^2}{3} \dot{\theta} \right) = \frac{m\ell^2}{3} \ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= -mg \frac{\ell}{2} \cos \theta. \end{aligned}$$

Therefore, the Euler-Lagrange equation becomes

$$\frac{m\ell^2}{3} \ddot{\theta} + mg \frac{\ell}{2} \cos \theta = \tau - b\dot{\theta}. \quad (3.1)$$

A Python class that implements the dynamics of the single link robot arm is shown below.

```
1 import numpy as np
2 import random
3 import armParam as P
4
5
6 class armDynamics:
7     def __init__(self, alpha=0.0):
8         # Initial state conditions
9         self.state = np.array([
10             [P.theta0],           # initial angle
11             [P.thetadot0]
12         ]) # initial angular rate
13
14         # Mass of the arm, kg
15         self.m = P.m * (1.+alpha*(2.*np.random.rand()-1.))
16
17         # Length of the arm, m
18         self.ell = P.ell * (1.+alpha*(2.*np.random.rand()-1.))
19
20         # Damping coefficient, Ns
21         self.b = P.b * (1.+alpha*(2.*np.random.rand()-1.))
22
23         # the gravity constant is well known, so we don't change it.
24         self.g = P.g
25
26         # sample rate at which the dynamics are propagated
27         self.Ts = P.Ts
28         self.torque_limit = P.tau_max
29
30     def update(self, u):
31         # This is the external method that takes the input u at time
32         # t and returns the output y at time t.
33         # saturate the input torque
34         u = self.saturate(u, self.torque_limit)
35
36         self.rk4_step(u) # propagate the state by one time sample
37         y = self.h() # return the corresponding output
38
39         return y
40
41     def f(self, state, tau):
42         # Return xdot = f(x,u), the system state update equations
43         # re-label states for readability
44         theta = state.item(0)
45         thetadot = state.item(1)
46         thetaddot = (3.0/self.m/self.ell**2) * \
47                     (tau - self.b*thetadot \
48                      - self.m*self.g*self.ell/2.0*np.cos(theta))
49         xdot = np.array([[thetadot],
50                         [thetaddot]])
51
52         return xdot
53
54     def h(self):
55         # return the output equations
56         # could also use input u if needed
```

```

57     theta = self.state.item(0)
58     y = np.array([[theta]])
59
60     return y
61
62     def rk4_step(self, u):
63         # Integrate ODE using Runge-Kutta RK4 algorithm
64         F1 = self.f(self.state, u)
65         F2 = self.f(self.state + self.Ts / 2 * F1, u)
66         F3 = self.f(self.state + self.Ts / 2 * F2, u)
67         F4 = self.f(self.state + self.Ts * F3, u)
68         self.state += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
69
70     def saturate(self, u, limit):
71         if abs(u) > limit:
72             u = limit*np.sign(u)
73
74     return u

```

Listing 3.1: armDynamics.py

Code that simulates the dynamics is given below.

```

1 import matplotlib.pyplot as plt
2 import sys
3 sys.path.append('..')  # add parent directory
4 import armParam as P
5 from hw2.signalGenerator import signalGenerator
6 from hw2.armAnimation import armAnimation
7 from hw2.dataPlotter import dataPlotter
8 from armDynamics import armDynamics
9
10 # instantiate arm, controller, and reference classes
11 arm = armDynamics()
12 reference = signalGenerator(amplitude=0.01, frequency=0.02)
13 torque = signalGenerator(amplitude=0.2, frequency=0.05)
14
15 # instantiate the simulation plots and animation
16 dataPlot = dataPlotter()
17 animation = armAnimation()
18
19 t = P.t_start  # time starts at t_start
20 while t < P.t_end:  # main simulation loop
21     # Propagate dynamics in between plot samples
22     t_next_plot = t + P.t_plot
23
24     # updates control and dynamics at faster simulation rate
25     while t < t_next_plot:
26
27         # Get referenced inputs from signal generators
28         r = reference.square(t)
29         u = torque.square(t)
30         y = arm.update(u)  # Propagate the dynamics
31         t = t + P.Ts  # advance time by Ts
32
33     # update animation and data plots
34     animation.update(arm.state)
35     dataPlot.update(t, r, arm.state, u)

```

```

35
36     # the pause causes the figure to be displayed during the
37     # simulation
38     plt.pause(0.0001)
39
40 # Keeps the program from closing until the user presses a button.
41 print('Press key to close')
42 plt.waitforbuttonpress()
43 plt.close()

```

Listing 3.2: armSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### 3.3 Design Study B. Inverted Pendulum



#### Example Problem B.3

- (a) Find the potential energy for the system.
- (b) Define the generalized coordinates.
- (c) Find the generalized forces and damping forces.
- (d) Derive the equations of motion using the Euler-Lagrange equations.
- (e) Referring to Appendices P.1, P.2, and P.3, write a class for s-function that implements the equations of motion. Simulate the system using a variable force on the cart as an input. The output should connect to the animation function developed in homework B.2.

#### Solution

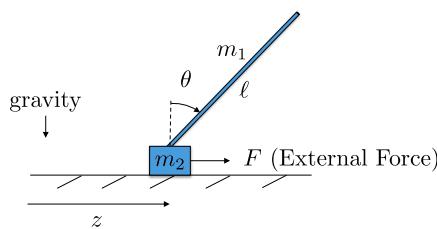


Figure 3-4: Pendulum on a cart.

Let  $P_0$  be the potential energy when  $\theta = 0$  and  $z = 0$ . Then the potential energy of the pendulum system is given by

$$P = P_0 + m_1 g \frac{\ell}{2} (\cos \theta - 1),$$

where  $\frac{\ell}{2} \cos \theta$  is the height of the center of mass of the rod. Note that the potential energy decreases as  $\theta$  increases.

The generalized coordinates for the system are the horizontal position  $z$  and the angle  $\theta$ . Therefore, let  $\mathbf{q} = (z, \theta)^\top$ .

The external force acting in the direction of  $z$  is  $\tau_1 = F$ , and the external torque acting in the direction of  $\theta$  is  $\tau_2 = 0$ . Therefore, the generalized forces are  $\boldsymbol{\tau} = (F, 0)^\top$ . A damping term acts in the direction of  $z$ , which implies that  $-B\dot{\mathbf{q}} = (-b\dot{z}, 0)^\top$ .

Using Equation (2.4), the kinetic energy can be written in terms of the generalized coordinates as

$$\begin{aligned} K(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}(m_1 + m_2)\dot{z}^2 + \frac{1}{2}m_1 \frac{\ell^2}{3}\dot{\theta}^2 + m_1 \frac{\ell}{2}\dot{z}\dot{\theta} \cos \theta \\ &= \frac{1}{2}(m_1 + m_2)\dot{q}_1^2 + \frac{1}{2}m_1 \frac{\ell^2}{3}\dot{q}_2^2 + m_1 \frac{\ell}{2}\dot{q}_1\dot{q}_2 \cos q_2, \end{aligned}$$

and the potential energy can be written as

$$\begin{aligned} P(\mathbf{q}) &= P_0 + m_1 g \frac{\ell}{2} (\cos \theta - 1) \\ &= P_0 + m_1 g \frac{\ell}{2} (\cos q_2 - 1). \end{aligned}$$

The Lagrangian is therefore given by

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}(m_1 + m_2)\dot{q}_1^2 + \frac{1}{2}m_1 \frac{\ell^2}{3}\dot{q}_2^2 + m_1 \frac{\ell}{2}\dot{q}_1\dot{q}_2 \cos q_2 - P_0 - m_1 g \frac{\ell}{2} (\cos q_2 - 1).$$

Therefore

$$\begin{aligned} \frac{\partial L}{\partial \dot{\mathbf{q}}} &= \begin{pmatrix} (m_1 + m_2)\dot{z} + m_1 \frac{\ell}{2}\dot{\theta} \cos \theta \\ m_1 \frac{\ell^2}{3}\dot{\theta} + m_1 \frac{\ell}{2}\dot{z} \cos \theta \end{pmatrix} \\ \frac{\partial L}{\partial \mathbf{q}} &= \begin{pmatrix} 0 \\ -m_1 \frac{\ell}{2}\dot{z}\dot{\theta} \sin \theta + m_1 g \frac{\ell}{2} \sin \theta \end{pmatrix}. \end{aligned}$$

Differentiating  $\frac{\partial L}{\partial \dot{\mathbf{q}}}$  with respect to time gives

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = \begin{pmatrix} (m_1 + m_2)\ddot{z} + m_1 \frac{\ell}{2}\ddot{\theta} \cos \theta - m_1 \frac{\ell}{2}\dot{\theta}^2 \sin \theta \\ m_1 \frac{\ell^2}{3}\ddot{\theta} + m_1 \frac{\ell}{2}\ddot{z} \cos \theta - m_1 \frac{\ell}{2}\dot{z}\dot{\theta} \sin \theta \end{pmatrix}.$$

Therefore the Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} - B\dot{\mathbf{q}}$$

gives

$$(m_1 + m_2)\ddot{z} + m_1 \frac{\ell}{2}\ddot{\theta} \cos \theta - m_1 \frac{\ell}{2}\dot{\theta}^2 \sin \theta = F - b\dot{z}$$

$$m_1 \frac{\ell^2}{3}\ddot{\theta} + m_1 \frac{\ell}{2}\ddot{z} \cos \theta - m_1 \frac{\ell}{2}\dot{z}\dot{\theta} \sin \theta + m_1 \frac{\ell}{2}\dot{z}\dot{\theta} \sin \theta - m_1 g \frac{\ell}{2} \sin \theta = 0.$$

Simplifying and moving all second order derivatives to the left-hand side, and all other terms to the right-hand side gives

$$\begin{pmatrix} (m_1 + m_2)\ddot{z} + m_1 \frac{\ell}{2}\ddot{\theta} \cos \theta \\ m_1 \frac{\ell^2}{3}\ddot{\theta} + m_1 \frac{\ell}{2}\ddot{z} \cos \theta \end{pmatrix} = \begin{pmatrix} m_1 \frac{\ell}{2}\dot{\theta}^2 \sin \theta + F - b\dot{z} \\ m_1 g \frac{\ell}{2} \sin \theta \end{pmatrix}.$$

Using matrix notation, this equation can be rearranged to isolate the second order derivatives on the left-hand side as

$$\begin{pmatrix} (m_1 + m_2) & m_1 \frac{\ell}{2} \cos \theta \\ m_1 \frac{\ell}{2} \cos \theta & m_1 \frac{\ell^2}{3} \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} m_1 \frac{\ell}{2} \dot{\theta}^2 \sin \theta + F - b\dot{z} \\ m_1 g \frac{\ell}{2} \sin \theta \end{pmatrix}. \quad (3.2)$$

Equation (3.2) represents the simulation model for the pendulum on a cart system.

A Python class that implements the dynamics of the pendulum is shown below

```

1 import numpy as np
2 import pendulumParam as P
3
4 class pendulumDynamics:
5     def __init__(self, alpha=0.0):
6         # Initial state conditions
7         self.state = np.array([
8             [P.z0], # z initial position
9             [P.theta0], # Theta initial orientation
10            [P.zdot0], # zdot initial velocity
11            [P.thetadot0], # Thetadot initial velocity
12        ])
13
14     # simulation time step
15     self.Ts = P.Ts
16
17     # Mass of the pendulum, kg
18     self.m1 = P.m1 * (1.+alpha*(2.*np.random.rand()-1.))
19
20     # Mass of the cart, kg
21     self.m2 = P.m2 * (1.+alpha*(2.*np.random.rand()-1.))
22
23     # Length of the rod, m
24     self.ell = P.ell * (1.+alpha*(2.*np.random.rand()-1.))
25
26     # Damping coefficient, Ns
27     self.b = P.b * (1.+alpha*(2.*np.random.rand()-1.))
28
29     # gravity constant is well known, don't change.
30     self.g = P.g
31     self.force_limit = P.F_max

```

```

32
33     def update(self, u):
34         # This is the external method that takes the input u at time
35         # t and returns the output y at time t.
36         # saturate the input force
37         u = self.saturate(u, self.force_limit)
38
39         self.rk4_step(u) # propagate the state by one time sample
40         y = self.h() # return the corresponding output
41
42         return y
43
44     def f(self, state, u):
45         # Return xdot = f(x,u)
46         z = state.item(0)
47         theta = state.item(1)
48         zdot = state.item(2)
49         thetadot = state.item(3)
50         F = u
51
52         # The equations of motion.
53         M = np.array([[self.m1 + self.m2,
54                         self.m1 * (self.ell/2.0) * np.cos(theta)],
55                         [self.m1 * (self.ell/2.0) * np.cos(theta),
56                         self.m1 * (self.ell**2/3.0)]])
57         C = np.array([[self.m1 * (self.ell/2.0)
58                         * thetadot**2 * np.sin(theta)
59                         + F - self.b*zdot],
60                         [self.m1 * self.g * (self.ell/2.0)
61                         * np.sin(theta)]])
62         tmp = np.linalg.inv(M) @ C
63         zddot = tmp.item(0)
64         thetaddot = tmp.item(1)
65
66         # build xdot and return
67         xdot = np.array([zdot, thetadot, zddot, thetaddot])
68
69         return xdot
70
71     def h(self):
72         # return y = h(x)
73         z = self.state.item(0)
74         theta = self.state.item(1)
75         y = np.array([z, theta])
76
77         return y
78
79     def rk4_step(self, u):
80         # Integrate ODE using Runge-Kutta RK4 algorithm
81         F1 = self.f(self.state, u)
82         F2 = self.f(self.state + self.Ts / 2 * F1, u)
83         F3 = self.f(self.state + self.Ts / 2 * F2, u)
84         F4 = self.f(self.state + self.Ts * F3, u)
85         self.state += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
86
87     def saturate(self, u, limit):
88         if abs(u) > limit:

```

```

89     u = limit*np.sign(u)
90
91     return u

```

Listing 3.3: pendulumDynamics.py

The Python code that simulates the dynamics is given below.

```

1 import matplotlib.pyplot as plt
2 import sys
3 sys.path.append('..') # add parent directory
4 import pendulumParam as P
5 from hw2.signalGenerator import signalGenerator
6 from hw2.pendulumAnimation import pendulumAnimation
7 from hw2.dataPlotter import dataPlotter
8 from pendulumDynamics import pendulumDynamics
9
10 # instantiate pendulum, controller, and reference classes
11 pendulum = pendulumDynamics(alpha=0.0)
12 reference = signalGenerator(amplitude=0.5, frequency=0.02)
13 force = signalGenerator(amplitude=1, frequency=1)
14
15 # instantiate the simulation plots and animation
16 dataPlot = dataPlotter()
17 animation = pendulumAnimation()
18
19 t = P.t_start # time starts at t_start
20 while t < P.t_end: # main simulation loop
21
22     # Propagate dynamics at rate Ts
23     t_next_plot = t + P.t_plot
24     while t < t_next_plot:
25         r = reference.square(t)
26         u = force.sin(t)
27         y = pendulum.update(u) # Propagate the dynamics
28         t = t + P.Ts # advance time by Ts
29
30     # update animation and data plots at rate t_plot
31     animation.update(pendulum.state)
32     dataPlot.update(t, r, pendulum.state, u)
33
34     # the pause causes figure to be displayed during simulation
35     plt.pause(0.0001)
36
37 # Keeps the program from closing until the user presses a button.
38 print('Press key to close')
39 plt.waitforbuttonpress()
40 plt.close()

```

Listing 3.4: pendulumSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### 3.4 Design Study C. Satellite Attitude Control



#### Example Problem C.3

- (a) Find the potential energy for the system.
- (b) Define the generalized coordinates.
- (c) Find the generalized forces and damping forces.
- (d) Derive the equations of motion using the Euler-Lagrange equations.
- (e) Referring to Appendices P.1, P.2, and P.3, write a class or s-function that implements the equations of motion. Simulate the system using a variable torque on the body as an input. The output should connect to the animation function developed in homework C.2.

#### Solution

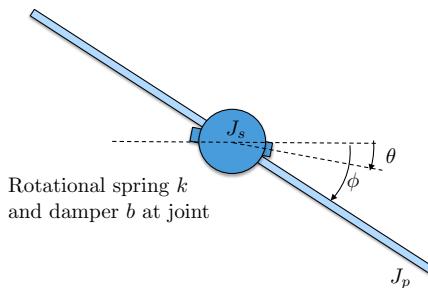


Figure 3-5: Satellite with flexible solar panels.

The generalized coordinates for the satellite attitude control problem are the angles  $\theta$  and  $\phi$ . Therefore, we define the generalized coordinates as  $\mathbf{q} = (\theta, \phi)^\top$ . We will assume the ability to apply torque on the main body ( $\theta$ ) of the satellite, but not on the solar panel. Therefore, the generalized force is  $\boldsymbol{\tau} = (\tau, 0)^\top$ . We will model the dissipation (friction) forces as proportional to the relative motion between the main body and the solar panel. Therefore, the damping term is given by

$$-B\dot{\mathbf{q}} = \begin{pmatrix} -b(\dot{\theta} - \dot{\phi}) \\ -b(\dot{\phi} - \dot{\theta}) \end{pmatrix} = - \begin{pmatrix} b & -b \\ -b & b \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}.$$

Using Equation (2.5), the kinetic energy can be written in terms of the generalized coordinates as

$$\begin{aligned} K(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2} J_s \dot{\theta}^2 + \frac{1}{2} J_p \dot{\phi}^2 \\ &= \frac{1}{2} J_s \dot{q}_1^2 + \frac{1}{2} J_p \dot{q}_2^2. \end{aligned}$$

The potential energy of the system is due to the spring force between the base and the solar panel, and is given by

$$\begin{aligned} P(\mathbf{q}) &= \frac{1}{2} k(\phi - \theta)^2 \\ &= \frac{1}{2} k(q_2 - q_1)^2, \end{aligned}$$

where  $k$  is the spring constant. The Lagrangian is therefore given by

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} J_s \dot{q}_1^2 + \frac{1}{2} J_p \dot{q}_2^2 - \frac{1}{2} k(q_2 - q_1)^2.$$

Therefore

$$\begin{aligned} \frac{\partial L}{\partial \dot{\mathbf{q}}} &= \begin{pmatrix} J_s \dot{\theta} \\ J_p \dot{\phi} \end{pmatrix} \\ \frac{\partial L}{\partial \mathbf{q}} &= \begin{pmatrix} k(\phi - \theta) \\ -k(\phi - \theta) \end{pmatrix}. \end{aligned}$$

Differentiating  $\frac{\partial L}{\partial \dot{\mathbf{q}}}$  with respect to time gives

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = \begin{pmatrix} J_s \ddot{\theta} \\ J_p \ddot{\phi} \end{pmatrix}.$$

Therefore the Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} - B\dot{\mathbf{q}}$$

gives

$$\begin{pmatrix} J_s \ddot{\theta} - k(\phi - \theta) \\ J_p \ddot{\phi} + k(\phi - \theta) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau} \\ 0 \end{pmatrix} + \begin{pmatrix} -b(\dot{\theta} - \dot{\phi}) \\ -b(\dot{\phi} - \dot{\theta}) \end{pmatrix}.$$

Simplifying and moving all second order derivatives to the left-hand side, and all other terms to the right-hand side gives

$$\begin{pmatrix} J_s \ddot{\theta} \\ J_p \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau} - b(\dot{\theta} - \dot{\phi}) - k(\theta - \phi) \\ -b(\dot{\phi} - \dot{\theta}) - k(\phi - \theta) \end{pmatrix}.$$

Using matrix notation, this equation can be rearranged to isolate the second order derivatives on the left-hand side as

$$\begin{pmatrix} J_s & 0 \\ 0 & J_p \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau} - b(\dot{\theta} - \dot{\phi}) - k(\theta - \phi) \\ -b(\dot{\phi} - \dot{\theta}) - k(\phi - \theta) \end{pmatrix}. \quad (3.3)$$

Equation (3.3) represents the simulation model for the simplified satellite system.

A Python class that implements the dynamics of the satellite is shown below

```

1 import numpy as np
2 import random
3 import satelliteParam as P
4
5 class satelliteDynamics:
6     def __init__(self, alpha=0.0):
7         # Initial state conditions
8         self.state = np.array([
9             [P.theta0], # initial base angle
10            [P.phi0], # initial panel angle
11            [P.thetadot0], # initial angular velocity of base
12            [P.phidot0], # initial angular velocity of panel
13        ])
14
15     # simulation time step
16     self.Ts = P.Ts
17
18     # inertia of base
19     self.Js = P.Js * (1.+alpha*(2.*np.random.rand()-1.))
20
21     # inertia of panel
22     self.Jp = P.Jp * (1.+alpha*(2.*np.random.rand()-1.))
23
24     # spring coefficient
25     self.k = P.k * (1.+alpha*(2.*np.random.rand()-1.))
26
27     # Damping coefficient, Ns
28     self.b = P.b * (1.+alpha*(2.*np.random.rand()-1.))
29     self.torque_limit = P.tau_max
30
31     def update(self, u):
32         # This is the external method that takes the input u at time
33         # t and returns the output y at time t.
34         # saturate the input torque
35         u = self.saturate(u, self.torque_limit)
36
37         self.rk4_step(u) # propagate the state by one time sample
38         y = self.h() # return the corresponding output
39
40         return y
41
42     def f(self, state, u):
43         # Return xdot = f(x,u)
44         theta = state.item(0)
45         phi = state.item(1)
46         thetadot = state.item(2)
47         phidot = state.item(3)
48         tau = u
49         # The equations of motion.
50         M = np.array([[self.Js, 0],
51                      [0, self.Jp]])
52         C = np.array([[tau -
53                         self.b*(thetadot-phidot)-self.k*(theta-phi)],
54                     [-self.b*(phidot-thetadot)-self.k*(phi-theta)]])

```

```

55         []])
56     tmp = np.linalg.inv(M) @ C
57     thetaddot = tmp.item(0)
58     phiddot = tmp.item(1)
59
60     # build xdot and return
61     xdot = np.array([[thetaddot], [phidot], [thetaddot],
62                      [phiddot]])
63     return xdot
64
65 def h(self):
66     # return y = h(x)
67     theta = self.state.item(0)
68     phi = self.state.item(1)
69     y = np.array([[theta], [phi]])
70
71     return y
72
73 def rk4_step(self, u):
74     # Integrate ODE using Runge-Kutta RK4 algorithm
75     F1 = self.f(self.state, u)
76     F2 = self.f(self.state + self.Ts / 2 * F1, u)
77     F3 = self.f(self.state + self.Ts / 2 * F2, u)
78     F4 = self.f(self.state + self.Ts * F3, u)
79     self.state += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
80
81 def saturate(self, u, limit):
82     if abs(u) > limit:
83         u = limit*np.sign(u)
84     return u

```

Listing 3.5: pendulumDynamics.py

Code that simulates the dynamics is given below.

```

1 import matplotlib.pyplot as plt
2 import sys
3 sys.path.append('..') # add parent directory
4 import satelliteParam as P
5 from hw2.signalGenerator import signalGenerator
6 from hw2.satelliteAnimation import satelliteAnimation
7 from hw2.dataPlotter import dataPlotter
8 from satelliteDynamics import satelliteDynamics
9
10 # instantiate satellite, controller, and reference classes
11 satellite = satelliteDynamics()
12 reference = signalGenerator(amplitude=0.5, frequency=0.1)
13 torque = signalGenerator(amplitude=0.1, frequency=0.1)
14
15 # instantiate the simulation plots and animation
16 dataPlot = dataPlotter()
17 animation = satelliteAnimation()
18
19 t = P.t_start # time starts at t_start
20 while t < P.t_end: # main simulation loop
21
22     # Propagate dynamics in between plot samples

```

```
23 t_next_plot = t + P.t_plot
24
25 # updates control and dynamics at faster simulation rate
26 while t < t_next_plot:
27     r = reference.square(t)
28     u = torque.sin(t)
29     y = satellite.update(u) # Propagate the dynamics
30     t = t + P.Ts # advance time by Ts
31
32 # update animation and data plots
33 animation.update(satellite.state)
34 dataPlot.update(t, r, satellite.state, u)
35
36 # the pause causes the figure to be displayed during the
37 # simulation
38 plt.pause(0.0001)
39
40 # Keeps the program from closing until the user presses a button.
41 print('Press key to close')
42 plt.waitforbuttonpress()
43 plt.close()
```

Listing 3.6: pendulumSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## Important Concepts:

- Generalized coordinates are the minimum number of independent coordinates that define the system's configuration.
- Generalized forces are nonconservative forces and torques that act on generalized coordinates.
- Damping forces/torques are proportional to the linear/angular velocity of the generalized coordinates.
- The Lagrangian is defined as the difference between a system's kinetic and potential energy.
- A system's equations of motion can be derived using the Euler-Lagrange equation.

## Notes and References

Derivation of the Lagrange-Euler equations can be found in many sources including [1, 2]. A simple derivation using only calculus can be found in [3].

# Part II

## Design Models

The equations of motion that we derived in the previous section are often too complicated to facilitate effective engineering design. Accordingly, the standard approach is to simplify the models into so-called *design models* that capture the essential features of the system. The design techniques studied in this class will require that the design models are linear and time-invariant. In addition, the PID method described in Part III will require that the design models are second-order systems.

Our approach to developing linear time-invariant design models will be to linearize the simulation models developed in Part I and then to convert the linearized models into two standard forms, namely

- Transfer function models and
- State space models.

Both forms will have advantages and disadvantages that will be explained throughout the remaining parts of the book. In Chapter 4 we will define the important concept of equilibrium and show how the equations of motion can be linearized about the equilibrium. Chapter 5 will show how to transform the linearized equations of motion into transfer function models, and Chapter 6 will show how to put the equations of motion in state space form.

The remaining parts of the book will use the design models to develop feed-back controllers for the system of interest. In Part III we will describe the Proportional-Integral-Derivative (PID) controllers based on second-order transfer function models. Part IV derives observer based controllers using state space models. Finally, Part V describes loopshaping design strategies using general transfer function models.



# 4

## Equilibria and Linearization

### Learning Objectives:

- Identify the equilibrium point(s) of a dynamical system.
- Compute the Jacobian linearization of a system about its equilibrium point(s).
- Feedback linearize a dynamical system.

## 4.1 Theory

In this chapter we will describe two methods for linearizing the system, namely Jacobian linearization and feedback linearization. Jacobian linearization is based on the idea of approximating the nonlinear terms in the equations of motion by the first two terms in their Taylor series expansion. Feedback linearization is based on the idea of using the input signal to artificially remove the nonlinear terms.

### 4.1.1 Jacobian Linearization

If the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a nonlinear function that can be differentiated an infinite number of times, then its Taylor series expansion about the point  $w_0$  is given by

$$g(w) = g(w_0) + \left. \frac{\partial g}{\partial w} \right|_{w_0} (w - w_0) + \frac{1}{2!} \left. \frac{\partial^2 g}{\partial w^2} \right|_{w_0} (w - w_0)^2 + \dots$$

A pictorial representation of Taylor series expansion is shown in [Fig. 4-1](#), where it can be seen that in a small neighborhood around  $w = w_0$ , the function  $g(w)$  is

well approximated by the first two terms in the Taylor series expansion, namely

$$g(w) \approx g(w_0) + \frac{\partial g}{\partial w} \Big|_{w_0} (w - w_0).$$

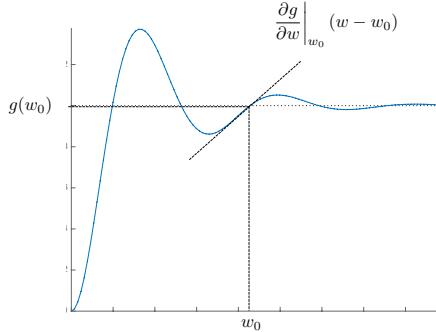


Figure 4-1: Taylor series expansion of  $g(w)$  about  $w = w_0$ .

If  $g$  is a scalar function of several variables, then the Taylor series about  $w_1 = w_{10}, w_2 = w_{20}, \dots, w_m = w_{m0}$  is given by

$$\begin{aligned} g(w_1, w_2, \dots, w_m) &= g(w_{10}, w_{20}, \dots, w_{m0}) + \frac{\partial g}{\partial w_1} \Big|_{w_{10}, w_{20}, \dots, w_{m0}} (w_1 - w_{10}) \\ &\quad + \frac{\partial g}{\partial w_2} \Big|_{w_{10}, w_{20}, \dots, w_{m0}} (w_2 - w_{20}) + \dots + \frac{\partial g}{\partial w_m} \Big|_{w_{10}, w_{20}, \dots, w_{m0}} (w_m - w_{m0}) + \dots \end{aligned} \quad (4.1)$$

For example, given the function  $g(v, w) = v \cos w$ , the constant and linear terms in the Taylor series, expanded around  $v_0$  and  $w_0$  are

$$\begin{aligned} g(v, w) &\approx g(v_0, w_0) + \frac{\partial g}{\partial v} \Big|_{v_0, w_0} (v - v_0) + \frac{\partial g}{\partial w} \Big|_{v_0, w_0} (w - w_0) \\ &= v_0 \cos w_0 + (\cos w)|_{v_0, w_0} (v - v_0) + (-v \sin w)|_{v_0, w_0} (w - w_0) \\ &= v_0 \cos w_0 + (\cos w_0)(v - v_0) - (v_0 \sin w_0)(w - w_0). \end{aligned}$$

Since approximating a nonlinear function by the first two terms in its Taylor series is only a good approximation in a small neighborhood of  $w_0$ , it is important to select  $w_0$  judiciously. In particular, we will be interested in linearizing about equilibrium points of differential equations.

**Definition.** Given the differential equation  $\dot{x} = f(x, u)$ , the pair  $(x_e, u_e)$  is an *equilibrium point* if

$$f(x_e, u_e) = 0.$$

In other words, an equilibrium point is a state-input pair where there is not any motion in the system.

For example, suppose that the nonlinear differential equation of a second-order system is given by

$$\ddot{y} + a\dot{y} + by + g(y, \dot{y}) = u, \quad (4.2)$$

where  $a$  and  $b$  are constants and  $g(y, \dot{y})$  is a known nonlinear function of  $y$  and  $\dot{y}$ , and  $u$  is the input signal. Defining  $x = (y, \dot{y})^\top$ , we get

$$\dot{x} = \begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ u - a\dot{y} - by - g(y, \dot{y}) \end{pmatrix} \triangleq f(x, u).$$

The equilibrium is when  $f(x_e, u_e) = 0$  or in other words, when

$$y_e = \text{anything}, \quad \dot{y}_e = 0, \quad u_e = by_e + g(y_e, 0).$$

Equivalently, at equilibrium there is no motion in the system, which implies that  $\ddot{y}_e = \dot{y}_e = 0$ , which from Equation (4.2) implies that  $u_e = by_e + g(y_e, 0)$ .

Jacobian linearization then proceeds by replacing each term in the nonlinear differential equations describing the system by the first two terms in the Taylor's series expansion about the equilibrium point.

For example, since the first term in Equation (4.2) is already linear, it can be expanded exactly as

$$\ddot{y} = \ddot{y}_e + \frac{\partial \ddot{y}}{\partial \ddot{y}} \Big|_e (\ddot{y} - \ddot{y}_e) = \ddot{\tilde{y}},$$

where we have defined the linearized quantity  $\tilde{y} \triangleq y - y_e$ , and we have used the fact that  $\ddot{y}_e = 0$ . Similarly, the remaining terms in Equation (4.2) are linearized as

$$\begin{aligned} a\dot{y} &= a\dot{y}_e + a \frac{\partial \dot{y}}{\partial \dot{y}} \Big|_e (\dot{y} - \dot{y}_e) = a\dot{\tilde{y}} \\ b\dot{y} &= b\dot{y}_e + b \frac{\partial \dot{y}}{\partial \dot{y}} \Big|_e (y - y_e) = b\dot{y}_e + b\tilde{y} \\ g(y, \dot{y}) &\approx g(y_e, \dot{y}_e) + \frac{\partial g}{\partial y} \Big|_e (y - y_e) + \frac{\partial g}{\partial \dot{y}} \Big|_e (\dot{y} - \dot{y}_e) \\ &= g(y_e, 0) + \frac{\partial g}{\partial y} \Big|_e \tilde{y} + \frac{\partial g}{\partial \dot{y}} \Big|_e \dot{\tilde{y}} \\ u &= u_e + \frac{\partial u}{\partial u} \Big|_e (u - u_e) = u_e + \tilde{u}. \end{aligned}$$

Substituting these expressions into Equation (4.2) the resulting linearized equations of motion are

$$[\ddot{\tilde{y}}] + [a\dot{\tilde{y}}] + [b\dot{y}_e + b\tilde{y}] + \left[ g(y_e, 0) + \frac{\partial g}{\partial y} \Big|_e \tilde{y} + \frac{\partial g}{\partial \dot{y}} \Big|_e \dot{\tilde{y}} \right] = [\tilde{u}]$$

Using the equilibrium  $u_e = by_e + g(y_e, 0)$  gives

$$\ddot{\tilde{y}} + \left( a + \frac{\partial g}{\partial \dot{y}} \Big|_e \right) \dot{\tilde{y}} + \left( b + \frac{\partial g}{\partial y} \Big|_e \right) \tilde{y} = \tilde{u}.$$

A block diagram that shows the design model for Equation (4.2) using Jacobian linearization is shown in [Fig. 4-2](#).

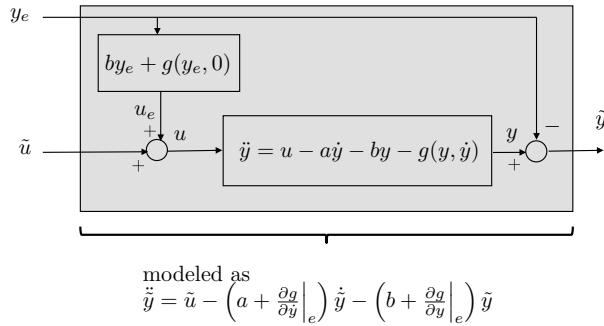


Figure 4-2: Linear model using Jacobian linearization.

### 4.1.2 Feedback Linearization

It is often the case that the nonlinearities in the system can be directly canceled by a judicious choice for the input variable. Consider again the nonlinear equations of motion in Equation (4.2). If  $u$  is selected as

$$u = g(y, \dot{y}) + \tilde{u}$$

then the resulting equation of motion is

$$\ddot{y} + a\dot{y} + by = \tilde{u},$$

which is linear. This technique is called *feedback linearization* because the feedback signal  $u$  has been used to linearize the system by directly canceling the nonlinearities. The advantage of this method is that the equations of motion are still globally defined and are not restricted to a small neighborhood of an equilibrium point. The disadvantage is that  $g(y, \dot{y})$  may not be precisely known, and therefore may not be completely canceled by  $u$ . A block diagram that shows the design model for Equation (4.2) using feedback linearization is shown in [Fig. 4-3](#).

## 4.2 Design Study A. Single Link Robot Arm



### Example Problem A.4

For the single link robot arm:

- (a) Find the equilibria of the system.

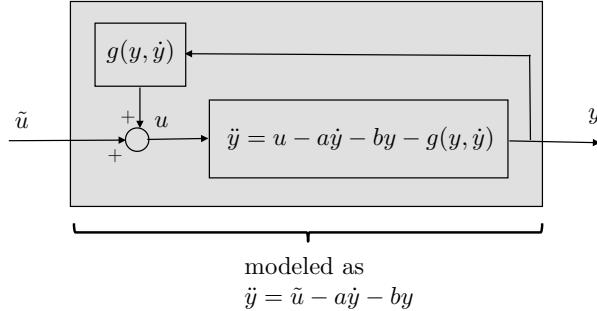


Figure 4-3: Linear model using feedback linearization.

(b) Linearize the system about the equilibria using Jacobian linearization.

(c) Linearize the system using feedback linearization.

### Solution

The differential equation describing the single link robot arm derived in HW A.3 is

$$\frac{m\ell^2}{3}\ddot{\theta} + \frac{mg\ell}{2}\cos\theta = \tau - b\dot{\theta}. \quad (4.3)$$

Defining  $x = (\theta, \dot{\theta})$ , and  $u = \tau$  we get

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \frac{3}{m\ell^2}\tau - \frac{3b}{m\ell^2}\dot{\theta} - \frac{3g}{2\ell}\cos\theta \end{pmatrix} \triangleq f(x, u).$$

The equilibrium is when  $f(x_e, u_e) = 0$ , or in other words, when

$$\theta_e = \text{anything}, \quad \dot{\theta}_e = 0, \quad \tau_e = \frac{mg\ell}{2}\cos\theta_e. \quad (4.4)$$

Equivalently, at equilibrium there is no motion in the system, which implies that  $\ddot{\theta}_e = \dot{\theta}_e = 0$ , which from Equation (4.3) implies that  $\tau_e = \frac{mg\ell}{2}\cos\theta_e$ .

Therefore, any pair  $(\theta_e, \tau_e)$  satisfying Equation (4.4) is an equilibria. Jacobian linearization then proceeds by replacing each term in the nonlinear differential equations describing the system by the first two terms in the Taylor's series expansion about the equilibrium point. Defining  $\tilde{\theta} \triangleq \theta - \theta_e$ ,  $\dot{\tilde{\theta}} \triangleq \dot{\theta} - \dot{\theta}_e = \dot{\theta}$ ,  $\ddot{\tilde{\theta}} = \ddot{\theta} - \ddot{\theta}_e = \ddot{\theta}$ , and  $\tilde{\tau} = \tau - \tau_e$ , each term in Equation (4.3) can be expanded

about the equilibrium as follows:

$$\begin{aligned}\frac{m\ell^2}{3}\ddot{\theta} &= \frac{m\ell^2}{3}\ddot{\theta}_e + \frac{m\ell^2}{3}\left.\frac{\partial\ddot{\theta}}{\partial\ddot{\theta}}\right|_e(\ddot{\theta} - \ddot{\theta}_e) = \frac{m\ell^2}{3}\ddot{\tilde{\theta}}, \\ \frac{mg\ell}{2}\cos\theta &\approx \frac{mg\ell}{2}\cos\theta_e + \frac{mg\ell}{2}\left.\frac{\partial}{\partial\theta}(\cos\theta)\right|_{\theta_e}(\theta - \theta_e) \\ &= \frac{mg\ell}{2}\cos\theta_e - \frac{mg\ell}{2}\sin\theta_e\dot{\theta} \\ \tau &= \tau_e + \left.\frac{\partial\tau}{\partial\tau}\right|_e(\tau - \tau_e) = \tau_e + \tilde{\tau} \\ b\dot{\theta} &= b\dot{\theta}_e + b\left.\frac{\partial\dot{\theta}}{\partial\dot{\theta}}\right|_e(\dot{\theta} - \dot{\theta}_e) = b\dot{\tilde{\theta}}.\end{aligned}$$

Substituting into Equation (4.3) gives

$$\frac{m\ell^2}{3}\left[\ddot{\theta}_e + \ddot{\tilde{\theta}}\right] + \frac{mg\ell}{2}\left[\cos\theta_e - \sin\theta_e\tilde{\theta}\right] = [\tau_e + \tilde{\tau}] - b\left[\dot{\theta}_e + \dot{\tilde{\theta}}\right].$$

Simplifying this expression using Equation (4.4) gives

$$\frac{m\ell^2}{3}\ddot{\tilde{\theta}} - \frac{mg\ell}{2}(\sin\theta_e)\tilde{\theta} = \tilde{\tau} - b\dot{\tilde{\theta}}, \quad (4.5)$$

which are the linearized equations of motion using Jacobian linearization.

Feedback linearization proceeds by using the feedback linearizing control

$$\tau = \frac{mg\ell}{2}\cos\theta + \tilde{\tau} \quad (4.6)$$

in Equation (4.3) to obtain the feedback linearized equations of motion

$$\frac{m\ell^2}{3}\ddot{\theta} = \tilde{\tau} - b\dot{\theta}, \quad (4.7)$$

where we emphasize that the equations of motion are valid for any  $\theta$  and  $\dot{\theta}$  and not just in a small region about an equilibrium. It is interesting to compare Equation (4.6) to the control signal using Jacobian linearization which is

$$\tau = \tau_e + \tilde{\tau} = \frac{mg\ell}{2}\cos\theta_e + \tilde{\tau}. \quad (4.8)$$

The control signal in (4.8) uses the equilibrium angle  $\theta_e$  whereas the control signal in (4.6) uses the actual angle  $\theta$ . For the single link robot arm we will use the design model obtained using feedback linearization.

### 4.3 Design Study B. Inverted Pendulum



#### Example Problem B.4

For the inverted pendulum,

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria using Jacobian linearization.

Note that for the inverted pendulum, since the nonlinearities are not all contained in the same channel as the control force  $F$ , the system cannot be feedback linearized.

## Solution

The differential equations describing the inverted pendulum derived in HW B.3 are

$$\begin{aligned} (m_1 + m_2)\ddot{z} + m_1 \frac{\ell}{2}\ddot{\theta} \cos \theta &= m_1 \frac{\ell}{2}\dot{\theta}^2 \sin \theta - b\dot{z} + F \\ m_1 \frac{\ell}{2}\ddot{z} \cos \theta + m_1 \frac{\ell^2}{3}\ddot{\theta} &= m_1 g \frac{\ell}{2} \sin \theta. \end{aligned} \quad (4.9)$$

The equilibria values  $(z_e, \theta_e, F_e)$  are found by setting  $\dot{z} = \ddot{z} = \dot{\theta} = \ddot{\theta} = 0$  in Equation (4.9) to obtain

$$F_e = 0 \quad (4.10)$$

$$m_1 g \frac{\ell}{2} \sin \theta_e = 0. \quad (4.11)$$

Therefore, any triple  $(z_e, \theta_e, F_e)$  satisfying Equations (4.10) and (4.11) is an equilibria, or in other words  $z_e$  can be any value,  $F_e = 0$  and  $\theta_e = k\pi$ , where  $k$  is an integer.

To linearize around  $(z_e, \theta_e, F_e)$  where  $k$  is an even integer, note that

$$\begin{aligned}\ddot{\theta} \cos \theta &\approx \ddot{\theta}_e \cos \theta_e + \frac{\partial}{\partial \theta}(\ddot{\theta} \cos \theta) \Big|_{(\ddot{\theta}_e, \theta_e)} (\theta - \theta_e) + \frac{\partial}{\partial \ddot{\theta}}(\ddot{\theta} \cos \theta) \Big|_{(\ddot{\theta}_e, \theta_e)} (\ddot{\theta} - \ddot{\theta}_e) \\&= \ddot{\theta}_e \cos \theta_e - (\ddot{\theta}_e \sin \theta_e) \tilde{\theta} + (\cos \theta_e) \ddot{\theta} \\&= \ddot{\tilde{\theta}} \\[10pt]\dot{\theta}^2 \sin \theta &\approx \dot{\theta}_e^2 \sin \theta_e + \frac{\partial}{\partial \theta}(\dot{\theta}^2 \sin \theta) \Big|_{(\theta_e, \dot{\theta}_e)} (\theta - \theta_e) + \frac{\partial}{\partial \dot{\theta}}(\dot{\theta}^2 \sin \theta) \Big|_{(\theta_e, \dot{\theta}_e)} (\dot{\theta} - \dot{\theta}_e) \\&= \dot{\theta}_e^2 \sin \theta_e + \dot{\theta}_e^2 \cos \theta_e \tilde{\theta} + 2\dot{\theta}_e \sin \theta_e \dot{\tilde{\theta}} \\&= 0, \\[10pt]\ddot{z} \cos \theta &\approx \ddot{z}_e \cos \theta_e + \frac{\partial}{\partial \theta}(\ddot{z} \cos \theta) \Big|_{(\ddot{z}_e, \theta_e)} (\theta - \theta_e) + \frac{\partial}{\partial z}(\ddot{z} \cos \theta) \Big|_{(\ddot{z}_e, \theta_e)} (\ddot{z} - \ddot{z}_e) \\&= \ddot{z}_e \cos \theta_e - (\ddot{z}_e \sin \theta_e) \tilde{\theta} + (\cos \theta_e) \ddot{\tilde{z}} \\&= \ddot{\tilde{z}} \\[10pt]\sin \theta &\approx \sin \theta_e + \frac{\partial}{\partial \theta}(\sin \theta) \Big|_{\theta_e} (\theta - \theta_e) \\&= \sin \theta_e + (\cos \theta_e) \tilde{\theta} \\&= \tilde{\theta},\end{aligned}$$

where we have defined  $\tilde{\theta} \triangleq \theta - \theta_e$  and  $\tilde{z} \triangleq z - z_e$ . Also defining  $\tilde{F} \triangleq F - F_e$ , and noting that

$$\begin{aligned}\theta &= \theta_e + \tilde{\theta} = \tilde{\theta} \\ \dot{\theta} &= \dot{\theta}_e + \dot{\tilde{\theta}} = \dot{\tilde{\theta}} \\ z &= z_e + \tilde{z} \\ \dot{z} &= \dot{z}_e + \dot{\tilde{z}} = \dot{\tilde{z}} \\ \ddot{z} &= \ddot{z}_e + \ddot{\tilde{z}} = \ddot{\tilde{z}} \\ F &= F_e + \tilde{F} = \tilde{F},\end{aligned}$$

we can write Equation (4.9) in its linearized form as

$$\begin{aligned}(m_1 + m_2)[\ddot{z}_e + \ddot{\tilde{z}}] + m_1 \frac{\ell}{2}[\ddot{\tilde{\theta}}] &= m_1 \frac{\ell}{2}[0] - b[\dot{z}_e + \dot{\tilde{z}}] + [F_e + \tilde{F}] \\ m_1 \frac{\ell}{2}[\ddot{\tilde{z}}] + m_1 \frac{\ell^2}{3}[\ddot{\theta}_e + \ddot{\tilde{\theta}}] &= m_1 g \frac{\ell}{2}[\tilde{\theta}],\end{aligned}$$

which simplifies to

$$\begin{pmatrix} (m_1 + m_2) & m_1 \frac{\ell}{2} \\ m_1 \frac{\ell}{2} & m_1 \frac{\ell^2}{3} \end{pmatrix} \begin{pmatrix} \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} -b\dot{\tilde{z}} + \tilde{F} \\ m_1 g \frac{\ell}{2} \tilde{\theta} \end{pmatrix}, \quad (4.12)$$

which are the linearized equations of motion.

## 4.4 Design Study C. Satellite Attitude Control



### Example Problem C.4

For the satellite system:

- (a) Find the equilibria of the system.

### Solution

The differential equations describing the motion of the satellite system derived in HW C.3 are

$$J_s \ddot{\theta} + b(\dot{\theta} - \dot{\phi}) + k(\theta - \phi) = \tau \quad (4.13)$$

$$J_p \ddot{\phi} + b(\dot{\phi} - \dot{\theta}) + k(\phi - \theta) = 0. \quad (4.14)$$

Defining  $x = (\theta, \phi, \dot{\theta}, \dot{\phi})$ , and  $u = \tau$  we get

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ -\frac{b}{J_s}(\dot{\theta} - \dot{\phi}) - \frac{k}{J_s}(\theta - \phi) + \frac{1}{J_s}\tau \\ -\frac{b}{J_p}(\dot{\phi} - \dot{\theta}) - \frac{k}{J_p}(\phi - \theta) \end{pmatrix} \triangleq f(x, u).$$

The equilibrium is when  $f(x_e, u_e) = 0$ , or in other words when

$$\theta_e = \phi_e = \text{anything}, \quad \dot{\theta}_e = \dot{\phi}_e = 0, \quad \tau_e = 0. \quad (4.15)$$

The equations of motion for this system are linear and do not require linearization. With that said, the information about the system equilibria can be helpful for designing the control system. In this case, understanding that the commanded angular position of the satellite and the panel should be the same to keep the system in equilibrium will influence our control design.

## Important Concepts:

- The equilibrium point of a differential equation is where it is stationary, i.e.  $f(x_e, u_e) = 0$ , where  $\dot{x} = f(x, u)$ .
- Jacobian linearization uses the first two terms of a Taylor series expansion to create a linear representation of a system about its equilibrium point.
- Feedback linearization uses the input or control variable  $u$  to directly cancel out nonlinear terms in the differential equations.
- Feedback linearization is valid globally, unlike Jacobian linearization which is only accurate in a region near the equilibrium point.
- The disadvantages of feedback linearization include (a) it assumes perfect knowledge of the terms that are being canceled out, and (b) it is not always feasible or may require a transformation of the system before being able to cancel out all the nonlinearities.

## Notes and References

Jacobian linearization is a standard technique that is well documented in most introductory textbooks on control. For example, see [4, 5, 6, 7]. Feedback linearization is a much deeper topic than can be covered in an introductory textbook on control. A discussion that is similar to ours, that assumes that the nonlinearity is in the input channel, is contained in [5]. Feedback linearization is a fairly standard technique for fully actuated mechanical systems modeled by Euler-Lagrange equations, where the model of the system can be reduced to

$$M(q)\ddot{q} + c(q, \dot{q})\dot{q} + g(q) = \tau,$$

where  $q \in \mathbb{R}^m$  is the generalized coordinate and  $\tau \in \mathbb{R}^m$  is the generalized force, and where  $M(q)$  is an invertible matrix. In that case we can write

$$\ddot{q} = M^{-1}(q) (\tau - c(q, \dot{q})\dot{q} - g(q)).$$

Selecting  $\tau$  as

$$\tau = c(q, \dot{q})\dot{q} + g(q) + M(q)\nu, \quad (4.16)$$

results in the linear system

$$\ddot{q} = \nu,$$

where the new control variable  $\nu$  can be designed to stabilize the system. The control law in Equation (4.16) is the feedback linearizing control. This is a standard technique for robot manipulators and is often called the method of *computed torque* [8, 9, 10].

When the nonlinearity is not in the input channel, the system can still often be feedback linearized through a suitable change of variables. A good introduction to the topic is contained in [11]. More advanced coverage of the topic can be found in [12, 13, 14, 15].

# 5

## Transfer Function Models

### Learning Objectives:

- Compute the transfer function of dynamical systems using the Laplace transform.
- Approximate high-order transfer functions with a cascade of lower-order transfer functions.

## 5.1 Theory

The (one-sided) Laplace transform of a time domain signal  $y(t)$  is defined as

$$\mathcal{L}\{y(t)\} \triangleq Y(s) = \int_0^{\infty} y(t)e^{-st} dt.$$

In essence, the Laplace transform indicates the content of the complex signal  $e^{-st}$  that is contained in  $y(t)$ . The Laplace transform of the time derivative of  $y(t)$  is given by

$$\mathcal{L}\{\dot{y}\} = \int_0^{\infty} \dot{y}e^{-st} dt.$$

Using integration by parts  $\int u dv = uv - \int v du$ , where  $u = e^{-st}$  and  $dv = \dot{y}dt$ , and assuming that  $\lim_{t \rightarrow \infty} y(t)e^{-st} = 0$ , or that  $s$  is in the region of convergence

of the Laplace transform, we get

$$\begin{aligned}\mathcal{L}\{\dot{y}\} &= \int_0^\infty \dot{y}e^{-st} dt \\ &= y(t)e^{-st} \Big|_0^\infty - \int_0^\infty y(t) (-se^{-st} dt) \\ &= \lim_{t \rightarrow \infty} y(t)e^{-st} - y(0) + s \int_0^\infty y(t)e^{-st} dt \\ &= sY(s) - y(0).\end{aligned}$$

Similarly, the Laplace transform of the second derivative of  $y(t)$  is given by

$$\begin{aligned}\mathcal{L}\{\ddot{y}\} &= s\mathcal{L}\{\dot{y}\} - \dot{y}(0) \\ &= s(sY(s) - y(0)) - \dot{y}(0) \\ &= s^2Y(s) - sy(0) - \dot{y}(0).\end{aligned}$$

The transfer function is typically defined as the relationship between the input and the output of the system. In that setting the transfer function is found when all initial conditions are set to zero. This is important to remember in the context of modeling. We will see that state space models explicitly account for initial conditions, but that transfer functions do not.

Given the  $n^{th}$  order differential equation

$$\begin{aligned}y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0y \\ = b_mu^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0u,\end{aligned}$$

the transfer function from  $u$  to  $y$  is found by taking the Laplace transform of both sides of the equation and setting all initial conditions to zero to obtain

$$\begin{aligned}s^nY(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_1sY(s) + a_0Y(s) \\ = b_ms^mU(s) + b_{m-1}s^{m-1}U(s) + \cdots + b_1sU(s) + b_0U(s).\end{aligned}$$

Factoring  $Y(s)$  and  $U(s)$  we get

$$\begin{aligned}(s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0) Y(s) \\ = (b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0) U(s).\end{aligned}$$

Solving for  $Y(s)$  gives

$$Y(s) = \left( \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} \right) U(s),$$

where

$$H(s) = \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}$$

is the transfer function from  $u$  to  $y$ .

Given the transfer function  $H(s)$ , the *poles* of the transfer function are the roots of the denominator polynomial

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0.$$

Similarly, the *zeros* of the transfer function are the roots of the numerator polynomial

$$b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0 = 0.$$

As a concrete example, consider the differential equation given by

$$y^{(3)} + 2\ddot{y} + 3\dot{y} + 4y = 5\dot{u} + 6u.$$

Taking the Laplace transform of both sides, setting the initial conditions to zero, and factoring  $Y(s)$  and  $U(s)$  gives

$$(s^3 + 2s^2 + 3s + 4)Y(s) = (5s + 6)U(s).$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{5s + 6}{s^3 + 2s^2 + 3s + 4}U(s).$$

The transfer function from  $u$  to  $y$  is therefore

$$H(s) = \frac{5s + 6}{s^3 + 2s^2 + 3s + 4}.$$

The poles of the transfer function are the roots of  $s^3 + 2s^2 + 3s + 4 = 0$  or  $p_1 = -1.6506$ ,  $p_2 = -0.1747 + j1.5469$ , and  $p_3 = -0.1747 - j1.5469$ . The zeros of the transfer function are the roots of  $5s + 6 = 0$  or  $z_1 = -6/5$ .

## 5.2 Design Study A. Single Link Robot Arm



### Example Problem A.5

For the single link robot arm, find the transfer function of the system from the torque  $\tau$  to the angle  $\theta$ .

#### Solution

As shown in Section 4.2, the feedback linearized model for the single link robot arm is given by Equation (4.7) as

$$\frac{m\ell^2}{3}\ddot{\theta} = \tilde{\tau} - b\dot{\theta}. \quad (5.1)$$

Taking the Laplace transform of Equation (5.1) and setting all initial conditions to zero we get

$$\frac{m\ell^2}{3}s^2\Theta(s) + bs\Theta(s) = \tilde{\tau}(s).$$

Solving for  $\Theta(s)$  gives

$$\Theta(s) = \left( \frac{1}{\frac{m\ell^2}{3}s^2 + bs} \right) \tilde{\tau}(s).$$

The canonical form for transfer functions is for the leading coefficient in the denominator polynomial to be unity. This is called monic form. Putting the transfer function in monic form results in

$$\Theta(s) = \left( \frac{\frac{3}{m\ell^2}}{s^2 + \frac{3b}{m\ell^2}s} \right) \tilde{\tau}(s), \quad (5.2)$$

where the expression in the parenthesis is the transfer function from  $\tilde{\tau}$  to  $\theta$ , where  $\tilde{\tau}$  indicates that we are working with the feedback linearized control in Equation (4.6). The block diagram associated with Equation (5.2) is shown in [Fig. 5-1](#).

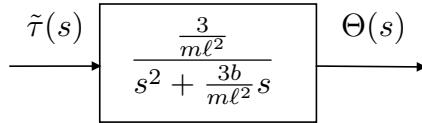


Figure 5-1: A block diagram of the single link robot arm.

### 5.3 SIMO Systems and Cascade Approximations

There are many systems that have multiple inputs and/or multiple outputs. In this case, a transfer function exists from each input to each output. For example, suppose that the system is given by the coupled differential equations

$$\begin{aligned} \ddot{y}_1 + a_{11}\dot{y}_1 + \ddot{y}_2 + a_{12}\dot{y}_2 &= b_{11}u_1 + b_{12}u_2 + b_{13}\dot{u}_2 \\ \ddot{y}_2 + a_{22}y_2 &= b_{21}u_2. \end{aligned}$$

Taking the Laplace transform and arranging in matrix notation gives

$$\left( \begin{array}{c|c} s^2 + a_{11}s & s^2 + a_{12}s \\ \hline 0 & s^2 + a_{22} \end{array} \right) \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \left( \begin{array}{c|c} b_{11} & (b_{13}s + b_{12}) \\ \hline 0 & b_{21} \end{array} \right) \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}.$$

To find the relevant transfer functions, we need to isolate the outputs on the left-hand side by inverting the matrix on the left. Using the inverse formula for a  $2 \times 2$

matrix derived in Appendix P.7

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad - bc}$$

we get

$$\begin{aligned} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} &= \left( \begin{array}{c|c} s^2 + a_{11}s & s^2 + a_{12}s \\ 0 & s^2 + a_{22} \end{array} \right)^{-1} \left( \begin{array}{c|c} b_{11} & b_{13}s + b_{12} \\ 0 & b_{21} \end{array} \right) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \\ &= \frac{\left( \begin{array}{c|c} s^2 + a_{22} & -s^2 - a_{12}s \\ 0 & s^2 + a_{11}s \end{array} \right)}{(s^2 + a_{11}s)(s^2 + a_{22})} \left( \begin{array}{c|c} b_{11} & b_{13}s + b_{12} \\ 0 & b_{21} \end{array} \right) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \\ &= \frac{\left( \begin{array}{c|c} b_{11}(s^2 + a_{22}) & (s^2 + a_{22})(b_{13}s + b_{12}) - b_{21}(s^2 + a_{12}s) \\ 0 & b_{21}(s^2 + a_{11}s) \end{array} \right)}{(s^2 + a_{11}s)(s^2 + a_{22})} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \\ &= \left( \begin{array}{c|c} \frac{b_{11}}{s^2 + a_{11}s} & \frac{b_{13}s^3 + (b_{12} - b_{21})s^2 + (b_{13}a_{22} - b_{21}a_{12})s + a_{22}b_{12}}{(s^2 + a_{11}s)(s^2 + a_{22})} \\ 0 & \frac{b_{21}}{s^2 + a_{22}} \end{array} \right) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}. \end{aligned}$$

Therefore, the transfer function from  $u_1$  to  $y_1$  is  $\frac{b_{11}}{s^2 + a_{11}s}$ , the transfer function from  $u_2$  to  $y_1$  is  $\frac{b_{13}s^3 + (b_{12} - b_{21})s^2 + (b_{13}a_{22} - b_{21}a_{12})s + a_{22}b_{12}}{(s^2 + a_{11}s)(s^2 + a_{22})}$ , the transfer function from  $u_1$  to  $y_2$  is 0, and the transfer function from  $u_2$  to  $y_2$  is  $\frac{b_{21}}{s^2 + a_{22}}$ .

For many of the case studies that we will look at in this book, there is a single input, but we are also interested in controlling two or more measurements or outputs. State-space control techniques are able to create control laws to control multiple outputs in a single design step, as we will see in later chapters. However, we also want to be able to create single-input, multiple-output controllers using more classical transfer-function based approaches. Doing so will require more intuition about the systems we are controlling, as well as a clear understanding of our control objectives so that we can make appropriate design decisions and simplifying assumptions.

The majority of our design studies (specifically studies B, C, E, and F) have coupled, higher-order dynamics (greater than order two) with multiple outputs. Instead of coming up with a single feedback loop to control the multiple outputs of interest, we will form a cascade of two lower-order transfer functions and design two independent control loops. Forming the cascade of lower-order transfer functions requires insights into the dynamic behavior of the system, because it may require simplifying assumptions about the dynamic coupling between the two subsystems. Instead of a two-way coupling, the cascade may make the dynamic coupling one-way. To reduce the complexity of the system, we may make simplifying assumptions about the dynamics of one of the subsystems. It is essential for these simplifications to be justifiable given the parameters of the system and our control objectives. For example, in design study C, we form a cascade of two transfer functions: the first is from the torque input to the satellite angular position and the second is from the satellite angular position to the solar panel

angular position. Because the satellite is more massive than the solar panels, we will make a valid assumption that while the satellite significantly influences the dynamic behavior of the solar panels, the solar panels do not significantly influence the dynamic behavior of the satellite. As we will see, this will simplify the design of the control system significantly and result in a viable control design that provides good performance.

In forming cascades of transfer functions, other factors may influence the form of the cascade. Our control priorities may play a significant role. For example, in design study B, the balancing of the inverted pendulum is the highest priority and therefore, we will use the force input to control the angle of the pendulum. Of importance, but lower priority, is controlling the cart. These considerations will lead us to form the cascade using the transfer functions from force to pendulum angle and from pendulum angle to cart position. In other situations, the causality of the dynamic behavior is more transparent. In the ball-beam system, the force on the beam causes the beam angle to change and the change in beam angle causes the ball to roll and translate. We will form the transfer function cascade for the ball-beam system accordingly. In the planar VTOL aircraft system, torque from the differential thrust causes the aircraft to pitch, while thrust combined with non-zero pitch causes the aircraft to translate. It is natural, therefore, to think of torque causing pitch, and pitch causing translational motion. We will thus form a cascade of two transfer functions: from torque to pitch angle and from pitch angle to horizontal position.

In forming a cascade of two (or more) transfer functions to represent the dynamics of a single system, it is critically important that we consider the physics of the system and our control objectives as we make simplifying assumptions. Important questions to consider include:

- What is the control input to the system?
- What outputs of the system are to be controlled?
- What outputs can be measured?
- What physically justifiable simplifications can be made about the dynamic coupling between subsystems?
- What physically justifiable simplifying assumptions can be made about the dynamics of the system?

Asking and answering these questions is fundamental to the design of successful control systems. The more insight you have into the physical behavior of the system you are controlling, the better your control system designs will be.

## 5.4 Design Study B. Inverted Pendulum



### Example Problem B.5

For the inverted pendulum:

- (a) Start with the linearized equations and use the Laplace transform to convert the equations of motion to the s-domain.
- (b) Find the transfer functions from the input  $\tilde{F}(s)$  to the outputs  $\tilde{Z}(s)$  and  $\tilde{\Theta}(s)$ . Assume that the damping constant is negligible. How does this simplify the transfer functions? Is this a reasonable assumption?
- (c) From the simplified  $\tilde{Z}(s)/\tilde{F}(s)$  and  $\tilde{\Theta}(s)/\tilde{F}(s)$  transfer functions, compute the  $\tilde{Z}(s)/\tilde{\Theta}(s)$  transfer function. Draw a block diagram of the system as a series of two transfer functions from  $\tilde{F}(s)$  to  $\tilde{\Theta}(s)$  and from  $\tilde{\Theta}(s)$  to  $\tilde{Z}(s)$ .
- (d) Describe how this transfer-function cascade makes sense physically. How does force influence the pendulum angle? How does the pendulum angle influence the cart position?

### Solution

From HW B.4, the linearized equations of motion are given by

$$\begin{pmatrix} (m_1 + m_2) & m_1 \frac{\ell}{2} \\ m_1 \frac{\ell}{2} & m_1 \frac{\ell^2}{3} \end{pmatrix} \begin{pmatrix} \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} -b\dot{\tilde{z}} + \tilde{F} \\ m_1 g \frac{\ell}{2} \tilde{\theta} \end{pmatrix}.$$

The second equation in this matrix formulation can be simplified by dividing both sides of the second equation by  $m_1 \frac{\ell}{2}$  to give

$$\begin{pmatrix} (m_1 + m_2) & m_1 \frac{\ell}{2} \\ 1 & \frac{2\ell}{3} \end{pmatrix} \begin{pmatrix} \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} -b\dot{\tilde{z}} + \tilde{F} \\ g\tilde{\theta} \end{pmatrix}.$$

We can write these equations with states and state derivatives on the left and inputs on the right as

$$\begin{aligned} (m_1 + m_2)\ddot{\tilde{z}} + m_1 \frac{\ell}{2}\ddot{\tilde{\theta}} + b\dot{\tilde{z}} &= \tilde{F} \\ \ddot{\tilde{z}} + \frac{2\ell}{3}\ddot{\tilde{\theta}} - g\tilde{\theta} &= 0. \end{aligned}$$

Taking the Laplace transform gives

$$\begin{aligned} [(m_1 + m_2)s^2 + bs]\tilde{Z}(s) + m_1 \frac{\ell}{2}s^2\tilde{\Theta}(s) &= \tilde{F}(s) \\ s^2\tilde{Z}(s) + \left(\frac{2\ell}{3}s^2 - g\right)\tilde{\Theta}(s) &= 0. \end{aligned}$$

These equations can be expressed in matrix form as

$$\begin{pmatrix} (m_1 + m_2)s^2 + bs & m_1 \frac{\ell}{2}s^2 \\ s^2 & \frac{2\ell}{3}s^2 - g \end{pmatrix} \begin{pmatrix} \tilde{Z}(s) \\ \tilde{\Theta}(s) \end{pmatrix} = \begin{pmatrix} \tilde{F}(s) \\ 0 \end{pmatrix}.$$

Inverting the matrix on the left hand side and solving for the transfer functions gives

$$\begin{aligned} \tilde{Z}(s) &= \left( \frac{\frac{2\ell}{3}s^2 - g}{(m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3})s^4 + b \frac{2\ell}{3}s^3 - (m_1 + m_2)gs^2 - bgs} \right) \tilde{F}(s) \\ \tilde{\Theta}(s) &= \left( \frac{-s^2}{(m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3})s^4 + b \frac{2\ell}{3}s^3 - (m_1 + m_2)gs^2 - bgs} \right) \tilde{F}(s). \end{aligned}$$

If we make the assumption that  $b \approx 0$ , then these transfer functions will simplify further and be even easier to work with from a control design perspective. This is a reasonable and conservative assumption because the damping force  $b\dot{z}$  is small relative to the other forces acting on the system. Furthermore, by underestimating the damping in the system, our control design will be conservative because it assumes there is no damping provided by the physics of the system and thus all the damping in the system must come from the feedback control. Assuming  $b = 0$ , we get

$$\begin{aligned} \tilde{Z}(s) &= \left( \frac{\frac{2\ell}{3}s^2 - g}{s^2 [(m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3})s^2 - (m_1 + m_2)g]} \right) \tilde{F}(s) \\ \tilde{\Theta}(s) &= \left( \frac{-1}{(m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3})s^2 - (m_1 + m_2)g} \right) \tilde{F}(s). \end{aligned}$$

From a controls perspective, we will be interested in how the pendulum angle  $\theta$  influences the cart position  $z$ , and so we want to find the transfer function from  $\tilde{\Theta}(s)$  to  $\tilde{Z}(s)$ . This can be easily calculated by recognizing that

$$\tilde{Z}(s) = \frac{\tilde{Z}(s)}{\tilde{F}(s)} \frac{\tilde{F}(s)}{\tilde{\Theta}(s)} \tilde{\Theta}(s).$$

Accordingly,

$$\tilde{Z}(s) = \left( \frac{-\frac{2\ell}{3}s^2 + g}{s^2} \right) \tilde{\Theta}(s).$$

The block diagram for the approximate system is shown in Figure 5-2.

This transfer function cascade makes sense physically. With the pendulum balanced vertically, a positive force on the cart would cause the pendulum to fall in the negative direction as indicated by the minus sign in the numerator. The pendulum falls in an unstable motion due to the right-half-plane pole in the transfer function. The angle of the rod influences the position of the cart as shown

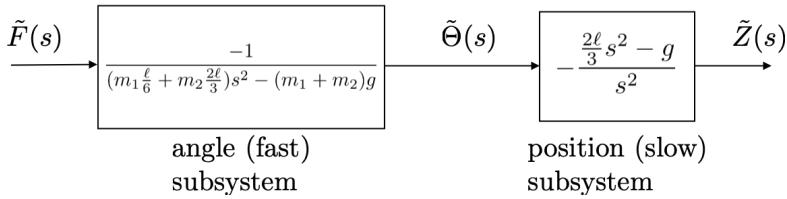


Figure 5-2: The inverted pendulum dynamics are approximated by a cascade of fast and slow subsystems. The fast subsystem is the transfer function from the force to the angle, and the slow subsystem is the transfer function from the angle to the position.

in the second transfer function. If the pendulum were balanced vertically with no input force applied and a small disturbance caused the pendulum to fall in the positive direction, the cart would shoot off in the negative direction. Again, this can be seen from the negative sign in the numerator of the transfer function and the right-half-plane pole in the denominator. Keep in mind that these equations are linearized about the vertical position of the pendulum. When the pendulum is hanging down, the equations are different. How are they different?

The PID and loopshaping techniques that we will discuss in Parts III and V will use the approximate transfer function models of the plant derived above. The state space methods discussed in Part IV will use a state space model, which does not depend on neglecting the coupling between subsystems and does not depend on a separation into fast and slow subsystems.

## 5.5 Design Study C. Satellite Attitude Control



### Example Problem C.5

For the satellite attitude control problem:

- Start with the linearized equations for the satellite attitude problem and use the Laplace transform to convert the equations of motion to the s-domain.
- Find the full transfer matrix from the input  $\tau(s)$  to the outputs  $\Phi(s)$  and  $\Theta(s)$ .
- From the transfer matrix, find the second-order transfer function from  $\Theta(s)$  to  $\Phi(s)$ .
- Under the assumption that the panel moment of inertia  $J_p$  is significantly smaller than the spacecraft moment of inertia  $J_s$  (specifically,  $(J_s + J_p)/J_s \approx 1$ ), find the second-order approximation for the transfer function from  $\tau(s)$  to  $\Theta(s)$ .

- (e) From your results on parts (c) and (d), form the approximate transfer function cascade for the satellite/panel system and justify why it makes sense physically.

### Solution

From HW C.3, the linearized equations of motion are given by

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} -\frac{b}{J_s}(\dot{\theta} - \dot{\phi}) - \frac{k}{J_s}(\theta - \phi) + \frac{1}{J_s}\tau \\ -\frac{b}{J_p}(\dot{\phi} - \dot{\theta}) - \frac{k}{J_p}(\phi - \theta) \end{pmatrix}$$

or in other words, the coupled differential equations

$$\begin{aligned} \ddot{\theta} + \frac{b}{J_s}\dot{\theta} + \frac{k}{J_s}\theta &= \frac{b}{J_s}\dot{\phi} + \frac{k}{J_s}\phi + \frac{1}{J_s}\tau \\ \ddot{\phi} + \frac{b}{J_p}\dot{\phi} + \frac{k}{J_p}\phi &= \frac{b}{J_p}\dot{\theta} + \frac{k}{J_p}\theta. \end{aligned}$$

Taking the Laplace transform with initial conditions set to zero and rearranging gives

$$(s^2 + \frac{b}{J_s}s + \frac{k}{J_s})\Theta(s) = (\frac{b}{J_s}s + \frac{k}{J_s})\Phi(s) + \frac{1}{J_s}\tau(s) \quad (5.3)$$

$$(s^2 + \frac{b}{J_p}s + \frac{k}{J_p})\Phi(s) = (\frac{b}{J_p}s + \frac{k}{J_p})\Theta(s). \quad (5.4)$$

To find the transfer matrix from  $\tau$  to  $(\Theta, \Phi)^\top$ , write Equation (5.3) and (5.4) in matrix form as

$$\left( \begin{array}{c|c} s^2 + \frac{b}{J_s}s + \frac{k}{J_s} & -\frac{b}{J_s}s - \frac{k}{J_s} \\ \hline -\frac{b}{J_p}s - \frac{k}{J_p} & s^2 + \frac{b}{J_p}s + \frac{k}{J_p} \end{array} \right) \begin{pmatrix} \Theta(s) \\ \Phi(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{J_s} \\ 0 \end{pmatrix} \tau(s),$$

and invert the matrix on the left hand side to obtain

$$\begin{pmatrix} \Theta(s) \\ \Phi(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{J_s}s^2 + \frac{b}{J_s}J_p s + \frac{k}{J_s}J_p \\ \frac{s^2 + \frac{b(J_s+J_p)}{J_s}s + \frac{k(J_s+J_p)}{J_s}}{s^2 + \frac{b(J_s+J_p)}{J_p}s + \frac{k(J_s+J_p)}{J_p}} \end{pmatrix} \tau(s).$$

By dividing the bottom transfer function of the transfer matrix  $\Phi(s)/\tau(s)$  by the top transfer function of the transfer matrix  $\Theta(s)/\tau(s)$ , we can find the transfer function from the satellite angular position to the panel angular position

$$\frac{\Phi(s)}{\Theta(s)} = \frac{\frac{b}{J_p}s + \frac{k}{J_p}}{s^2 + \frac{b}{J_p}s + \frac{k}{J_p}}. \quad (5.5)$$

From the transfer matrix, the transfer function from  $\tau(s)$  to  $\Theta(s)$  is given by

$$\frac{\Theta(s)}{\tau(s)} = \frac{\frac{1}{J_s}s^2 + \frac{b}{J_s J_p}s + \frac{k}{J_s J_p}}{s^2 \left( s^2 + \frac{b(J_s + J_p)}{J_s J_p}s + \frac{k(J_s + J_p)}{J_s J_p} \right)} \quad (5.6)$$

Under the assumption that the moment of inertia of the panel is significantly smaller than the inertia of the satellite (which is true for this problem), we can infer that  $(J_s + J_p)/J_s \approx 1$ . Taking this into account, we can simplify  $\Theta(s)/\tau(s)$  as

$$\begin{aligned} \frac{\Theta(s)}{\tau(s)} &= \frac{\frac{1}{J_s} \left( s^2 + \frac{b}{J_p}s + \frac{k}{J_p} \right)}{s^2 \left( \frac{J_s + J_p}{J_s} \right) \left( \frac{J_s}{J_s + J_p}s^2 + \frac{b}{J_p}s + \frac{k}{J_p} \right)} \\ &\approx \frac{1}{(J_s + J_p)s^2}. \end{aligned}$$

The block diagram for the approximate system is shown in Figure 5-3

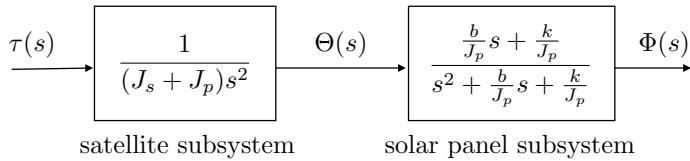


Figure 5-3: The satellite attitude dynamics are approximated by a cascade of the satellite and panel subsystems.

The cascade approximation has the exact transfer function for the panel subsystem, but it has an approximate transfer function for the satellite subsystem. The approximation implies that the dynamics of the satellite affects the dynamics of the panel, but the dynamics of the panel do not affect the dynamics of the satellite. Notice that the transfer function for the satellite has the dynamics of a rigid body with a moment of inertia of  $J_s + J_p$ . As long as the moment of inertia of the satellite is significantly greater than the moment of inertia of the panel, this assumption is reasonable and approximates the fully coupled dynamics of the system with acceptable accuracy.

**Important Concepts:**

- A transfer function captures the relationship between the input and output of the system in the Laplace domain.
- Transfer functions assume that all the initial conditions are set to zero.
- The roots of the transfer function's denominator polynomial are the poles of the dynamical system.
- The roots of the transfer function's numerator polynomial are the zeros of the dynamical system.
- For multiple-input and/or multiple-output systems, there will be a matrix of transfer functions representing the relationship between every input-output pairing.
- Some higher-order systems may be decoupled and represented as a cascade of two or more lower-order transfer functions.

**Notes and References**

The Laplace transform can be found in introductory textbooks on signals and systems like [16] and in introductory textbooks on differential equations. The transfer function of a linear time-invariant system is also covered in introductory textbooks on signals and systems, and in most introductory textbooks on control like [4].

# 6

## State Space Models

### Learning Objectives:

- Use Jacobian linearization to put dynamical systems into a state space model.
- Convert state space models to their transfer function representation.

## 6.1 Theory

The transfer function models discussed in the previous chapter are a frequency domain representation of the system because they describe how the system responds to sinusoidal inputs at specific frequencies. In contrast, differential equation models are time-domain representations of the systems that directly model the time-evolution of the system. The *state space* model is also a time-domain representation that is essentially a reformatting of the differential equations. In essence, the state space model represents the system by an input, an output, and a memory element, called the state. In this book, the input will always be denoted as  $u(t)$ , the output as  $y(t)$ , and the state as  $x(t)$ . The state represents all of the memory elements in the system. For example, the state may represent a storage register, the altitude of an aircraft, the velocity of a car, the voltage across a capacitor, or the current through an inductor.

The state space model is composed of two equations. The first equation is called the *state evolution equation* and represents how the memory elements, or states, change as a function of the current state and the inputs to the system. The second equation is called the *output equation* and describes how the current output of the system depends on the current state and the current input. The general state

space equations for a continuous time system are written as

$$\dot{x} = f(x, u) \quad (6.1)$$

$$y = h(x, u), \quad (6.2)$$

where Equation (6.1) is the state evolution equation and Equation (6.2) is the output equation. Note that for continuous time systems the state evolution equation is a system of coupled first order nonlinear differential equations. The general state space equations for a discrete time system are written as

$$x[k+1] = f(x[k], u[k]) \quad (6.3)$$

$$y[k] = h(x[k], u[k]), \quad (6.4)$$

where again Equation (6.3) is the state evolution equation, this time given by a system of coupled first order difference equations, and Equation (6.4) is the output equation.

Any nonlinear  $n^{th}$  order differential equation can be transformed into state space form. For example, consider the nonlinear differential equation

$$\ddot{y} + y^2 \dot{y} + \sin(y\dot{y}) + e^y = u,$$

where  $u$  is the input and  $y$  is the output. Define the state as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \triangleq \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix}.$$

Then the state evolution equation can be derived as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \ddot{y} \\ -y^2 \dot{y} - \sin(y\dot{y}) - e^y + u \end{pmatrix} \\ &= \begin{pmatrix} x_2 \\ x_3 \\ -x_1^2 x_3 - \sin(x_1 x_2) - e^{x_1} + u \end{pmatrix} \triangleq f(x, u), \end{aligned}$$

and the output equation is

$$y = x_1 \triangleq h(x, u).$$

As another example, consider the coupled differential equations

$$\begin{aligned} \ddot{y}_1 + \dot{y}_1 y_2 + y_2 &= u \\ \ddot{y}_2 + \dot{y}_2 \cos(y_1) + \dot{y}_1 &= 0. \end{aligned}$$

Defining the output as  $y = (y_1, y_2)^\top$  and the state as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \triangleq \begin{pmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix},$$

the state evolution equation can be derived as

$$\begin{aligned}\dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ -\dot{y}_1 y_2 - y_2 + u \\ -\dot{y}_2 \cos(y_1) - y_1 \end{pmatrix} \\ &= \begin{pmatrix} x_3 \\ x_4 \\ -x_3 x_2 - x_2 + u \\ -x_4 \cos(x_1) - x_3 \end{pmatrix} \triangleq f(x, u),\end{aligned}$$

and the output equation is given by

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \triangleq h(x, u).$$

### 6.1.1 Jacobian Linearization Revisited

Jacobian linearization of the system is more straightforward when the equations are in state space form. Given the nonlinear state space equations

$$\dot{x} = f(x, u) \quad (6.5)$$

$$y = h(x, u) \quad (6.6)$$

an equilibrium point is defined as any  $(x_e, u_e)$  such that  $f(x_e, u_e) = 0$ . In other words, equilibrium points are combinations of states and inputs where the state stops evolving in time. To linearize about an equilibrium point, Equations (6.5) and (6.6) can be expanded in a Taylor series about  $(x_e, u_e)$  as

$$f(x, u) \approx f(x_e, u_e) + \left. \frac{\partial f}{\partial x} \right|_e (x - x_e) + \left. \frac{\partial f}{\partial u} \right|_e (u - u_e) + H.O.T. \quad (6.7)$$

$$h(x, u) \approx h(x_e, u_e) + \left. \frac{\partial h}{\partial x} \right|_e (x - x_e) + \left. \frac{\partial h}{\partial u} \right|_e (u - u_e) + H.O.T., \quad (6.8)$$

where the Jacobian matrices are defined as

$$\begin{aligned}\frac{\partial f}{\partial x} &\triangleq \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \\ \frac{\partial f}{\partial u} &\triangleq \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{pmatrix} \\ \frac{\partial h}{\partial x} &\triangleq \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{pmatrix} \\ \frac{\partial h}{\partial u} &\triangleq \begin{pmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{pmatrix}.\end{aligned}$$

Therefore  $\frac{\partial f}{\partial x}$  is an  $n \times n$  matrix,  $\frac{\partial f}{\partial u}$  is an  $n \times m$  matrix,  $\frac{\partial h}{\partial x}$  is a  $p \times n$  matrix, and  $\frac{\partial h}{\partial u}$  is a  $p \times m$  matrix.

Defining  $\tilde{x} \triangleq x - x_e$  and  $\tilde{u} = u - u_e$  and letting  $A = \left. \frac{\partial f}{\partial x} \right|_e$  and  $B = \left. \frac{\partial f}{\partial u} \right|_e$ , and noting that  $\dot{\tilde{x}} = \dot{x} - \dot{x}_e = \dot{x}$ , results in the linearized state evolution equation

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}.$$

At the equilibria, the output may not necessarily be zeros. If we define the equilibrium output to be  $y_e = h(x_e, u_e)$  and the linearized output as  $\tilde{y} = y - y_e$ , and define  $C \triangleq \left. \frac{\partial h}{\partial x} \right|_e$  and  $D \triangleq \left. \frac{\partial h}{\partial u} \right|_e$ , then using Equation (6.8) in (6.6) gives

$$\begin{aligned}\tilde{y} &= y - y_e \\ &\approx h(x_e, u_e) + C\tilde{x} + D\tilde{u} - h(x_e, u_e) \\ &= C\tilde{x} + D\tilde{u}.\end{aligned}$$

The linearized state space equations are therefore

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y} &= C\tilde{x} + D\tilde{u}.\end{aligned}$$

### 6.1.2 Converting State Space Models to Transfer Function Models

Linear state space models can be converted to transfer function models using the following technique. Suppose that the linear state space model is given by

$$\dot{x} = Ax + Bu \quad (6.9)$$

$$y = Cx + Du. \quad (6.10)$$

Defining the Laplace transform of a vector to be the Laplace transform of each element, i.e.,

$$\mathcal{L} \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\} = \begin{pmatrix} \mathcal{L}\{x_1\} \\ \vdots \\ \mathcal{L}\{x_n\} \end{pmatrix},$$

and noting that the Laplace transform is a linear operator, which implies that

$$\mathcal{L}\{Ax\} = A\mathcal{L}\{x\},$$

taking the Laplace transform of Equations (6.9) and (6.10), and setting the initial condition to zeros, gives

$$sX(s) = AX(s) + BU(s) \quad (6.11)$$

$$Y(s) = CX(s) + DU(s). \quad (6.12)$$

Solving (6.11) for  $X(s)$  gives

$$\begin{aligned} sX(s) - AX(s) &= BU(s) \\ \implies (sI - A)X(s) &= BU(s) \\ \implies X(s) &= (sI - A)^{-1}BU(s), \end{aligned} \quad (6.13)$$

where  $I$  is the  $n \times n$  identity matrix. Substituting (6.13) into (6.12) gives

$$\begin{aligned} Y(s) &= C(sI - A)^{-1}BU(s) + DU(s) \\ \implies Y(s) &= [C(sI - A)^{-1}B + D]U(s). \end{aligned}$$

The transfer function from  $u$  to  $y$  is therefore

$$H(s) = C(sI - A)^{-1}B + D. \quad (6.14)$$

As an example, suppose that the state space equations are given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -4 & -3 \end{pmatrix}x + \begin{pmatrix} 0 \\ 2 \end{pmatrix}u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix}x + (0)u, \end{aligned}$$

then the transfer function from  $u$  to  $y$  is given by

$$\begin{aligned}
 H(s) &= C(sI - A)^{-1}B + D \\
 &= (1 \ 0) \left( s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -4 & -3 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 0 \\
 &= (1 \ 0) \begin{pmatrix} s & -1 \\ 4 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\
 &= (1 \ 0) \frac{\begin{pmatrix} s+3 & 1 \\ -4 & s \end{pmatrix}}{s^2 + 3s + 4} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\
 &= \frac{(s+3 \ 1)}{s^2 + 3s + 4} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\
 &= \frac{2}{s^2 + 3s + 4}.
 \end{aligned}$$

The form of the transfer function in Equation (6.14) allows us to derive a simple relationship between the poles of the transfer function  $H(s)$  and the eigenvalues of  $A$ . Recall (see Appendix P.7) that for any square invertible matrix  $M$ , we have that

$$M^{-1} = \frac{\text{adj}(M)}{\det(M)},$$

where  $\text{adj}(M)$  is the adjugate of  $M$  defined as the transpose of the cofactors of  $M$ , and  $\det(M)$  is the determinant of  $M$ . Therefore

$$\begin{aligned}
 H(s) &= C(sI - A)^{-1}B + D \\
 &= \frac{C\text{adj}(sI - A)B}{\det(sI - A)} + D \\
 &= \frac{C\text{adj}(sI - A)B + D\det(sI - A)}{\det(sI - A)}.
 \end{aligned}$$

This expression implies that the zeros of  $H(s)$  are given by the roots of the polynomial

$$C\text{adj}(sI - A)B + D\det(sI - A) = 0,$$

and that the poles of  $H(s)$  are given by the roots of the polynomial

$$\det(sI - A) = 0. \tag{6.15}$$

Recall from linear algebra (see Appendix P.7) that Equation (6.15) also defines the eigenvalues of  $A$ . Therefore, the poles of  $H(s)$  are equivalent to the eigenvalues of  $A$ .

In Chapters 11 and 13 we will show two general techniques for converting SISO transfer function models into state space models.

## 6.2 Design Study A. Single Link Robot Arm



### Example Problem A.6

For the feedback linearized equations of motion for the single link robot arm given in Equation (4.7), define the states as  $x = (\theta, \dot{\theta})^\top$ , the input as  $\tilde{u} = \tilde{\tau}$ , and the measured output as  $y = \theta$ . Find the linear state space equations in the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du.\end{aligned}$$

### Solution

The linear state space equations can be derived in two different ways: (1) directly from the linearized equations of motion and (2) by linearizing the nonlinear equations of motion.

Starting with the feedback linearized equation in Equation (4.7) and solving for  $\ddot{\theta}$  gives

$$\ddot{\theta} = \frac{3}{m\ell^2} \tilde{\tau} - \frac{3b}{m\ell^2} \dot{\theta}.$$

Therefore,

$$\begin{aligned}\dot{x} &\triangleq \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \frac{3}{m\ell^2} \tilde{\tau} - \frac{3b}{m\ell^2} \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{x}_2 \\ \frac{3}{m\ell^2} \tilde{u} - \frac{3b}{m\ell^2} x_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & -\frac{3b}{m\ell^2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \tilde{u}.\end{aligned}$$

Assuming that the measured output of the system is  $y = \theta$ , the linearized output is given by

$$y = \theta = x_1 = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (0) \tilde{u}.$$

Therefore, the linearized state space equations are given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -\frac{3b}{m\ell^2} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \tilde{u} \\ y &= (1 \quad 0) x.\end{aligned}\tag{6.16}$$

Alternatively, the state space equations can be found directly from the nonlinear equations of motion given in Equation (3.1), by forming the nonlinear state space model as

$$\dot{x} \triangleq \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \left( \frac{3}{m\ell^2} \tau - \frac{3b}{m\ell^2} \dot{\theta} - \frac{3g}{2\ell} \cos \theta \right) \triangleq f(x, u).$$

Evaluating the Jacobians at the equilibrium  $(\theta_e, \dot{\theta}_e, \tau_e) = (0, 0, \frac{mg\ell}{2})$  gives

$$\begin{aligned} A &= \frac{\partial f}{\partial x} \Big|_e = \begin{pmatrix} \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \dot{\theta}} \\ \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \dot{\theta}} \end{pmatrix} \Big|_e = \begin{pmatrix} 0 & 1 \\ \frac{3g}{2\ell} \sin \theta & -\frac{3b}{m\ell^2} \end{pmatrix} \Big|_e \\ &= \begin{pmatrix} 0 & 1 \\ 0 & -\frac{3b}{m\ell^2} \end{pmatrix} \\ B &= \frac{\partial f}{\partial u} \Big|_e = \begin{pmatrix} \frac{\partial f_1}{\partial \tau} \\ \frac{\partial f_2}{\partial \tau} \end{pmatrix} \Big|_e = \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \Big|_e \\ &= \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix}. \end{aligned}$$

Similarly, the output is given by  $y = \theta = x_1 \stackrel{\triangle}{=} h(x, u)$ , which implies that

$$\begin{aligned} C &= \frac{\partial h}{\partial x} \Big|_e = \begin{pmatrix} \frac{\partial h_1}{\partial \theta} & \frac{\partial h_1}{\partial \dot{\theta}} \end{pmatrix} \Big|_e = (1 \ 0) \Big|_e \\ &= (1 \ 0) \\ D &= \frac{\partial h}{\partial u} \Big|_e = 0. \end{aligned}$$

The linearized state space equations are therefore given by

$$\begin{aligned} \dot{\tilde{x}} &= \begin{pmatrix} 0 & 1 \\ \frac{3g \sin \theta_e}{2\ell} & -\frac{3b}{m\ell^2} \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \tilde{u} \\ \tilde{y} &= (1 \ 0) \tilde{x}, \end{aligned}$$

which is similar to Equation (6.16) but with linearized state and output.

The transfer function can be found from the linearized state space equations using Equation (6.14) as

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B + D \\ &= (1 \ 0) \left( s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -\frac{3b}{m\ell^2} \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \\ &= (1 \ 0) \begin{pmatrix} s & -1 \\ 0 & s + \frac{3b}{m\ell^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \\ &= \frac{(1 \ 0) \begin{pmatrix} s + \frac{3b}{m\ell^2} & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix}}{s^2 + \frac{3b}{m\ell^2}s} \\ &= \frac{\left( s + \frac{3b}{m\ell^2} \ 1 \right) \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix}}{s^2 + \frac{3b}{m\ell^2}s} \\ &= \frac{\frac{3}{m\ell^2}}{s^2 + \frac{3b}{m\ell^2}s}, \end{aligned}$$

which is identical to Equation (5.2).

### 6.3 Design Study B. Inverted Pendulum



#### Example Problem B.6

Defining the states as  $\tilde{x} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$ , the input as  $\tilde{u} = \tilde{F}$ , and the measured output as  $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$ , find the linear state space equations in the form

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

$$\tilde{y} = C\tilde{x} + D\tilde{u}.$$

#### Solution

The linear state space equations can be derived in two different ways: (1) directly from the linearized equations of motion and (2) by linearizing the nonlinear equations of motion.

Starting with the linear state space equation in Equation (4.12) and solving for  $(\ddot{\tilde{z}}, \ddot{\tilde{\theta}})^\top$  gives

$$\begin{aligned} \begin{pmatrix} \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} &= \frac{\begin{pmatrix} m_1 \frac{\ell^2}{3} & -m_1 \frac{\ell}{2} \\ -m_1 \frac{\ell}{2} & (m_1 + m_2) \end{pmatrix}}{m_1^2 \frac{\ell^2}{3} + m_1 m_2 \frac{\ell^2}{3} - m_1^2 \frac{\ell^2}{4}} \begin{pmatrix} -b\dot{\tilde{z}} + \tilde{F} \\ m_1 g \frac{\ell}{2} \tilde{\theta} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{b}{\frac{1}{4}m_1+m_2}\dot{\tilde{z}} + \frac{1}{\frac{1}{4}m_1+m_2}\tilde{F} - \frac{\frac{3}{4}m_1g}{\frac{1}{4}m_1+m_2}\tilde{\theta} \\ \frac{3b}{2(\frac{1}{4}m_1+m_2)\ell}\dot{\tilde{z}} - \frac{3}{2(\frac{1}{4}m_1+m_2)\ell}\tilde{F} + \frac{3(m_1+m_2)g}{2(\frac{1}{4}m_1+m_2)\ell}\tilde{\theta} \end{pmatrix}. \end{aligned}$$

Therefore, defining  $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)^\top \triangleq (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$ , and  $\tilde{u} \triangleq \tilde{F}$  gives

$$\begin{aligned} \dot{\tilde{x}} &\triangleq \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{pmatrix} = \begin{pmatrix} \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \\ \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} \tilde{x}_3 \\ \tilde{x}_4 \\ -\frac{b}{\frac{1}{4}m_1+m_2}\tilde{x}_3 + \frac{1}{\frac{1}{4}m_1+m_2}\tilde{u} - \frac{\frac{3}{4}m_1g}{\frac{1}{4}m_1+m_2}\tilde{x}_2 \\ \frac{3b}{2(\frac{1}{4}m_1+m_2)\ell}\tilde{x}_3 - \frac{3}{2(\frac{1}{4}m_1+m_2)\ell}\tilde{u} + \frac{3(m_1+m_2)g}{2(\frac{1}{4}m_1+m_2)\ell}\tilde{x}_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\frac{3}{4}m_1g}{\frac{1}{4}m_1+m_2} & -\frac{b}{\frac{1}{4}m_1+m_2} & 0 \\ 0 & \frac{3(m_1+m_2)g}{2(\frac{1}{4}m_1+m_2)\ell} & \frac{3b}{2(\frac{1}{4}m_1+m_2)\ell} & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\frac{1}{4}m_1+m_2} \\ \frac{-3}{2(\frac{1}{4}m_1+m_2)\ell} \end{pmatrix} \tilde{u}. \end{aligned}$$

Assuming that the measured output of the system is  $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$ , the linearized output is given by

$$\tilde{y} = \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tilde{u}.$$

Therefore, the linearized state space equations are given by

$$\begin{aligned}\dot{\tilde{x}} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\frac{3}{4}m_1g}{\frac{1}{4}m_1+m_2} & -\frac{b}{\frac{1}{4}m_1+m_2} & 0 \\ 0 & \frac{3(m_1+m_2)g}{2(\frac{1}{4}m_1+m_2)\ell} & \frac{3b}{2(\frac{1}{4}m_1+m_2)\ell} & 0 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\frac{1}{4}m_1+m_2} \\ \frac{-3}{2(\frac{1}{4}m_1+m_2)\ell} \end{pmatrix} \tilde{u} \\ \tilde{y} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tilde{x}.\end{aligned}\quad (6.17)$$

Alternatively, the linearized state space equations can be derived from the nonlinear equations of motion. Starting with Equation (3.2) and solving for  $(\ddot{z}, \ddot{\theta})^\top$  gives

$$\begin{aligned}\begin{pmatrix} \ddot{z} \\ \ddot{\theta} \end{pmatrix} &= \frac{\begin{pmatrix} m_1 \frac{\ell^2}{3} & -m_1 \frac{\ell}{2} \cos \theta \\ -m_1 \frac{\ell}{2} \cos \theta & (m_1 + m_2) \end{pmatrix}}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 \theta)} \begin{pmatrix} m_1 \frac{\ell}{2} \dot{\theta}^2 \sin \theta - b \dot{z} + F \\ m_1 g \ell \sin \theta \end{pmatrix} \\ &= \frac{\begin{pmatrix} m_1^2 \frac{\ell^3}{6} \dot{\theta}^2 \sin \theta - b m_1 \frac{\ell^2}{3} \dot{z} + m_1 \frac{\ell^2}{3} F - m_1^2 \frac{\ell^2}{4} g \sin \theta \cos \theta \\ -m_1^2 \frac{\ell^2}{4} \dot{\theta}^2 \sin \theta \cos \theta + b m_1 \frac{\ell}{2} \dot{z} \cos \theta - m_1 \frac{\ell}{2} \cos \theta F + (m_1 + m_2) m_1 g \frac{\ell}{2} \sin \theta \end{pmatrix}}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 \theta)}.\end{aligned}$$

Defining  $x \triangleq (z, \theta, \dot{z}, \dot{\theta})^\top$ ,  $u \triangleq F$ , and  $y \triangleq (z, \theta)^\top$  results in the nonlinear state space equations

$$\begin{aligned}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} &= \begin{pmatrix} x_3 \\ x_4 \\ \frac{m_1^2 \frac{\ell^3}{6} x_4^2 \sin x_2 - b m_1 \frac{\ell^2}{3} x_3 + m_1 \frac{\ell^2}{3} u - m_1^2 \frac{\ell^2}{4} g \sin x_2 \cos x_2}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)} \\ \frac{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)} \end{pmatrix} \\ &\triangleq f(x, u) \\ y &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \triangleq h(x, u).\end{aligned}$$

Taking the Jacobians about the equilibrium  $x_e = (z_e, 0, 0, 0)^\top$  and  $u_e = 0$  gives

$$\begin{aligned}
 A &\triangleq \frac{\partial f}{\partial x}\Big|_e \\
 &= \left( \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\partial f_3}{\partial x_2} & \frac{-bm_1 \frac{\ell^2}{3}}{m_1 m_2 \frac{\ell^2}{4} + m_1 \frac{\ell^2}{4} \sin^2 x_2} & \frac{2m_1^2 \frac{\ell^3}{6} x_4 \sin x_2}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)} \\ 0 & \frac{\partial f_4}{\partial x_2} & \frac{bm_1 \frac{\ell}{2} \cos x_2}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)} & \frac{-2m_1^2 \frac{\ell^2}{4} x_4 \sin x_2 \cos x_2}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)} \end{array} \right)\Big|_e \\
 &= \left( \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{3}{4}m_1 g & -\frac{b}{\frac{1}{4}m_1 + m_2} & 0 \\ 0 & \frac{3(m_1 + m_2)g}{2(\frac{1}{4}m_1 + m_2)\ell} & \frac{3b}{2(\frac{1}{4}m_1 + m_2)\ell} & 0 \end{array} \right) \\
 B &\triangleq \frac{\partial f}{\partial u}\Big|_e = \left( \begin{array}{c} 0 \\ 0 \\ \frac{m_1 \frac{\ell^2}{3}}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)} \\ -\frac{m_1 \frac{\ell}{2} \cos x_2}{m_1 m_2 \frac{\ell^2}{3} + m_1^2 \ell^2 (\frac{1}{3} - \frac{1}{4} \cos^2 x_2)} \end{array} \right)\Big|_e = \left( \begin{array}{c} 0 \\ 0 \\ \frac{1}{\frac{1}{4}m_1 + m_2} \\ \frac{-3}{2(\frac{1}{4}m_1 + m_2)\ell} \end{array} \right) \\
 C &\triangleq \frac{\partial h}{\partial x}\Big|_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
 D &\triangleq \frac{\partial h}{\partial u}\Big|_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
 \end{aligned}$$

resulting in the linearized state space equations given in Equation (6.17).

## 6.4 Design Study C. Satellite Attitude Control



### Example Problem C.6

Suppose that a star tracker is used to measure  $\theta$  and a strain gauge is used to approximate  $\dot{\phi} - \theta$ . Defining the states as  $x = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$ , the input as  $u = \tau$ , and the measured output as  $y = (\theta, \phi - \theta)^\top$ , find the linear state space equations in the form

$$\begin{aligned}
 \dot{x} &= Ax + Bu \\
 y &= Cx + Du.
 \end{aligned}$$

## Solution

The equations of motion from HW C.3 are given by

$$\begin{aligned}\ddot{\theta} &= \frac{1}{J_s}\tau - \frac{b}{J_s}\dot{\theta} + \frac{b}{J_s}\dot{\phi} - \frac{k}{J_s}\theta + \frac{k}{J_s}\phi \\ \ddot{\phi} &= \frac{b}{J_p}\dot{\theta} - \frac{b}{J_p}\dot{\phi} + \frac{k}{J_p}\theta - \frac{k}{J_p}\phi.\end{aligned}$$

Defining the state  $x \triangleq (x_1, x_2, x_3, x_4)^\top = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$ , the input  $u \triangleq \tau$ , and the output  $y \triangleq (y_1, y_2)^\top = (\theta, \phi - \theta)^\top$ , the state space equations are given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ \frac{1}{J_s}u - \frac{b}{J_s}x_3 + \frac{b}{J_s}x_4 - \frac{k}{J_s}x_1 + \frac{k}{J_s}x_2 \\ \frac{b}{J_p}x_3 - \frac{b}{J_p}x_4 + \frac{k}{J_p}x_1 - \frac{k}{J_p}x_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{J_s} & \frac{k}{J_s} & -\frac{b}{J_s} & \frac{b}{J_s} \\ \frac{k}{J_p} & -\frac{k}{J_p} & \frac{b}{J_p} & -\frac{b}{J_p} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J_s} \\ 0 \end{pmatrix} u.\end{aligned}$$

Similarly

$$\begin{aligned}y &= \begin{pmatrix} \theta \\ \phi - \theta \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - x_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u.\end{aligned}$$

Therefore, the state space model is

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du,\end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{J_s} & \frac{k}{J_s} & -\frac{b}{J_s} & \frac{b}{J_s} \\ \frac{k}{J_p} & -\frac{k}{J_p} & \frac{b}{J_p} & -\frac{b}{J_p} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J_s} \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

### Important Concepts:

- State space models are composed of the state evolution equation and an output equation.
- The state evolution equation describes how the states change over time as a function of the current state and the input.
- The output equation describes how the variables that can be seen or measured are related to the states and input.
- Any  $n^{th}$  order differential equation can be written as  $n$  first order state evolution equations.
- Jacobian linearization will linearize nonlinear state space equations.
- The transfer function of a state space realization is  $H(s) = C(sI - A)^{-1}B + D$ .

### Notes and References

There are numerous resources that derive state space models from differential equations and transfer functions. For example [4, 17, 6, 5].

# Part III

## PID Control Design

In this part of the book we will introduce feedback control by studying the most commonly used industrial control law, the so-called proportional-integral-derivative control, or PID control. PID controllers are widely used because they are intuitive and easy to understand, they can be designed without any understanding of the underlying physics of the system, and a model of the system is not required, which is especially important in the chemical processing industry. However, the stability and performance of the closed-loop system can only be guaranteed when using PID control on second order systems. In Part IV on observer-based control and Part V on loopshaping, we will extend the basic ideas covered in this Part to higher-order systems.

**Fig. 6-1** shows a basic proportional control strategy. The output  $y$  of the physical system is subtracted from the commanded reference output  $y_r$  to produce the error  $e = y_r - y$ . Proportional control determines the input to the system  $u$  to be proportional to the error, i.e.,  $u = k_P e$ . The constant gain  $k_P$  is called the proportional gain. Proportional control can be augmented with integral control

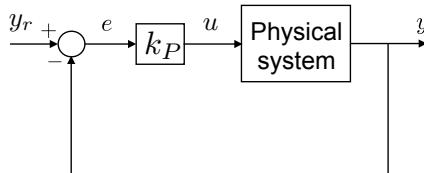


Figure 6-1: Proportional Control. The feedback control is a gain  $k_p$  multiplied by the error.

to produce PI control, as shown in **Fig. 6-2**. The basic idea is to integrate the error so that the system responds to error accrued in the past. The constant gain  $k_I$  is called the integral gain. Alternatively, proportional control can be augmented with a derivative control as shown in **Fig. 6-3**. The idea is to respond to how fast

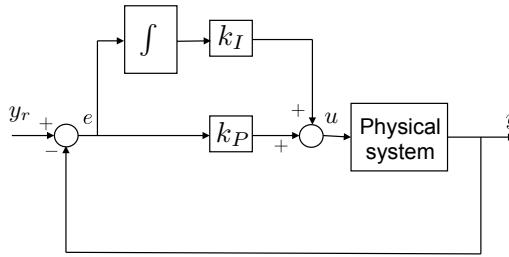


Figure 6-2: Proportional plus Integral (PI) Control. The feedback control is a linear combination of proportional and integral control.

the error is changing, or the derivative of the error. The constant gain  $k_D$  is called the derivative gain. When all three elements are used, as shown in Fig. 6-4, the

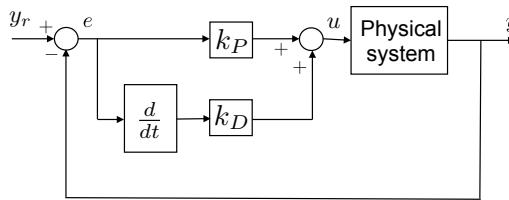


Figure 6-3: Proportional plus Derivative (PD) Control. The feedback control is a linear combination of proportional and derivative control.

resulting controller is called a PID control. In Part III of this book, we will explore various aspects of PID control. Many of these concepts can be generalized to other control schemes in later chapters, and so our study of PID control will also help to motivate and understand more advanced concepts.

Intuition for PID control can be gained by considering the step response shown in Fig. 6-5. At the beginning of the response, there is significant error between the current output  $y(t)$  and the reference command  $y_r(t)$ . The proportional control term will tend to push the system so as to reduce the gap between  $y(t)$  and  $y_r(t)$ . As  $y(t)$  begins to change, a fast response may lead to significant overshoot due to the momentum in the system. Therefore, if it is moving too quickly, we may want to slow down the response, or if it is moving too slowly, we may want to speed up the response. Derivative control is used to affect these changes. After the response has settled into steady state, there may be significant steady state error, as shown in Fig. 6-5. Integral control adds up this steady error and will eventually act to reduce the steady state error. A good example is an aircraft in a constant crosswind. An integrator can be used to correct the response of the aircraft so that it crabs into the wind to maintain its desired flight path.

Part III of the book is organized as follows. In Chapter 7 we will show how PD control can be used to specify the desired closed-loop poles of a second order system. In Chapter 8 we introduce the concepts of rise time, natural frequency,

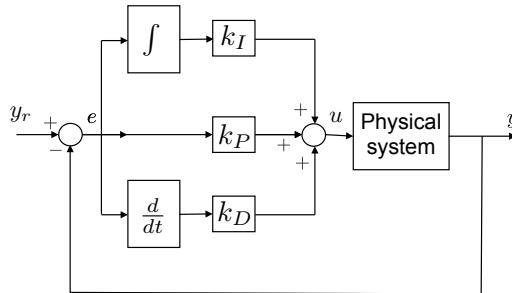


Figure 6-4: Proportional plus Integral plus Derivative (PID) Control. The feedback control is a linear combination of proportional and integral and derivative control.

and damping ratio for second order systems and show how these concepts relate to the location of the closed-loop poles. We also show how these quantities can be used to design PD controllers for second order systems and cascades of second order systems. In Chapter 9 we use the final value theorem to show the effect of integrators on the closed-loop system. We introduce the notion of *system type* in the context of reference tracking and disturbance rejection. Finally, in Chapter 10 we show how to write computer code to implement PID controllers.

### Important Concepts:

- PID controllers are widely used for a variety of applications. They act on the error between the reference input and the system's output.
- A proportional controller sets the input to the system to be proportional to the error.
- Integral control corrects for accrued, or integrated, error in the past.
- Derivative control changes the input based upon the error's slope.
- Combinations of control types, such as proportional (P), proportional-integral (PI), proportional-derivative (PD), and proportional-integral-derivative (PID) are used in feedback control systems.

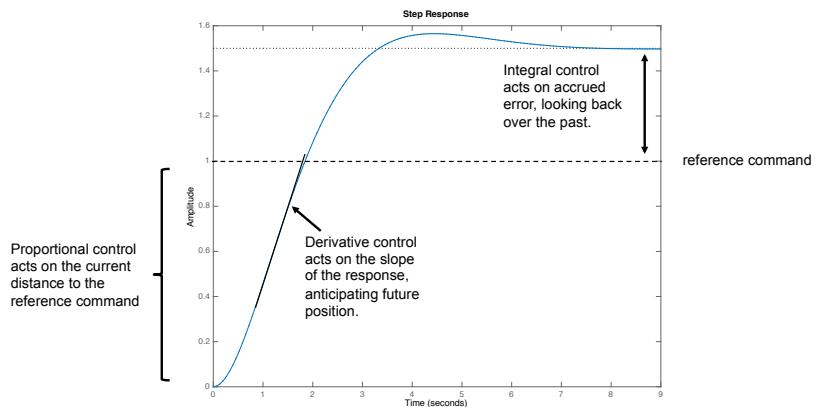


Figure 6-5: Intuitive interpretation of PID control. Proportional control acts on the current error, integral control acts on accrued past error, and derivative control acts on anticipated future error.



## Pole Placement for Second Order Systems

### Learning Objectives:

- Gain intuition on how PD control works.
- Choose proportional and derivative gains that will place the closed-loop poles of a 2nd order system at user desired locations.

## 7.1 Theory

Suppose that the model for the linearized physical system is given by the second order transfer function

$$P(s) = \frac{b_0}{s^2 + a_1 s + a_0}. \quad (7.1)$$

The *open loop poles* of the plant  $P(s)$  are defined to be the roots of the *open loop characteristic polynomial*

$$\Delta_{ol}(s) = s^2 + a_1 s + a_0,$$

which are given by

$$p_{ol} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}.$$

Note that the open loop poles are determined by the physical parameters and that the poles may be in the right half plane. If we use PD control to regulate the output  $y$  to the reference command  $y_r$ , then the block diagram is shown in [Fig. 7-1](#). The transfer function for the closed-loop system can be derived by noting from the

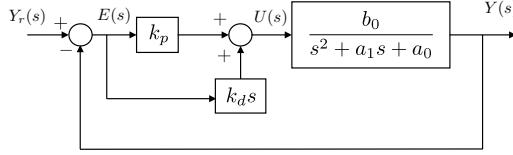


Figure 7-1: PD control of a second order system.

block diagram that

$$\begin{aligned} Y(s) &= \left( \frac{b_0}{s^2 + a_1 s + a_0} \right) (k_P + k_D s) (Y_r(s) - Y(s)) \\ \implies (s^2 + a_1 s + a_0) Y(s) &= (b_0 k_P + b_0 k_D s) (Y_r(s) - Y(s)) \\ \implies [(s^2 + a_1 s + a_0) + (b_0 k_P + b_0 k_D s)] Y(s) &= (7.2) \\ &= (b_0 k_P + b_0 k_D s) Y_r(s) \end{aligned}$$

$$\begin{aligned} \implies (s^2 + (a_1 + b_0 k_D) s + (a_0 + b_0 k_P)) Y(s) &= (7.3) \\ &= (b_0 k_P + b_0 k_D s) Y_r(s) \end{aligned}$$

$$\implies Y(s) = \frac{b_0 k_D s + b_0 k_P}{s^2 + (a_1 + b_0 k_D) s + (a_0 + b_0 k_P)} Y_r(s). \quad (7.4)$$

Equation (7.4) constitutes the closed-loop transfer function for the system under the influence of PD control. The *closed-loop poles* are given by the roots of the *closed-loop characteristic polynomial*

$$\Delta_{cl}(s) = s^2 + (a_1 + b_0 k_D) s + (a_0 + b_0 k_P),$$

which are

$$p_{cl} = -\frac{a_1 + b_0 k_D}{2} \pm \sqrt{\left( \frac{(a_1 + b_0 k_D)}{2} \right)^2 - (a_0 + b_0 k_P)}.$$

Note that the closed-loop poles can be specified by the designer by selecting  $k_P$  and  $k_D$ . The basic idea is to select desired closed-loop poles  $-p_1^d$  and  $-p_2^d$  and then to form the desired characteristic polynomial

$$\Delta_{cl}^d(s) = (s + p_1^d)(s + p_2^d) \stackrel{\triangle}{=} s^2 + \alpha_1 s + \alpha_0.$$

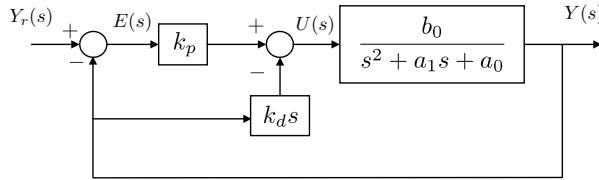
By setting  $\Delta_{cl}(s) = \Delta_{cl}^d(s)$  we can equate the leading coefficients of each term of the polynomial and solve for the gains  $k_P$  and  $k_D$  as

$$\begin{aligned} k_P &= \frac{\alpha_0 - a_0}{b_0} \\ k_D &= \frac{\alpha_1 - a_1}{b_0}. \end{aligned}$$

One of the disadvantages to the PD architecture shown in [Fig. 7-1](#) is the introduction of a zero in the closed-loop transfer function. Note that while the open loop system [\(7.1\)](#) does not have a zero, the closed-loop system [\(7.4\)](#) has a zero at

$$z_{cl} = -\frac{k_P}{k_D}.$$

The presence of the zero will modify the closed-loop response of the system. A simple trick that removes the zero is to use the control structure shown in [Fig. 7-2](#). In this particular case, the differentiator acts only on the output  $y$  and not on the error  $e$ . Notice the change of sign on the signal coming from the differentiator. In



[Figure 7-2](#): PD control of a second order system, where the derivative control only differentiates the output  $y$  and not the error  $e$ .

in this case the closed-loop transfer function is calculated as

$$\begin{aligned} Y(s) &= \left( \frac{b_0}{s^2 + a_1 s + a_0} \right) (k_P(Y_r(s) - Y(s)) - k_D s Y(s)) \\ &\Rightarrow (s^2 + a_1 s + a_0)Y(s) = (b_0 k_P(Y_r(s) - Y(s)) - b_0 k_D s Y(s)) \\ &\Rightarrow [(s^2 + a_1 s + a_0) + (b_0 k_P + b_0 k_D s)] Y(s) = b_0 k_P Y_r(s) \\ &\Rightarrow (s^2 + (a_1 + b_0 k_D)s + (a_0 + b_0 k_P))Y(s) = b_0 k_P Y_r(s) \\ &\Rightarrow Y(s) = \frac{b_0 k_P}{s^2 + (a_1 + b_0 k_D)s + (a_0 + b_0 k_P)} Y_r(s). \end{aligned} \quad (7.5)$$

Note that the closed-loop transfer function [\(7.5\)](#) has the same poles as Equation [\(7.4\)](#), but that the zero has been removed. An added benefit to the structure shown in [Fig. 7-2](#) is that when  $y_r(t)$  is a step input, the derivative can introduce a large signal spike in  $u(t)$ . These large spikes are removed by only differentiating  $y(t)$  instead of  $e(t) = y_r(t) - y(t)$ .

## 7.2 Design Study A. Single Link Robot Arm



### Example Problem A.7

- (a) Given the open loop transfer function found in problem [A.6](#), find the open loop poles of the system, when the equilibrium angle is  $\theta_e = 0$ .

- (b) Using the PD control architecture shown in [Fig. 7-2](#), find the closed loop transfer function from  $\theta_r$  to  $\theta$  and find the closed loop poles as a function of  $k_P$  and  $k_D$ .
- (c) Select  $k_P$  and  $k_D$  to place the closed loop poles at  $p_1 = -3$  and  $p_2 = -4$ .
- (d) Using the gains from part (c), implement the PD control for the single link robot arm in simulation and plot the step response.

### Solution

The open loop transfer function from homework A.6 is

$$\Theta(s) = \left( \frac{\frac{3}{m\ell^2}}{s^2 + \frac{3b}{m\ell^2}s} \right) \tilde{\tau}(s).$$

Using the parameters from Section A gives

$$\Theta(s) = \left( \frac{66.67}{s^2 + 0.667s} \right) \tilde{\tau}(s).$$

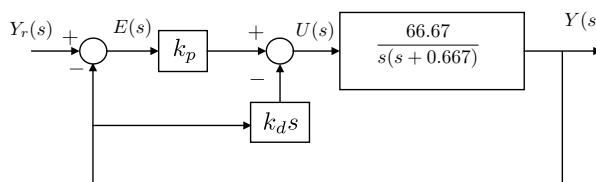
The open loop poles are therefore the roots of the open loop polynomial

$$\Delta_{ol}(s) = s^2 + 0.667s,$$

which are given by

$$p_{ol} = 0, -0.667.$$

Using PD control, the closed loop system is therefore shown in [Fig. 7-3](#). The



[Figure 7-3](#): PD control of the single link robot arm.

transfer function of the closed loop system is given by

$$\begin{aligned} \Theta(s) &= \left( \frac{66.67}{s^2 + 0.667s} \right) (k_P(\Theta^c(s) - \Theta(s)) - k_D s \Theta(s)) \\ \implies (s^2 + 0.667s)\Theta(s) &= (66.67k_P(\Theta^c(s) - \Theta(s)) - 66.67k_D s \Theta(s)) \\ \implies (s^2 + (0.667 + 66.67k_D)s + 66.67k_P)\Theta(s) &= 66.67k_P\Theta^c(s) \\ \implies \Theta(s) &= \frac{66.67k_P}{s^2 + (0.667 + 66.67k_D)s + 66.67k_P}\Theta^c(s). \end{aligned}$$

Therefore, the closed loop poles are given by the roots of the closed loop characteristic polynomial

$$\Delta_{cl}(s) = s^2 + (0.667 + 66.67k_D)s + 66.67k_P$$

which are given by

$$p_{cl} = -\frac{(0.667 + 66.67k_D)}{2} \pm \sqrt{\left(\frac{(0.667 + 66.67k_D)}{2}\right)^2 - 66.67k_P}$$

If the desired closed loop poles are at  $-3$  and  $-4$ , then the desired closed loop characteristic polynomial is

$$\begin{aligned}\Delta_{cl}^d &= (s + 3)(s + 4) \\ &= s^2 + 7s + 12.\end{aligned}$$

Equating the actual closed loop characteristic polynomial  $\Delta_{cl}$  with the desired characteristic polynomial  $\Delta_{cl}^d$  gives

$$s^2 + (0.667 + 66.67k_D)s + 66.67k_P = s^2 + 7s + 12,$$

or by equating each term we get

$$\begin{aligned}0.667 + 66.67k_D &= 7 \\ 66.67k_P &= 12.\end{aligned}$$

Solving for  $k_P$  and  $k_D$  gives

$$\begin{aligned}k_P &= 0.18 \\ k_D &= 0.095.\end{aligned}$$

A Python class that implements a PD controller for the single link robot arm is shown below.

```

1 import numpy as np
2 import armParamHW7 as P
3 import sys
4 sys.path.append('..') # add parent directory
5 import armParam as P0
6
7 class armController:
8     def __init__(self):
9
10         # Instantiates the PD object
11         self.kp = P.kp
12         self.kd = P.kd
13         self.limit = P0.tau_max
14
15     def update(self, theta_r, x):
16         theta = x.item(0)

```

```

17     thetadot = x.item(1)
18
19     # feedback linearized torque
20     tau_fl = P0.m * P0.g * (P0.ell / 2.0) * np.cos(theta)
21
22     # equilibrium torque around theta_e = 0
23     theta_e = 0.0
24     tau_e = P0.m * P0.g * P0.ell/2.0 * np.cos(theta_e)
25
26     # compute the linearized torque using PD control
27     tau_tilde = self.kp * (theta_r - theta) - self.kd * thetadot
28
29     # compute total torque
30     #tau = tau_fl + tau_tilde
31
32     tau = tau_e + tau_tilde
33     # always saturate to protect hardware
34     tau = self.saturate(tau)
35
36     return tau
37
38 def saturate(self, u):
39     if abs(u) > self.limit:
40         u = self.limit*np.sign(u)
41     return u

```

Listing 7.1: armController.py

Code that simulates closed-loop system is given below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import armParam as P
6 from hw3.armDynamics import armDynamics
7 from armController import armController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.armAnimation import armAnimation
10 from hw2.dataPlotter import dataPlotter
11
12 # instantiate arm, controller, and reference classes
13 arm = armDynamics()
14 controller = armController()
15 reference = signalGenerator(amplitude=5*np.pi/180.0, frequency=0.05)
16 disturbance = signalGenerator(amplitude=0.01)
17 noise = signalGenerator(amplitude=0.01)
18
19 # instantiate the simulation plots and animation
20 dataPlot = dataPlotter()
21 animation = armAnimation()
22
23 t = P.t_start # time starts at t_start
24 y = arm.h() # output of system at start of simulation
25 while t < P.t_end: # main simulation loop
26     # Get referenced inputs from signal generators
27     # Propagate dynamics in between plot samples

```

```

28 t_next_plot = t + P.t_plot
29
30 # updates control and dynamics at faster simulation rate
31 while t < t_next_plot:
32     r = reference.square(t)
33     d = 0#disturbance.step(t) # input disturbance
34     n = noise.random(t) # simulate sensor noise
35     x = arm.state
36     u = controller.update(r, x) # update controller
37     y = arm.update(u + d) # propagate system
38     t = t + P.Ts # advance time by Ts
39
40 # update animation and data plots
41 animation.update(arm.state)
42 dataPlot.update(t, r, arm.state, u)
43
44 # the pause causes the figure to be displayed for simulation
45 plt.pause(0.0001)
46
47 # Keeps the program from closing until the user presses a button.
48 print('Press key to close')
49 plt.waitforbuttonpress()
50 plt.close()

```

Listing 7.2: armSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### Important Concepts:

- The PD controller gains can place the poles of a second order closed-loop characteristic equation to a desired location.
- PD gains can be chosen to move unstable poles, ones in the right side of the complex plane, to stable locations in the closed-loop feedback system.
- A PD controller introduces a zero into the closed-loop system. It may be eliminated by modifying the derivative portion of the controller so that it acts on the output (rather than the error).

## Notes and References

The concept of using feedback to place the poles of the system at desired locations is a much deeper topic than is covered in this chapter. While the method discussed in this chapter is only applicable to second order systems, it is usually possible to arbitrarily assign the poles for an  $n^{th}$  order system. In general, to assign  $n$  poles will require  $n$  gains. We will discuss pole placement in more detail in Chapter 11 of this book.



# 8

## Design Strategies for Second Order Systems

### Learning Objectives:

- Understand how pole locations affect the step response of a system.
- Choose pole locations (for second order systems) that satisfy desired design requirements for the rise time, natural frequency, and damping ratio.

## 8.1 Theory

In the previous section we saw that for second order systems, PD control could be used to exactly specify the pole location of the closed-loop system. This chapter examines the design problem of how to select the pole locations to achieve desired behavior. In Section 8.1.1 we derive the relationship between a single pole location and the step response of the system. In Section 8.1.2 we show that the time domain response for a second order system can be understood in terms of natural frequency and damping ratio. Finally in Section 8.1.3 we show the effect that a zero has on the step response of a second order system.

### 8.1.1 First Order Response

Consider a first order system given by

$$Y(s) = \frac{p}{s + p} U(s),$$

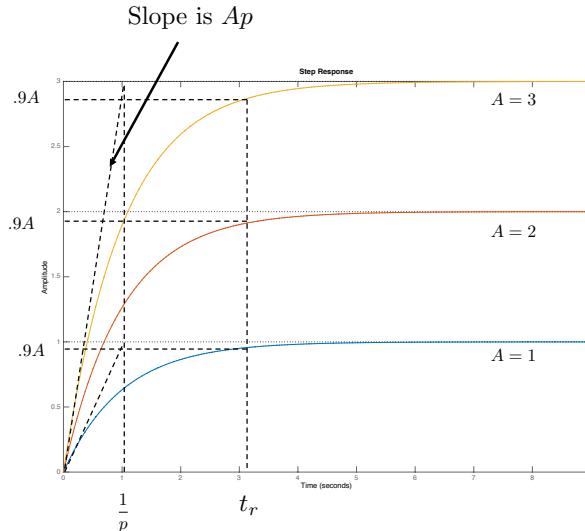
where  $p > 0$ . The pole location of the first order system is at  $-p$ . If  $U(s) = \frac{A}{s}$  is a step of magnitude  $A$ , then

$$Y(s) = \frac{Ap}{s(s + p)}.$$

As shown in Appendix P.5, the inverse Laplace transform is given by

$$y(t) = \begin{cases} A(1 - e^{-pt}), & t \geq 0 \\ 0, & \text{otherwise} \end{cases}. \quad (8.1)$$

The response is shown in **Fig. 8-1**, which shows the response to steps of size  $A = 1$ ,  $A = 2$ , and  $A = 3$ , respectively. If we define the rise time  $t_r$  of the



**Figure 8-1:** Step response of a first order system with pole  $p$ . Three step responses are shown, where the step size is  $A = 1$ ,  $A = 2$ , and  $A = 3$ , respectively.

system to be the time after the step when the response reaches 90% of its final value, then we note from **Fig. 8-1** that the rise time is independent of the step size  $A$ . Also, by differentiating Equation (8.1) at time  $t = 0$ , it is straightforward to show that the slope of the response at  $t = 0$  is equal to  $Ap$ . This is also shown graphically in **Fig. 8-1**.

When  $A = 1$ , the step response for different pole locations is shown in **Fig. 8-2**. Pole locations corresponding to  $p = 1$ ,  $p = 2$ , and  $p = 4$  are shown. Note that as the pole location moves further into the left half of the complex plane, the rise time decreases.

An important concept for the steady state response of a system is the *DC-gain*, which is defined as follows.

**Definition** *The DC-gain of a transfer function  $H(s)$  is*

$$H(s)|_{\text{DC-gain}} = \lim_{s \rightarrow 0} H(s).$$

An important fact is that if the poles of  $H(s)$  are in the open left half plane, then the response of  $H(s)$  to a step of size  $A$  approaches  $A H(s)|_{\text{DC-gain}}$  as  $t \rightarrow \infty$ .

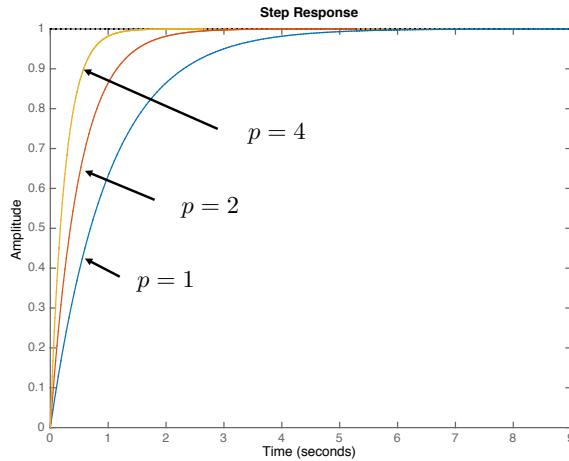


Figure 8-2: Step response of a first order system with poles  $p = 1$ ,  $p = 2$ , and  $p = 4$ . The rise time decreases as the pole location moves further into the left half of the complex plane.

This is true for first order systems, as well as general  $H(s)$ . For example, the DC-gain of the first order system  $H(s) = \frac{q}{s+p}$  is  $\frac{q}{p}$ , as long as  $p > 0$ .

### 8.1.2 Second Order Response

Suppose that the closed-loop system has a second order transfer function with two real poles and no zeros:

$$Y(s) = \frac{K}{(s + p_1)(s + p_2)} Y_r(s).$$

To find the output of the system when the input is a step of magnitude  $A$ , we find the inverse Laplace transform of

$$Y(s) = \frac{K}{(s + p_1)(s + p_2)} \frac{A}{s}.$$

Using partial fraction expansion we get

$$Y(s) = \frac{\frac{KA}{p_1 p_2}}{s} + \frac{\frac{KA}{(-p_1)(p_2 - p_1)}}{s + p_1} + \frac{\frac{KA}{(-p_2)(p_1 - p_2)}}{s + p_2}.$$

Taking the inverse Laplace transform gives

$$y(t) = \begin{cases} \frac{KA}{p_1 p_2} + \frac{KA}{(-p_1)(p_2 - p_1)} e^{-p_1 t} + \frac{KA}{(-p_2)(p_1 - p_2)} e^{-p_2 t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Note that as  $t \rightarrow \infty$  that  $y(t) \rightarrow \frac{KA}{p_1 p_2}$  which is equal to the step size  $A$  times the DC-gain of the system.

Now suppose that the closed-loop system has two complex poles and no zeros. We will write a generic second order system with poles in the left half plane using the notation

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where  $K$  is the DC-gain of the system (typically  $K = 1$ ),  $\omega_n > 0$  is called the natural frequency, and  $\zeta > 0$  is called the damping ratio. The poles are given by the roots of the characteristic polynomial

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

in other words, the poles are

$$\begin{aligned} p_{1,2} &= -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} \\ &= -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}. \end{aligned}$$

If  $0 < \zeta < 1$ , then the roots are complex and given by

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2}. \quad (8.2)$$

Using the Python code

```

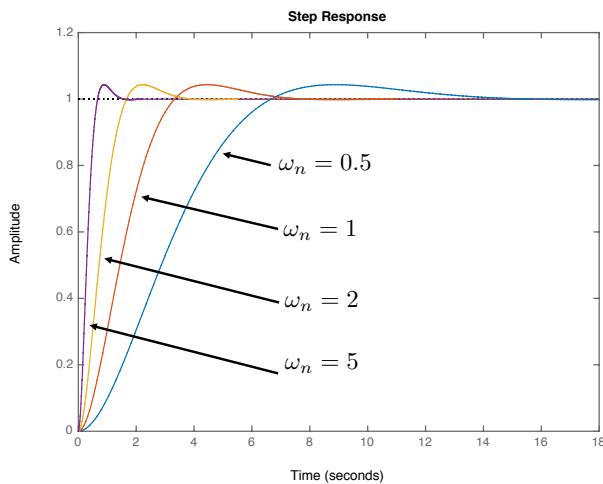
1 import control as cnt
2 import matplotlib.pyplot as plt
3
4 #start by varying the natural frequency
5 plt.figure(1), plt.clf()
6 zeta = 0.707
7
8 for wn in [0.5, 1.0, 2.0, 5.0]:
9     G = cnt.tf(wn**2, [1, 2*zeta*wn, wn**2])
10    t,y = cnt.step_response(G)
11    plt.plot(t, y)
12
13 plt.plot([0, 20.0] , [1, 1], 'k--')
14 plt.xlabel('Time (seconds)')
15 plt.ylabel('Amplitude')
16 plt.title('Step Response')
17 plt.legend(('wn=0.5', 'wn=1.0', 'wn=2.0', 'wn=5.0'))
18
19 #now we'll see the effect of varying zeta
20 plt.figure(2), plt.clf()
21 wn = 1.0
22
23 for zeta in [0.2, 0.4, 0.707, 1.0, 2.0]:
24     G = cnt.tf(wn**2, [1, 2*zeta*wn, wn**2])
25     t,y = cnt.step_response(G)
26     plt.plot(t, y)
27
28 plt.plot([0, 35.0] , [1, 1], 'k--')
```

```

29 plt.xlabel('Time (seconds)')
30 plt.ylabel('Amplitude')
31 plt.title('Step Response')
32 plt.legend(['zeta=0.2', 'zeta=0.4', 'zeta=0.707',
33             'zeta=1.0', 'zeta=2.0'])
34
35 plt.show() # necessary to make plots appear

```

we obtain second order response plots similar to the ones shown in Figures 8-3 and 8-4. Note from **Fig. 8-3** that as the natural frequency increases, the rise time decreases. We should note that a similar phenomena to **Fig. 8-1** also occurs, where the rise time is independent of the size of the step. Note from **Fig. 8-4** that the damping ratio affects the amount of ringing in the system. For a small damping ratio, e.g.,  $\zeta = 0.2$  there is a large overshoot and significant ringing in the system. At the other extreme, when  $\zeta$  is larger than one, the poles are real and the response is highly damped. The sweet spot is when  $\zeta = 0.707$  where the rise time is small, but the overshoot is minimal.



**Figure 8-3:** Second order response for  $\omega_n = 0.5, 1, 2, 5$ , when the damping ratio is fixed at  $\zeta = 0.707$ .

Note from Equation (8.2) that when  $\zeta = 0.707 = \frac{1}{\sqrt{2}}$ , the poles are at

$$p_{1,2} = -\frac{\omega_n}{\sqrt{2}} \pm j \frac{\omega_n}{\sqrt{2}}.$$

In the complex plane, the pole locations are shown in **Fig. 8-5**, where  $\omega_n$  is the distance from the origin to the poles. When  $\zeta = \frac{1}{\sqrt{2}}$  the angle  $\theta$  equals 45 degrees.

In rectangular coordinates, the poles are given by

$$p_{1,2} = -\sigma \pm j\omega_d, \quad (8.3)$$

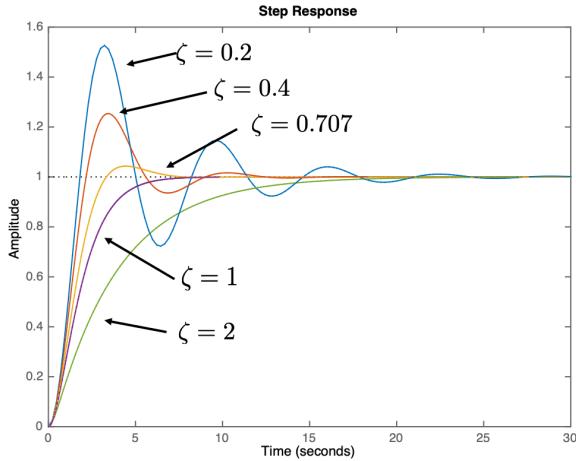


Figure 8-4: Second order response  $\zeta = 0.2, 0.4, 0.707, 1, 2$  when the natural frequency is fixed at  $\omega_n = 1$ .

as shown in **Fig. 8-5**. Equations (8.2) and (8.3) give

$$\begin{aligned}\sigma &= \zeta\omega_n \\ \omega_d &= \omega_n\sqrt{1 - \zeta^2}.\end{aligned}$$

From **Fig. 8-5** it is also clear that

$$\sin \theta = \frac{\sigma}{\omega_n} = \zeta.$$

Therefore

$$\theta = \sin^{-1} \zeta.$$

Since the pole locations are at  $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ , the poles are both real if  $\zeta \geq 1$ . When  $\zeta = 1$  the poles are both located at  $-\omega_n$ . When  $0 < \zeta < 1$  the poles are imaginary and occur in complex conjugate pairs. As shown in **Fig. 8-6**, we say that the system is *over damped* when  $\zeta > 1$ , we say that the system is *critically damped* when  $\zeta = 1$ , and we say that the system is *under damped* when  $0 < \zeta < 1$ .

The response of  $H(s)$  to a step of size  $A$  is the inverse Laplace transform of

$$Y(s) = \frac{A\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

which, as shown in Appendix P.5, when the poles are complex results in

$$y(t) = A \left[ 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\sigma t} \cos \left( \omega_d t - \tan^{-1} \left( \frac{\sigma}{\omega_d} \right) \right) \right]. \quad (8.4)$$

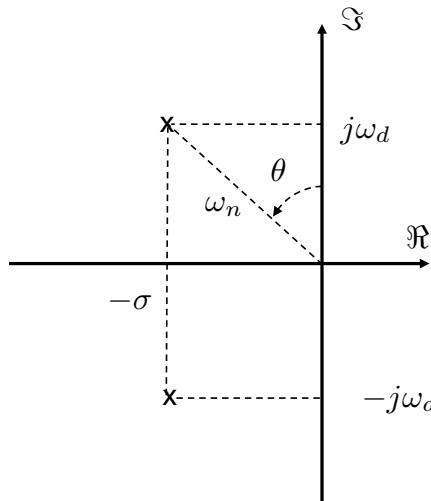


Figure 8-5: Complex conjugate poles in the left half plane.

A plot of this signal is shown in Fig. 8-7.

The rise time is defined to be the time that it takes the output to transition from 10% to 90% of its final value  $A$ . A suitable approximation is that the rise time is one half of the peak time, which is the time when  $y(t)$  reaches its first peak. To find the peak time  $t_p$ , differentiate  $y(t)$  in Equation (8.4) and then solve for the first instance when the derivative is zero. The result is

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}.$$

Therefore, the rise time is approximately

$$t_r \approx \frac{1}{2} t_p = \frac{1}{2} \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}.$$

When  $\zeta = 0.707$  we get

$$t_r \approx \frac{2.2}{\omega_n}. \quad (8.5)$$

Specifications for second order systems are usually given either in terms of the rise time  $t_r$  and the damping ratio  $\zeta$ , or in terms of the natural frequency  $\omega_n$  and the damping ratio  $\zeta$ .

### 8.1.3 Effect of a Zero on the Step Response

In the previous section, we looked at the step response when there were two complex poles and no zeros. In this section we will briefly discuss the effect of zeros on the step response of the system. This is a topic that we could treat at a much deeper level.

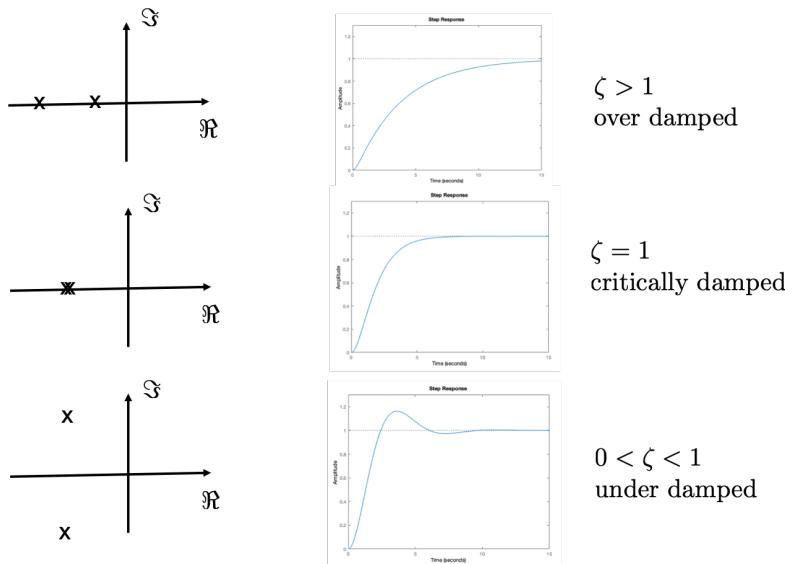


Figure 8-6: Second order response as a function of the damping ratio  $\zeta$ .

A transfer function is called *rational* if it is the ratio of two polynomials in  $s$ . For example,

$$H(s) = \frac{s+1}{s^3 + 3s^2 + s + 4}$$

is a rational transfer function, whereas

$$H(s) = \frac{(s+1)e^{-s}}{s^3 + 3s^2 + s + 4}$$

is not. A rational transfer function can be written as

$$H(s) = \frac{N(s)}{D(s)},$$

where  $N(s)$  is the numerator polynomial and  $D(s)$  is the denominator polynomial. The *zeros* of the transfer function are roots of the equation  $N(s) = 0$ , and the poles of the system are the roots of the equation  $D(s) = 0$ .

One way to understand the effect of zeros is to consider the effect that zeros have on the partial fraction expansion of a transfer function. For a review of partial fraction expansion, see Appendix P.5. First consider a two pole system with characteristic equation given by  $\Delta(s) = (s+2)(s+3)$ . The transfer function with DC-gain equal to one and without zeros is given by

$$H(s) = \frac{6}{(s+2)(s+3)}.$$

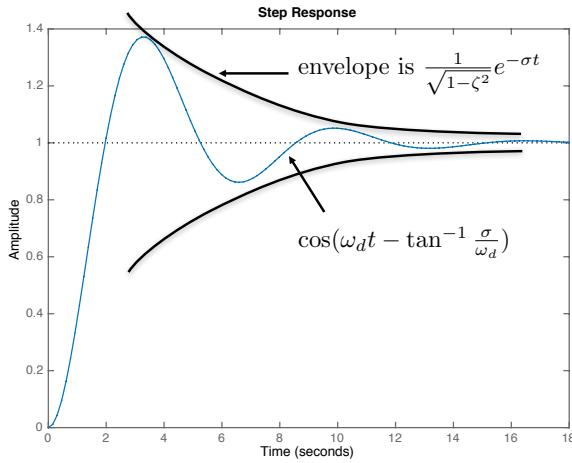


Figure 8-7: Second order response. The steady state value is  $A$ . The envelope is  $\frac{1}{\sqrt{1-\zeta^2}}e^{-\sigma t}$ , and the frequency and phase shift are given by  $\omega_d$  and  $\tan^{-1}\left(\frac{\sigma}{\omega_d}\right)$  respectively.

The response to a unit step is given by

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1} \left\{ \frac{6}{s(s+2)(s+3)} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{-3}{s+2} + \frac{2}{s+3} \right\} \\
 &= 1 - 3e^{-2t} + 2e^{-3t}, \quad t \geq 0.
 \end{aligned} \tag{8.6}$$

Now consider the step response when a zero has been added at  $z = -2.1$  and the gain adjusted so that the DC-gain is still one, i.e., when

$$H(s) = \frac{\frac{6}{2.1}(s+2.1)}{(s+2)(s+3)}.$$

The response to a unit step is given by

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1} \left\{ \frac{\frac{6}{2.1}(s+2.1)}{s(s+2)(s+3)} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{-0.1429}{s+2} + \frac{-0.8571}{s+3} \right\} \\
 &= 1 - 0.1429e^{-2t} - 0.8571e^{-3t}, \quad t \geq 0.
 \end{aligned}$$

Note that  $y(t)$  still settles to one, but the response of the pole at  $-2$  has almost been removed. If however, the zero is far away from the poles, then the effect is

much less. Suppose that the zero is at  $z = -60$  and the gain is adjusted so that the DC-gain is still one, i.e.,

$$H(s) = \frac{\frac{6}{60}(s + 60)}{(s + 2)(s + 3)}.$$

The response to a unit step is given by

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{\frac{6}{60}(s + 60)}{s(s + 2)(s + 3)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{-2.9}{s + 2} + \frac{1.9}{s + 3} \right\} \\ &= 1 - 2.9e^{-2t} + 1.9e^{-3t}, \quad t \geq 0. \end{aligned}$$

In this case, the response is similar to Equation (8.6).

Another way to understand the effect of zeros on the system is to recall that an  $s$  in the numerator is essentially a differentiator. Therefore

$$\begin{aligned} H(s) &= \frac{\omega_n^2(\tau s + 1)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \tau \frac{\omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \end{aligned}$$

Therefore the step response is given by

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \tau \frac{\omega_n^2 s}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \right\} \\ &= \tau \frac{d}{dt} \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \right\} + \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \right\}. \end{aligned}$$

Therefore, the effect of the zero is to add  $\tau$  times the derivative of the step response to what the step response would have been without the zero. **Fig. 8-8** shows the step response for different values of  $\tau$ , where it can be seen that adding a zero has a strong effect on the rise time and the damping in the response. Note also that when  $\tau < 0$ , i.e., when zeros are in the right-half plane, that the system response first goes negative, before going positive. Right half plane zeros are often called *nonminimum phase zeros* and the associated response is often called nonminimum phase response. Physical systems that have right-half plane zeros include the inverted pendulum and aircraft altitude control. For the inverted pendulum, in order to move the cart to the right from a stopped position the cart must first move left, so that the rod leans right, before cart can move to its desired position. Similarly, aircraft pitch up to climb altitude. However, when the aircraft pitches up it loses speed which causes it to lose lift, which causes the altitude to drop. When the altitude drops it gains speed and produces the lift necessary to increase altitude.

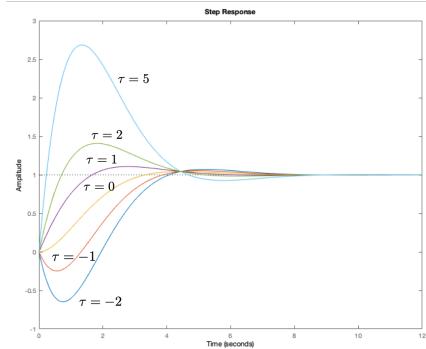


Figure 8-8: Step response of a second order system to a single zero.

### 8.1.4 Successive Loop Closure

The natural frequency  $\omega_n$  and the damping ratio  $\zeta$  are only defined for second order systems, and therefore, the PD design technique based on  $\omega_n$  and  $\zeta$  are only valid for second order systems. However, these concepts can be extended to higher-order systems through a technique called successive loop closure, or inner-loop, outer-loop design. When we derived the Laplace transform in Chapter 5, we saw that for many systems, the transfer functions can be arranged in cascade form as shown in Fig. 8-9 where the input to the system is  $u_1$  and the output is  $y_2$ , but where  $y_1$  and  $u_2$  are intermediary values that represent the natural physical coupling between internal variables. Many physical systems can be represented by a cascade of systems.

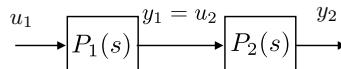


Figure 8-9: Open-loop transfer function modeled as a cascade of two transfer functions.

When the system is represented as a cascade, the control strategy can be designed using *successive loop closure*. The basic idea is to close feedback loops in succession around the open-loop plant dynamics rather than designing a single control system. To illustrate this approach, consider the open-loop system shown in Fig. 8-9. The open-loop dynamics are given by the product of two transfer functions in series:  $P(s) = P_1(s)P_2(s)$ . We assume that each of the transfer functions has an output ( $y_1, y_2$ ) that can be measured and used for feedback. Typically, each of the transfer functions,  $P_1(s)$  and  $P_2(s)$ , is of relatively low order – usually first or second order. In this case, we are interested in controlling the output  $y_2$ . Instead of closing a single feedback loop around  $y_2$ , we will close feedback loops around  $y_1$  and  $y_2$  in succession as shown in Fig. 8-10. We will design the compensators  $C_1(s)$  and  $C_2(s)$  in succession. A necessary condition in the design process is that

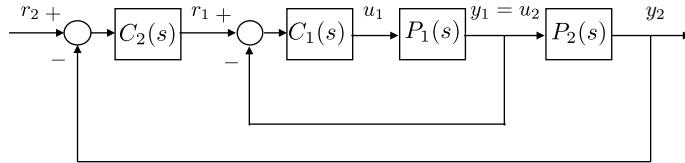


Figure 8-10: Successive loop closure design.

the inner loop must be much faster than each successive loop. As a rule of thumb, if the rise time of the inner loop is  $t_{r_1}$ , the rise time of the next loop should be 5 to 10 times longer, i.e.,  $t_{r_2} > W_1 t_{r_1}$ , where  $W_1$  is a design parameter, usually on the order of 5 – 10. In part V of this book, we will talk about the frequency response of the system. At that time, we will provide a deeper explanation that can be understood in terms of the bandwidth of each successive loop.

Examining the inner loop shown in Fig. 8-10, the goal is to design a closed-loop system from  $r_1$  to  $y_1$  having a rise time of  $t_{r_1}$ . If the rise time of the inner loop is significantly faster than the rise time of the outer loop, and if the DC-gain of the inner loop is  $k_{DC_1}$ , then relative to the outer loop the inner loop can be effectively modeled as the DC-gain. This is depicted schematically in Fig. 8-11. With the inner-loop transfer function modeled as its DC-gain, design of the outer loop is simplified because it only includes the plant transfer function  $P_2(s)$  and the compensator  $C_2(s)$  and the DC-gain  $k_{DC_1}$ . Because each of the plant models

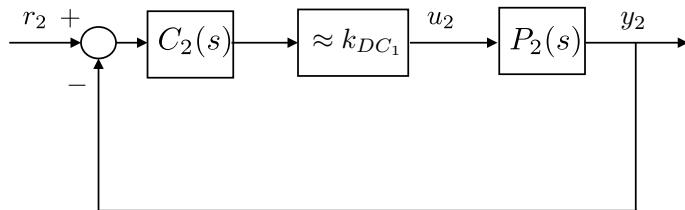


Figure 8-11: Successive loop closure design with inner loop modeled as a unity gain.

$P_1(s)$  and  $P_2(s)$  are first or second order, conventional PID compensators can be employed. The loopshaping techniques discussed in Part V can also be used for successive loop closure of higher-order systems.

### 8.1.5 Input Saturation and Limits of Performance

A natural question is to ask if the rise time can be made arbitrarily small. Unfortunately, saturation limits on the actuators place a lower bound on the achievable rise time. All physical systems have saturation limits on their actuators. For example, in the robot arm problem, the torque that can be applied to the arm is limited by the motor that is used to apply that torque. Force and torque will always be limited

by physical constraints like current limits. Similarly, for an airplane, the rudder command is limited by how far the rudder can physically move, which is typically on the order of  $\pm 40$  degrees. A natural way to model input constraints is to add a saturation block preceding the physical plant. **Fig. 8-12** shows a saturation block in conjunction with a PD control scheme. Mathematically, the saturation block is

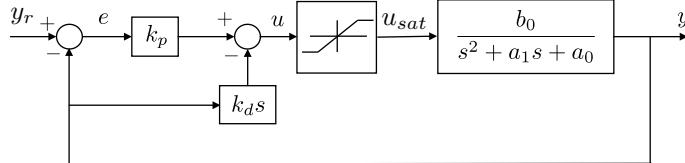


Figure 8-12: Control system with input saturation. The input saturation places a lower bound on the achievable rise time.

given by

$$u_{sat} = \begin{cases} u_{max}, & \text{if } u \geq u_{max} \\ u, & \text{if } -u_{max} \leq u \leq u_{max} \\ -u_{max}, & \text{otherwise} \end{cases}$$

where the input to the block is  $u$ , the output is  $u_{sat}$ , and the saturation limit is  $u_{max}$ . Note that the saturation block is non-linear. Therefore, one design strategy is to select the feedback gains so that the system remains linear by keeping the input out of saturation.

To see how this might be done, consider the second-order system shown in **Fig. 8-12** with proportional feedback on the output error and derivative feedback on the output. When the input is not saturated, the closed-loop transfer function is

$$Y(s) = \left( \frac{b_0 k_p}{s^2 + (a_1 + b_0 k_d)s + (a_0 + b_0 k_p)} \right) Y_r(s). \quad (8.7)$$

We can see that the closed-loop poles of the system are defined by the selection of the control gains  $k_p$  and  $k_d$ . Note also that the actuator effort  $u$  can be expressed as  $u = k_p e - k_d \dot{y}$ . When  $\dot{y}$  is zero or small, the size of the actuator effort  $u$  is primarily governed by the size of the control error  $e$  and the control gain  $k_p$ . If the system is stable, the largest control effort in response to a step input will occur immediately after the step when  $\dot{y} = 0$ . If the system is to be kept just out of saturation, then immediately after the step we have  $|u| = u_{max} = k_p e_{max}$ . Rearranging this expression, we find that the proportional control gain can be determined from the maximum anticipated output error and the saturation limits of the actuator as

$$k_p = \pm \frac{u_{max}}{e_{max}}, \quad (8.8)$$

where  $u_{max}$  is the maximum control effort the system can provide,  $e_{max}$  is the step error that results from a step input of nominal size, and the sign of  $k_p$  is determined by the physics of the system.

Suppose that the desired closed-loop characteristic polynomial is given by

$$\Delta_{cl}^d(s) = s^2 + 2\zeta\omega_n s + \omega_n^2,$$

then by comparing with the actual closed-loop characteristic polynomial

$$\Delta_{cl}(s) = s^2 + (a_1 + b_0 k_d)s + (a_0 + b_0 k_p)$$

we have that

$$\omega_n = \sqrt{a_0 + b_0 k_p} \leq \sqrt{a_0 + b_0 \frac{u_{\max}}{e_{\max}}},$$

which is an upper limit on the natural frequency of the closed-loop system, ensuring that saturation of the actuator is avoided. The corresponding rise time constraint is

$$t_r \geq \frac{2.2}{\omega_n} = \frac{2.2}{\sqrt{a_0 + b_0 \frac{u_{\max}}{e_{\max}}}},$$

which is the smallest possible rise time for an error step of  $e_{\max}$ .

If the input has an equilibrium value, then the saturation constraint used in the above calculation must be modified to account for the equilibrium. For example, suppose that  $u = u_e + \tilde{u}$  where  $u_e$  is the equilibrium value and  $\tilde{u}$  is the output of the PID controller. Also suppose that the input saturation constraint is given by  $u_{\min} \leq u \leq u_{\max}$ . The objective is to find a value for  $\tilde{u}_{\max}$  where  $|\tilde{u}| \leq \tilde{u}_{\max}$  guarantees that  $u_{\min} \leq u \leq u_{\max}$ . We have that

$$\begin{aligned} u_{\min} &\leq u \leq u_{\max} \\ \implies u_{\min} &\leq u_e + \tilde{u} \leq u_{\max} \\ \implies u_{\min} - u_e &\leq \tilde{u} \leq u_{\max} - u_e. \end{aligned}$$

The desired value for  $\tilde{u}_{\max}$  is therefore the minimum (in absolute value) of the right and left hand sides of this expression. The situation is shown in [Fig. 8-13](#), where it is clear that the value for  $\tilde{u}_{\max}$  can be written as

$$\tilde{u}_{\max} = \begin{cases} |u_{\max} - u_e|, & \text{if } u_e \geq \frac{u_{\max} + u_{\min}}{2} \\ |u_e - u_{\min}|, & \text{if } u_e \leq \frac{u_{\max} + u_{\min}}{2}. \end{cases}$$

The takeaway from this discussion is that the rise time is limited by the input saturation constraint and cannot be made arbitrarily small. In practice, the control parameters are tuned so that the closed-loop system has the fastest possible rise time without driving the plant input into saturation.

## 8.2 Design Study A. Single Link Robot Arm



### Example Problem A.8

For the single link robot arm, do the following:

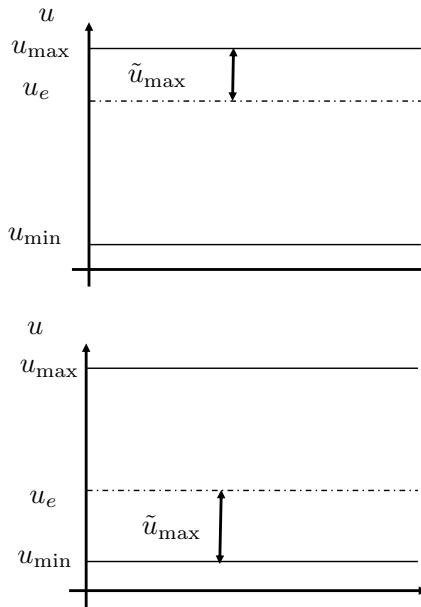


Figure 8-13: The saturation constraint for  $\tilde{u}$  depends on the value of the equilibrium input  $u_e$ .

- (a) Suppose that the design requirements are that the rise time is  $t_r \approx 0.8$  seconds, with a damping ratio of  $\zeta = 0.707$ . Find the desired closed loop characteristic polynomial  $\Delta_{cl}^d(s)$  and the associated pole locations. Find the proportional and derivative gains  $k_P$  and  $k_D$  to achieve these specifications, and modify the simulation from HW A.8 to verify that the step response satisfies the requirements.
- (b) Suppose that the input torque for the robot arm is limited to  $|\tau| \leq \tau_{\max} = 1$  Nm. Modify the simulation to include a saturation on the torque  $\tau$ . Using the rise time  $t_r$  and damping ratio  $\zeta$  as tuning parameters, tune the PD control law so that the input just saturates when a step of size 50 degrees is placed on  $\tilde{\theta}^r$ .

### Solution

A rise time of  $t_r \approx 0.8$  seconds, implies that the natural frequency is

$$\omega_n = 2.2/0.8 = 2.75.$$

Therefore, the desired characteristic polynomial is

$$\Delta_{cl}^d = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 3.889s + 7.5625. \quad (8.9)$$

From HW A.7 the actual closed loop characteristic polynomial is

$$\Delta_{cl}(s) = s^2 + (0.667 + 66.67k_D)s + 66.67k_P. \quad (8.10)$$

Equating Equations (8.9) and (8.10) and solving for the gains gives

$$\begin{aligned} k_P &= 0.1134 \\ k_D &= 0.0483. \end{aligned}$$

A Python class that implements a PD controller for the single link robot arm is shown below.

```

1 import numpy as np
2 import armParamHW8 as P
3 import sys
4 sys.path.append('..') # add parent directory
5 import armParam as P0
6
7
8 class armController:
9
10    def __init__(self):
11        # Instantiates the PD object
12        self.kp = P.kp
13        self.kd = P.kd
14        self.tau_max = P.tau_max # Maximum torque
15
16    def update(self, theta_r, state):
17        theta = state.item(0)
18        thetadot = state.item(1)
19
20        # compute feedback linearizing torque tau_fl
21        tau_fl = P0.m * P0.g * (P0.ell / 2.0) * np.cos(theta)
22
23        # compute the linearized torque using PD
24        tau_tilde = self.kp * (theta_r - theta) - self.kd * thetadot
25
26        # compute total torque
27        tau = tau_fl + tau_tilde
28        tau = self.saturate(tau, self.tau_max)
29        return tau
30
31    def saturate(self, u, limit):
32        if abs(u) > limit:
33            u = limit*np.sign(u)
34        return u

```

Listing 8.1: armController.py

Code that computes the control gains is given below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np

```

```
5 import armParam as P
6 from hw3.armDynamics import armDynamics
7 from armController import armController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.armAnimation import armAnimation
10 from hw2.dataPlotter import dataPlotter
11
12 # instantiate arm, controller, and reference classes
13 arm = armDynamics()
14 controller = armController()
15 reference = signalGenerator(amplitude=30*np.pi/180.0, frequency=0.05)
16 disturbance = signalGenerator(amplitude=0.0)
17
18 # instantiate the simulation plots and animation
19 dataPlot = dataPlotter()
20 animation = armAnimation()
21
22 t = P.t_start # time starts at t_start
23 y = arm.h() # output of system at start of simulation
24 while t < P.t_end: # main simulation loop
25     # Get referenced inputs from signal generators
26     # Propagate dynamics in between plot samples
27     t_next_plot = t + P.t_plot
28
29     # updates control and dynamics at faster simulation rate
30     while t < t_next_plot:
31         r = reference.square(t)
32         d = disturbance.step(t) # input disturbance
33         n = 0.0 #noise.random(t) # simulate sensor noise
34         x = arm.state
35         u = controller.update(r, x) # update controller
36         y = arm.update(u + d) # propagate system
37         t = t + P.Ts # advance time by Ts
38
39     # update animation and data plots
40     animation.update(arm.state)
41     dataPlot.update(t, r, arm.state, u)
42
43     # the pause causes the figure to be displayed for simulation
44     plt.pause(0.0001)
45
46 # Keeps the program from closing until the user presses a button.
47 print('Press key to close')
48 plt.waitforbuttonpress()
49 plt.close()
```

Listing 8.2: armSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### 8.3 Design Study B. Inverted Pendulum



#### Example Problem B.8

For the inverted pendulum, do the following:

- Using the principle of successive loop closure, draw a block diagram that uses PD control for both inner loop control and outer loop control. The input to the outer loop controller is the desired cart position  $z^d$  and the output of the controller is the desired pendulum angle  $\theta^d$ . The input to the inner loop controller is the desired pendulum angle  $\theta^d$  and the output is the force  $F$  on the cart.
- Focusing on the inner loop, find the PD gains  $k_{P_\theta}$  and  $k_{D_\theta}$  so that the rise time of the inner loop is  $t_{r_\theta} = 0.5$  seconds, and the damping ratio is  $\zeta_\theta = 0.707$ .
- Find the DC gain  $k_{DC_\theta}$  of the inner loop.
- Replace the inner loop by its DC-gain and consider the outer loop. Add a low-pass filter that cancels the left-half plane zero, and then design a PD control strategy to stabilize the position of the cart.
- Implement the control design to control on the inverted pendulum. Modify the simulation to include saturation on the force  $F$  to limit the size of the input force on the cart to  $F_{\max} = 5$  N. Give the pendulum angle an initial condition of 10 deg and simulate the response of the system for 10 seconds to verify that the cart is able to balance the pendulum.

#### Solution

The block diagram for the inner loop is shown in **Fig. 8-14**. The closed loop

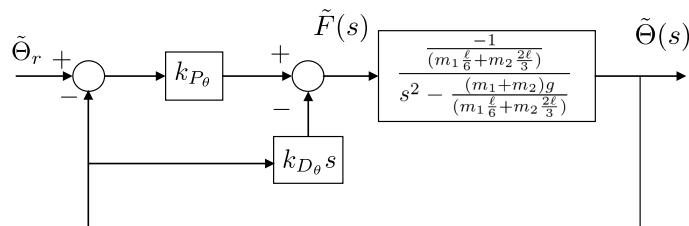


Figure 8-14: Block diagram for inner loop of inverted pendulum control

transfer function from  $\tilde{\Theta}_r$  to  $\tilde{\Theta}$  is given by

$$\tilde{\Theta}(s) = \frac{k_{P_\theta}}{s^2 - \frac{k_{D_\theta}}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}} s - \left( \frac{(m_1 + m_2)g}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}} + \frac{k_{P_\theta}}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}} \right)} \tilde{\Theta}_r(s).$$

Therefore the closed loop characteristic equation is

$$\Delta_{cl}(s) = s^2 - \frac{k_{D_\theta}}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} s - \left( \frac{(m_1 + m_2)g}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} + \frac{k_{P_\theta}}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} \right)$$

The desired closed loop characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2,$$

where

$$\begin{aligned}\omega_{n_\theta} &= \frac{2.2}{t_{r_\theta}} = 4.4 \\ \zeta_\theta &= 0.707.\end{aligned}$$

Therefore

$$\begin{aligned}k_{P_\theta} &= -(m_1 + m_2)g - (m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3})\omega_{n_\theta}^2 = -25.96 \\ k_{D_\theta} &= -2\zeta_\theta \omega_{n_\theta} (m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}) = -4.407.\end{aligned}$$

The DC gain of the inner loop is given by

$$k_{DC_\theta} = \frac{k_{P_\theta}}{(m_1 + m_2)g + k_{P_\theta}} = 1.893.$$

Replacing the inner loop by its DC gain, the block diagram for the outer loop is shown in **Fig. 8-15**. The closed loop transfer function from  $\tilde{Z}_r$  to  $\tilde{Z}$  is given by

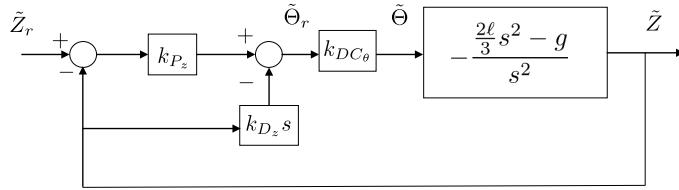


Figure 8-15: Block diagram for outer loop of inverted pendulum control.

$$\tilde{Z}(s) = \frac{\frac{k_{P_z}}{k_{D_z}}(s^2 - \frac{3g}{2\ell})}{s^3 - s^2(\frac{3}{2k_{DC_\theta}k_{D_z}\ell} + \frac{k_{P_z}}{k_{D_z}}) - s\frac{3g}{2\ell} - \frac{3k_{P_z}g}{2k_{D_z}\ell}} \tilde{Z}_r.$$

Note that the DC gain for the outer loop is equal to one. The closed-loop characteristic equation is

$$\Delta_{cl}(s) = s^3 - s^2(\frac{3}{2k_{DC_\theta}k_{D_z}\ell} + \frac{k_{P_z}}{k_{D_z}}) - s\frac{3g}{2\ell} - \frac{3k_{P_z}g}{2k_{D_z}\ell}.$$

Since the desired characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2, \quad (8.11)$$

it is not possible to pick PD gain values that will create a match between the actual characteristic equation and the desired characteristic equation.

To resolve this problem, note that the transfer function of the outer loop can be written as

$$H(s) = \frac{-\frac{2\ell}{3}s^2 + g}{s^2} = \frac{-\frac{2\ell}{3}(s + \sqrt{\frac{3g}{2\ell}})(s - \sqrt{\frac{3g}{2\ell}})}{s^2},$$

where we see that the system has two zeros, one in the right-half plane and one in the left-half plane. The strategy for making the numerator polynomial have degree one is to add a low-pass filter after the PD controller that attempts to cancel the left-half plane zero, as shown in Figure 8-16. After accounting for the cancellations

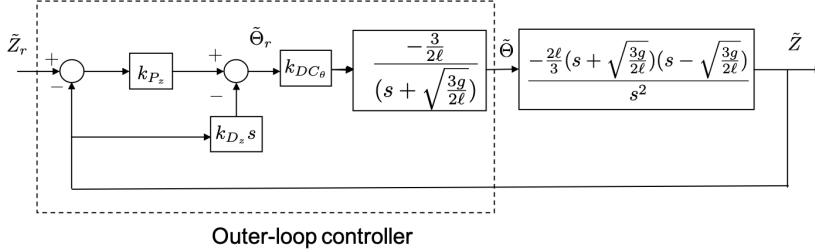


Figure 8-16: Block diagram for outer loop of inverted pendulum control, with low-pass filter to cancel the LHP zero.

introduced by the low-pass filter, the equivalent closed-loop systems is shown in Figure 8-17. In this case, the closed-loop characteristic equation from  $\tilde{Z}_r$  to  $\tilde{Z}$  is

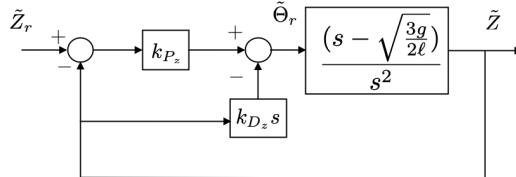


Figure 8-17: Block diagram for the equivalent outer loop of inverted pendulum control with low-pass filter.

given by

$$\tilde{Z}(s) = \left( \frac{\left(\frac{k_{p_z}}{1+k_{d_z}}\right)(s - \sqrt{\frac{3g}{2\ell}})}{s^2 - \left(\frac{k_{p_z} - k_{d_z}\sqrt{\frac{3g}{2\ell}}}{1+k_{d_z}}\right)s - \frac{k_{p_z}\sqrt{\frac{3g}{2\ell}}}{1+k_{d_z}}} \right) \tilde{Z}_r.$$

Setting the closed-loop characteristic equation equal to the desire closed-loop characteristic equation in Equation (8.11) gives

$$\frac{k_{p_z}}{1 + k_{d_z}} = -\omega_{n_z}^2 \sqrt{\frac{2\ell}{3g}} \triangleq a \quad (8.12)$$

$$\frac{k_{d_z}}{1 + k_{d_z}} = \left[ \frac{k_{p_z}}{1 + k_{d_z}} - 2\zeta_z \omega_{n_z} \right] \sqrt{\frac{2\ell}{3g}} \triangleq b. \quad (8.13)$$

Then

$$k_{d_z} = \frac{b}{1 - b}$$

$$k_{p_z} = a(1 + k_{d_z}).$$

A Python class that implements a PD controller for the inverted pendulum is shown below.

```

1 import numpy as np
2 import pendulumParam as P
3 import pendulumParamHW8 as P8
4
5 class pendulumController:
6     def __init__(self):
7         self.kp_z = P8.kp_z
8         self.kd_z = P8.kd_z
9         self.kp_th = P8.kp_th
10        self.kd_th = P8.kd_th
11        self.filter = zeroCancelingFilter()
12
13    def update(self, z_r, state):
14        z = state.item(0)
15        theta = state.item(1)
16        zdot = state.item(2)
17        thetadot = state.item(3)
18
19        # the reference angle for theta comes from the
20        # outer loop PD control
21        tmp = self.kp_z * (z_r - z) - self.kd_z * zdot
22
23        # low pass filter the outer loop to cancel
24        # left-half plane zero and DC-gain
25        theta_r = self.filter.update(tmp)
26
27        # the force applied to the cart comes from the
28        # inner loop PD control
29        F = self.kp_th * (theta_r - theta) - self.kd_th * thetadot
30        return F
31
32 class zeroCancelingFilter:
33     def __init__(self):
34         self.a = -3.0/(2.0*P.ell*P8.DC_gain)
35         self.b = np.sqrt(3.0*P.g/(2.0*P.ell))
36         self.state = 0.0

```

```

37
38     def update(self, input):
39         # integrate using RK1
40         self.state = self.state \
41             + P.Ts * (-self.b*self.state + self.a*input)
42

```

Listing 8.3: pendulumController.py

Python code that implements the closed loop system is given below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import pendulumParam as P
5 from hw3.pendulumDynamics import pendulumDynamics
6 from pendulumController import pendulumController
7 from hw2.signalGenerator import signalGenerator
8 from hw2.pendulumAnimation import pendulumAnimation
9 from hw2.dataPlotter import dataPlotter
10
11 # instantiate pendulum, controller, and reference classes
12 pendulum = pendulumDynamics()
13 controller = pendulumController()
14 reference = signalGenerator(amplitude=0.5, frequency=0.02)
15 disturbance = signalGenerator(amplitude=0)
16
17 # instantiate the simulation plots and animation
18 dataPlot = dataPlotter()
19 animation = pendulumAnimation()
20
21 t = P.t_start # time starts at t_start
22 y = pendulum.h() # output of system at start of simulation
23
24 while t < P.t_end: # main simulation loop
25     # Propagate dynamics in between plot samples
26     t_next_plot = t + P.t_plot
27
28     while t < t_next_plot:
29         r = reference.sin(t) # reference input
30         d = disturbance.step(t) # input disturbance
31         n = 0.0 #noise.random(t) # simulate sensor noise
32         x = pendulum.state # use state instead of output
33         u = controller.update(r, x) # update controller
34         y = pendulum.update(u + d) # propagate system
35         t = t + P.Ts # advance time by Ts
36
37     # update animation and data plots
38     animation.update(pendulum.state)
39     dataPlot.update(t, r, pendulum.state, u)
40     plt.pause(0.0001)
41
42 # Keeps the program from closing until the user presses a button.
43 print('Press key to close')
44 plt.waitforbuttonpress()
45 plt.close()

```

---

 Listing 8.4: pendulumSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 8.4 Design Study C. Satellite Attitude Control



### Example Problem C.8

- (a) For this problem and future problems, change the satellite spring constant to  $k = 0.1 \text{ N m}$ . For the simple satellite system, using the principle of successive loop closure, draw a block diagram that uses PD control for the inner and the outer loop control. The input to the outer loop controller is the desired angle of the solar panel  $\phi_r$  and the output is the desired angle of the satellite  $\theta_r$ . The input to the inner loop controller is  $\theta_r$  and the output is the torque on the satellite  $\tau$ .
- (b) Focusing on the inner loop, find the PD gains  $k_{P_\theta}$  and  $k_{D_\theta}$  so that the rise time of the inner loop is  $t_{r_\theta} = 1$  second, and the damping ratio is  $\zeta_\theta = 0.9$ .
- (c) Find the DC gain  $k_{DC_\theta}$  of the inner loop.
- (d) Replacing the inner loop by its DC-gain, find the PD gains  $k_{P_\phi}$  and  $k_{D_\phi}$  so that the rise time of the outer loop is  $t_{r_\phi} = 10t_{r_\theta}$  with damping ratio  $\zeta_\phi = 0.9$ .
- (e) Implement the successive loop closure design for the satellite system in simulation where the commanded solar panel angle is given by a square wave with magnitude 15 degrees and frequency 0.015 Hz.
- (f) Suppose that the size of the input torque on the satellite is limited to  $\tau_{\max} = 5 \text{ Nm}$ . Modify the simulation to include saturation on the torque  $\tau$ . Using the rise time of the outer loop as a tuning parameter, tune the PD control law to get the fastest possible response without input saturation when a step of size 30 degrees is placed on  $\phi^r$ .

### Solution

The block diagram for the inner loop is shown in **Fig. 8-18**. The closed loop transfer function from  $\Theta^d$  to  $\Theta$  is given by

$$\Theta(s) = \frac{\frac{k_{P_\theta}}{J_s + J_p}}{s^2 + \left(\frac{k_{D_\theta}}{J_s + J_p}\right)s + \left(\frac{k_{P_\theta}}{J_s + J_p}\right)} \Theta^d(s).$$

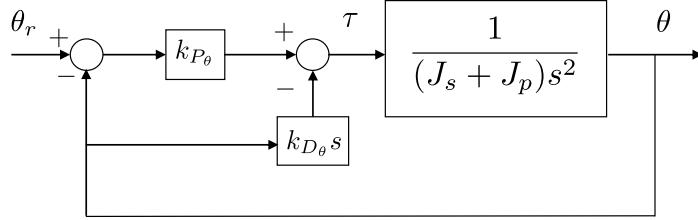


Figure 8-18: Block diagram for inner loop of satellite control

Therefore the closed loop characteristic equation is

$$\Delta_{cl}(s) = s^2 + \left(\frac{k_{D_\theta}}{J_s + J_p}\right)s + \left(\frac{k_{P_\theta}}{J_s + J_p}\right).$$

The desired closed loop characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_\theta\omega_{n_\theta}s + \omega_{n_\theta}^2,$$

where

$$\begin{aligned}\omega_{n_\theta} &= \frac{1}{2} \frac{\pi}{t_{r_\theta} \sqrt{1 - \zeta_\theta^2}} \\ \zeta_\theta &= 0.9.\end{aligned}$$

Therefore

$$\begin{aligned}k_{P_\theta} &= \omega_{n_\theta}^2 (J_s + J_p) = 77.9 \\ k_{D_\theta} &= 2\zeta_\theta\omega_{n_\theta}(J_s + J_p) = 38.9.\end{aligned}$$

The DC gain of the inner loop is given by

$$k_{DC_\theta} = 1.$$

Replacing the inner loop by its DC gain, the block diagram for the outer loop is shown in Fig. 8-19. To find the closed loop transfer function from  $\phi^d$  to  $\phi$ ,

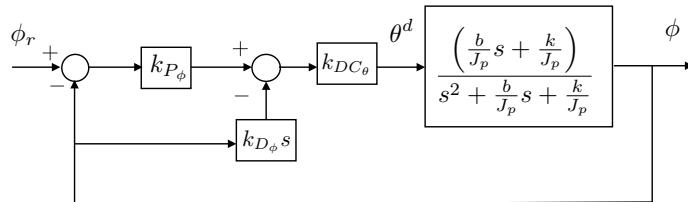


Figure 8-19: Block diagram for outer loop of satellite attitude control

follow the loop backwards from  $\phi$  to obtain

$$\Phi(s) = \left( \frac{\frac{b}{J_p}s + \frac{k}{J_p}}{s^2 + \frac{b}{J_p}s + \frac{k}{J_p}} \right) [k_{DC_\theta} k_{P_\phi} (\Phi_r - \Phi) - k_{DC_\theta} k_{D_\phi} s \Phi].$$

After some manipulation we get

$$\begin{aligned} [(J_p + bk_{DC_\theta} k_{D_\phi})s^2 + (b + bk_{DC_\theta} k_{P_\phi} + kk_{DC_\theta} k_{D_\phi})s + (k + kk_{DC_\theta} k_{P_\phi})] \Phi \\ = [bk_{DC_\theta} k_{P_\phi} s + kk_{DC_\theta} k_{P_\phi}] \Phi_r, \end{aligned} \quad (8.14)$$

which results in the monic transfer function

$$\Phi(s) = \frac{\left(\frac{bk_{DC_\theta} k_{P_\phi}}{J_p + bk_{DC_\theta} k_{D_\phi}}\right)s + \left(\frac{kk_{DC_\theta} k_{P_\phi}}{J_p + bk_{DC_\theta} k_{D_\phi}}\right)}{s^2 + \left(\frac{b + bk_{DC_\theta} k_{P_\phi} + kk_{DC_\theta} k_{D_\phi}}{J_p + bk_{DC_\theta} k_{D_\phi}}\right)s + \left(\frac{k + kk_{DC_\theta} k_{P_\phi}}{J_p + bk_{DC_\theta} k_{D_\phi}}\right)} \Phi_r.$$

The desired closed loop characteristic equation for the outer loop is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_\phi \omega_{n_\phi} s + \omega_{n_\phi}^2,$$

where

$$\begin{aligned} t_{r_\phi} &= 10 t_{r_\theta} = 10 \\ \omega_{n_\phi} &= \frac{1}{2} \frac{\pi}{t_{r_\phi} \sqrt{1 - \zeta_\phi^2}} \\ \zeta_\phi &= 0.9. \end{aligned}$$

Therefore the gains  $k_{P_\phi}$  and  $k_{P_\theta}$  satisfy

$$\begin{aligned} \frac{k + kk_{DC_\theta} k_{P_\phi}}{J_p + bk_{DC_\theta} k_{D_\phi}} &= \omega_{n_\phi}^2 \\ \frac{b + bk_{DC_\theta} k_{P_\phi} + kk_{DC_\theta} k_{D_\phi}}{J_p + bk_{DC_\theta} k_{D_\phi}} &= 2\zeta_\phi \omega_{n_\phi}. \end{aligned}$$

Expressing these equations in matrix form gives

$$\begin{pmatrix} kk_{DC_\theta} & -bk_{DC_\theta} \omega_{n_\phi}^2 \\ bk_{DC_\theta} & kk_{DC_\theta} - 2bk_{DC_\theta} \zeta_\phi \omega_{n_\phi} \end{pmatrix} \begin{pmatrix} k_{P_\phi} \\ k_{D_\phi} \end{pmatrix} = \begin{pmatrix} -k + J_p \omega_{n_\phi}^2 \\ -b + 2J_p \zeta_\phi \omega_{n_\phi} \end{pmatrix},$$

implying that

$$\begin{aligned} \begin{pmatrix} k_{P_\phi} \\ k_{D_\phi} \end{pmatrix} &= \begin{pmatrix} kk_{DC_\theta} & -bk_{DC_\theta} \omega_{n_\phi}^2 \\ bk_{DC_\theta} & kk_{DC_\theta} - 2bk_{DC_\theta} \zeta_\phi \omega_{n_\phi} \end{pmatrix}^{-1} \begin{pmatrix} -k + J_p \omega_{n_\phi}^2 \\ -b + 2J_p \zeta_\phi \omega_{n_\phi} \end{pmatrix} \\ &= \begin{pmatrix} 0.835 \\ 8.254 \end{pmatrix} \end{aligned}$$

The DC gain of the outer loop is given by

$$k_{DC_\phi} = \frac{kk_{DC_\theta}k_{P_\phi}}{k + kk_{DC_\theta}k_{P_\phi}} = 0.455.$$

Note that the DC gain of the outer loop is not equal to one, so there will be significant steady state error. To remedy this, a feedforward term can be used to ensure that  $\theta$  and  $\phi$  are made to be equal in steady state, resulting in an overall DC gain of one. This configuration is depicted in **Fig. 8-20**. Feedforward control will be discussed in a later chapter.

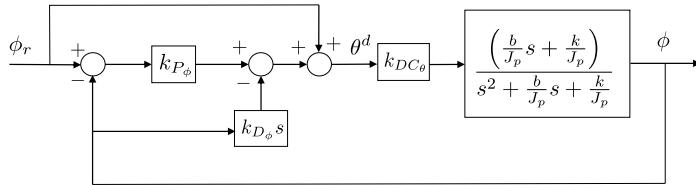


Figure 8-20: Block diagram for outer loop of satellite attitude control with a feed-forward term.

A Python class that implements a PD controller using state feedback is shown below.

```

1 import satelliteParamHW8 as P
2
3 class satelliteController:
4     def __init__(self):
5         self.kp_phi = P.kp_phi
6         self.kd_phi = P.kd_phi
7         self.kp_th = P.kp_th
8         self.kd_th = P.kd_th
9
10    def update(self, phi_r, state):
11
12        theta = state.item(0)
13        phi = state.item(1)
14        thetadot = state.item(2)
15        phidot = state.item(3)
16
17        # outer loop: outputs the reference angle for theta
18        # note that book recommends a feed forward term because
19        # of poor DC gain on the outer loop which
20        # is why we add an addition "phi_r" at the end
21        theta_r = self.kp_phi * (phi_r - phi) \
22                    - self.kd_phi * phidot + phi_r
23
24        # inner loop: outputs the torque applied to the base
25        tau = self.kp_th * (theta_r - theta) \
26                    - self.kd_th * thetadot
27
28        return tau

```

---

Listing 8.5: satelliteController.py

Python code that implements the closed loop system is given below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import satelliteParam as P
6 from hw3.satelliteDynamics import satelliteDynamics
7 from satelliteController import satelliteController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.satelliteAnimation import satelliteAnimation
10 from hw2.dataPlotter import dataPlotter
11
12 # instantiate satellite, controller, and reference classes
13 satellite = satelliteDynamics()
14 controller = satelliteController()
15 reference = signalGenerator(amplitude=15.0*np.pi/180.0,
16                               frequency=0.02)
17 disturbance = signalGenerator(amplitude=1.0)
18
19 # instantiate the simulation plots and animation
20 dataPlot = dataPlotter()
21 animation = satelliteAnimation()
22
23 t = P.t_start # time starts at t_start
24 y = satellite.h() # output of system at start of simulation
25
26 while t < P.t_end: # main simulation loop
27     # Propagate dynamics in between plot samples
28     t_next_plot = t + P.t_plot
29
30     # updates control and dynamics at faster simulation rate
31     while t < t_next_plot:
32         r = reference.square(t) # reference input
33         d = disturbance.step(t) # input disturbance
34         n = 0.0 #noise.random(t) # simulate sensor noise
35         x = satellite.state
36         u = controller.update(r, x) # update controller
37         y = satellite.update(u + d) # propagate system
38         t = t + P.Ts # advance time by Ts
39
40     # update animation and data plots
41     animation.update(satellite.state)
42     dataPlot.update(t, r, satellite.state, u)
43
44     # the pause causes the figure to display for the simulation.
45     plt.pause(0.0001)
46
47 # Keeps the program from closing until the user presses a button.
48 print('Press key to close')
49 plt.waitforbuttonpress()
50 plt.close()
```

Listing 8.6: satelliteSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### Important Concepts:

- The DC-gain of a transfer function is equal to the limit as  $s \rightarrow 0$  of the transfer function.
- If a transfer function contains only stable poles (those in the left half plane) then the steady-state response to a unit step input will equal the DC-gain.
- The time response  $y(t)$  to a step input is found by taking the inverse Laplace transform of the transfer function multiplied by the step input. It is useful to use partial fraction expansion to get the combined system into a form that is easily invertible to the time domain.
- The characteristic equation of a second order system can be written in terms of its natural frequency and damping ratio.
- The natural frequency most strongly affects the system's rise time, while the damping ratio mainly affects how much overshoot and oscillation occur in the response.
- Zeros can affect the system's overshoot and rise time. They will most strongly affect the poles that are positioned nearby.

### Notes and References

# 9

## System Type and Integrators

### Learning Objectives:

- Compute a system's type with respect to reference tracking, disturbances, and noise.
- Understand how the system type will affect the system's ability to track different types of inputs such as steps, ramps, and parabolas.

## 9.1 Theory

In the previous chapter we explored design techniques for second order systems where the feedback controller was a PD-type control. Since second order systems have two degrees of freedom, their transient response can be completely determined by the proportional and derivative gains. However, from the simulation results it can easily be observed that if there is a disturbance on the system, or if the controller does not know the system parameters exactly, then there will be a steady state error in the output. The steady state error can be effectively removed using an integrator in the control law. In this chapter we will explore these ideas in more detail and explain when and how integral control can be used to enhance steady state tracking and to reduce the effect of disturbances.

### 9.1.1 Final Value Theorem

The key theoretical tool to understanding the steady state error and the effect of integrators is the final value theorem, which we state and prove below.

#### Final Value Theorem.

Suppose that  $z(t)$  and  $Z(s)$  are Laplace transform pairs, and suppose that  $\lim_{t \rightarrow \infty} z(t)$  is finite, then

$$\lim_{t \rightarrow \infty} z(t) = \lim_{s \rightarrow 0} sZ(s).$$

*Proof.*

From the fundamental theorem of calculus we have that for  $t \geq 0$

$$z(t) - z(0) = \int_0^t dz.$$

Therefore

$$\begin{aligned} \lim_{t \rightarrow \infty} z(t) &= z(0) + \lim_{t \rightarrow \infty} \int_0^t \frac{dz}{d\tau} d\tau \\ &= z(0) + \lim_{t \rightarrow \infty} \int_0^t \dot{z}(\tau) d\tau \\ &= z(0) + \lim_{t \rightarrow \infty} \int_0^t \lim_{s \rightarrow 0} \dot{z}(\tau) e^{-s\tau} d\tau \\ &= z(0) + \lim_{s \rightarrow 0} \lim_{t \rightarrow \infty} \int_0^t \dot{z}(\tau) e^{-s\tau} d\tau \\ &= z(0) + \lim_{s \rightarrow 0} \int_0^{\infty} \dot{z}(\tau) e^{-s\tau} d\tau \\ &= z(0) + \lim_{s \rightarrow 0} (sZ(s) - z(0)) \\ &= \lim_{s \rightarrow 0} sZ(s), \end{aligned}$$

which completes the proof.

### 9.1.2 System Type for Reference Tracking

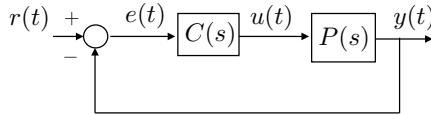


Figure 9-1: Block diagram for reference tracking.

Consider the closed-loop system shown in Fig. 9-1. The transfer function from the reference  $r$  to the error  $e = r - y$  is computed as

$$\begin{aligned} E(s) &= R(s) - P(s)C(s)E(s) \\ \implies E(s) &= \frac{1}{1 + P(s)C(s)}R(s) \end{aligned}$$

#### Step Input.

Suppose that  $r(t)$  is a step of size one, then  $R(s) = \frac{1}{s}$  and

$$E(s) = \frac{1}{1 + P(s)C(s)} \frac{1}{s}.$$

Assuming that the closed-loop system is stable and therefore that  $\lim_{t \rightarrow \infty} e(t)$  is finite, then the final value theorem gives

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} s \frac{1}{1 + P(s)C(s)} \frac{1}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + P(s)C(s)} \\ &= \frac{1}{1 + \lim_{s \rightarrow 0} P(s)C(s)}.\end{aligned}$$

Let

$$M_p \triangleq \lim_{s \rightarrow 0} P(s)C(s),$$

then

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{1 + M_p}.$$

Now suppose that

$$P(s)C(s) = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)},$$

then

$$M_p = \lim_{s \rightarrow 0} P(s)C(s) = \frac{Kz_1z_2 \cdots z_m}{p_1p_2 \cdots p_n}.$$

In this case, the steady state error is  $\frac{1}{1+M_p}$  which is finite. Note also that the DC-gain of the closed-loop system is

$$k_{DC} = \frac{1}{1 + M_p}.$$

Now suppose that  $P(s)C(s)$  contains a pole at the origin, or in other words, one free integrator, i.e.,

$$P(s)C(s) = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{s(s + p_1)(s + p_2) \cdots (s + p_n)}.$$

In this case we have

$$M_p = \lim_{s \rightarrow 0} P(s)C(s) = \infty,$$

and the steady state error is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{1 + \infty} = 0.$$

Therefore, one free integrator in  $P(s)C(s)$  implies that the steady state error to a step is zero. Note that multiple free integrators, i.e.,

$$P(s)C(s) = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{s^q(s + p_1)(s + p_2) \cdots (s + p_n)},$$

gives that same result for  $q \geq 1$ .

**Ramp Input.**

Now suppose that  $r(t)$  is a unit ramp, or in other words that  $R(s) = \frac{1}{s^2}$ . In this case the error is given by

$$E(s) = \frac{1}{1 + P(s)C(s)} \frac{1}{s^2}.$$

Again assuming that the closed-loop system is stable, the final value theorem gives

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} s \frac{1}{1 + P(s)C(s)} \frac{1}{s^2} \\ &= \lim_{s \rightarrow 0} \frac{1}{s + sP(s)C(s)} \\ &= \frac{1}{\lim_{s \rightarrow 0} sP(s)C(s)}. \end{aligned}$$

Defining

$$M_v = \lim_{s \rightarrow 0} sP(s)C(s),$$

gives

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_v}.$$

If  $P(s)C(s)$  has no free integrators, i.e.,

$$P(s)C(s) = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

then

$$M_v = \lim_{s \rightarrow 0} s \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} = 0.$$

Therefore

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{0} = \infty.$$

If on the other hand  $P(s)C(s)$  has one free integrator, i.e.,

$$P(s)C(s) = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{s(s + p_1)(s + p_2) \cdots (s + p_n)}$$

then

$$M_v = \lim_{s \rightarrow 0} s \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{s(s + p_1)(s + p_2) \cdots (s + p_n)} = \frac{Kz_1z_2 \cdots z_m}{p_1p_2 \cdots p_n},$$

therefore the steady state error is finite. If  $P(s)C(s)$  has two or more free integrators, then

$$M_v = \lim_{s \rightarrow 0} s \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{s^q(s + p_1)(s + p_2) \cdots (s + p_n)} = \infty,$$

when  $q \geq 2$ , therefore

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{\infty} = 0.$$

Therefore, the number of free integrators in  $P(s)C(s)$  is the key parameter for reference tracking with zero steady state error.

For the reference tracking problem, we say that the system is *type q* if the steady state error to input  $R(s) = \frac{1}{s^{q+1}}$  is finite and if the steady state error to input  $R(s) = \frac{1}{s^p}$  is zero for  $1 \leq p \leq q$ . In general, for the feedback system shown in **Fig. 9-1**, the steady state tracking error when  $R(s) = \frac{1}{s^{q+1}}$  is given by

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \left( \frac{1}{1 + P(s)C(s)} \right) \left( \frac{1}{s^q} \right).$$

Table 9-1 summarizes the tracking error for different inputs as a function of system type, where

$$M_p = \lim_{s \rightarrow 0} P(s)C(s)$$

$$M_v = \lim_{s \rightarrow 0} sP(s)C(s)$$

$$M_a = \lim_{s \rightarrow 0} s^2 P(s)C(s).$$

Table 9-1: Reference tracking verse system type.

System Type	Step ( $\frac{1}{s}$ )	Ramp ( $\frac{1}{s^2}$ )	Parabola ( $\frac{1}{s^3}$ )	...
0	$\frac{1}{1+M_p}$	$\infty$	$\infty$	...
1	0	$\frac{1}{M_v}$	$\infty$	...
2	0	0	$\frac{1}{M_a}$	...
3	0	0	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

As an example, consider the feedback system shown in **Fig. 9-2**. The open

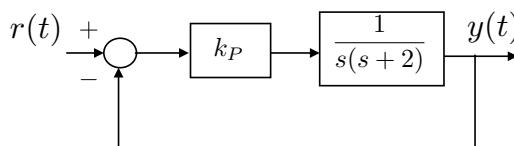


Figure 9-2: The system type for reference tracking for this system is type 1.

loop system

$$P(s)C(s) = \frac{k_P}{s(s+2)}$$

has one free integrator (pole at zero) and therefore the system is type 1. From Table 9-1 the tracking error when the input is a step is zero, and the tracking error when the input is a ramp of slope one is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_v} = \frac{1}{\lim_{s \rightarrow 0} sP(s)C(s)} = \frac{1}{\frac{k_P}{2}} = \frac{2}{k_P}.$$

The tracking error when the input is a parabola, or higher-order polynomial, is  $\infty$ , meaning that  $y(t)$  and  $r(t)$  diverge as  $t \rightarrow \infty$ .

As another example, consider the system shown in Fig. 9-3.

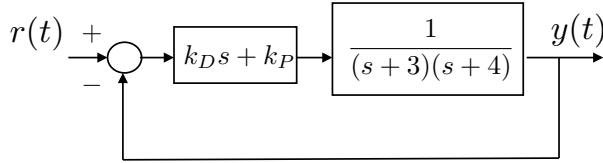


Figure 9-3: The system type for reference tracking for this system is type 0.

In this case, the open loop system

$$P(s)C(s) = \frac{k_Ds + k_P}{(s + 3)(s + 4)}$$

does not have any free integrators (poles at zero) and therefore the system is type 0. From Table 9-1 the tracking error when the input is a step is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{1 + M_p} = \frac{1}{1 + \lim_{s \rightarrow 0} P(s)C(s)} = \frac{1}{1 + \frac{k_P}{12}} = \frac{12}{12 + k_P}.$$

The tracking error when the input is a ramp, or higher-order polynomial, is  $\infty$ , meaning that  $y(t)$  and  $r(t)$  diverge as  $t \rightarrow \infty$ .

If on the other hand, a PID controller is used as shown in Fig. 9-4 where

$$U(s) = \left( k_P + \frac{k_I}{s} + k_Ds \right) E(s) = \left( \frac{k_Ds^2 + k_Ps + k_I}{s} \right) E(s),$$

then the open loop system

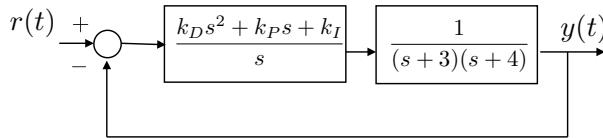


Figure 9-4: The system type for reference tracking for this system is type 1.

$$P(s)C(s) = \frac{k_Ds^2 + k_Ps + k_I}{s(s + 3)(s + 4)}$$

has one free integrator (pole at zero) and therefore the system becomes type 1 by design. From Table 9-1 the tracking error when the input is a step is zero, and the tracking error when the input is a ramp is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_v} = \frac{1}{\lim_{s \rightarrow 0} sP(s)C(s)} = \frac{1}{\frac{k_I}{12}} = \frac{12}{k_I}.$$

The tracking error when the input is a parabola, or higher-order polynomial, is  $\infty$ , meaning that  $y(t)$  and  $r(t)$  diverge as  $t \rightarrow \infty$ .

### 9.1.3 System Type for Input Disturbances

Consider the general feedback loop shown in Fig. 9-5. There are in general four inputs to the system, namely the reference input  $r(t)$ , the input disturbance  $d_{in}(t)$ , the output disturbance  $d_{out}(t)$ , and the sensor noise  $n(t)$ . In general  $r(t)$ ,  $d_{in}(t)$ , and  $d_{out}(t)$  are signals with low frequency content, and the sensor noise  $n(t)$  has high frequency content. The negative sign on the disturbance and noise terms is for notational convenience.

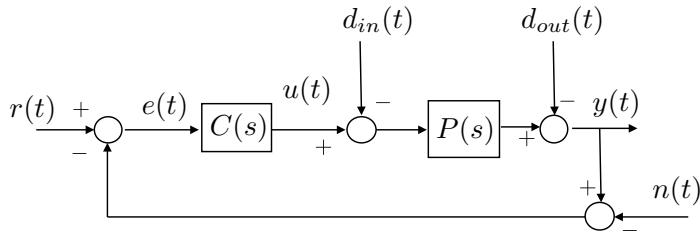


Figure 9-5: General feedback loop with reference input  $r(t)$ , input disturbance  $d_{in}(t)$ , output disturbance  $d_{out}(t)$ , and sensor noise  $n(t)$ .

We can compute the transfer function from each input to the error signal by following the signal flow to obtain

$$E(s) = R(s) + N(s) + D_{out}(s) + P(s)D_{in}(s) - P(s)C(s)E(s).$$

Solving for  $E(s)$  gives

$$\begin{aligned} E(s) &= \frac{1}{1 + P(s)C(s)}R(s) + \frac{P(s)C(s)}{1 + P(s)C(s)}N(s) \\ &\quad + \frac{1}{1 + P(s)C(s)}D_{out}(s) + \frac{P(s)}{1 + P(s)C(s)}D_{in}(s). \end{aligned}$$

Since the transfer function from  $N(s)$  and  $D_{out}(s)$  to  $E(s)$  is identical to the transfer function from  $R(s)$  to  $E(s)$ , the system type as defined in the previous section for reference inputs, also characterizes the steady state response to output disturbances and noise. However, the transfer function from the input disturbance

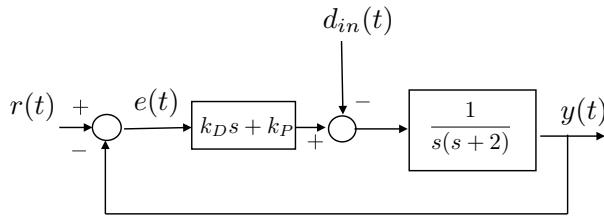
to the error is different, and so we need to further analyze system type with respect to input disturbances.

Suppose that the input disturbance is given by  $D_{in}(s) = \frac{1}{s^{q+1}}$ , then by the final value theorem, the steady state error due to the input disturbance is

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \frac{P(s)}{1 + P(s)C(s)} D_{in}(s) = \lim_{s \rightarrow 0} \left( \frac{P(s)}{1 + P(s)C(s)} \right) \left( \frac{1}{s^q} \right).$$

The system type with respect to input disturbance is the value of  $q$  such that the steady state error is finite, and the steady state error is zero for  $D_{in}(s) = \frac{1}{s^p}$  for  $1 \leq p \leq q$ .

For example, consider the system shown in **Fig. 9-6**. Recall from the discussion above, that the system type with respect to the reference input is type 1. For



**Figure 9-6:** The system type for input disturbances for this system is type 0.

the input disturbance we have

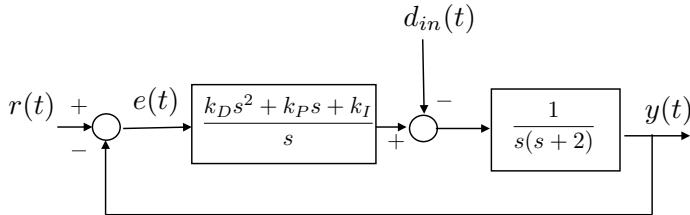
$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} \left( \frac{P(s)}{1 + P(s)C(s)} \right) \left( \frac{1}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{\frac{1}{s(s+2)}}{1 + \left( \frac{1}{s(s+2)} \right) (k_D s + k_P)} \right) \left( \frac{1}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{1}{s(s+2) + (k_D s + k_P)} \right) \left( \frac{1}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{1}{k_P} \right) \left( \frac{1}{s^q} \right) \end{aligned}$$

which is finite when  $q = 0$ . Therefore, the system type with respect to input disturbances is type 0 and the steady state error to a step input disturbance is  $\frac{1}{k_P}$  whereas the steady state error with respect to a ramp and higher-order polynomials is infinite.

If on the other hand, we add an integrator as in [Fig. 9-7](#), then

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} \left( \frac{P(s)}{1 + P(s)C(s)} \right) \left( \frac{1}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{\frac{1}{s(s+2)}}{1 + \left( \frac{1}{s(s+2)} \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right)} \right) \left( \frac{1}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{s}{s^2(s+2) + (k_D s^2 + k_P s + k_I)} \right) \left( \frac{1}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{1}{k_I} \right) \left( \frac{1}{s^{q-1}} \right)\end{aligned}$$

which is finite when  $q = 1$ . Therefore, the system type with respect to input disturbances is type 1 and the steady state error to a step on the input disturbance is zero. The steady state error with respect to a ramp on the input disturbance is  $\frac{1}{k_I}$  whereas the steady state error with respect to a parabola and higher-order polynomials on the input disturbance is infinite. Therefore, the system type with respect



[Figure 9-7](#): The system type for input disturbances for this system is type 1.

to input disturbances is related to the number of free integrators in the controller  $C(s)$  and not in the open loop system  $P(s)C(s)$ . As a consequence, while an integrator may not be necessary for steady state tracking, it is often necessary to reject constant input disturbances.

## 9.2 Design Study A. Single Link Robot Arm



### Example Problem A.9

- (a) When the controller for the single link robot arm is PD control, what is the system type? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

- (b) Consider the case where a constant disturbance acts at the input to the plant (for example gravity in this case). What is the steady state error to a constant input disturbance when the integrator is not present and when it is present?

### Solution

The closed loop system is shown in [Fig. 9-8](#). Without the integrator, the open-

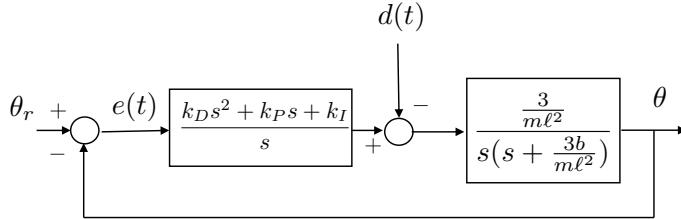


Figure 9-8: Closed loop system for problem HW A.9.

loop transfer function is given by

$$P(s)C(s) = \left( \frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2})} \right) (k_D s + k_P).$$

The system has one free integrator and is therefore type 1, which, from Table 9-1 implies that the tracking error when the input is a step is zero, and the tracking error when the input is a ramp is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_v} = \frac{1}{\lim_{s \rightarrow 0} sP(s)C(s)} = \frac{1}{\frac{k_P}{b}} = \frac{b}{k_P}.$$

The tracking error when the input is a parabola, or higher order polynomial, is  $\infty$ , meaning that  $\theta(t)$  and  $\theta_r(t)$  diverge as  $t \rightarrow \infty$ .

With the integrator, the open loop transfer function is

$$P(s)C(s) = \left( \frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2})} \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right).$$

which has two free integrators and is therefore type 2. Therefore, from Table 9-1 the tracking error when the input is either a step or a ramp is zero, and the tracking error when the input is a parabola is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_a} = \frac{1}{\lim_{s \rightarrow 0} s^2 P(s)C(s)} = \frac{1}{\frac{k_I}{b}} = \frac{b}{k_I}.$$

The tracking error when the input is  $t^3$  or a higher order polynomial, is  $\infty$ .

For the input disturbance, the transfer function from  $D(s)$  to  $E(s)$  is given by

$$E(s) = \frac{P(s)}{1 + P(s)C(s)}D(s).$$

Without the integrator, and when  $D(s) = \frac{A}{s^{q+1}}$ , the steady state error is given by

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} \frac{P}{1 + PC} \frac{A}{s^q} \\ &= \lim_{s \rightarrow 0} \left( \frac{\left( \frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2})} \right)}{1 + \left( \frac{\frac{3}{m\ell^2}(k_D s + k_P)}{s(s + \frac{3b}{m\ell^2})} \right)} \right) \left( \frac{A}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2}) + \frac{3}{m\ell^2}(k_D s + k_P)} \right) \left( \frac{A}{s^q} \right) \\ &= \frac{A \frac{3}{m\ell^2}}{\frac{3k_P}{m\ell^2}} = \frac{A}{k_P}, \end{aligned}$$

if  $q = 0$ . Therefore, to an input disturbance the system is type 0. The steady state error when a constant step of size  $A$  is placed on  $d(t)$  is  $\frac{A}{k_P}$ .

With the integrator, and when  $D(s) = \frac{A}{s^{q+1}}$ , the steady state error is given by

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} \frac{P}{1 + PC} \frac{A}{s^q} \\ &= \lim_{s \rightarrow 0} \left( \frac{\left( \frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2})} \right)}{1 + \left( \frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2})} \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right)} \right) \left( \frac{A}{s^q} \right) \\ &= \lim_{s \rightarrow 0} \left( \frac{s}{s^2(s + \frac{3b}{m\ell^2}) + \frac{3}{m\ell^2}(k_D s^2 + k_P s + k_I)} \right) \left( \frac{A}{s^q} \right) \\ &= \frac{A \frac{3}{m\ell^2}}{\frac{3k_I}{m\ell^2}} = \frac{A}{k_I}, \end{aligned}$$

if  $q = 1$ . Therefore, to an input disturbance the system is type 1. The steady state error when  $d(t)$  is a constant step is zero, and the steady state error when  $d(t)$  is a ramp of slope size  $A$  is  $\frac{A}{k_I}$ .

### 9.3 Design Study B. Inverted Pendulum



#### Example Problem B.9

- (a) When the inner loop controller for the inverted pendulum is PD control, what is the system type with respect to tracking of the inner loop? Characterize the steady state error when  $\tilde{\theta}^r$  is a step, a ramp, and a parabola. What is the system type with respect to an input disturbance?
- (b) When the outer loop controller for the inverted pendulum is PD control, what is the system type of the outer loop with respect to tracking? Characterize the steady state error when  $z^r$  is a step, a ramp, and a parabola. How does this change if you add an integrator? What is the system type with respect to an input disturbance for both PD and PID control?

### Solution

The block diagram for the inner loop is shown in **Fig. 9-9**. The open loop system

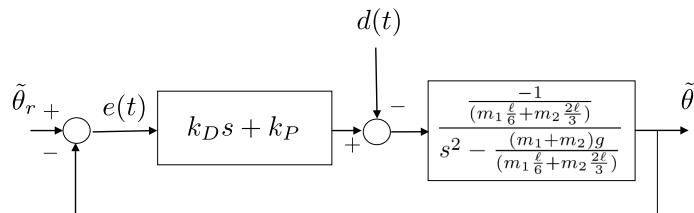


Figure 9-9: Inner loop system for problem HW B.9.

is given by

$$P(s)C(s) = \left( \frac{-\frac{1}{m_1\frac{\ell}{6}+m_2\frac{2\ell}{3}}}{s^2 - \frac{(m_1+m_2)g}{m_1\frac{\ell}{6}+m_2\frac{2\ell}{3}}} \right) (k_D s + k_P).$$

Since there are no free integrators, the system is type 0, and from Table 9-1 the tracking error when the input is a step is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{1 + M_p} = \frac{1}{1 + \lim_{s \rightarrow 0} P(s)C(s)} = \frac{1}{1 + \frac{k_P}{(m_1+m_2)g}}.$$

The tracking error when the input is a ramp, or higher order polynomial, is  $\infty$ .

For the disturbance input, the steady state error to a step on  $d(t)$  is

$$\begin{aligned}
 \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} s \frac{P(s)}{1 + P(s)C(s)} \frac{1}{s} \\
 &= \lim_{s \rightarrow 0} \frac{\left( \frac{-\frac{1}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}}}{s^2 - \frac{(m_1 + m_2)g}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}}} \right)}{1 + \left( \frac{-\frac{1}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}}}{s^2 - \frac{(m_1 + m_2)g}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}}} \right) (k_D s + k_P)} \\
 &= \lim_{s \rightarrow 0} \frac{\left( -\frac{1}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} \right)}{\left( s^2 - \frac{(m_1 + m_2)g}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} \right) + \left( -\frac{1}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} \right) (k_D s + k_P)} \\
 &= \frac{\left( -\frac{1}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} \right)}{\left( -\frac{(m_1 + m_2)g}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} \right) + \left( -\frac{k_p}{m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3}} \right)} = \frac{1}{(m_1 + m_2)g + k_P}.
 \end{aligned}$$

The system is type 0 with respect to the input disturbance.

The block diagram for the outer loop is shown in **Fig. 9-10**. The open loop

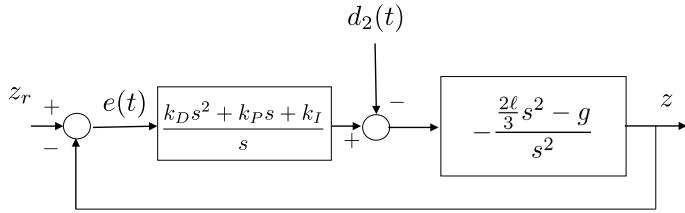


Figure 9-10: Outer loop system for problem HW B.9.

system is given by

$$P(s)C(s) = \left( \frac{-2\ell s^2}{s^2} + g \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right).$$

When  $k_I = 0$  there are two free integrators in  $P(s)C(s)$  and the system is type 2, and from Table 9-1 the tracking error when the input is a parabola is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_a} = \frac{1}{\lim_{s \rightarrow 0} s^2 P(s)C(s)} = \frac{1}{-k_P g}.$$

When  $k_I \neq 0$ , there are three free integrators in  $P(s)C(s)$  and the system is type 3, with zero tracking error for a step, ramp, and parabola input.

For the disturbance input, the steady state error when  $D(s) = \frac{1}{s^{q+1}}$  is

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} s \frac{P(s)}{1 + P(s)C(s)} \frac{1}{s^{q+1}} \\ &= \lim_{s \rightarrow 0} \frac{\left( \frac{\frac{2\ell s^2}{3} - g}{s^2} \right)}{1 + \left( \frac{\frac{2\ell s^2}{3} - g}{s^2} \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right)} \frac{1}{s^q}.\end{aligned}$$

Without the integrator, i.e., when  $k_I = 0$  we have

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} \frac{\frac{2\ell s^2}{3} - g}{s^2 + \left( \frac{2\ell s^2}{3} - g \right) (k_D s + k_P)} \frac{1}{s^q} = \lim_{s \rightarrow 0} \frac{g}{g(k_P)} \frac{1}{s^q}$$

which is finite when  $q = 0$ . Therefore, the system is type 0 with respect to the input disturbance and the steady state error when  $d_2(t)$  is a unit step is  $1/k_P$ .

When  $k_I \neq 0$ , we have

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} \frac{s \left( \frac{2\ell s^2}{3} - g \right)}{s^3 + \left( \frac{2\ell s^2}{3} - g \right) (k_D s^2 + k_P s + k_I)} \frac{1}{s^q} = \lim_{s \rightarrow 0} \frac{g}{g(k_I)} \frac{1}{s^{q-1}}$$

which is finite when  $q = 1$ . Therefore, the system is type 1 with respect to the input disturbance and the steady state error when  $d_2(t)$  is a unit ramp is  $1/k_I$ .

## 9.4 Design Study C. Satellite Attitude Control



### Example Problem C.9

- (a) When the inner loop controller of the satellite system is PD control, what is the system type of the inner loop with respect to the reference input and with respect to the disturbance input? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. Characterize the steady state error when the input disturbance is a step, a ramp, and a parabola.
- (b) With PD control for the outer loop, what is the system type of the outer loop with respect to the reference input and with respect to the disturbance input? Characterize the steady state error when the reference input and disturbance input is a step, a ramp, and a parabola. How does this change if you add an integrator?

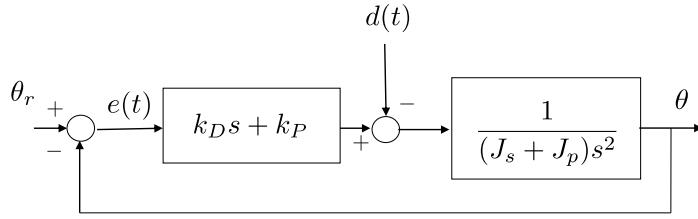


Figure 9-11: Inner loop system for problem HW C.9.

## Solution

The block diagram for the inner loop is shown in [Fig. 9-11](#). The open loop system is given by

$$P(s)C(s) = \left( \frac{1}{(J_s + J_p)s^2} \right) (k_D s + k_P).$$

Since there are two free integrators, the system is type 2, and from [Table 9-1](#) the tracking error when the input is a step is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{1 + M_p} = \frac{1}{1 + \lim_{s \rightarrow 0} P(s)C(s)} = 0.$$

The tracking error when the input is a ramp is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_v} = \frac{1}{\lim_{s \rightarrow 0} sP(s)C(s)} = 0.$$

Finally, the tracking error when the input is a parabola is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_a} = \frac{1}{\lim_{s \rightarrow 0} s^2 P(s)C(s)} = \frac{J_p + J_s}{k_p}.$$

The tracking error for higher order polynomials is  $\infty$ . For a disturbance input of  $D(s) = 1/s^{q+1}$ , the steady state error is

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \frac{P(s)}{1 + P(s)C(s)} \frac{1}{s^{q+1}} = \lim_{s \rightarrow 0} \frac{1}{(J_s + J_p)s^2 + k_d s + k_p} \frac{1}{s^q}$$

which is finite and equal to  $1/k_P$  when  $q = 0$ . Therefore, the system is type 0 with respect to the input disturbance, and the steady state error to a step on  $d(t)$  is  $1/k_P$  which is infinite to a ramp and higher order polynomials.

The block diagram for the outer loop is shown in [Fig. 9-12](#). The open loop system is given by

$$P(s)C(s) = \left( \frac{\frac{b}{J_p}s + \frac{k}{J_p}}{s^2 + \frac{b}{J_p}s + \frac{k}{J_p}} \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right).$$

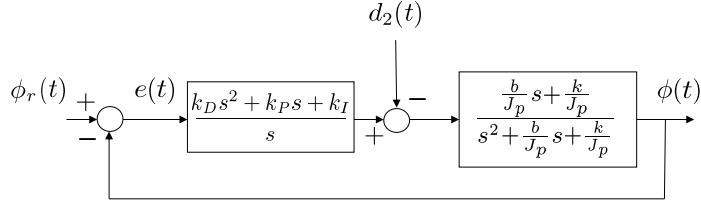


Figure 9-12: Outer loop system for problem HW C.9.

When  $k_I = 0$  there are no free integrators in  $P(s)C(s)$  and the system is type 0, and from Table 9-1 the tracking error when the input is a step is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{1 + M_p} = \frac{1}{1 + \lim_{s \rightarrow 0} P(s)C(s)} = \frac{1}{1 + k_P}.$$

When the reference is a ramp or higher order polynomial, the tracking error is infinite. When  $k_I \neq 0$ , there is one free integrator in  $P(s)C(s)$  and the system is type 1. From Table 9-1 the tracking error when the input is a ramp is

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{M_v} = \frac{1}{\lim_{s \rightarrow 0} sP(s)C(s)} = \frac{1}{k_I}.$$

The tracking error when  $\phi_r$  is a step is zero, and it is infinite when  $\phi_r$  is a parabola or higher order polynomial.

For the disturbance input, the steady state error when  $D(s) = \frac{1}{s^{q+1}}$  is

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} s \frac{P(s)}{1 + P(s)C(s)} \frac{1}{s^{q+1}} \\ &= \lim_{s \rightarrow 0} \frac{\left( \frac{\frac{b}{J_p} s + \frac{k}{J_p}}{s^2 + \frac{b}{J_p} s + \frac{k}{J_p}} \right)}{1 + \left( \frac{\frac{b}{J_p} s + \frac{k}{J_p}}{s^2 + \frac{b}{J_p} s + \frac{k}{J_p}} \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right)} \frac{1}{s^q}. \end{aligned}$$

Without the integrator, i.e., when  $k_I = 0$  we have

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} \frac{\frac{b}{J_p} s + \frac{k}{J_p}}{(s^2 + \frac{b}{J_p} s + \frac{k}{J_p}) + (\frac{b}{J_p} s + \frac{k}{J_p})(k_D s + k_P)} \frac{1}{s^q} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + k_P} \frac{1}{s^q} \end{aligned}$$

which is finite when  $q = 0$ . Therefore, the system is type 0 with respect to the input disturbance and the steady state error when  $d_2(t)$  is a unit step is  $1/(1+k_P)$ .

When  $k_I \neq 0$ , we have

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} \frac{s \left( \frac{b}{J_p} s + \frac{k}{J_p} \right)}{s \left( s^2 + \frac{b}{J_p} s + \frac{k}{J_p} \right) + \left( \frac{b}{J_p} s + \frac{k}{J_p} \right) (k_D s^2 + k_P s + k_I)} \frac{1}{s^q} \\ &= \lim_{s \rightarrow 0} \frac{1}{k_I} \frac{1}{s^{q-1}}\end{aligned}$$

which is finite when  $q = 1$ . Therefore, the system is type 1 with respect to the input disturbance and the steady state error when  $d_2(t)$  is a unit ramp is  $1/k_I$ . It is zero when  $d_2$  is a step, and it is infinite when  $d_2$  is a parabola or higher order polynomial.

### Important Concepts:

- The final value theorem relates a transfer function's frequency-domain behavior to its steady-state time-domain behavior.
- The system type describes the type of input (step, ramp, parabola) that produces a finite steady-state error. For input types of higher-order the steady-state error will be infinite. For input types of lower-order the steady-state error will be zero.
- In reference tracking the system type is equal to the number of free integrators in the open loop.

## Notes and References

The PID structure analyzed in this chapter uses a differentiator on the error as shown in [Fig. 7-1](#) instead of differentiating the output variable as shown in [Fig. 7-2](#). It is straightforward to show, using the final value theorem, that the system type does not change under the two configurations. However, the value of the steady state error does change and is not necessarily given by the simple formulas in [Table 9-1](#).



# 10

## Digital Implementation of PID Controllers

### Learning Objectives:

- Implement a discrete PID controller.

### 10.1 Theory

In this book we work primarily with continuous time systems and continuous time controllers. However, almost all modern control strategies are implemented digitally on a micro-controller or a micro-processor, typically implemented using C-code. In this chapter, we describe how PID controllers can be implemented in a sampled data system using computer code.

A general PID control signal is given by

$$u(t) = k_P e(t) + k_I \int_{-\infty}^t e(\tau) d\tau + k_D \frac{de}{dt}(t),$$

where  $e(t) = y_r(t) - y(t)$  is the error between the desired reference output  $y_r(t)$  and the current output  $y(t)$ . Define  $u_P(t) = e(t)$ ,  $u_I(t) = \int_{-\infty}^t e(\tau) d\tau$ , and  $u_D(t) = \frac{de}{dt}(t)$ . The PID controller is therefore given by  $u(t) = k_p u_P(t) + k_I u_I(t) + k_D u_D(t)$ . In the following paragraphs we will derive discrete sampled-time equivalents for  $u_P(t)$ ,  $u_I(t)$ , and  $u_D(t)$ .

First consider a sampled time implementation of  $u_P(t)$ . When  $t = nT_s$ , where  $T_s$  is the sample period and  $n$  is the sample time, we have  $u_P(nT_s) = e(nT_s)$ , or using discrete time notation

$$u_P[n] = e[n].$$

To derive the sample-time implementation of an integrator, note that

$$\begin{aligned} u_I(nT_s) &= \int_{-\infty}^{nT_s} e(\tau) d\tau \\ &= \int_{-\infty}^{(n-1)T_s} e(\tau) d\tau + \int_{(n-1)T_s}^{nT_s} e(\tau) d\tau \\ &= u_I((n-1)T_s) + \int_{(n-1)T_s}^{nT_s} e(\tau) d\tau. \end{aligned}$$

As shown in **Fig. 10-1** a trapezoidal approximation of the integral between  $(n-1)T_s$  and  $nT_s$  is given by

$$\int_{(n-1)T_s}^{nT_s} e(\tau) d\tau \approx \frac{T_s}{2} (e(nT_s) + e((n-1)T_s)).$$

Therefore

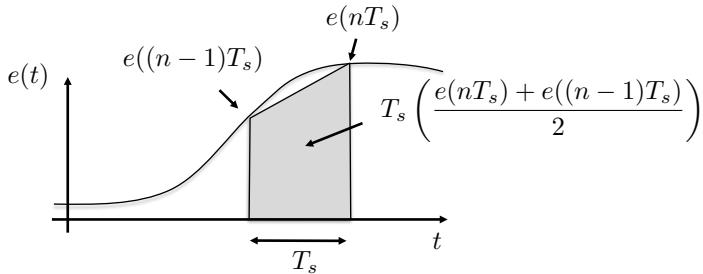


Figure 10-1: Trapezoidal rule for digital integration.

$$u_I(nT_s) \approx u_I((n-1)T_s) + \frac{T_s}{2} (e(nT_s) + e((n-1)T_s)).$$

Taking the z-transform we get

$$U_I(z) = z^{-1}U_I(z) + \frac{T_s}{2} (E(z) + z^{-1}E(z)).$$

Solving for  $U_I(z)$  gives

$$U_I(z) = \frac{T_s}{2} \left( \frac{1+z^{-1}}{1-z^{-1}} \right) E(z). \quad (10.1)$$

Since in the Laplace domain we have

$$U_I(s) = \frac{1}{s} E(s),$$

we have derived the so-called Tustin approximation where  $s$  in the Laplace domain is replaced with the  $z$ -transform approximation

$$s \mapsto \frac{2}{T_s} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right), \quad (10.2)$$

where  $T_s$  is the sample period [18]. Taking the inverse  $z$ -transform of Equation (10.1) gives the discrete time implementation of an integrator as

$$u_I[n] = u_I[n - 1] + \frac{T_s}{2} (e[n] + e[n - 1]). \quad (10.3)$$

In the Laplace domain, a pure differentiator is given by  $U_D(s) = sE(s)$ . However, since a pure differentiator is not causal, the standard approach is to use the band-limited, or dirty derivative

$$U_D(s) = \frac{s}{\sigma s + 1} E(s),$$

where  $\sigma$  is small, and  $\frac{1}{\sigma}$  defines the bandwidth of the differentiator. In practical terms, the dirty derivative differentiates signals with frequency content less than  $\frac{1}{\sigma}$  radians per second. Using the Tustin approximation (10.2), the dirty derivative in the  $z$ -domain becomes

$$U_D(z) = \frac{\frac{2}{T_s} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)}{\frac{2\sigma}{T_s} \left( \frac{1-z^{-1}}{1+z^{-1}} \right) + 1} E(z) = \frac{\left( \frac{2}{2\sigma+T_s} \right) (1 - z^{-1})}{1 - \left( \frac{2\sigma-T_s}{2\sigma+T_s} \right) z^{-1}} E(z).$$

Transforming to the time domain, we have

$$u_D[n] = \left( \frac{2\sigma - T_s}{2\sigma + T_s} \right) u_D[n - 1] + \left( \frac{2}{2\sigma + T_s} \right) (e[n] - e[n - 1]). \quad (10.4)$$

### 10.1.1 Integrator Anti-Windup

A potential problem with a straight-forward implementation of PID controllers is integrator wind up. When the error  $y_r - y$  is large and a large error persists for an extended period of time, the value of the integrator can become large, or “wind up.” A large integrator will cause  $u$  to saturate, which will cause the system to push with maximum effort in the direction needed to correct the error. Since the value of the integrator will continue to wind up until the error signal changes sign, the control signal may not come out of saturation until well after the error has changed sign, which can cause a large overshoot and may potentially destabilize the system.

Since integrator wind up can destabilize the system, it is important to add an anti-windup scheme. A number of different anti-windup schemes are possible. In

in this section we will discuss two schemes that are commonly used in practice. The first scheme is based on the fact that integrators are typically used to correct for steady state error and the fact that overshoot is caused by integrating error during the transients of the step response. Therefore, the integrator can be turned off when the derivative of the error is large. This anti-windup scheme is implemented by an `if`-statement conditioned on the size of  $\dot{e}$ .

The second anti-windup scheme is to subtract from the integrator exactly the amount needed to keep  $u$  at the saturation bound. In particular, let

$$u_{\text{unsat}} = k_P e + k_D u_D + k_I u_I$$

denote the unsaturated control value before applying the anti-windup scheme. After the saturation block we get

$$u = \text{sat}(u_{\text{unsat}}) = \text{sat}(k_P e + k_D u_D + k_I u_I).$$

The idea of anti-windup is to modify the output of the integrator from  $u_I$  to  $u_I^+$  so that

$$k_P e + k_D u_D + k_I u_I^+ = \text{sat}(k_P e + k_D u_D + k_I u_I) = u.$$

Subtracting  $u_{\text{unsat}}$  from  $u$  gives

$$\begin{aligned} u - u_{\text{unsat}} &= (k_P e + k_D u_D + k_I u_I^+) - (k_P e + k_D u_D + k_I u_I) \\ &= k_I(u_I^+ - u_I). \end{aligned}$$

Solving for  $u_I^+$  gives

$$u_I^+ = u_I + \frac{1}{k_I}(u - u_{\text{unsat}}),$$

which is the updated value of the integrator that ensures that the control signal remains just out of saturation. Note that if the control value is not in saturation then  $u = u_{\text{unsat}}$  and the integrator value remains unchanged.

### 10.1.2 Implementation

Python code that implements a general class for a PID loop is shown below.

```

1 import numpy as np
2
3 class PIDControl:
4     def __init__(self, kp, ki, kd, limit, sigma, Ts, flag=True):
5         self.kp = kp # Proportional control gain
6         self.ki = ki # Integral control gain
7         self.kd = kd # Derivative control gain
8         self.limit = limit # The output saturates at this limit
9         self.sigma = sigma # dirty derivative bandwidth is 1/sigma
10        self.beta = (2.0*sigma-Ts)/(2.0*sigma+Ts)
11        self.Ts = Ts # sample rate
12        self.flag = flag
13        # if flag == True, then returns
14        #     u = kp * error + ki * integral(error) + kd * error_dot.

```

```

15     # else returns
16     # u = kp * error + ki * integral(error) - kd * y_dot.
17     self.y_dot = 0.0 # estimated derivative of y
18     self.y_d1 = 0.0 # Signal y delayed by one sample
19     self.error_dot = 0.0 # estimated derivative of error
20     self.error_d1 = 0.0 # Error delayed by one sample
21     self.integrator = 0.0 # integrator
22
23 def PID(self, y_r, y):
24     # Compute the current error
25     error = y_r - y
26     # integrate error using trapazoidal rule
27     self.integrator = self.integrator \
28         + (self.Ts/2) * (error + self.error_d1)
29     # PID Control
30     if flag is True:
31         # differentiate error
32         self.error_dot = self.beta * self.error_dot \
33             + (1-self.beta)/self.Ts * (error - self.error_d1)
34     # PID control
35     u_unsat = self.kp*error \
36         + self.ki*self.integrator \
37         + self.kd*self.error_dot
38 else:
39     # differentiate y
40     self.y_dot = self.beta * self.y_dot \
41         + (1-self.beta)/self.Ts * (y - self.y_d1)
42     # PID control
43     u_unsat = self.kp*error \
44         + self.ki*self.integrator \
45         - self.kd*self.y_dot
46     # return saturated control signal
47     u_sat = self.saturate(u_unsat)
48     # integrator anti - windup
49     if self.ki != 0.0:
50         self.integrator = self.integrator \
51             + 1.0 / self.ki * (u_sat - u_unsat)
52     # update delayed variables
53     self.error_d1 = error
54     self.y_d1 = y
55     return u_sat
56
57 def PD(self, y_r, y):
58     # Compute the current error
59     error = y_r - y
60     # PD Control
61     if flag is True:
62         # differentiate error
63         self.error_dot = self.beta * self.error_dot \
64             + (1-self.beta)/self.Ts * (error - self.error_d1)
65     # PD control
66     u_unsat = self.kp*error \
67         + self.kd*self.error_dot
68 else:
69     # differentiate y
70     self.y_dot = self.beta * self.y_dot \
71         + (1-self.beta)/self.Ts * (y - self.y_d1)

```

```

72     # PD control
73     u_unsat = self.kp*error \
74             - self.kd*self.y_dot
75     # return saturated control signal
76     u_sat = self.saturate(u_unsat)
77     # update delayed variables
78     self.error_d1 = error
79     self.y_d1 = y
80     return u_sat
81
82 def saturate(self,u):
83     if abs(u) > self.limit:
84         u = self.limit*np.sign(u)
85     return u

```

Listing 10.1: PIDControl.py

The PID controller class is initialized on lines 4–21, where  $k_p$ ,  $k_i$ , and  $k_d$  are the control gains,  $\text{limit}$  is the saturation limit for the control signal,  $\sigma$  is the dirty derivative gain,  $T_s$  is the sample rate, and  $\text{flag}$  is a flag that indicates whether the controller differentiates the error or the output. The internal memory used to implement the differentiators and integrators are initialized to zero in lines 17–21, where  $y_{\text{d}1}$  is the signal  $y$  delayed by one sample and  $\text{error}_{\text{d}1}$  is the error delayed by one signal. PID control is implemented in lines 23–55 and PID control is implemented in lines 57–80. The integrator update is shown in lines 50–51.

### 10.1.3 Gain Selection for PID control

Gain selection for PID control usually proceeds in the following manner. The proportional and derivative gains  $k_P$  and  $k_D$  are first selected according to the methods discussed in Chapter 8. The integral gain  $k_I$  is then tuned by starting with zero and then slowly increasing the gain until the steady state error is removed. The parameters associated with the anti-windup scheme, especially  $\bar{v}$ , are also adjusted during this phase. The proportional and derivative gains can then be adjusted if necessary by retuning  $\omega_n$  and  $\zeta$ . In the case of successive loop closure, integrators are only added on the outer loop.

The addition of an integrator has a destabilizing effect on the feedback system. This effect is most easily understood by using the root locus method described in Appendix P.6, where it can be seen that for second order systems, a large  $k_I$  will always result in closed-loop poles in the right half plane. The root locus can also be used to aid in the selection of the integral gain  $k_I$ , where the value of  $k_I$  is selected so that integrator has minimal impact on the closed-loop poles designed using PD control. A more effective method for selecting integral gains will be discussed in Chapter 12.

## 10.2 Design Study A. Single Link Robot Arm



### Example Problem A.10

The objective of this problem is to implement the PID controller using only measured outputs of the system.

- (a) Modify the system dynamics file so that the parameters vary by up to 20% each time they are run (uncertainty parameter = 0.2).
- (b) Change the simulation files so that the input to the controller is the output and not the state. The controller should only assume knowledge of the angle  $\theta$  and the reference angle  $\theta_r$ .
- (c) Implement the PID controller designed in Problems A.8. Use the dirty derivative gain of  $\sigma = 0.05$ . Tune the integrator to remove the steady state error caused by the uncertain parameters.

### Solution

A Python class that implements a PID controller for the single link robot arm is shown below.

```

1 import numpy as np
2 import armParamHW10 as P
3 import sys
4 sys.path.append('..') # add parent directory
5 import armParam as P0
6 from PIDControl import PIDControl
7
8
9 class armController:
10
11     def __init__(self):
12         # Instantiates the PD object
13         self.thetaCtrl = PIDControl(P.kp, P.ki, P.kd,
14                                     P0.tau_max, P.sigma, P.Ts)
15         self.limit = P0.tau_max
16
17     def update(self, theta_r, y):
18         theta = y.item(0)
19
20         # compute feedback linearized torque tau_fl
21         #tau_e = P0.m * P0.g * (P0.ell / 2.0) * np.cos(0.0)
22
23         # compute feedback linearized torque tau_fl
24         tau_fl = P0.m * P0.g * (P0.ell / 2.0) * np.cos(theta)
25
26         # compute the linearized torque using PD
27         tau_tilde = self.thetaCtrl.PID(theta_r, theta, False)
28

```

```

29     # compute total torque
30     tau = tau_fl + tau_tilde
31     tau = self.saturate(tau)
32
33     return tau
34
35 def saturate(self, u):
36     if abs(u) > self.limit:
37         u = self.limit*np.sign(u)
38
39     return u

```

Listing 10.2: armController.py

Python code that implements the complete simulation is shown below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import armParam as P
6 from hw3.armDynamics import armDynamics
7 from armController import armController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.armAnimation import armAnimation
10 from hw2.dataPlotter import dataPlotter
11
12 # instantiate arm, controller, and reference classes
13 arm = armDynamics(alpha=0.2)
14 controller = armController()
15 reference = signalGenerator(amplitude=30*np.pi/180.0,
16                               frequency=0.05)
17 disturbance = signalGenerator(amplitude=0)
18
19 # instantiate the simulation plots and animation
20 dataPlot = dataPlotter()
21 animation = armAnimation()
22
23 t = P.t_start # time starts at t_start
24 y = arm.h() # output of system at start of simulation
25
26 while t < P.t_end: # main simulation loop
27
28     # Get referenced inputs from signal generators
29     # Propagate dynamics in between plot samples
30     t_next_plot = t + P.t_plot
31
32     # updates control and dynamics at faster simulation rate
33     while t < t_next_plot:
34         r = reference.square(t)
35         d = disturbance.step(t) # input disturbance
36         n = 0.0 #noise.random(t) # simulate sensor noise
37         u = controller.update(r, y + n) # update controller
38         y = arm.update(u + d) # propagate system
39         t = t + P.Ts # advance time by Ts
40
41     # update animation and data plots

```

```

42     animation.update(arm.state)
43     dataPlot.update(t, r, arm.state, u)
44
45     # the pause causes the figure to display during simulation
46     plt.pause(0.0001)
47
48 # Keeps the program from closing until the user presses a button.
49 print('Press key to close')
50 plt.waitforbuttonpress()
51 plt.close()

```

Listing 10.3: armSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### 10.3 Design Study B. Inverted Pendulum



#### Example Problem B.10

The objective of this problem is to implement the PID controller using only measured outputs of the system.

- (a) Modify the system dynamics file so that the parameters  $m_1$ ,  $m_2$ ,  $\ell$  and  $b$  vary by up to 20% of their nominal value each time they are run (uncertainty parameter = 0.2).
- (b) Change the simulation files so that the input to the controller is the output and not the state. The controller should only assume knowledge of the position  $z$  and the angle  $\theta$ , as well as the reference position  $z_r$ .
- (c) Implement the nested PID loops designed in Problems B.8. Use the dirty derivative gain of  $\sigma = 0.05$ . Tune the integrator to remove the steady state error caused by the uncertain parameters.

#### Solution

A Python class that implements a PID controller for the inverted pendulum is shown below.

```

1 import numpy as np
2 import pendulumParam as P
3 import pendulumParamHW10 as P10
4 from PIDControl import PIDControl
5
6 class pendulumController:
7     def __init__(self):
8         # Instantiates the SS_ctrl object
9         self.zCtrl = PIDControl(P10.kp_z, P10.ki_z, P10.kd_z,

```

```

10          P10.theta_max, P10.sigma, P10.Ts)
11      self.thetaCtrl = PIDControl(P10.kp_th, 0.0, P10.kd_th,
12                                  P10.F_max, P10.sigma, P10.Ts)
13      self.filter = zeroCancelingFilter()
14
15  def update(self, z_r, y):
16      z = y.item(0)
17      theta = y.item(1)
18
19      # the reference angle for theta comes from
20      # the outer loop PID control
21      theta_r = self.zCtrl.PID(z_r, z, flag=False)
22
23      # low pass filter the outer loop to cancel
24      # left-half plane zero and DC-gain
25      theta_r = self.filter.update(theta_r)
26
27      # the force applied to the cart comes from
28      # the inner loop PD control
29      F = self.thetaCtrl.PD(theta_r, theta, flag=False)
30
31      return F
32
33 class zeroCancelingFilter:
34     def __init__(self):
35         self.a = -3.0/(2.0*P.ell*P10.DC_gain)
36         self.b = np.sqrt(3.0*P.g/(2.0*P.ell))
37         self.state = 0.0
38
39     def update(self, input):
40         # integrate using RK1
41         self.state = self.state \
42                     + P.Ts * (-self.b*self.state + self.a*input)
43
44     return self.state

```

Listing 10.4: pendulumController.py

Python code for the closed loop system is shown below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import pendulumParam as P
5 from hw3.pendulumDynamics import pendulumDynamics
6 from pendulumController import pendulumController
7 from hw2.signalGenerator import signalGenerator
8 from hw2.pendulumAnimation import pendulumAnimation
9 from hw2.dataPlotter import dataPlotter
10
11 # instantiate pendulum, controller, and reference classes
12 pendulum = pendulumDynamics()
13 controller = pendulumController()
14 reference = signalGenerator(amplitude=0.5, frequency=0.06)
15 disturbance = signalGenerator(amplitude=0.1)
16 noise = signalGenerator(amplitude=0.01)
17
18 # instantiate the simulation plots and animation
19 dataPlot = dataPlotter()

```

```

20 animation = pendulumAnimation()
21
22 t = P.t_start # time starts at t_start
23 y = pendulum.h() # output of system at start of simulation
24
25 while t < P.t_end: # main simulation loop
26     t_next_plot = t + P.t_plot
27
28     while t < t_next_plot:
29         r = reference.sin(t) # reference input
30         d = disturbance.step(t) # input disturbance
31         n = noise.random(t) # simulate sensor noise
32         u = controller.update(r, y + n) # update controller
33         y = pendulum.update(u + d) # propagate system
34         t = t + P.Ts # advance time by Ts
35
36     # update animation and data plots
37     animation.update(pendulum.state)
38     dataPlot.update(t, r, pendulum.state, u)
39     plt.pause(0.0001)
40
41 # Keeps the program from closing until the user presses a button.
42 print('Press key to close')
43 plt.waitforbuttonpress()
44 plt.close()

```

Listing 10.5: pendulumSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 10.4 Design Study C. Satellite Attitude Control



### Example Problem C.10

The objective of this problem is to implement the PID controller using only measured outputs of the system.

- (a) Modify the system dynamics file so that the parameters  $J_s$ ,  $J_p$ ,  $k$  and  $b$  vary by up to 20% of their nominal value each time they are run (uncertainty parameter = 0.2).
- (b) Change the simulation files so that the input to the controller is the output and not the state. Assume that the controller only has knowledge of the angles  $\phi$  and  $\theta$  as well as the reference angle  $\phi_r$ .
- (c) Implement the nested PID loops designed in Problems C.8. Use the dirty derivative gain of  $\sigma = 0.05$ . Tune the integrator to remove the steady state error caused by the uncertain parameters.

## Solution

A Python class that implements a PID controller for the satellite is shown below.

```

1 import satelliteParamHW10 as P
2 from PIDControl import PIDControl
3
4 class satelliteController:
5     def __init__(self):
6         # Instantiates the SS_ctrl object
7         self.phiCtrl = PIDControl(P.kp_phi, P.ki_phi, P.kd_phi,
8                                     P.theta_max, P.sigma, P.Ts)
9         self.thetaCtrl = PIDControl(P.kp_th, 0.0, P.kd_th,
10                                     P.tau_max, P.sigma, P.Ts)
11
12     def update(self, phi_r, y):
13         theta = y.item(0)
14         phi = y.item(1)
15
16         # the reference angle for theta comes from
17         # the outer loop PD control
18         theta_r = self.phiCtrl.PID(phi_r, phi, flag=False)
19
20         # the torque applied to the base comes from
21         # the inner loop PD control
22         tau = self.thetaCtrl.PD(theta_r, theta, flag=False)
23
24         return tau

```

Listing 10.6: satelliteController.py

Python code that implements the complete simulation is shown below

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import satelliteParam as P
6 from hw3.satelliteDynamics import satelliteDynamics
7 from satelliteController import satelliteController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.satelliteAnimation import satelliteAnimation
10 from hw2.dataPlotter import dataPlotter
11
12 # instantiate satellite, controller, and reference classes
13 satellite = satelliteDynamics(alpha=0.2)
14 controller = satelliteController()
15 reference = signalGenerator(amplitude=15.0*np.pi/180.0,
16                               frequency=0.02)
17 disturbance = signalGenerator(amplitude=0.0)
18
19 # instantiate the simulation plots and animation
20 dataPlot = dataPlotter()
21 animation = satelliteAnimation()
22
23 t = P.t_start # time starts at t_start
24 y = satellite.h() # output of system at start of simulation

```

```

25
26 while t < P.t_end: # main simulation loop
27     # Propagate dynamics in between plot samples
28     t_next_plot = t + P.t_plot
29
30     # updates control and dynamics at faster simulation rate
31     while t < t_next_plot:
32         r = reference.square(t) # reference input
33         d = disturbance.step(t) # input disturbance
34         n = 0.0 #noise.random(t) # simulate sensor noise
35         u = controller.update(r, y + n) # update controller
36         y = satellite.update(u + d) # propagate system
37         t = t + P.Ts # advance time by Ts
38     # update animation and data plots
39     animation.update(satellite.state)
40     dataPlot.update(t, r, satellite.state, u)
41
42     # the pause causes the figure to display during simulation
43     plt.pause(0.0001)
44
45 # Keeps the program from closing until the user presses a button.
46 print('Press key to close')
47 plt.waitforbuttonpress()
48 plt.close()

```

Listing 10.7: satelliteSim.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### Important Concepts:

- The Tustin approximation provides a relationship between the Laplace s-domain and the Z-domain.
- The trapezoid rule can be used to approximate the integrator.
- The band-limited or dirty derivative is used in place of a pure differentiator to avoid non-causal implementations (i.e. current outputs depend on future inputs).
- Integrator anti-windup prevents overshoot and destabilization that may arise when large integrator errors exist.

## Notes and References

A standard reference for digital implementation of PID controllers is [18]. Simple anti-wind-up schemes are discussed in [18, 4].

# Part IV

## Observer Based Control Design

In Part III we focused on designing feedback control systems for second order systems. Since second order systems have two poles, we saw that it is possible to use feedback control with two gains to place the closed-loop poles at any desired location. In Chapter 11 of this book we will see that for an  $n^{th}$  order system it is possible to arbitrarily place the closed-loop poles, but that this will require  $n$  feedback gains. In that sense, the full state feedback controllers that we will derive in Chapter 11 are a generalization of PD control to higher order systems. This generalization is most easily accomplished using the state space model of the system derived in Chapter 6. In fact, the state space model leads to efficient numerical routines for designing and implementing the feedback controller.

In Chapter 12 we show how integral control can be added to the state feedback controllers designed in Chapter 11. The method introduced in Chapter 12 has a significant advantage over the integrator tuning method discussed in Chapter 10 in the sense that tuning the integral gain will not impact the other closed-loop poles and therefore allows for greater design flexibility.

We saw in Chapter 10 that in order to implement PD/PID control, the derivative of the output needs to be reconstructed using a digital differentiator. The generalization of this idea to state feedback control is that the full state of the system must be reconstructed. The tool that we will use to reconstruct the system state is called an *observer*. The observer design problem will be discussed in Chapter 13.

It turns out that observers can be used in a variety of different contexts outside of feedback control. In particular, observers can be used to solve target tracking problems and to estimate the system parameters of complex dynamic systems. In the setting of feedback control where unknown input and output disturbances act on the system, observers can be used to estimate the unknown disturbances. In Chapter 14 we will show how to design the observer to estimate both the system

state and the unknown constant input disturbance. We will also show that estimating the input disturbance is mathematically and functionally similar to adding an integrator.

The design tools developed in Part IV of this book use state space models and require the use of linear algebra and matrix analysis. A brief review of linear algebra concepts that will be used in Part IV are given in Appendix P.7.



# 11

## Full State Feedback

### Learning Objectives:

- Design a full-state feedback controller.
- Evaluate the controllability of a system.
- Compute the feedforward gain that ensures that the DC-gain is equal to one.

### 11.1 Theory

In this chapter we will show how to use full state feedback to stabilize a linear time-invariant system represented in state space form. Recall from Section 6.1.2, that for state space equations given by

$$\dot{x} = Ax + Bu \quad (11.1)$$

$$y = Cx + Du, \quad (11.2)$$

the poles of the system are the eigenvalues of  $A$ , or in other words, the roots of the open-loop characteristic equation

$$\Delta_{ol}(s) = \det(sI - A) = 0.$$

The matrix  $A$  is called the state-interaction matrix because it specifies how the states interact with each other. The basic idea behind full state feedback is to use the feedback signal to change the interaction matrix so that the poles are in specified locations. A full state feedback controller is given by

$$u = -Kx + \nu, \quad (11.3)$$

where  $K \in \mathbb{R}^{m \times n}$  are the feedback gains, and  $\nu$  will be a signal that is specified later. Using (11.3) in Equation (11.1) results in the closed-loop state space

equations

$$\dot{x} = Ax + B(-Kx + \nu) = (A - BK)x + B\nu \quad (11.4)$$

$$y = Cx, \quad (11.5)$$

where the interaction matrix is now  $A - BK$ , and so the closed-loop poles will be given by the eigenvalues of  $A - BK$ , or in other words, the roots of the closed-loop characteristic equation

$$\Delta_{cl}(s) = \det(sI - (A - BK)) = 0.$$

A natural question that will be addressed in the remainder of this section, is how to select  $K$  so that the closed-loop characteristic equation is equal to a desired closed-loop characteristic  $\Delta_{cl}^d(s)$ , where the designer selects the roots of  $\Delta_{cl}^d(s) = 0$  to achieve desired closed-loop performance.

Section 6.1.2 showed that the state space model given by Equations (11.1)–(11.2) is equivalent to the transfer function model given by

$$Y(s) = C(sI - A)^{-1}BU(s). \quad (11.6)$$

While the mapping from state-space model to transfer function model is unique, the reverse is not true. In fact, there are an infinite number of state space models that correspond to a particular transfer function model. Accordingly, we say that a state space model is a *realization* of the transfer function model, if it produces the same input-output behavior when the initial conditions are zero.

In particular, define the change of variables

$$z = P^{-1}x, \quad (11.7)$$

where  $P$  is an invertible matrix. Then

$$\dot{z} = P^{-1}\dot{x} = P^{-1}(Ax + Bu) = P^{-1}APz + P^{-1}Bu \quad (11.8)$$

$$y = Cx = CPz \quad (11.9)$$

is an alternative state space representation of system (11.1)–(11.2). To show that these two representations have the same input-output behavior, or in other words, the same transfer function, note from Equation (11.6) that the transfer function of system (11.8)–(11.9) is

$$\begin{aligned} Y(s) &= (CP(sI - P^{-1}AP)^{-1}P^{-1}B)U(s) \\ &= (CP(sP^{-1}P - P^{-1}AP)^{-1}P^{-1}B)U(s) \\ &= (CP(P^{-1}(sI - A)P)^{-1}P^{-1}B)U(s) \\ &= (CPP^{-1}(sI - A)^{-1}PP^{-1}B)U(s) \\ &= (C(sI - A)^{-1}B)U(s), \end{aligned}$$

which is identical to Equation (11.6). Therefore Equations (11.1)–(11.2) and Equations (11.8)–(11.9) are realizations of the same system.

### 11.1.1 State Space Realization in Control Canonic Form

In this section, we will derive one specific state space realization for single-input single-output transfer functions, where the state feedback problem is particularly simple. The realization that we derive below is called the *control canonic form* of the system.

Consider the general SISO monic transfer function model given by

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} U(s), \quad (11.10)$$

where  $m < n$ . The block diagram of the system is shown in [Fig. 11-1](#). The first

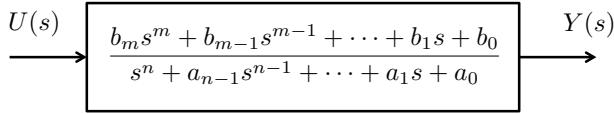


Figure 11-1: General transfer function for a SISO system.

step in deriving the control canonic state space equations is to artificially decompose the transfer function as shown in [Fig. 11-2](#), where the numerator polynomial and the denominator polynomial appear in separate cascaded blocks, and where  $Z(s)$  is an artificial intermediate variable. In the  $s$ -domain, the equations corre-

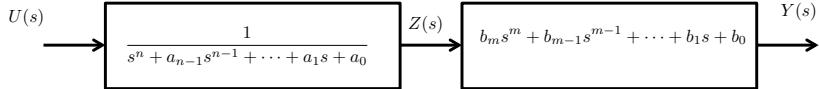


Figure 11-2: General transfer function for a SISO system decomposed into state dynamics and output dynamics.

sponding to [Fig. 11-2](#) can be written as

$$\begin{aligned} Z(s) &= \frac{1}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} U(s) \\ Y(s) &= (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) Z(s). \end{aligned}$$

Taking the inverse Laplace transform of these equations gives

$$\frac{d^n z}{dt^n} = u(t) - a_{n-1} \frac{d^{n-1} z}{dt^{n-1}} - \cdots - a_1 \dot{z} - a_0 z(t) \quad (11.11)$$

$$y(t) = b_m \frac{d^m z}{dt^m} + b_{m-1} \frac{d^{m-1} z}{dt^{m-1}} + \cdots + b_1 \dot{z} + b_0 z(t). \quad (11.12)$$

The next step in the derivation of the state space equations is to draw the analog computer implementation of Equations (11.11) and (11.12), which is shown in [Fig. 11-3](#). Labeling the output of the integrators as the state variables  $x_1, \dots, x_n$ ,

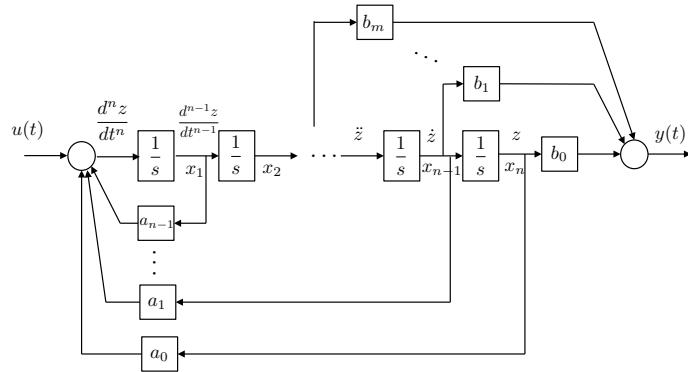


Figure 11-3: Block diagram showing the analog computer implementation of Equations (11.11) and (11.12).

as shown in **Fig. 11-3**, we can write the state space equations as

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} &= \begin{pmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \quad (11.13) \\ y &= (0 \quad \dots \quad 0 \quad b_m \quad \dots \quad b_1 \quad b_0) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + 0 \cdot u. \end{aligned}$$

In the case when  $m = n$  it is straightforward to show that the output equation becomes

$$y = (b_{n-1} - b_n a_{n-1}, \dots, b_1 - b_n a_1, b_0 - b_n a_0) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + b_n \cdot u.$$

We will define the control canonic realization as

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u \\ y &= C_c x_c + D u, \end{aligned}$$

where

$$A_c \triangleq \begin{pmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (11.14)$$

$$B_c \triangleq \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (11.15)$$

$$C_c \triangleq \begin{cases} \begin{pmatrix} 0 & \dots & 0 & b_m & \dots & b_1 & b_0 \end{pmatrix} & m < n \\ \begin{pmatrix} b_{n-1} - b_n a_{n-1}, & \dots & b_1 - b_n a_1, & b_0 - b_n a_0 \end{pmatrix} & m = n \end{cases} \quad (11.16)$$

$$D \triangleq \begin{cases} 0 & m < n \\ b_n & m = n \end{cases}. \quad (11.17)$$

Note that the negative of the coefficients of the denominator of (11.10) constitute the first row of  $A_c$ , and that the remainder of  $A_c$  has ones on the lower off diagonal and zeros in all other elements. Matrices with this structure are said to be in *companion form*. The  $B_c$  vector has a single one as its first element, with the remaining elements equal to zero. The coefficients of the numerator of Equation (11.10) are the last  $p$ -elements of  $C_c$ .

As an example, suppose that the transfer function model of the system is given by

$$Y(s) = \frac{9s + 20}{s^3 + 6s^2 - 11s + 8} U(s),$$

then the control canonic realization of the system is given by

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} -6 & 11 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \\ y &= (0 \quad 9 \quad 20) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \end{aligned} \quad (11.18)$$

### 11.1.2 Full State Feedback using Control Canonic Form

The design of a state feedback controller is particularly easy when the state space equations are in control canonic form. The key insight is that there is a direct relationship between matrices in companion form, as shown in Equation (11.14)

and their characteristic polynomial

$$\Delta_{ol}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0.$$

Suppose that we let

$$u = -K_c x_c + \nu = -\begin{pmatrix} k_1 & k_2 & \dots & k_{n-1} & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \nu. \quad (11.19)$$

Substituting into Equation (11.13) and rearranging gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} -(a_{n-1} + k_1) & -(a_{n-2} + k_2) & \dots & -(a_1 + k_{n-1}) & -(a_0 + k_n) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \nu.$$

Since the state interaction matrix is still in companion form, the characteristic polynomial of the closed-loop system is given by

$$\Delta_{cl}(s) = s^n + (a_{n-1} + k_1)s^{n-1} + \cdots + (a_1 + k_{n-1})s + (a_0 + k_n).$$

If the desired characteristic polynomial is given by

$$\Delta_{cl}^d(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0, \quad (11.20)$$

then we have that

$$\begin{aligned} a_{n-1} + k_1 &= \alpha_{n-1} \\ a_{n-2} + k_2 &= \alpha_{n-2} \\ &\vdots \\ a_1 + k_{n-1} &= \alpha_1 \\ a_0 + k_n &= \alpha_0. \end{aligned}$$

Defining the row vectors

$$\begin{aligned} \mathbf{a}_A &= (a_{n-1}, a_{n-2}, \dots, a_1, a_0) \\ \boldsymbol{\alpha} &= (\alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_1, \alpha_0) \\ K_c &= (k_1, k_2, \dots, k_{n-1}, k_n), \end{aligned}$$

we have that the feedback gains satisfy

$$K_c = \boldsymbol{\alpha} - \mathbf{a}_A, \quad (11.21)$$

where the subscript “c” is used to emphasize that the gains are when the state equations are in control canonic form.

As an example, consider the state equations given in Equation (11.18). The open-loop characteristic polynomial is given by

$$\Delta_{ol}(s) = s^3 + 6s^2 - 11s + 8,$$

with roots at  $-7.5885$ , and  $0.7942 \pm j0.6507$ . Suppose that the desired closed-loop characteristic polynomial is

$$\Delta_{cl}^d(s) = s^3 + 5s^2 + 12s + 8,$$

with roots at  $-1$  and  $-2 \pm j2$ , then  $\mathbf{a}_A = (6, -11, 8)$ , and  $\boldsymbol{\alpha} = (5, 12, 8)$ . The feedback gains that place the poles at the desired locations are therefore given by

$$K_c = \boldsymbol{\alpha} - \mathbf{a}_A = (5, 12, 8) - (6, -11, 8) = (-1, 23, 0).$$

### 11.1.3 Full State Feedback: General Case

The formula given in Equation (11.21) assumes that the state space equations are in control canonic form. It turns out that the closed-loop eigenvalues cannot always be arbitrarily assigned to desired locations. When they can, the system  $\dot{x} = Ax + Bu$  is said to be *controllable*. In this section we will derive an expression for the state feedback gains, and we will simultaneously derive a simple test for determining when the system is controllable.

Recall from Equations (11.7)–(11.9) that state space equations can be transformed using the change of variables in Equation (11.7) without modifying the input-output properties of the system. This implies that the change of variables does not change the characteristic polynomial. We can verify this fact explicitly as follows:

$$\begin{aligned} \det(sI - P^{-1}AP) &= \det(sP^{-1}P - P^{-1}AP) \\ &= \det(P^{-1}(sI - A)P) \\ &= \det(P^{-1})\det(sI - A)\det(P) \\ &= \det(sI - A). \end{aligned}$$

Therefore, to derive full state feedback gains for the general state space equations

$$\dot{x} = Ax + Bu, \quad (11.22)$$

the idea is to find a state transformation matrix  $P$  so that the transformed system is in control canonic form, i.e., find  $P$  so that

$$x_c = P^{-1}x, \quad (11.23)$$

and

$$\dot{x}_c = P^{-1}APx_c + P^{-1}Bu = A_cx_c + B_cu, \quad (11.24)$$

where  $A_c$  and  $B_c$  have the form given by Equations (11.14) and (11.15), respectively. Therefore, we need to find  $P$  so that  $P^{-1}B = B_c$ , and  $P^{-1}AP = A_c$ , which are satisfied if and only if  $B = PB_c$  and  $AP = PA_c$ . Let  $P = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$ , where  $\mathbf{p}_i \in \mathbf{R}^n$  is the  $i^{th}$  column of  $P$ . The requirement that  $B = PB_c$  implies that

$$B = (\mathbf{p}_1 \quad \mathbf{p}_2 \quad \dots \quad \mathbf{p}_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{p}_1.$$

Therefore,  $\mathbf{p}_1 = B$ . The requirement that  $PA_c = AP$  implies that

$$(\mathbf{p}_1 \quad \mathbf{p}_2 \quad \dots \quad \mathbf{p}_n) \begin{pmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \vdots & 1 & 0 \end{pmatrix} = A(\mathbf{p}_1 \quad \mathbf{p}_2 \quad \dots \quad \mathbf{p}_n).$$

After multiplication, the first  $n - 1$  columns can be equated to get

$$-a_{n-1}\mathbf{p}_1 + \mathbf{p}_2 = A\mathbf{p}_1 \quad (11.25)$$

$$-a_{n-2}\mathbf{p}_1 + \mathbf{p}_3 = A\mathbf{p}_2 \quad (11.26)$$

$$\vdots \quad (11.27)$$

$$-a_1\mathbf{p}_1 + \mathbf{p}_n = A\mathbf{p}_{n-1}. \quad (11.28)$$

Equation (11.25), and the previous conclusion that  $\mathbf{p}_1 = B$ , implies that

$$\mathbf{p}_2 = a_{n-1}B + AB.$$

Using this results in Equation (11.26) gives

$$\mathbf{p}_3 = a_{n-2}B + a_{n-1}AB + A^2B.$$

We can continue to iteratively solve for each column of  $P$  until the  $(n - 1)^{th}$  column, which from Equation (11.28) is

$$\mathbf{p}_n = a_1B + a_2AB + \dots + a_{n-2}A^{n-2}B + A^{n-1}B.$$

Notice that each column in  $P$  is a linear combination of the elements  $B, AB, A^2B, \dots, A^{n-1}B$ . Accordingly, defining

$$\mathcal{C}_{A,B} = (B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B), \quad (11.29)$$

gives

$$\begin{aligned}\mathbf{p}_1 &= \mathcal{C}_{A,B} (1, 0, \dots, 0, 0)^\top, \\ \mathbf{p}_2 &= \mathcal{C}_{A,B} (a_{n-1}, 1, \vdots, 0, 0)^\top, \\ &\vdots \quad \vdots \\ \mathbf{p}_n &= \mathcal{C}_{A,B} (a_1, a_2, \vdots, a_{n-1}, 1)^\top.\end{aligned}$$

Defining the matrix

$$\mathcal{A}_A = \begin{pmatrix} 1 & a_{n-1} & a_{n-2} & \dots & a_2 & a_1 \\ 0 & 1 & a_{n-1} & \vdots & a_3 & a_2 \\ 0 & 0 & 1 & \vdots & a_4 & a_3 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad (11.30)$$

gives that the transformation matrix that converts the system into control canonic form is

$$P = \mathcal{C}_{A,B} \mathcal{A}_A.$$

Notice that since  $\mathcal{A}_A$  is an upper triangular matrix with ones on the diagonal, that  $\det(\mathcal{A}_A) = 1$ , implying that  $\mathcal{A}_A^{-1}$  always exists. It is clear that the matrix  $P$  is invertible with

$$P^{-1} = \mathcal{A}_A^{-1} \mathcal{C}_{A,B}^{-1}$$

if and only if  $\mathcal{C}_{A,B}^{-1}$  exists. The matrix  $\mathcal{C}_{A,B}$  in Equation (11.29) is called the *controllability matrix*, and a single input system is said to be *controllable* if  $\mathcal{C}$  is invertible. Therefore, when the original system (11.22) is controllable, there exists an invertible state transformation matrix  $P$  so that the change of variables in Equation (11.23) is well defined and that results in the transformed equations being in control canonic form via Equations (11.24).

In control canonic form, the state feedback equation is

$$u = -K_c x_c + \nu = -(\boldsymbol{\alpha} - \mathbf{a}_A)x_c + \nu.$$

Using Equation (11.23) the state feedback control can be expressed in the original states  $x$  as

$$\begin{aligned}u &= -(\boldsymbol{\alpha} - \mathbf{a}_A)P^{-1}x + \nu \\ &= -(\boldsymbol{\alpha} - \mathbf{a}_A)\mathcal{A}_A^{-1}\mathcal{C}_{A,B}^{-1}x + \nu \\ &= -Kx + \nu,\end{aligned} \quad (11.31)$$

where

$$K \triangleq (\alpha - \mathbf{a}_A) \mathcal{A}_A^{-1} \mathcal{C}_{A,B}^{-1}. \quad (11.32)$$

Equation (11.32) is called *Ackermann's formula* for the feedback gains. Since  $\mathcal{C}_{A,B}$  is only square in the single-input case, Ackermann's formula only holds for single input systems.

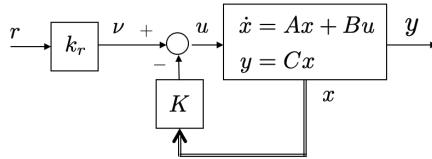
Using the full state feedback control in Equation (11.31) in the state equations (11.1)-(11.2) results in the closed-loop system

$$\begin{aligned}\dot{x} &= (A - BK)x + B\nu \\ y &= Cx.\end{aligned}$$

The resulting transfer function from  $\nu$  to  $y$  is given by

$$Y(s) = (C(sI - (A - BK))^{-1}B) \nu(s). \quad (11.33)$$

A block diagram of the resulting closed-loop system is shown in [Fig. 11-4](#) where the double line associated with  $x$  is meant to imply that the state  $x$  is not really available. The block diagram also shows the new input  $\nu$ . Since the objective is for the output  $y$  to track the reference input  $r$ , we let  $\nu(t) = k_r r(t)$ , as shown in [Fig. 11-4](#). The feedforward gain  $k_r$  is selected so that the DC-gain from  $r$  to  $y$  is equal to one.



[Figure 11-4:](#) Full state feedback with reference input  $r$ .

From Equation (11.33), the transfer function from  $R$  to  $Y$  is given by

$$Y(s) = (C(sI - (A - BK))^{-1}Bk_r) R(s), \quad (11.34)$$

and the DC-gain is

$$k_{DC} = \lim_{s \rightarrow 0} [C(sI - (A - BK))^{-1}Bk_r] = -C(A - BK)^{-1}Bk_r. \quad (11.35)$$

Therefore, the DC-gain is equal to one if

$$k_r = -\frac{1}{C(A - BK)^{-1}B}.$$

The matrix  $A - BK$  will be invertible when there are no eigenvalues at zero. Since  $K$  is selected to place the eigenvalues in the open left half plane, the matrix will always be invertible, and  $k_r$  exists. Note that in many applications, the output

measured by the sensors is not the same as the desired reference output. In other words, the state space equations may be given by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y_r &= C_r x \\ y_m &= C_m x,\end{aligned}$$

where  $y_r$  is the reference output and  $y_m$  is the measured output. In this case, the reference gain  $k_r$  is selected to make the DC-gain from  $r$  to  $y_r$  equal to one, or in other words

$$k_r = -\frac{1}{C_r(A - BK)^{-1}B}. \quad (11.36)$$

Chapter 13 will describe observers that use the measured output  $y_m$  to reconstruct the state  $x$ .

The feedforward gain computed in Equation (11.36) is for single input systems. More generally, if the size of the reference output  $y_r$  is equal to the size of the input vector  $u \in \mathbb{R}^m$ , then  $C_r \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times m}$ , which implies that the transfer matrix in Equation (11.34) is square and has dimension  $m \times m$ . Similarly, the resulting matrix gain  $K_r \in \mathbb{R}^{m \times m}$  will also be square and from Equation (11.35) will be equal to

$$K_r = -[C_r(A - BK)^{-1}B]^{-1}.$$

#### 11.1.4 State Feedback for Linearized Systems

Suppose that the state space model is from a linearized system, i.e, suppose that  $x \in \mathbb{R}^n$  is the state and that  $x_e \in \mathbb{R}^n$  is the equilibrium state, and that  $\tilde{x} = x - x_e$  is the state linearized around the equilibrium. Similarly, suppose that  $u \in \mathbb{R}^m$  is the input, that  $u_e \in \mathbb{R}^m$  is the input at equilibrium, and that  $\tilde{u} = u - u_e$  is the input linearized around equilibrium, and that  $y_r \in \mathbb{R}^p$  is the reference output,  $y_{re} = C_r x_e \in \mathbb{R}^p$  is the reference output at equilibrium, and  $\tilde{y}_r = y_r - y_{re}$  is the reference output linearized about equilibrium, then the linearized state space model is given by

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y}_r &= C_r \tilde{x}.\end{aligned}$$

Now suppose that the state feedback design process is used to find the state feedback controller

$$\tilde{u} = -K\tilde{x} + k_r \tilde{r} \quad (11.37)$$

where  $\tilde{r} = r - y_{re}$  is the reference input around the equilibrium output, and where the eigenvalues of  $(A - BK)$  are in the open left half plane, and the DC-gain from  $\tilde{r}$  to  $\tilde{y}_r$  is one. The controller input to the plant is therefore

$$u = u_e - K(x - x_e) + k_r(r - y_{re}). \quad (11.38)$$

## 11.2 Summary of Design Process - State Feedback

The state feedback design process can be summarized as follows.

Given the plant

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y_r &= C_r x\end{aligned}$$

and the desired pole locations  $p_1, p_2, \dots, p_n$ , find the gains  $K$  and  $k_r$  so that the feedback controller

$$u = -Kx + k_r r$$

places the poles at the desired location and results in a DC-gain equal to one.

**Step 1.** Check to see if the system is controllable by computing the controllability matrix

$$\mathcal{C}_{A,B} = [B, AB, \dots, A^{n-1}B]$$

and checking to see if  $\text{rank}(\mathcal{C}) = n$ .

**Step 2.** Find the open-loop characteristic polynomial

$$\Delta_{ol}(s) = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0,$$

and construct the row vector

$$\mathbf{a}_A = (a_{n-1}, a_{n-2}, \dots, a_1, a_0)$$

and the matrix

$$\mathcal{A}_A = \begin{pmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\ 0 & 1 & a_{n-1} & \cdots & a_3 & a_2 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & a_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

**Step 3.** Select the desired closed-loop polynomial

$$\begin{aligned}\Delta_{cl}^d(s) &= (s - p_1)(s - p_2) \cdots (s - p_n) \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0,\end{aligned}$$

and construct the row vector

$$\boldsymbol{\alpha} = (\alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_1, \alpha_0).$$

**Step 4.** Compute the desired gains as

$$\begin{aligned}K &= (\boldsymbol{\alpha} - \mathbf{a}_A)\mathcal{A}_A^{-1}\mathcal{C}_{A,B}^{-1} \\ k_r &= \frac{-1}{C_r(A - BK)^{-1}B}.\end{aligned}$$

Using Python, these steps can be implemented as follows:

```

1 import control as cnt
2 import numpy as np
3
4 CC = cnt.ctrb(A,B)
5
6 # check for controllability
7 if np.linalg.matrix_rank(CC) == np.size(A,1):
8     print('the system is controllable')
9 else:
10    print('the system is not controllable')
11
12 # place the poles at p1, p2, ..., pn
13 K = cnt.place(A, B, [p1, p2, p3])
14
15 # compute reference gain
16 kr = -1/(C @ np.linalg.inv(A - B @ K) @ B)

```

### 11.2.1 Simple example

Given the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 4 \end{pmatrix} u \\ y_r &= \begin{pmatrix} 5 & 0 \end{pmatrix} x\end{aligned}$$

find the feedback gain  $K$  to place the closed-loop poles at  $-1 \pm j$ , and the reference gain  $k_r$  so that the DC-gain is one.

**Step 1.** The controllability matrix is given by

$$\mathcal{C}_{A,B} = [B, AB] = \begin{pmatrix} 0 & 4 \\ 4 & 12 \end{pmatrix}.$$

The determinant is  $\det(\mathcal{C}) = -16 \neq 0$ , therefore the system is controllable.

**Step 2.** The open-loop characteristic polynomial is

$$\Delta_{ol}(s) = \det(sI - A) = \det \begin{pmatrix} s & -1 \\ -2 & s-3 \end{pmatrix} = s^2 - 3s - 2,$$

which implies that

$$\begin{aligned}\mathbf{a}_A &= (-3, -2) \\ \mathcal{A}_A &= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

**Step 3.** The desired closed-loop polynomial is

$$\Delta_{cl}^d(s) = (s+1-j)(s+1+j) = s^2 + 2s + 2,$$

which implies that

$$\boldsymbol{\alpha} = (2, \quad 2).$$

**Step 4.** The gains are therefore given as

$$\begin{aligned} K &= (\alpha - \mathbf{a}_A) \mathcal{A}_A^{-1} \mathcal{C}_{A,B}^{-1} \\ &= ((2, 2) - (-3, -2)) \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} \\ &= (5 \quad 4) \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} \\ &= (1 \quad \frac{5}{4}) \end{aligned}$$

$$\begin{aligned} k_r &= \frac{-1}{C_r(A - BK)^{-1}B} \\ &= \frac{-1}{(5 \quad 0) \left( \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} (1 \quad \frac{5}{4}) \right)^{-1} \begin{pmatrix} 0 \\ 4 \end{pmatrix}} \\ &= \frac{-1}{(5 \quad 0) \left( \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 4 & 5 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 4 \end{pmatrix}} \\ &= \frac{-1}{(5 \quad 0) \left( \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 4 \end{pmatrix}} \\ &= \frac{-1}{-10} = \frac{1}{10}. \end{aligned}$$

### 11.3 Design Study A. Single Link Robot Arm



#### Example Problem A.11

The objective of this problem is to implement a state feedback controller using the full state. Start with the simulation files developed in Homework A.10.

- (a) Select the closed loop poles as the roots of the equation  $s^2 + 2\zeta\omega_n + \omega_n^2 = 0$  where  $t_r = 0.4890$ , and  $\zeta = 0.707$ , which may be different than parameters found in Homework A.8.
- (b) Add the state space matrices  $A$ ,  $B$ ,  $C$ ,  $D$  derived in Homework A.6 to your param file.
- (c) Verify that the state space system is controllable by checking that  $\text{rank}(\mathcal{C}_{A,B}) = n$ .
- (d) Find the feedback gain  $K$  so that the eigenvalues of  $(A - BK)$  are equal to the desired closed loop poles. Find the reference gain  $k_r$  so that the DC-gain

from  $\theta_r$  to  $\theta$  is equal to one. Note that if you used the exact same pole values from Homework A.8,  $K = (k_p, k_d)$  where  $k_p$  and  $k_d$  are the proportional and derivative gains found in that problem. Why?

- (e) Modify the control code to implement the state feedback controller. To construct the state  $x = (\theta, \dot{\theta})^\top$  use a digital differentiator to estimate  $\dot{\theta}$ .

## Solution

From HW A.6, the state space equations for the single link robot arm are given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1.0000 \\ 0 & -0.667 \end{pmatrix} x + \begin{pmatrix} 0 \\ 66.667 \end{pmatrix} u \\ y &= (1 \quad 0) x.\end{aligned}$$

**Step 1.** The controllability matrix is therefore

$$\mathcal{C}_{A,B} = [B, AB] = \begin{pmatrix} 0 & 66.6667 \\ 66.6667 & -44.44440 \end{pmatrix}.$$

The determinant is  $\det(\mathcal{C}_{A,B}) = -4444 \neq 0$ , therefore the system is controllable.

**Step 2.** The open loop characteristic polynomial is

$$\Delta_{ol}(s) = \det(sI - A) = \det \begin{pmatrix} s & -1 \\ 0 & s + 0.667 \end{pmatrix} = s^2 + 0.667s,$$

which implies that

$$\begin{aligned}\mathbf{a}_A &= (0.667, 0) \\ \mathcal{A}_A &= \begin{pmatrix} 1 & 0.667 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

**Step 3.** The desired closed loop polynomial is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 6.3616s + 20.2408$$

which implies that

$$\boldsymbol{\alpha} = (6.3616, 20.2408).$$

**Step 4.** The gains are therefore given as

$$\begin{aligned}K &= (\boldsymbol{\alpha} - \mathbf{a}_A)\mathcal{A}_A^{-1}\mathcal{C}_{A,B}^{-1} = (0.3036 \quad 0.0854) \\ k_r &= \frac{-1}{C(A - BK)^{-1}B} = 0.3036.\end{aligned}$$

A Python class that implements state feedback control for the single link robot arm is shown below.

```

1 import numpy as np
2 import armParamHW11 as P
3
4 class armController:
5     # dirty derivatives to estimate thetadot
6     def __init__(self):
7         self.K = P.K # state feedback gain
8         self.kr = P.kr # Input gain
9         self.limit = P.tau_max # Maximum torque
10        self.Ts = P.Ts # sample rate of controller
11
12    def update(self, theta_r, x):
13        theta = x.item(0)
14        thetadot = x.item(1)
15
16        # compute feedback linearizing torque tau_fl
17        tau_fl = P.m * P.g * (P.ell / 2.0) * np.cos(theta)
18
19        # Compute the state feedback controller
20        tau_tilde = -self.K @ x + self.kr * theta_r
21
22        # compute total torque
23        tau = self.saturate(tau_fl + tau_tilde)
24
25        return tau
26
27    def saturate(self,u):
28        if abs(u) > self.limit:
29            u = self.limit*np.sign(u)
30
31        return u

```

Listing 11.1: armController.py

Python code that computes the control gains is given below.

```

1 # Single link arm Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import armParam as P
7
8 Ts = P.Ts # sample rate of the controller
9 tau_max = P.tau_max # limit on control signal
10 m = P.m
11 ell = P.ell
12 g = P.g
13
14 # tuning parameters
15 tr = 0.4
16 zeta = 0.707
17
18 # State Space Equations
19 # xdot = A*x + B*u

```

```

20 # y = C*x
21 A = np.array([[0.0, 1.0],
22                 [0.0, -1.0*P.b/P.m/(P.ell**2)]])
23
24 B = np.array([[0.0],
25                 [3.0/P.m/(P.ell**2)]])
26
27 C = np.array([[1.0, 0.0]])
28
29 # gain calculation
30 wn = 2.2/tr # natural frequency
31 des_char_poly = [1, 2*zeta*wn, wn**2]
32 des_poles = np.roots(des_char_poly)
33
34 # Compute the gains if the system is controllable
35 if np.linalg.matrix_rank(cnt.ctrb(A, B)) != 2:
36     print("The system is not controllable")
37
38 else:
39     K = (cnt.acker(A, B, des_poles))
40     kr = -1.0/(C @ np.linalg.inv(A - B @ K) @ B)
41
42 print('K: ', K)
43 print('kr: ', kr)
44 print(des_poles)

```

Listing 11.2: armParamHW11.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 11.4 Design Study B. Inverted Pendulum



### Example Problem B.11

The objective of this problem is to implement a state feedback controller using the full state. Start with the simulation files developed in Homework B.10.

- (a) Find the desired closed loop poles using the values for  $\zeta_z$ ,  $\omega_{n_\theta}$ , and  $\zeta_\theta$  selected in Homework B.8 and an  $\omega_{n_z}$  that gives  $t_z = 3t_\theta$ .
- (b) Add the state space matrices  $A$ ,  $B$ ,  $C$ ,  $D$  derived in Homework B.6 to your param file.
- (c) Verify that the state space system is controllable by checking that:  $\text{rank}(\mathcal{C}) = n$ .
- (d) Find the feedback gain  $K$  so that the eigenvalues of  $(A - BK)$  are equal to the desired closed loop poles. Find the reference gain  $k_r$  so that the DC-gain from  $z_r$  to  $z$  is equal to one.

- (e) Implement the state feedback scheme and tune the closed loop poles to get a good response. You should be able to get a much faster response using state space methods.

### Solution

From HW B.6, the state space equations for the inverted pendulum are given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -1.7294 & -0.0471 & 0 \\ 0 & 34.5882 & 0.1412 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0.9412 \\ -2.8235 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x,\end{aligned}\quad (11.39)$$

where Equation (12.2) represents the measured outputs. The reference output is the position  $z$ , which implies that

$$y_r = (1 \ 0 \ 0 \ 0) x.$$

**Step 1.** The controllability matrix is therefore

$$\begin{aligned}\mathcal{C}_{A,B} &= [B, AB, A^2B, A^3B] \\ &= \begin{pmatrix} 0 & 0.9412 & -0.0443 & 4.8851 \\ 0 & -2.8235 & 0.1329 & -97.6672 \\ 0.9412 & -0.0443 & 4.8851 & -0.4597 \\ -2.8235 & 0.1329 & -97.6672 & 5.2855 \end{pmatrix}.\end{aligned}$$

The determinant is  $\det(\mathcal{C}_{A,B}) = -6104.1 \neq 0$ , implying that the system is controllable.

**Step 2.** The open loop characteristic polynomial is

$$\Delta_{ol}(s) = \det(sI - A) = s^4 + 0.0471s^3 - 34.59s^2 - 1.384s$$

which implies that

$$\begin{aligned}\mathbf{a}_A &= (0.0471, -34.59, -1.384, 0) \\ \mathcal{A}_A &= \begin{pmatrix} 1 & 0.0471 & -34.59 & -1.384 \\ 0 & 1 & 0.0471 & -34.59 \\ 0 & 0 & 1 & 0.0471 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

**Step 3.** The desired closed loop polynomial is

$$\begin{aligned}\Delta_{cl}^d(s) &= (s^2 + 2\zeta_\theta\omega_{n_\theta}s + \omega_{n_\theta}^2)(s^2 + 2\zeta_z\omega_{n_z}s + \omega_{n_z}^2) \\ &= s^4 + 8.2955s^3 + 34.4139s^2 + 53.5334s + 41.6455\end{aligned}$$

which implies that

$$\boldsymbol{\alpha} = (8.2955, 34.4139, 53.5334, 41.6455).$$

**Step 4.** The gains are therefore given as

$$K = (\alpha - \mathbf{a}_A) \mathcal{A}_A^{-1} \mathcal{C}_{A,B}^{-1} = (-1.5050, -24.9399, -1.9847, -3.5829)$$

The feedforward reference gain  $k_r = -1/C_r(A - BK)^{-1}B$  is computed using  $C_r = (1, 0, 0, 0)$ , which gives

$$k_r = \frac{-1}{C_r(A - BK)^{-1}B} = -1.5050.$$

A Python class that implements a state feedback controller for the inverted pendulum is shown below.

```

1 import numpy as np
2 import pendulumParamHW11 as P
3
4 class pendulumController:
5     def __init__(self):
6         self.K = P.K # state feedback gain
7         self.kr = P.kr # Input gain
8         self.limit = P.F_max # Maximum force
9         self.Ts = P.Ts # sample rate of controller
10
11    def update(self, z_r, x):
12        # Compute the state feedback controller
13        F_unsat = -self.K @ x + self.kr * z_r
14        F_sat = self.saturate(F_unsat)
15        return F_sat.item(0)
16
17    def saturate(self,u):
18        if abs(u) > self.limit:
19            u = self.limit*np.sign(u)
20        return u

```

Listing 11.3: pendulumController.py

The gains are computed with the following Python script.

```

1 # Inverted Pendulum Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import pendulumParam as P
7
8 # sample rate of the controller
9 Ts = P.Ts
10
11 # saturation limits
12 F_max = 5.0 # Max Force, N
13 #####
14 ##### State Space #####
15 #####
16 ##### Tuning Parameters #####
17 # tuning parameters

```

```

18 tr_z = 1.5      # rise time for position
19 tr_theta = 0.5   # rise time for angle
20 zeta_z = 0.707  # damping ratio position
21 zeta_th = 0.707 # damping ratio angle
22
23 # State Space Equations
24 # xdot = A*x + B*u
25 # y = C*x
26 A = np.array([[0.0, 0.0, 1.0, 0.0],
27               [0.0, 0.0, 0.0, 1.0],
28               [0.0, -3*P.m1*P.g/4/(.25*P.m1+P.m2),
29                -P.b/(.25*P.m1+P.m2), 0.0],
30               [0.0, 3*(P.m1+P.m2)*P.g/2/(.25*P.m1+P.m2)/P.ell,
31                3*P.b/2/(.25*P.m1+P.m2)/P.ell, 0.0]])
32 B = np.array([[0.0],
33               [0.0],
34               [1/(.25*P.m1+P.m2)],
35               [-3.0/2/(.25*P.m1+P.m2)/P.ell]])
36
37 C = np.array([[1.0, 0.0, 0.0, 0.0],
38               [0.0, 1.0, 0.0, 0.0]])
39
40 # gain calculation
41 wn_th = 2.2/tr_theta # natural frequency for angle
42 wn_z = 2.2/tr_z # natural frequency for position
43 des_char_poly = np.convolve([1, 2*zeta_z*wn_z, wn_z**2],
44                             [1, 2*zeta_th*wn_th, wn_th**2])
45 des_poles = np.roots(des_char_poly)
46
47 # Compute the gains if the system is controllable
48 if np.linalg.matrix_rank(cnt.ctrb(A, B)) != 4:
49     print("The system is not controllable")
50 else:
51     K = cnt.acker(A, B, des_poles)
52     Cr = np.array([[1.0, 0.0, 0.0, 0.0]])
53     kr = -1.0/(Cr @ np.linalg.inv(A - B @ K) @ B)
54
55 print('K: ', K)
56 print('kr: ', kr)

```

Listing 11.4: pendulumParamHW11.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 11.5 Design Study C. Satellite Attitude Control



### Example Problem C.11

The objective of this problem is to implement a state feedback controller using the full state. Start with the simulation files developed in Homework C.10.

- (a) Using the values for  $\omega_{n_\phi}$ ,  $\zeta_\phi$ ,  $\omega_{n_\theta}$ , and  $\zeta_\theta$  selected in Homework C.8, find the desired closed loop poles.
- (b) Add the state space matrices  $A$ ,  $B$ ,  $C$ ,  $D$  derived in Homework C.6 to your param file.
- (c) Verify that the state space system is controllable by checking that  $\text{rank}(\mathcal{C}_{A,B}) = n$ .
- (d) Find the feedback gain  $K$  so that the eigenvalues of  $(A - BK)$  are equal to the desired closed loop poles. Find the reference gain  $k_r$  so that the DC-gain from  $\phi_r$  to  $\phi$  is equal to one.
- (e) Implement the state feedback scheme and tune the closed loop poles to get a good response. You should be able to get a much faster response using state space methods.

### Solution

From HW C.6, the state space equations for the satellite are given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -0.0300 & 0.0300 & -0.0100 & 0.0100 \\ 0.1500 & -0.1500 & 0.0500 & -0.0500 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0.2 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x\end{aligned}\quad (11.40)$$

where Equation (11.40) represents the measured outputs. The reference output is the angle  $\phi$ , which implies that

$$y_r = (0 \ 1 \ 0 \ 0) x.$$

**Step 1.** The controllability matrix is

$$\begin{aligned}\mathcal{C}_{A,B} &= [B, AB, A^2B, A^3B] \\ &= \begin{pmatrix} 0 & 0.2000 & -0.0020 & -0.0059 \\ 0 & 0 & 0.0100 & 0.0294 \\ 0.2000 & -0.0020 & -0.0059 & 0.0007 \\ 0 & 0.0100 & 0.0294 & -0.0036 \end{pmatrix}.\end{aligned}$$

The determinant is  $\det(\mathcal{C}_{A,B}) = -36,000 \neq 0$ , therefore the system is controllable.

**Step 2.** The open loop characteristic polynomial is

$$\Delta_{ol}(s) = \det(sI - A) = s^4 + 0.0600s^3 + 0.18s^2$$

which implies that

$$\mathbf{a}_A = (0.06, 0.18, 0, 0)$$

$$\mathcal{A}_A = \begin{pmatrix} 1 & 0.06 & 0.18 & 0 \\ 0 & 1 & 0.06 & 0.18 \\ 0 & 0 & 1 & 0.06 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Step 3.** The desired closed loop polynomial is

$$\Delta_{cl}^d(s) = (s^2 + 2\zeta_\theta\omega_{n_\theta}s + \omega_{n_\theta}^2)(s^2 + 2\zeta_\phi\omega_{n_\phi}s + \omega_{n_\phi}^2)$$

$$= s^4 + 4.9275s^3 + 12.1420s^2 + 14.6702s + 8.8637,$$

when  $\omega_\theta = 1.9848$ ,  $\zeta_\theta = 0.707$ ,  $\omega_\phi = 1.5$ ,  $\zeta_\phi = 0.707$ , which implies that

$$\boldsymbol{\alpha} = (4.9275, 12.1420, 14.6702, 8.8637).$$

**Step 4.** The gains are therefore given by

$$K = (\boldsymbol{\alpha} - \mathbf{a}_A)\mathcal{A}_A^{-1}\mathcal{C}_{A,B}^{-1} = (40.2842, 255.1732, 24.3375, 366.1825)$$

The feedforward reference gain  $k_r = -1/C_r(A - BK)^{-1}B$  is computed using  $C_r = (0, 1, 0, 0)$ , which gives

$$k_r = \frac{-1}{C_r(A - BK)^{-1}B} = 295.4573.$$

A Python class that implements a PID controller for the satellite is shown below.

```

1 import numpy as np
2 import satelliteParamHW11 as P
3
4 class satelliteController:
5     def __init__(self):
6         self.K = P.K # state feedback gain
7         self.kr = P.kr # Input gain
8         self.limit = P.tau_max # Maximum torque
9         self.Ts = P.Ts # sample rate of controller
10
11    def update(self, phi_r, x):
12        # Compute the state feedback controller
13        tau_unsat = -self.K @ x + self.kr * phi_r
14        tau = self.saturate(tau_unsat.item(0))
15        return tau
16
17    def saturate(self,u):
18        if abs(u) > self.limit:
19            u = self.limit*np.sign(u)
20        return u

```

Listing 11.5: satelliteController.py

The gains are computed with the following Python script.

```

1 # satellite Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import satelliteParam as P
7
8 # import variables from satelliteParam
9 Ts = P.Ts
10 tau_max = P.tau_max
11
12 # tuning parameters
13 tr_phi = 2
14 tr_th = 3.66
15 zeta_phi = 0.707 # damping ratio position
16 zeta_th = 0.707 # damping ratio angle
17
18 # State Space Equations
19 # xdot = A*x + B*u
20 # y = C*x
21 A = np.array([[0.0, 0.0, 1.0, 0.0],
22               [0.0, 0.0, 0.0, 1.0],
23               [-P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js],
24               [P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp]])
25
26 B = np.array([[0.0],
27               [0.0],
28               [1.0/P.Js],
29               [0.0]])
30
31 C = np.array([[1.0, 0.0, 0.0, 0.0],
32               [0.0, 1.0, 0.0, 0.0]])
33
34 # gain calculation
35 wn_th = 2.2/tr_th
36 wn_phi = 2.2/tr_phi
37 des_char_poly = np.convolve([1, 2*zeta_th*wn_th, wn_th**2],
38                             [1, 2*zeta_phi*wn_phi, wn_phi**2])
39 des_poles = np.roots(des_char_poly)
40
41 # Compute the gains if the system is controllable
42 if np.linalg.matrix_rank(cnt.ctrb(A, B)) != 4:
43     print("The system is not controllable")
44 else:
45     K = cnt.acker(A, B, des_poles)
46     Cr = np.array([[1.0, 0.0, 0.0, 0.0]])
47     kr = -1.0/(Cr @ np.linalg.inv(A - B @ K) @ B)
48
49 print('K: ', K)
50 print('kr: ', kr)

```

Listing 11.6: satelliteParamHW11.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### Important Concepts:

- Ackerman's formula will put the closed-loop poles of a controllable SISO system at any desired locations.
- The columns of the controllability matrix are equal to  $A^i B$  with  $i$  ranging from zero to  $n - 1$ , where  $n$  is equal to the number of states.
- If a system is controllable then the associated controllability matrix will be full rank.
- Control canonical form is a specific state-space representation where the  $A$  matrix contains the negative coefficients of the characteristic polynomial in the top row. The  $B$  vector has a single one in its first row and all other entries as zeros. And the  $C$  matrix contains the coefficients of the transfer function's numerator.
- In closed-loop control canonical form each control gain directly modifies one of the characteristic polynomial coefficients. Since it is possible to individually modify every coefficient in the characteristic polynomial, the system's poles may be moved to any desired location.

### Notes and References

Controllability is typically defined as follows. The state space system

$$\dot{x} = Ax + Bu$$

is said to be controllable from initial state  $x_0$ , if there exists a control function  $u(t)$  defined on some interval  $t \in [0, T]$  that transitions the state to the origin. It can be shown that for linear time-invariant systems, the system is controllable if and only if the rank of the controllability matrix  $\mathcal{C}_{A,B}$  is equal to the size of the state  $n$ . For single input systems, this is equivalent to  $\mathcal{C}_{A,B}$  being invertible. Controllability is an interesting and deep topic that extends nicely to nonlinear and multi-input systems. For further information see [5, 6, 12].

Ackermann's formula given in Equation (11.32) is only valid for single input systems. In the multi-input case, the controllability matrix  $\mathcal{C}_{A,B}$  is not square and therefore cannot be inverted. However, a more general formula can be derived if  $\mathcal{C}_{A,B}$  has rank  $n$ . The more general formula is implemented by the `place` command in both Matlab and Python. For additional information on the derivation of the feedback gain in the MIMO case see [6].

# 12

## Integrator with Full State Feedback

### Learning Objectives:

- Eliminate the steady-state error by incorporating an integrator into the full-state feedback controller.

### 12.1 Theory

In the previous chapter we derived a full state feedback controller that stabilized the system and ensured that the DC-gain from the reference  $r(t)$  to the reference output  $y_r(t)$  is equal to one. If the model is precise meaning that we know all of the parameters exactly and we have not neglected any dynamics, or if there are no disturbances on the system, then the output will track the reference as time goes to infinity. However, models are never precise, and there are almost always disturbances on the system, which implies that there will be steady state error in the response. In Chapter 9 we showed that adding an integrator to the controller is an effective way of eliminating steady state error caused by model mismatch and input/output disturbances. In this section we will show how to add an integrator to the full state feedback controller derived in Chapter 11.

A block diagram of the system with full state feedback and an integrator is shown in [Fig. 12-1](#). Let the state of the integrator be defined as  $x_I$ , as shown in [Fig. 12-1](#), then

$$\dot{x}_I = r - y_r = r - C_r x.$$

In Appendix P.6 we argue using the root locus, that adding an integrator after the other gains are selected, will change all of the pole locations. Since the pole placement design methods described in Part III were limited to second order systems, we did not have an effective method for placing the system poles and the integrator pole simultaneously. On the other hand, the pole placement technique

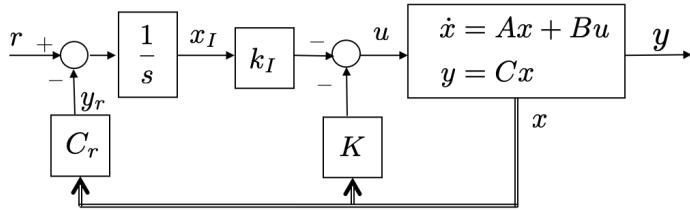


Figure 12-1: Full state feedback with integrator.

described in Chapter 11 was not limited to second order systems. As such, there should be a way to use the methods of Chapter 11 to simultaneously place both the system poles and the integrator pole. To do so, we augment the states of the original system with the integrator state  $x_I$ . Since the dynamics of the original system are given by  $\dot{x} = Ax + Bu$ , the dynamics of the augmented system are given by

$$\begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} = \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r, \quad (12.1)$$

where the state of the augmented system is  $(x^\top, x_I)^\top$ , the interaction matrix, or the “A-matrix,” of the augmented system is

$$A_1 = \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix},$$

and the “B-matrix” of the augmented system is

$$B_1 = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}.$$

The first question that might be asked is whether the augmented system  $(A_1, B_1)$  is controllable when the original system  $(A, B)$  is controllable. Since the original system is controllable, we know that the controllability matrix

$$\mathcal{C}_{A,B} = (B \ AB \ \dots \ A^{n-1}B)$$

has rank  $n$ . For the augmented system, the controllability matrix is

$$\mathcal{C}_{A_1,B_1} = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B & A^nB \\ 0 & -C_rB & -C_rAB & \dots & -C_rA^{n-2}B & -C_rA^{n-1}B \end{pmatrix},$$

which is guaranteed to have rank  $n+1$  if  $C_rA^{n-1}B \neq 0$ . When the augmented system is controllable, we can use the pole placement technique described in Chapter 11 to place the system poles and the pole of the integrator simultaneously using the augmented system. For example, letting  $A_1$  and  $B_1$  represent the augmented system, and using the Python place command

```
1 import control as cnt
2 K1 = cnt.place(A1,B1, [p1,...,pn,pI])
```

where  $p_I$  is the integrator pole. The resulting controller is

$$\begin{aligned} u(t) &= -K_1 \begin{pmatrix} x \\ x_I \end{pmatrix} = -\begin{pmatrix} K & k_I \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} = -Kx - k_I x_I \\ &= \underbrace{-Kx(t)}_{\text{state feedback}} - \underbrace{k_I \int_0^t (r(\tau) - y_r(\tau)) d\tau}_{\text{integral feedback}}. \end{aligned}$$

We can show that adding the integrator ensures that the DC-gain of the closed-loop system is equal to one. Indeed from Equation (12.1) the augmented closed-loop system is

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} &= \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} - \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} (K \quad k_I) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r \\ &= \begin{pmatrix} A - BK & -Bk_I \\ -C_r & 0 \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r, \end{aligned}$$

where the reference output equation is

$$y_r = (C_r \quad 0) \begin{pmatrix} x \\ x_I \end{pmatrix}.$$

Therefore, the transfer function from  $R(s)$  to  $Y_r(s)$  is

$$Y_r(s) = (C_r \quad 0) \left( sI - \begin{pmatrix} A - BK & -Bk_I \\ -C_r & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} R(s),$$

which implies that the DC-gain is

$$k_{DC} = - (C_r \quad 0) \left( \begin{pmatrix} A - BK & -Bk_I \\ -C_r & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Defining  $M \triangleq (A - BK)^{-1}$ , and using the Matrix inversion formula for block matrices given in Appendix P.7 Equation (P.7.2) we get

$$\begin{aligned} k_{DC} &= - (C_r \quad 0) \begin{pmatrix} M - MBk_I(C_r MBk_I)^{-1} C_r M & -MBk_I(C_r MBk_I)^{-1} \\ -(C_r MBk_I)^{-1} C_r M & (C_r MBk_I)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= C_r MBk_I(C_r MBk_I)^{-1} \\ &= 1. \end{aligned}$$

### 12.1.1 Integral Feedback for Linearized Systems

Suppose that the state space model is from a linearized system, i.e. suppose that  $x \in \mathbb{R}^n$  is the state,  $x_e \in \mathbb{R}^n$  is the equilibrium state, and that  $\tilde{x} = x - x_e$  is

the state linearized around the equilibrium. Similarly, suppose that  $u \in \mathbb{R}^m$  is the input, that  $u_e \in \mathbb{R}^m$  is the input at equilibrium, and that  $\tilde{u} = u - u_e$  is the input linearized around equilibrium, and that  $y_r \in \mathbb{R}^p$  is the reference output,  $y_{re} = C_r x_e \in \mathbb{R}^p$  is the reference output at equilibrium,  $\tilde{y}_r = y_r - y_{re}$  is the reference output linearized about equilibrium, and the reference input is given by  $\tilde{r} = r - y_{re}$ . In this case the integral state is selected as  $\tilde{x}_I = \int (\tilde{r} - \tilde{y}_r) d\tau$ , and the augmented system is given by

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_I \end{pmatrix} = \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_I \end{pmatrix} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \tilde{u} + \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \tilde{r}.$$

The pole placement design proceeds as in the previous section with the resulting controller being

$$\tilde{u} = -K\tilde{x} - k_I \int_{-\infty}^t \tilde{e}(\tau) d\tau.$$

Noting that

$$\tilde{e} = \tilde{r} - \tilde{y} = (r - y_{re}) - (y_r - y_{re}) = r - y_r$$

we get that the controller for the linearized system is given by

$$u = u_e - K(x - x_e) - k_I \int_{-\infty}^t (r(\tau) - y_r(\tau)) d\tau.$$

## 12.2 Summary of Design Process - State Feedback with Integrator

The design process for adding an integrator to state feedback can be summarized as follows. Given the plant

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y_r &= C_r x, \end{aligned}$$

where we desire that  $y_r$  tracks the reference  $r$  using integral action using the controller

$$u(t) = -Kx(t) - k_I \int_{-\infty}^t (r(\tau) - y_r(\tau)) d\tau,$$

with desired closed-loop gains at  $p_1, p_2, \dots, p_n, p_I$ .

**Step 1.** Augment the state evolution equation as

$$\begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} = \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} u + \begin{pmatrix} \mathbf{0} \\ I \end{pmatrix} r,$$

and let

$$A_1 = \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}$$

**Step 2.** Follow the pole placement procedure outlined in Chapter 11 to find  $K_1$  that places the poles of the augmented system at  $p_1, p_2, \dots, p_n$ , and  $p_I$ .

**Step 3.** Find the gains  $K$  and  $k_I$  as

$$K = K_1(1 : n) \\ k_I = K_1(n + 1).$$

Using Python, these steps can be implemented using the following script.

```

1 import numpy as np
2 import control as cnt
3
4 # augment the system matrices
5 A1 = np.vstack((
6     np.hstack((A, np.zeros((np.size(A,1),1)))),
7     np.hstack((-Cr, 0)) ))
8
9 B1 = np.vstack((
10    B, 0))
11
12 # Compute the gains if the system is controllable
13 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != np.size(A1,1):
14     print("The system is not controllable")
15 else:
16     K1 = cnt.acker(A1, B1, [p1, p2, ..., pn, pI])
17
18 #extract the state feedback gains and the integral gain
19 K = K1[0,0:-1]
20 ki = K1[0,-1]
21
22 print('K: ', K)
23 print('ki: ', ki)
```

### 12.2.1 Simple example

Given the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 4 \end{pmatrix} u \\ y_r = \begin{pmatrix} 5 & 0 \end{pmatrix} x$$

find the feedback gain  $K$  to place the closed-loop poles at  $-1 \pm j$ , and the reference gain  $k_r$  so that the DC-gain is one, and add an integrator with gain at  $p_I = -0.1$ .

**Step 1.** Form the augmented system

$$A_1 = \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & 0 \\ -5 & 0 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

**Step 2.** To place the poles at  $p_1 = -1 + j$ ,  $p_2 = -1 - j$ ,  $p_I = -0.1$ , we first form the controllability matrix

$$\mathcal{C}_{A_1, B_1} = [B_1, A_1 B_1, A_1^2 B_1] = \begin{pmatrix} 0 & 4 & 12 \\ 4 & 12 & 44 \\ 0 & 0 & -20 \end{pmatrix}.$$

The determinant is  $\det(\mathcal{C}_{A_1, B_1}) = 320 \neq 0$ , therefore the system is controllable.

The open-loop characteristic polynomial

$$\Delta_{ol}(s) = \det(sI - A_1) = \det \begin{pmatrix} s & -1 & 0 \\ -2 & s-3 & 0 \\ 5 & 0 & s \end{pmatrix}$$

$$= s^3 - 3s^2 - 2s,$$

which implies that

$$\mathbf{a}_{A_1} = (-3, -2, 0)$$

$$\mathcal{A}_{A_1} = \begin{pmatrix} 1 & -3 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The desired closed-loop polynomial

$$\Delta_{cl}^d(s) = (s + 1 - j)(s + 1 + j)(s + 0.1)$$

$$= s^3 + 2.1s^2 + 2.2s + 0.2,$$

which implies that

$$\boldsymbol{\alpha} = (2.1, 2.2, 0.2).$$

The augmented gains are therefore given as

$$K_1 = (\boldsymbol{\alpha} - \mathbf{a}_{A_1}) \mathcal{A}_{A_1, B_1}^{-1} \mathcal{C}_{A_1, B_1}^{-1}$$

$$= ((2.1, 2.2, 0.2) - (-3, -2, 0)) \cdot$$

$$\begin{pmatrix} 1 & -3 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 4 & 12 \\ 4 & 12 & 44 \\ 0 & 0 & -20 \end{pmatrix}^{-1}$$

$$= (1.05, 1.275, -0.01)$$

**Step 3.** The feedback gains are therefore given by

$$K = K_1(1 : 2) = \begin{pmatrix} 1.05 & 1.275 \end{pmatrix}$$

$$k_I = K_1(3) = -0.01$$

Alternatively, we could have used the following Python script

```

1 import numpy as np
2 import control as cnt
3
4 # define original state space system and desired poles
5 A = np.array([[0.0, 1.0], [2.0, 3.0]])
6 B = np.array([[0.0], [4.0]])
7 Cr = np.array([5.0, 0])
8
9 p1 = -1.0 - 1.j
10 p2 = -1.0 + 1.j
11 pI = -0.1
12
13 # augment the system matrices
14 A1 = np.vstack((
15     np.hstack((A, np.zeros((np.size(A,1),1)))),
16     np.hstack((-Cr, 0)) ))
17 B1 = np.vstack((
18     B, 0))
19
20 # Compute the gains if the system is controllable
21 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != np.size(A1,1):
22     print("The system is not controllable")
23 else:
24     K1 = cnt.acker(A1, B1, [p1, p2, pI])
25
26 #extract the state feedback gains and the integral gain
27 K = K1[0,0:-1]
28 ki = K1[0,-1]
```

## 12.3 Design Study A. Single Link Robot Arm



### Example Problem A.12

- (a) Modify the state feedback solution developed in Homework A.11 to add an integrator with anti-windup.
- (b) Add a disturbance to the system and allow the system parameters to change up to 20%.
- (c) Tune the integrator pole (and other gains if necessary) to get good tracking performance.

## Solution

**Step 1.** The original state space equations are

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1.0000 \\ 0 & -0.667 \end{pmatrix} x + \begin{pmatrix} 0 \\ 66.667 \end{pmatrix} u \\ y &= (1 \ 0) x,\end{aligned}$$

therefore the augmented system is

$$\begin{aligned}A_1 &= \begin{pmatrix} A & \mathbf{0} \\ -C & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & 1.0000 & 0 \\ 0 & -0.667 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ B_1 &= \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 \\ 66.667 \\ 0 \end{pmatrix}\end{aligned}$$

**Step 2.** After the design in HW A.11, the closed loop poles were located at  $p_{1,2} = -3.1191 \pm j3.1200$ . We will add the integrator pole at  $p_I = -5$ . The new controllability matrix is

$$\begin{aligned}\mathcal{C}_{A_1, B_1} &= [B_1, A_1 B_1, A_1^2 B_1] \\ &= \begin{pmatrix} 0 & 66.6670 & -44.4669 \\ 66.6670 & -44.4669 & 29.6594 \\ 0 & 0 & -66.6670 \end{pmatrix}.\end{aligned}$$

The determinant is  $\det(\mathcal{C}_{A_1, B_1}) = 2.9630e+05 \neq 0$ , therefore the system is controllable.

The open loop characteristic polynomial

$$\begin{aligned}\Delta_{ol}(s) &= \det(sI - A_1) = \det \begin{pmatrix} s & -1.0000 & 0 \\ 0 & s + 0.667 & 0 \\ 1 & 0 & s \end{pmatrix} \\ &= s^3 + 0.6174s^2,\end{aligned}$$

which implies that

$$\begin{aligned}\mathbf{a}_{A_1} &= (0.6174, \ 0, \ 0) \\ \mathcal{A}_{A_1} &= \begin{pmatrix} 1 & 0.6174 & 0 \\ 0 & 1 & 0.6174 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

The desired closed loop polynomial

$$\begin{aligned}\Delta_{cl}^d(s) &= (s + 3.8885 - j3.8897)(s + 3.8885 + j3.8897)(s + 5) \\ &= s^3 + 12.7770s^2 + 69.1350s + 151.2500,\end{aligned}$$

which implies that

$$\alpha = (12.7770, \quad 69.1350, \quad 151.2500).$$

The augmented gains are therefore given as

$$\begin{aligned} K_1 &= (\alpha - \mathbf{a}_{A_1}) \mathcal{A}_{A_1}^{-1} \mathcal{C}_{A_1, B_1}^{-1} \\ &= (1.0370, \quad 0.1817, \quad -2.2687) \end{aligned}$$

**Step 3.** The feedback gains are therefore given by

$$\begin{aligned} K &= K_1(1 : 2) = (1.0370, \quad 0.1817) \\ k_I &= K_1(3) = -2.2687 \end{aligned}$$

A Python class that implements state feedback control with an integrator for the single link robot arm is shown below.

```

1 import numpy as np
2 import armParamHW12 as P
3
4 class armController:
5     def __init__(self):
6         self.integrator = 0.0 # integrator
7         self.error_d1 = 0.0 # error signal delayed by 1 sample
8         self.K = P.K # state feedback gain
9         self.ki = P.ki # Input gain
10        self.limit = P.tau_max # Maximum torque
11        self.Ts = P.Ts # sample rate of controller
12
13    def update(self, theta_r, x):
14        theta = x.item(0)
15
16        # integrate error
17        error = theta_r - theta
18        self.integrateError(error)
19
20        # compute feedback linearizing torque tau_f1
21        tau_f1 = P.m * P.g * (P.ell / 2.0) * np.cos(theta)
22
23        # Compute the state feedback controller
24        tau_tilde = -self.K @ x - self.ki*self.integrator
25
26        # compute total torque
27        tau = self.saturate(tau_f1 + tau_tilde)
28
29        return tau
30
31    def integrateError(self, error):
32        self.integrator = self.integrator \
33                            + (self.Ts/2.0)*(error + self.error_d1)
34        self.error_d1 = error
35
36    def saturate(self,u):
37        if abs(u) > self.limit:

```

```

37     u = self.limit*np.sign(u)
38     return u

```

Listing 12.1: armController.py

Python code that computes the control gains is given below.

```

1 # Single link arm Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import armParam as P
7
8 Ts = P.Ts # sample rate of the controller
9 #beta = P.beta # dirty derivative gain
10 tau_max = P.tau_max # limit on control signal
11 m = P.m
12 ell = P.ell
13 g = P.g
14
15 # tuning parameters
16 tr = 0.4
17 zeta = 0.707
18 integrator_pole = np.array([-0.5])
19
20 # State Space Equations
21 # xdot = A*x + B*u
22 # y = C*x
23 A = np.array([[0.0, 1.0],
24               [0.0, -1.0*P.b/P.m/(P.ell**2)]])
25
26 B = np.array([[0.0],
27               [3.0/P.m/(P.ell**2)]])
28
29 C = np.array([[1.0, 0.0]])
30
31 # form augmented system
32 A1 = np.array([[0.0, 1.0, 0.0],
33                 [0.0, -1.0*P.b/P.m/(P.ell**2), 0.0],
34                 [-1.0, 0.0, 0.0]])
35
36 B1 = np.array([[0.0],
37                 [3.0/P.m/(P.ell**2)],
38                 [0.0]])
39
40 # gain calculation
41 #wn = 2.2/tr # natural frequency if zeta = 0.707
42 wn = 0.5*np.pi/(tr*np.sqrt(1-zeta**2)) # natural frequency
43
44 des_char_poly = np.convolve(
45     [1, 2*zeta*wn, wn**2],
46     np.poly(integrator_pole))
47 des_poles = np.roots(des_char_poly)
48
49 # Compute the gains if the system is controllable
50 if np.linalg.matrix_rank(cnt ctrb(A1, B1)) != 3:

```

```

51     print("The system is not controllable")
52 else:
53     K1 = cnt.place(A1, B1, des_poles)
54     K = np.array([[K1.item(0), K1.item(1)]])
55     ki = K1.item(2)
56
57 print('K: ', K)
58 print('ki: ', ki)

```

Listing 12.2: armParamHW12.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 12.4 Design Study B. Inverted Pendulum



### Example Problem B.12

- (a) Modify the state feedback solution developed in Homework B.11 to add an integrator with anti-windup to the position feedback.
- (b) Add a constant input disturbance of 0.5 Newtons to the input of the plant and allow the system parameters to change up to 20%.
- (c) Tune the integrator pole (and other gains if necessary) to get good tracking performance.

### Solution

**Step 1.** The original state space equations are

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -1.7294 & -0.0471 & 0 \\ 0 & 34.5882 & 0.1412 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0.9412 \\ -2.8235 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x, \end{aligned} \quad (12.2)$$

The integrator will only be on  $z = x_1$ , therefore

$$C_r = (1 \ 0 \ 0 \ 0).$$

The augmented system is therefore

$$\begin{aligned} A_1 &= \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & -1.7294 & -0.0471 & 0 & 0 \\ 0 & 34.5882 & 0.1412 & 0 & 0 \\ -1.0000 & 0 & 0 & 0 & 0 \end{pmatrix} \\ B_1 &= \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.9412 \\ -2.8235 \\ 0 \end{pmatrix} \end{aligned}$$

**Step 2.** After the design in HW B.11, the closed loop poles were located at  $p_{1,2} = -1.4140 \pm j1.4144$ ,  $p_{3,4} = -0.8000 \pm j0.6000$ . We will add the integrator pole at  $p_I = -10$ . The new controllability matrix

$$\begin{aligned} \mathcal{C}_{A_1, B_1} &= [B_1, A_1 B_1, A_1^2 B_1, A_1^3 B_1, A_1^4 B_1] \\ &= \begin{pmatrix} 0.94118 & -0.044291 & 4.8851 & -0.45968 & \\ 0 & -2.8235 & 0.13287 & -97.667 & 5.2855 \\ 0.94118 & -0.044291 & 4.8851 & -0.45968 & 168.93 \\ -2.8235 & 0.13287 & -97.667 & 5.2855 & -3378.2 \\ 0 & 0 & -0.94118 & 0.044291 & -4.8851 \end{pmatrix}. \end{aligned}$$

The determinant is nonzero, therefore the system is controllable.

The open loop characteristic polynomial

$$\begin{aligned} \Delta_{ol}(s) &= \det(sI - A_1) \\ &= \det \begin{pmatrix} s & 0 & -1 & 0 & 0 \\ 0 & s & 0 & -1 & 0 \\ 0 & 1.7294 & s + 0.0471 & 0 & 0 \\ 0 & -34.588 & -0.14118 & s & 0 \\ 1 & 0 & 0 & 0 & s \end{pmatrix} \\ &= s^5 + 0.0471s^4 - 34.588s^3 - 1.3835s^2, \end{aligned}$$

which implies that

$$\begin{aligned} \mathbf{a}_{A_1} &= (0.0471, -34.588, 1.3835, 0, 0) \\ \mathcal{A}_{A_1} &= \begin{pmatrix} 1 & 0.0471 & -34.588 & -1.3835 & 0 \\ 0 & 1 & 0.0471 & -34.588 & -1.3835 \\ 0 & 0 & 1 & 0.0471 & -34.588 \\ 0 & 0 & 0 & 1 & 0.0471 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The desired closed loop polynomial

$$\begin{aligned}\Delta_{cl}^d(s) &= (s + 1.0369 - j1.0372)(s + 1.0369 + j1.0372) \dots \\ &\quad (s + 3.1108 - j3.1117)(s + 3.1108 + j3.1117)(s + 2) \\ &= s^5 + 10.2955s^4 + 51.0048s^3 + 122.3612s^2 \\ &\quad + 148.7123s + 83.2910,\end{aligned}$$

which implies that

$$\boldsymbol{\alpha} = (10.2955, \quad 51.0048, \quad 122.3612, \quad 148.7123, \quad 83.2910).$$

The augmented gains are therefore given as

$$\begin{aligned}K_1 &= (\boldsymbol{\alpha} - \mathbf{a}_{A_1})\mathcal{A}_{A_1, B_1}^{-1} \\ &= (-5.3744, \quad -32.1057, \quad -4.5745, \quad -5.1545, \quad 3.0101)\end{aligned}$$

**Step 3.** The feedback gains are therefore given by

$$\begin{aligned}K &= K_1(1 : 4) = (-5.3744, \quad -32.1057, \quad -4.5745, \quad -5.1545) \\ k_I &= K_1(5) = 3.0101\end{aligned}$$

A Python class that implements a state feedback controller with integrator for the inverted is shown below.

```

1 import numpy as np
2 import pendulumParamHW12 as P
3
4 class pendulumController:
5     def __init__(self):
6         self.integrator = 0.0 # integrator
7         self.error_d1 = 0.0 # error signal delayed by 1 sample
8         self.K = P.K # state feedback gain
9         self.ki = P.ki # integrator gain
10        self.limit = P.F_max # Maximum force
11        self.Ts = P.Ts # sample rate of controller
12
13    def update(self, z_r, x):
14        z = x.item(0)
15        # integrate error
16        error = z_r - z
17        self.integrateError(error)
18        # Compute the state feedback controller
19        F_unsat = -self.K @ x - self.ki * self.integrator
20        F_sat = self.saturate(F_unsat)
21        return F_sat.item(0)
22
23    def integrateError(self, error):
24        self.integrator = self.integrator \
25            + (self.Ts / 2.0) * (error + self.error_d1)
```

```

26     self.error_d1 = error
27
28     def saturate(self, u):
29         if abs(u) > self.limit:
30             u = self.limit*np.sign(u)
31         return u

```

Listing 12.3: pendulumController.py

The gains are computed with the following Python script.

```

1 # Inverted Pendulum Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import pendulumParam as P
7
8 # sample rate of the controller
9 Ts = P.Ts
10
11 # saturation limits
12 F_max = 5.0          # Max Force, N
13 theta_max = 30.0*np.pi/180.0 # Max theta, rads
14
15 ##### State Space #####
16 # State Space
17 #####
18 # tuning parameters
19 tr_z = 1.5           # rise time for position
20 tr_theta = 0.5        # rise time for angle
21 zeta_z = 0.707       # damping ratio position
22 zeta_th = 0.707      # damping ratio angle
23 integrator_pole = -2 # integrator pole
24
25 # State Space Equations
26 # xdot = A*x + B*u
27 # y = C*x
28 A = np.array([[0.0, 0.0, 1.0, 0.0],
29               [0.0, 0.0, 0.0, 1.0],
30               [0.0, -3*P.m1*P.g/4/(.25*P.m1+P.m2),
31                 -P.b/(.25*P.m1+P.m2), 0.0],
32               [0.0, 3*(P.m1+P.m2)*P.g/2/(.25*P.m1+P.m2)/P.ell,
33                 3*P.b/2/(.25*P.m1+P.m2)/P.ell, 0.0]])
34
35 B = np.array([[0.0],
36               [0.0],
37               [1/(.25*P.m1+P.m2)],
38               [-3.0/2/(.25*P.m1+P.m2)/P.ell]])
39
40 C = np.array([[1.0, 0.0, 0.0, 0.0],
41               [0.0, 1.0, 0.0, 0.0]])
42
43 # form augmented system
44 Cout = np.array([[1.0, 0.0, 0.0, 0.0]])
45 A1 = np.array([[0.0, 0.0, 1.0, 0.0, 0.0],
46               [0.0, 0.0, 0.0, 1.0, 0.0],

```

```

47     [0.0, -3*P.m1*P.g/4/(.25*P.m1+P.m2),
48      -P.b/(.25*P.m1+P.m2), 0.0, 0.0],
49     [0.0, 3*(P.m1+P.m2)*P.g/2/(.25*P.m1+P.m2)/P.ell,
50      3*P.b/2/(.25*P.m1+P.m2)/P.ell, 0.0, 0.0],
51     [-1.0, 0.0, 0.0, 0.0, 0.0])
52 B1 = np.array([[0.0],
53                [0.0],
54                [1/(.25*P.m1+P.m2)],
55                [-3.0/2/(.25*P.m1+P.m2)/P.ell],
56                [0.0]])
57
58 # gain calculation
59 wn_th = 2.2/tr_theta # natural frequency for angle
60 wn_z = 2.2/tr_z # natural frequency for position
61 des_char_poly = np.convolve(
62     np.convolve([1, 2*zeta_z*wn_z, wn_z**2],
63                [1, 2*zeta_th*wn_th, wn_th**2]),
64     np.poly([integrator_pole]))
65 des_poles = np.roots(des_char_poly)
66
67 # Compute the gains if the system is controllable
68 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 5:
69     print("The system is not controllable")
70 else:
71     K1 = cnt.acker(A1, B1, des_poles)
72     K = np.array([[K1.item(0), K1.item(1),
73                   K1.item(2), K1.item(3)]])
74     ki = K1.item(4)
75
76 print('K: ', K)
77 print('ki: ', ki)

```

Listing 12.4: pendulumParamHW12.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 12.5 Design Study C. Satellite Attitude Control



### Example Problem C.12

- (a) Modify the state feedback solution developed in Homework C.11 to add an integrator with anti-windup to the feedback loop for  $\phi$ .
- (b) Add a constant input disturbance of 1 Newtons-meters to the input of the plant and allow the plant parameters to vary up to 20%.
- (c) Tune the integrator pole (and other gains if necessary) to get good tracking performance.

## Solution

**Step 1.** The original state space equations are

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.03 & 0.03 & -0.01 & 0.01 \\ 0.14 & -0.14 & 0.05 & -0.05 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0.21 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x.\end{aligned}$$

The integrator will only be on  $\phi = x_2$ , therefore

$$C_r = (0 \ 1 \ 0 \ 0).$$

The augmented system is therefore

$$\begin{aligned}A_1 &= \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.03 & 0.03 & -0.01 & 0.01 & 0 \\ 0.14 & -0.14 & 0.05 & -0.05 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \\ B_1 &= \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.21 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

**Step 2.** After the design in HW C.11, the closed loop poles were located at  $p_{1,2} = -1.0605 \pm j1.0608$ ,  $p_{3,4} = -0.7016 \pm j0.7018$ . We will add the integrator pole at  $p_I = -1$ . The new controllability matrix

$$\begin{aligned}\mathcal{C}_{A_1, B_1} &= [B_1, A_1 B_1, A_1^2 B_1, A_1^3 B_1, A_1^4 B_1] \\ &= \begin{pmatrix} 0 & 0.21 & -0.0023 & -0.0062 & 0.0008 \\ 0 & 0 & 0.011 & 0.0279 & -0.0034 \\ 0.21 & -0.0023 & -0.0062 & 0.0008 & 0.0010 \\ 0 & 0.011 & 0.0279 & -0.0034 & -0.0044 \\ 0 & 0 & 0 & 0.0105 & 0.0279 \end{pmatrix}.\end{aligned}$$

The determinant is nonzero, therefore the system is controllable.

The open loop characteristic polynomial

$$\begin{aligned}\Delta_{ol}(s) &= \det(sI - A_1) \\ &= s^5 + 0.0611s^4 + 0.1657s^3,\end{aligned}$$

which implies that

$$\mathbf{a}_{A_1} = (0.0611, \quad 0.1657, \quad 0, \quad 0, \quad 0)$$

$$\mathcal{A}_{A_1} = \begin{pmatrix} 1 & 0.0611 & 0.1657 & 0 & 0 \\ 0 & 1 & 0.0611 & 0.1657 & 0 \\ 0 & 0 & 1 & 0.0611 & 0.1657 \\ 0 & 0 & 0 & 1 & 0.0611 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The desired closed loop polynomial

$$\begin{aligned} \Delta_{cl}^d(s) &= (s + 1.0605 - j1.0608)(s + 1.0605 + j1.0608) \dots \\ &\quad (s + 0.7016 - j0.7018)(s + 0.7016 + j0.7018)(s + 1) \\ &= s^5 + 3.4038s^4 + 5.2934s^3 + 4.4761s^2 \\ &\quad + 2.0221s + 0.4356, \end{aligned}$$

which implies that

$$\boldsymbol{\alpha} = (3.4038, \quad 5.2934 \quad 4.4761, \quad 2.0221, \quad 0.4356).$$

The augmented gains are therefore given as

$$\begin{aligned} K_1 &= (\boldsymbol{\alpha} - \mathbf{a}_{A_1})\mathcal{A}_{A_1}^{-1}\mathcal{C}_{A_1, B_1}^{-1} \\ &= (19.15, \quad 43.41, \quad 16.72, \quad 111.63, \quad -14.52). \end{aligned}$$

**Step 3.** The feedback gains are therefore given by

$$\begin{aligned} K &= K_1(1 : 4) = (19.15, \quad 43.41, \quad 16.72, \quad 111.63) \\ k_I &= K_1(5) = -14.5200. \end{aligned}$$

A Python class that implements a PID controller for the satellite is shown below.

```

1 import numpy as np
2 import satelliteParamHW12 as P
3
4 class satelliteController:
5     def __init__(self):
6         self.integrator = 0.0          # integrator
7         self.error_d1 = 0.0            # error delayed by 1 sample
8         self.K = P.K                  # state feedback gain
9         self.ki = P.ki                # integrator gain
10        self.limit = P.tau_max       # Maximum torque
11        self.Ts = P.Ts                # sample rate of controller
12
13    def update(self, phi_r, x):
14        theta = x.item(0)
15        phi = x.item(1)

```

```

16
17     # integrate error
18     error = phi_r - phi
19     self.integrateError(error)
20
21     # Compute the state feedback controller
22     tau_unsat = -self.K @ x - self.ki*self.integrator
23     tau = self.saturate(tau_unsat.item(0))
24
25     return tau
26
27 def integrateError(self, error):
28     self.integrator = self.integrator + (self.Ts / 2.0) \
29         * (error + self.error_d1)
30
31     self.error_d1 = error
32
33 def saturate(self,u):
34     if abs(u) > self.limit:
35         u = self.limit*np.sign(u)
36     return u

```

Listing 12.5: satelliteController.py

The gains are computed with the following Python script.

```

1 # satellite Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import satelliteParam as P
7
8 # import variables from satelliteParam
9 Ts = P.Ts
10 sigma = P.sigma
11 beta = P.beta
12 tau_max = P.tau_max
13
14 # tuning parameters
15 wn_th = 0.6
16 wn_phi = 1.1    # rise time for angle
17 zeta_phi = 0.707 # damping ratio position
18 zeta_th = 0.707 # damping ratio angle
19 integrator_pole = np.array([-1])
20
21
22 # State Space Equations
23 # xdot = A*x + B*u
24 # y = C*x
25 A = np.array([[0.0, 0.0, 1.0, 0.0],
26               [0.0, 0.0, 0.0, 1.0],
27               [-P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js],
28               [P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp]])
29
30 B = np.array([[0.0],
31               [0.0],
32               [0.0],
33               [0.0]])

```

```

32             [1.0/P.Js],
33             [0.0]])
34
35 C = np.array([[1.0, 0.0, 0.0, 0.0],
36               [0.0, 1.0, 0.0, 0.0]])
37
38 # form augmented system
39 Cout = np.array([[0.0, 1.0, 0.0, 0.0]])
40 A1 = np.array([[0.0, 0.0, 1.0, 0.0, 0.0],
41                 [0.0, 0.0, 0.0, 1.0, 0.0],
42                 [-P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js, 0.0],
43                 [P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp, 0.0],
44                 [0.0, -1.0, 0.0, 0.0, 0.0]])
45
46 B1 = np.array([[0.0],
47                 [0.0],
48                 [1.0/P.Js],
49                 [0.0],
50                 [0.0]])
51
52
53 # gain calculation
54 des_char_poly = np.convolve(
55     np.convolve([1, 2*zeta_phi*wn_phi, wn_phi**2],
56                 [1, 2*zeta_th*wn_th, wn_th**2]),
57     np.poly(integrator_pole))
58 des_poles = np.roots(des_char_poly)
59
60 # Compute the gains if the system is controllable
61 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 5:
62     print("The system is not controllable")
63 else:
64     K1 = cnt.place(A1, B1, des_poles)
65     K = np.array([K1.item(0), K1.item(1), K1.item(2), K1.item(3)])
66     ki = K1.item(4)
67
68 print('K: ', K)
69 print('ki: ', ki)

```

Listing 12.6: satelliteParamHW12.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### Important Concepts:

- Including an integrator will eliminate steady-state error.
- Integrator gains should be selected simultaneously with the controller gains to avoid changing the closed-loop pole locations.
- The state-space equations can be augmented to include the integrator as an extra state. The techniques for pole placement may then be used on the augmented state-space system to find all the gains concurrently.

## Notes and References

Integral control for linear state space systems is a well established technique. Additional examples can be found in [19, 5, 4]. A nice introduction to integral control for nonlinear systems is given in [20].

# 13

## Observers

### Learning Objectives:

- Create an observer to estimate unknown states.
- Determine when the system is observable.
- Choose observer gains that ensure that the state estimation error converges exponentially fast.

## 13.1 Theory

In the previous two chapters we assumed that the full state  $x$  was available to the controller for full state feedback. In this chapter we remove that assumption and show how the state can be reconstructed given the measurements. We will assume that the state space model is given by

$$\dot{x} = Ax + Bu \quad (13.1)$$

$$y = Cx \quad (13.2)$$

$$y_r = C_r x,$$

where  $y_r \in \mathbb{R}^r$  is the reference output as discussed in the previous two chapters, and  $y \in \mathbb{R}^p$  is the measured output, or the signals measured by the sensors. While the objective of the controller is to force  $y_r(t)$  to converge to the reference  $r(t)$ , the objective of the observer is to estimate the state  $x(t)$  given a history of the input  $u(t)$  and the measured output  $y(t)$ . In this book, the estimate of the state will be denoted as  $\hat{x}(t)$ .

A block diagram of the closed-loop system with observer based control is shown in [Fig. 13-1](#). Comparing [Fig. 13-1](#) to the case of full state feedback in [Fig. 11-4](#) note that the controller uses the estimated state  $\hat{x}$  instead of the full state

x. In other words, the controller in Equation (11.37) becomes

$$u = -K\hat{x} + k_r r.$$

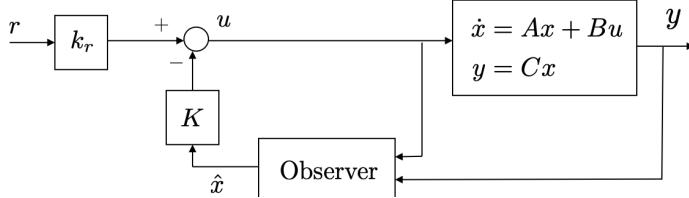


Figure 13-1: Closed-loop system with observer based feedback.

In this chapter we will design a so-called Luenberger observer that has the structure given by

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{predictor}} + \underbrace{L(y - C\hat{x})}_{\text{corrector}}. \quad (13.3)$$

The first term in Equation (13.3), i.e.,  $\dot{\hat{x}} = A\hat{x} + Bu$  is a copy of the equations of motion for the plant given in Equation (13.1). The second term given by  $L(y - C\hat{x})$  is a correction term, where  $L$  is the observer gain, and  $(y - C\hat{x})$  is the difference between the measured output  $y = Cx$  and the predicted value of the measured output  $C\hat{x}$ . The term  $y - C\hat{x}$  is called the *innovation* signal.

Defining the observer error as  $e = x - \hat{x}$  and differentiating with respect to time gives

$$\begin{aligned}\dot{e} &= \dot{\hat{x}} - \dot{x} \\ &= Ax + Bu - (A\hat{x} + Bu + Ly - LC\hat{x}) \\ &= A(x - \hat{x}) - LCx + LC\hat{x} \\ &= (A - LC)(x - \hat{x}) \\ &= (A - LC)e.\end{aligned}$$

Therefore, the poles that govern the time-evolution of the observation error are given by the eigenvalues of  $A - LC$ . The observer design problem is therefore to select  $L$  to place the eigenvalues of  $A - LC$  at locations specified by the control engineer. The derivation of a formula for the observer gain  $L$  proceeds in a manner similar to the derivation of the feedback gain  $K$  in Chapter 11.

### 13.1.1 Observer Design using Observer Canonic Form

Given the general SISO monic transfer function

$$Y(s) = \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} U(s), \quad (13.4)$$

where  $n > m$ , and taking the inverse Laplace transform gives

$$\begin{aligned} \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \dot{y} + a_0 y(t) \\ = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \dot{u} + b_0 u(t). \end{aligned}$$

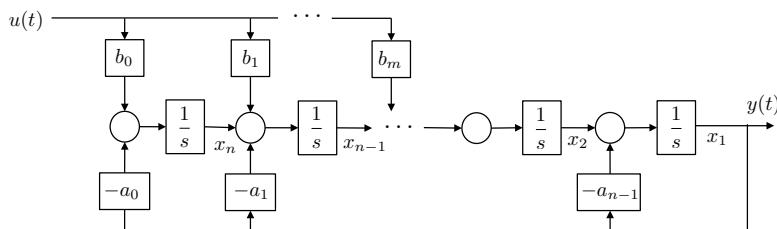
Solving for the  $n^{th}$  derivative of  $y$  and collecting terms with similar order derivatives gives

$$\begin{aligned} \frac{d^n y}{dt^n} = -a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} - a_{m+1} \frac{d^{m+1} y}{dt^{m+1}} \\ + \left( b_m \frac{d^m u}{dt^m} - a_m \frac{d^m y}{dt^m} \right) + \cdots + (b_1 \dot{u} - a_1 \dot{y}) + (b_0 u - a_0 y). \end{aligned}$$

Integrating  $n$  times gives

$$\begin{aligned} y(t) = \int \left[ -a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} - \cdots - a_{m+1} \frac{d^{m+1} y}{dt^{m+1}} \right. \\ \left. + \int \left[ \left( b_m \frac{d^m u}{dt^m} - a_m \frac{d^m y}{dt^m} \right) + \dots \right. \right. \\ \left. \left. + \int \left[ (b_1 \dot{u} - a_1 \dot{y}) + \int (b_0 u - a_0 y) \right] \dots \right] \dots \right]. \quad (13.5) \end{aligned}$$

The analog computer implementation of Equation (13.5) is shown in [Fig. 13-2](#). Labeling the states as the output of the integrators, as shown in [Fig. 13-2](#) and



[Figure 13-2](#): Block diagram showing the analog computer implementation of Equation (13.5).

writing the equations in linear state space form gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} -a_{n-1} & 1 & \dots & 0 & 0 \\ -a_{n-2} & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -a_1 & 0 & \dots & 0 & 1 \\ -a_0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} u \quad (13.6)$$

$$= (1 \ 0 \ \dots \ 0 \ 0) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}. \quad (13.7)$$

We define the observer canonic realization as

$$\begin{aligned} \dot{x}_o &= A_o x_o + B_o u \\ y &= C_o x_o, \end{aligned}$$

where

$$A_o \stackrel{\triangle}{=} \begin{pmatrix} -a_{n-1} & 1 & \dots & 0 & 0 \\ -a_{n-2} & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -a_1 & 0 & \dots & 0 & 1 \\ -a_0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (13.8)$$

$$B_o \stackrel{\triangle}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} \quad (13.9)$$

$$C_o \stackrel{\triangle}{=} (1 \ 0 \ \dots \ 0 \ 0). \quad (13.10)$$

Comparing the state space equations in observer canonic form in Equations (13.8)–(13.10) with the state space equations for control canonic form in Equations (11.14)–(11.16), we see that

$$\begin{aligned} A_o &= A_c^\top \\ B_o &= C_c^\top \\ C_o &= B_c^\top. \end{aligned}$$

Comparing the state equation in Equation (13.6) with the transfer function in Equation (13.4) we see that the eigenvalues of  $A_o$  are the roots of the open-loop characteristic polynomial

$$\Delta_{ol}(s) = \det(sI - A_o) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0.$$

Since  $C_o$  in Equation (13.7) has the structure of one in the first element followed by zeros, we see that the structure of  $A_o - L_o C_o$  is given by

$$A_o - L_o C_o = \begin{pmatrix} -(a_{n-1} + \ell_1) & 1 & \dots & 0 & 0 \\ -(a_{n-2} + \ell_2) & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -(a_1 + \ell_{n-1}) & 0 & \dots & 0 & 1 \\ -(a_0 + \ell_n) & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial for the closed-loop observation error is therefore given by

$$\begin{aligned} \Delta_{cl,obs}(s) &= \det(sI - (A_o - L_o C_o)) \\ &= s^n + (a_{n-1} + \ell_1)s^{n-1} + \cdots + (a_1 + \ell_{n-1})s + (a_0 + \ell_n). \end{aligned}$$

If the desired characteristic polynomial for the closed-loop observation error is

$$\Delta_{cl,obs}^d(s) = s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1s + \beta_0, \quad (13.11)$$

then we have that

$$\begin{aligned} a_{n-1} + \ell_1 &= \beta_{n-1} \\ a_{n-2} + \ell_2 &= \beta_{n-2} \\ &\vdots \\ a_1 + \ell_{n-1} &= \beta_1 \\ a_0 + \ell_n &= \beta_0. \end{aligned}$$

Defining the vectors

$$\begin{aligned} \mathbf{a}_A &\stackrel{\triangle}{=} (a_{n-1}, a_{n-2}, \dots, a_1, a_0) \\ \boldsymbol{\beta} &\stackrel{\triangle}{=} (\beta_{n-1}, \beta_{n-2}, \dots, \beta_1, \beta_0) \\ L_o &\stackrel{\triangle}{=} (\ell_1, \ell_2, \dots, \ell_{n-1}, \ell_n)^\top, \end{aligned}$$

we have that the observer gain satisfies

$$L_o = (\boldsymbol{\beta} - \mathbf{a}_A)^\top, \quad (13.12)$$

where the subscript “o” is used to emphasize what the gains are when the state equations are in observer canonic form. As an example, suppose that the transfer function model of the system is given by

$$Y(s) = \frac{9s + 20}{s^3 + 6s^2 - 11s + 8} U(s),$$

then the state space equations in observer canonic form are given by

$$\begin{aligned}\dot{x}_o &= \begin{pmatrix} -6 & 1 & 0 \\ 11 & 0 & 1 \\ -8 & 0 & 0 \end{pmatrix} x_o + \begin{pmatrix} 0 \\ 9 \\ 20 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x_o.\end{aligned}$$

The open-loop characteristic polynomial is

$$\Delta_{ol}(s) = s^3 + 6s^2 - 11s + 8,$$

with roots at  $-7.5885$ , and  $0.7942 \pm j0.6507$ . Suppose that the desired poles for the observation error are at  $-10$  and  $-20 \pm j20$ , then the desired closed-loop polynomial for the observation error is

$$\begin{aligned}\Delta_{cl,obs}^d(s) &= (s + 10)(s + 20 - j20)(s + 20 + j20) \\ &= s^3 + 50s^2 + 1200s + 8000,\end{aligned}$$

then  $\mathbf{a}_{A_o} = (6, -11, 8)$ ,  $\boldsymbol{\beta} = (50, 1200, 8000)$ , and the observer gain that places the eigenvalues of  $A_o - L_o C_o$  at the desired locations is given by

$$L_o = (\boldsymbol{\beta} - \mathbf{a}_{A_o})^\top = \begin{pmatrix} 50 \\ 1200 \\ 8000 \end{pmatrix} - \begin{pmatrix} 6 \\ -11 \\ 8 \end{pmatrix} = \begin{pmatrix} 44 \\ 1211 \\ 7992 \end{pmatrix}.$$

### 13.1.2 Observer Design: General State Space Equations

The formula given in Equation (13.12) assumes that the state space equations are in observer canonic form. Similar to the general full state feedback case discussed in Section 11.1.3, a formula for the observer gains can be derived for general state space equations. Following the method used in Section 11.1.3, the strategy is to find a state transformation matrix that converts the general state space equations into observer canonic form. Rather than repeating the steps in Section 11.1.3 we will derive the formula for the observer gain by noting the similarities between the design problem for full state feedback and the design problem for the observer gain.

For full state feedback, the feedback gains  $K$  that place the eigenvalues of  $A - BK$  at the roots of the desired characteristic equation given in Equation (11.20) are given by Equation (11.32) or

$$K = (\boldsymbol{\alpha} - \mathbf{a}_A)\mathcal{A}_A^{-1}\mathcal{C}_{A,B}^{-1}, \quad (13.13)$$

where

$$\mathcal{A}_A = \begin{pmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\ 0 & 1 & a_{n-1} & \vdots & a_3 & a_2 \\ 0 & 0 & 1 & \vdots & a_4 & a_3 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$\mathcal{C}_{A,B} = (B \ AB \ A^2B \ \cdots \ A^{n-1}B),$$

and where the subscripts are used to indicate that  $\mathcal{A}_A$  is a function of the matrix  $A$ , and  $\mathcal{C}_{A,B}$  is a function of the matrices  $A$  and  $B$ .

The observer design problem is to find the observer gain  $L$  that places the eigenvalues of  $A - LC$  at the roots of Equation (13.11). Recalling from Appendix P.7 that for a general square matrix  $M$ , the eigenvalues of  $M$  equal the eigenvalues of  $M^\top$ , the observer problem can be posed as finding  $L$  that places the eigenvalues of  $A^\top - C^\top L^\top$  at the roots of Equation (13.11). This problem is identical to the full state feedback gain problem with  $A$  replaced by  $A^\top$ ,  $B$  replaced by  $C^\top$ , and  $K$  replaced by  $L^\top$ . Making the appropriate substitutions in Ackermann's formula (13.13) gives

$$L^\top = (\beta - \mathbf{a}_{Ap}) \mathcal{A}_{A^\top}^{-1} \mathcal{C}_{A^\top, C^\top}^{-1} \quad (13.14)$$

Since  $\det(sI - A) = \det(sI - A^\top)$ , the characteristic equations for  $A$  and  $A^\top$  are identical, which implies that  $\mathbf{a}_A = \mathbf{a}_{A^\top}$  and  $\mathcal{A}_A = \mathcal{A}_{A^\top}$ . Taking the transpose of Equation (13.14) gives

$$L = \mathcal{C}_{A^\top, C^\top}^{-\top} \mathcal{A}_A^{-\top} (\beta - \mathbf{a}_A)^\top, \quad (13.15)$$

where we have used the notation

$$M^{-\top} \triangleq (M^{-1})^\top = (M^\top)^{-1}.$$

Defining the *observability matrix*

$$\begin{aligned} \mathcal{O}_{A,C} &\triangleq \mathcal{C}_{A^\top, C^\top}^\top \\ &= (C^\top \ AB^\top C^\top \ (A^\top)^2 C^\top \ \cdots \ (A^\top)^{n-1} C^\top)^\top \\ &= \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}, \end{aligned}$$

the observer gain (13.15) can be written as

$$L = \mathcal{O}_{A,C}^{-1} \mathcal{A}_A^{-\top} (\beta - \mathbf{a}_A)^\top, \quad (13.16)$$

where  $L$  is well defined if the observability matrix  $\mathcal{O}_{A,C}$  is invertible. Accordingly, the system  $(A, C)$  is said to be *observable* if  $\text{rank}(\mathcal{O}_{A,C}) = n$ . Equation (13.16) is Ackermann's formula for the observer gain  $L$ . Ackermann's formula is only valid for single output systems. For multi-output systems there are more advanced methods that are implemented in the Matlab and Python place commands. The observer gain  $L$  that places the eigenvalues of  $A - LC$  at locations  $q_1, q_2, \dots, q_n$  can be found using the following Python commands

```

1 import numpy as np
2 import control as cnt
3
4 if np.linalg.matrix_rank(cnt.observ(A, C)) != n:
5     print('System Not Observable')
6 else:      # if so, compute observer gains
7     L = cnt.place(A.T, C.T, [q1, q2, ..., qn]).T

```

where the first operation checks to see if the system is observable by checking whether  $\text{rank}(\mathcal{O}_{A,C}) = n$ . When finding the gain  $L$  it is important to note the three transposes which correspond to the transposes in Equation (13.14).

If an integrator is used in the control law, as discussed in Chapter 12, then the complete observer based controller is shown in Fig. 13-3. Note that integral feedback uses the reference output  $y_r$ , whereas the observer uses the measured output  $y$ . There are many systems for which the measured output is identical to the reference output.

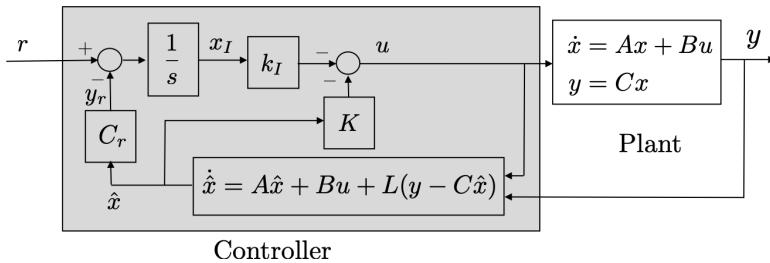


Figure 13-3: closed-loop system with observer based feedback, including integral feedback.

### 13.1.3 Separation Principle

The observer based feedback design consists of two steps:

- Find the state feedback gains  $K$  to place the eigenvalues of  $A - BK$  at specified locations  $p_1, \dots, p_n$ ,

- Find the observer gains  $L$  to place the eigenvalues of  $A - LC$  at specified locations  $q_1, \dots, q_n$ .

Since the closed-loop system includes both the observer and the state feedback, it is important to analyze whether the closed-loop system is stable and to determine the closed-loop poles. If the integrator is excluded in Fig. 13-3 then the closed-loop system has  $2n$  states:  $n$  states associated with the open-loop plant  $x$  and  $n$  states associated with the observer  $\hat{x}$ . Since the reference input doesn't affect the internal stability of the systems, we can set  $r = 0$  and the equations of motion for the closed-loop system can be written as

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ u &= -K\hat{x} \\ y &= Cx,\end{aligned}$$

which implies that

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - BK - LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

Recalling that a change of variables does not change the eigenvalues of the system, and defining the observation error as  $e = x - \hat{x}$ , and noting that

$$\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix},$$

we have that

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} &= \begin{pmatrix} I & \mathbf{0} \\ I & -I \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} \\ &= \begin{pmatrix} I & \mathbf{0} \\ I & -I \end{pmatrix} \begin{pmatrix} A & -BK \\ LC & A - BK - LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \\ &= \begin{pmatrix} I & \mathbf{0} \\ I & -I \end{pmatrix} \begin{pmatrix} A & -BK \\ LC & A - BK - LC \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ I & -I \end{pmatrix}^{-1} \begin{pmatrix} x \\ e \end{pmatrix},\end{aligned}$$

and noting that

$$\begin{pmatrix} I & \mathbf{0} \\ I & -I \end{pmatrix}^{-1} = \begin{pmatrix} I & \mathbf{0} \\ I & -I \end{pmatrix}$$

gives

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ \mathbf{0} & A - LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}.$$

Using the determinant properties for block diagonal matrices discussed in Appendix P.7, it is straightforward to show that

$$\text{eig} \begin{pmatrix} A - BK & BK \\ \mathbf{0} & A - LC \end{pmatrix} = \text{eig}(A - BK) \cup \text{eig}(A - LC).$$

Therefore the poles of the closed-loop system are precisely the poles selected during the design of the feedback gains  $K$  in addition to the poles selected during the design of the observer gains  $L$ . This happy circumstance is called the *separation principle*, which in essence implies that the controller and the observer can be designed separately and then combined without affecting the stability of the system. Unfortunately, the separation principle does not hold in general for nonlinear systems, and even more unfortunately, while the separation principle guarantees stability, it turns out that the addition of an observer can negatively impact the robustness of a state feedback design.

### 13.1.4 Observer Design for Linearized Systems

Suppose that the state space model is from a linearized system, i.e, suppose that  $x \in \mathbb{R}^n$  is the state,  $x_e \in \mathbb{R}^n$  is the equilibrium state, and that  $\tilde{x} = x - x_e$  is the state linearized around the equilibrium. Similarly, suppose that  $u \in \mathbb{R}^m$  is the input, that  $u_e \in \mathbb{R}^m$  is the input at equilibrium, and that  $\tilde{u} = u - u_e$  is the input linearized around equilibrium, and that  $y \in \mathbf{R}^p$  is the measured output,  $y_{me} = Cx_e \in \mathbf{R}^p$  is the measured output at equilibrium, and  $\tilde{y} = y - y_e$  is the measured output linearized about equilibrium, then the linearized state space model is given by

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y} &= C\tilde{x}.\end{aligned}$$

The observer design problem then finds an observer gain so that the eigenvalues of  $A - LC$  are at specified locations, and the resulting observer is

$$\dot{\hat{x}} = A\hat{x} + B\tilde{u} + L(\tilde{y} - C\hat{x}),$$

where  $\hat{x} = \hat{x} - x_e$ . Writing the observer in terms of  $\hat{x}$  gives

$$\dot{\hat{x}} - \dot{x}_e = A(\hat{x} - x_e) + B(u - u_e) + L((y - y_e) - C(\hat{x} - x_e)).$$

Noting that  $\dot{x}_e = 0$  and that  $y_e = Cx_e$  gives

$$\dot{\hat{x}} = A(\hat{x} - x_e) + B(u - u_e) + L(y - C\hat{x}).$$

## 13.2 Summary of Design Process - Observer Design

The procedure for designing an observer is summarized as follows.

Given the plant

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

find the observer gain  $L$  so that the state observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

has stable estimation error dynamics with poles located at  $q_1, q_2, \dots, q_n$ .

**Step 1.** Check to see if the system is observable by computing the observability matrix

$$\mathcal{O}_{A,C} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

and checking to see if  $\text{rank}(\mathcal{O}_{A,C}) = n$ .

**Step 2.** Find the open-loop characteristic polynomial

$$\Delta_{ol}(s) = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0,$$

and construct the row vector

$$\mathbf{a}_A = (a_{n-1}, a_{n-2}, \dots, a_1, a_0)$$

and the matrix

$$\mathcal{A}_A = \begin{pmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\ 0 & 1 & a_{n-1} & \cdots & a_3 & a_2 \\ \dots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & a_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

**Step 3.** Find the desired characteristic polynomial for the observation error

$$\Delta_{obs}^d(s) = (s - q_1)(s - q_2) \cdots (s - q_n) = s^n + \beta_{n-1}s^{n-1} + \dots + \beta_1s + \beta_0,$$

and construct the row vector

$$\boldsymbol{\beta} = (\beta_{n-1}, \beta_{n-2}, \dots, \beta_1, \beta_0).$$

**Step 4.** Compute the desired observer gains as

$$L = \mathcal{O}_{A,C}^{-1}(\mathcal{A}_A^\top)^{-1}(\boldsymbol{\beta} - \mathbf{a}_A)^\top$$

Using Python, these steps can be implemented using the following script.

```

1 import numpy as np
2 import control as cnt
3
4 if np.linalg.matrix_rank(cnt.obsv(A,C)) != n:
5     print('System Not Observable')
6 else: # if so, compute observer gains
7     L = cnt.place(A.T,C.T,[q1,q2,...,qn]).T

```

### 13.2.1 Simple example

Given the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 4 \end{pmatrix} u \\ y &= \begin{pmatrix} 5 & 0 \end{pmatrix} x\end{aligned}$$

find the observer gain  $L$  so that the state observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

has stable estimation error dynamics with poles located at  $-10 \pm j10$ .

**Step 1.** The observability matrix is given by

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

The determinant is  $\det(\mathcal{O}) = 25 \neq 0$ , therefore the system is observable.

**Step 2.** The open-loop characteristic polynomial

$$\Delta_{oi}(s) = \det(sI - A) = \det \begin{pmatrix} s & -1 \\ -2 & s - 3 \end{pmatrix} = s^2 - 3s - 2,$$

implies that

$$\mathbf{a} = (-3, -2)$$

$$\mathcal{A} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

**Step 3.** The desired characteristic polynomial for the observation error is

$$\Delta_{obs}^d(s) = (s + 10 - j10)(s + 10 + j10) = s^2 + 20s + 200,$$

which implies that

$$\boldsymbol{\beta} = (20, 200).$$

**Step 4.** The observation gains are therefore given as

$$\begin{aligned}L &= \mathcal{O}^{-1}(\mathcal{A}^\top)^{-1}(\boldsymbol{\beta} - \mathbf{a})^\top \\ &= \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} ((20, 200) - (-3, -2))^\top \\ &= \begin{pmatrix} 1/5 & 0 \\ 3/5 & 1/5 \end{pmatrix} \begin{pmatrix} 23 \\ 202 \end{pmatrix} = \begin{pmatrix} 23/5 \\ 271/5 \end{pmatrix}\end{aligned}$$

Alternatively, we could have used the following Python script

```

1 import numpy as np
2 import control as cnt
3
4 # original state space system
5 A = np.array([[0.0, 1.0], [2.0, 3.0]])
6 B = np.array([[0.0], [4.0]])
7 C = np.array([[5.0, 0.0]])
8
9 # desired poles for observer
10 q1 = -10 + 10.j
11 q2 = -10 - 10.j
12
13 # is the system observable?
14 if np.linalg.matrix_rank(cnt.observ(A,C)) != np.size(A,1):
15     print('System Not Observable')
16 else: # if so, compute observer gains
17     L = cnt.place(A.T,C.T,[q1,q2]).T

```

The corresponding observer is given by

$$\dot{\hat{x}} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} u + \begin{pmatrix} 23/5 \\ 271/5 \end{pmatrix} (y - (5 \ 0) \hat{x}), \quad (13.17)$$

and can be implemented as member functions within a Python class as follows (see the solutions for case studies A-C for class-specific implementation examples):

```

1 def update_observer(self, y_m):
2     # update the observer using RK4 integration
3     F1 = self.observer_f(self.x_hat, y_m)
4     F2 = self.observer_f(self.x_hat + self.Ts / 2 * F1, y_m)
5     F3 = self.observer_f(self.x_hat + self.Ts / 2 * F2, y_m)
6     F4 = self.observer_f(self.x_hat + self.Ts * F3, y_m)
7
8     self.x_hat += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
9
10    return self.x_hat
11
12 def observer_f(self, x_hat, y_m):
13     # xhatdot = A*xhat + B*u + L(y-C*xhat)
14     # assumes that all matrices below are np.arrays
15     xhat_dot = self.A @ x_hat \
16                 + self.B @ self.u \
17                 + self.L @ (y_m - self.C @ x_hat)
18
19    return xhat_dot

```

Digital implementation of the observer in Equation (13.17) requires two persistent (or member) variables, one for  $\hat{x}$  and one for  $u$ . The observer is updated using the RK4 algorithm in lines 3–8. The differential equation that describes the observer is given on lines 15–17.

### 13.3 Design Study A. Single Link Robot Arm



#### Example Problem A.13

The objective of this problem is to design an observer that estimates the state of the system and uses the estimated state in the controller designed in Homework A.12.

- (a) For the sake of understanding the function of the observer, for this problem we will use exact parameters, without an input disturbance. Modify the arm dynamics so that the parameters known to the controller are the actual plant parameters (uncertainty parameter  $\alpha = 0$ ).
- (b) Verify that the state space system is observable by checking that  $\text{rank}(\mathcal{O}_{A,C}) = n$ .
- (c) Modify the controller to add an observer to estimate the state  $\hat{x}$ , and then use the estimate of the state  $\hat{x}$  in your feedback controller. Tune the poles of the controller and observer to obtain good performance.
- (d) Modify the simulation files so that the controller outputs both  $u$  and  $\hat{x}$ . Add a plotting routine to plot both the state and the estimated state of the system on the same graph.
- (e) As motivation for the next chapter, add an input disturbance to the system of 0.01 and observe that there is steady state error in the response even though there is an integrator. This is caused by a steady state error in the observation error. In the next chapter we will show how to remove the steady state error in the observation error.

#### Solution

The single link robot arm is simulated using the following Python code.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import armParam as P
6 from hw3.armDynamics import armDynamics
7 from armController import armController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.armAnimation import armAnimation
10 from hw2.dataPlotter import dataPlotter
11 from dataPlotterObserver import dataPlotterObserver
12
13 # instantiate arm, controller, and reference classes
14 arm = armDynamics(alpha=0.0)

```

```

15 controller = armController()
16 reference = signalGenerator(amplitude=30*np.pi/180.0,
17                               frequency=0.05)
18 disturbance = signalGenerator(amplitude=0.25)
19 noise = signalGenerator(amplitude=0.1)
20
21 # instantiate the simulation plots and animation
22 dataPlot = dataPlotter()
23 dataPlotObserver = dataPlotterObserver()
24 animation = armAnimation()
25
26 t = P.t_start # time starts at t_start
27 y = arm.h() # output of system at start of simulation
28
29 while t < P.t_end: # main simulation loop
30     # Get referenced inputs from signal generators
31     # Propagate dynamics in between plot samples
32     t_next_plot = t + P.t_plot
33
34     # updates control and dynamics at faster simulation rate
35     while t < t_next_plot:
36         r = reference.square(t)
37         d = disturbance.step(t) # input disturbance
38         n = noise.random(t) # simulate sensor noise
39         u, xhat = controller.update(r, y + n) # update controller
40         y = arm.update(u + d) # propagate system
41         t = t + P.Ts # advance time by Ts
42
43     # update animation and data plots
44     animation.update(arm.state)
45     dataPlot.update(t, r, arm.state, u)
46     dataPlotObserver.update(t, arm.state, xhat, d, 0.0)
47
48     # the pause causes the figure to display during simulation
49     plt.pause(0.0001)
50
51 # Keeps the program from closing until user presses a button.
52 print('Press key to close')
53 plt.waitforbuttonpress()
54 plt.close()

```

Listing 13.1: armSim.py

Note that the input to the controller is the output  $y$  corrupted by noise  $n$  and that the input disturbance  $d$  has been added to  $u$ . The dataPlotObserver plots the state  $x$  and the estimated state  $\hat{x}$ , as well as the disturbance  $d$  and estimated disturbance  $\hat{d}$ , as will be discussed in the next chapter.

A Python class that implements observer-based control with an integrator for the single link robot arm is shown below.

```

1 import numpy as np
2 import armParamHW13 as P
3
4 class armController:
5     def __init__(self):
6         self.x_hat = np.array([

```

```

7         [0.0], # theta_hat_0
8         [0.0], # thetadot_hat_0
9     ])
10    self.tau_d1 = 0.0 # control torque, delayed 1 sample
11    self.integrator = 0.0 # integrator
12    self.error_d1 = 0.0 # error signal, delayed 1 sample
13    self.K = P.K # state feedback gain
14    self.ki = P.ki # Input gain
15    self.L = P.L # observer gain
16    self.A = P.A # system model
17    self.B = P.B
18    self.C = P.C
19    self.limit = P.tau_max # Maximum torque
20    self.Ts = P.Ts # sample rate of controller
21
22 def update(self, theta_r, y):
23     # update the observer and extract theta_hat
24     x_hat = self.update_observer(y)
25     theta_hat = x_hat.item(0)
26
27     # integrate error
28     error = theta_r - theta_hat
29     self.integrateError(error)
30
31     # feedback linearizing torque tau_fl
32     tau_fl = P.m * P.g * (P.ell / 2.0) * np.cos(theta_hat)
33
34     # Compute the state feedback controller
35     tau_tilde = -self.K @ x_hat - self.ki * self.integrator
36
37     # compute total torque
38     tau = self.saturate(tau_fl + tau_tilde.item(0))
39     self.tau_d1 = tau
40
41     return tau, x_hat
42
43 def integrateError(self, error):
44     self.integrator = self.integrator \
45         + (self.Ts/2.0)*(error + self.error_d1)
46     self.error_d1 = error
47
48 def update_observer(self, y_m):
49     # update the observer using RK4 integration
50     F1 = self.observer_f(self.x_hat, y_m)
51     F2 = self.observer_f(self.x_hat + self.Ts / 2 * F1, y_m)
52     F3 = self.observer_f(self.x_hat + self.Ts / 2 * F2, y_m)
53     F4 = self.observer_f(self.x_hat + self.Ts * F3, y_m)
54     self.x_hat += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
55
56     return self.x_hat
57
58 def observer_f(self, x_hat, y_m):
59     # compute feedback linearizing torque tau_fl
60     theta_hat = x_hat.item(0)
61     tau_fl = P.m * P.g * (P.ell / 2.0) * np.cos(theta_hat)
62
63     # xhatdot = A*(xhat-xe) + B*(u-ue) + L(y-C*xhat)
64     xhat_dot = self.A @ x_hat\
```

```

64         + self.B * (self.tau_d1 - tau_fl) \
65         + self.L * (y_m - self.C @ x_hat)
66     return xhat_dot
67
68     def saturate(self,u):
69         if abs(u) > self.limit:
70             u = self.limit*np.sign(u)
71     return u

```

Listing 13.2: armController.py

Notice that we must update the observer dynamics and that  $\hat{x}$  is used in the controller instead of  $x$ , but otherwise this controller is identical to the controller in Chapter 12. The observer is updated using the RK4 algorithm (see line 48). The differential equations for the observer are also specified on lines 64–66.

Python code that computes the control gains is given below.

```

1 # Single link arm Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import armParam as P
7
8 # tuning parameters
9 tr = 0.4
10 zeta = 0.707
11 integrator_pole = 9
12 wn_obs = 10 # natural frequency for observer
13 zeta_obs = 0.707 # damping ratio for observer
14
15 Ts = P.Ts # sample rate of the controller
16 tau_max = P.tau_max # limit on control signal
17 m = P.m
18 ell = P.ell
19 g = P.g
20
21
22 # State Space Equations
23 A = np.array([[0.0, 1.0],
24               [0.0, -1.0*P.b/P.m/(P.ell**2)]])
25 n = A.shape[0]
26
27 B = np.array([[0.0],
28               [3.0/P.m/(P.ell**2)]])
29
30 C = np.array([[1.0, 0.0]])
31
32 # control design
33 # form augmented system
34 A1 = np.array([[0.0, 1.0, 0.0],
35                [0.0, -1.0*P.b/P.m/(P.ell**2), 0.0],
36                [-1.0, 0.0, 0.0]])
37
38 B1 = np.array([[0.0],
39               [3.0/P.m/(P.ell**2)],

```

```

40             [0.0]))
41
42 # gain calculation
43 wn = 2.2/tr # natural frequency
44 des_char_poly = np.convolve(
45     [1, 2*zeta*wn, wn**2],
46     [1, integrator_pole])
47 des_poles = np.roots(des_char_poly)
48
49 # Compute the gains if the system is controllable
50 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 3:
51     print("The system is not controllable")
52 else:
53     K1 = cnt.acker(A1, B1, des_poles)
54     K = np.array([K1.item(0), K1.item(1)])
55     ki = K1.item(2)
56
57 # observer design
58 #des_observ_char_poly = [1, 2*zeta_obs*wn_obs, wn_obs**2]
59 des_observ_poles = des_poles[0:n]*2. #np.roots(des_observ_char_poly)
60
61 # Compute the gains if the system is controllable
62 if np.linalg.matrix_rank(cnt.ctrb(A.T, C.T)) != 2:
63     print("The system is not observable")
64 else:
65     L = cnt.acker(A.T, C.T, des_observ_poles).T
66
67 print('K: ', K)
68 print('ki: ', ki)
69 print('L^T: ', L.T)

```

Listing 13.3: armParamHW13.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 13.4 Design Study B. Inverted Pendulum



### Example Problem B.13

The objective of this problem is to design an observer that estimates the state of the system and to use the estimated state in the controller designed in Homework B.12.

- (a) For the sake of understanding the function of the observer, for this problem we will use exact parameters, without an input disturbance. Modify the pendulum dynamics so that the parameters known to the controller are the actual plant parameters (uncertainty parameter  $\alpha = 0$ ).

- (b) Verify that the state space system is observable by checking that  $\text{rank}(\mathcal{O}_{A,C}) = n$ .
- (c) In the control block, add an observer to estimate the state  $\hat{x}$ , and use the estimate of the state in your feedback controller. Tune the poles of the controller and observer to obtain good performance.
- (d) Modify the simulation files so that the controller outputs both  $u$  and  $\hat{x}$ . Add a plotting routine to plot both the state and the estimated state of the system on the same graph.
- (e) As motivation for the next chapter, add an input disturbance to the system of 0.05 and observe that there is steady state error in the response even though there is an integrator. This is caused by a steady state error in the observation error. In the next chapter we will show how to remove the steady state error in the observation error.

## Solution

The inverted pendulum is simulated using the following Python code.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import pendulumParam as P
6 from hw3.pendulumDynamics import pendulumDynamics
7 from pendulumController import pendulumController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.pendulumAnimation import pendulumAnimation
10 from hw2.dataPlotter import dataPlotter
11 from dataPlotterObserver import dataPlotterObserver
12
13 # instantiate pendulum, controller, and reference classes
14 pendulum = pendulumDynamics(alpha=0.0)
15 controller = pendulumController()
16 reference = signalGenerator(amplitude=0.5, frequency=0.05)
17 disturbance = signalGenerator(amplitude=0.5)
18 noise_z = signalGenerator(amplitude=0.01)
19 noise_th = signalGenerator(amplitude=0.01)
20
21 # instantiate the simulation plots and animation
22 dataPlot = dataPlotter()
23 dataPlotObserver = dataPlotterObserver()
24 animation = pendulumAnimation()
25
26 t = P.t_start # time starts at t_start
27 y = pendulum.h() # output of system at start of simulation
28
29 while t < P.t_end: # main simulation loop
30     # Get referenced inputs from signal generators
31     # Propagate dynamics in between plot samples
32     t_next_plot = t + P.t_plot

```

```

33
34     while t < t_next_plot:
35         r = reference.square(t)
36         d = disturbance.step(t)
37         n = np.array([[0.0], [0.0]])
38         #n = np.array([[noise_z.random(t)], [noise_th.random(t)]])
39         u, xhat = controller.update(r, y + n)
40         y = pendulum.update(u + d) # propagate system
41         t = t + P.Ts # advance time by Ts
42
43     # update animation and data plots
44     animation.update(pendulum.state)
45     dataPlot.update(t, r, pendulum.state, u)
46     dataPlotObserver.update(t, pendulum.state, xhat, d, 0.0)
47     plt.pause(0.0001)
48
49 # Keeps the program from closing until the user presses a button.
50 print('Press key to close')
51 plt.waitforbuttonpress()
52 plt.close()

```

Listing 13.4: pendulumSim.py

Note in line 38, that the input to the controller is the output  $y$  corrupted by noise  $n$  and that the input disturbance  $d$  has been added to  $u$  in line 39. The dataPlotObserver plots the state  $x$  and the estimated state  $\hat{x}$ , as well as the disturbance  $d$  and estimated disturbance  $\hat{d}$ , as will be discussed in the next chapter.

A Python class that implements observer-based control with an integrator for the inverted pendulum is shown below.

```

1 import numpy as np
2 import pendulumParamHW13 as P
3
4 class pendulumController:
5     def __init__(self):
6         self.x_hat = np.array([
7             [0.0], # initial estimate for z_hat
8             [0.0], # initial estimate for theta_hat
9             [0.0], # initial estimate for z_hat_dot
10            [0.0]]) # initial estimate for theta_hat_dot
11         self.F_d1 = 0.0 # Computed Force, delayed by one sample
12         self.integrator = 0.0 # integrator
13         self.error_d1 = 0.0 # error signal delayed by 1 sample
14         self.K = P.K # state feedback gain
15         self.ki = P.ki # integrator gain
16         self.L = P.L # observer gain
17         self.A = P.A # system model
18         self.B = P.B
19         self.C = P.C
20         self.limit = P.F_max # Maximum force
21         self.Ts = P.Ts # sample rate of controller
22
23     def update(self, z_r, y):
24         # update the observer and extract z_hat
25         x_hat = self.update_observer(y)
26         z_hat = x_hat.item(0)

```

```

27
28     # integrate error
29     error = z_r - z_hat
30     self.integrate_error(error)
31
32     # Compute the state feedback controller
33     F_unsat = -self.K @ x_hat - self.ki * self.integrator
34     F_sat = self.saturate(F_unsat.item(0))
35     self.F_d1 = F_sat
36
37     return F_sat, x_hat
38
39 def update_observer(self, y_m):
40     # update the observer using RK4 integration
41     F1 = self.observer_f(self.x_hat, y_m)
42     F2 = self.observer_f(self.x_hat + self.Ts / 2 * F1, y_m)
43     F3 = self.observer_f(self.x_hat + self.Ts / 2 * F2, y_m)
44     F4 = self.observer_f(self.x_hat + self.Ts * F3, y_m)
45     self.x_hat += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
46     return self.x_hat
47
48 def observer_f(self, x_hat, y_m):
49     # xhatdot = A*xhat + B*u + L(y-C*xhat)
50     xhat_dot = self.A @ x_hat \
51                 + self.B * self.F_d1 \
52                 + self.L @ (y_m - self.C @ x_hat)
53
54     return xhat_dot
55
56 def integrate_error(self, error):
57     self.integrator = self.integrator \
58                         + (self.Ts / 2.0) * (error + self.error_d1)
59     self.error_d1 = error
60
61 def saturate(self, u):
62     if abs(u) > self.limit:
63         u = self.limit * np.sign(u)
64     return u

```

Listing 13.5: pendulumController.py

The observer is updated on line 25, and  $\hat{x}$  is used in the controller instead of  $x$  but otherwise is identical to the controller in Chapter 12. The observer is updated on lines 40–48 using the RK4 algorithm. The differential equations for the observer are specified on lines 50–56.

Python code that computes the observer and control gains is given below.

```

1 # Inverted Pendulum Parameter File
2 import sys
3 sys.path.append('..') # add parent directory
4 import pendulumParam as P
5 import numpy as np
6 from scipy import signal
7 import control as cnt
8
9 # sample rate of the controller

```

```

10 Ts = P.Ts
11
12 # saturation limits
13 F_max = 5.0 # Max Force, N
14
15 ######
16 # State Space
17 #####
18 # tuning parameters
19 tr_z = 1.5 # rise time for position
20 tr_theta = 0.5 # rise time for angle
21 zeta_z = 0.707 # damping ratio position
22 zeta_th = 0.707 # damping ratio angle
23 integrator_pole = -2.0 # integrator pole
24 tr_z_obs = tr_z/5.0 # rise time for observer - position
25 tr_theta_obs = tr_theta/5.0 # rise time for observer - angle
26
27 # State Space Equations
28 # xdot = A*x + B*u
29 # y = C*x
30
31 A = np.matrix([[0.0, 0.0, 1.0, 0.0],
32                 [0.0, 0.0, 0.0, 1.0],
33                 [0.0, -3*P.m1*P.g/4/(.25*P.m1+P.m2),
34                  -P.b/(.25*P.m1+P.m2), 0.0],
35                 [0.0, 3*(P.m1+P.m2)*P.g/2/(.25*P.m1+P.m2)/P.ell,
36                  3*P.b/2/(.25*P.m1+P.m2)/P.ell, 0.0]])
37
38 B = np.matrix([[0.0],
39                 [0.0],
40                 [1/(.25*P.m1+P.m2)],
41                 [-3.0/2/(.25*P.m1+P.m2)/P.ell]])
42
43 C = np.matrix([[1.0, 0.0, 0.0, 0.0],
44                 [0.0, 1.0, 0.0, 0.0]])
45
46 # form augmented system
47 Cr = C[0, :]
48
49 # Augmented Matrices
50 A1 = np.concatenate((
51     np.concatenate((A, np.zeros((4, 1))), axis=1),
52     np.concatenate((-Cr, np.matrix([[0.0]])), axis=1)),
53     axis=0)
54 B1 = np.concatenate((B, np.matrix([[0.0]])), axis=0)
55
56 # computer control gains
57 wn_th = 2.2/tr_theta # natural frequency for angle
58 wn_z = 2.2/tr_z # natural frequency for position
59 des_char_poly = np.convolve(
60     np.convolve([1, 2*zeta_z*wn_z, wn_z**2],
61                [1, 2*zeta_th*wn_th, wn_th**2]),
62                np.poly([integrator_pole]))
63 des_poles = np.roots(des_char_poly)
64
65 # Compute the gains if the system is controllable
66 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 5:

```

```

67     print("The system is not controllable")
68 else:
69     K1 = cnt.acker(A1, B1, des_poles)
70     K = K1[0, 0:4]
71     ki = K1[0, 4]
72
73 # computer observer gains
74 wn_z_obs = 2.2/tr_z_obs
75 wn_th_obs = 2.2/tr_theta_obs
76 des_obs_char_poly = np.convolve(
77     [1, 2*zeta_z*wn_z_obs, wn_z_obs**2],
78     [1, 2*zeta_th*wn_th_obs, wn_th_obs**2])
79 des_obs_poles = np.roots(des_obs_char_poly)
80
81 # Compute the gains if the system is observable
82 if np.linalg.matrix_rank(cnt ctrb(A.T, C.T)) != 4:
83     print("The system is not observable")
84 else:
85     L = signal.place_poles(A.T, C.T, des_obs_poles).gain_matrix.T
86
87 print('K: ', K)
88 print('ki: ', ki)
89 print('L^T: ', L.T)

```

Listing 13.6: pendulumParamHW13.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 13.5 Design Study C. Satellite Attitude Control



### Example Problem C.13

The objective of this problem is to design an observer that estimates the state of the system and to use the estimated state in the controller designed in Homework C.12.

- (a) For the sake of understanding the function of the observer, for this problem we will use exact parameters, without an input disturbance. Modify the satellite dynamics so that the parameters known to the controller are the actual plant parameters (uncertainty parameter  $\alpha = 0$ ).
- (b) Verify that the state space system is observable by checking that  $\text{rank}(\mathcal{O}_{A,C}) = n$ .
- (c) In the control block, add an observer to estimate the state  $\hat{x}$ , and use the estimate of the state in your feedback controller. Tune the poles of the controller and observer to obtain good performance.

- (d) Modify the simulation files so that the controller outputs both  $u$  and  $\hat{x}$ . Add a plotting routine to plot both the state and the estimated state of the system on the same graph.
- (e) As motivation for the next chapter, add an input disturbance to the system of 1.0 and observe that there is steady state error in the response even though there is an integrator. This is caused by a steady state error in the observation error. In the next chapter we will show how to remove the steady state error in the observation error.

## Solution

The satellite is simulated using the following Python code.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import satelliteParam as P
6 from hw3.satelliteDynamics import satelliteDynamics
7 from satelliteController import satelliteController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.satelliteAnimation import satelliteAnimation
10 from hw2.dataPlotter import dataPlotter
11 from dataPlotterObserver import dataPlotterObserver
12
13 # instantiate satellite, controller, and reference classes
14 satellite = satelliteDynamics(alpha=0.0)
15 controller = satelliteController()
16 reference = signalGenerator(amplitude=15.0*np.pi/180.0,
17                               frequency=0.03)
18 disturbance = signalGenerator(amplitude=1.0)
19 noise_phi = signalGenerator(amplitude=0.01)
20 noise_th = signalGenerator(amplitude=0.01)
21
22 # instantiate the simulation plots and animation
23 dataPlot = dataPlotter()
24 dataPlotObserver = dataPlotterObserver()
25 animation = satelliteAnimation()
26
27 t = P.t_start # time starts at t_start
28 y = satellite.h() # output of system at start of simulation
29
30 while t < P.t_end: # main simulation loop
31     # Propagate dynamics in between plot samples
32     t_next_plot = t + P.t_plot
33
34     # updates control and dynamics at faster simulation rate
35     while t < t_next_plot:
36         r = reference.square(t) # reference input
37         d = disturbance.step(t) # input disturbance
38
39         # simulate sensor noise -
40         # #np.array([[noise_phi.random(t)], [noise_th.random(t)]])
```

```

1   n = np.array([[noise_phi.random(t)],[noise_th.random(t)]])
2
3   u, xhat = controller.update(r, y + n) # update controller
4   y = satellite.update(u + d) # propagate system
5   t = t + P.Ts # advance time by Ts
6
7   # update animation and data plots
8   animation.update(satellite.state)
9   dataPlot.update(t, r, satellite.state, u)
10  dataPlotObserver.update(t, satellite.state, xhat, d, 0.0)
11
12  # the pause causes the figure to display during simulation
13  plt.pause(0.0001)
14
15  # Keeps the program from closing until the user presses a button.
16  print('Press key to close')
17  plt.waitforbuttonpress()
18  plt.close()

```

Listing 13.7: satelliteSim.py

Note in line 43, that the input to the controller is the output  $y$  corrupted by noise  $n$  and that the input disturbance  $d$  has been added to  $u$  in line 44. The dataPlotObserver plots the state  $x$  and the estimated state  $\hat{x}$ , as well as the disturbance  $d$  and estimated disturbance  $\hat{d}$  which will be discussed in the next chapter.

A Python class that implements observer-based control with an integrator for the satellite is shown below.

```

1 import numpy as np
2 import satelliteParamHW13 as P
3
4 class satelliteController:
5     def __init__(self):
6         self.x_hat = np.array([
7             [0.0], # initial estimate for theta_hat
8             [0.0], # initial estimate for phi_hat
9             [0.0], # initial estimate for theta_hat_dot
10            [0.0], # initial estimate for phi_hat_dot
11        ])
12        self.tau_d1 = 0.0 # Computed torque delayed 1 sample
13        self.integrator = 0.0 # integrator
14        self.error_d1 = 0.0 # error signal delayed 1 sample
15        self.K = P.K # state feedback gain
16        self.ki = P.ki # integrator gain
17        self.L = P.L # observer gain
18        self.A = P.A # system model
19        self.B = P.B
20        self.C = P.C
21        self.limit = P.tau_max # Maximum torque
22        self.Ts = P.Ts # sample rate of controller
23
24    def update(self, phi_r, y):
25        # update the observer and extract z_hat
26        x_hat = self.update_observer(y)
27        phi_hat = x_hat.item(1)
28

```

```

29     # integrate error
30     error = phi_r - phi_hat
31     self.integrateError(error)
32
33     # Compute the state feedback controller
34     tau_unsat = -self.K @ x_hat \
35                 - self.ki*self.integrator
36     tau = self.saturate(tau_unsat.item(0))
37     self.tau_d1 = tau
38
39     return tau, x_hat
40
41 def update_observer(self, y_m):
42     # update the observer using RK4 integration
43     F1 = self.observer_f(self.x_hat, y_m)
44     F2 = self.observer_f(self.x_hat + self.Ts / 2 * F1, y_m)
45     F3 = self.observer_f(self.x_hat + self.Ts / 2 * F2, y_m)
46     F4 = self.observer_f(self.x_hat + self.Ts * F3, y_m)
47     self.x_hat += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
48
49     return self.x_hat
50
51 def observer_f(self, x_hat, y_m):
52     # xhatdot = A*xhat + B*u + L(y-C*xhat)
53     xhat_dot = self.A @ x_hat \
54                 + self.B * self.tau_d1 \
55                 + self.L @ (y_m - self.C @ x_hat)
56
57     return xhat_dot
58
59 def integrateError(self, error):
60     self.integrator = self.integrator + (self.Ts / 2.0) * \
61                               (error + self.error_d1)
62     self.error_d1 = error
63
64 def saturate(self,u):
65     if abs(u) > self.limit:
66         u = self.limit*np.sign(u)
67     return u

```

Listing 13.8: satelliteController.py

The observer is updated on line 26, and  $\hat{x}$  is used in the controller instead of  $x$  but otherwise is identical to the controller in Chapter 12. The observer is updated on lines 41–49 using the RK4 algorithm. The differential equations for the observer are specified on lines 51–57.

Python code that computes the observer and control gains is given below.

```

1 # satellite Parameter File
2 import numpy as np
3 import control as cnt
4 from scipy import signal
5 import sys
6 sys.path.append('..') # add parent directory
7 import satelliteParam as P
8

```

```

9 Ts = P.Ts
10 #sigma = P.sigma
11 #beta = P.beta
12 tau_max = P.tau_max
13
14 # tuning parameters
15 wn_th = 0.6
16 wn_phi = 1.1      # rise time for angle
17 zeta_phi = 0.707  # damping ratio position
18 zeta_th = 0.707  # damping ratio angle
19 integrator_pole = [-1.0]
20 # pick observer poles
21 wn_th_obs = 10.0*wn_th
22 wn_phi_obs = 10.0*wn_phi
23
24
25 # State Space Equations
26 # xdot = A*x + B*u
27 # y = C*x
28 A = np.array([[0.0, 0.0, 1.0, 0.0],
29                 [0.0, 0.0, 0.0, 1.0],
30                 [-P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js],
31                 [P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp]])
32
33 B = np.array([[0.0],
34                 [0.0],
35                 [1.0/P.Js],
36                 [0.0]])
37
38 C = np.array([[1.0, 0.0, 0.0, 0.0],
39                 [0.0, 1.0, 0.0, 0.0]])
40
41 # form augmented system
42 Cout = np.array([[0.0, 1.0, 0.0, 0.0]])
43 A1 = np.array([[0.0, 0.0, 1.0, 0.0, 0.0],
44                 [0.0, 0.0, 0.0, 1.0, 0.0],
45                 [-P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js, 0.0],
46                 [P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp, 0.0],
47                 [0.0, -1.0, 0.0, 0.0, 0.0]])
48
49 B1 = np.array([[0.0],
50                 [0.0],
51                 [1.0/P.Js],
52                 [0.0],
53                 [0.0]])
54
55 # gain calculation
56 des_char_poly = np.convolve(
57     np.convolve([1, 2*zeta_phi*wn_phi, wn_phi**2],
58                 [1, 2*zeta_th*wn_th, wn_th**2]),
59     np.poly(integrator_pole))
60 des_poles = np.roots(des_char_poly)
61
62 # Compute the gains if the system is controllable
63 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 5:
64     print("The system is not controllable")
65 else:

```

```

66 K1 = cnt.acker(A1, B1, des_poles)
67 K = np.matrix([K1.item(0), K1.item(1), K1.item(2), K1.item(3)])
68 ki = K1.item(4)
69
70 # computer observer gains
71 des_obs_char_poly = np.convolve([1, 2*zeta_phi*wn_phi_obs,
72                                 wn_phi_obs**2],
73                                 [1, 2*zeta_th*wn_th_obs,
74                                 wn_th_obs**2])
75 des_obs_poles = np.roots(des_obs_char_poly)
76
77 # Compute the gains if the system is observable
78 if np.linalg.matrix_rank(cnt.ctrb(A.T, C.T)) != 4:
79     print("The system is not observable")
80 else:
81     # place_poles returns an object with various properties.
82     # The gains are accessed through .gain_matrix
83     # .T transposes the matrix
84     L = cnt.place(A.T, C.T, des_obs_poles).T
85
86 print('K: ', K)
87 print('ki: ', ki)
88 print('L^T: ', L.T)

```

Listing 13.9: satelliteParamHW13.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## Important Concepts:

- Typically only a subset of the states are known. To implement a state feedback controller the unknown states must be estimated.
- An observer estimates the system states using the measured outputs. A typical observer will have a predictor term, that is based upon the state-space equation, and a corrector term that modifies the prediction based on the difference between the actual and expected measurements.
- When a system is observable, we are able to (1) estimate all of the states using measured outputs, (2) transform the state space equations into observer canonical form, (3) invert the Observability matrix (i.e. the Observability matrix is full rank), and (4) place the observer poles in any desired location.
- There is a duality relationship between controllability and observability that allows for analogous steps in finding the observer gains as was shown in finding the controller gains.
- The separation principle indicates that the controller gains and observer gains may be chosen independently without affecting the closed-loop stability.

## Notes and References

The original source for Luenberger observers is in [21]. Observer design using pole placement is covered in most introductory textbooks on control, see for example [4, 22, 23], as well as introductions to state space design [5, 6]. The most common definition of observability is that a dynamic system is said to be *observable* if the initial state  $x_0$  can be reconstructed by observing the output over any finite interval. Linear time-invariant systems are observable if and only if the observability matrix is full rank [5]. Observer theory and design is an interesting topic in its own right. The design of nonlinear observers is currently an active area of research. The most commonly used observer for nonlinear systems is the extended Kalman filter [24, 25].



# 14

## Disturbance Observers

### Learning Objectives:

- Use the observer to estimate the input disturbance in addition to the state.

## 14.1 Theory

In previous sections we have discussed that modeling errors and other real-world effects result in an input disturbance to the system as shown in **Fig. 14-1**. The

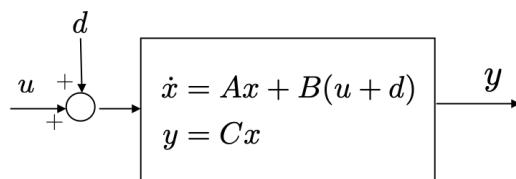


Figure 14-1: Input disturbance to plant modeled by state space equations.

objective of this section is to explain the effect of the disturbance on an observer. It turns out that the disturbance creates a steady state bias in the observer states. We will also describe how the observer can be augmented, the dual of adding an integrator, to estimate the disturbance and to remove the steady state observation error.

To understand the effect of the disturbance on the observer, note that when the disturbance is present, the state space equations for the system are

$$\begin{aligned}\dot{x} &= Ax + B(u + d) \\ y &= Cx.\end{aligned}$$

The standard equation for the observer is

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}). \quad (14.1)$$

Defining the observation error as  $e = x - \hat{x}$  and taking the derivative with respect to time gives

$$\dot{e} = \dot{x} - \dot{\hat{x}} \quad (14.2)$$

$$\begin{aligned} &= Ax + Bu + Bd - A\hat{x} - Bu - LCx + LC\hat{x} \\ &= (A - LC)e + Bd. \end{aligned} \quad (14.3)$$

Taking the Laplace transform of both sides while accounting for the initial condition, and solving for  $E(s)$  gives

$$E(s) = (sI - (A - LC))^{-1}e(0) + (sI - (A - LC))^{-1}BL\{d(t)\}.$$

Therefore, the error is due to two terms. The first term involves the initial conditions  $e(0)$  and decays to zero if the eigenvalues of  $A - LC$  are in the open left half of the complex plane. The second term is driven by the disturbance and for a constant bounded disturbance, it will result in a constant bounded observation error.

If the disturbance  $d(t)$  were perfectly known, then we could modify the observer in Equation (14.1) as

$$\dot{\hat{x}} = A\hat{x} + B(u + d) + L(y - C\hat{x}), \quad (14.4)$$

so that the evolution of the observation error becomes

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + B(u + d) - A\hat{x} - B(u + d) - LCx + LC\hat{x} \\ &= (A - LC)e, \end{aligned}$$

which, in contrast to Equation (14.3) is not driven by the disturbance  $d$ . However, since  $d$  is not known it needs to be estimated. The basic idea in this chapter is to estimate  $d$  using an augmented observer.

If we assume that  $d$  is constant, then the differential equation that governs the evolution of  $d$  is given by  $\dot{d} = 0$ . Therefore, if we augment the state  $x$  of the system with the disturbance, then the equations of motion can be written as

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{d} \end{pmatrix} &= \begin{pmatrix} A & B \\ \mathbf{0} & 0 \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix}. \end{aligned}$$

Define the augmented state space matrices as

$$\begin{aligned} A_2 &= \begin{pmatrix} A & B \\ \mathbf{0} & 0 \end{pmatrix} \\ C_2 &= \begin{pmatrix} C & 0 \end{pmatrix}. \end{aligned}$$

If  $(A_2, C_2)$  are observable, then using the techniques discussed in Chapter 13 we can design an observer for the augmented system, where the augmented observer equations are

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{pmatrix} = \begin{pmatrix} A & B \\ \mathbf{0} & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + \begin{pmatrix} L \\ L_d \end{pmatrix} \left( y - (C \quad 0) \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} \right) + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

or in component form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B(u + \hat{d}) + L(y - C\hat{x}) \\ \dot{\hat{d}} &= L_d(y - C\hat{x}), \end{aligned}$$

where the gain  $L_2 = (L^\top, L_d)^\top$  is obtained using the Ackerman formula

$$L_2 = \mathcal{O}_{A_2, C_2}^{-1}(\mathcal{A}_{A_2}^\top)^{-1}(\beta_d - \mathbf{a}_{A_2})^\top,$$

where  $\mathbf{a}_{A_2}$  is obtained from the characteristic polynomial of  $A_2$ , and  $\beta_d$  is obtained from the desired characteristic polynomial for  $A_2 - L_2 C_2$ . Alternatively,  $L_2$  can be found using the Python place command:

```
1 import control as cnt
2 L2 = cnt.place(A2.T, C2.T, desired_observer_poles).T
```

### 14.1.1 Simple Example

Consider the example in Sections 11.2.1, 12.2.1, and 13.2.1. A block diagram for the system is shown in [Fig. 14-2](#), where a constant input disturbance has been added to the system.

Suppose that a standard feedback controller without an integrator and without a disturbance observer is designed for this system. An example of potential Python code is

```
1 import numpy as np
2 import control as cnt
3
4 P.A = np.array([[0., 1.0], [2.0, 3.0]])
5 P.B = np.array([[0.], [4.0]])
6 P.C = np.array([[5.0, 0.]])
7 P.D = 0
8
9 P.Ts = 0.01 #sampling time
10
11 # pick poles for the controller
12 wn_ctrl = 1
13 zeta_ctrl = 0.707
14 charpoly = [1., 2.0*zeta_ctrl*wn_ctrl, wn_ctrl**2]
15 des_ctrl_poles = np.roots(charpoly)
16
17 # pick poles for the observer
```

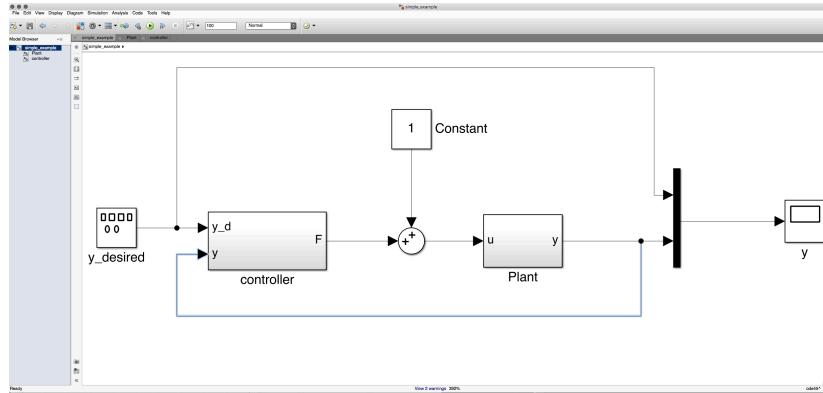


Figure 14-2: Block diagram for a simple example, where a constant input disturbance has been added to the system.

```

18 wn_obsv = 10.0
19 zeta_obsv = 0.707
20 charpoly = [1., 2.0*zeta_obsv*wn_obsv, wn_obsv**2]
21 des_obsv_poles = np.roots(charpoly)
22
23 # is the system controllable?
24 if np.linalg.matrix_rank(cnt.ctrb(P.A, P.B)) != np.size(P.A, 1):
25     print('System Not Controllable')
26 else:
27     P.K = cnt.place(P.A, P.B, des_ctrl_poles)
28     P.kr = -1.0/(P.C*np.linalg.inv(P.A-P.B*P.K)*P.B)
29
30 # is the system observable?
31 if np.linalg.matrix_rank(cnt.obsv(P.A, P.C)) != np.size(P.A, 1):
32     print('System Not Observable')
33 else:
34     P.L = cnt.place(P.A.T, P.C.T, des_obsv_poles).T

```

The associated control code is

```

1 # variables 'xhat' and 'u' must be declared as global, or
2 # class member variables for the sake of persistence, and the
3 # measurement 'y' and reference input 'r' must be passed in.
4 N = 10
5
6 # solve the observer differential equations using N steps of

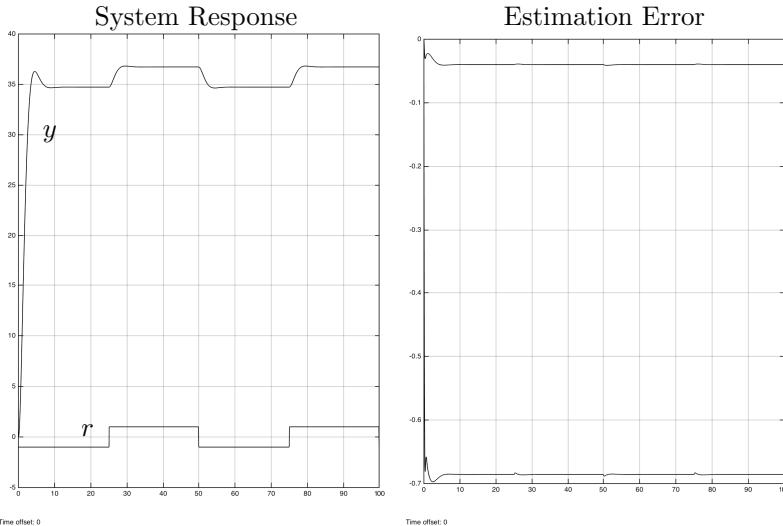
```

```

7 # RK1 for i = 1 to N
8 for i in range(N):
9     self.xhat = self.xhat + P.Ts/N*(P.A*self.xhat + \
10                                P.B*self.u + \
11                                P.L*(y-P.C*self.xhat))
12
13 # observer-based update with feedback controller
14 self.u = P.kr*r - P.K*self.xhat

```

The step response for this controller is shown in **Fig. 14-3**, where it is obvious that the input disturbance is causing a large steady state error in the system response, as well as in the estimation error. If an integrator is added to the system using the



**Figure 14-3:** Step response for standard observer based control without integrator and without disturbance observer.

following design file

```

1 # design file with integrator
2 integrator_pole = -1.0
3
4 # augment system to add integrator
5 P.A1 = np.block([[P.A, np.zeros((2,1))], [-P.C, 0]])
6 P.B1 = np.block([[P.B], [0]])
7
8 # is this system controllable?
9 if np.linalg.matrix_rank(cnt.ctrb(P.A1, P.B1)) != np.size(P.A1, 1):
10     print('System Not Controllable')
11 else:
12     K1 = cnt.place(P.A1, P.B1, np.concatenate((des_ctrl_poles,
13                                                 [integrator_pole])))
14     P.K = K1[0,0:-1]
15     P.ki = K1[0, -1]

```

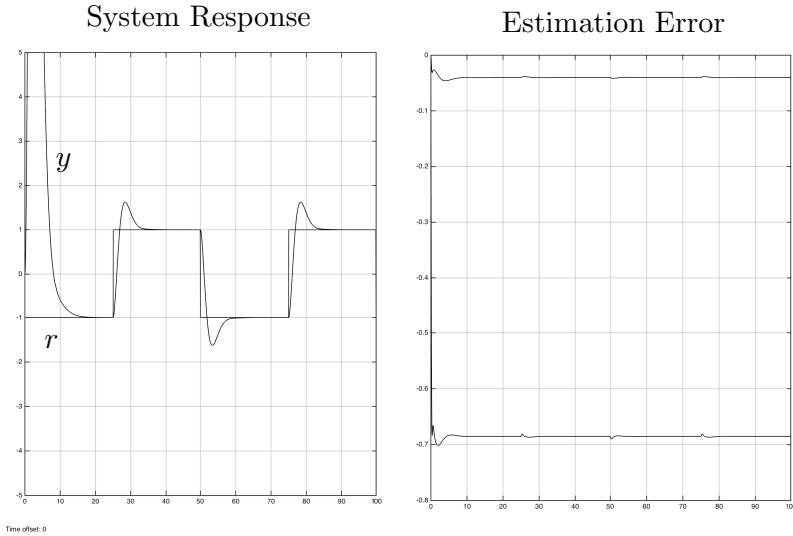
and the control code listed below:

```

1 # variables 'xhat,' 'error_d1,' 'integrator,' and 'u' must be
2 # declared as global, or class member variables for the sake
3 # of persistence, and the measurement 'y' and reference input
4 # 'r' must be passed in.
5 N = 10
6
7 # solve the observer differential equations using N steps of
8 # RK1 for i = 1 to N
9 for i in range(N):
10     self.xhat = self.xhat + P.Ts/N*(P.A*self.xhat + \
11                                     P.B*self.u + \
12                                     P.L*(y-P.C*self.xhat))
13
14 # implement the integrator
15 error = r-y
16 self.integrator = self.integrator + \
17     (P.Ts/2.0)*(error+self.error_d1)
18 self.error_d1 = error
19
20 # observer-based feedback controller
21 self.u = -P.K*self.xhat - P.ki*self.integrator

```

The response of the system with the integrator is shown in [Fig. 14-4](#), where it is clear that the steady state error in the response has been removed, but where there is now a steady state response in the estimation error.



[Figure 14-4](#): Step response for observer based control with integrator but without disturbance observer.

Now consider the case where the integrator is not present, but we add a distur-

bance observer. The design code is listed below.

```

1 # is the system controllable?
2 if np.linalg.matrix_rank(cnt.ctrb(P.A, P.B)) != np.size(P.A, 1):
3     print('System Not Controllable')
4 else:
5     P.K = cnt.place(P.A, P.B, des_ctrl_poles)
6     P.kr = -1.0/(P.C*np.linalg.inv(P.A-P.B*P.K)*P.B)
7
8 # augment system for disturbance observer
9 disturb_obsrv_pole = -2.0
10 P.A2 = np.block([[P.A, P.B], [np.zeros((1,2)), 0]])
11 P.B2 = np.block([[P.B], [0]])
12 P.C2 = np.block([P.C, 0])
13
14 # is the system observable?
15 if np.linalg.matrix_rank(cnt.obsrv(P.A2, P.C2)) != np.size(P.A2, 1):
16     print('System Not Observable')
17 else:
18     P.L2 = cnt.place(P.A2.T, P.C2.T,
19                     np.concatenate((des_obsrv_poles,
20                                     [disturb_obsrv_pole])).T

```

The associated control code is listed below.

```

1 # variables 'x_obsrv' and 'u' must be declared as global, or
2 # class member variables for the sake of persistence, and the
3 # measurement 'y' and reference input 'r' must be passed in.
4 N = 10
5
6 # solve the observer differential equations using N steps of
7 # RK1 for i = 1 to N
8 for i in range(N):
9     self.x_obsrv = self.x_obsrv + P.Ts/N*(P.A2*self.x_obsrv + \
10                                         P.B2*self.u + \
11                                         P.L2*(y-P.C2*self.x_obsrv))
12 xhat = self.x_obsrv[0, 0:-1]
13 dhat = self.x_obsrv[0, -1]
14
15 # observer-based feedback controller with disturbance obsverver
16 self.u = P.kr*r - P.K*self.xhat - dhat

```

Note the presence of  $\hat{d}$  on Line 13. The disturbance estimate is subtracted from the control input on Line 16. The resulting system response is shown in [Fig. 14-5](#). Note that in this case, the steady state error in both the system response and the estimation error has been removed. At this point, the integrator can be added back into the system to remove the effect of model mismatch and other disturbances on the system.

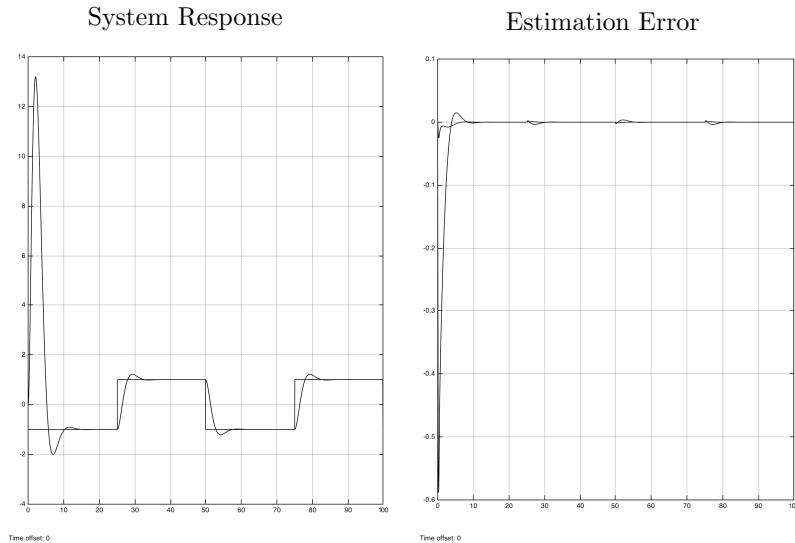


Figure 14-5: Step response for observer based control with a disturbance observer rather than an integrator.

## 14.2 Design Study A. Single Link Robot Arm



### Example Problem A.14

- (a) Modify the simulation from HW A.13 so that the uncertainty parameter in `arm_dynamics.m` is  $\alpha = 0.2$ , representing 20% inaccuracy in the knowledge of the system parameters and so that the input disturbance is 0.5 Newton-meters. Also, add noise to the output channel  $\theta_m$  with a standard deviation of 0.001. Before adding the disturbance observer, run the simulation and note that the controller is not robust enough to handle the large input disturbance.
- (b) Add a disturbance observer to the controller, and verify that the steady state error in the estimator has been removed. Tune the system to get good response.

### Solution

The single link robot arm is simulated using the following Python code.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np

```

```

5 import armParam as P
6 from hw3.armDynamics import armDynamics
7 from armController import armController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.armAnimation import armAnimation
10 from hw2.dataPlotter import dataPlotter
11 from hw13.dataPlotterObserver import dataPlotterObserver
12
13 # instantiate arm, controller, and reference classes
14 arm = armDynamics(alpha=0.2)
15 controller = armController()
16 reference = signalGenerator(amplitude=30*np.pi/180.0,
17                               frequency=0.05)
18 disturbance = signalGenerator(amplitude=0.25)
19 noise = signalGenerator(amplitude=0.01)
20
21 # instantiate the simulation plots and animation
22 dataPlot = dataPlotter()
23 dataPlotObserver = dataPlotterObserver()
24 animation = armAnimation()
25
26 t = P.t_start # time starts at t_start
27 y = arm.h() # output of system at start of simulation
28
29 while t < P.t_end: # main simulation loop
30     # Get referenced inputs from signal generators
31     # Propagate dynamics in between plot samples
32     t_next_plot = t + P.t_plot
33
34     # updates control and dynamics at faster simulation rate
35     while t < t_next_plot:
36         r = reference.square(t)
37         d = disturbance.step(t) # input disturbance
38         n = noise.random(t) # simulate sensor noise
39         # update controller
40         u, xhat, dhat = controller.update(r, y + n)
41         y = arm.update(u + d) # propagate system
42         t = t + P.Ts # advance time by Ts
43
44     # update animation and data plots
45     animation.update(arm.state)
46     dataPlot.update(t, r, arm.state, u)
47     dataPlotObserver.update(t, arm.state, xhat, d, dhat)
48     plt.pause(0.0001)
49
50 # Keeps the program from closing until the user presses a button.
51 print('Press key to close')
52 plt.waitforbuttonpress()
53 plt.close()

```

Listing 14.1: armSim.py

Note that in this simulation, the uncertainty parameter is  $\alpha = 0.2$  implying 20% variation in the plant parameters, and that the noise and disturbance are not zero.

A Python class that implements observer-based control with an integrator and disturbance observer for the single link robot arm is shown below.

```

1 import numpy as np
2 import armParamHW14 as P
3
4 class armController:
5     def __init__(self):
6         self.observer_state = np.array([
7             [0.0], # estimate of theta
8             [0.0], # estimate of theta_hat
9             [0.0], # estimate of disturbance
10            ])
11        self.tau_d1 = 0.0      # control torque, delayed 1 sample
12        self.integrator = 0.0 # integrator
13        self.error_d1 = 0.0   # error signal delayed by 1 sample
14        self.K = P.K          # state feedback gain
15        self.ki = P.ki         # Input gain
16        self.L = P.L          # observer gain
17        self.Ld = P.Ld
18        self.L2 = P.L2
19        self.A2 = P.A2          # system model
20        self.B2 = P.B2
21        self.C2 = P.C2
22        self.limit = P.tau_max # Maximum torque
23        self.Ts = P.Ts          # sample rate of controller
24
25    def update(self, theta_r, y_m):
26        # update the observer and extract theta_hat
27        x_hat, d_hat = self.update_observer(y_m)
28        theta_hat = x_hat.item(0)
29
30        # integrate error
31        error = theta_r - theta_hat
32        self.integrateError(error)
33
34        # compute feedback linearizing torque tau_fl
35        tau_fl = P.m * P.g * (P.ell / 2.0) * np.cos(theta_hat)
36
37        # Compute the state feedback controller
38        tau_tilde = -self.K @ x_hat \
39                    - self.ki * self.integrator - d_hat
40
41        # compute total torque
42        tau = self.saturate(tau_fl + tau_tilde.item(0))
43        self.tau_d1 = tau
44
45        return tau, x_hat, d_hat
46
47    def update_observer(self, y_m):
48        # update the observer using RK4 integration
49        F1 = self.observer_f(self.observer_state, y_m)
50        F2 = self.observer_f(self.observer_state \
51                            + self.Ts / 2 * F1, y_m)
52        F3 = self.observer_f(self.observer_state \
53                            + self.Ts / 2 * F2, y_m)
54        F4 = self.observer_f(self.observer_state \
55                            + self.Ts * F3, y_m)
56        self.observer_state += self.Ts / 6 * \

```

```

57                                     (F1 + 2 * F2 + 2 * F3 + F4)
58     x_hat = np.array([[self.observer_state.item(0)],
59                         [self.observer_state.item(1)]])
60     d_hat = self.observer_state.item(2)
61
62     return x_hat, d_hat
63
64 def observer_f(self, x_hat, y_m):
65     # compute feedback linearizing torque tau_f1
66     theta_hat = x_hat.item(0)
67     tau_f1 = P.m * P.g * (P.ell / 2.0) * np.cos(theta_hat)
68     # xhatdot = A*xhat + B*(u-ue) + L(y-C*xhat)
69     xhat_dot = self.A2 @ x_hat \
70                 + self.B2 * (self.tau_d1 - tau_f1) \
71                 + self.L2 * (y_m - self.C2 @ x_hat)
72
73     return xhat_dot
74
75 def integrateError(self, error):
76     self.integrator = self.integrator \
77                     + (self.Ts/2.0)*(error + self.error_d1)
78     self.error_d1 = error
79
80 def saturate(self, u):
81     if abs(u) > self.limit:
82         u = self.limit*np.sign(u)
83     return u

```

Listing 14.2: armController.py

Python code that computes the control gains is given below.

```

1 # Single link arm Parameter File
2 import numpy as np
3 import control as cnt
4 import sys
5 sys.path.append('..') # add parent directory
6 import armParam as P
7
8 # tuning parameters
9 tr = 0.4
10 zeta = 0.707
11 integrator_pole = -5
12 wn_obs = 10           # natural frequency for observer
13 zeta_obs = 0.707      # damping ratio for observer
14 dist_observ_pole = -5.5 # pole for disturbance observer
15
16
17 Ts = P.Ts # sample rate of the controller
18 tau_max = P.tau_max # limit on control signal
19 m = P.m
20 ell = P.ell
21 g = P.g
22
23
24 # State Space Equations
25 A = np.array([[0.0, 1.0],

```

```

26             [0.0, -1.0*P.b/P.m/(P.ell**2)]])
27
28 B = np.array([[0.0,
29                 [3.0/P.m/(P.ell**2)]])
30
31 C = np.array([[1.0, 0.0]])
32
33 # control design
34 # form augmented system
35 A1 = np.array([[0.0, 1.0, 0.0],
36                 [0.0, -1.0*P.b/P.m/(P.ell**2), 0.0],
37                 [-1.0, 0.0, 0.0]])
38
39 B1 = np.array([[0.0],
40                 [3.0/P.m/(P.ell**2)],
41                 [0.0]])
42
43 # gain calculation
44 wn = 2.2/tr # natural frequency
45
46 #des_char_poly = np.convolve(
47 #    [1, 2*zeta*wn, wn**2],
48 #    np.poly(integrator_pole))
49 #des_poles = np.roots(des_char_poly)
50 des_char = np.array([1., 2.*zeta*wn_obs, wn_obs**2.])
51 des_char_augmented = np.convolve(des_char,
52                                 np.poly([integrator_pole]))
53 des_poles = np.roots(des_char_augmented)
54
55 # Compute the gains if the system is controllable
56 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 3:
57     print("The system is not controllable")
58 else:
59     K1 = cnt.acker(A1, B1, des_poles)
60     K = np.array([K1.item(0), K1.item(1)])
61     ki = K1.item(2)
62
63 # observer design
64 # Augmented Matrices
65 A2 = np.concatenate((
66     np.concatenate((A, B), axis=1),
67     np.zeros((1, 3))),
68     axis=0)
69 B2 = np.concatenate((B, np.zeros((1, 1))), axis=0)
70 C2 = np.concatenate((C, np.zeros((1, 1))), axis=1)
71
72 des_char_est = np.array([1., 2.*zeta*wn_obs, wn_obs**2.])
73 des_char_est_aug = np.convolve(des_char_est,
74                                 np.poly([dist_observ_pole]))
75 des_observ_poles = np.roots(des_char_est_aug)
76
77 # Compute the gains if the system is controllable
78 if np.linalg.matrix_rank(cnt.ctrb(A2.T, C2.T)) != 3:
79     print("The system is not observable")
80 else:
81     L2 = cnt.acker(A2.T, C2.T, des_observ_poles).T
82     L = np.array([[L2.item(0)], [L2.item(1)]])

```

```

83     Ld = L2.item(0)
84
85 print('K: ', K)
86 print('ki: ', ki)
87 print('L^T: ', L.T)
88 print('Ld: ', Ld)

```

Listing 14.3: armParamHW14.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

### 14.3 Design Study B. Inverted Pendulum



#### Example Problem B.14

- (a) Modify the simulation from HW B.13 so that the uncertainty parameter in `pendulum_dynamics.m` is  $\alpha = 0.2$ , representing 20% inaccuracy in the knowledge of the system parameters, and so that the input disturbance is 0.5. Also, add noise to the output channels  $z_m$  and  $\theta_m$  with a standard deviation of 0.001. Before adding the disturbance observer, run the simulation and note that the controller is not robust to the large input disturbance.
- (b) Add a disturbance observer to the controller, and verify that the steady state error in the estimator has been removed. Tune the system to get good response.

#### Solution

The inverted pendulum is simulated using the following Python code.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import pendulumParam as P
6 from hw3.pendulumDynamics import pendulumDynamics
7 from pendulumController import pendulumController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.pendulumAnimation import pendulumAnimation
10 from hw2.dataPlotter import dataPlotter
11 from hw13.dataPlotterObserver import dataPlotterObserver
12
13 # instantiate pendulum, controller, and reference classes
14 pendulum = pendulumDynamics(alpha = 0.2)
15 controller = pendulumController()
16 reference = signalGenerator(amplitude=0.5, frequency=0.05)
17 disturbance = signalGenerator(amplitude=0.5, frequency=.1)
18 noise_z = signalGenerator(amplitude=0.01)
19 noise_th = signalGenerator(amplitude=0.01)

```

```

20
21 # instantiate the simulation plots and animation
22 dataPlot = dataPlotter()
23 dataPlotObserver = dataPlotterObserver()
24 animation = pendulumAnimation()
25
26 t = P.t_start # time starts at t_start
27 y = pendulum.h() # output of system at start of simulation
28
29 while t < P.t_end: # main simulation loop
30     # Get referenced inputs from signal generators
31     # Propagate dynamics in between plot samples
32     t_next_plot = t + P.t_plot
33
34     while t < t_next_plot:
35         r = reference.square(t)
36         d = disturbance.sin(t) # input disturbance
37         #n = np.array([[0.], [0.]])
38         n = np.array([[noise_z.random(t)], [noise_th.random(t)]])
39         u, xhat, dhat = controller.update(r, y + n)
40         y = pendulum.update(u + d) # propagate system
41         t = t + P.Ts # advance time by Ts
42
43     # update animation and data plots
44     animation.update(pendulum.state)
45     dataPlot.update(t, r, pendulum.state, u)
46     dataPlotObserver.update(t, pendulum.state, xhat, d, dhat)
47     plt.pause(0.0001)
48
49 # Keeps the program from closing until the user presses a button.
50 print('Press key to close')
51 plt.waitforbuttonpress()
52 plt.close()

```

Listing 14.4: pendulumSim.py

Note that in this simulation, the uncertainty parameter is  $\alpha = 0.2$  implying 20% variation in the plant parameters, and that the noise and disturbance are not zero.

A Python class that implements observer-based control with an integrator and disturbance observer for the inverted pendulum is shown below.

```

1 import numpy as np
2 import pendulumParamHW14 as P
3
4 class pendulumController:
5     def __init__(self):
6         self.obsv_state = np.array([
7             [0.0], # initial estimate for z_hat
8             [0.0], # initial estimate for theta_hat
9             [0.0], # initial estimate for z_hat_dot
10            [0.0], # initial estimate for theta_hat_dot
11            [0.0], # estimate of the disturbance
12        ])
13        self.F_d1 = 0.0 # Computed Force, delayed by 1 sample
14        self.integrator = 0.0 # integrator
15        self.error_d1 = 0.0 # error signal delayed by 1 sample

```

```

16     self.K = P.K # state feedback gain
17     self.ki = P.ki # integrator gain
18     self.L = P.L2 # observer gain
19     self.A = P.A2 # system model
20     self.B = P.B1
21     self.C = P.C2
22     self.limit = P.F_max # Maximum force
23     self.Ts = P.Ts # sample rate of controller
24
25     def update(self, z_r, y):
26         # update the observer and extract z_hat
27         x_hat, d_hat = self.update_observer(y)
28         z_hat = x_hat.item(0)
29
30         # integrate error
31         error = z_r - z_hat
32         self.integrateError(error)
33
34         # Compute the state feedback controller
35         F_unsat = -self.K @ x_hat \
36                     - self.ki * self.integrator \
37                     - d_hat
38         F = self.saturate(F_unsat.item(0))
39         self.F_d1 = F
40
41         return F, x_hat, d_hat
42
43     def update_observer(self, y):
44         # update the observer using RK4 integration
45         F1 = self.obsv_f(self.obsv_state, y)
46         F2 = self.obsv_f(self.obsv_state + self.Ts / 2 * F1, y)
47         F3 = self.obsv_f(self.obsv_state + self.Ts / 2 * F2, y)
48         F4 = self.obsv_f(self.obsv_state + self.Ts * F3, y)
49         self.obsv_state += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
50         x_hat = np.array([[self.obsv_state.item(0)],
51                           [self.obsv_state.item(1)],
52                           [self.obsv_state.item(2)],
53                           [self.obsv_state.item(3)]])
54
55         d_hat = self.obsv_state.item(4)
56
57         return x_hat, d_hat
58
59     def obsv_f(self, x_hat, y_m):
60         # xhatdot = A*xhat + B*(u-ue) + L(y-C*xhat)
61         xhat_dot = self.A @ x_hat \
62                     + self.B * self.F_d1 \
63                     + self.L @ (y_m - self.C @ x_hat)
64
65         return xhat_dot
66
67     def integrateError(self, error):
68         self.integrator = self.integrator \
69                         + (self.Ts/2.0)*(error + self.error_d1)
70         self.error_d1 = error
71
72     def saturate(self,u):
73         if abs(u) > self.limit:
74             u = self.limit*np.sign(u)

```

```
73     return u
```

Listing 14.5: pendulumController.py

Python code that computes the observer and control gains is given below.

```

1 # Inverted Pendulum Parameter File
2 import sys
3 sys.path.append('..')  # add parent directory
4 import pendulumParam as P
5 import numpy as np
6 from scipy import signal
7 import control as cnt
8
9 # sample rate of the controller
10 Ts = P.Ts
11
12 # saturation limits
13 F_max = P.F_max # Max Force, N
14
15 # tuning parameters
16 tr_z = 1.5 # rise time for position
17 tr_theta = 0.5 # rise time for angle
18 zeta_z = 0.707 # damping ratio position
19 zeta_th = 0.707 # damping ratio angle
20 integrator_pole = -10.0 # integrator pole
21 tr_z_obs = tr_z/10.0 # rise time for observer - position
22 tr_theta_obs = tr_theta/10.0 # rise time for observer - angle
23 dist_obsrv_pole = -1.0 # pole for disturbance observer
24
25 # State Space Equations
26 # xdot = A*x + B*u
27 # y = C*x
28 A = np.array([[0.0, 0.0, 1.0, 0.0],
29                 [0.0, 0.0, 0.0, 1.0],
30                 [0.0, -3*P.m1*P.g/4/(.25*P.m1+P.m2),
31                  -P.b/(.25*P.m1+P.m2), 0.0],
32                 [0.0, 3*(P.m1+P.m2)*P.g/2/(.25*P.m1+P.m2)/P.ell,
33                  3*P.b/2/(.25*P.m1+P.m2)/P.ell, 0.0]])
34 B = np.array([[0.0],
35                 [0.0],
36                 [1/(.25*P.m1+P.m2)],
37                 [-3.0/2/(.25*P.m1+P.m2)/P.ell]])
38 C = np.array([[1.0, 0.0, 0.0, 0.0],
39                 [0.0, 1.0, 0.0, 0.0]])
40
41 # form augmented system
42 Cr = np.array([[1.0, 0.0, 0.0, 0.0]])
43
44 # Augmented Matrices
45 A1 = np.concatenate((
46     np.concatenate((A, np.zeros((4, 1))), axis=1),
47     np.concatenate((-Cr, np.array([[0.0]])), axis=1)),
48     axis=0)
49 B1 = np.concatenate((B, np.array([[0.0]])), axis=0)
50
51 # computer control gains
```

```

52 wn_th = 2.2/tr_theta # natural frequency for angle
53 wn_z = 2.2/tr_z # natural frequency for position
54 des_char_poly = np.convolve(
55     np.convolve([1, 2*zeta_z*wn_z, wn_z**2],
56                 [1, 2*zeta_th*wn_th, wn_th**2]),
57                 np.poly([integrator_pole]))
58 des_poles = np.roots(des_char_poly)
59
60 # Compute the gains if the system is controllable
61 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 5:
62     print("The system is not controllable")
63 else:
64     K1 = cnt.acker(A1, B1, des_poles)
65     K = K1[0, 0:4]
66     ki = K1[0, 4]
67
68 # compute observer gains
69 # Augmented Matrices
70 A2 = np.concatenate((
71     np.concatenate((A, B), axis=1),
72     np.zeros((1, 5))),
73     axis=0)
74 C2 = np.concatenate((C, np.zeros((2, 1))), axis=1)
75
76 wn_z_obs = 2.2/tr_z_obs
77 wn_th_obs = 2.2/tr_theta_obs
78 des_obs_char_poly = np.convolve(
79     np.convolve([1, 2*zeta_z*wn_z_obs, wn_z_obs**2],
80                 [1, 2*zeta_th*wn_th_obs, wn_th_obs**2]),
81                 np.poly([dist_observ_pole]))
82 des_obs_poles = np.roots(des_obs_char_poly)
83
84 # Compute the gains if the system is observable
85 if np.linalg.matrix_rank(cnt.ctrb(A2.T, C2.T)) != 5:
86     print("The system is not observable")
87 else:
88     L2 = signal.place_poles(A2.T, C2.T, des_obs_poles).gain_matrix.T
89
90 print('K: ', K)
91 print('ki: ', ki)
92 print('L^T: ', L2.T)

```

Listing 14.6: pendulumParamHW14.py

Complete simulation code for Matlab, Python, and Simulink can be downloaded at <http://controlbook.byu.edu>.

## 14.4 Design Study C. Satellite Attitude Control



### Example Problem C.14

- (a) Modify the simulation from HW C.13 so that the uncertainty parameter in `satellite_dynamics.m` is  $\alpha = 0.2$ , representing 20% inaccuracy in the knowledge of the system parameters and so that the input disturbance is 1.0 Newton-meters. Also, add noise to the output channels  $\phi_m$  and  $\theta_m$  with a standard deviation of 0.001. Before adding the disturbance observer, run the simulation and note that the controller is not robust to the large input disturbance.
- (b) Add a disturbance observer to the controller, and verify that the steady state error in the estimator has been removed. Tune the system to get good response.

### Solution

The satellite is simulated using the following Python code.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import matplotlib.pyplot as plt
4 import numpy as np
5 import satelliteParam as P
6 from hw3.satelliteDynamics import satelliteDynamics
7 from satelliteController import satelliteController
8 from hw2.signalGenerator import signalGenerator
9 from hw2.satelliteAnimation import satelliteAnimation
10 from hw2.dataPlotter import dataPlotter
11 from hw13.dataPlotterObserver import dataPlotterObserver
12
13 # instantiate satellite, controller, and reference classes
14 satellite = satelliteDynamics(alpha=0.2)
15 controller = satelliteController()
16 reference = signalGenerator(amplitude=15.0*np.pi/180.0,
17                               frequency=0.03)
18 disturbance = signalGenerator(amplitude=1.0)
19 noise_phi = signalGenerator(amplitude=0.01)
20 noise_th = signalGenerator(amplitude=0.01)
21
22 # instantiate the simulation plots and animation
23 dataPlotObserver = dataPlotterObserver()
24 dataPlot = dataPlotter()
25 animation = satelliteAnimation()
26
27 t = P.t_start # time starts at t_start
28 y = satellite.h() # output of system at start of simulation
29
30 while t < P.t_end: # main simulation loop
31     # Propagate dynamics in between plot samples

```

```

32 t_next_plot = t + P.t_plot
33
34 # updates control and dynamics at faster simulation rate
35 while t < t_next_plot:
36     r = reference.square(t)    # reference input
37     d = disturbance.step(t)   # input disturbance
38
39     # simulate sensor noise
40     n = np.array([[noise_phi.random(t)],
41                   [noise_th.random(t)]])
42
43     # update controller
44     u, xhat, dhat = controller.update(r, y + n)
45     y = satellite.update(u + d)  # propagate system
46     t = t + P.Ts   # advance time by Ts
47
48     # update animation and data plots
49     animation.update(satellite.state)
50     dataPlot.update(t, r, satellite.state, u)
51     dataPlotObserver.update(t, satellite.state, xhat, d, dhat)
52     plt.pause(0.0001)
53
54 # Keeps the program from closing until user presses a button.
55 print('Press key to close')
56 plt.waitforbuttonpress()
57 plt.close()

```

Listing 14.7: satelliteSim.py

Note that in this simulation, the uncertainty parameter is  $\alpha = 0.2$  implying 20% variation in the plant parameters, and that the noise and disturbance are not zero.

A Python class that implements observer-based control with an integrator and disturbance observer for the satellite is shown below.

```

1 import numpy as np
2 import satelliteParamHW14 as P
3
4 class satelliteController:
5     def __init__(self):
6         self.observer_state = np.array([
7             [0.0],    # initial estimate for theta_hat
8             [0.0],    # initial estimate for phi_hat
9             [0.0],    # initial estimate for theta_hat_dot
10            [0.0],    # initial estimate for phi_hat_dot
11            [0.0],    # estimate of the disturbance
12        ])
13         self.tau_d1 = 0.0      # Computed torque delayed 1 sample
14         self.integrator = 0.0  # integrator
15         self.error_d1 = 0.0   # error signal delayed by 1 sample
16         self.K = P.K          # state feedback gain
17         self.ki = P.ki         # integrator gain
18         self.L = P.L2          # observer gain
19         self.Ld = P.Ld         # gain for disturbance observer
20         self.A = P.A2          # system model
21         self.B = P.B1
22         self.C = P.C2

```

```

23     self.limit = P.tau_max # Maximum torque
24     self.Ts = P.Ts          # sample rate of controller
25
26     def update(self, phi_r, y):
27         # update the observer and extract z_hat
28         x_hat, d_hat = self.update_observer(y)
29         phi_hat = x_hat.item(1)
30
31         # integrate error
32         error = phi_r - phi_hat
33         self.integrateError(error)
34
35         # Compute the state feedback controller
36         tau_unsat = -self.K @ x_hat \
37             - self.ki*self.integrator \
38             - d_hat
39         tau = self.saturate(tau_unsat.item(0))
40         self.tau_d1 = tau
41
42         return tau, x_hat, d_hat
43
44     def update_observer(self, y_m):
45         # update the observer using RK4 integration
46         F1 = self.observer_f(self.observer_state, y_m)
47         F2 = self.observer_f(self.observer_state \
48             + self.Ts / 2 * F1, y_m)
49         F3 = self.observer_f(self.observer_state \
50             + self.Ts / 2 * F2, y_m)
51         F4 = self.observer_f(self.observer_state \
52             + self.Ts * F3, y_m)
53         self.observer_state += self.Ts / 6 \
54             * (F1 + 2 * F2 + 2 * F3 + F4)
55         x_hat = np.array([[self.observer_state.item(0)],
56                         [self.observer_state.item(1)],
57                         [self.observer_state.item(2)],
58                         [self.observer_state.item(3)]])
59         d_hat = self.observer_state.item(4)
60
61     def observer_f(self, x_hat, y_m):
62         # xhatdot = A*xhat + B*(u-ue) + L(y-C*xhat)
63         xhat_dot = self.A @ x_hat \
64             + self.B * self.tau_d1 \
65             + self.L @ (y_m - self.C @ x_hat)
66
67     def integrateError(self, error):
68         self.integrator = self.integrator \
69             + (self.Ts / 2.0) * (error + self.error_d1)
70         self.error_d1 = error
71
72     def saturate(self,u):
73         if abs(u) > self.limit:
74             u = self.limit*np.sign(u)
75
76         return u

```

Listing 14.8: satelliteController.py

Python code that computes the control gains is given below.

```

1 # satellite Parameter File
2 import numpy as np
3 import control as cnt
4 from scipy import signal
5 import sys
6 sys.path.append('..') # add parent directory
7 import satelliteParam as P
8
9 Ts = P.Ts
10 sigma = P.sigma
11 beta = P.beta
12 tau_max = P.tau_max
13
14 # tuning parameters
15 wn_th = 0.6
16 wn_phi = 1.1    # rise time for angle
17 zeta_phi = 0.707 # damping ratio position
18 zeta_th = 0.707 # damping ratio angle
19 integrator_pole = np.array([-1.0])
20 # pick observer poles
21 wn_th_obs = 10.0*wn_th
22 wn_phi_obs = 10.0*wn_phi
23 dist_observ_pole = np.array([-1.0]) # disturbance observer pole
24
25 # State Space Equations
26 # xdot = A*x + B*u
27 # y = C*x
28 A = np.array([[0.0, 0.0, 1.0, 0.0],
29               [0.0, 0.0, 0.0, 1.0],
30               [-P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js],
31               [P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp]])
32
33 B = np.array([[0.0],
34               [0.0],
35               [1.0/P.Js],
36               [0.0]])
37
38 C = np.array([[1.0, 0.0, 0.0, 0.0],
39               [0.0, 1.0, 0.0, 0.0]])
40
41 # form augmented system
42 Cout = np.array([[0.0, 1.0, 0.0, 0.0]])
43 A1 = np.array([[0.0, 0.0, 1.0, 0.0, 0.0],
44               [0.0, 0.0, 0.0, 1.0, 0.0],
45               [-P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js, 0.0],
46               [P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp, 0.0],
47               [0.0, -1.0, 0.0, 0.0, 0.0]])
48
49 B1 = np.array([[0.0],
50               [0.0],
51               [1.0/P.Js],
52               [0.0],
53               [0.0]])
54
55

```

```

56 # gain calculation
57 des_char_poly = np.convolve(
58     np.convolve([1, 2*zeta_phi*wn_phi, wn_phi**2],
59                 [1, 2*zeta_th*wn_th, wn_th**2]),
60             np.poly(integrator_pole))
61 des_poles = np.roots(des_char_poly)
62
63 # Compute the gains if the system is controllable
64 if np.linalg.matrix_rank(cnt.ctrb(A1, B1)) != 5:
65     print("The system is not controllable")
66 else:
67     K1 = cnt.acker(A1, B1, des_poles)
68     K = np.array([K1.item(0),
69                   K1.item(1),
70                   K1.item(2),
71                   K1.item(3)])
72     ki = K1.item(4)
73
74 # compute observer gains
75 # Augmented Matrices
76 A2 = np.concatenate((
77     np.concatenate((A, B), axis=1),
78     np.zeros((1, 5))),
79     axis=0)
80 C2 = np.concatenate((C, np.zeros((2, 1))), axis=1)
81 des_obs_char_poly = np.convolve(
82     np.convolve([1, 2*zeta_phi*wn_phi_obs, wn_phi_obs**2],
83                 [1, 2*zeta_th*wn_th_obs, wn_th_obs**2]),
84             np.poly(dist_observ_pole))
85 des_obs_poles = np.roots(des_obs_char_poly)
86
87 # Compute the gains if the system is observable
88 if np.linalg.matrix_rank(cnt.ctrb(A2.T, C2.T)) != 5:
89     print("The system is not observable")
90 else:
91     # place_poles returns an object with various properties.
92     # The gains are accessed through .gain_matrix
93     # .T transposes the matrix
94     L2 = signal.place_poles(A2.T,
95                             C2.T,
96                             des_obs_poles).gain_matrix.T
97     L = L2[0:4, 0:2]
98     Ld = L2[4:5, 0:2]
99
100 print('K: ', K)
101 print('ki: ', ki)
102 print('L^T: ', L.T)
103 print('Ld: ', Ld)

```

Listing 14.9: satelliteParamHW14.py

Simulation code for Matlab, Python, and Simulink can be downloaded at  
<http://controlbook.byu.edu>.

**Important Concepts:**

- Input disturbances create a steady-state bias in the observer states.
- Augmenting the observer with a disturbance state provides a method to estimate and remove the disturbance from the control input.

**Notes and References**

Disturbance observers are one example of how observers can be used to estimate a variety of different parameters in the system. In general, observers can be used to estimate model parameters like mass and inertia, but in this case, nonlinear observers are required. The most commonly used nonlinear observer is the extended Kalman filter. Derivations of the extended Kalman filter can be found in [25, 24, 26]. In robotics, observers are used to estimate the system state, but also the biases in gyroscopes and accelerometers [27], which are a form of input disturbances.

# Part V

## Loopshaping Control Design

This part of the book introduces the loopshaping design method based on the frequency response characteristics of the system. In Part II we derived transfer function and state space models based on the linearized system. In Part III we showed that PID controllers can be used to regulate the output of second order systems. For higher order systems, PID can be used if the original system can be written as a cascade, or series, of second order systems. In Part IV we showed that controllers can be derived for higher order systems using state space models. The state space design method is a time-domain method that focuses on shaping the time-response of the system. In this part of the book we extend PID control using frequency domain ideas where we shape the frequency response of the system. When the transfer function is available, it provides a compact representation of the frequency response of the open loop system. Therefore, in this part we will focus on transfer function models of the system. The methods in this part are not constrained to second order systems and do not depend on a cascade structure. In fact, they do not even require a transfer function of the system, as long as the frequency response, or Bode plot, is known.

Chapter 15 is a review of frequency response models of linear time-invariant systems and develops intuition concerning the Bode plot of the input-output response. In Chapter 16 we show how performance specifications for reference tracking, disturbance rejection, and noise attenuation can be related to the frequency response of the open loop control-plant pair. Chapter 17 addresses stability of the closed-loop system from a frequency response point of view. Chapter 18 represents the culminating chapter in this part of the book and describes how to design the controller to achieve closed-loop stability and to satisfy design specifications on reference tracking, disturbance rejection, and noise attenuation.

# 15

## Frequency Response of LTI Systems

### Learning Objectives:

- Convert complex numbers between rectangular coordinates and polar form.
- Identify how transfer functions change the magnitude and phase of an input signal.
- Draw straight-line approximations for Bode plots.

## 15.1 Theory

### 15.1.1 Manipulating Complex Numbers

Any complex number  $z$  can be represented in rectangular coordinates as

$$z = \Re\{z\} + j\Im\{z\},$$

where  $\Re\{z\}$  is the real part of  $z$ , and  $\Im\{z\}$  is the imaginary part of  $z$ . Similarly,  $z$  can be represented in polar form as

$$z = |z| e^{j\angle z}, \quad (15.1)$$

where  $|z|$  denotes the magnitude of  $z$  and  $\angle z$  denotes the angle or phase of  $z$ . The relationship between the real and imaginary parts, and the magnitude and phase of  $z$ , are depicted in [Fig. 15-1](#). From the geometry shown in [Fig. 15-1](#) we can see that

$$\begin{aligned}|z| &= \sqrt{\Re\{z\}^2 + \Im\{z\}^2} \\ \angle z &= \tan^{-1} \frac{\Im\{z\}}{\Re\{z\}}.\end{aligned}$$

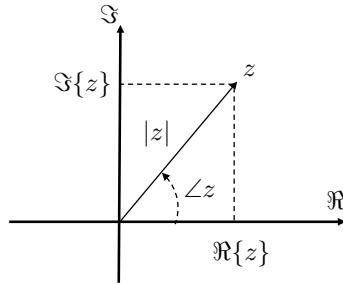


Figure 15-1: Relationship between the real and imaginary parts of  $z$ , and the magnitude and phase of  $z$ .

Similarly, from Euler's relationship

$$e^{j\theta} = \cos \theta + j \sin \theta,$$

we have from Equation (15.1) that

$$z = |z| e^{j\angle z} = |z| \cos \angle z + j |z| \sin \angle z$$

which implies that

$$\begin{aligned}\Re\{z\} &= |z| \cos \angle z \\ \Im\{z\} &= |z| \sin \angle z.\end{aligned}$$

The conjugate of  $z$ , denoted  $\bar{z}$ , is given by

$$\bar{z} = \Re\{z\} - j\Im\{z\} = |z| e^{-j\angle z}.$$

Note that  $z\bar{z} = |z|^2$ .

Suppose that  $z_1 = |z_1| e^{j\angle z_1}$  and  $z_2 = |z_2| e^{j\angle z_2}$  are two complex numbers, then

$$\begin{aligned}z_1 z_2 &= |z_1| e^{j\angle z_1} |z_2| e^{j\angle z_2} \\ &= |z_1| |z_2| e^{j\angle z_1} e^{j\angle z_2} \\ &= |z_1| |z_2| e^{j(\angle z_1 + \angle z_2)},\end{aligned}$$

therefore when multiplying complex numbers, their magnitudes multiply, but their phases add. Similarly, we have

$$\frac{z_1}{z_2} = \frac{|z_1| e^{j\angle z_1}}{|z_2| e^{j\angle z_2}} = \frac{|z_1|}{|z_2|} \frac{e^{j\angle z_1}}{e^{j\angle z_2}} = \frac{|z_1|}{|z_2|} e^{j(\angle z_1 - \angle z_2)},$$

therefore when dividing complex numbers, their magnitudes divide, but their phases subtract.

A transfer function  $H(s)$  is a complex number for any specific value of  $s$ . Therefore  $H(s)$  can be expressed in both rectangular and polar forms as

$$H(s) = \Re\{H(s)\} + j\Im\{H(s)\} = |H(s)| e^{j\angle H(s)}.$$

For example, if  $H(s) = \frac{1}{s+1}$ , then when  $s = 2 + j3$

$$\begin{aligned} H(2 + j3) &= \frac{1}{2 + j3 + 1} = \frac{1}{3 + j3} = \frac{1}{3 + j3} \frac{3 - j3}{3 - j3} \\ &= \frac{1}{6} - j \frac{1}{6} = \sqrt{\frac{1}{18}} e^{-j\frac{\pi}{4}}. \end{aligned}$$

If  $H(s)$  has multiple poles and zeros, then the magnitude and phase of  $H(s)$  can be represented in terms of the magnitude and phase of the poles and zeros. For example, suppose that

$$H(s) = \frac{K(s + z_1)(s + z_2) \dots (s + z_m)}{(s + p_1)(s + p_2) \dots (s + p_n)}.$$

Since  $(s + a)$  is a complex number for any  $s$ , it can be written in polar form as  $s + a = |s + a| e^{j\angle(s+a)}$ , therefore

$$\begin{aligned} H(s) &= \frac{K(s + z_1)(s + z_2) \dots (s + z_m)}{(s + p_1)(s + p_2) \dots (s + p_n)} \\ &= \frac{|K| e^{j\angle K} |s + z_1| e^{j\angle(s+z_1)} |s + z_2| e^{j\angle(s+z_2)} \dots |s + z_m| e^{j\angle(s+z_m)}}{|s + p_1| e^{j\angle(s+p_1)} |s + p_2| e^{j\angle(s+p_2)} \dots |s + p_n| e^{j\angle(s+p_n)}} \\ &= \frac{|K| |s + z_1| |s + z_2| \dots |s + z_m|}{|s + p_1| |s + p_2| \dots |s + p_n|} \frac{e^{j\angle K} e^{j\angle(s+z_1)} e^{j\angle(s+z_2)} \dots e^{j\angle(s+z_m)}}{e^{j\angle(s+p_1)} e^{j\angle(s+p_2)} \dots e^{j\angle(s+p_n)}} \\ &= \frac{|K| |s + z_1| |s + z_2| \dots |s + z_m|}{|s + p_1| |s + p_2| \dots |s + p_n|} e^{j(\angle K + \sum_{i=1}^m \angle(s+z_i) - \sum_{i=1}^n \angle(s+p_i))}. \end{aligned}$$

Therefore

$$|H(s)| = \frac{|K| |s + z_1| |s + z_2| \dots |s + z_m|}{|s + p_1| |s + p_2| \dots |s + p_n|} \quad (15.2)$$

$$\angle H(s) = \angle K + \sum_{i=1}^m \angle(s + z_i) - \sum_{i=1}^n \angle(s + p_i). \quad (15.3)$$

### 15.1.2 Frequency Response of LTI Systems

In systems theory, the magnitude and phase representations of the transfer function  $H(s)$  are used because they have important physical meaning. In particular, suppose that  $H(s)$  represents a physical system with input  $u(t)$  and output  $y(t)$  as shown in **Fig. 15-2**. If the input to the system is given by a sinusoid of magnitude  $A$  and frequency  $\omega_0$ , i.e.,

$$u(t) = A \sin(\omega_0 t),$$

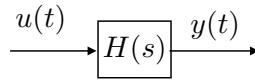


Figure 15-2: LTI system represented by the transfer function  $H(s)$  with input  $u(t)$  and output  $y(t)$ .

then the output is given by

$$y(t) = A |H(j\omega_0)| \sin(\omega_0 t + \angle H(j\omega_0)). \quad (15.4)$$

Therefore, the transfer function  $H(s)$  is an elegant and compact representation of the frequency response of the system, in that the magnitude of  $H(j\omega_0)$  describes the gain of the system for input signals with frequency  $\omega_0$ , and the phase of  $H(j\omega_0)$  describes the phase shift imposed by the system on input signals with frequency  $\omega_0$ . For example, Fig. 15-3 shows the input-output response of an LTI system with transfer function  $H(s) = \frac{1}{s+1}$ . The input signal, which is shown in grey, is  $u(t) = \sin(\omega_0 t)$  where  $\omega_0$  changes in each subplot. The output signal is shown in blue and is given by Equation (15.4). Note that in subplot (a) the time scale is different than the other subplots. When  $\omega_0 = 0.1$ ,  $H(j\omega_0) \approx 0.995e^{j5^\circ}$ . Accordingly, in subplot (a), the output is only slightly attenuated, with very little phase shift. In contrast, when  $\omega_0 = 5$ ,  $H(j\omega_0) \approx 0.2e^{j80^\circ}$ , and in subplot (f) we see that the output is approximately 20% of the input with a phase shift approaching 90 degrees.

Therefore, the response of an LTI system to inputs at different frequencies can be visualized by plotting the magnitude and phase of  $H(j\omega)$  as a function of  $\omega$ . The frequency response of  $H(s) = 1/(s + 1)$  is plotted in two different ways in Fig. 15-4. In subfigure (a) a linear scale is used for both  $|H(j\omega)|$  and  $\angle H(j\omega)$ , as well as the frequency scale  $\omega$ . In subfigure (b), the scale for magnitude is in dB:  $20 \log |H(j\omega)|$  whereas the scale for  $\angle H(j\omega)$  is linear, however, in both cases a log scale for  $\omega$  is used on the  $x$ -axis. The plot in subfigure (b) is called a Bode plot. It should be clear from Fig. 15-4 that the Bode plot provides much more useful information than a linear plot since it more effectively reveals the behavior of the system at lower frequencies. For this reason, Bode plots are the most common method to display the frequency response of the system.

As a note, since the magnitude of  $H(j\omega)$  is displayed on the Bode plot as  $20 \log |H(j\omega)|$  in units of dB, note that  $20 \log |H(j\omega)| = 0$  dB implies that  $|H(j\omega)| = 1$ . Similarly  $20 \log |H(j\omega)| = 20$  dB implies that  $|H(j\omega)| = 10$  and  $20 \log |H(j\omega)| = 40$  dB implies that  $|H(j\omega)| = 100$ . Going in the other direction,  $20 \log |H(j\omega)| = -20$  dB implies that  $|H(j\omega)| = 0.1$ , and  $20 \log |H(j\omega)| = -40$  dB implies that  $|H(j\omega)| = 0.01$ . In general,

$$20 \log |H(j\omega)| = 20n \text{ dB} \implies |H(j\omega)| = 10^n.$$

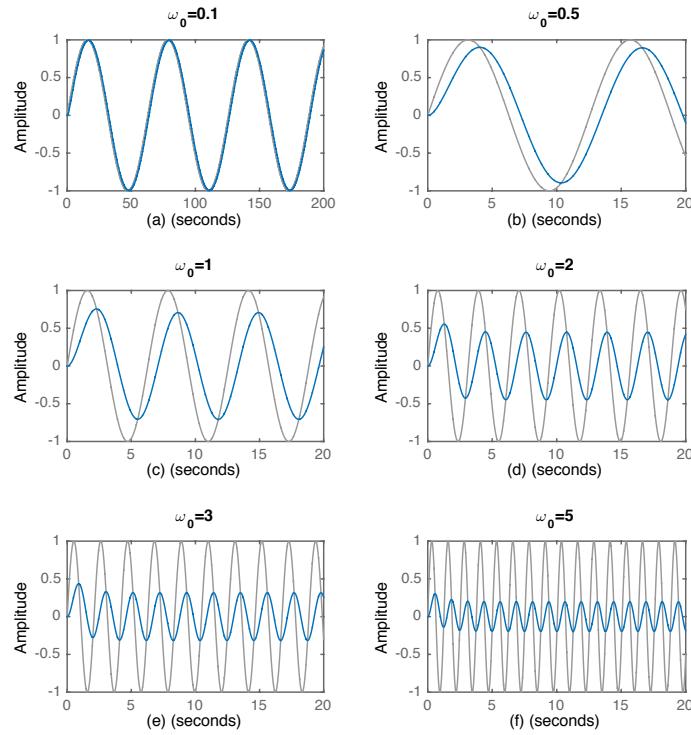


Figure 15-3: Frequency response to LTI system with transfer function  $H(s) = 1/(s + 1)$ .

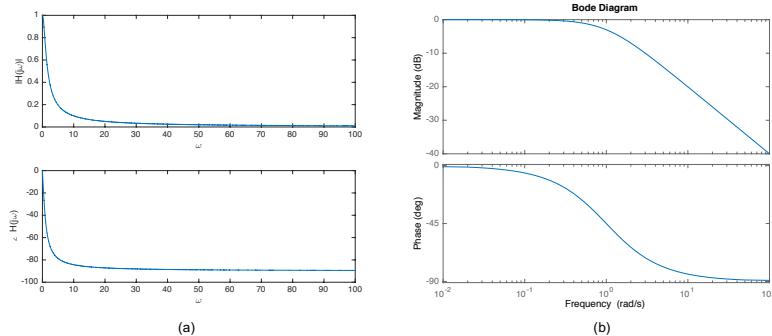


Figure 15-4: Frequency response of  $H(s) = 1/(s + 1)$  on a linear scale (a), and a logarithmic scale (b).

### 15.1.3 Straight-line Approximations for Bode Plots

While the Bode plot for a system can be easily generated using the Matlab or Python `bode` command, it is still useful to understand how to approximately draw a Bode plot by hand. The reason for this is that in the loopshaping design methodology, we add elements to the control transfer function  $C(s)$  to "shape" the Bode plot of the loop gain  $P(s)C(s)$  where  $P(s)$  is the transfer function of the plant. Learning to approximately draw the Bode plot by hand will provide helpful insights into this process. Therefore, in this section we provide a brief tutorial on how to approximate the Bode plot for a transfer function  $H(s)$ .

We start by putting the transfer function into *Bode canonical form* by factoring out the poles and zeros as follows:

$$\begin{aligned} H(s) &= \frac{K(s + z_1)(s + z_2) \dots (s + z_m)}{(s + p_1)(s + p_2) \dots (s + p_n)} \\ &= \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_n} \frac{(1 + s/z_1)(1 + s/z_2) \dots (1 + s/z_m)}{(1 + s/p_1)(1 + s/p_2) \dots (1 + s/p_n)}. \end{aligned}$$

Therefore, following Equations (15.2) and (15.3) we have

$$|H(j\omega)| = \left| \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_m} \right| \frac{|1 + j\omega/z_1| |1 + j\omega/z_2| \dots |1 + j\omega/z_m|}{|1 + j\omega/p_1| |1 + j\omega/p_2| \dots |1 + j\omega/p_n|} \quad (15.5)$$

$$\angle H(j\omega) = \angle \left( \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_m} \right) + \sum_{i=1}^m \angle(1 + j\omega/z_i) - \sum_{i=1}^n \angle(1 + j\omega/p_i). \quad (15.6)$$

Since  $20 \log |AB| = 20 \log |A| + 20 \log |B|$ , Equation (15.5) gives

$$\begin{aligned} 20 \log |H(j\omega)| &= 20 \log \left| \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_m} \right| \\ &\quad + \sum_{i=1}^m 20 \log |1 + j\omega/z_i| - \sum_{i=1}^n 20 \log |1 + j\omega/p_i|. \quad (15.7) \end{aligned}$$

Drawing the Bode plot for a general transfer function with real poles and zeros, can be decomposed into drawing the Bode plot for each pole and zero, and then graphically adding them to get the general Bode plot. Note first that

$$20 \log \left| \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_m} \right|$$

is not a function of  $\omega$  and is therefore a constant line on the Bode plot. Also note that

$$\angle \left( \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_m} \right)$$

is also a constant and is either 0 degrees if  $\left( \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_m} \right) > 0$ , or 180 degrees if  $\left( \frac{Kz_1 z_2 \dots z_m}{p_1 p_2 \dots p_m} \right) < 0$ .

We start by drawing the Bode plot when  $H(s) = s + z$  has a single zero at  $s = -z$  and  $z > 0$ . The first step is to put the transfer function in Bode canonical form as

$$H(j\omega) = z(1 + j\frac{\omega}{z}).$$

Therefore, from Equation (15.7) we have

$$\begin{aligned} 20 \log |H(j\omega)| &= 20 \log |z| + 20 \log \left| 1 + j\frac{\omega}{z} \right| \\ &= 20 \log |z| + 20 \log \sqrt{1 + (\omega/z)^2}. \end{aligned}$$

When  $\omega/z \ll 1$  we have that  $\sqrt{1 + (\omega/z)^2} \approx 1$ , implying that

$$20 \log |H(j\omega)| \approx 20 \log |z|.$$

Note that since this term is not a function of  $\omega$ , it is a constant value on the Bode plot. On the other hand, when  $\omega/z \gg 1$  we have that  $\sqrt{1 + (\omega/z)^2} \approx |\omega/z|$ , giving

$$20 \log |H(j\omega)| \approx 20 \log |z| + 20 \log |\omega/z|.$$

Note that the second term is a straight line on a Bode plot that intersects the 0 dB axis at  $\omega = z$  and increases 20 dB for every factor of 10 increase in  $\omega$ . Therefore, on the Bode plot, the slope of the second term is +20 dB/decade. A straight-line approximation is shown in Fig. 15-5, along with the Bode plot produced by Matlab. Note that the largest mismatch between the straight-line approximation

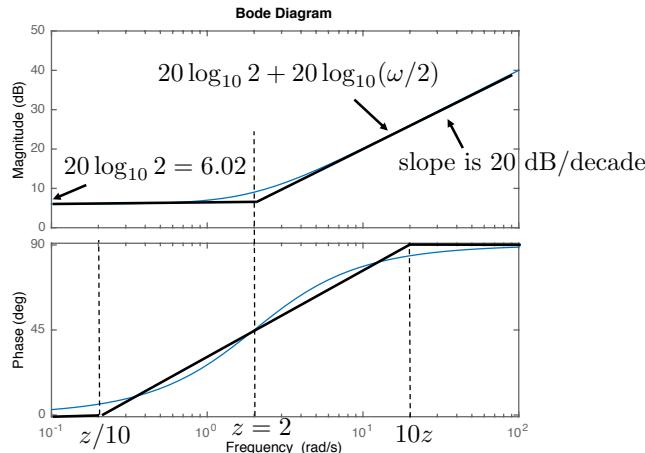


Figure 15-5: Straight-line approximation for a single zero.

and the actual Bode plot is at  $\omega = z$ . In that case we have

$$20 \log |H(jz)| \approx 20 \log |z| + 20 \log \sqrt{2} = 20 \log |z| + 3 \text{ dB}.$$

Therefore, at the corner frequency  $z$  the Bode plot is 3 dB above the low frequency response.

To approximate the phase plot, note that

$$\angle H(j\omega) = \angle z(1 + j\omega/z) = \tan^{-1} \frac{\omega}{z}. \quad (15.8)$$

When  $\omega = z$  we have that  $\tan^{-1} 1 = 45$  degrees. When  $\omega \ll z$ , we have that  $\tan^{-1}(\omega/z) \approx 0$  degrees, and when  $\omega \gg z$ , we have that  $\tan^{-1}(\omega/z) \approx 90$  degrees. The phase plot as calculated by Matlab is shown in Fig. 15-5. An adequate straight-line approximation is obtained by setting the phase to zero when  $\omega < z/10$ , i.e., one decade less than  $z$ , and setting the phase to 90 degrees when  $\omega > 10z$ , i.e., one decade greater than  $z$ , and drawing a straight line between those points so that the phase at  $z$  is 45 degrees. The straight-line approximation for the phase is also shown in black in Fig. 15-5.

The straight-line approximation for phase as described above assumes that  $z > 0$ , i.e., that the zero is in the left half of the complex plane. For right half plane zeros when  $z < 0$ , the phase as calculated in Equation (15.6) is

$$\angle H(j\omega) = \angle z + \angle(1 + j\omega/z) = 180^\circ - \tan^{-1} \frac{\omega}{|z|}.$$

Therefore, the phase starts at 180 degrees and decreases by 90 degrees as  $\omega$  gets large rather than increases. Fig. 15-6 shows the Bode plot and its straight-line approximation when the zero is in the right half plane. When a system has a zero in the right half plane, it is called non-minimum phase because of the 180 degree phase increase at low frequencies.

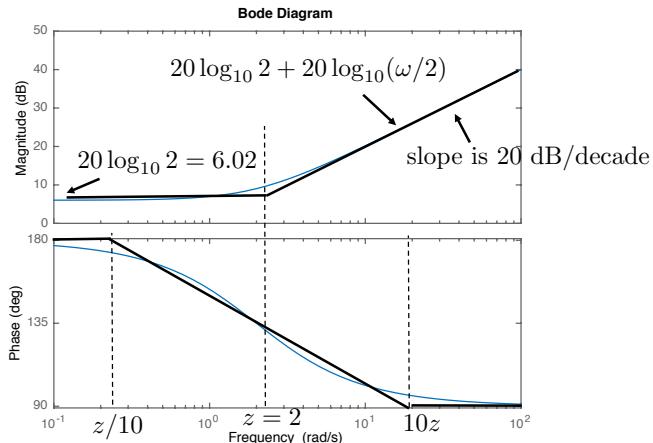


Figure 15-6: Straight-line approximation for a zero in the right half of the complex plane.

Now consider drawing the Bode plot for a single pole, i.e.,  $H(s) = \frac{p}{s+p}$ , which has a single pole at  $s = -p$  which is in the right half plane when  $p > 0$ . In Bode canonical form  $H(s)$  becomes

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{p}}.$$

Therefore, from Equation (15.7) we have

$$\begin{aligned} 20 \log |H(j\omega)| &= 20 \log |1| - 20 \log \left| 1 + j\frac{\omega}{p} \right| \\ &= -20 \log \sqrt{1 + (\omega/p)^2}. \end{aligned}$$

When  $\omega/p \ll 1$ , we have that  $\sqrt{1 + (\omega/p)^2} \approx 1$ , implying that

$$20 \log |H(j\omega)| \approx 0.$$

Note that since this term is not a function of  $\omega$ , it is a constant value at 0 dB on the Bode plot. On the other hand, when  $\omega/p \gg 1$ , we have that  $\sqrt{1 + (\omega/p)^2} \approx |\omega/p|$ , giving

$$20 \log |H(j\omega)| \approx -20 \log |\omega/p|.$$

Note that the second term is a straight line on a Bode plot that intersects the 0 dB axis when  $\omega = p$  and decreases at  $-20$  dB for every factor of 10 increase in  $\omega$ . Therefore, on the Bode plot, the slope of the second term is  $-20$  dB/decade. A straight-line approximation is shown in Fig. 15-7, along with the Bode plot produced by Matlab. Note that the largest mismatch between the straight-line

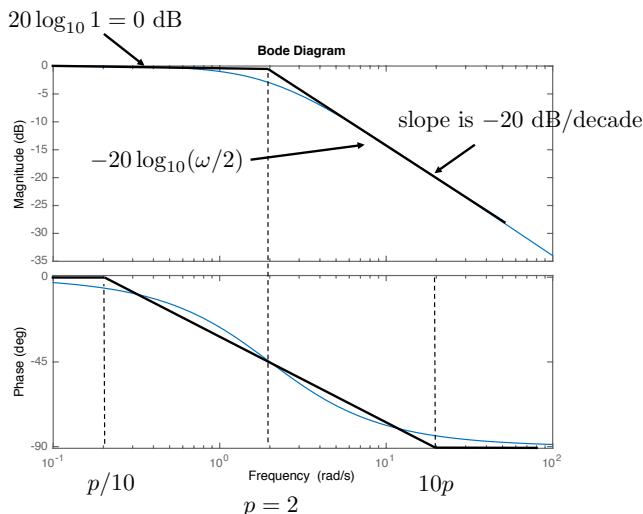


Figure 15-7: Straight-line approximation for a single pole.

approximation and the actual Bode is at  $\omega = p$ . In that case, we have

$$20 \log |H(jp)| \approx -20 \log \sqrt{2} = -3 \text{ dB}.$$

Therefore, at the corner frequency  $p$  the Bode plot is 3 dB below the low frequency response.

To approximate the phase plot, note that

$$\angle H(j\omega) = -\angle(1 + j\frac{\omega}{p}) = -\tan^{-1} \frac{\omega}{p}. \quad (15.9)$$

When  $\omega = p$  we have that  $-\tan^{-1} 1 = -45$  degrees. When  $\omega \ll p$ , we have that  $-\tan^{-1}(\omega/p) \approx 0$  degrees, and when  $\omega \gg p$ ,  $-\tan^{-1}(\omega/p) \approx -90$  degrees. The phase plot as calculated by Matlab is shown in [Fig. 15-7](#). An adequate straight-line approximation is obtained by setting the phase to zero when  $\omega < p/10$ , i.e., one decade less than  $p$ , and setting the phase to  $-90$  degrees when  $\omega > 10p$ , i.e., one decade greater than  $p$ , and drawing a straight line between those points so that the phase at  $p$  is  $-45$  degrees. The straight-line approximation for the phase is also shown in [Fig. 15-7](#).

The straight-line approximation for phase as described above assumes that  $p > 0$ , i.e., that the pole is in the left half of the complex plane. For right half plane poles when  $p < 0$ , the phase as calculated in Equation (15.6) is

$$\angle H(j\omega) = -\angle(1 + j\omega/p) = \tan^{-1} \frac{\omega}{|p|}.$$

Therefore, the phase starts at 0 degrees and increases by 90 degrees as  $\omega$  gets large. [Fig. 15-8](#) shows the Bode plot and its straight-line approximation when the pole is in the right half plane. Note that stability and instability are not directly evident from the shapes of the magnitude and phase plots.

Suppose now that the poles are complex. Let

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

In Bode canonical form we have

$$\begin{aligned} H(j\omega) &= \frac{1}{\omega_n^2} \frac{\omega_n^2}{\frac{(j\omega)^2}{\omega_n^2} + 2\zeta\omega_n \left(\frac{j\omega}{\omega_n^2}\right) + 1} \\ &= \frac{1}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right) + j2\zeta\left(\frac{\omega}{\omega_n}\right)} \\ &= \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \frac{\omega^2}{\omega_n^2}}} e^{-j \tan^{-1} \left( \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}} \right)}. \end{aligned}$$

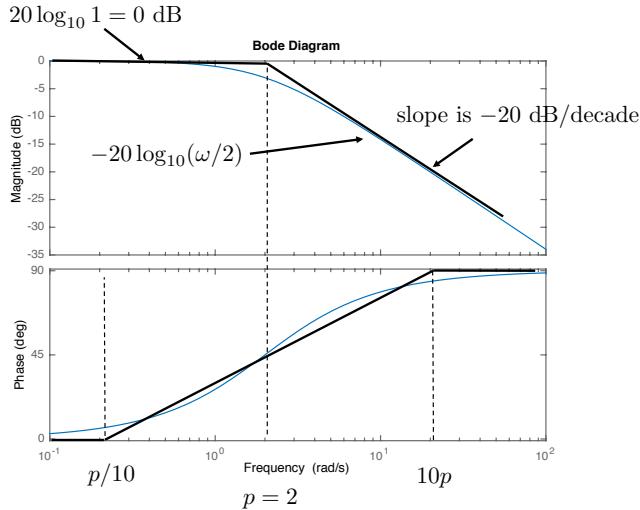


Figure 15-8: Straight-line approximation for a pole in the right half of the complex plane.

Therefore

$$|H(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \frac{\omega^2}{\omega_n^2}}}$$

$$\angle H(j\omega) = -\tan^{-1} \left( \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}} \right).$$

When \$\omega \ll \omega\_n\$ we have that \$20 \log\_{10} |H(j\omega)| \approx 20 \log\_{10} |1| = 0\$ dB. Also, \$\angle H(j\omega) \approx -\tan^{-1} \frac{0}{1} = 0\$ degrees. Alternatively, when \$\omega \gg \omega\_n\$ we have

$$20 \log_{10} |H(j\omega)| \approx -20 \log \sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \frac{\omega^2}{\omega_n^2}}$$

$$\approx -20 \log \sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2}$$

$$\approx -20 \log \left| \frac{\omega}{\omega_n} \right|^2$$

$$= -40 \log \left| \frac{\omega}{\omega_n} \right|,$$

which is a straight line that crosses the 0 dB axis at \$\omega\_n\$ with a slope of \$-40\$ dB/decade. When \$\omega = \omega\_n\$ we have \$20 \log\_{10} |H(j\omega)| = -20 \log\_{10} \sqrt{4\zeta^2} = -20 \log\_{10} |2\zeta|\$. When \$\zeta = 0.707 = \frac{\sqrt{2}}{2}\$, then \$20 \log\_{10} |H(j\omega)| = -20 \log\_{10} 1/\sqrt{2} = -3\$ dB. For

the angle plot, when  $\omega \ll \omega_n$  we have  $\angle H(j\omega) \approx \tan^{-1} \frac{0}{1} = 0$  degrees. When  $\omega \gg \omega_n$  we have  $\angle H(j\omega) \approx \tan^{-1} (2\zeta \frac{\omega_n}{\omega}) \approx -180$  degrees, and when  $\omega = \omega_n$  we have  $\angle H(j\omega) \approx -\tan^{-1} \frac{2\zeta}{0} = -90$  degrees.

The Bode plot for different values of  $\zeta$  are shown in **Fig. 15-9**

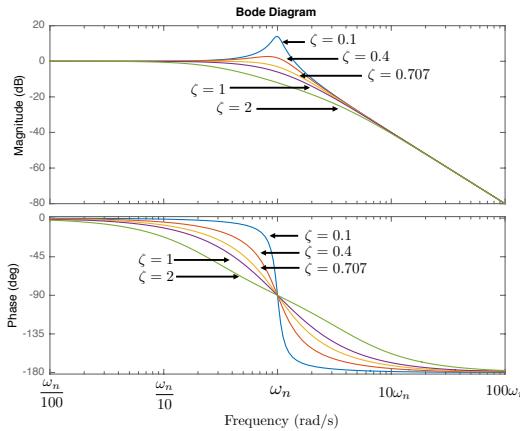


Figure 15-9: The Bode plot for a second order system for various values of  $\zeta$ .

#### 15.1.4 Example

Find the straight-line approximation for the Bode plot of the transfer function

$$H(s) = \frac{20}{s(s+10)}. \quad (15.10)$$

In Bode canonical form we have

$$H(j\omega) = \frac{20}{10} \frac{1}{j\omega(1 + j\frac{\omega}{10})}.$$

Therefore

$$\begin{aligned} 20 \log_{10} |H(j\omega)| = \\ 20 \log_{10} 2 - 20 \log_{10} |\omega| - 20 \log_{10} \sqrt{1 + \left(\frac{\omega}{10}\right)^2}. \end{aligned} \quad (15.11)$$

Note that the term due to the pole at zero, namely  $-20 \log_{10} |\omega|$  is a straight line that intercepts the 0 dB line at  $\omega = 1$ , and has a slope of  $-20$  dB per decade. Therefore, the Bode plot for magnitude will be the graphical addition of three terms:

1. A constant term at  $20 \log_{10} 2 = 2$  dB,

2. A straight line with slope of  $-20$  dB/decade, passing through the  $0$  dB line at  $\omega = 1$ , and
3. A first order low-pass filter with a cut-off frequency at  $\omega = 10$ .

Similarly, the phase is given by

$$\begin{aligned}\angle H(j\omega) &= \angle 2 - \angle j\omega - \angle(1 + j\frac{\omega}{10}) \\ &= 0 - \tan^{-1} \frac{\omega}{0} - \tan^{-1} \frac{\omega}{10} \\ &= -90^\circ - \tan^{-1} \frac{\omega}{10}.\end{aligned}$$

The straight-line approximation as well as the Bode plot generated by Matlab are shown in **Fig. 15-10**.

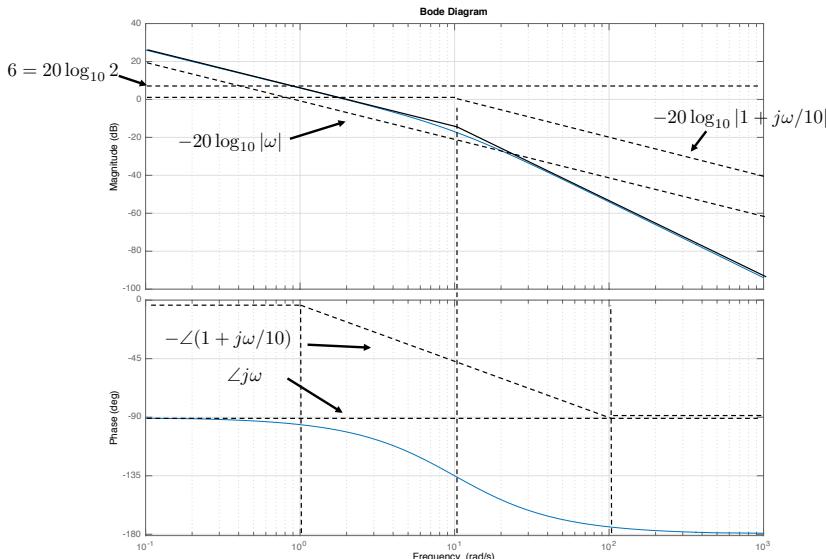


Figure 15-10: Bode plot for the transfer function given in Equation (15.10).

### 15.1.5 Example

Find the straight-line approximation for the Bode plot of the transfer function

$$H(s) = \frac{8000(s+1)(s+10)}{(s+2)(s+20)(s+200)}. \quad (15.12)$$

In Bode canonical form we have

$$\begin{aligned} H(j\omega) &= \frac{8000 \cdot 10}{2 \cdot 20 \cdot 200} \frac{(1+j\omega)(1+j\frac{\omega}{10})}{(1+j\frac{\omega}{2})(1+j\frac{\omega}{20})(1+j\frac{\omega}{200})} \\ &= 10 \frac{(1+j\omega)(1+j\frac{\omega}{10})}{(1+j\frac{\omega}{2})(1+j\frac{\omega}{20})(1+j\frac{\omega}{200})}. \end{aligned}$$

Therefore

$$\begin{aligned} 20 \log_{10} |H(j\omega)| &= 20 \log_{10} 10 + 20 \log_{10} |1+j\omega| + \\ &\quad 20 \log_{10} \left| 1+j\frac{\omega}{10} \right| - 20 \log_{10} \left| 1+j\frac{\omega}{2} \right| \\ &\quad - 20 \log_{10} \left| 1+j\frac{\omega}{20} \right| - 20 \log_{10} \left| 1+j\frac{\omega}{200} \right|. \quad (15.13) \end{aligned}$$

Therefore, the Bode plot for magnitude will be the graphical addition of a constant gain, two zeros, and three poles. Similarly, the phase is given by

$$\begin{aligned} \angle H(j\omega) &= \angle 10 + \angle(1+j\omega) + \angle(1+j\frac{\omega}{10}) \\ &\quad - \angle(1+j\frac{\omega}{2}) - \angle(1+j\frac{\omega}{20}) - \angle(1+j\frac{\omega}{200}). \end{aligned}$$

The straight-line approximation as well as the Bode plot generated by Matlab are shown in **Fig. 15-11**.

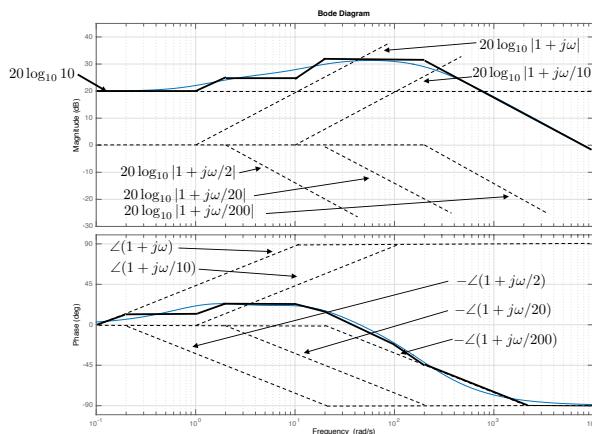


Figure 15-11: Bode plot for the transfer function given in Equation (15.12).

## 15.2 Design Study A. Single Link Robot Arm



### Example Problem A.15

Draw by hand the Bode plot of the single link robot arm from torque  $\tilde{\tau}$  to angle  $\tilde{\theta}$  given that the equilibrium angle is  $\theta_e = 0$ . Use the `bode` command (from Matlab or Python) and compare your results.

#### Solution

From HW A.5, the transfer function for the single link robot arm is

$$P(s) = \frac{3/m\ell^2}{s(s + 3b/m\ell^2)} = \frac{44.44}{s(s + 0.4444)}. \quad (15.14)$$

In Bode canonical form we have

$$P(j\omega) = \frac{100}{(j\omega)(1 + j\frac{\omega}{0.4444})}$$

Therefore

$$\begin{aligned} 20 \log_{10} |P(j\omega)| &= 20 \log_{10} 100 \\ &\quad - 20 \log_{10} |j\omega| - 20 \log_{10} \left| 1 + j\frac{\omega}{0.4444} \right|. \end{aligned} \quad (15.15)$$

Therefore, the Bode plot for magnitude will be the graphical addition of a constant gain, an integrator, and a pole. Similarly, the phase is given by

$$\angle P(j\omega) = \angle 100 - \angle(j\omega) - \angle\left(1 + j\frac{\omega}{0.4444}\right).$$

Python commands to generate the Bode plots are:

```

1 # Single link arm Parameter File
2 import sys
3 sys.path.append('..') # add parent directory
4 import armParam as P
5 from control import tf, bode
6 import matplotlib.pyplot as plt
7
8 # flag to define if using dB or absolute scale for M(omega)
9 dB_flag = True
10
11 # Compute plant transfer functions
12 th_e = 0
13 Plant = tf([3.0/P.m/P.ell**2],
14             [1, 3.0*P.b/P.m/P.ell**2, 0.0])
15

```

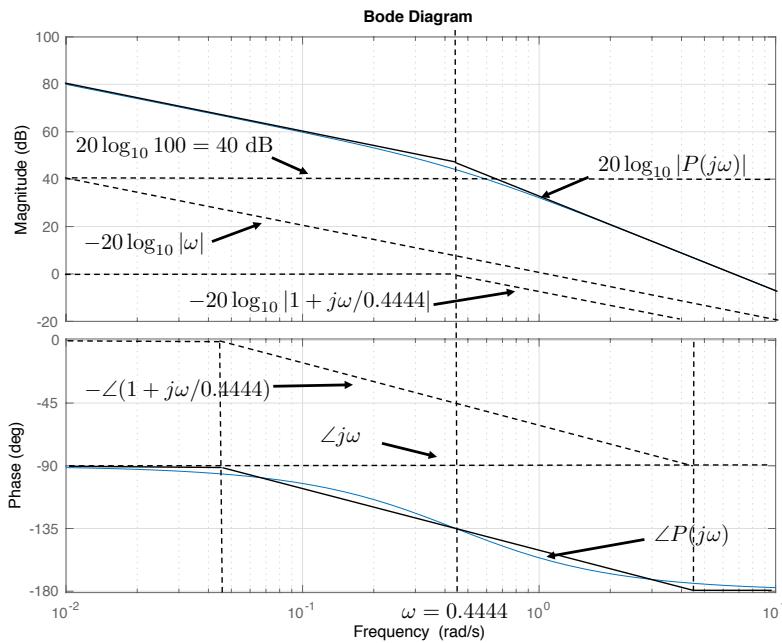


Figure 15-12: Bode plot for the transfer function given in Equation (15.14).

```

16
17 if __name__=="__main__":
18
19     # Bode plot for the plant
20     fig = plt.figure()
21     bode(Plant, dB=dB_flag, margins=False)
22     fig.axes[0].set_title('P(s) for arm')
23
24     # if you want specific values at specific frequencies, you can
25     # do the following (but the magnitudes are absolute, not dB)
26     mag, phase, omega = bode(Plant, plot=False,
27                               omega = [0.3, 10.0, 1000.0])
28
29     print('Close window to end program')
30     plt.show()

```

Listing 15.1: armParamHW15.py

### 15.3 Design Study B. Inverted Pendulum



#### Example Problem B.15

- (a) Draw by hand the Bode plot of the inner loop transfer function from force  $F$  to angle  $\theta$  for the inverted pendulum. Use the `bode` command (from Matlab or Python) and compare your results.
- (b) Draw by hand the Bode plot of the outer loop transfer function from angle  $\theta$  to position  $z$  for the inverted pendulum. Use the `bode` (from Matlab or Python) command and compare your results.

#### Solution

From HW B.5, the transfer function for the inner loop of the inverted pendulum is

$$P_{in}(s) = \frac{\frac{-1}{(m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3})}}{s^2 - \frac{(m_1 + m_2)g}{(m_1 \frac{\ell}{6} + m_2 \frac{2\ell}{3})}} = \frac{-2.823}{s^2 - 34.58} = \frac{-2.823}{(s + 5.881)(s - 5.881)}. \quad (15.16)$$

In Bode canonical form we have

$$P_{in}(j\omega) = \frac{0.0816}{(1 + j\frac{\omega}{5.881})(1 - j\frac{\omega}{5.881})}$$

Therefore

$$\begin{aligned} 20 \log_{10} |P_{in}(j\omega)| &= 20 \log_{10} 0.0816 \\ &\quad - 20 \log_{10} \left| 1 + j\frac{\omega}{5.881} \right| - 20 \log_{10} \left| 1 - j\frac{\omega}{5.881} \right|. \end{aligned} \quad (15.17)$$

Therefore, the Bode plot for magnitude will be the graphical addition of a constant gain, a right half plane pole, and a left half plane pole. Similarly, the phase is given by

$$\angle P_{in}(j\omega) = \angle 0.0816 - \angle \left( 1 + j\frac{\omega}{5.881} \right) - \angle \left( 1 - j\frac{\omega}{5.881} \right).$$

The straight-line approximation as well as the Bode plot generated by Matlab are shown in **Fig. 15-13**.

$$\tilde{Z}(s) = \left( \frac{-\frac{2\ell}{3}s^2 + g}{s^2} \right) \tilde{\Theta}(s).$$

From HW B.5, the transfer function for the outer loop of the inverted pendulum is

$$P_{out}(s) = -\frac{\frac{2\ell}{3}s^2 - g}{s^2} = -\frac{\frac{2}{3}\ell(s^2 - \frac{g}{\ell}\frac{3}{2})}{s^2} = -\frac{\frac{2}{3}(s - 3.83)(s + 3.83)}{s^2}. \quad (15.18)$$

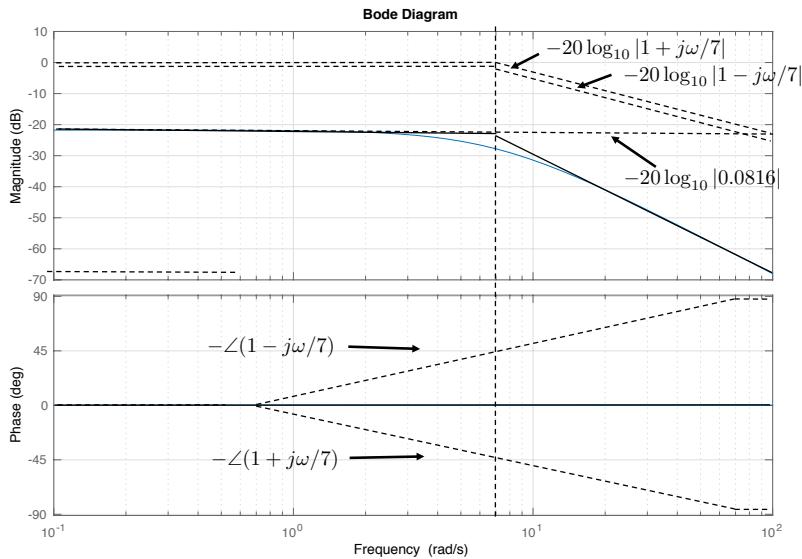


Figure 15-13: Bode plot for the transfer function given in Equation (15.17).

In Bode canonical form we have

$$P_{out}(j\omega) = \frac{0.174(j\omega/3.83 + 1)(-j\omega/3.83 + 1)}{(j\omega)^2}.$$

Therefore

$$\begin{aligned} 20 \log_{10} |P_{out}(j\omega)| &= \\ &20 \log_{10} 0.174 + 20 \log_{10} \left| 1 + j \frac{\omega}{3.83} \right| \\ &+ 20 \log_{10} \left| 1 + j \frac{\omega}{-3.83} \right| - 20 \log_{10} |\omega|^2. \quad (15.19) \end{aligned}$$

Similarly, the phase is given by

$$\angle P_{out}(j\omega) = \angle 0.174 + \angle \left( 1 + j \frac{\omega}{3.83} \right) + \angle \left( 1 - j \frac{\omega}{3.83} \right) - \angle(j\omega) - \angle(j\omega).$$

The Bode plot generated by Python is shown in Fig. 15-14.

Python code to generate Bode plots for the previous two figures is as follows:

```

1 # Inverted Pendulum Parameter File
2 import sys
3 sys.path.append('..') # add parent directory
4 import pendulumParam as P
5 #from control.matlab import *
6 from control import tf, bode
7 import matplotlib.pyplot as plt

```

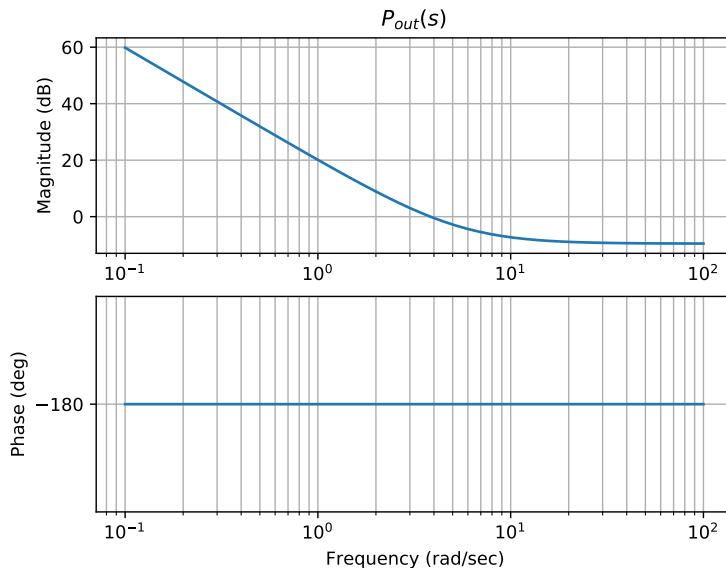


Figure 15-14: Bode plot for the transfer function given in Equation (15.19).

```

8
9 # flag to define if using dB or absolute scale for M(omega)
10 dB_flag = True
11
12 # Compute inner and outer open-loop transfer functions
13 temp = (P.m1*P.ell/6.0+P.m2*2*P.ell/3.0)
14 P_in = tf([-1/temp],
15 [1, 0, -(P.m1+P.m2)*P.g/temp])
16 P_out = tf([-2*P.ell/3.0, 0, P.g], [1, 0, 0])
17
18 if __name__=="__main__":
19
20     # Plot the open loop bode plots for the inner loop
21     fig1 = plt.figure()
22     bode(P_in, dB=dB_flag)
23     fig1.axes[0].set_title('$P_{in}(s)$')
24
25     fig2 = plt.figure()
26     bode(P_out, dB=dB_flag)
27     fig2.axes[0].set_title('$P_{out}(s)$')
28
29     print('Close window to end program')
30     plt.show()

```

Listing 15.2: pendulumParamHW15.py

## 15.4 Design Study C. Satellite Attitude Control



### Example Problem C.15

- (a) Draw by hand the Bode plot of the inner loop transfer function from torque  $\tau$  to angle  $\theta$  for the satellite attitude problem. Use the `bode` command (from Matlab or Python) and compare your results.
- (b) Draw by hand the Bode plot of the outer loop transfer function from body angle  $\theta$  to panel angle  $\phi$  for the satellite attitude problem. Use the `bode` command (from Python or Matlab) and compare your results.

### Solution

From HW C.5, the transfer function for the inner loop of the satellite attitude problem is

$$P_{in}(s) = \frac{1}{(J_s + J_p)s^2} = \frac{1}{6s^2}. \quad (15.20)$$

In Bode canonical form we have

$$P_{in}(j\omega) = \frac{1}{6} \frac{1}{(j\omega)(j\omega)}.$$

Therefore,

$$20 \log_{10} |P_{in}(j\omega)| = 20 \log_{10} 1/6 - 20 \log_{10} |\omega| - 20 \log_{10} |\omega| \quad (15.21)$$

Therefore, the Bode plot for magnitude will be the graphical addition of a constant gain and two straight lines passing through the 0 dB line at  $\omega = 1$  with a slope of -20 dB/decade. Similarly, the phase is given by

$$\angle P_{in}(j\omega) = \angle 1/6 - \angle j\omega - \angle j\omega.$$

The Bode plot generated by Matlab is shown in Fig. 15-15. As can be seen from the figure, the straight-line approximation corresponds directly with the Bode plot in this case.

From HW C.5, the transfer function for the outer loop of the satellite is

$$P_{out}(s) = \frac{\frac{b}{J_p}s + \frac{k}{J_p}}{s^2 + \frac{b}{J_p}s + \frac{k}{J_p}} = \frac{0.05s + 0.15}{s^2 + 0.05s + 0.15}. \quad (15.22)$$

In Bode canonical form we have

$$P_{out}(j\omega) = \frac{1 + j\frac{\omega}{3}}{1 + j\frac{\omega}{3} + (j\frac{\omega}{0.3873})^2}$$

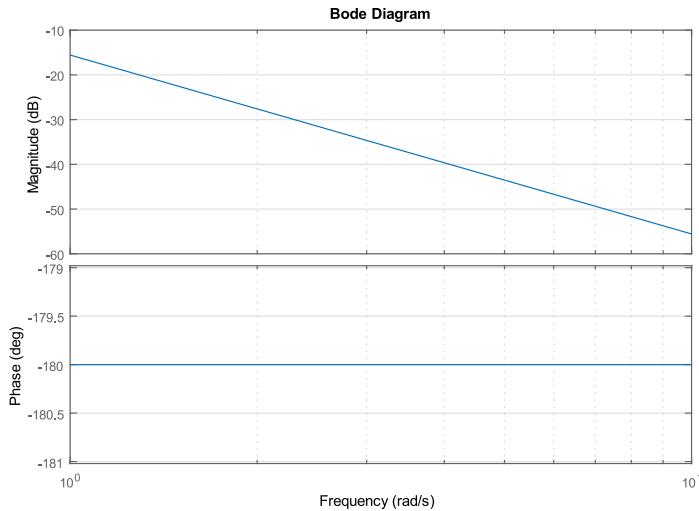


Figure 15-15: Bode plot for the transfer function given in Equation (15.21).

Therefore

$$\begin{aligned} 20 \log_{10} |P_{out}(j\omega)| &= 20 \log_{10} |1 + j\omega/3| \\ &\quad - 20 \log_{10} \left| 1 + j\omega/3 + \left( j \frac{\omega}{0.3873} \right)^2 \right| \end{aligned} \quad (15.23)$$

Therefore, the Bode plot for magnitude will be the graphical addition of a zero, and a complex pole. Similarly, the phase is given by

$$\angle P_{out}(j\omega) = \angle(1 + j\omega/3) - \angle \left( 1 + j\omega/3 + \left( j \frac{\omega}{0.3873} \right)^2 \right).$$

The straight-line approximation as well as the Bode plot generated by Matlab are shown in **Fig. 15-16**.

The Python code to generate similar Bode plots is

```

1 # Inverted Pendulum Parameter File
2 import sys
3 sys.path.append('..') # add parent directory
4 import satelliteParam as P
5 from control import tf, bode
6 import matplotlib.pyplot as plt
7
8 # flag to define if using dB or absolute scale for M(omega)
9 dB_flag = True
10
11 # Compute inner and outer open-loop transfer functions
12 P_in = tf([1],[P.Js+P.Jp,0,0])
13 P_out = tf([P.b/P.Jp, P.k/P.Jp], [1, P.b/P.Jp, P.k/P.Jp])
14

```

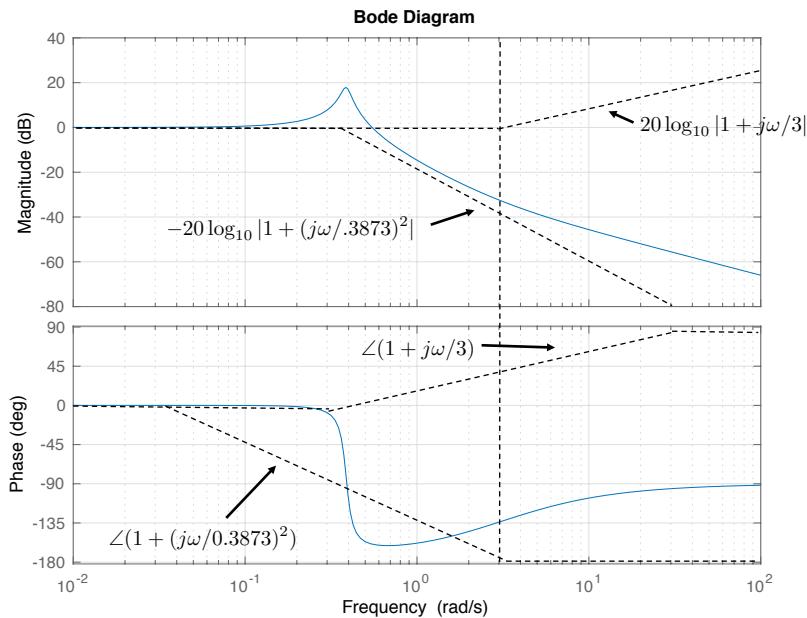


Figure 15-16: Bode plot for the transfer function given in Equation (15.23).

```

15 if __name__=="__main__":
16
17     # Plot the open loop bode plots for the inner loop
18     fig1 = plt.figure()
19     bode(P_in, dB=dB_flag)
20     fig1.axes[0].set_title('$P_{in}(s)$')
21
22     fig2 = plt.figure()
23     bode(P_out, dB=dB_flag)
24     fig2.axes[0].set_title('$P_{out}(s)$')
25
26     print('Close window to end program')
27     plt.show()

```

Listing 15.3: satelliteParamHW15.py

## Notes and References

A good introduction to frequency response for linear time-invariant system is [16]. A good overview of the Bode plot and straight-line approximations to the Bode plot is contained in [4] and many other introductory textbooks on feedback control.

# 16

## Frequency Domain Specifications

### Learning Objectives:

- Use the open-loop gain at low frequencies to evaluate a system's ability to track reference signals and reject output disturbances.
- Use the open-loop gain at high frequencies to evaluate a system's ability to attenuate noise.
- Use the open-loop gain and plant at low frequencies to evaluate a system's ability to reject input disturbances.
- Identify a system's type based upon the open-loop gain.

### 16.1 Theory

In this chapter we show how the magnitude of the frequency response of the open loop system can be used to determine the performance of the closed-loop feedback system. A general feedback loop is shown in [Fig. 16-1](#). In this feedback system,  $r(t)$  is the reference input to be tracked by  $y(t)$ ,  $d_{in}(t)$  is an unknown input disturbance,  $d_{out}$  is an unknown output disturbance, and  $n(t)$  is an unknown noise signal. Note that because of the presence of  $n$ , the error signal  $e = r - y$  is no longer the input to  $C(s)$ . The input to  $C(s)$  is now  $r - y - n$ . Note that this is a different definition than was used in Chapter 9 and will result in a different transfer function between the error and the noise. To find the transfer function from the inputs  $r$ ,  $d_{in}$ ,  $d_{out}$ , and  $n$  to the error  $e(t)$  we start by finding the transfer functions to  $y$ . Starting at  $Y(s)$  and following the loop backward until returning

to  $Y$  we get

$$\begin{aligned} Y(s) &= -D_{out}(s) + P(-D_{in}(s) + C(R(s) - N(s) - Y(s))) \\ \implies (1+PC)Y(s) &= PCR(s) - PCN(s) - D_{out}(s) - PD_{in}(s) \\ Y(s) &= \frac{PC}{1+PC}R(s) - \frac{PC}{1+PC}N(s) - \frac{1}{1+PC}D_{out}(s) - \frac{P}{1+PC}D_{in}(s). \end{aligned} \quad (16.1)$$

Defining the error to be  $E(s) \triangleq R(s) - Y(s)$  we get

$$E(s) = R(s) - Y(s) \quad (16.2)$$

$$= \frac{1}{1+PC}R(s) + \frac{PC}{1+PC}N(s) + \frac{1}{1+PC}D_{out}(s) + \frac{P}{1+PC}D_{in}(s), \quad (16.3)$$

Therefore,

1. The transfer function from the reference  $r$  to the error  $e$  is  $\frac{1}{1+P(s)C(s)}$ ,
2. The transfer function from the noise  $n$  to the error  $e$  is  $\frac{P(s)C(s)}{1+P(s)C(s)}$ ,
3. The transfer function from the output disturbance  $d_{out}$  to the error  $e$  is  $\frac{1}{1+P(s)C(s)}$ , and
4. The transfer function from the input disturbance  $d_{in}$  to the error  $e$  is  $\frac{P(s)}{1+P(s)C(s)}$ .

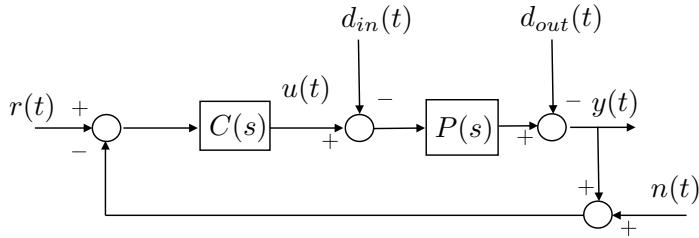


Figure 16-1: General feedback loop.

For typical applications, we have that

1. The reference signal  $r(t)$  is a low frequency signal with frequency content below  $\omega_r$ ,
2. The noise signal  $n(t)$  is a high frequency signal with frequency content above  $\omega_{no}$ ,
3. The output disturbance signal  $d_{out}(t)$  is a low frequency signal with frequency content below  $\omega_{d_{out}}$ ,

4. The input disturbance signal  $d_{in}(t)$  is a low frequency signal with frequency content below  $\omega_{d_{in}}$ .

The objective for the control design is to find  $C(s)$  so that the following specifications are satisfied:

1. **Tracking:** Track reference signals  $r(t)$  with frequency content below  $\omega_r$  so that the error due to the reference signal satisfies

$$|e(t)| \leq \gamma_r |r(t)|,$$

2. **Noise Attenuation:** Attenuate noise signals  $n(t)$  with frequency content above  $\omega_{no}$  so that the error due to the noise signal satisfies

$$|e(t)| \leq \gamma_n |n(t)|,$$

3. **Reject Output Disturbances:** Reject output disturbance signals  $d_{out}(t)$  with frequency content below  $\omega_{d_{out}}$  so that the error due to the output disturbance signal satisfies

$$|e(t)| \leq \gamma_{d_{out}} |d_{out}(t)|,$$

4. **Reject Input Disturbances:** Reject input disturbance signals  $d_{in}(t)$  with frequency content below  $\omega_{d_{in}}$  so that the error due to the input disturbance signal satisfies

$$|e(t)| \leq \gamma_{d_{in}} |d_{in}(t)|.$$

The objective in the remainder of this chapter is to show how each of these specifications can be translated into a requirement on the magnitude of the (open) loop gain  $20 \log_{10} |P(j\omega)C(j\omega)|$ .

### 16.1.1 Tracking

The transfer function from the reference signal  $r(t)$  to the error  $e(t)$  is

$$E(s) = \frac{1}{1 + P(s)C(s)} R(s).$$

Therefore,

$$|E(j\omega)| \leq \left| \frac{1}{1 + P(j\omega)C(j\omega)} \right| |R(j\omega)|.$$

The objective is to track reference signals  $r(t)$  with frequency content below  $\omega_r$  so that the error due to the reference signal satisfies

$$|e(t)| \leq \gamma_r |r(t)|.$$

Therefore, the controller  $C(s)$  needs to be designed so that

$$\left| \frac{1}{1 + P(j\omega)C(j\omega)} \right| \leq \gamma_r,$$

or equivalently so that

$$|1 + P(j\omega)C(j\omega)| \geq \frac{1}{\gamma_r}.$$

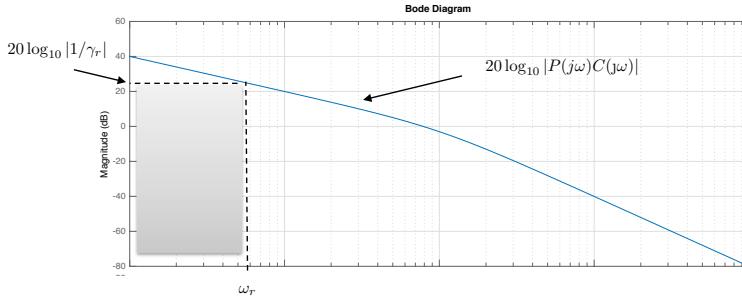
If  $|P(j\omega)C(j\omega)| \gg 1$ , then a sufficient condition is that

$$|P(j\omega)C(j\omega)| \geq \frac{1}{\gamma_r},$$

or equivalently

$$20 \log_{10} |P(j\omega)C(j\omega)| \geq 20 \log_{10} 1/\gamma_r, \quad (16.4)$$

which needs to be satisfied for all  $\omega \leq \omega_r$ . On a Bode plot, the constraint represented by Equation (16.4) is shown graphically in **Fig. 16-2**.



**Figure 16-2:** Bode plot representation of the loopshaping constraint corresponding to tracking  $r(t)$  so that the corresponding error satisfies  $|e(t)| \leq \gamma_r |r(t)|$ .

Alternatively, from the Bode plot of  $PC$  we can compute the tracking characteristics of the closed-loop system. For example, suppose that on the Bode plot

$$20 \log_{10} |P(j\omega)C(j\omega)| > B_r \text{ dB}$$

for all  $\omega < \omega_r$ , then

$$|P(j\omega)C(j\omega)| > 10^{B_r/20},$$

for all  $\omega < \omega_r$ . If  $|P(j\omega)C(j\omega)| \gg 1$  for all  $\omega < \omega_r$ , then

$$\left| \frac{1}{1 + P(j\omega)C(j\omega)} \right| \lesssim 10^{-B_r/20},$$

which implies that

$$|E(j\omega)| \leq 10^{-B_r/20} |R(j\omega)|$$

for all  $\omega < \omega_r$ , or in other words that

$$|e(t)| \leq 10^{-B_r/20} |r(t)|, \quad (16.5)$$

for all  $r(t)$  with frequency content below  $\omega_r$ , which implies that

$$\gamma_r = 10^{-B_r/20}.$$

### 16.1.2 Noise Attenuation

The transfer function from the noise signal  $n(t)$  to the error  $e(t)$  is

$$E(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} N(s).$$

Therefore,

$$|E(j\omega)| \leq \left| \frac{P(j\omega)C(j\omega)}{1 + P(j\omega)C(j\omega)} \right| |N(j\omega)|.$$

The objective is to attenuate noise signals  $n(t)$  with frequency content above  $\omega_{no}$  so that the error due to the noise signal satisfies

$$|e(t)| \leq \gamma_n |n(t)|.$$

Therefore, the controller  $C(s)$  needs to be designed so that

$$\left| \frac{P(j\omega)C(j\omega)}{1 + P(j\omega)C(j\omega)} \right| \leq \gamma_n.$$

If  $|P(j\omega)C(j\omega)| \ll 1$ , then a sufficient condition is that

$$|P(j\omega)C(j\omega)| \leq \gamma_n,$$

or equivalently

$$20 \log_{10} |P(j\omega)C(j\omega)| \leq 20 \log_{10} \gamma_n, \quad (16.6)$$

which needs to be satisfied for all  $\omega \geq \omega_{no}$ . On a Bode plot, the constraint represented by Equation (16.6) is shown graphically in [Fig. 16-3](#).

Alternatively, from the Bode plot of  $PC$  we can compute the noise attenuation characteristics of the closed-loop system. For example, suppose that on the Bode plot

$$20 \log_{10} |P(j\omega)C(j\omega)| < -B_n \text{ dB}$$

for all  $\omega > \omega_{no}$ , then

$$|P(j\omega)C(j\omega)| < 10^{-B_n/20},$$

for all  $\omega > \omega_{no}$ . If  $P(j\omega)C(j\omega) \ll 1$  for all  $\omega > \omega_{no}$ , then

$$\left| \frac{P(j\omega)C(j\omega)}{1 + P(j\omega)C(j\omega)} \right| \lesssim 10^{-B_n/20},$$

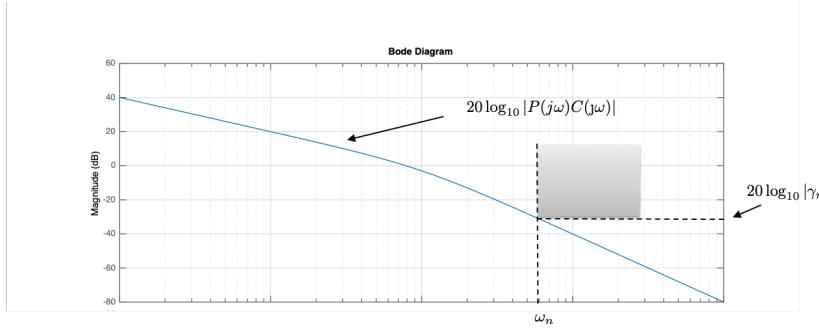


Figure 16-3: Bode plot representation of the loopshaping constraint corresponding to attenuating the noise  $n(t)$  so that the corresponding error satisfies  $|e(t)| \leq \gamma_n |n(t)|$ .

which implies that

$$|E(j\omega)| \leq 10^{-B_n/20} |N(j\omega)|$$

for all  $\omega > \omega_{no}$ , or in other words that

$$|e(t)| \leq 10^{-B_n/20} |n(t)|,$$

for all  $n(t)$  with frequency content above  $\omega_{no}$ , which implies that

$$\gamma_n = 10^{-B_n/20}.$$

### 16.1.3 Reject Output Disturbances

The transfer function from the output disturbance signals  $d_{out}(t)$  to the error  $e(t)$  is

$$E(s) = \frac{1}{1 + P(s)C(s)} D_{out}(s).$$

Therefore,

$$|E(j\omega)| \leq \left| \frac{1}{1 + P(j\omega)C(j\omega)} \right| |D_{out}(j\omega)|.$$

The objective is to attenuate output disturbance signals  $d_{out}(t)$  with frequency content below  $\omega_{d_{out}}$  so that the error due to the output disturbance signal satisfies

$$|e(t)| \leq \gamma_{d_{out}} |d_{out}(t)|.$$

Therefore, the controller  $C(s)$  needs to be designed so that

$$\left| \frac{1}{1 + P(j\omega)C(j\omega)} \right| \leq \gamma_{d_{out}},$$

or equivalently so that

$$|1 + P(j\omega)C(j\omega)| \geq \frac{1}{\gamma_{d_{out}}}.$$

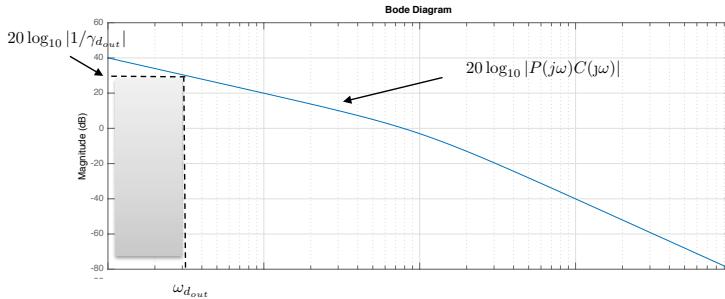
If  $|P(j\omega)C(j\omega)| \gg 1$ , then a sufficient condition is that

$$|P(j\omega)C(j\omega)| \geq \frac{1}{\gamma_{d_{out}}},$$

or equivalently

$$20 \log_{10} |P(j\omega)C(j\omega)| \geq 20 \log_{10} 1/\gamma_{d_{out}}, \quad (16.7)$$

which needs to be satisfied for all  $\omega \leq \omega_{d_{out}}$ . On a Bode plot, the constraint represented by Equation (16.7) is shown graphically in **Fig. 16-4**.



**Figure 16-4:** Bode plot representation of the loopshaping constraint corresponding to rejecting the output disturbance  $d_{out}(t)$  so that the corresponding error satisfies  $|e(t)| \leq \gamma_{d_{out}} |d_{out}(t)|$ .

Alternatively, from the Bode plot of  $PC$  we can compute the output disturbance rejection characteristics of the closed-loop system. For example, suppose that on the Bode plot

$$20 \log_{10} |P(j\omega)C(j\omega)| > B_{d_{out}} \text{ dB}$$

for all  $\omega < \omega_{d_{out}}$ , then

$$|P(j\omega)C(j\omega)| > 10^{B_{d_{out}}/20},$$

for all  $\omega < \omega_{d_{out}}$ . If  $|P(j\omega)C(j\omega)| \gg 1$  for all  $\omega < \omega_{d_{out}}$ , then

$$\left| \frac{1}{1 + P(j\omega)C(j\omega)} \right| \lesssim 10^{-B_{d_{out}}/20},$$

which implies that

$$|E(j\omega)| \leq 10^{-B_{d_{out}}/20} |D_{out}(j\omega)|$$

for all  $\omega < \omega_{d_{out}}$ , or in other words that

$$|e(t)| \leq 10^{-B_{d_{out}}/20} |d_{out}(t)|,$$

for all  $d_{out}(t)$  with frequency content below  $\omega_{d_{out}}$ , which implies that

$$\gamma_{d_{out}} = 10^{-B_{d_{out}}/20}.$$

### 16.1.4 Reject Input Disturbances

The transfer function from the input disturbance signals  $d_{in}(t)$  to the error  $e(t)$  is

$$E(s) = \frac{P(s)}{1 + P(s)C(s)} D_{in}(s).$$

Therefore,

$$|E(j\omega)| \leq \left| \frac{P(j\omega)}{1 + P(j\omega)C(j\omega)} \right| |D_{in}(j\omega)|.$$

The objective is to attenuate input disturbance signals  $d_{in}(t)$  with frequency content below  $\omega_{d_{in}}$  so that the error due to the input disturbance signal satisfies

$$|e(t)| \leq \gamma_{d_{in}} |d_{in}(t)|.$$

Therefore, the controller  $C(s)$  needs to be designed so that

$$\left| \frac{P(j\omega)}{1 + P(j\omega)C(j\omega)} \right| \leq \gamma_{d_{in}},$$

or equivalently so that

$$\frac{|1 + P(j\omega)C(j\omega)|}{|P(j\omega)|} \geq \frac{1}{\gamma_{d_{in}}}.$$

If  $|P(j\omega)C(j\omega)| \gg 1$ , then a sufficient condition is that

$$\frac{|P(j\omega)C(j\omega)|}{|P(j\omega)|} \geq \frac{1}{\gamma_{d_{in}}},$$

or equivalently

$$20 \log_{10} |P(j\omega)C(j\omega)| - 20 \log_{10} |P(j\omega)| \geq 20 \log_{10} 1/\gamma_{d_{in}}, \quad (16.8)$$

which needs to be satisfied for all  $\omega \leq \omega_{d_{in}}$ . On a Bode plot, the constraint represented by Equation (16.8) is shown graphically in **Fig. 16-5**, where it can be seen that the loop gain  $|PC|$  must be above the magnitude of the plant  $|P|$  by the specified amount.

Alternatively, from the Bode plot of  $PC$  we can compute the input disturbance rejection characteristics of the closed-loop system. For example, suppose that on the Bode plot

$$20 \log_{10} |P(j\omega)C(j\omega)| - 20 \log_{10} |P(j\omega)| > B_{d_{in}} \text{ dB}$$

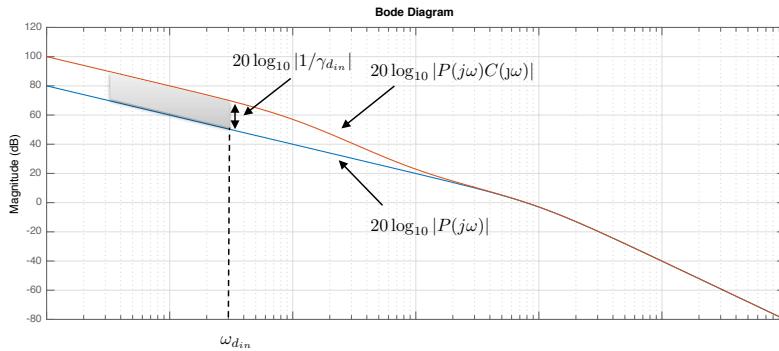


Figure 16-5: Bode plot representation of the loopshaping constraint corresponding to rejecting the input disturbance  $d_{in}(t)$  so that the corresponding error satisfies  $|e(t)| \leq \gamma_{d_{in}} |d_{in}(t)|$ .

for all  $\omega < \omega_{d_{in}}$ , then

$$\frac{|P(j\omega)C(j\omega)|}{|P(j\omega)|} > 10^{B_{d_{in}}/20},$$

for all  $\omega < \omega_{d_{in}}$ . If  $|P(j\omega)C(j\omega)| \gg 1$  for all  $\omega < \omega_{d_{in}}$ , then

$$\left| \frac{P(j\omega)}{1 + P(j\omega)C(j\omega)} \right| \lesssim 10^{-B_{d_{in}}/20},$$

which implies that

$$|E(j\omega)| \leq 10^{-B_{d_{in}}/20} |D_{in}(j\omega)|$$

for all  $\omega < \omega_{d_{in}}$ , or in other words that

$$|e(t)| \leq 10^{-B_{d_{in}}/20} |d_{in}(t)|, \quad (16.9)$$

for all  $d_{in}(t)$  with frequency content below  $\omega_{d_{in}}$ , which implies that

$$\gamma_{d_{in}} = 10^{-B_{d_{in}}/20}.$$

### 16.1.5 Frequency Response Characterization of System Type

Recall that the loop gain  $P(s)C(s)$  is said to be

- Type 0 if there are no free integrators,
- Type 1 if there is one free integrator,
- Type 2 if there are two free integrators, etc..

### Type 0 System

Note that for a type 0 system

$$\begin{aligned}\lim_{s \rightarrow 0} P(s)C(s) &= \text{constant} \\ \Rightarrow \lim_{\omega \rightarrow 0} |P(j\omega)C(j\omega)| &= \text{constant} \\ \Rightarrow \lim_{\omega \rightarrow 0} 20 \log_{10} |P(j\omega)C(j\omega)| &= \text{constant.}\end{aligned}$$

In other words, at low frequency, the Bode magnitude plot is as shown in [Fig. 16-6](#).

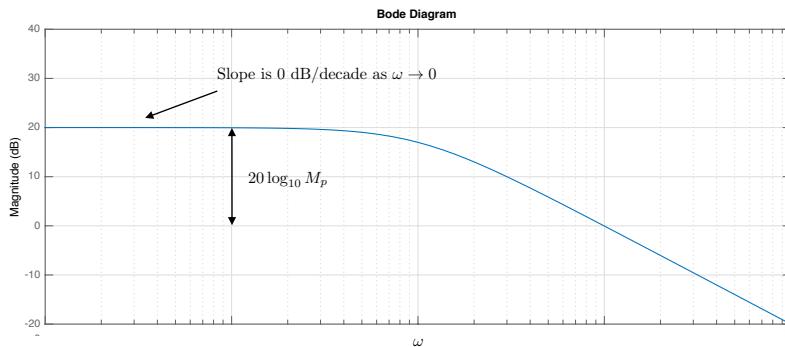


Figure 16-6: Bode plot of a type 0 system.

Recall that for a step input to a type 0 system where  $R(s) = A/s$ , the steady state error is

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{A}{1 + P(s)C(s)} = \frac{A}{1 + M_p},$$

where

$$M_p = \lim_{s \rightarrow 0} P(s)C(s) = \lim_{\omega \rightarrow 0} |P(j\omega)C(j\omega)|.$$

As shown in [Fig. 16-6](#),  $20 \log_{10} M_p$  is the constant limit of the loop gain as  $\omega \rightarrow 0$ .

Therefore, from the Bode plot of  $PC$  for a type 0 system, we can compute the tracking error for a step input of size  $A$ . Suppose that on the Bode plot

$$\lim_{\omega \rightarrow 0} 20 \log_{10} |P(j\omega)C(j\omega)| = B_0 \text{ dB.}$$

Then

$$M_p = 10^{B_0/20},$$

and

$$\lim_{t \rightarrow \infty} |e(t)| \leq \frac{1}{1 + M_p} A. \quad (16.10)$$

### Type 1 System

For a type 1 system since  $P(s)C(s)$  has one free integrator

$$\begin{aligned} \lim_{s \rightarrow 0} sP(s)C(s) &= \text{constant} \\ \implies \lim_{\omega \rightarrow 0} |(j\omega)P(j\omega)C(j\omega)| &= \text{constant} \\ \implies \lim_{\omega \rightarrow 0} \frac{|P(j\omega)C(j\omega)|}{\left| \frac{1}{j\omega} \right|} &= \text{constant} \\ \implies \lim_{\omega \rightarrow 0} \left[ 20 \log_{10} |P(j\omega)C(j\omega)| - 20 \log_{10} \left| \frac{1}{j\omega} \right| \right] &= \text{constant}. \end{aligned}$$

In other words, at low frequency, the Bode magnitude plot of  $20 \log_{10} |P(j\omega)C(j\omega)|$  has a slope of  $-20$  dB/decade as  $\omega \rightarrow 0$ , as shown in **Fig. 16-7**.

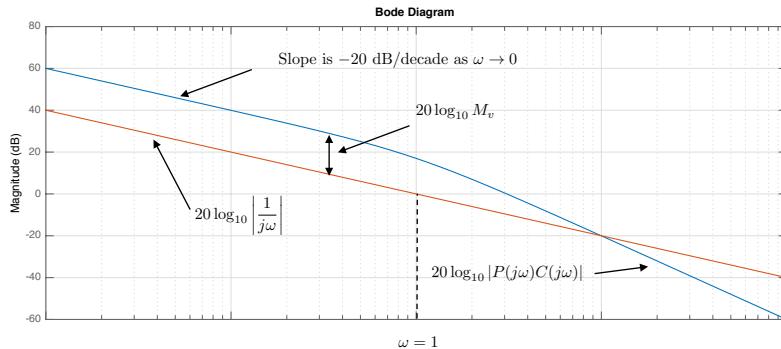


Figure 16-7: Bode plot of a type 1 system.

Recall that for a ramp input to a type 1 system where  $R(s) = A/s^2$ , the steady state error is

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{A}{s + sP(s)C(s)} = \frac{A}{M_v},$$

where

$$M_v = \lim_{s \rightarrow 0} sP(s)C(s) = \lim_{\omega \rightarrow 0} |(j\omega)P(j\omega)C(j\omega)|.$$

As shown in **Fig. 16-7**,  $20 \log_{10} M_v$  is the amount that the loop gain exceeds the Bode plot of  $20 \log_{10} |1/j\omega|$  as  $\omega \rightarrow 0$ .

Therefore, from the Bode plot of  $PC$  for a type 1 system, we can compute the tracking error for a ramp input with slope  $A$ . Suppose that on the Bode plot

$$\lim_{\omega \rightarrow 0} \left[ 20 \log_{10} |P(j\omega)C(j\omega)| - 20 \log_{10} \left| \frac{1}{j\omega} \right| \right] = B_1 \text{ dB},$$

then

$$M_v = 10^{B_1/20},$$

and

$$\lim_{t \rightarrow \infty} |e(t)| \leq \frac{1}{M_v} A. \quad (16.11)$$

### Type 2 System

For a type 2 system where  $P(s)C(s)$  has two free integrators we have

$$\begin{aligned} & \lim_{s \rightarrow 0} s^2 P(s)C(s) = \text{constant} \\ \implies & \lim_{\omega \rightarrow 0} |(j\omega)^2 P(j\omega)C(j\omega)| = \text{constant} \\ \implies & \lim_{\omega \rightarrow 0} \frac{|P(j\omega)C(j\omega)|}{\left| \frac{1}{j\omega} \right|^2} = \text{constant} \\ \implies & \lim_{\omega \rightarrow 0} \left[ 20 \log_{10} |P(j\omega)C(j\omega)| - 20 \log_{10} \left| \frac{1}{j\omega} \right|^2 \right] = \text{constant}. \end{aligned}$$

In other words, at low frequency, the Bode magnitude plot of  $20 \log_{10} |P(j\omega)C(j\omega)|$  has a slope of  $-40 as  $\omega \rightarrow 0$ , as shown in **Fig. 16-8**.$

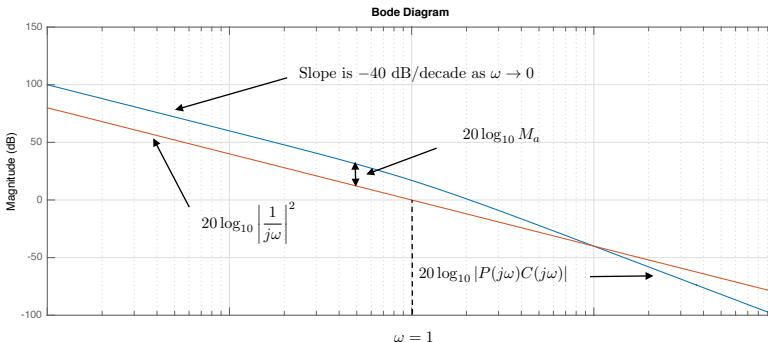


Figure 16-8: Bode plot of a type 2 system.

Recall that for a parabolic input to a type 2 system where  $R(s) = A/s^3$ , the steady state error is

$$e_{ss} = \lim_{t \rightarrow 0} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{A}{s^2 + s^2 P(s)C(s)} = \frac{A}{M_a},$$

where

$$M_a = \lim_{s \rightarrow 0} s^2 P(s)C(s) = \lim_{\omega \rightarrow 0} |(j\omega)^2 P(j\omega)C(j\omega)|.$$

As shown in **Fig. 16-8**,  $20 \log_{10} M_a$  is the amount that the loop gain exceeds the Bode plot of  $20 \log_{10} |1/j\omega|^2$  as  $\omega \rightarrow 0$ .

Therefore, from the Bode plot of  $PC$  for a type 2 system, we can compute the tracking error for a parabola input with curvature  $A$ . Suppose that on the Bode plot

$$\lim_{\omega \rightarrow 0} \left[ 20 \log_{10} |P(j\omega)C(j\omega)| - 20 \log_{10} \left| \frac{1}{j\omega} \right|^2 \right] = B_2 \text{ dB}.$$

Then

$$M_a = 10^{B_2/20},$$

and

$$\lim_{t \rightarrow \infty} |e(t)| \leq \frac{1}{M_a} A. \quad (16.12)$$

## 16.2 Design Study A. Single Link Robot Arm



### Example Problem A.16

For the single link robot arm, use the `bode` command (from Matlab or Python) to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PID control, using the control gains calculated in HW A.10.

- (a) To what percent error can the closed loop system under PID control track the desired input if all of the frequency content of  $\theta_r(t)$  is below  $\omega_r = 0.4$  radians per second?
- (b) If the reference input is  $\theta_r(t) = 5t^2$  for  $t \geq 0$ , what will the steady state tracking error to this input be when using PID control?
- (c) If all of the frequency content of the input disturbance  $d_{in}(t)$  is below  $\omega_{d_{in}} = 0.01$  radians per second, what percentage of the input disturbance shows up in the output  $\theta$  under PID control?
- (d) If all of the frequency content of the noise  $n(t)$  is greater than  $\omega_{no} = 100$  radians per second, what percentage of the noise shows up in the output signal  $\theta$ ?

### Solution

- (a) The Bode plot of the plant  $P(s)$ , and the loop gain with PID control  $P(s)C_{PID}(s)$  is shown in [Fig. 16-9](#). From [Fig. 16-9](#) we see that below  $\omega_r = 0.4$  rad/sec, the loop gain is above  $B_r = 44.5$  dB. Therefore, from Equation (16.5) we have that

$$|e(t)| \leq \gamma_r |r(t)|,$$

where  $\gamma_r = 10^{-44.5/20} = 0.006$ , which implies that the tracking error will be 0.60% of the magnitude of the input.

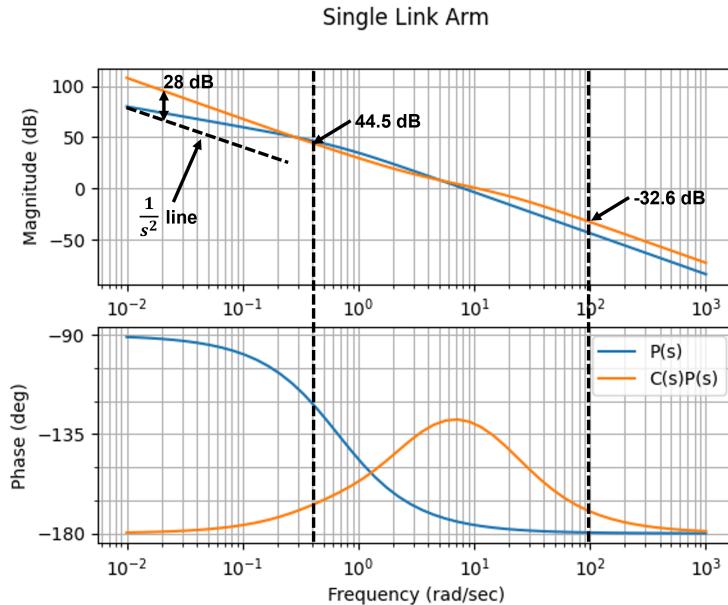


Figure 16-9: Bode plot for single link robot arm, plant only and under PID control.

**(b)** Suppose now that the desired reference input is  $\theta_r(t) = 5t^2$ . Under PID control, the slope of the loop gain as  $\omega \rightarrow 0$  is  $-40$  dB/dec, which implies that the system is type 2 and will track a step and a ramp with zero steady state error. For a parabola, there will be a finite error. Since the loop gain under PID control is  $B_2 = 28$  dB above the  $1/s^2$  line, from Equation (16.12) the steady state error satisfies

$$\lim_{t \rightarrow \infty} |e(t)| \leq \frac{A}{M_a} = 5 \cdot 10^{-28/20} = 0.2.$$

**(c)** If the input disturbance is below  $\omega_{d_{in}} = 0.01$  rad/s, then the difference between the loop gain and the plant is  $B_{d_{in}} = 28$  dB at  $\omega_{d_{in}} = 0.01$  rad/s, therefore, from Equation (16.9) we see that

$$\gamma_{d_{in}} = 10^{-28/20} = 0.04,$$

implying that 4% of the input disturbance will show up in the output.

**(d)** For noise greater than  $\omega_{no} = 100$  rad/sec, we see from Fig. 16-9 that  $B_n = 32.6$  dB. Therefore,  $\gamma_n = 10^{-32.6/20} = 0.023$  which implies that 2.3% of the noise will show up in the output signal.

Python code to generate the relevant plots can be downloaded at <http://controlbook.byu.edu>.

## 16.3 Design Study B. Inverted Pendulum



### Example Problem B.16

For the inner loop of the inverted pendulum, use the `bode` command (in Python or Matlab) to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework B.10.

- (a) To what percent error can the closed loop system track the desired input if all of the frequency content of  $\theta_r(t)$  is below  $\omega_r = 1.0$  radians per second?
- (b) If all of the frequency content of the sensor noise on the inner  $n(t)$  is greater than  $\omega_{no} = 200$  radians per second, what percentage of the noise shows up in the output signal  $\theta$ ?

For the outer loop of the inverted pendulum, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for the plant, and the plant under PID control, using the control gains calculated in Homework B.10.

- (c) If the reference signal  $y_r(t)$  has frequency content below  $\omega_r = 0.001$  radian-second, what is the tracking error under PID control if  $|r(t)| \leq 50$ ?

### Solution

- (a) The Bode plot of the plant  $P(s)$ , and the loop gain with PD control  $P(s)C_{PD}(s)$  are shown in Fig. 16-10. From Fig. 16-10 we see that below  $\omega_r = 1.0$  rad/sec, the loop gain is above  $B_r = 6.5$  dB. Therefore, from Equation (16.5) we have that

$$|e(t)| \leq \gamma_r |r(t)|,$$

where  $\gamma_r = 10^{-6.5/20} = 0.47$ , which implies that the tracking error will be 47% of the magnitude of the input.

- (b) For noise greater than  $\omega_{no} = 200$  rad/sec, we see from Fig. 16-10 that  $B_n = 32.2$  dB. Therefore,  $\gamma_n = 10^{-32.2/20} = 0.0245$  which implies that 2.45% of the noise will show up in the output signal.

- (c) The Bode plot of the outer loop  $P(s)$ , and the loop gain with PID control  $P(s)C_{PID}(s)$  is shown in Fig. 16-11. From Fig. 16-11 it can be seen that the loop gain under PID control for the outer loop is above  $B_r = 154$  dB for all  $\omega < \omega_r = 0.001$  radians/second. Therefore, from Equation (16.5) the tracking error satisfies

$$|e(t)| \leq \gamma_r |r(t)|,$$

where  $\gamma_r = 10^{-154/20} = 2.0e - 08$ . Therefore, if  $|r(t)| \leq 50$ , then  $|e(t)| \leq 100e - 08$ .

Python code to generate the relevant plots can be downloaded at <http://controlbook.byu.edu>.

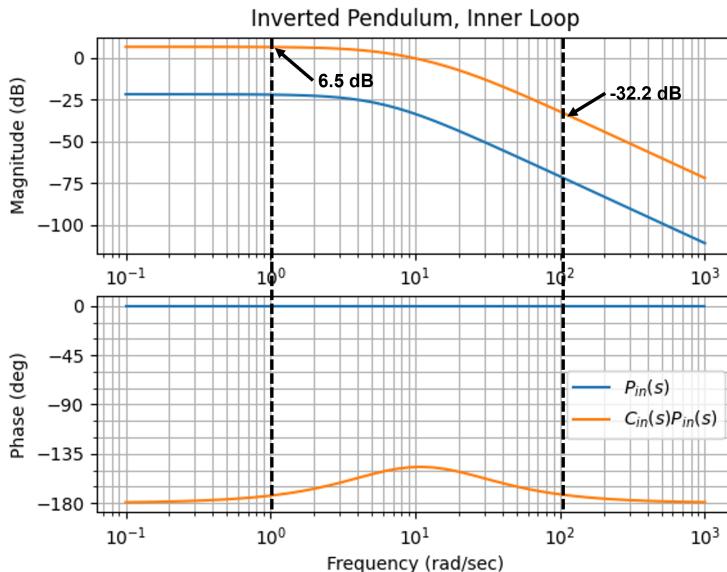


Figure 16-10: Bode plot for the inner loop of the inverted pendulum for (1) plant only (blue), and (2) under PD control (orange).

## 16.4 Design Study C. Satellite Attitude Control



### Example Problem C.16

For the inner loop of the satellite attitude control, use the `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework C.10.

- (a) To what percent error can the closed loop system under PD control track a step in  $\theta_r(t) = 20t^2$  for  $t \geq 0$ ?
- (b) If the input disturbance has frequency content below  $\omega_{din} = 0.1$  radians per second, what percentage of the input disturbance appears in the output?

For the outer loop of the satellite attitude control, use the `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PID control using the control gains calculated in Homework C.10.

- (c) If all of the frequency content of the noise  $n(t)$  is greater than  $\omega_{no} = 10$  radians per second, what percentage of the noise shows up in the output signal  $\phi$ , using PI control?

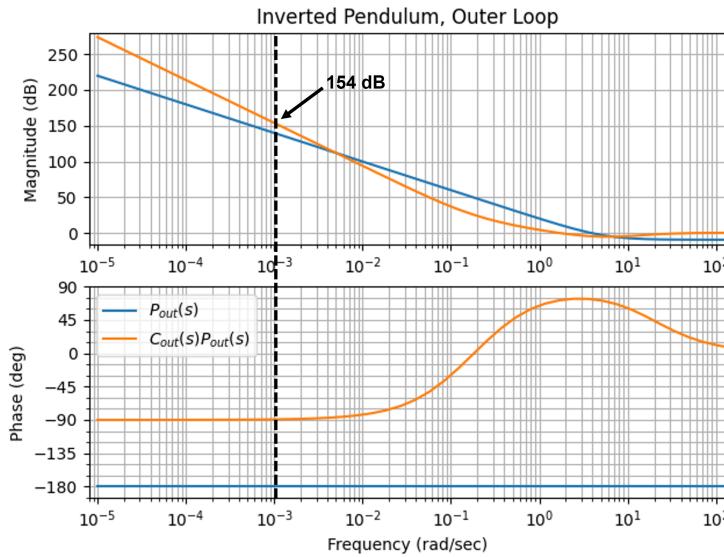


Figure 16-11: Bode plot for the outer loop of the inverted pendulum for (1) plant only (blue), and (2) under PID control (orange).

## Solution

(a)

The Bode plot of the plant  $P(s)$  and the loop gain with PD control  $P(s)C_{PD}(s)$  are shown in Fig. 16-12. Suppose that the desired reference input is  $\theta^r(t) = 20t^2$ . Under PD control, the slope of the loop gain as  $\omega \rightarrow 0$  is  $-40$  dB/dec, which implies that the system is type 2 and will track a step and a ramp with zero steady state error. For a parabola, there will be a finite error. Since the loop gain under PID control is  $B_2 = 20$  dB above the  $1/s^2$  line, from Equation (16.12) the steady state error satisfies

$$\lim_{t \rightarrow \infty} |e(t)| \leq \frac{A}{M_a} = 20 \cdot 10^{-22.3/20} = 1.53.$$

(b) If the input disturbance is below  $\omega_{d_{in}} = 0.1$  rad/s, then the difference between the loop gain and the plant is  $B_{d_{in}} = 38$  dB, therefore, from Equation (16.9) we see that

$$\gamma_{d_{in}} = 10^{-38/20} = 0.0126,$$

implying that 1.26% of the input disturbance will show up in the output.

(c) The Bode plot of outer loop  $P_{out}(s)$ , and the loop gain with PI control  $P_{out}(s)C_{PI}(s)$  are shown in Fig. 16-13. For noise greater than  $\omega_{no} = 10$  rad/sec, we see from Fig. 16-13 that  $B_n = 8.4$  dB. Therefore,  $\gamma_n = 10^{-8.4/20} = 0.380$  which implies that 38% of the noise will show up in the output signal.

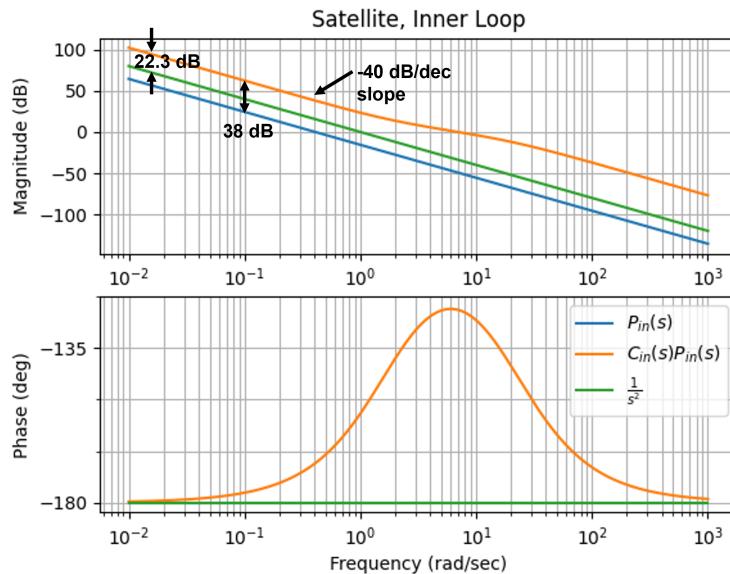


Figure 16-12: Bode plot for the inner loop of the satellite attitude control for (a) plant only (blue), and (b) under PD control (orange).

Python code to generate the relevant plots can be downloaded at <http://controlbook.byu.edu>.

### Important Concepts:

- The Bode magnitude plot is used to identify the error bounds due to tracking, disturbances, and noise.
- The system type can be determined from the magnitude slope of the open-loop gain at low frequencies.
- The steady-state error when tracking an input signal is calculated by measuring the dB difference, at low frequencies, between the open-loop gain and the gain for system's number of free integrators.

### Notes and References

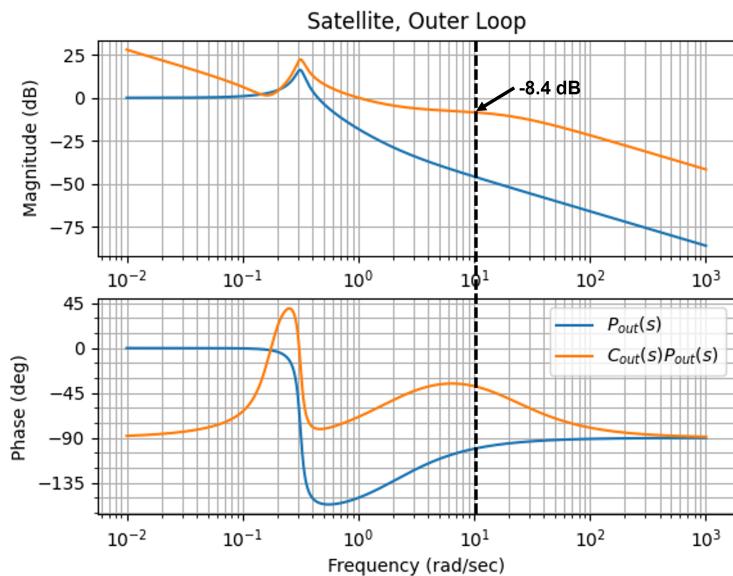


Figure 16-13: Bode plot for the outer loop of the satellite attitude control for (a) plant only (blue), and (b) PI control (orange).



# 17

## Stability and Robustness Margins

### Learning Objectives:

- Calculate the gain margin and phase margin for a system.

## 17.1 Theory

### 17.1.1 Phase and Gain Margins

To achieve the objectives discussed in Chapter 16, the closed-loop system must be stable. Recall that tracking performance and disturbance rejection characteristics are determined by the loopshape at low frequencies when  $|P(j\omega)C(j\omega)| \gg 1$ . Similarly, noise attenuation properties are determined by the loopshape at high frequencies when  $|P(j\omega)C(j\omega)| \ll 1$ . In both cases we ignored the phase of  $P(j\omega)C(j\omega)$ . It turns out that stability of the closed-loop system is determined by the phase of  $P(j\omega)C(j\omega)$  when  $P(j\omega)C(j\omega) = 1$ , as depicted in Fig. 17-1.

Define the *crossover frequency*  $\omega_{co}$  to be the frequency where

$$|P(j\omega_{co})C(j\omega_{co})| = 1,$$

or equivalently where

$$20 \log_{10} |P(j\omega_{co})C(j\omega_{co})| = 0 \text{ dB}.$$

The phase  $\angle P(j\omega_{co})C(j\omega_{co})$  at the crossover frequency plays a critical role in determining the stability of the system. To better understand why this is the case, consider the transfer functions from  $R$ ,  $D_{in}$ ,  $D_{out}$ , and  $N$ , to  $E$  in Equation (16.3). The denominator in each transfer function is  $1 + P(s)C(s)$ . Therefore, the closed-loop poles are given by the closed-loop characteristic equation

$$1 + P(s)C(s) = 0,$$

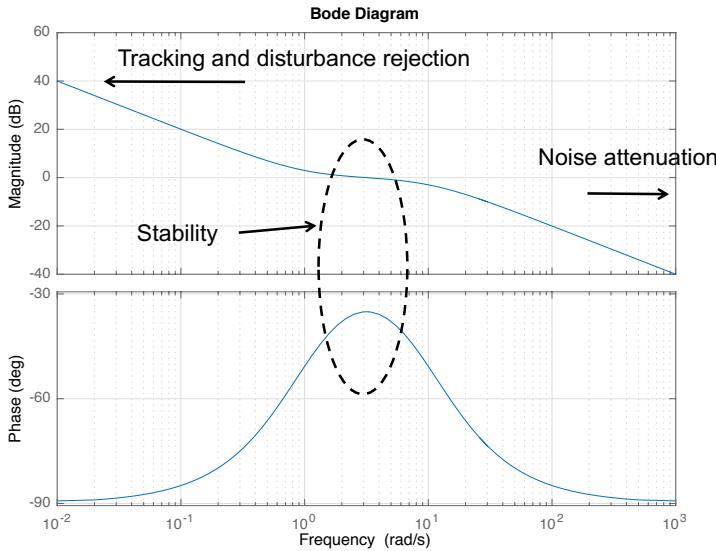


Figure 17-1: The shape of the magnitude plot primarily determines the tracking and disturbance rejection characteristics at low frequencies and the noise attenuation at high frequencies. Stability is primarily determined by the phase at the crossover frequency.

or equivalently where

$$P(s)C(s) = -1.$$

Recall that for any  $s$ ,  $P(s)C(s)$  is a complex number with magnitude and phase. Therefore, given any  $s_0$  we can draw  $1 + P(s_0)C(s_0)$  in the complex plane as shown in Fig. 17-2. Since the physical response of the system is determined by  $P(s)C(s)$  evaluated on the  $j\omega$ -axis, there are stability problems if there exists an  $\omega_0$  such that  $P(j\omega_0)C(j\omega_0) = -1$ , or in other words when

$$|P(j\omega_0)C(j\omega_0)| = 1 \quad \text{and} \quad \angle P(j\omega_0)C(j\omega_0) = -180^\circ,$$

as shown in Fig. 17-3. Since the magnitude of the loop gain is equal to one at the crossover frequency  $\omega_{co}$ , it turns out that the quality of stability is determined by the phase of the loop gain at the crossover frequency. Define the *phase margin*  $PM$  as

$$PM = \angle P(j\omega_{co})C(j\omega_{co}) - (-180)^\circ.$$

Fig. 17-4 depicts the phase margin on the Bode plot. Another way to understand the phase margin is shown in Fig. 17-5. The phase margin indicates the angular distance that  $P(j\omega_{co})C(j\omega_{co})$  is from  $-1$ .

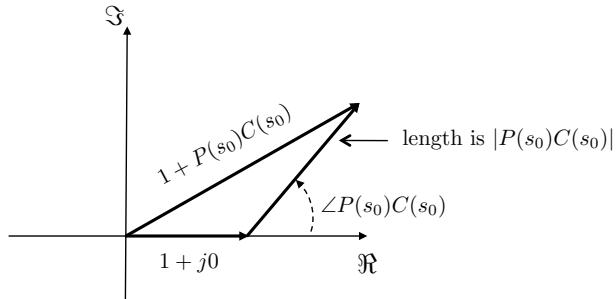


Figure 17-2: The plot of  $1 + P(s_0)C(s_0)$  in the complex plane, is the sum of the complex number  $1 + j0$  and  $P(s_0)C(s_0)$ .

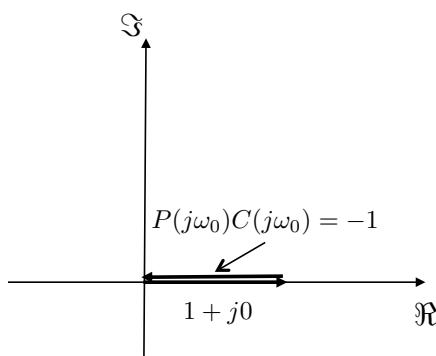


Figure 17-3: The plot of  $1 + P(j\omega_0)C(j\omega_0)$  in the complex plane when  $P(j\omega_0)C(j\omega_0) = -1$ .

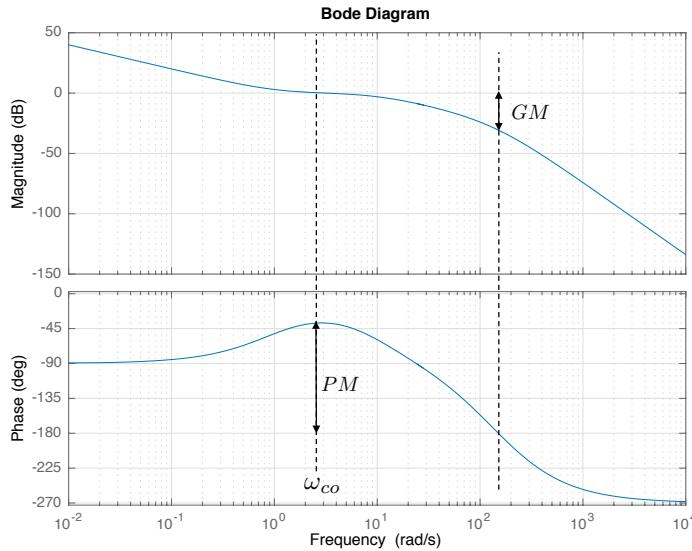


Figure 17-4: The phase margin  $PM$  is the difference between the phase of  $P(j\omega)C(j\omega)$  and  $-180$  degrees at the crossover frequency  $\omega_{co}$ . The gain margin  $GM$  is the magnitude of  $P(j\omega)C(j\omega)$  when the phase crosses  $-180$  degrees.

A related quantity is the *Gain Margin*  $GM$ , which measures how far the magnitude  $|P(j\omega)C(j\omega)|$  is from unity when the phase  $\angle P(j\omega)C(j\omega) = -180$  degrees. The gain margin is depicted on the Bode plot in Fig. 17-4. An alternative view is shown in Fig. 17-6. A good control design will have large phase and gain margins. A phase margin of  $PM = 60$  degrees is considered to be very good and is optimal for many applications. A gain margin of  $GM = 10$  dB would be considered very good for most applications.

### 17.1.2 Open and Closed Loop Frequency Response

From Equation (16.1), the transfer function from the reference  $R(s)$  to the output  $Y(s)$  is

$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}R(s)$$

Note that

- when  $|P(j\omega)C(j\omega)| \gg 1$ , then  $\left| \frac{P(j\omega)C(j\omega)}{1+P(j\omega)C(j\omega)} \right| \approx 1$ ,
- when  $|P(j\omega)C(j\omega)| \ll 1$ , then  $\left| \frac{P(j\omega)C(j\omega)}{1+P(j\omega)C(j\omega)} \right| \approx |P(j\omega)C(j\omega)|$ .

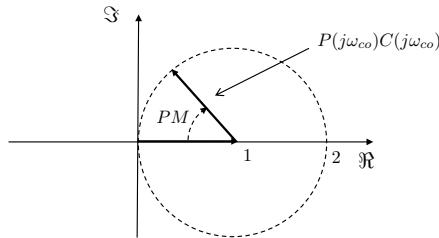


Figure 17-5: Alternative graphical view of the phase margin  $PM$ .

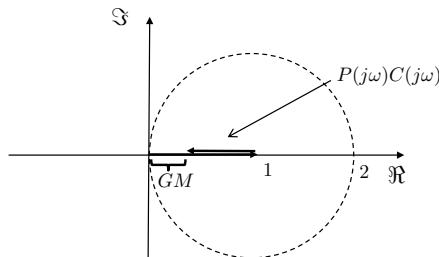


Figure 17-6: Alternative graphical view of the gain margin  $GM$ .

When  $|P(j\omega)C(j\omega)| = 1$ , i.e., when  $\omega = \omega_{co}$ , then  $\left| \frac{P(j\omega_{co})C(j\omega_{co})}{1+P(j\omega_{co})C(j\omega_{co})} \right|$  is a complex number divided by a complex number, and the magnitude will depend on the phase of the  $P(j\omega_{co})C(j\omega_{co})$ , or in other words the phase margin  $PM$ . As shown in Figures 17-5,  $PM$  will determine the magnitude of  $1+P(j\omega_{co})C(j\omega_{co})$ . When  $PM$  is small,  $|1+P(j\omega_{co})C(j\omega_{co})|$  will be small and  $\left| \frac{P(j\omega_{co})C(j\omega_{co})}{1+P(j\omega_{co})C(j\omega_{co})} \right|$  will be larger than one. When  $PM$  is larger than 90 degrees,  $|1+P(j\omega_{co})C(j\omega_{co})|$  will be larger than  $|P(j\omega_{co})C(j\omega_{co})|$  and  $\left| \frac{P(j\omega_{co})C(j\omega_{co})}{1+P(j\omega_{co})C(j\omega_{co})} \right|$  will be less than one.

**Fig. 17-7** shows the closed-loop Bode plot superimposed on the open loop Bode plot for different values of the phase margin. Note that when the phase margin is  $PM = 60$  degrees, the closed-loop frequency response looks very similar to the closed-loop frequency response for a second order system when  $\zeta = 0.707$ , as shown in **Fig. 15-9**. Phase margins smaller than 60 degrees have a peaking response similar to second order system with  $\zeta < 0.707$ . Note also that the bandwidth of the closed-loop system is approximately the crossover frequency of the open loop system. In general, the crossover frequency  $\omega_{co}$  plays a similar role for general systems that the natural frequency  $\omega_n$  plays for second order systems.

**Fig. 17-7** also shows the corresponding step response for the closed-loop system. The step response looks similar to the step response for second order systems shown in **Fig. 8-3**, where again, the step response for  $PM = 60$  degrees in **Fig. 17-7** is roughly equivalent to the step response for  $\zeta = 0.707$  in **Fig. 8-4**.

For a given reference signal, the size of the control signal  $u(t)$  is determined by the crossover frequency. A large crossover frequency will result in a larger

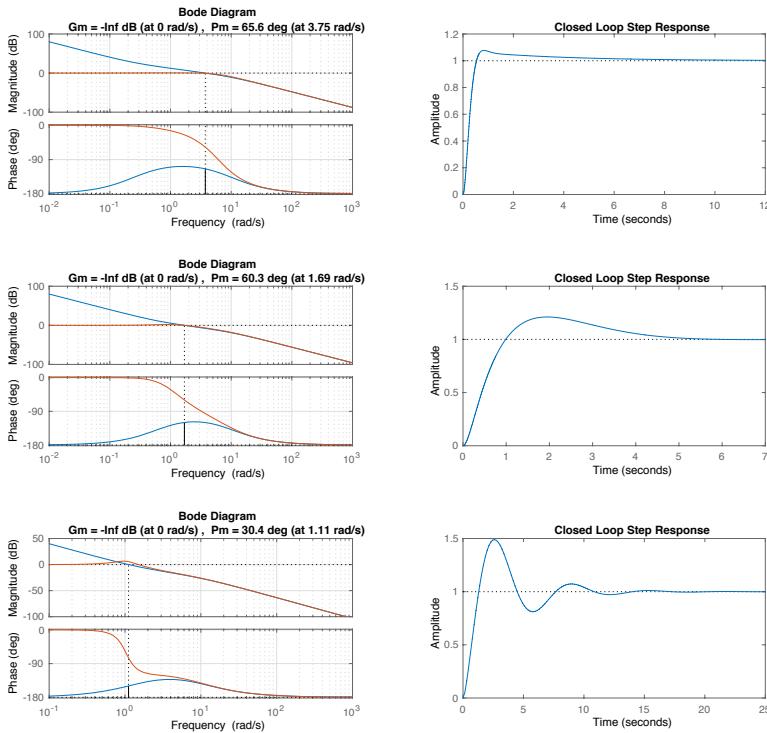


Figure 17-7: closed-loop Bode plot superimposed on the open loop Bode plot, together with the closed-loop step response, for different phase margins.

control signal  $u(t)$ . Fig. 17-8 shows the open and closed-loop transfer functions for a plant equal to  $P(s) = 1/s(s + 1)$  and for three different controllers that are tuned for equivalent phase margins but with cross over frequency at 2, 20, and 200 respectively. The corresponding control signal when the reference is a unit step is also shown in Fig. 17-8. Note that as  $\omega_{co}$  increases, the magnitude of the control signal increases and the speed of the response also increases. In fact, the integral of the control signals will be the same for each  $\omega_{co}$ . Therefore, saturation constraints on  $u(t)$  will necessarily limit the size of the cross over frequency  $\omega_{co}$ .

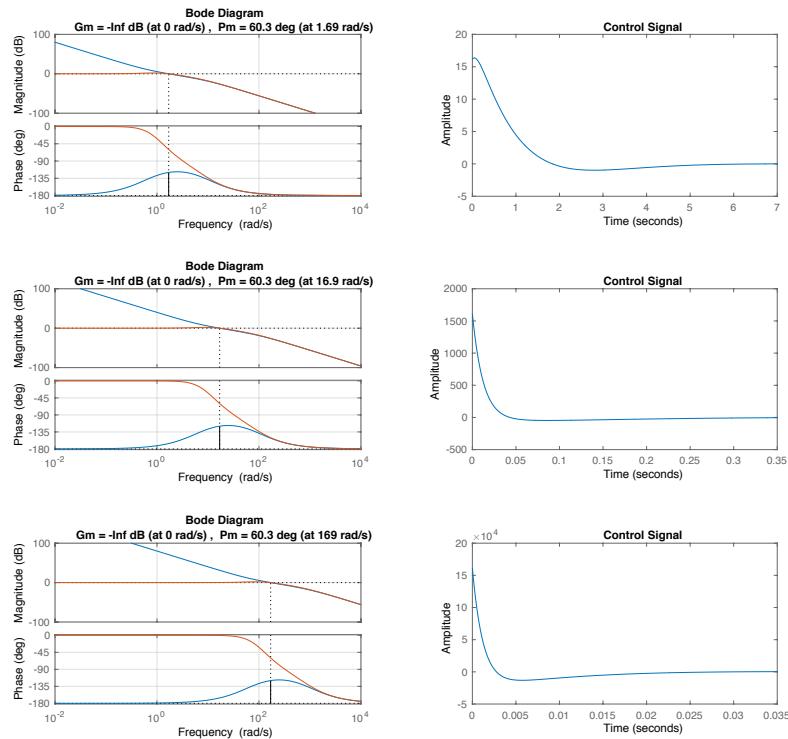


Figure 17-8: closed-loop Bode plot superimposed on the open loop Bode plot, together with the corresponding control signal, for different cross-over frequencies.

### 17.1.3 Bode Phase Gain Relationship

In Bode's original work, he proved the following theorem that relates the slope of the magnitude plot to the phase.

**Bode's Phase-Gain Theorem.** *For any stable, minimum phase (i.e., zeros in the LHP), linear time-invariant system  $G(s)$ , the phase of  $G(j\omega)$  is uniquely related to  $20 \log_{10} |G(j\omega)|$  by*

$$\angle G(j\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{dM}{du} \right) W(u) du, \quad (17.1)$$

where

$$\begin{aligned} M &= 20 \log_{10} |G(j\omega)|, \\ u &= \log_{10} \left| \frac{\omega}{\omega_0} \right|, \\ W(u) &= \log_{10} \left( \coth \frac{|u|}{2} \right). \end{aligned}$$

The term  $\frac{dM}{du}$  in Equation (17.1) is the slope of the magnitude of  $G(j\omega)$  on the Bode plot. A plot of  $W(u)$  is shown in Fig. 17-9, where it can be seen that  $W(u)$  heavily weights the slope of the Bode magnitude around  $\omega_0$ , but also includes bleed-over from the slope for roughly a decade before and after  $\omega_0$ . This implies that the phase of  $P(j\omega)C(j\omega)$  at  $\omega = \omega_0$  is determined by the slope of the Bode magnitude plot in a region about  $\omega_0$ . Recall that the phase for an integrator

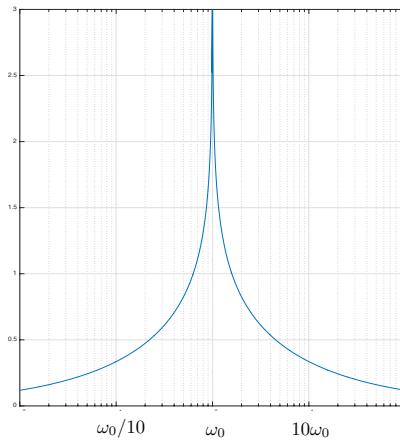


Figure 17-9: The weighting function  $W(u)$  in the Bode gain-phase relationship heavily weights the slope around  $\omega_0$ .

$1/s$  is  $-90$  degrees and that the phase for a differentiator is  $90$  degrees. Also, recall that the phase for a constant is  $0$  degrees. Therefore, we can approximate Equation (17.1) as follows:

- Slope of  $20 \log_{10} |G(j\omega_0)| \approx +20$  dB/dec  
 $\Rightarrow \angle G(j\omega_0) \approx +90$  deg,
- Slope of  $20 \log_{10} |G(j\omega_0)| \approx 0$  dB/dec  
 $\Rightarrow \angle G(j\omega_0) \approx 0$  deg,
- Slope of  $20 \log_{10} |G(j\omega_0)| \approx -20$  dB/dec  
 $\Rightarrow \angle G(j\omega_0) \approx -90$  deg,
- Slope of  $20 \log_{10} |G(j\omega_0)| \approx -40$  dB/dec  
 $\Rightarrow \angle G(j\omega_0) \approx -180$  deg,
- Slope of  $20 \log_{10} |G(j\omega_0)| \approx -60$  dB/dec  
 $\Rightarrow \angle G(j\omega_0) \approx -270$  deg.

Of course, these values are approximate and require that the slope around  $\omega_0$  persist for approximately a decade before and after  $\omega_0$ .

While we do not directly use Equation (17.1) in practice, it has clear implications for feedback design. In particular, the takeaway message is that to have a good phase margin, the slope of the Bode magnitude plot at the crossover frequency  $\omega_{co}$  cannot be too steep, and that it needs to be reasonably shallow for roughly a decade before and after crossover. As a rule of thumb, a phase margin of  $PM = 60$  degrees requires that the slope at crossover is between  $-20$  dB/dec and  $-40$  dB/dec. In other words, the ideal loopshape will need to look something like **Fig. 17-10**. It is also important to understand that the Bode phase-gain theorem requires that there is adequate frequency separation between the frequency content of signals that we are tracking or rejecting and the frequency content of the sensor noise that is to be attenuated.

## 17.2 Design Study A. Single Link Robot Arm



### Example Problem A.17

For the single link robot arm, use the `bode` and `margin` commands (from Python or Matlab) to find the phase and gain margin for the closed loop system under PID control. On the same graph, plot the open loop Bode plot and the closed loop Bode plot. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency? Use the gains found in HW A.10.

### Solution

Python code to generate the plots from this problem is included below. However, similar Matlab code is available on the book website (see <http://controlbook.byu.edu>).

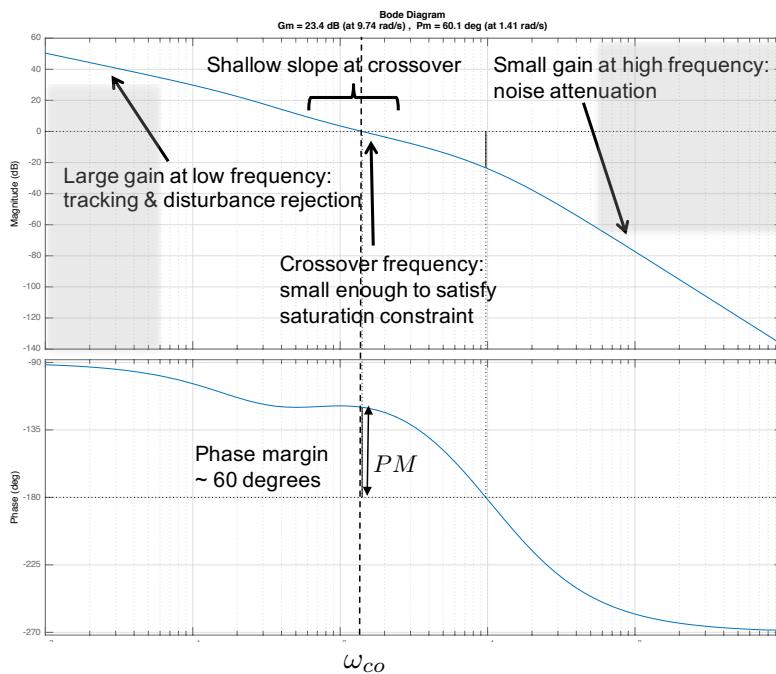


Figure 17-10: The ideal loopshape is large at low frequencies to achieve good tracking and disturbance rejection, is small at high frequencies to attenuate sensor noise, has a shallow slope at the crossover frequency to achieve a good phase margin, and has a small enough crossover frequency to satisfy the input saturation constraints.

```

1 # Single link arm Parameter File
2 import sys
3 sys.path.append('..') # add parent directory
4 sys.path.append('../hw16')
5 import armParamHW16 as P16
6 from control import bode, margin, mag2db
7 import matplotlib.pyplot as plt
8
9 # flag to define if using dB or absolute scale for M(omega)
10 dB_flag = P16.dB_flag
11
12 # assigning plant and controller from past HW (to make sure
13 # we don't introduce additional errors)
14 Plant = P16.Plant
15 C_pid = P16.C_pid
16
17 if __name__=="__main__":
18     # Calculate the phase and gain margins
19     if dB_flag:

```

```

20     gm, pm, Wcg, Wcp = margin(Plant*C_pid)
21     gm = mag2db(gm)
22     print("Inner Loop:", "gm: ", gm,
23           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
24
25 else:
26     gm, pm, Wcg, Wcp = margin(Plant*C_pid)
27     print("Inner Loop:", "gm: ", gm,
28           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
29
30 # display bode plots of transfer functions
31 fig1 = plt.figure()
32
33 # this makes two bode plots for open and closed loop
34 bode(Plant * C_pid, dB=dB_flag,
35       label='C(s)P(s) - Open-loop')
36 bode(Plant*C_pid/(1+Plant*C_pid), dB=dB_flag,
37       label=r'$\frac{P(s)C(s)}{1+P(s)C(s)}$ - Closed-loop')
38
39 # now we can add lines to show where we calculated the GM and PM
40 gm_line = fig1.axes[0].plot([Wcg, Wcg],
41                            plt.ylim(), 'k--', label='GM')
42 gm_line[0].set_label('GM')
43 fig1.axes[0].legend()
44 fig1.axes[1].plot([Wcp, Wcp], plt.ylim(), 'b--', label='PM')
45 plt.legend()
46
47 # setting axis title
48 fig1.axes[0].set_title('Single Link Robot Arm - ' +
49                       ' GM:' + str(round(gm, 2)) +
50                       ', PM:' + str(round(pm, 2)))
51
52 plt.show()

```

Listing 17.1: armParamHW17.py

The transfer function for the plant is defined in Line 14. The transfer function for the PID controller is

$$\begin{aligned}
 C_{PID}(s) &= k_P + \frac{k_I}{s} + \frac{k_D s}{\sigma s + 1} = \frac{s(\sigma s + 1)k_P + (\sigma s + 1)k_I + k_D s^2}{s(\sigma s + 1)} \\
 &= \frac{(k_D + \sigma k_P)s^2 + (k_P + \sigma k_I)s + k_I}{s(\sigma s + 1)},
 \end{aligned}$$

and is defined in Line 15. In Python the `margin` command returns information about the phase margin, gain margin, and frequencies where those are calculated (like the crossover frequency). The `bode` command can be used to make plots of transfer functions (and will show the margin information if sent the "margin=True" argument). The results of the Python code is shown in [Fig. 17-11](#). As seen in [Fig. 17-11](#) the bandwidth for PID control is approximately 18 rad/sec, which is slightly larger than the cross over frequency of 10.8 rad/sec. The larger bandwidth is due to the small phase margin of  $PM = 49.0$  degrees.

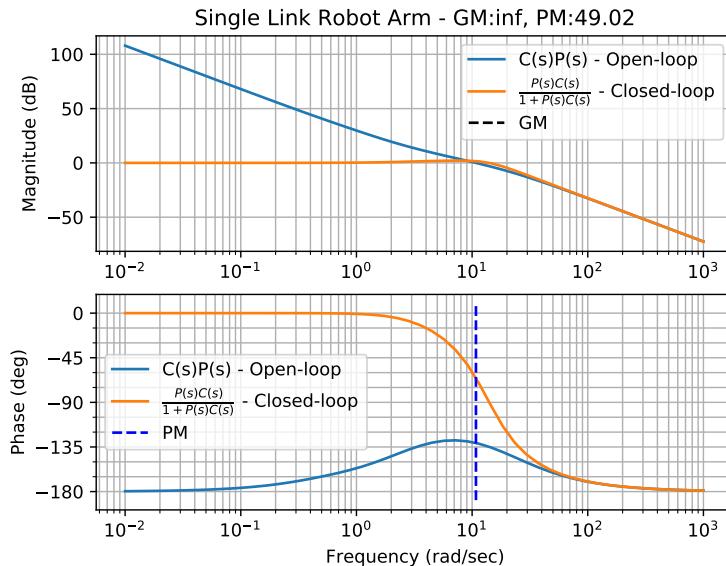


Figure 17-11: Bode plots of the open loop and closed loop system for the single link robot arm under PID control.

### 17.3 Design Study B. Inverted Pendulum



#### Example Problem B.17

For this problem, use the gains found in HW B.10.

- For the inner loop of the inverted pendulum, use the Python or Matlab `bode` and `margin` commands to find the phase and gain margin for the inner loop system under PD control. On the same graph, plot the open loop Bode plot and the closed loop Bode plot. What is the bandwidth of the inner loop, and how does this relate to the crossover frequency?
- For the outer loop of the inverted pendulum, use the `bode` and `margin` commands to find the phase and gain margin for the outer loop system under PID control. Plot the open and closed loop Bode plots for the outer loop on the same plot as the open and closed loop for the inner loop. What is the bandwidth of the outer loop, and how does this relate to the crossover frequency?

- (c) What is the bandwidth separation between the inner (fast) loop, and the outer (slow) loop? For this design, is successive loop closure justified?

## Solution

The Python code that can be used to generate the plots for this solution is shown below.

```

1 # Inverted Pendulum Parameter File
2 import sys
3 sys.path.append('..')
4 sys.path.append('../hw16')
5 import pendulumParam as P
6 import pendulumParamHW16 as P16
7 from control import bode, margin, mag2db
8 import matplotlib.pyplot as plt
9
10 # flag to define if using dB or absolute scale for M(omega)
11 dB_flag = P16.dB_flag
12
13 # Assign inner and outer open-loop transfer functions from
14 # previous HW solution
15 P_in = P16.P_in
16 P_out = P16.P_out
17
18 # Compute inner and outer closed-loop transfer functions
19 C_in = P16.C_in
20 C_out = P16.C_out
21
22 if __name__=="__main__":
23
24 ##### Inner loop #####
25 # Calculate the phase and gain margins
26 if dB_flag:
27     gm, pm, Wcg, Wcp = margin(P_in*C_in)
28     gm = mag2db(gm)
29     print("Inner Loop:", "gm: ", gm,
30           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
31
32 else:
33     gm, pm, Wcg, Wcp = margin(P_in * C_in)
34     print("Inner Loop:", "gm: ", gm,
35           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
36
37 # display bode plots of transfer functions
38 fig1 = plt.figure()
39
40 # this makes two bode plots for open and closed loop
41 bode(P_in * C_in, dB=dB_flag,
42       label='$C_{in}P_{in}$ - Open-loop')
43 bode(P_in*C_in/(1+P_in*C_in), dB=dB_flag,
44       label=r'$\frac{P_{in}C_{in}}{1+P_{in}C_{in}}$'
45             +'- Closed-loop')
46
47

```

```

48 # now we can add lines to show where we calculated the GM and PM
49 gm_line = fig1.axes[0].plot([Wcg, Wcg],
50                             plt.ylim(), 'k--', label='GM')
51 gm_line[0].set_label('GM')
52 fig1.axes[0].legend()
53 fig1.axes[1].plot([Wcp, Wcp], plt.ylim(), 'b--', label='PM')
54 plt.legend()
55
56 # setting axis title
57 fig1.axes[0].set_title('Pendulum Inner Loop - ' +
58                       'GM:' + str(round(gm, 2)) +
59                       ', PM:' + str(round(pm, 2)))
60
61 ##### Outer loop #####
62 # Calculate the phase and gain margins
63 if dB_flag:
64     gm, pm, Wcg, Wcp = margin(P_out * C_out)
65     gm = mag2db(gm)
66     print("Outer Loop:", "gm: ", gm,
67           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
68
69 else:
70     gm, pm, Wcg, Wcp = margin(P_out * C_out)
71     print("Outer Loop:", "gm: ", gm,
72           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
73
74 # display bode plots of transfer functions
75 fig2 = plt.figure()
76
77 # this makes two bode plots for open and closed loop
78 bode(P_out * C_out, dB=dB_flag,
79       label='\$C_{out}P_{out}\$ - Open-loop')
80 bode(P_out*C_out/(1+P_out*C_out), dB=dB_flag,
81       label=r'\frac{P_{out}C_{out}}{1+P_{out}C_{out}}\$' +
82             ' - Closed-loop')
83
84
85 # now we can add lines to show where we calculated the GM and PM
86 gm_line = fig2.axes[0].plot([Wcg, Wcg],
87                             plt.ylim(), 'k--', label='GM')
88 gm_line[0].set_label('GM')
89 fig2.axes[0].legend()
90 fig2.axes[1].plot([Wcp, Wcp], plt.ylim(), 'b--', label='PM')
91 plt.legend()
92
93 # setting axis title
94 fig2.axes[0].set_title('Pendulum Outer Loop - ' +
95                       'GM:' + str(round(gm, 2)) +
96                       ', PM:' + str(round(pm, 2)))
97
98 print('Close window to end program')
99 plt.show()

```

Listing 17.2: pendulumParamHW17.py

The transfer functions for the inner and outer loop plants and controller are defined in Lines 14–19. For this problem, we first calculate the gain and phase

margins (see Lines 26–35). We do this so that we can then plot the open-loop and closed-loop bode plots with stability margin information shown on the plots (see Lines 38–56). If all we want is the open-loop Bode plot with stability margins, we can simply pass the argument “margins=True” to the `bode` function. The results of this code are shown in [Fig. 17-12](#). We follow a similar process for the outer loop with results shown in [Fig. 17-13](#).

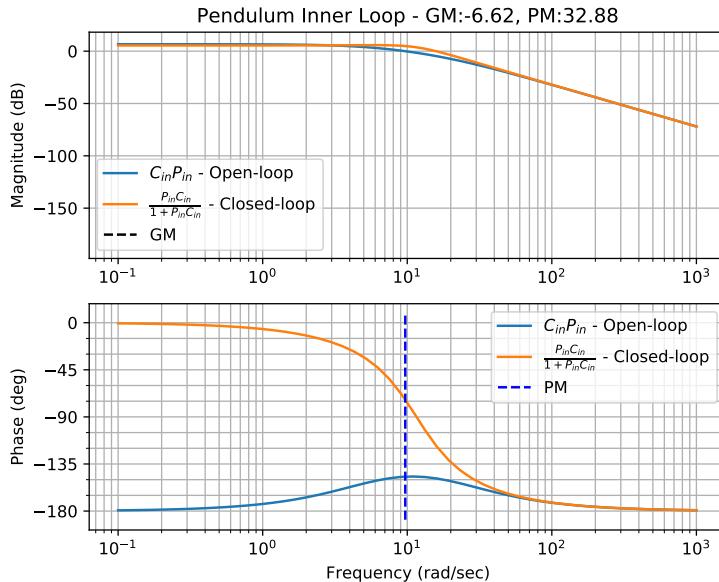


Figure 17-12: Bode plots with stability margins for the open and closed-loop systems of the inner loop of the inverted pendulum system.

As seen from [Fig. 17-12](#), the bandwidth of the inner loop is approximately 19 rad/sec, which is slightly larger than the cross over frequency of 10 rad/sec. Similarly, [Fig. 17-13](#) indicates that the bandwidth of the outer loop is approximately 1.3 rad/sec, which is slightly larger than the cross over frequency of 1.0 rad/sec with a phase margin of  $PM = 72$  degrees (the plot says -108, but this is off by 180, so that  $PM = 180 - 108$ ).

For a second order system, making the step response 10 times faster implies that  $t_r$  is divided by 10, or equivalently that  $\omega_n$  is multiplied by 10. A similar principle holds for high order systems. To be 10 times faster, the closed loop bandwidth of the inner loop should be a decade higher on the Bode plot than the closed loop bandwidth of the outer loop. The bandwidth separation between the inner and outer loop is about one decade justifying the successive loop closure design approach.

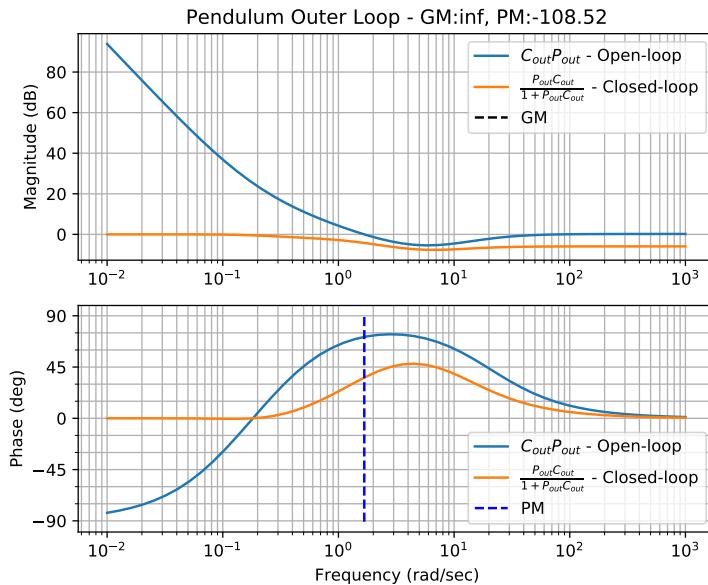


Figure 17-13: Bode plots with stability margins for the open and closed-loop systems of the outer loop of the inverted pendulum system.

## 17.4 Design Study C. Satellite Attitude Control



### Example Problem C.17

For this problem, use the gains found in HW C.10.

- For the inner loop of the satellite attitude controller, use the `bode` and `margin` commands (from Matlab or Python) to find the phase and gain margin for the inner loop system under PD control. On the same graph, plot the open loop Bode plot and the closed loop Bode plot. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency?
- For the outer loop of the satellite attitude controller, use the `bode` and `margin` commands to find the phase and gain margin for the outer loop system under PID control. Plot the open and closed loop Bode plots for the outer loop on the same plot as the open and closed loop for the inner loop.

What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency?

- (c) What is the bandwidth separation between the inner (fast) loop, and the outer (slow) loop? For this design, is successive loop closure justified?

## Solution

Python code to generate the plots in this solutions is included below.

```

1 # Satellite Parameter File
2 import sys
3 sys.path.append('..') # add parent directory
4 import satelliteParam as P
5 sys.path.append('../hw16') # add parent directory
6 import satelliteParamHW16 as P16
7 from control import *
8 import matplotlib.pyplot as plt
9
10 # flag to define if using dB or absolute scale for M(omega)
11 dB_flag = P16.dB_flag
12
13 # Compute inner and outer open-loop transfer functions
14 P_in = P16.P_in
15 P_out = P16.P_out
16
17 # Compute the controller transfer functions from HW10
18 # (to make sure we don't introduce additional errors)
19 C_in = P16.C_in
20 C_out = P16.C_out
21
22 if __name__=="__main__":
23
24     # we have to define the frequencies, or we get non-smooth
25     # bode plots.
26     omegas = np.logspace(-2, 3, 1000)
27
28 ##### Inner loop #####
29 ##### Calculate the phase and gain margins
30 if dB_flag:
31     gm, pm, Wcg, Wcp = margin(P_in * C_in)
32     gm = mag2db(gm)
33     print("Inner Loop:", "gm: ", gm,
34           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
35
36 else:
37     gm, pm, Wcg, Wcp = margin(P_in * C_in)
38     print("Inner Loop:", "gm: ", gm,
39           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
40
41 # display bode plots of transfer functions
42 fig1 = plt.figure()
43
44 # this makes two bode plots for open and closed loop

```

```

46 bode(P_in * C_in, omega=omegas, dB=dB_flag,
47       label='\$C_{in}P_{in}\$ - Open-loop')
48 bode(P_in * C_in / (1 + P_in * C_in),
49       omega=omegas, dB=dB_flag,
50       label=r'\frac{P_{in}C_{in}}{1+P_{in}C_{in}}\$'
51       + ' - Closed-loop')
52
53 # now we can add lines to show where we calculated the GM and PM
54 gm_line = fig1.axes[0].plot([Wcg, Wcg],
55                            plt.ylim(), 'k--', label='GM')
56 gm_line[0].set_label('GM')
57 fig1.axes[0].legend()
58 fig1.axes[1].plot([Wcp, Wcp], plt.ylim(), 'b--', label='PM')
59 plt.legend()
60
61 # setting axis title
62 fig1.axes[0].set_title(r'Satellite Inner Loop ($\theta$) - ' +
63                       'GM:' + str(round(gm, 2)) +
64                       ', PM:' + str(round(pm, 2)))
65
66 ##### Outer loop #####
67 # Calculate the phase and gain margins
68 if dB_flag:
69     gm, pm, Wcg, Wcp = margin(P_out * C_out)
70     gm = mag2db(gm)
71     print("Outer Loop:", "gm: ", gm,
72           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
73
74 else:
75     gm, pm, Wcg, Wcp = margin(P_out * C_out)
76     print("Outer Loop:", "gm: ", gm,
77           " pm: ", pm, " Wcg: ", Wcg, " Wcp: ", Wcp)
78
79 # display bode plots of transfer functions
80 fig2 = plt.figure()
81
82 # this makes two bode plots for open and closed loop
83 bode(P_out * C_out, omega=omegas, dB=dB_flag,
84       label='\$C_{out}P_{out}\$ - Open-loop')
85 bode(P_out * C_out / (1 + P_out * C_out),
86       omega=omegas, dB=dB_flag,
87       label=r'\frac{P_{out}C_{out}}{1+P_{out}C_{out}}\$'
88       + ' - Closed-loop')
89
90 # now we can add lines to show where we calculated the GM and PM
91 gm_line = fig2.axes[0].plot([Wcg, Wcg],
92                            plt.ylim(), 'k--', label='GM')
93 gm_line[0].set_label('GM')
94 fig2.axes[0].legend()
95 fig2.axes[1].plot([Wcp, Wcp], plt.ylim(), 'b--', label='PM')
96 plt.legend()
97
98 # setting axis title
99 fig2.axes[0].set_title(r'Satellite Outer Loop ($\phi$) - ' +
100                      'GM:' + str(round(gm, 2)) +
101                      ', PM:' + str(round(pm, 2)))
102

```

```

103
104     print('Close window to end program')
105     plt.show()

```

Listing 17.3: satelliteParamHW17.py

Similar to the last two case studies, the transfer functions for the inner and outer loop plants are calculated first. Then we find the stability margins and generate bode plots for the inner loop (see [Fig. 17-14](#)). We follow the same procedure for the outer loop plots shown in [Fig. 17-15](#).

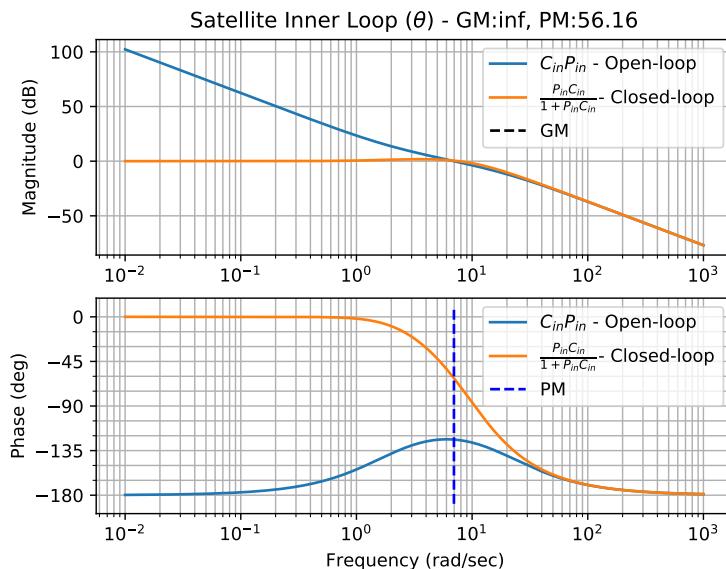


Figure 17-14: Bode plots with stability margins for the open and closed-loop systems of the inner loop of the satellite system.

As seen from [Fig. 17-14](#), the bandwidth of the inner loop is approximately 11 rad/sec, which is slightly larger than the cross over frequency of 7 rad/sec, with a phase margin of  $PM = 56$  degrees. Similarly, [Fig. 17-15](#) indicates that the bandwidth of the outer loop is approximately the cross over frequency of 1 rad/sec. The bandwidth separation between the inner and outer loop is close to a decade and the closed-loop Bode plot of the inner-loop is approximately one for frequencies far beyond the cross over frequency of the outer loop.

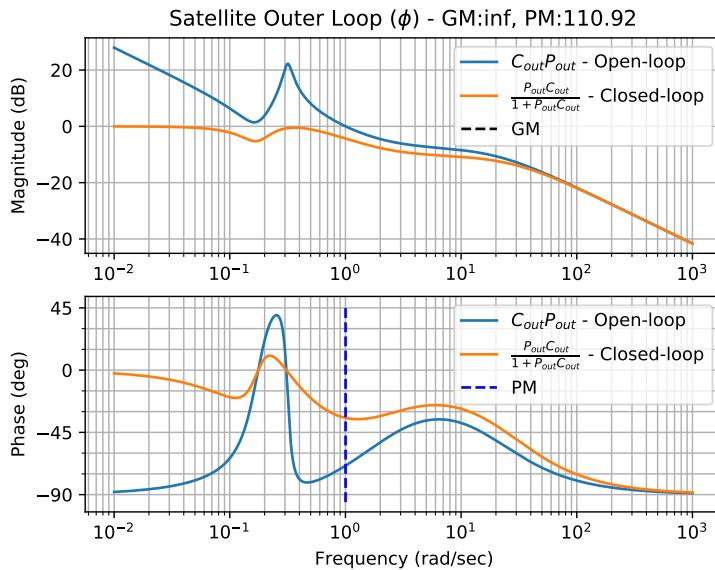


Figure 17-15: Bode plots with stability margins for the open and closed-loop systems of the outer loop of the satellite system.

### Important Concepts:

- The gain/phase margins describe the amount that the gain/phase can be shifted before the system becomes unstable.
- The crossover frequency is the frequency where the gain is equal to unity (zero dB).
- The gain margin measures how far the gain is from unity (zero dB) when the phase equals  $-180$  degrees.
- The phase margin measures how far the phase is from  $-180$  degrees when the gain is at the crossover frequency.
- The bandwidth of a system may be approximated by the crossover frequency. A large crossover frequency will result in a large control input.
- A good phase margin, requires that the slope of the Bode magnitude plot be relatively shallow (between  $-20$ –  $-40$  dB/decade for roughly a decade before and after the crossover frequency).

### Notes and References

# 18

## Compensator Design

### Important Concepts:

- Design controllers that satisfy specified frequency criteria and result in good stability margins.
- Understand how different controller building blocks help to satisfy each design/stability specification.

## 18.1 Theory

The loopshaping design methodology is to add elements to the compensator  $C(s)$  to achieve a loop shape that satisfies the design specifications and that results in good stability margins, ideally  $PM \approx 60$  degrees, all while satisfying input saturation constraints. The idea is that the controller  $C(s)$  will be composed of several building blocks:

$$C(s) = C_1(s)C_2(s) \cdots C_q(s).$$

Since the loop gain is  $P(s)C(s)$ , on the Bode plot we have

$$\begin{aligned} 20 \log_{10} |PC| &= 20 \log_{10} |P| + \sum_{i=1}^q 20 \log_{10} |C_i| \\ \angle PC &= \angle P + \sum_{i=1}^q \angle C_i. \end{aligned}$$

Therefore, the total design can be realized by adding different elements to the Bode plot until the design specifications are satisfied. In this chapter we will discuss the following five basic building blocks for loopshaping design:

- **Proportional Gain.** A proportional gain does not have phase (assuming  $k_P > 0$ ) and only affects the magnitude plot, by moving it up and down on the Bode magnitude plot. A proportional gain is used to affect the location of the cross over frequency.
- **Integral (PI) Control.** An integrator adds  $-20$  dB of slope and  $-90$  degrees of phase at all frequencies. The addition of an integrator changes the system type. A modified PI controller can be used to affect only the low frequency loop gain, thereby enhancing reference tracking and input/output disturbance rejection.
- **Low-pass Filter.** Low-pass filters are used to decrease the loop gain at high frequencies to achieve noise attenuation.
- **phase-lag Filter.** A phase-lag filter can be used to increase the loop gain at low frequencies thereby enhancing reference tracking and input/output disturbance rejection. A phase-lag filter differs from integral control in that it does not change the system type.
- **phase-lead Filter.** A phase-lead filter can be used to add phase at the cross over frequency and can thereby stabilize the system and increase the phase margin.

Each of these basic building blocks will be described in more detail in the sections that follow.

### 18.1.1 Building Block #1: Proportional Gain

The first basic building block for loopshaping design is proportional control where

$$C_P(s) = k_P.$$

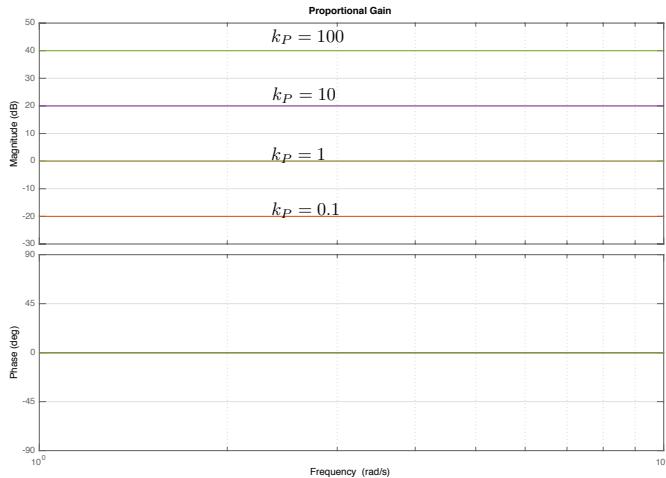
Proportional control is used to change the loop gain at all frequencies, without affecting the phase. A Bode plot for proportional control is shown in [Fig. 18-1](#) for different values of  $k_P$ .

### 18.1.2 Building Block #2: Integral Control

The second basic building block for loopshaping design is proportional-integral (PI) control where  $k_P = 1$ , i.e.,

$$C_{PI}(s) = 1 + \frac{k_I}{s} = \frac{s + k_I}{s}.$$

The Bode plot of  $C_{PI}$  for different values of  $k_I$  is shown in [Fig. 18-2](#). Integral control is used to change the system type by changing the slope of the loop gain by  $-20$  dB/dec at low frequencies. The Bode plot for integral control is shown in [Fig. 18-2](#) for different values of  $k_I$ . It is clear that integral control changes the



**Figure 18-1:** The Bode plot for proportional gain at different values of  $k_P$ . The proportional gain adds a constant to the loopshape at all frequencies and does not affect the phase at any frequency.

slope for frequencies  $\omega < k_I$  while not affecting the loop gain when  $\omega > k_I$ . Note also that integral control decreases the phase at low frequencies, which negatively impacts stability. Therefore, integral control is usually applied at low frequencies, where  $k_I \ll \omega_{co}$ .

### 18.1.3 Building Block #3: Low-Pass Filter

The third basic building block for loopshaping design is a low-pass filter where

$$C_{LPF}(s) = \frac{p}{s + p}.$$

Note that the low-pass filter is constructed so that the DC-gain is one. The Bode plot for a low-pass filter is shown in [Fig. 18-3](#) for different values of  $p$ . It is evident from [Fig. 18-3](#) that a low-pass filter decreases the loop gain when  $\omega > p$  while leaving the loop gain unaffected when  $\omega < p$ . Low-pass filters are used at high frequencies to enhance noise attenuation.

### 18.1.4 Building Block #4: Phase-Lag Filter

The fourth basic building block for loopshaping design is a phase-lag filter where

$$C_{Lag}(s) = \frac{s + z}{s + p}, \quad (18.1)$$

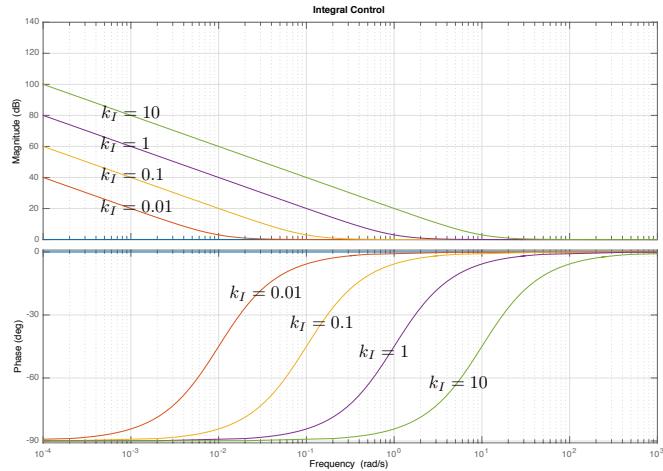


Figure 18-2: The Bode plot for integral control at different values of  $k_I$ . Integral control changes the system type by decreasing the slope of the loopshape by  $-20$  dB/dec for frequencies below  $k_I$ , while leaving high frequencies unaffected.

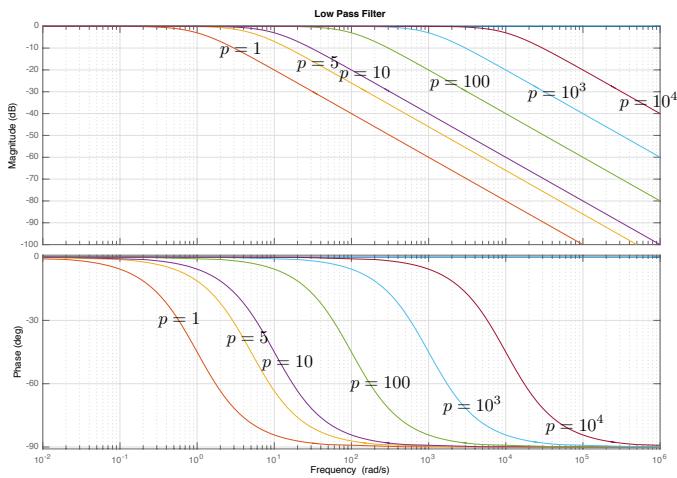


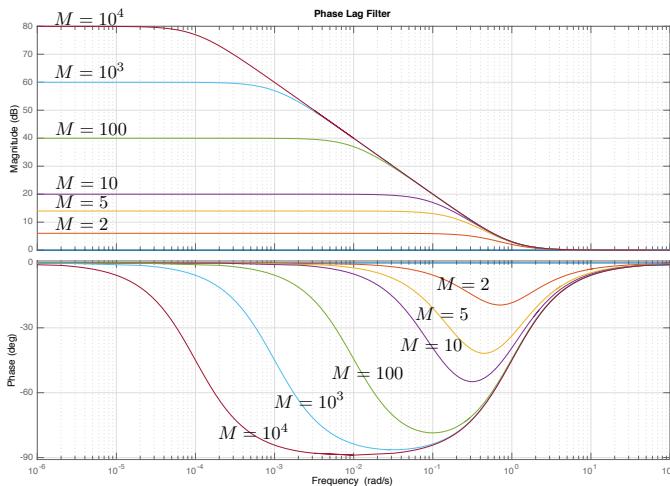
Figure 18-3: The Bode plot for a low-pass filter at different values of the pole  $p$ . A low-pass filter decreasing the slope of the loopshape by  $-20$  dB/dec for frequencies above  $p$ , while leaving low frequencies unaffected.

where  $z > p > 0$ . Note that the phase-lag filter is constructed so that

$$\lim_{s \rightarrow \infty} \frac{s+z}{s+p} = \lim_{s \rightarrow \infty} \frac{1 + \frac{z}{s}}{1 + \frac{p}{s}} = 1.$$

Therefore, phase-lag filters change the loop gain at low frequencies while leaving high frequencies unaffected. The Bode plot for a phase-lag filter is shown in **Fig. 18-4** when  $z = 1$  and when  $p = z/M$  for different values of pole-zero separation  $M$ . Note that phase-lag filters increase the gain at low frequencies and that the total amount of gain is equal to the separation between the zero and the pole. Specifically, the increased gain on the Bode plot is equal to  $20 \log_{10} M$ . Phase-lag filters are used to enhance tracking performance and input/output disturbance rejection. Unfortunately, phase-lag filters decrease the phase between the zero and the pole and will therefore negatively affect stability if the zero is too close to the crossover frequency. Therefore, phase-lag filters are typically added at low frequencies and are characterized by the zero  $z$  where the effect of the lag ends and the pole-zero separation  $M$  which indicates the maximum gain at low frequencies. Accordingly, the lag filter (18.1) can be written as

$$C_{Lag}(s) = \frac{s+z}{s+z/M}.$$



**Figure 18-4:** The Bode plot for a phase-lag filter when the zero is at  $z = 1$  and the pole is at  $p = z/M$ , for different values of  $M$ . A phase-lag filter increases the loop gain at low frequencies while leaving high frequencies unaffected.

### 18.1.5 Building Block #5: phase-lead Filter

The final basic building block for loopshaping design that we will discuss is a phase-lead filter where

$$C_{Lead}(s) = \left(\frac{p}{z}\right) \left(\frac{s+z}{s+p}\right), \quad (18.2)$$

where  $p > z > 0$ . Note that the phase-lag filter is constructed so that

$$\lim_{s \rightarrow 0} \frac{p}{z} \frac{s+z}{s+p} = 1.$$

Therefore, phase-lead filters change the loop gain at high frequencies while leaving low frequencies unaffected. The Bode plot for a phase-lead filter is shown in **Fig. 18-5** when  $z = 1$  and when  $p = Mz$  for different values of pole-zero separation  $M$ . Note that phase-lead filters increase the phase for frequencies between the zero and the pole. It can be shown that the peak of the phase plot shown in **Fig. 18-5** is at

$$\omega_{Lead} = \sqrt{zp}$$

which is the midpoint between  $z$  and  $p$  on the Bode plot. It can also be shown that the increase in phase at  $\omega_{Lead}$  is

$$\phi_{Lead} = \tan^{-1} \sqrt{\frac{p}{z}} - \tan^{-1} \sqrt{\frac{z}{p}}.$$

Phase-lead filters are used to increase the phase margin and thereby enhance the stability and robustness of the system. Typically  $\omega_{Lead}$  is selected to be the cross over frequency  $\omega_{co}$  to get the maximum benefit to the phase margin. Unfortunately, phase-lead filters also increase the loop gain at high frequencies as shown in **Fig. 18-5**, which will make the closed-loop system more sensitive to noise. Therefore, phase-lead filters are typically accompanied by a low-pass filter at higher frequencies. Since phase-lead filters are typically added at the frequency giving the desired bump in phase, it is convenient to write the transfer function (18.2) in terms of the center frequency  $\omega_{Lead}$  and the lead ratio  $M$ . Using the relationship  $p = Mz$  and  $\omega_{Lead} = \sqrt{pz}$  we can write Equation (18.2) as

$$C_{Lead}(s) = M \left( \frac{s + \frac{\omega_{Lead}}{\sqrt{M}}}{s + \omega_{Lead}\sqrt{M}} \right)$$

$$C_{Lead}(s) = \frac{\frac{\sqrt{M}}{\omega_{Lead}} s + 1}{\frac{1}{\omega_{Lead}\sqrt{M}} s + 1}.$$

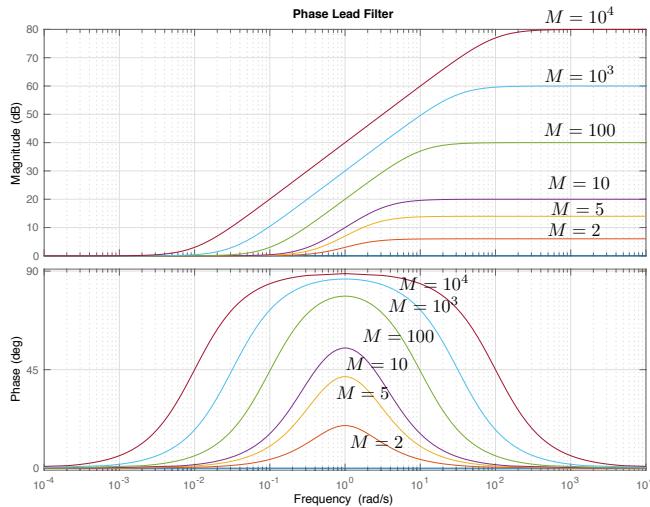


Figure 18-5: The Bode plot for a phase-lead filter when  $\omega_{Lead} = 1$ , for different values of  $M$ . A phase-lead filter increases the phase around  $\omega_{Lead}$ . It also increases the loop gain at high frequencies.

### 18.1.6 Simple Example 1

Given the open loop plant

$$P(s) = \frac{1}{s+1},$$

design a controller  $C(s)$  so that the closed-loop system satisfies the following constraints:

- (1) Tracks a ramp within  $\gamma_1 = 0.03$ .
- (2) Attenuate noise with frequency content above  $\omega_n = 10$  rad/sec by  $\gamma_n = 0.1$ .
- (3) Reject input disturbances with frequencies content below  $\omega_{din} = 0.1$  rad/sec by  $\gamma_{din} = 0.1$ .
- (4) Phase margin that is approximately 60 degrees.

The first three design specifications can be displayed on the Bode magnitude plot as shown in Fig. 18-6. To track a ramp within  $\gamma_1 = 0.03$  requires that the loop gain be above

$$B_1 = 20 \log_{10} \left| \frac{1}{0.03} \right| - 20 \log_{10} |\omega|$$

as  $\omega \rightarrow 0$ . Therefore, to satisfy this specification, we need to find  $C$  so that the Bode plot for  $PC$  rises above the green line on the top left of Fig. 18-6.

To attenuate noise with frequency content above  $\gamma_n = 10 \text{ rad/sec}$  by  $\gamma_n = 0.1$ , the loop gain should be below

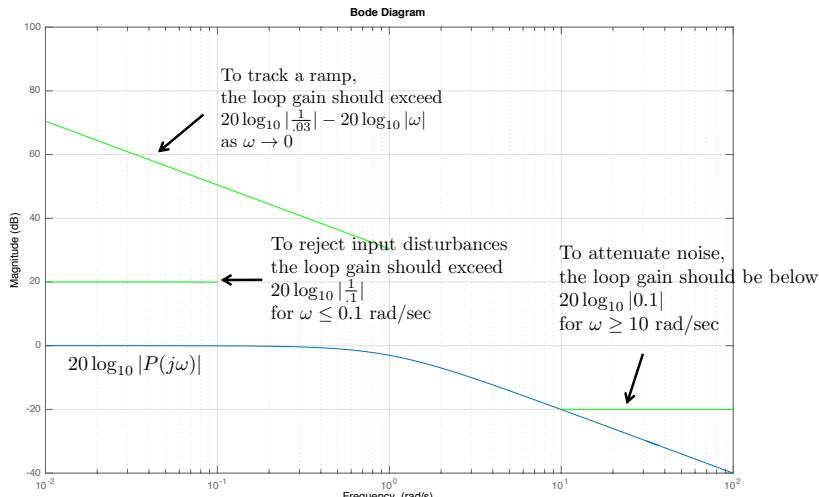
$$B_n = 20 \log_{10} |0.1| = -20 \text{ dB}$$

for  $\omega > 10 \text{ rad/sec}$ . Therefore, we need to design  $C$  so that the Bode plot for  $PC$  is below the green line in the bottom right of [Fig. 18-6](#).

To reject input disturbances with frequency content below  $\omega_{d_{in}} = 0.1 \text{ rad/sec}$  by  $\gamma_{d_{in}} = 0.1$ , requires that the loop gain is above

$$20 \log_{10} \left| \frac{1}{0.1} \right| + 20 \log_{10} |P(j\omega_{d_{in}})| = 20 - 0 = 20 \text{ dB}$$

for  $\omega < 0.1 \text{ rad/sec}$ . Therefore we need to design  $C$  so that the Bode plot for  $PC$  is above the green line in the left-middle of [Fig. 18-6](#).



[Figure 18-6](#): The Bode plot for the open loop plant in Example 1, together with the design specifications.

To increase the type of the system so that the slope of  $PC$  is  $-20 \text{ dB/dec}$  as  $\omega \rightarrow 0$ , integral control can be added. [Fig. 18-7](#) shows the loop gain of  $PC$  when

$$C(s) = C_P(s)C_{PI}(s) = (2) \left( \frac{s + 0.4}{s} \right).$$

To increase the loop gain at low frequencies to get above the  $1/s$  line as  $\omega \rightarrow 0$ , a phase-lag filter can be added. [Fig. 18-8](#) shows the loop gain of  $PC$  when

$$C(s) = C_P(s)C_{PI}(s)C_{Lag}(s) = (2) \left( \frac{s + 0.4}{s} \right) \left( \frac{s + 0.8}{s + 0.8/50} \right).$$

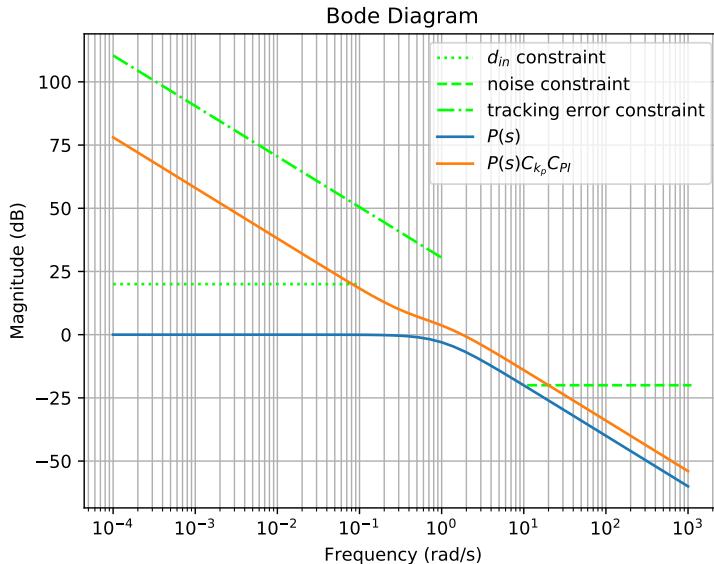


Figure 18-7: The Bode gain plot of  $P$  and  $PC$  when  $C$  includes proportional and integral control.

From Fig. 18-8 we see that the closed-loop system will satisfy the tracking and input disturbance specifications.

To decrease the loop gain at high frequencies to achieve the noise attenuation specification, a low-pass filter can be added at  $p = 5$ . Fig. 18-9 shows the loop gain of  $PC$  when

$$\begin{aligned} C(s) &= C_P(s)C_{P_I}(s)C_{Lag}(s)C_{LPF}(s) \\ &= (2) \left( \frac{s + 0.4}{s} \right) \left( \frac{s + 0.8}{s + 0.8/50} \right) \left( \frac{5}{s + 5} \right). \end{aligned}$$

The phase margin for this design is  $PM = 63$  degrees and so all of the design specifications are satisfied, as shown in Fig. 18-9. Python code that implements this example is shown below.

```

1 from control import *
2 import matplotlib.pyplot as plt
3 import numpy as np
4
5 Plant = tf([1],[1,1]) #transfer function for plant
6
7 # initialize frequency response for plant only
8 plt.figure()
9 plt.grid(which='both')
10

```

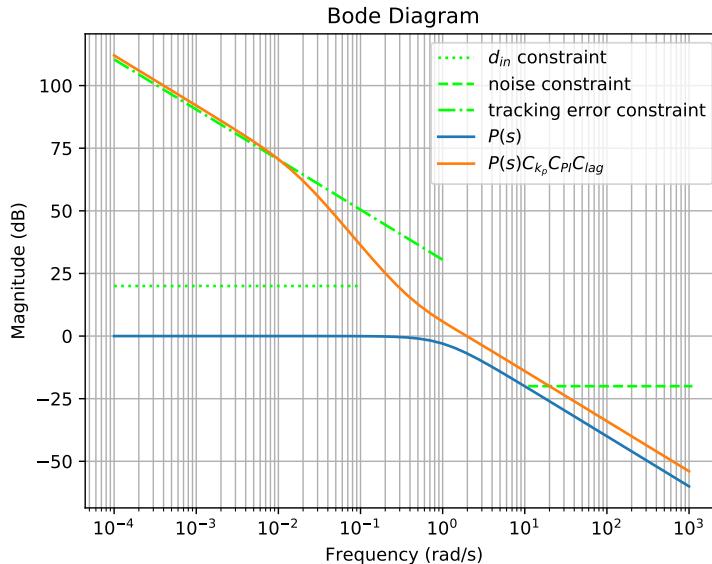


Figure 18-8: The Bode gain plot of  $P$  and  $PC$  when  $C$  includes proportional, integral, and lag filters.

```

11 ######
12 # input disturbance specification #
13 #####
14 # reject dist below this frequency
15 omega_din = 10**-1
16 omega = np.logspace(-4, np.log10(omega_din))
17 Pmag, phase, _ = bode(Plant, omega, Plot=False)
18
19 # amount of input disturbance in output
20 gamma_din = 0.1
21 mag_din = 20*np.log10(1/gamma_din)*np.ones(omega.size)+\
22     20*np.log10(Pmag)
23 plt.semilogx(omega, mag_din, color=[0, 1, 0], ls=':',
24             label='$d_{in}$ constraint')
25
26 #####
27 # noise specification #
28 #####
29 # attenuate noise above this frequency
30 omega_n = 10^1
31
32 #attenuate noise by this amount
33 gamma_n = .1
34 omega = np.logspace(np.log10(omega_n),2+np.log10(omega_n))
35 plt.semilogx(omega, 20*np.log10(gamma_n)*np.ones(omega.size),
36             color=[0, 1, 0], ls='--', label='noise constraint')

```

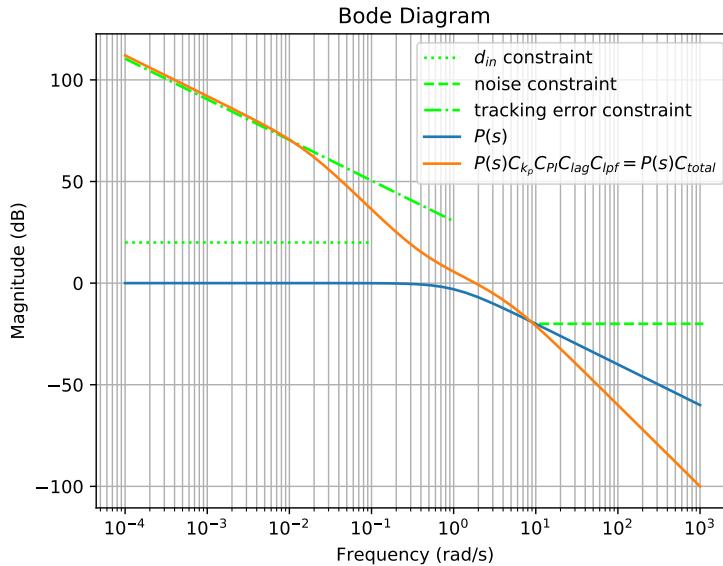


Figure 18-9: The Bode gain plot of  $P$  and  $PC$  when  $C$  includes proportional, integral, lag, and low-pass filters.

```

37 #####
38 # steady state error #
39 ######
40 ######
41 #steady state tracking of ramp
42 gamma_1 = .03
43 omega = np.logspace(-4, 0)
44 plt.semilogx(omega, 20*np.log10(1/gamma_1)-20*np.log10(omega),
45 color=[0, 1, 0], ls='-.',
46 label='tracking error constraint')
47
48 # open-loop freq response for plant only
49 omega = np.logspace(-4, 3)
50 Pmag, phase, omega = bode(Plant,omega, Plot=False)
51 Pmag_dB = 20*np.log10(Pmag)
52 plt.semilogx(omega, Pmag_dB, label='$P(s)$')
53
54 #####
55 # control design #
56 ######
57 C = 1.0
58
59 #proportional control
60 C_kp = 2.0
61
62 #integral control

```

```

63 k_I = 0.4
64 C_PI = tf([1, k_I], [1, 0])
65
66 #phase lag
67 z = 0.8
68 M = 50.0
69 C_lag = tf([1, z], [1, z/M])
70
71 #low-pass filter
72 p = 5
73 C_lpf = tf([p], [1, p])
74
75 #total filter:
76 # for figure 18-7
77 C1 = C*C_kp*C_PI
78
79 # for figure 18-8
80 C2 = C*C_kp*C_PI*C_lag
81
82 # for figure 18-9
83 C3 = C*C_kp*C_PI*C_lag*C_lpf
84
85 # open-loop freq response of the controller + plant
86 omega = np.logspace(-4, 3)
87 PC_mag, phase, _ = bode(Plant*C3, omega, Plot=False)
88 plt.semilogx(omega, 20*np.log10(PC_mag),
89             label='$P(s)C_{k_p}C_{PI}C_{lag}C_{lpf}= P(s)C(s)$')
90
91 plt.title('Bode Diagram')
92 plt.xlabel('Frequency (rad/s)')
93 plt.ylabel('Magnitude (dB)')
94 plt.legend()
95
96 plt.show()

```

Listing 18.1: example\_18\_1\_6.py

### 18.1.7 Controller Implementation

The loopshaping design procedure produces a controller  $C(s)$  that will have multiple elements including integrators, lead and lag filters, and low-pass filters. Each of these elements are straightforward to implement using analog circuits. However, if the controller is to be implemented on digital hardware using computer code, what is the best way to proceed? While there are many possibilities, one option that is straightforward to implement and that fits well with the previous chapters, is to implement the controller using state space equations. The controller

$$U(s) = C(s)E(s)$$

is first converted to continuous time state space equations using, for example, control canonical form, to produce the equations

$$\dot{z}_C = A_C z_C + B_C e \quad (18.3)$$

$$u = C_C z_C + D_C e \quad (18.4)$$

where  $e(t) = r(t) - y(t)$  is the input to the state space equations, and the control signal  $u(t)$  is the output. In the example of the previous section we get

$$C(s) = \frac{10s^2 + 12s + 3.2}{s^3 + 5.016s^2 + 0.08s}.$$

Therefore, using control canonic form, the controller implementation is

$$\begin{aligned}\dot{z}_C &= \begin{pmatrix} -5.016 & -0.08 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} z_C + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e \\ u &= (10 \quad 12 \quad 3.2) z_C.\end{aligned}$$

In Python, the command to find the state space equations from the controller transfer function ‘C’ is given by

```

1 C_ss = tf2ss(C)
2 A_C = C_ss.A
3 B_C = C_ss.B
4 C_C = C_ss.C
5 D_C = C_ss.D

```

The controller is then implemented by integrating the differential equations between sample times. If  $T_s$  is the sample time, and  $N = 10$  Euler steps are used between each sample, then Python code that implements the controller is given by

Python code that could be used for the design of the controller is shown below:

```

1 # assuming that self.z_C (the controller state) has been
2 # initialized elsewhere and that the controller ss equations
3 # are available in the Python class
4 def u = updateController(self, r, y):
5     # calc error signal
6     error = r - y
7
8     # Euler integration steps per control calculation
9     N = 10
10
11    # integrate differential equation for controller
12    for i in range(N):
13        self.z_C = self.z_C + Ts / N * (self.A_C * self.z_C \
14                                         + self.B_C * error)
15    u = self.C_C * self.z_C + self.D_C * error
16
17    return u

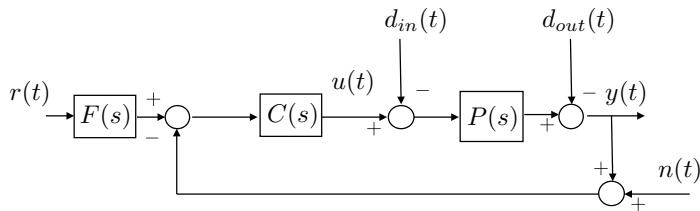
```

Listing 18.2: ss\_control\_example.py

The inputs to the controller are the reference signal  $r$ , the output  $y$ , and the current time  $t$ . The error signal is computed on Line 5. The control dynamics (18.3) are integrated using  $N$  Euler steps in Lines 11–13, and the control output equation is computed in Line 14.

### 18.1.8 Prefilter Design

There are applications where it is very difficult to meet the design specifications at both low and high frequencies and still achieve a phase margin that is close to 60 degrees. If for example, the phase margin is close to 45 degrees, then there will be a peak in the closed-loop Bode plot. As shown in [Fig. 17-7](#), the peak in the closed-loop Bode plot produces overshoot and ringing in the closed-loop step response. One method for reducing the overshoot in the closed-loop step response is to add a prefilter  $F(s)$  as shown in [Fig. 18-10](#). The role of the prefilter  $F(s)$



[Figure 18-10](#): The closed-loop system with a prefilter  $F(s)$  on the reference command  $r$ .

is to attenuate frequencies in  $r(t)$  that resonate with the closed-loop system and produce the overshoot. From a Bode plot perspective, the goal of the prefilter is to flatten the closed-loop Bode plot so that the peak is removed. There are several standard forms used for the prefilter. These include a notch filter and a low-pass filter. For simplicity, we will discuss the use of a low-pass filter below. As explained previously, the transfer function for a low-pass filter is given by

$$F(s) = \frac{p}{s + p}, \quad (18.5)$$

where the DC-gain is equal to one.

Just as the controller  $C(s)$  was implemented using state space equations, the prefilter can also be implemented using state space equations. The prefilter

$$R_f(s) = F(s)R(s)$$

is converted to continuous time state space equations using, for example, control canonical form, to produce the equations

$$\dot{z}_F = A_F z_F + B_F r \quad (18.6)$$

$$r_f = C_F z_F + D_F r \quad (18.7)$$

where the reference  $r(t)$  is the input to the state space equations, and the filtered reference signal  $r_f(t)$  is the output. For the transfer function given in Equation (18.5) we get

$$\begin{aligned}\dot{z}_F &= -pz_F + r \\ r_f &= pz_F.\end{aligned}$$

In Python, the command to find the state space equations are given by (where  $F$  is the transfer function object representing the prefilter)

```
1 F_ss = tf2ss(F)
2 A_F = F_ss.A
3 B_F = F_ss.B
4 C_F = F_ss.C
5 D_F = F_ss.D
```

The prefilter is then implemented by integrating the differential equations between sample times. If  $T_s$  is the sample time, and  $N = 10$  Euler steps are used between each sample, then Python code that implements the controller plus prefilter is given by

```
1 # assuming that self.z_C and self.z_F have been initialized
2 # elsewhere and that the controller AND prefilter ss equations
3 # (e.g. A_C, B_C, C_C, D_C, A_F, B_F, etc.) are available from
4 # the Python class
5
6 def u = updateController(self, r, y):
7
8     # Euler integration steps per filter calculation
9     N = 10
10
11    # integrate differential equation for controller
12    for i in range(N):
13        # prefilter the reference command
14        self.z_F = self.z_F + Ts / N * (self.A_F * self.z_F \
15                                         + self.B_F * r)
16        r_filtered = self.C_F * self.z_F + self.D_F * r
17
18        # error signal uses filtered reference
19        error = r_filtered - y
20        self.z_C = self.z_C + Ts / N * (self.A_C * self.z_C \
21                                         + self.B_C * error)
22
23    u = self.C_C * self.z_C + self.D_C * error
24
25    return u
```

Listing 18.3: ss\_prefilter\_example.py

### Prefilter Example

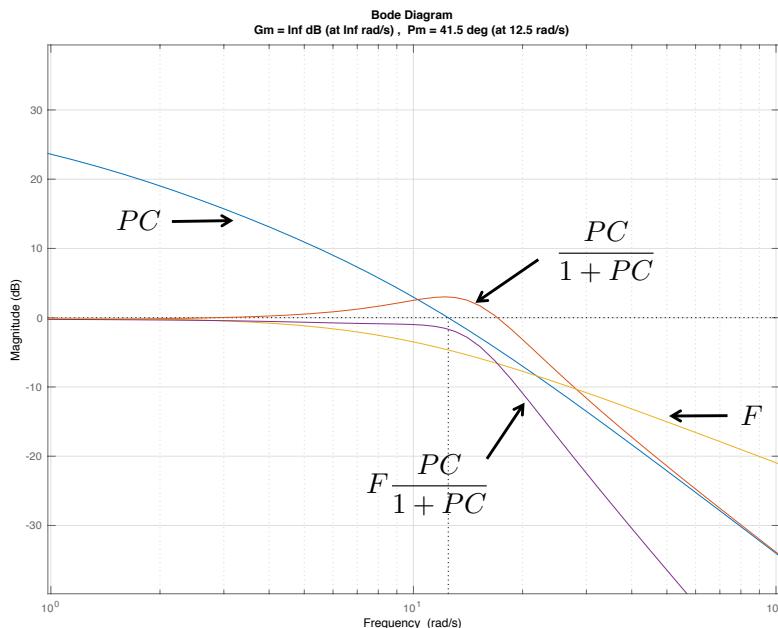
As an example, consider the open loop system  $PC$  shown in [Fig. 18-11](#). The phase margin is  $PM = 42.5$  degrees resulting in a peak in the closed-loop frequency response, as shown in [Fig. 18-11](#). The corresponding step response is

shown in [Fig. 18-12](#), where it can be seen that there is significant ringing in the step response.

To remove the peak in the closed-loop frequency response, a prefilter is added to the system as shown in [Fig. 18-10](#), where the low-pass filter pole is  $p = 9$ , resulting in

$$F(s) = \frac{9}{s + 9}.$$

The Bode magnitude of the prefilter  $F$  is shown in [Fig. 18-11](#), as well as the pre-filtered closed-loop response  $FPC/(1 + PC)$ . As can be seen from [Fig. 18-11](#), the prefilter removes the peak in the closed-loop frequency response. The corresponding step response is shown in [Fig. 18-12](#), where it can be seen that the ringing has been significantly reduced.



[Figure 18-11](#): The loop gain for the open loop system  $PC$ , the closed-loop system  $PC/(1 + PC)$ , the prefilter  $F$ , and the closed-loop system with the prefilter  $FPC/(1 + PC)$ .

### 18.1.9 Successive Loop Closure Design

When loopshaping is used to design an inner and outer loop controller in a successive loop closure design, the design of the outer loop can take into account the complete dynamics of the inner loop. This is in contrast to the design of PID controllers in Chapter 8 where the inner loop was replaced with the DC-gain.

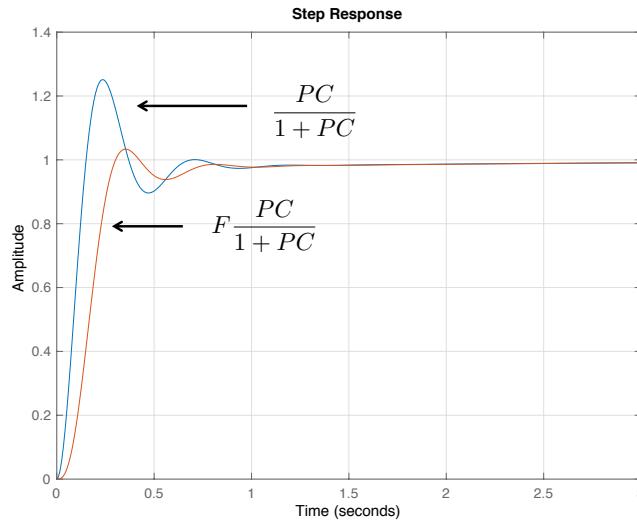


Figure 18-12: The step response of the closed-loop system  $PC/(1+PC)$  and the closed-loop system with the prefilter  $FPC/(1+PC)$ .

**Fig. 18-13** shows the system architecture for loopshaping design. The inner and outer loop controllers are accompanied by the prefilter  $F(s)$  which is only on the outer loop. The first step is to design the inner loop controller  $C_{in}(s)$  using the

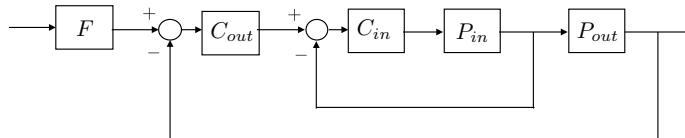


Figure 18-13: The control architecture for a successive loop closure design, with prefilter.

plant  $P_{in}(s)$ . However, for the design of the outer loop controller  $C_{out}(s)$ , we can use the outer loop plant in cascade with the inner closed-loop system. Therefore, the plant for the outer loop design is given by

$$P = P_{out} \left( \frac{P_{in}C_{in}}{1 + P_{in}C_{in}} \right).$$

Examples will be given in the sections below.

## 18.2 Design Study A. Single Link Robot Arm



### Example Problem A.18

For this homework assignment we will use loopshaping to improve the PID controllers developed in HW A.10. Let  $C_{pid}(s)$  be the PID controller designed in HW A.10. The final control will be  $C(s) = C_{pid}(s)C_l(s)$  where  $C_l$  is designed using loopshaping techniques.

- (a) Design  $C_l(s)$  to meet the following objectives:
  - (1) Improve tracking and disturbance rejection by a factor of 10 for reference signals and disturbances below 0.07 rad/sec,
  - (2) Improve noise attention by a factor of 10 for frequencies above 1000 radians/sec.
  - (3) Phase margin that is approximately  $PM = 60$  degrees.
- (b) Add zero mean Gaussian noise with standard deviation  $\sigma^2 = 0.01$  to the simulation developed in HW A.10.
- (c) Implement the controller  $C(s)$  in simulation using its state space equivalent.
- (d) Note that despite having a good phase margin, there is still significant overshoot, due in part to the windup effect in the phase-lag filter. This can be mitigated by adding a prefilter, that essentially modifies the hard step input into the system. Add a low pass filter for  $F(s)$  as a prefilter to flatten out the closed loop Bode response and implement in simulation using its discrete-time state space equivalent.

### Solution

The first two design specifications can be displayed on the Bode magnitude plot as shown in Fig. 18-14. To improve the tracking and disturbance rejection by a factor of 10 for reference signals and disturbances below  $\omega_r = 0.07$  rad/sec, the loop gain must be 20 dB above  $P(s)C_{PID}(s)$ , as shown by the green line in the top left of Fig. 18-14. To improve noise attenuation by a factor of 10 for frequencies above  $\omega_n = 1000$  rad/sec, the loop gain must be 20 dB below  $P(s)C_{PID}(s)$ , as shown by the green line in the bottom right of Fig. 18-14.

To increase the loop gain below  $\omega_r = 0.07$  rad/sec, a phase-lag filter can be added with zero at  $z = 1.5$  with a separation of  $M = 40$ . Fig. 18-15 shows the loop gain of  $PC$  when

$$\begin{aligned} C(s) &= C_{PID}(s)C_{Lag}(s) \\ &= \left( \frac{0.179s^2 + 0.554s + 0.25}{0.05s^2 + s} \right) \left( \frac{s + 1.5}{s + 0.0375} \right). \end{aligned}$$

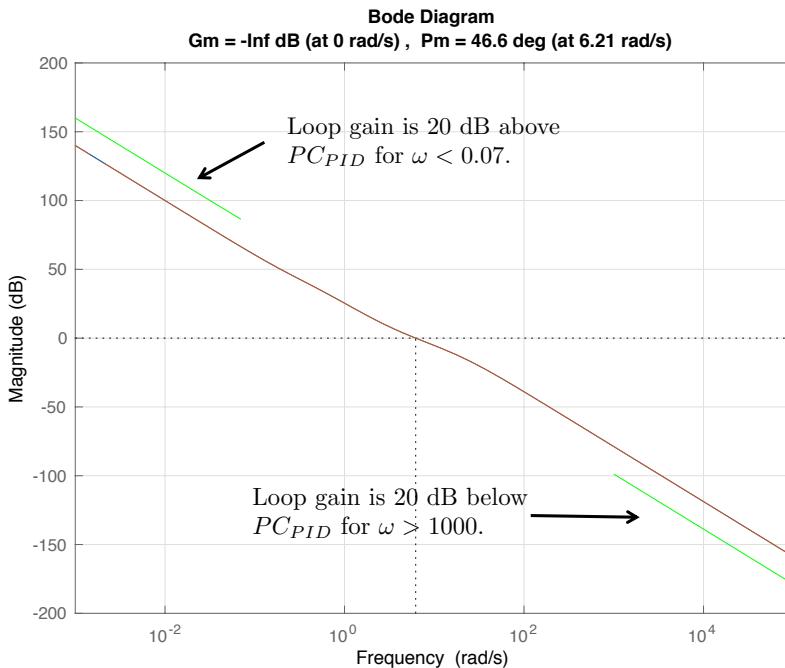


Figure 18-14: The Bode plot for the open loop plant in HW A.18, together with the design specifications.

From Fig. 18-15 we see that the closed loop system will satisfy the tracking and input disturbance specifications.

The phase margin after adding the lag filter is  $PM = 41.3$  degrees. To increase the phase margin, a phase-lead filter is added with center frequency  $\omega_c = 40$  rad/sec and a separation of  $M = 10$ . The center frequency for the lead filter is selected to be above the cross over frequency so as not to change the location of the cross over frequency, thereby keeping the closed loop bandwidth, and therefore the control effort, roughly the same as with the PID controller. After adding the lead filter, the controller is

$$\begin{aligned} C(s) &= C_{PID}(s)C_{Lag}(s)C_{Lead}(s) \\ &= \left( \frac{0.179s^2 + 0.554s + 0.25}{0.05s^2 + s} \right) \left( \frac{s + 1.5}{s + 0.0375} \right) \left( \frac{10s + 126.5}{s + 126.5} \right), \end{aligned}$$

and the corresponding loop gain is shown in Fig. 18-16.

In order to satisfy the noise attenuation specification, we start by adding a low pass filter at  $p = 50$ . The corresponding loop gain is shown in Fig. 18-17, from which we see that the noise specification is not yet satisfied. Therefore, we add an additional low pass filter at  $p = 150$  to obtain the loop gain in Fig. 18-18. The

$G_m = -21.37 \text{ dB}$  (at  $2.01 \text{ rad/s}$ ),  $P_m = 41.28 \text{ deg}$  (at  $10.87 \text{ rad/s}$ )

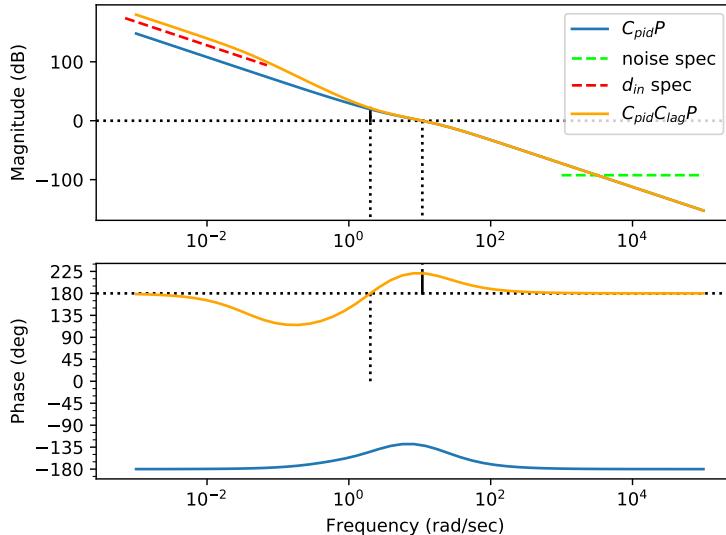


Figure 18-15: The Bode plot for HW A.18 where  $C = C_{PID}C_{lag}$ .

phase margin is  $PM = 64$  degrees. The corresponding controller is

$$\begin{aligned} C(s) &= C_{PID}(s)C_{Lag}(s)C_{Lead}(s)C_{lpf}(s)C_{lpf2}(s) \\ &= \left( \frac{0.179s^2 + 0.554s + 0.25}{0.05s^2 + s} \right) \left( \frac{s + 1.5}{s + 0.0375} \right) \left( \frac{s + 0.7}{s + 0.07} \right) \\ &\quad \cdot \left( \frac{10s + 126.5}{s + 126.5} \right) \left( \frac{50}{s + 50} \right) \left( \frac{150}{s + 150} \right). \end{aligned}$$

Note that we have selected the filter values to leave the cross over frequency the same as with the original PID controller.

The closed loop frequency response  $PC/(1 + PC)$  and the corresponding step response and control effort are shown by the blue lines in Fig. 18-19. The overshoot in the step response is caused by the small amount of peaking on the closed loop bode plot. The peaking can be reduced by adding a prefilter, which in this case is a low pass filter with pole  $p = 3$ . The prefiltered closed loop frequency response  $FPC/(1 + PC)$  and the corresponding step response and control effort are shown by the red lines in Fig. 18-19. As shown by Fig. 18-19, the prefilter reduces the overshoot and lowers the control effort.

Python code that could be used for the design of the controller is shown below:

```

1 import sys
2 sys.path.append('..') # add parent directory
3 sys.path.append('../hw16') # add parent directory

```

$G_m = -24.10 \text{ dB}$  (at  $1.73 \text{ rad/s}$ ),  $P_m = 81.50 \text{ deg}$  (at  $15.00 \text{ rad/s}$ )

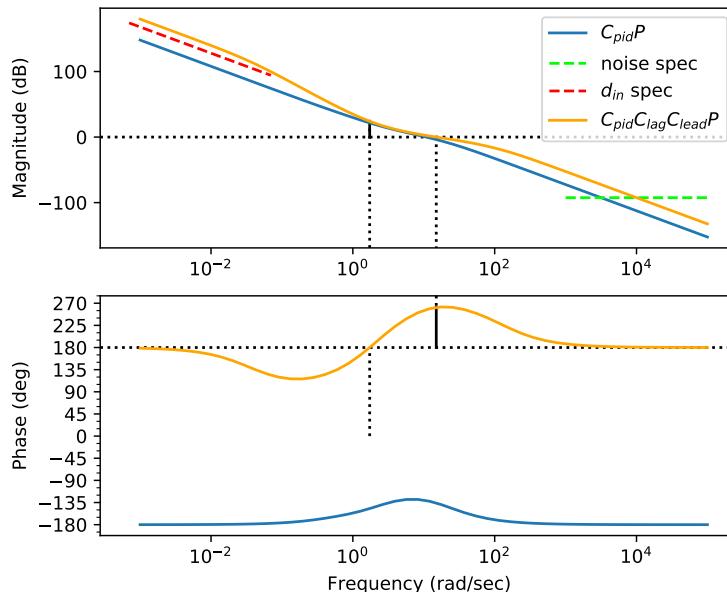


Figure 18-16: The Bode plot for HW A.18 where  $C = C_{PID}C_{lag}C_{lead}$ .

```

4 import armParamHW16 as P16
5 import matplotlib.pyplot as plt
6 from control import tf, bode, margin, step_response, mag2db
7 import numpy as np
8 import loopshape_tools as ls
9
10 # flag to define if using dB or absolute scale for M(omega)
11 dB_flag = P16.dB_flag
12
13 # assigning plant and controller from past HW
14 # (to make sure we don't introduce additional errors)
15 Plant = P16.Plant
16 C_pid = P16.C_pid
17
18 ##### Control Design #####
19 #   Control Design
20 #####
21 C = C_pid \
22     * ls.lead(w_L=10.0, M=3.0) \
23     * ls.lag(z=1.0, M=40.0) \
24     * ls.lpf(p=50.0) \
25     #* ls.lpf(p=150.0)
26
27 ##### add a prefilter to eliminate the overshoot #####
28 # add a prefilter to eliminate the overshoot

```

$G_m = 20.30 \text{ dB}$  (at  $82.99 \text{ rad/s}$ ),  $P_m = 65.09 \text{ deg}$  (at  $14.31 \text{ rad/s}$ )

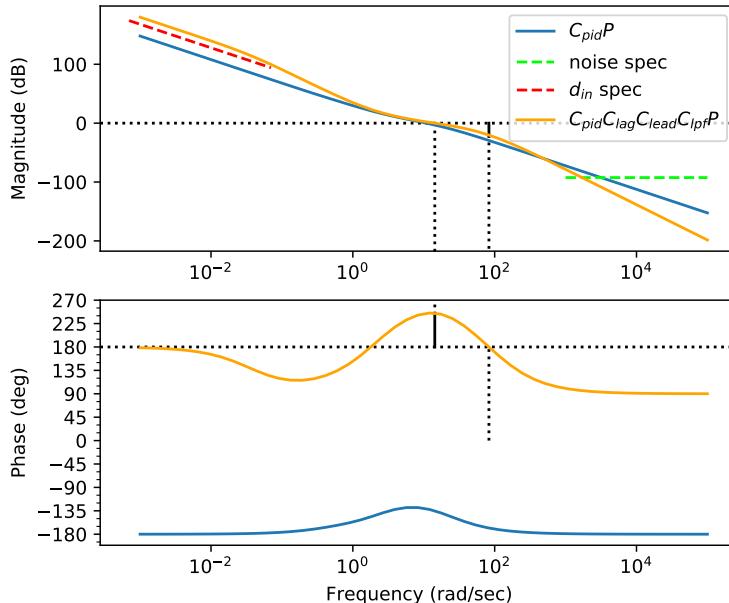


Figure 18-17: The Bode plot for HW A.18 where  $C = C_{PID}C_{lag}C_{lead}C_{lpf}$ .

```

29 #####
30 F = tf(1, 1) \
31     * ls.lpf(3.0)
32 #* ls.notch(p1=3.0, p2=50.0, M=10)
33
34
35 #####
36 # Convert Controller to State Space Equations if following
37 # method in 18.1.7
38 #####
39 # C_ss = tf2ss(C)    # convert to state space
40 # F_ss = tf2ss(F)    # convert to state space
41 C_num = np.asarray(C.num[0])
42 C_den = np.asarray(C.den[0])
43 F_num = np.asarray(F.num[0])
44 F_den = np.asarray(F.den[0])
45
46
47 if __name__=="__main__":
48     # calculate bode plot and gain and phase margin
49     # for original PID * plant dynamics
50     mag, phase, omega = bode(Plant * C_pid, dB=dB_flag,
51                               omega=np.logspace(-3, 5),
52                               plot=True, label="$C_{(pid)}P$")

```

$G_m = 14.69 \text{ dB}$  (at  $56.29 \text{ rad/s}$ ),  $P_m = 59.69 \text{ deg}$  (at  $14.24 \text{ rad/s}$ )

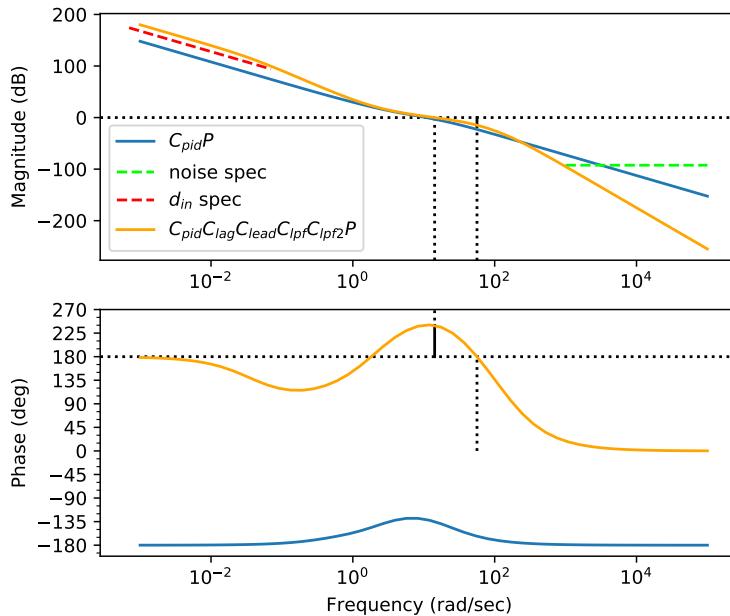


Figure 18-18: The Bode plot for HW A.18 where  $C = C_{PID}C_{lag}C_{lead}C_{lpf}C_{lpf2}$ .

```

53
54
55 gm, pm, Wcg, Wcp = margin(Plant * C_pid)
56 print("for original C_pid system:")
57 if dB_flag is True:
58     print(" pm: ", pm, " Wcp: ", Wcp,
59           "gm: ", mag2db(gm), " Wcg: ", Wcg)
60 elif dB_flag is False:
61     print(" pm: ", pm, " Wcp: ", Wcp,
62           "gm: ", gm, " Wcg: ", Wcg)
63
64 #####
65 # Define Design Specifications
66 #####----- noise specification -----#
67 omega_n = 1000    # attenuate noise above this frequency
68 mag, phase, omega = bode(Plant * C_pid, dB=dB_flag,
69                           omega=[omega_n])
70 gamma_n = 0.1*mag[0]    # attenuate noise by this amount
71 ls.spec_noise(gamma_n, omega_n, dB_flag)
72 #####----- general tracking specification -----#
73
74
75

```

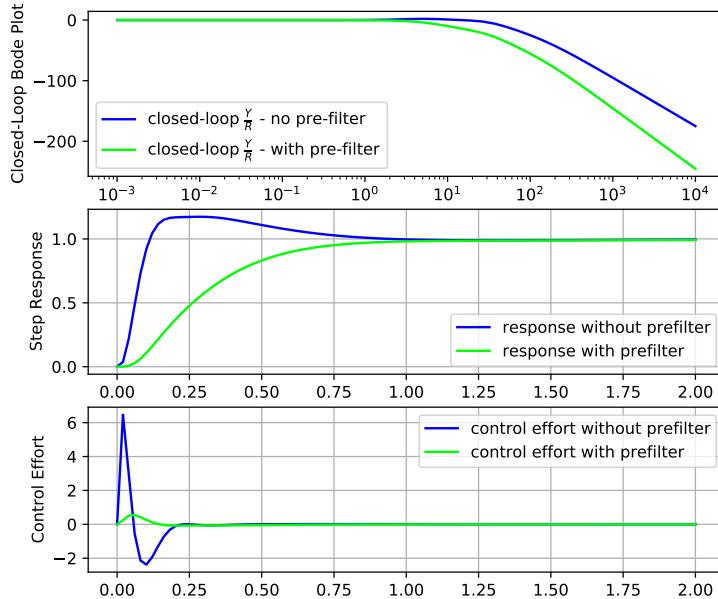


Figure 18-19: On the left is the closed loop frequency response in blue, and the prefiltered closed loop frequency response in red. In the middle is the associated closed-loop step response. On the right is the associated control effort.

```

76 omega_d = 0.07 # track signals below this frequency
77 gamma_d = 0.1 # tracking improvement over original system
78 ls.spec_input_disturbance(gamma_d, omega_d,
79     Plant*C_pid, dB_flag)
80
81 ## plot the effect of adding the new compensator terms
82 mag, phase, omega = bode(Plant * C, dB=dB_flag,
83     omega=np.logspace(-3, 5),
84     plot=True, label="$C_{final}(s)P(s)$",
85     color='orange')
86
87 gm, pm, Wcg, Wcp = margin(Plant * C)
88 print("for final C*P:")
89 if dB_flag is True:
90     print(" pm: ", pm, " Wcp: ", Wcp,
91           "gm: ", mag2db(gm), " Wcg: ", Wcg)
92 elif dB_flag is False:
93     print(" pm: ", pm, " Wcp: ", Wcp,
94           "gm: ", gm, " Wcg: ", Wcg)
95
96 fig = plt.gcf()

```

```

97     fig.axes[0].legend()
98     #plt.show()
99
100
101 ######
102 # now check the closed-loop response with prefilter
103 #####
104 # Closed loop transfer function from R to Y - no prefilter
105 CLOSED_R_to_Y = (Plant * C / (1.0 + Plant * C))
106 # Closed loop transfer function from R to Y - with prefilter
107 CLOSED_R_to_Y_with_F = (F * Plant * C / (1.0 + Plant * C))
108 # Closed loop transfer function from R to U - no prefilter
109 CLOSED_R_to_U = (C / (1.0 + Plant * C))
110 # Closed loop transfer function from R to U - with prefilter
111 CLOSED_R_to_U_with_F = (F*C / (1.0 + Plant * C))
112
113 plt.figure(4)
114 plt.clf()
115 plt.grid(True)
116 plt.subplot(311)
117 mag, phase, omega = bode(CLOSED_R_to_Y, dB=dB_flag, plot=False)
118 if dB_flag:
119     plt.semilogx(omega, mag2db(mag), color=[0, 0, 1],
120                  label='closed-loop $\frac{Y}{R}$ - no pre-filter')
121 else:
122     plt.loglog(omega, mag, color=[0, 0, 1],
123                label='closed-loop $\frac{Y}{R}$ - no pre-filter')
124 mag, phase, omega = bode(CLOSED_R_to_Y_with_F,
125                           dB=dB_flag, plot=False)
126 if dB_flag:
127     plt.semilogx(omega, mag2db(mag), color=[0, 1, 0],
128                  label='closed-loop $\frac{Y}{R}$ - with pre-filter')
129 else:
130     plt.loglog(omega, mag, color=[0, 1, 0],
131                 label='closed-loop $\frac{Y}{R}$ - with pre-filter')
132 plt.ylabel('Closed-Loop Bode Plot')
133 plt.grid(True)
134 plt.legend()
135
136 plt.subplot(312), plt.grid(True)
137 T = np.linspace(0, 2, 100)
138 _, yout_no_F = step_response(CLOSED_R_to_Y, T)
139 _, yout_F = step_response(CLOSED_R_to_Y_with_F, T)
140 plt.plot(T, yout_no_F, color=[0,0,1],
141           label='response without prefilter')
142 plt.plot(T, yout_F, color=[0,1,0],
143           label='response with prefilter')
144 plt.legend()
145 plt.ylabel('Step Response')
146
147
148 plt.subplot(313)
149 plt.grid(True)
150 _, Uout = step_response(CLOSED_R_to_U, T)
151 _, Uout_F = step_response(CLOSED_R_to_U_with_F, T)
152 plt.plot(T, Uout, color=[0, 0, 1],
153           label='control effort without prefilter')

```

```
154     plt.plot(T, Uout_F, color=[0, 1, 0],  
155                 label='control effort with prefilter')  
156     plt.ylabel('Control Effort')  
157     plt.legend()  
158  
159     plt.show()
```

Listing 18.4: loopshape\_arm.py

### 18.3 Design Study B. Inverted Pendulum



#### Example Problem B.18

For this homework assignment we will use the loopshaping design technique to design a successive loop closure controller for the inverted pendulum.

- (a) First consider the inner loop, where  $P_{in}(s)$  is the transfer function of the inner loop derived in HW B.5. Using a proportional and phase-lead controller, design  $C_{in}(s)$  to stabilize the system with a phase margin close to 60 deg, and to ensure that noise above  $\omega_{no} = 200$  rad/s is rejected by  $\gamma_{no} = 0.1$ . We want the closed-loop bandwidth of the inner loop to be approximately 40 rad/s. Note that since the gain on  $P_{in}(s)$  is negative, the proportional gain will also need to be negative.
- (b) Now consider the design of the controller for the outer loop system. The 'plant' for the design of the outer loop controller is

$$P = P_{out} \frac{P_{in} C_{in}}{1 + P_{in} C_{in}}.$$

Design the outer loop controller  $C_{out}$  so that the system is stable with phase margin close to  $PM = 60$  degrees, and so that reference signals with frequency below  $\omega_r = 0.0032$  radians/sec are tracked with error  $\gamma_r = 0.00001$ , and noise with frequency content above  $\omega_{no} = 1000$  radians/sec are rejected with  $\gamma_{no} = 0.0001$ . Design the crossover frequency so that the rise time of the system is about 2 sec.

#### Solution

**Fig. 18-20** shows the Bode plot of the plant  $P_{in}(s)$  together with the design specification on the noise attenuation.

The first step is to add a negative proportional gain of 1 to account for the negative sign in the plant transfer function. **Fig. 18-21** shows the corresponding Bode plot for a proportional gain of  $K = -1$ . Once this is applied the phase of the plant alone is  $-180$  deg over all frequencies.

We want the bandwidth of the inner loop to be approximately 40 rad/s and so we will set the crossover frequency to be 40 rad/s. To stabilize the system, we will add lead compensation centered at 40 rad/s with a lead ratio of 13.9 that corresponds to the desired phase margin of 60 deg. **Fig. 18-22** shows the addition of the phase-lead filter

$$C_{lead} = -155.12 \left( \frac{\frac{s}{10.72} + 1}{\frac{s}{120} + 1} \right),$$

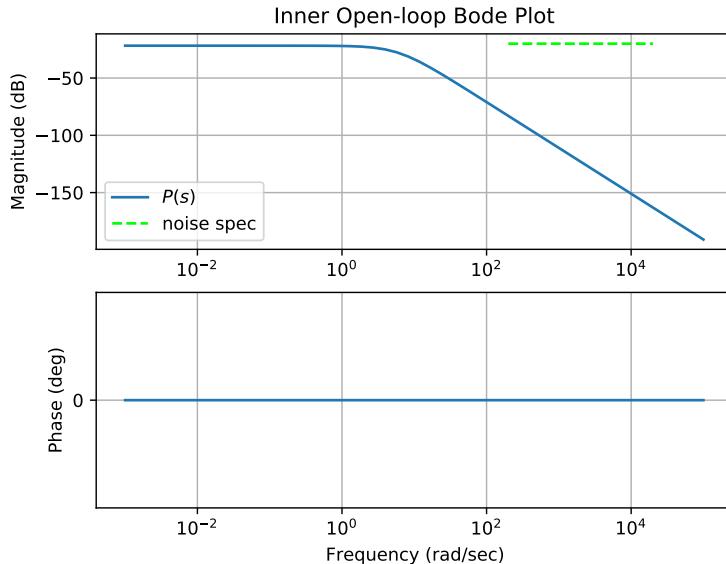


Figure 18-20: The Bode plot for the inner loop plant in HW B.18, together with the design specification.

which has a phase margin of  $PM = 60$  degrees. The gain 155.12 sets the crossover frequency to be 40 rad/s. The lead compensation alone also satisfies the noise specification.

Notice that the DC-gain is not equal to one. The closed-loop magnitude response for the inner-loop subsystem

$$\frac{P_{in}C_{in}}{1 + P_{in}C_{in}},$$

as well as the unit step response for the output and control signal are all shown in Fig. 18-23.

The Python code used to design the inner loop is shown below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 sys.path.append('../hw16') # add parent directory
4 import pendulumParam as P
5 import pendulumParamHW16 as P16
6 import matplotlib.pyplot as plt
7 from control import tf, step_response, bode, tf2ss, margin, mag2db
8 import numpy as np
9 import helper_functions as hf
10
11
12 # flag to define if using dB or absolute scale for M(omega)

```

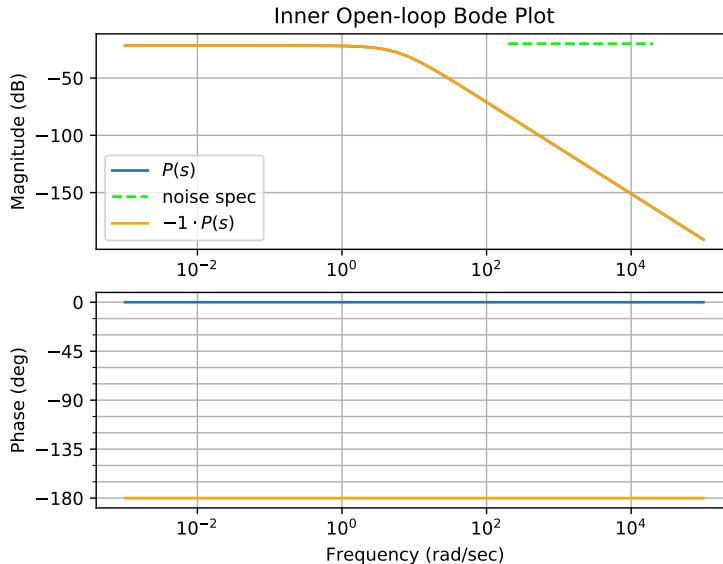


Figure 18-21: The Bode plot for the inner loop plant in HW B.18, with a gain of  $-1$  applied.

```

13 dB_flag = P16.dB_flag
14
15 # assigning plant and controller from past HW
16 # (to make sure we don't introduce additional errors)
17 P_in = P16.P_in
18 Plant = P_in
19
20 ##### Control Design #####
21 #   Control Design
22 #####
23 C = tf([1], [1])
24
25 # Proportional control: correct for negative sign in plant
26 K_neg = hf.get_control_proportional(-1)
27 C = C*K_neg
28
29 # phase lead: increase PM (stability)
30 w_max = 40 #location of maximum frequency bump
31 phi_max = 60*np.pi/180
32 M = (1 + np.sin(phi_max))/(1 - np.sin(phi_max)) # lead ratio
33 Lead = hf.get_control_lead(w_max, M)
34 C = C*Lead
35
36 # find gain to set crossover at w_max
37 mag, phase, omega = bode(Plant*C*Lead, dB=False,
38                           omega=[w_max], plot=False)

```

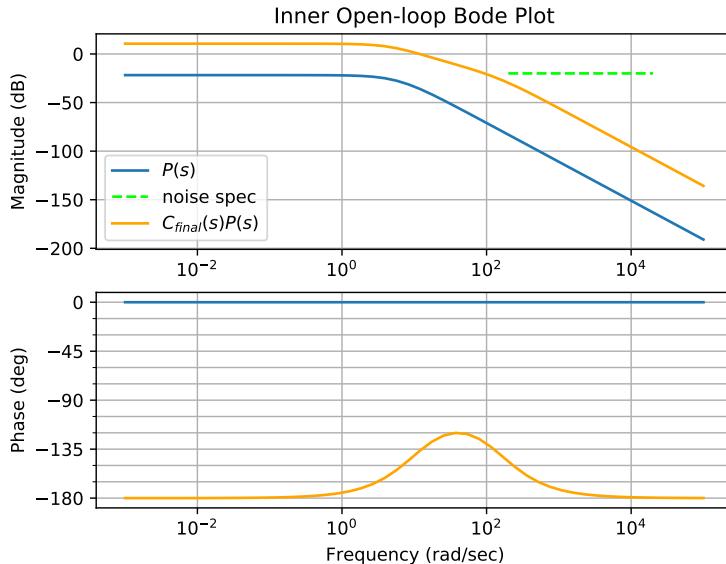


Figure 18-22: The Bode plot for the inner loop system in HW B.18, with proportional gain and phase-lead compensation.

```

39
40 # Proportional control: correct for negative sign in plant
41 K = hf.get_control_proportional(1/mag[0])
42 C = C*K
43
44 ######
45 # Convert Controller to State Space Equations
46 #####
47 C_ss = tf2ss(C) # convert to state space
48
49
50 if __name__=="__main__":
51     # calculate bode plot and gain and phase margin
52     # for original PID * plant dynamics
53     mag, phase, omega = bode(Plant, dB=dB_flag,
54                               omega=np.logspace(-3, 5),
55                               plot=True, label="$P(s)$")
56
57 gm, pm, Wcg, Wcp = margin(Plant)
58 print("for original system:")
59 if dB_flag == True:
60     print(" pm: ", pm, " Wcp: ", Wcp,
61           "gm: ", mag2db(gm), " Wcg: ", Wcg)
62 elif dB_flag == False:
63     print(" pm: ", pm, " Wcp: ", Wcp,
64           "gm: ", gm, " Wcg: ", Wcg)

```

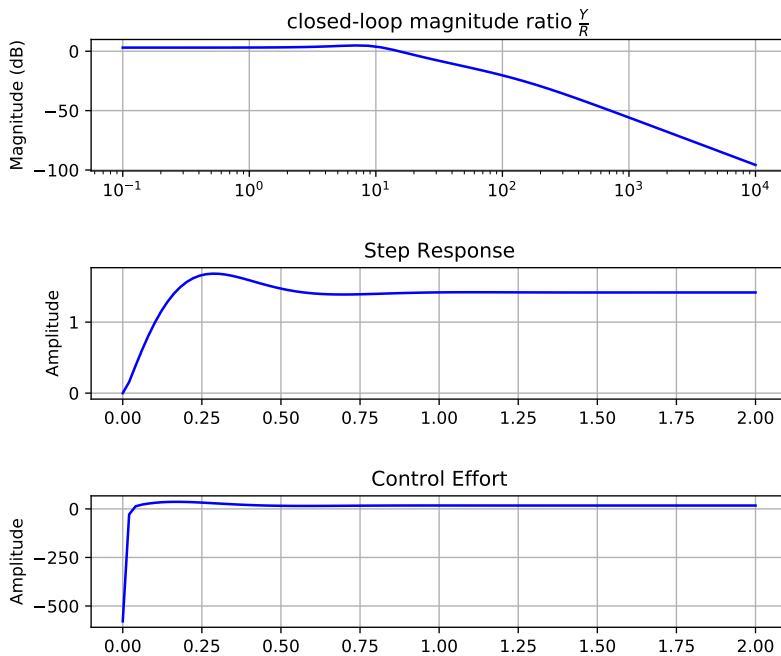


Figure 18-23: The closed loop bode response, the unit step response for the output, and the unit step response for the input of the inner loop design in HW B.18.

```

65
66
67 ##### Define Design Specifications #####
68 # Define Design Specifications
69 ##### noise specification #####
70 # ----- noise specification -----
71 omega_n = 200 # attenuate noise above this frequency
72 gamma_n = 0.1 # attenuate noise by this amount
73 hf.add_spec_noise(gamma_n, omega_n, dB_flag)
74
75 ## plot the effect of adding the new compensator terms
76 mag, phase, omega = bode(Plant * C, dB=dB_flag,
77                           omega=np.logspace(-3, 5),
78                           plot=True, label="$C_{final}(s)P(s)$",
79                           color='orange')
80
81 gm, pm, Wcg, Wcp = margin(Plant * C)
82 print("for final C*P:")
83 if dB_flag == True:
84     print(" pm: ", pm, " Wcp: ", Wcp,
85           "gm: ", mag2db(gm), " Wcg: ", Wcg)
86 elif dB_flag == False:
87     print(" pm: ", pm, " Wcp: ", Wcp,

```

```

88         "gm: ", gm, " Wcg: ", Wcg)
89
90     fig = plt.gcf()
91     fig.axes[0].legend()
92     fig.axes[0].set_title('Inner Open-loop Bode Plot')
93     plt.show()
94
95 ######
96 # now check the closed-loop response
97 ######
98 # Open-loop transfer function
99 OPEN = Plant*C
100 # Closed loop transfer function from R to Y
101 CLOSED_R_to_Y = (Plant*C/(1.0+Plant*C))
102 # Closed loop transfer function from R to U
103 CLOSED_R_to_U = (C/(1.0+Plant*C))
104
105 plt.figure()
106 plt.subplot(311)
107 mag, phase, omega = bode(CLOSED_R_to_Y,
108                             dB=dB_flag, plot=False)
109 if dB_flag:
110     plt.semilogx(omega, mag2db(mag), color=[0,0,1])
111 else:
112     plt.loglog(omega, mag, color=[0,0,1])
113 plt.grid(True)
114 plt.ylabel('Magnitude (dB)')
115 plt.title('closed-loop magnitude ratio $\frac{Y}{R}$')
116
117 plt.subplot(312), plt.grid(True)
118 T = np.linspace(0, 2, 100)
119 _, yout = step_response(CLOSED_R_to_Y, T)
120 plt.plot(T, yout, color=[0,0,1])
121 plt.ylabel('Amplitude')
122 plt.title('Step Response')
123
124 plt.subplot(313), plt.grid(True)
125 _, Uout = step_response(CLOSED_R_to_U, T)
126 plt.plot(T, Uout, color=[0,0,1])
127 plt.ylabel('Amplitude')
128 plt.title('Control Effort')
129
130 plt.tight_layout()
131 plt.show()

```

Listing 18.5: loopshape\_in.py

For the outer loop design, **Fig. 18-24** shows the Bode plot of the plant  $P = \frac{P_{in}C_{in}}{P_{out} + P_{in}C_{in}}$  together with the design specification on reference tracking and noise attenuation. We want a rise time of 2 sec for the system, so this leads us to select a crossover frequency of 1.1 rad/s. At 1.1 rad/s, the plant alone has a small negative phase margin (remember that 180 deg and -180 deg are equivalent), so to bring the phase margin up above 60 deg, we will use a lead compensator centered at 1.1 rad/s with a lead ratio of 32 to add 70 deg of phase. (A lead ratio of 32 is large. An alternative would be to add two leads that each contribute 35 deg

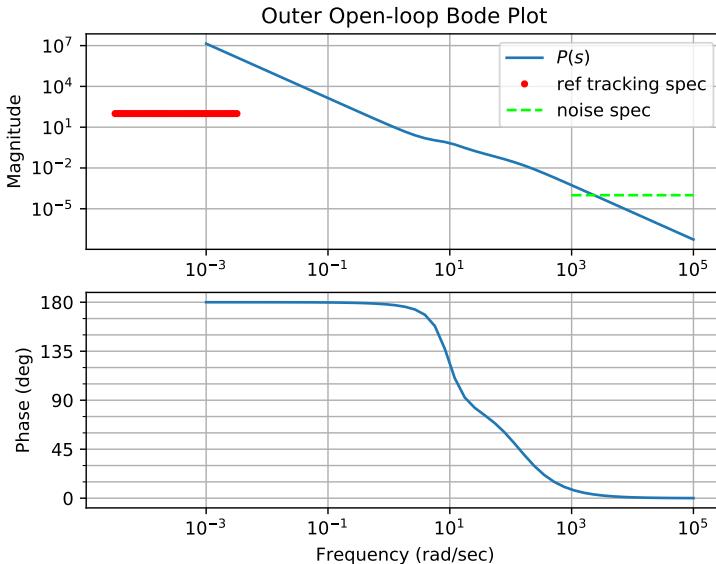


Figure 18-24: The Bode plot for the outer loop plant in HW B.18, together with the design specification.

of phase margin.) As shown in Fig. 18-25, with the lead applied, the crossover frequency is pushed out beyond 100 rad/s. By applying a proportional gain of 0.0146, we set the crossover at 1.1 rad/s. As can be seen from Fig. 18-25, with the lead applied and crossover set at 1.1 rad/s, neither the tracking constraint nor the noise attenuation constraint is satisfied. We can satisfy the tracking constraint by applying lag compensation to boost the low-frequency gain near the constraint. Checking the gain of the lead compensated system at the tracking constraint frequency ( $\omega_r = 0.0032$  rad/s), we can see that the gain is too low by a factor of 4.8. Applying a lag compensator with a lag ratio of 8 ensures that this tracking constraint is satisfied as shown in Fig. 18-26. Finally, we can apply a low-pass filter to reduce the effects of sensor noise in the system and satisfy the noise attenuation constraint. To satisfy the constraint, we need to drop the gain of the open-loop system by a factor of 2 at  $\omega_{no} = 1000$  rad/s. We can accomplish this with a low-pass filter with a cut-off frequency of 100 rad/s. This is shown in Fig. 18-26 below. The resulting compensator is

$$C_{out}(s) = 0.469 \left( \frac{s + 0.194}{s + 6.24} \right) \left( \frac{s + 0.0256}{s + 0.0032} \right) \left( \frac{100}{s + 100} \right).$$

The closed-loop response for both the inner and outer loop systems, as well as the unit-step response for the output and control signal of the outer loops, are all shown in Fig. 18-27. A prefilter

$G_m = 9.97 \text{ dB}$  (at  $7.67 \text{ rad/s}$ ),  $P_m = 67.59 \text{ deg}$  (at  $1.10 \text{ rad/s}$ )

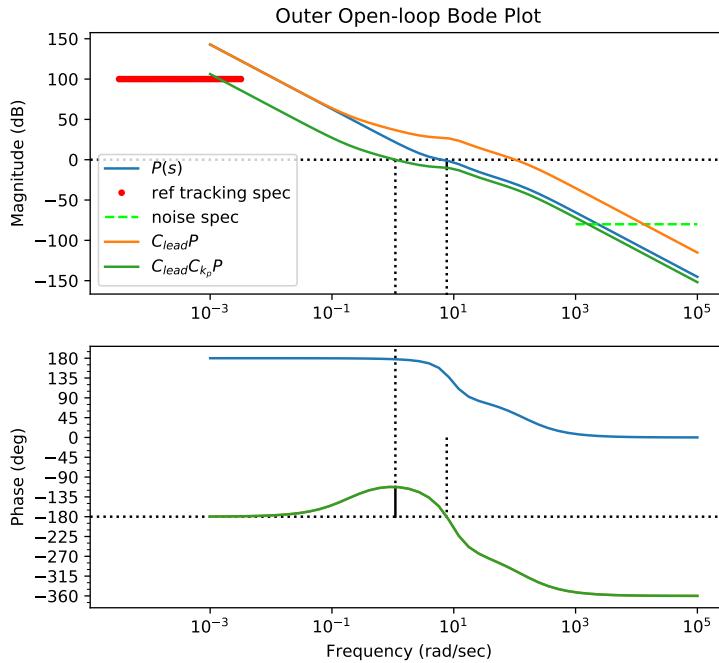


Figure 18-25: The Bode plot for the outer loop system in HW B.18, with lead compensation and proportional gain.

$$F(s) = \frac{2}{s+2}$$

has been added to the system. Note the significant decrease in the control effort required when prefiltering is used with the step input.

Python code that could be used to design the outer loop is shown below. The Python code used to design the inner loop is shown below.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 sys.path.append('../hw16') # add parent directory
4 import pendulumParamHW16 as P16
5 import matplotlib.pyplot as plt
6 from control import tf, step_response, bode, tf2ss, \
7     margin, mag2db, minreal
8 import numpy as np
9 import helper_functions as hf
10 import numpy as np
11 import loopshape_in as L_in
12
13 # flag to define if using dB or absolute scale for M(omega)

```

$G_m = 9.85 \text{ dB}$  (at  $7.31 \text{ rad/s}$ ),  $P_m = 65.79 \text{ deg}$  (at  $1.10 \text{ rad/s}$ )

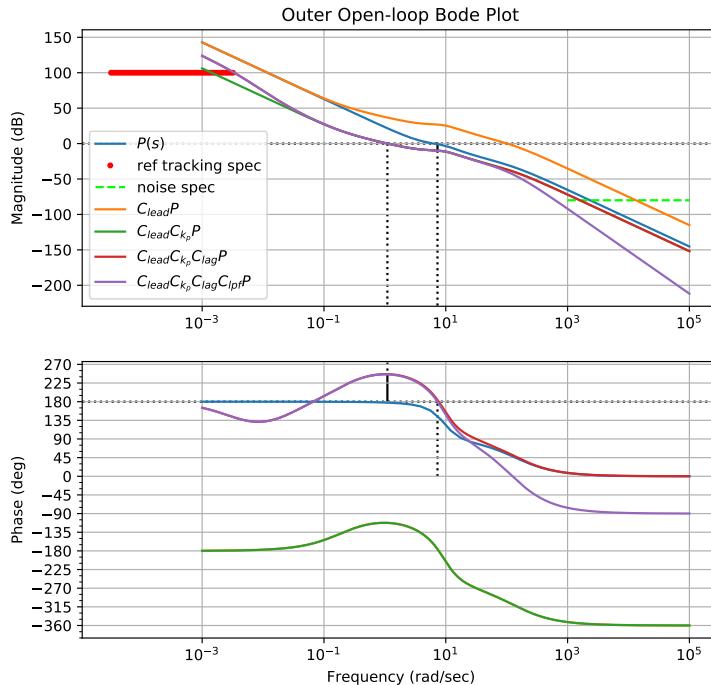


Figure 18-26: The Bode plot for the outer loop system in HW B.18, with lead, and lag control.

```

14 dB_flag = P16.dB_flag
15
16 P_out = P16.P_out
17
18 # construct plant as cascade of P_out and closed inner loop
19 Plant = minreal(P_out* \
20     (L_in.P_in*L_in.C/(1+L_in.P_in*L_in.C)))
21
22 ##### Control Design #####
23 # Control Design
24 #####
25 C = tf([1], [1])
26
27 # define specifications:
28 omega_r = 0.0032 # track signals below this frequency
29 gamma_r = 0.00001 # tracking error below this value
30
31 omega_n = 1000 # attenuate noise above this frequency
32 gamma_n = 0.0001 # attenuate noise by this amount
33

```

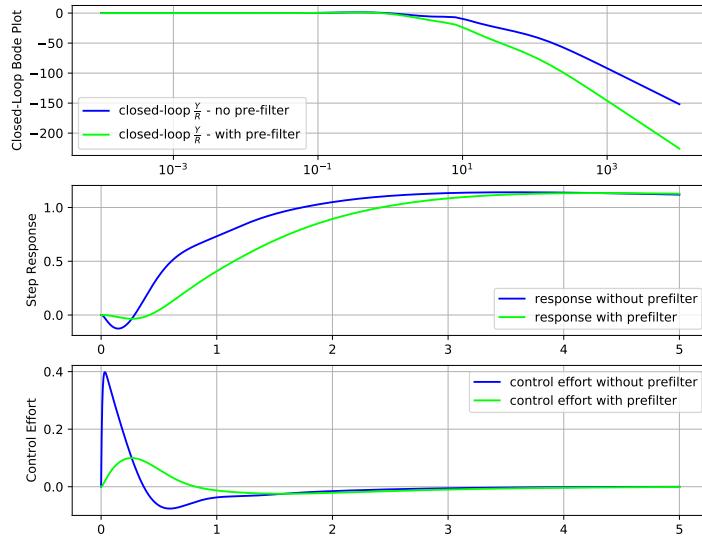


Figure 18-27: The closed-loop bode response, the unit-step response for the output, and the unit step response for the input of the inner loop design in HW B.18.

```

34
35 # phase lead: increase PM (stability)
36 # At desired crossover frequency, PM = -3
37 # Add 70 deg of PM with lead
38
39 #location of maximum frequency bump (desired crossover)
40 w_max = 1.1
41 M = 32    # lead ratio
42 Lead = hf.get_control_lead(w_max, M)
43 C = C*Lead
44
45 # find gain to set crossover at w_max = 1.1 rad/s
46 mag, phase, omega = bode(Plant*C, dB=False,
47                           omega=[w_max], Plot=False)
48 K = hf.get_control_proportional(1.0/mag[0])
49 C = K*C
50
51 ## boost low-frequency gain
52 # Find gain increase needed at omega_r
53 mag, phase, omega = bode(Plant*C, omega=[omega_r],
54                           Plot=False)
55 gain_increase_needed = 1/gamma_r/mag.item(0)
56
57 # minimum gain increase at low frequencies is 4.8 let lag ratio
58 # be 8.

```

```

59 M = 8
60 p = omega_r # set pole at omega_r
61 z = M*p # set zero at M*omega_r
62 Lag = hf.get_control_lag(z, M)
63 C = C*Lag
64
65
66 # Noise attenuation constraint not quite satisfied can be
67 # satisfied by reducing gain at 400 rad/s by a factor of 2 Use
68 # a low-pass filter
69 lpf = hf.get_control_lpf(100.0)
70 C = lpf*C
71
72 ######
73 # Prefilter Design
74 #####
75 F = tf([1], [1])
76
77 # low pass filter
78 p = 2
79 LPF = tf([p], [1, p])
80 F = F*LPF
81
82 #####
83 # Convert Controller to State Space Equations
84 #####
85 C_ss = tf2ss(C)
86 F_ss = tf2ss(F)
87
88
89 if __name__=="__main__":
90     # calculate bode plot and gain and phase margin
91     # for original PID * plant dynamics
92     mag, phase, omega = bode(Plant, dB=dB_flag,
93                             omega=np.logspace(-3, 5),
94                             plot=True, label="$P(s)$")
95
96 gm, pm, Wcg, Wcp = margin(Plant)
97 print("for original system:")
98 if dB_flag == True:
99     print(" pm: ", pm, " Wcp: ", Wcp,
100          "gm: ", mag2db(gm), " Wcg: ", Wcg)
101 elif dB_flag == False:
102     print(" pm: ", pm, " Wcp: ", Wcp,
103          "gm: ", gm, " Wcg: ", Wcg)
104
105 #####
106 # Define Design Specifications
107 #####
108 # ----- general tracking specification -----
109 hf.add_spec_ref_tracking(gamma_r, omega_r, dB_flag)
110
111 # ----- noise specification -----
112 hf.add_spec_noise(gamma_n, omega_n, dB_flag)
113
114 mag, phase, omega = bode(Plant * C, dB=dB_flag,
115                           omega=np.logspace(-3, 5),

```

```

116         plot=True, label="$C_{final}(s)P(s)$",
117         margins=True, color='orange')
118
119 gm, pm, Wcg, Wcp = margin(Plant * C)
120 print("for final C*P:")
121 if dB_flag == True:
122     print(" pm: ", pm, " Wcp: ", Wcp,
123           "gm: ", mag2db(gm), " Wcg: ", Wcg)
124 elif dB_flag == False:
125     print(" pm: ", pm, " Wcp: ", Wcp,
126           "gm: ", gm, " Wcg: ", Wcg)
127
128 fig = plt.gcf()
129 fig.axes[0].legend()
130 fig.axes[0].set_title('Outer Open-loop Bode Plot')
131 fig.axes[0].grid(True)
132 fig.axes[1].grid(True)
133 plt.show()
134
135
136 ######
137 # now check the closed-loop response with prefilter
138 #####
139 # Closed loop transfer function from R to Y - no prefilter
140 CLOSED_R_to_Y = (Plant * C / (1.0 + Plant * C))
141 # Closed loop transfer function from R to Y - with prefilter
142 CLOSED_R_to_Y_with_F = (F * Plant * C / (1.0 + Plant * C))
143 # Closed loop transfer function from R to U - no prefilter
144 CLOSED_R_to_U = (C / (1.0 + Plant * C))
145 # Closed loop transfer function from R to U - with prefilter
146 CLOSED_R_to_U_with_F = (F*C / (1.0 + Plant * C))
147
148 plt.figure()
149 plt.grid(True)
150 plt.subplot(311)
151 mag, phase, omega = bode(CLOSED_R_to_Y,
152                            dB=dB_flag, plot=False)
153 if dB_flag:
154     plt.semilogx(omega, mag2db(mag), color=[0,0,1],
155                  label='closed-loop $\frac{Y}{R}$ - no pre-filter')
156 else:
157     plt.loglog(omega, mag, color=[0,0,1],
158                label='closed-loop $\frac{Y}{R}$ - no pre-filter')
159 mag, phase, omega = bode(CLOSED_R_to_Y_with_F,
160                           dB=dB_flag, plot=False)
161 if dB_flag:
162     plt.semilogx(omega, mag2db(mag), color=[0,1,0],
163                  label='closed-loop $\frac{Y}{R}$ - with pre-filter')
164 else:
165     plt.loglog(omega, mag, color=[0,1,0],
166                label='closed-loop $\frac{Y}{R}$ - with pre-filter')
167 plt.ylabel('Closed-Loop Bode Plot')
168 plt.grid(True)
169 plt.legend()
170
171 plt.subplot(312), plt.grid(True)
172 T = np.linspace(0, 5, 5*200)

```

```
173 _, yout_no_F = step_response(CLOSED_R_to_Y, T)
174 _, yout_F = step_response(CLOSED_R_to_Y_with_F, T)
175 plt.plot(T, yout_no_F, color=[0,0,1],
176           label='response without prefilter')
177 plt.plot(T, yout_F, color=[0,1,0],
178           label='response with prefilter')
179 plt.legend()
180 plt.ylabel('Step Response')
181
182
183 plt.subplot(313), plt.grid(True)
184 _, Uout = step_response(CLOSED_R_to_U, T)
185 _, Uout_F = step_response(CLOSED_R_to_U_with_F, T)
186 plt.plot(T, Uout, color=[0,0,1],
187           label='control effort without prefilter')
188 plt.plot(T, Uout_F, color=[0,1,0],
189           label='control effort with prefilter')
190 plt.ylabel('Control Effort')
191 plt.legend()
192
193 plt.show()
```

Listing 18.6: loopshape\_out.py

See <http://controlbook.byu.edu> for the complete solution.

## 18.4 Design Study C. Satellite Attitude Control



### Example Problem C.18

For this homework assignment we will use the loopshaping design technique to design a successive loop closure controller for the satellite attitude problem.

- (a) First consider the inner loop, where  $P_{in}(s)$  is the transfer function of the inner loop derived in HW C.5. The Bode plot for this system shows an underdamped resonant mode. We have seen in previous chapters that this resonate mode can be removed by introducing rate feedback of the form

$$u = -k_D \dot{y} + u',$$

where  $y$  is the output signal and  $u'$  is the additional control signal to be designed. Using the derivative gain  $k_{D\theta}$  found in HW C.10, find the revised transfer function from  $\tau'$  to  $\theta$ , and use this transfer function for  $P_{in}(s)$ . Letting

$$\tau'(s) = C_{in}(s)E(s),$$

where  $e(t) = \theta_r(t) - \theta(t)$ , design  $C_{in}(s)$  to meet the following specifications:

- The inner loop will track  $\theta_r$  with frequency content below  $\omega_r = 0.01$  radians/sec to within  $\gamma_r = 0.01$ .
- The inner loop will attenuate noise on the measurement of  $\theta$  for all frequencies above  $\omega_n = 20$  rad/sec by  $\gamma_n = 0.01$ .
- The phase margin of the inner loop should be around  $PM = 60$  degrees.

- (b) Now consider the design of the controller for the outer loop system. As seen from HW C.5, the transfer function for the outer loop also has a strong resonate mode. Similar to the inner loop, use the derivative gain found in HW C.10 to introduce rate damping, and find the associated transfer function  $P_{out}(s)$ . The 'plant' for the design of the outer loop controller is

$$P = P_{out} \frac{P_{in}C_{in}}{1 + P_{in}C_{in}}.$$

Design the outer loop controller  $C_{out}$  to meet the following specs:

- Track steps in  $\phi_r$  with zero steady state error.
- Reject input disturbances with frequency content below  $\omega_{din} = 0.01$  by  $\gamma_{din} = 0.1$ .
- Attenuate noise on the measurement of  $\phi$  with frequency content above  $\omega_n = 10$  rad/sec by  $\gamma_n = 10^{-4}$ .

- The phase margin of the outer loop should be around  $PM = 60$  degrees.
- Use a prefilter to reduce any peaking in the closed loop response of the outer loop.

## Solution

From HW C.5, the transfer function model for the inner loop is given by

$$\Theta(s) = \frac{\frac{1}{J_s}}{s^2 + \frac{b}{J_s}s + \frac{k}{J_s}} \tau(s).$$

Rate damping is added by designing the control signal so that

$$\tau(s) = -k_D \frac{s}{\sigma s + 1} \Theta(s) + \tau'(s).$$

Solving for  $\Theta(s)$  in terms of  $\tau'(s)$  gives

$$\Theta(s) = \frac{\sigma s + 1}{\sigma J_s s^3 + (\sigma b + J_s)s^2 + (\sigma k + b + k_D)s + k} \tau'(s).$$

Therefore, for the inner loop we will set

$$P_{in} = \frac{\sigma s + 1}{\sigma J_s s^3 + (\sigma b + J_s)s^2 + (\sigma k + b + k_D)s + k}.$$

**Fig. 18-28** shows the Bode plot of the plant  $P_{in}(s)$  together with the design specification on input tracking and noise attenuation. The first step is to add a proportional gain to satisfy the tracking requirement. **Fig. 18-29** shows the corresponding Bode plot for a proportional gain of  $k_P = 45$ . Since the phase margin is over  $PM = 60$  degrees, a low pass filter is added. **Fig. 18-30** shows the loop gain with the addition of the low pass filter

$$C_{LPF} = \frac{8}{s + 8},$$

which has a phase margin of  $PM = 76.9$  degrees, and satisfies the noise specification. The resulting compensator is

$$C_{in}(s) = 45 \left( \frac{8}{s + 8} \right).$$

Notice that the DC-gain is not equal to one. The closed loop response for the inner subsystem

$$\frac{P_{in}C_{in}}{1 + P_{in}C_{in}},$$

as well as the unit step response for the output and control signal are all shown in **Fig. 18-31**.

The Python code used to design the inner loop is shown below.

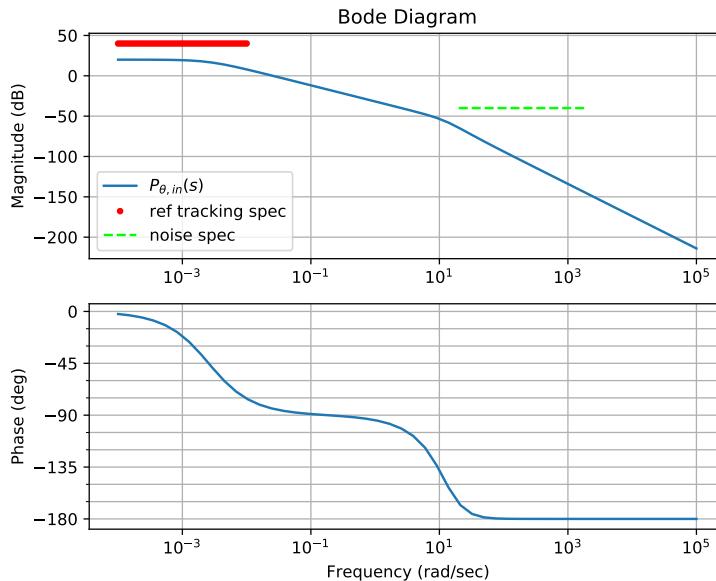


Figure 18-28: The Bode plot for the inner loop plant in HW C.18, together with the design specification.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import satelliteParam as P
4 sys.path.append('../hw10') # add parent directory
5 sys.path.append('../hw16') # add parent directory
6 import satelliteParamHW10 as P10
7 import satelliteParamHW16 as P16
8 import matplotlib.pyplot as plt
9 from control import tf, margin, bode, tf2ss, step_response, mag2db
10 import numpy as np
11 import helper_functions as hf
12
13 # flag to define if using dB or absolute scale for M(omega)
14 dB_flag = P16.dB_flag
15
16 # Compute open-loop transfer functions as described in Chapter 18
17 Plant = tf([P.sigma, 1],
18            [P.sigma*P.Js, P.sigma*P.b+P.Js,
19             P.sigma*P.k+P.b+P10.kd_th, P.k])
20
21 ##### Control Design #####
22 # Control Design
23 #####
24 C = tf([1], [1])
25

```

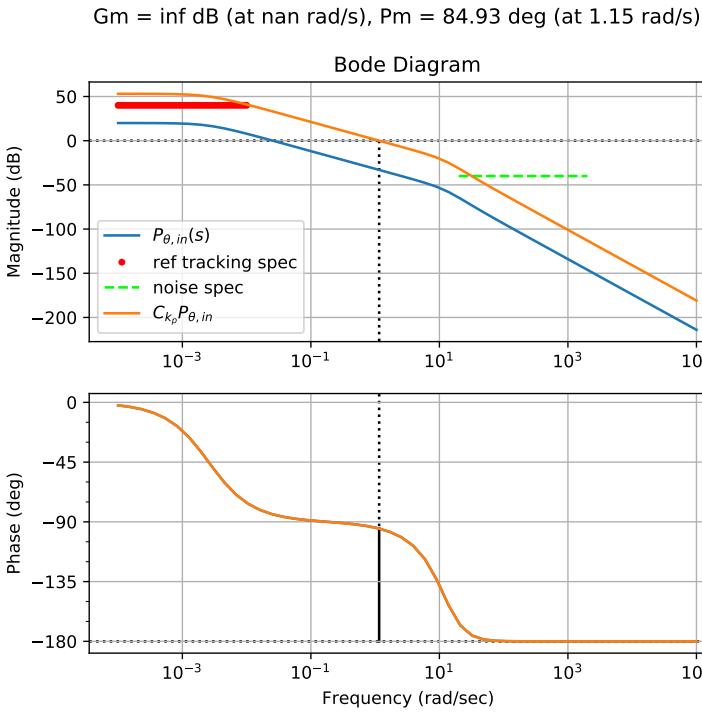


Figure 18-29: The Bode plot for the inner loop plant in HW C.18, with proportional gain.

```

26 # -----proportional control: change cross over frequency-----
27 kp = 45.0
28 C_k = hf.get_control_proportional(kp)
29
30 # -----low pass filter: decrease gain at high frequency (noise)-----
31 p = 8.0
32 C_lpf = hf.get_control_lpf(p)
33
34 # calculating final controller
35 C = C * C_lpf * C_k
36
37 ######
38 # Convert Controller to State Space Equations
39 #####
40 C_ss = tf2ss(C) # convert to state space
41
42
43 if __name__=="__main__":
44
45     # calculate bode plot and gain and phase margin
46     # for original PID * plant dynamics

```

$G_m = 22.17 \text{ dB}$  (at  $8.81 \text{ rad/s}$ ),  $P_m = 76.86 \text{ deg}$  (at  $1.14 \text{ rad/s}$ )

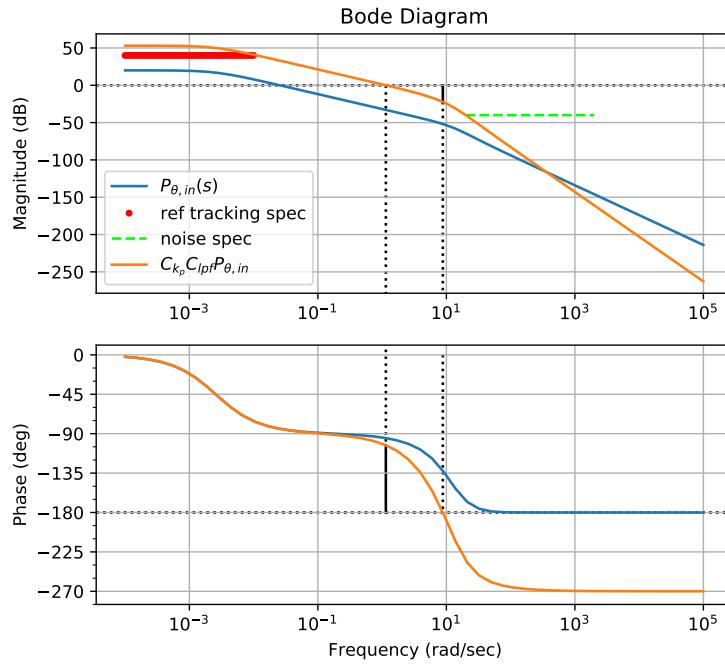


Figure 18-30: The Bode plot for the inner loop system in HW C.18, with proportional gain and low pass filter.

```

47 mag, phase, omega = bode(Plant, dB=dB_flag,
48                         omega=np.logspace(-4, 5),
49                         plot=True, label=r'$P_{\theta, in}(s)$')
50
51 gm, pm, Wcg, Wcp = margin(Plant)
52 print("for original system:")
53 print(" pm: ", pm, " Wcp: ", Wcp, "gm: ", gm, " Wcg: ", Wcg)
54
55 ##### Define Design Specifications #####
56 # Define Design Specifications
57 ##### general tracking specification #####
58 #----- general tracking specification -----
59 omega_r = 0.01 # track signals below this frequency
60 gamma_r = 0.01 # tracking error below this value
61 hf.add_spec_ref_tracking(gamma_r, omega_r, dB_flag)
62
63 ##### noise specification #####
64 omega_n = 20 # attenuate noise above this frequency
65 gamma_n = 0.01 # attenuate noise by this amount
66 hf.add_spec_noise(gamma_n, omega_n, dB_flag)
67

```

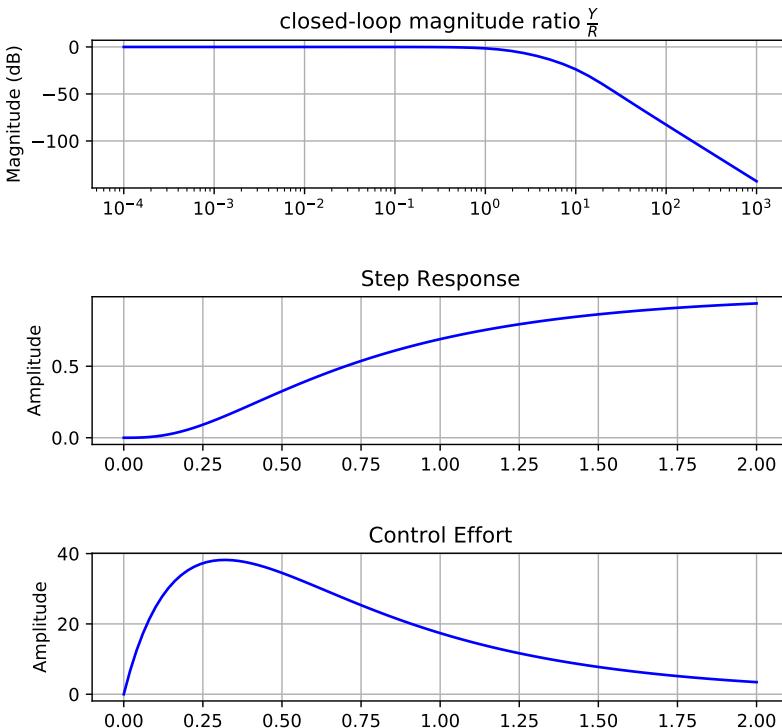


Figure 18-31: The closed loop bode response, the unit step response for the output, and the unit step response for the input of the inner loop design in HW C.18.

```

68 ## plot the effect of adding the new compensator terms
69 mag, phase, omega = bode(Plant * C, dB=dB_flag,
70                           omega=np.logspace(-4, 5),
71                           plot=True,
72                           label=r"$C_{\theta,in}(s)$"+
73                           r"$P_{\theta,in}(s)$",
74                           margins=True)
75
76 gm, pm, Wcg, Wcp = margin(Plant * C)
77 print("for final C*P:")
78 print(" pm: ", pm, " Wcp: ", Wcp, " gm: ", gm, " Wcg: ", Wcg)
79
80 fig = plt.gcf()
81 fig.axes[0].legend()
82 fig.axes[0].grid(True)
83 fig.axes[1].grid(True)
84 fig.axes[0].set_title('Bode Diagram')
85 plt.show()
86

```

```

87 #####
88 # now check the closed-loop response
89 #####
90 # Closed loop transfer function from R to Y
91 CLOSED_R_to_Y = (Plant * C / (1.0 + Plant * C))
92 # Closed loop transfer function from R to U
93 CLOSED_R_to_U = (C / (1.0 + Plant * C))
94
95 plt.figure()
96 plt.subplot(311)
97 mag, phase, omega = bode(CLOSED_R_to_Y,
98                             dB=dB_flag, plot=False)
99 if dB_flag:
100     plt.semilogx(omega, mag2db(mag), color=[0,0,1])
101 else:
102     plt.loglog(omega, mag, color=[0,0,1])
103 plt.grid(True)
104 plt.ylabel('Magnitude (dB)')
105 plt.title('closed-loop magnitude ratio $\frac{Y}{R}$')
106
107 plt.subplot(312), plt.grid(True)
108 T = np.linspace(0, 2, 100)
109 _, yout = step_response(CLOSED_R_to_Y, T)
110 plt.plot(T, yout, color=[0,0,1])
111 plt.ylabel('Amplitude')
112 plt.title('Step Response')
113
114 plt.subplot(313), plt.grid(True)
115 _, uout = step_response(CLOSED_R_to_U, T)
116 plt.plot(T, uout, color=[0,0,1])
117 plt.ylabel('Amplitude')
118 plt.title('Control Effort')
119
120 plt.tight_layout()
121 plt.show()

```

Listing 18.7: loopshape\_in.py

The transfer function model for the outer loop was derived in HW C.5 to be

$$\Phi(s) = \frac{\frac{b}{J_p}s + \frac{k}{J_p}}{s^2 + \frac{b}{J_p}s + \frac{k}{J_p}} \Theta^r(s).$$

Rate damping is added by designing the control signal so that

$$\Theta^r(s) = -k_D \frac{s}{\sigma s + 1} \Phi(s) + \Theta^{r'}(s).$$

Solving for  $\Phi(s)$  in terms of  $\Theta^{r'}(s)$  gives

$$\Phi(s) = \frac{\sigma bs^2 + (\sigma k + b)s + k}{\sigma J_p s^3 + (\sigma b + J_p + bk_D)s^2 + (\sigma k + b + kk_D)s + k} \Theta^{r'}(s).$$

The plant for the outer loop is therefore

$$P_{out} = \frac{\sigma s + 1}{\sigma J_p s^3 + (\sigma b + J_p)s^2 + (\sigma k + b + k_D)s + k} \frac{P_{in}C_{in}}{1 + P_{in}C_{in}}.$$

**Fig. 18-32** shows the Bode plot for  $P_{out}$  together with the design specification on rejecting input disturbances and attenuating the noise. The requirement that steady state error to a step requires that the slope as  $s \rightarrow 0$  is  $-20$  dB/dec, and is not shown in **Fig. 18-32**.

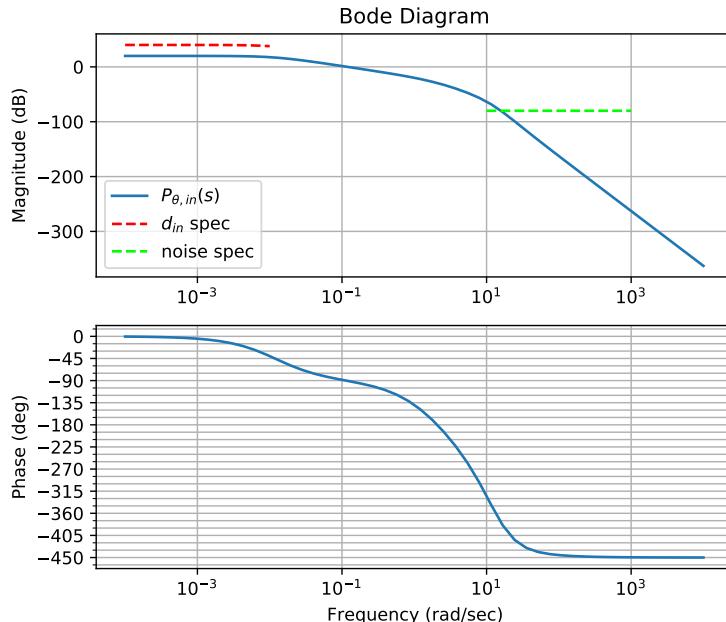


Figure 18-32: The Bode plot for the outer loop plant in HW C.18, together with the design specifications.

The first step is to add an integrator to meet the steady state tracking requirement. The PM was also too low ( $PM = 20.8$  degrees). So a lead compensator is added at  $0.15$  rad/sec with  $M = 120$ , at approximately the midpoint in frequency between the disturbance and noise constraints. This causes the disturbance constraint to be violated. For this reason, a lag compensator is also added to boost magnitude ratio at low frequencies. The lag compensator is defined with  $z = 2.0$  and  $M = 60.0$ . **Fig. 18-33** shows the loop gain after adding the integral control, lead compensator, and lag compensator.

$$C = C_{int} C_{lead} C_{lag} = \left( \frac{s + 0.1}{s} \right) \left( \frac{120s + 1.64}{s + 1.64} \right) \left( \frac{s + 0.5}{0.0083} \right).$$

Without adding an overall gain to shift the crossover frequency (i.e. to the desired  $0.15$  rad/sec for which the lead compensator was designed), the phase margin as shown in **Fig. 18-33** is  $PM = -36$  degrees. Therefore a proportional

$G_m = -6.65 \text{ dB}$  (at  $2.71 \text{ rad/s}$ ),  $P_m = -36.38 \text{ deg}$  (at  $4.03 \text{ rad/s}$ )

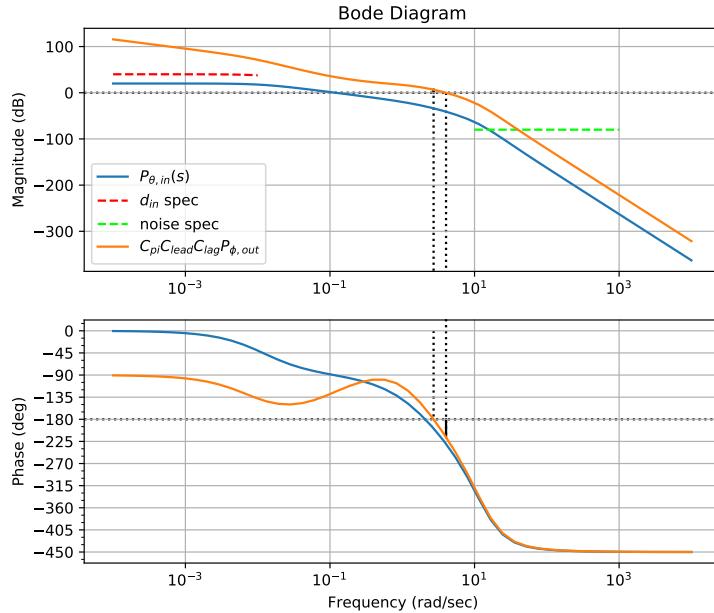


Figure 18-33: The Bode plot for the outer loop system in HW C.18, with integral, lead, and lag control.

gain of  $k_P = 0.0275$  is added to the compensator. Two low-pass filters with  $p = 1.5$  and  $p = 1.8$  are also added to meet the noise constraint and the result is shown in Fig. 18-34. The resulting system has  $PM = 52$  degrees which is a direct result of tradeoffs between the constraints in noise and tracking.

The final compensator is defined as:

$$\begin{aligned} C_{total} &= C_{k_p} C_{pi} C_{lead} C_{lag} C_{lpf1} C_{lpf2} \\ &= 0.0275 \left( \frac{s + 0.1}{s} \right) \left( \frac{120s + 1.64}{s + 1.64} \right) \left( \frac{s + 0.5}{0.0083} \right) \left( \frac{1.5}{s + 1.5} \right) \left( \frac{1.8}{s + 1.8} \right). \end{aligned}$$

The closed loop response for both the inner and outer loop systems, as well as the unit step response for the output and control signal of the outer loops, are all shown in Fig. 18-35, where the prefilter

$$F(s) = \frac{0.1}{s + 0.1}$$

has been added to reduce overshoot in the closed-loop response.

The time response of the outer loop is fairly slow, but this is because of the chosen crossover frequency.

The Python code used to design the outer loop is shown below.

$G_m = 17.45 \text{ dB}$  (at  $1.04 \text{ rad/s}$ ),  $P_m = 51.75 \text{ deg}$  (at  $0.15 \text{ rad/s}$ )

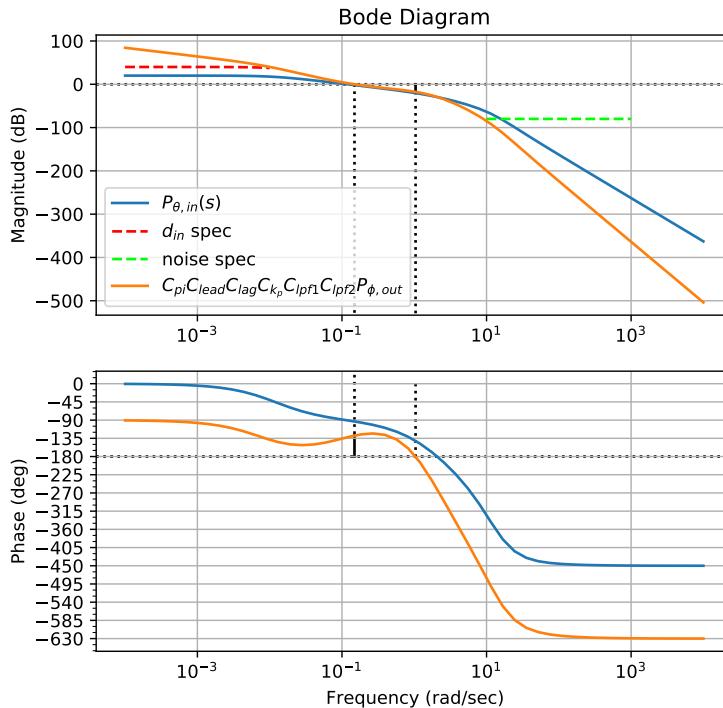


Figure 18-34: The Bode plot for the outer loop system in HW C.18, with the final controller.

```

1 import sys
2 sys.path.append('..') # add parent directory
3 import satelliteParam as P
4 sys.path.append('../hw10') # add parent directory
5 sys.path.append('../hw16') # add parent directory
6 import satelliteParamHW10 as P10
7 import satelliteParamHW16 as P16
8 import matplotlib.pyplot as plt
9 from control import tf, margin, bode, tf2ss, step_response, mag2db
10 import numpy as np
11 import loopshape_in as L_in
12 import helper_functions as hf
13
14 # flag to define if using dB or absolute scale for M(omega)
15 dB_flag = P16.dB_flag
16
17 # Compute outer loop transfer function as described in Chapter 18
18 num_out = [P.sigma, 1.0]

```

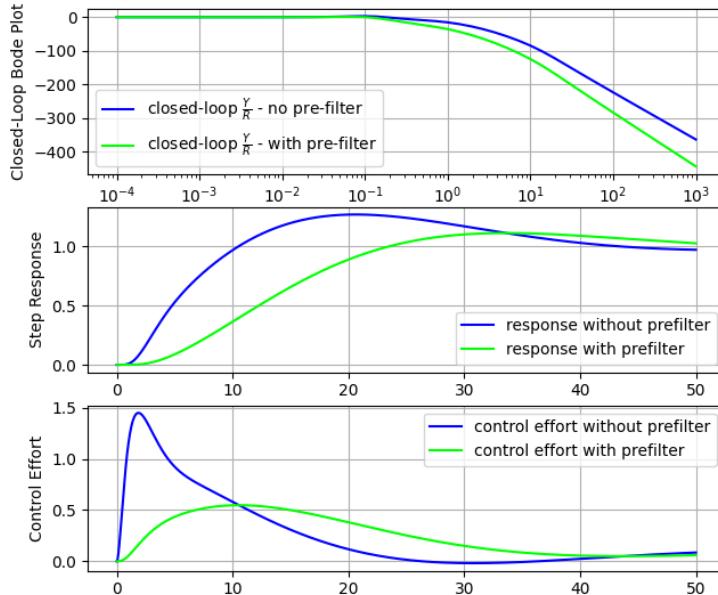


Figure 18-35: The closed loop bode response, the unit step response for the output, and the unit step response for the input of the inner loop design in HW C.18.

```

19 den_out = [P.sigma*P.Jp, P.sigma*P.b+P.Jp,
20           P.sigma*P.k+P.b+P10.kd_phi, P.k]
21 P_out = tf(num_out, den_out)
22
23 # construct plant as cascade of P_out and closed inner loop
24 Plant = P_out * (L_in.Plant * L_in.C /
25                   (1 + L_in.Plant * L_in.C))
26
27 ##### Control Design #####
28 #   Control Design
29 #####
30 C = tf([1], [1])
31
32 # -----integral control: change low freq. slope-----
33 ki = 0.1
34 C_int = hf.get_control_integral(ki)
35
36
37 # -----lead control: change the phase margin -----
38 omega_max = 0.15
39 M = 120.0
40 C_lead = hf.get_control_lead(omega_max, M)

```

```

41 # -----lag control: increase magnitude near required freq. -----
42 z = 2.0
43 M = 60.0
44 C_lag = hf.get_control_lag(0.5, 60)
45
46 # -----proportional control: change cross over frequency-----
47 mag, phase, omega = bode(Plant*C_int*C_lead*C_lag,
48                           omega=[omega_max],
49                           Plot=False)
50 C_k = hf.get_control_proportional(1.0/mag[0])
51
52 # -----low pass filter: decrease gain at high frequency (noise)-----
53 # we will have to use two to get the desired noise attenuation
54 p = 1.5
55 C_lpf1 = hf.get_control_lpf(p)
56 p = 1.8
57 C_lpf2 = hf.get_control_lpf(p)
58
59 # calculating final controller
60 C = C*C_int*C_lead*C_lag*C_k*C_lpf1*C_lpf2
61
62
63 #####
64 # Prefilter Design
65 #####
66 # low pass filter
67 p = 0.1
68 F = hf.get_control_lpf(p)
69
70
71 #####
72 # Convert Controller to State Space Equations
73 #####
74 C_ss = tf2ss(C)
75 F_ss = tf2ss(F)
76
77
78 if __name__=="__main__":
79
80     # calculate bode plot and gain and phase margin
81     # for original PID * plant dynamics
82     mag, phase, omega = bode(Plant, dB=dB_flag,
83                               omega=np.logspace(-4, 4),
84                               plot=True, label=r'$P_{\{\theta,\text{in}\}}(s)$')
85
86     gm, pm, Wcg, Wcp = margin(Plant)
87     print("for original system:")
88     print(" pm: ", pm, " Wcp: ", Wcp, "gm: ", gm, " Wcg: ", Wcg)
89
90 #####
91 # Define Design Specifications
92 #####
93 #----- input disturbance spec -----
94 omega_din = 10 ** (-2.0) # reject input disturbances below freq.
95 gamma_din = 0.1 # amount of input disturbance in output
96 hf.add_spec_input_disturbance(gamma_din, omega_din,
97

```

```

98                     Plant, dB_flag)
99
100 # ----- noise specification -----
101 omega_n = 10 # attenuate noise above this frequency
102 gamma_n = 10.0 ** (-4.0) # attenuate noise by this amount
103 hf.add_spec_noise(gamma_n, omega_n, dB_flag)
104
105 ## plot the effect of adding the new compensator terms
106 mag, phase, omega = bode(Plant * C, dB=dB_flag,
107                           omega=np.logspace(-4, 4),
108                           plot=True, margins=True,
109                           label=r"$C_{\{\phi_i, \text{out}\}}(s) +"
110                           "P_{\{\phi_i, \text{out}\}}(s)$")
111
112 gm, pm, Wcg, Wcp = margin(Plant * C)
113 print("for final C*P:")
114 print(" pm: ", pm, " Wcp: ", Wcp, "gm: ", gm, " Wcg: ", Wcg)
115
116 fig = plt.gcf()
117 fig.axes[0].legend()
118 fig.axes[0].set_title('Bode Diagram')
119 fig.axes[0].grid(True)
120 fig.axes[1].grid(True)
121 plt.show()
122
123 ######
124 # now check the closed-loop response with prefilter
125 #####
126 # Closed loop transfer function from R to Y - no prefilter
127 CLOSED_R_to_Y = (Plant * C / (1.0 + Plant * C))
128 # Closed loop transfer function from R to Y - with prefilter
129 CLOSED_R_to_Y_with_F = (F * Plant * C / (1.0 + Plant * C))
130 # Closed loop transfer function from R to U - no prefilter
131 CLOSED_R_to_U = (C / (1.0 + Plant * C))
132 # Closed loop transfer function from R to U - with prefilter
133 CLOSED_R_to_U_with_F = (F*C / (1.0 + Plant * C))
134
135 plt.figure()
136 plt.grid(True)
137 plt.subplot(311)
138 mag, phase, omega = bode(CLOSED_R_to_Y,
139                           dB=dB_flag, plot=False)
140 if dB_flag:
141     plt.semilogx(omega, mag2db(mag), color=[0,0,1],
142                  label='closed-loop $\frac{Y}{R} - \text{no pre-filter}')
143 else:
144     plt.loglog(omega, mag, color=[0,0,1],
145                label='closed-loop $\frac{Y}{R} - \text{no pre-filter}')
146 mag, phase, omega = bode(CLOSED_R_to_Y_with_F,
147                           dB=dB_flag, plot=False)
148 if dB_flag:
149     plt.semilogx(omega, mag2db(mag), color=[0,1,0],
150                  label='closed-loop $\frac{Y}{R} - \text{with pre-filter}')
151 else:
152     plt.loglog(omega, mag, color=[0,1,0],
153                label='closed-loop $\frac{Y}{R} - \text{with pre-filter}')
154 plt.ylabel('Closed-Loop Bode Plot')

```

```
155     plt.grid(True)
156     plt.legend()
157
158     plt.subplot(312), plt.grid(True)
159     T = np.linspace(0, 50, 50*100)
160     _, yout_no_F = step_response(CLOSED_R_to_Y, T)
161     _, yout_F = step_response(CLOSED_R_to_Y_with_F, T)
162     plt.plot(T, yout_no_F, color=[0,0,1],
163               label='response without prefilter')
164     plt.plot(T, yout_F, color=[0,1,0],
165               label='response with prefilter')
166     plt.legend()
167     plt.ylabel('Step Response')
168
169
170     plt.subplot(313), plt.grid(True)
171     _, Uout = step_response(CLOSED_R_to_U, T)
172     _, Uout_F = step_response(CLOSED_R_to_U_with_F, T)
173     plt.plot(T, Uout, color=[0,0,1],
174               label='control effort without prefilter')
175     plt.plot(T, Uout_F, color=[0,1,0],
176               label='control effort with prefilter')
177     plt.ylabel('Control Effort')
178     plt.legend()
179
180     plt.show()
```

Listing 18.8: loopshape\_outer.py

See <http://controlbook.byu.edu> for the complete solution.

**Important Concepts:**

- Loopshaping design may be achieved by multiplying different building block controllers together. On the Bode plot, this will result in the summation of the gain (dB) and phase of each control block.
- A proportional gain controller alters the open-loop system's magnitude. It will increase/decrease the magnitude if the gain is larger/less than one. If the proportional controller's gain is positive, the phase remains the same. Otherwise, 180 degrees of phase is added.
- At low frequencies, a modified integral controller will (1) increase the system type, (2) increase the gain, and (3) decrease the phase.
- A low-pass filter will decrease the gain at high frequencies to improve the system's noise attenuation. It also decreases the phase.
- A phase-lag filter uses a pole and zero (with the magnitude of the zero being greater than that of the pole) to increase the loop gain at low frequencies. This also decreases the phase between the zero and the pole. phase-lag filters are used to improve tracking and disturbance rejection at low frequencies without introducing a potentially destabilizing integrator.
- A phase-lead filter also uses a pole and zero (with the pole magnitude being greater than that of the zero) to increase the phase between the zero and the pole.
- Prefilters attenuate frequencies that will resonate in the closed-loop system.

**Notes and References**

# **Part VI**

# **Homework Problems**





## Design Study: Mass Spring Damper

Figure 19.1 shows the mass-spring-damper system. The mass slides along a frictionless level surface and is connected to a wall by a spring with spring constant  $k$  and a damper with damping constant  $b$ . The position of the mass is given by  $z$ , where  $z = 0$  is the position where the spring is not stretched. The speed of the mass is given by  $\dot{z}$ . An external force  $F$  is applied to the mass as shown in Fig. 19.1.

Assume the following physical constants:  $m = 5 \text{ kg}$ ,  $k = 3 \text{ N/m}$ ,  $b = 0.5 \text{ N-sec/m}$ .

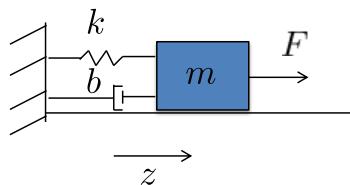


Figure 19-1: Mass Spring Damper

### Homework Problems:

- D.2 Kinetic energy.
- D.3 Equations of Motion.
- D.4 Linearize equations of motion.
- D.5 Transfer function model.
- D.6 State space model.
- D.7 Pole placement using PD.
- D.8 Second order design.
- D.9 Integrators and system type.
- D.P.6 Root locus PID.

- D.10 Digital PID.
- D.11 Full state feedback.
- D.12 Full state with integrator.
- D.13 Observer based control.
- D.14 Disturbance observer.
- D.15 Frequency Response.
- D.16 Loop gain.
- D.17 Stability margins.
- D.18 Loopshaping design.



## Homework D.2

- (a) Using the configuration variable  $z$ , write an expression for the kinetic energy of the system.
- (b) Create an animation of the mass-spring-damper system. The input should be a variable  $z$ . Turn in a screen capture of the animation.



## Homework D.3

- (a) Find the potential energy for the system.
- (b) Define the generalized coordinates.
- (c) Find the generalized forces and damping forces.
- (d) Derive the equations of motion using the Euler-Lagrange equations.
- (e) Referring to Appendices P.1, P.2, and P.3, write a class or s-function that implements the equations of motion. Simulate the system using a variable force input. The output should connect to the animation function developed in homework D.2.



## Homework D.4

For the mass spring damper:

- (a) Find the equilibria of the system.
- (b) Linearize the system about the equilibria using Jacobian linearization.
- (c) Linearize the system using feedback linearization.



## Homework D.5

For the equations of motion for the mass spring damper,

- (a) Using the Laplace transform, convert from time domain to the s-domain.
- (b) Find the transfer function from the input force  $F$  to the mass position  $z$ .
- (c) Draw the associated block diagram.



## Homework D.6

For the equations of motion for the mass spring damper, define the states as  $x = (z, \dot{z})^\top$ , and the input as  $u = F$ , and the measured output as  $y = z$ . Find the linear state space equations in the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du.\end{aligned}$$



## Homework D.7

- (a) Given the open loop transfer function from force to position in Homework D.5, find the open loop poles of the system, when the equilibrium position is  $z_e = 0$ .
- (b) Using PD control architecture shown in Figure 7-2, find the closed loop transfer function from  $z_r$  to  $z$  and find the closed loop poles as a function of  $k_P$  and  $k_D$ .
- (c) Select  $k_P$  and  $k_D$  to place the closed loop poles at  $p_1 = -1$  and  $p_2 = -1.5$ .
- (d) Using the gains from part (c), implement the PD control for the mass spring damper in simulation and plot the response of the system to a 1 m step input.



## Homework D.8

For the mass spring, do the following:

- (a) Suppose that the design requirements are that the rise time is  $t_r \approx 2$  seconds, with a damping ratio of  $\zeta = 0.7$ . Find the desired closed loop characteristic polynomial  $\Delta_{cl}^d(s)$ , and the associated pole locations. Find the proportional and derivative gains  $k_P$  and  $k_D$  to achieve these specifications, and modify the simulation from HW D.7 to verify that the step response satisfies the requirements.
- (b) Suppose that the size of the input force is limited to  $F_{\max} = 6$  N. Modify the simulation to include a saturation block on the force  $F$ . Using the rise time  $t_r$  as a tuning parameter, tune the PD control gains so that the input just saturates when a step of size of 1 meter is placed on  $z^r$ . Plot the step response showing that saturation does not occur for a 1 meter step input for the new control gains.



### Homework D.9

- (a) When the controller for the mass spring damper is PD control, what is the system type? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?
- (b) Consider the case where a constant disturbance acts at the input to the plant (for example an inaccurate knowledge of the spring constant in this case). What is the steady state error to a constant input disturbance when the integrator is not present, and when it is present?



### Homework D.P.6

Adding an integrator to obtain PID control for the mass spring damper, put the characteristic equation in Evan's form and use the Matlab `rlocus` command to plot the root locus versus the integrator gain  $k_I$ . Select a value for  $k_I$  that does not significantly change the other locations of the closed loop poles.



### Homework D.10

The objective of this problem is to implement the PID controller using only measured outputs of the system.

- (a) Modify the system dynamics file so that the parameters  $m$ ,  $k$ , and  $b$  vary by up to 20% of their nominal value each time they are run (uncertainty parameter = 0.2).
- (b) Change the simulation files so that the input to the controller is the output and not the state. The controller should only assume knowledge of the position  $z$  and the reference position  $z_r$ .
- (c) Implement the PID controller designed in Problems D.8 in simulation. Use the dirty derivative gain of  $\sigma = 0.05$ . Tune the integrator to remove the steady state error caused by the uncertain parameters.



### Homework D.11

The objective of this problem is to implement state feedback controller for the mass spring damper using the full state. Start with the simulation files developed

in Homework D.10.

- (a) Select the closed loop poles as the roots of the equations  $s^2 + 2\zeta\omega_n + \omega_n^2 = 0$  where  $\omega_n$ , and  $\zeta$  were found in Homework D.8.
- (b) Add the state space matrices  $A$ ,  $B$ ,  $C$ ,  $D$  derived in Homework D.6 to your param file.
- (c) Verify that the state space system is controllable by checking that  $\text{rank}(\mathcal{C}_{A,B}) = n$ .
- (d) Find the feedback gain  $K$  so that the eigenvalues of  $(A - BK)$  are equal to the desired closed loop poles. Find the reference gain  $k_r$  so that the DC-gain from  $z_r$  to  $z$  is equal to one. Note that  $K = (k_p, k_d)$  where  $k_p$  and  $k_d$  are the proportional and derivative gains found in Homework D.8. Why?
- (e) Modify the control simulation code to implement the state feedback controller. To construct the state  $x = (z, \dot{z})^\top$  use a digital differentiator to estimate  $\dot{z}$ .



## Homework D.12

- (a) Modify the state feedback solution developed in Homework D.11 to add an integrator with anti-windup to the feedback loop for  $z$ .
- (b) Add a constant input disturbance of 0.25 Newtons to the input of the plant and allow the plant parameters to vary up to 20%.
- (c) Tune the integrator pole (and other gains if necessary) to get good tracking performance.



## Homework D.13

The objective of this problem is to design an observer that estimates the state of the system and to use the estimated state in the controller designed in Homework D.12.

- (a) For the sake of understanding the function of the observer, for this problem we will use exact parameters, without an input disturbance. Modify the mass dynamics so that the parameters known to the controller are the actual plant parameters (uncertainty parameter  $\alpha = 0$ ).
- (b) Verify that the state space system is observable by checking that  $\text{rank}(\mathcal{O}_{A,C}) = n$ .

- (c) In the control block, add an observer to estimate the state  $\hat{x}$ , and use the estimate of the state in your feedback controller. Tune the poles of the controller and observer to obtain good performance.
- (d) Modify the simulation files so that the controller outputs both  $u$  and  $\hat{x}$ . Add a plotting routine to plot both the state and the estimated state of the system on the same graph.
- (e) As motivation for the next chapter, add an input disturbance to the system of 0.25 and observe that there is steady state error in the response even though there is an integrator. This is caused by a steady state error in the observation error. In the next chapter we will show how to remove the steady state error in the observation error.



## Homework D.14

- (a) Modify your solution from HW D.13 so that the uncertainty parameter is  $\alpha = 0.2$ , representing 20% inaccuracy in the knowledge of the system parameters, and so that the input disturbance is 0.25. Also, add noise to the output channels  $z_m$  and  $\theta_m$  with standard deviation of 0.001.
- (b) Add a disturbance observer to the controller, and verify that the steady state error in the estimator has been removed. Tune the system to get good response.



## Homework D.15

Draw by hand the Bode plot of the mass spring damper from force  $\tilde{F}$  to position  $\tilde{z}$ . Use the `bode` command in Matlab or Python and compare your results.



## Homework D.16

For the mass spring, use the `bode` command (from Python or Matlab) to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PID control, using the control gains calculated in Homework D.10. Use the dirty derivative when implementing the derivative portion of the PID control.

- (a) What is the tracking error to a unit ramp under PID control?

- (b) If the frequency content of the input disturbance  $d_{in}(t)$  is below  $\omega_{din} = 0.1$  radians per second, what percentage of the input disturbance shows up in the output  $z$  under PID control?
- (c) If all of the frequency content of the noise  $n(t)$  is greater than  $\omega_{no} = 100$  radians per second, what percentage of the noise shows up in the output signal  $z$ ?



### Homework D.17

For the mass spring damper, use the `bode` and `margin` commands (from Matlab or Python) to find the phase and gain margin for the closed loop system under PID control. On the same graph, plot the open loop Bode plot and the closed loop Bode plot. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency? Use the gains found in HW D.10.



### Homework D.18

For this homework assignment we will use the loopshaping design technique to design a controller for the mass spring damper problem with dynamics given by

$$Z(s) = \left( \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \right) F(s).$$

The controller designed in this problem will take the form

$$F = C(s)E(s),$$

where the error signal is  $e = z_f^r - z$ , where  $z_f^r$  is the prefiltered reference command. Find the controller  $C(s)$  so that the closed loop system meets the following specifications.

- Reject constant input disturbances.
- Track reference signals with frequency content below  $\omega_r = 0.1$  rad/s to within  $\gamma_r = 0.03$ .
- Attenuate noise on the measurement of  $z$  for all frequencies above  $\omega_n = 500$  rad/s by  $\gamma_n = 0.001$ .
- The phase margin is close to  $PM = 60$  degrees.
- Use a prefilter  $F(s)$  to reduce any peaking in the closed loop response.
- Implement a control strategy in simulation using the state space or digital filter equivalents of  $C(s)$  and  $F(s)$ .



# E

## Design Study: Ball on Beam

Figure 20-1 shows the system. The position of the ball measured from the pivot point is  $z$  and the speed of the ball along the direction of the beam is  $\dot{z}$ . The angle of the beam from level is  $\theta$  and the angular speed of the beam is  $\dot{\theta}$ . Gravity acts in the down direction. The mass of the ball is  $m_1$  and the mass of the beam is  $m_2$ . The length of the beam is  $\ell$ .

An external vertical force is applied at the end of the beam as shown in Fig. 20-1.

Use the following physical parameters:  $m_1 = 0.35 \text{ kg}$ ,  $m_2 = 2 \text{ kg}$ ,  $\ell = 0.5 \text{ m}$ ,  $g = 9.8 \text{ m/s}^2$ .

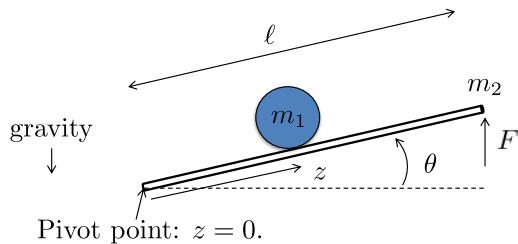


Figure 20-1: Ball on Beam Problem

### Homework Problems:

- E.2** Kinetic energy.
- E.3** Equations of Motion.
- E.4** Linearize equations of motion.
- E.5** Transfer function model.
- E.6** State space model.
- E.8** Successive loop closure.
- E.9** Integrators and system type.
- E.P.6** Root locus PID.
- E.10** Digital PID.

- E.11** Full state feedback.
- E.12** Full state with integrator.
- E.13** Observer based control.
- E.14** Disturbance observer.
- E.15** Frequency Response.
- E.16** Loop gain.
- E.17** Stability margins.
- E.18** Loopshaping design.



## Homework E.2

- (a) Using the configuration variables  $z$  and  $\theta$ , write an expression for the kinetic energy of the system. Assume that the ball is a point mass without rolling inertia. In general, this term will be small and will not have a large effect on the dynamics of the ball-beam system.
- (b) Create an animation of the ball on beam system. The inputs should be  $z$  and  $\theta$ . Turn in a screen capture of the animation.



## Homework E.3

- (a) Find the potential energy for the system.
- (b) Define the generalized coordinates.
- (c) Find the generalized forces and damping forces.
- (d) Derive the equations of motion using the Euler-Lagrange equations.
- (e) Referring to Appendices P.1, P.2, and P.3, write a class or s-function that implements the equations of motion. Simulate the system using a variable force input. The output should connect to the animation function developed in homework E.2.



## Homework E.4

For the ball on beam system:

- (a) Find the equilibria of the system.
- (b) Linearize the system about the equilibria using Jacobian linearization.
- (c) If possible, linearize the system using feedback linearization.



## Homework E.5

For the ball on beam system:

- (a) Start with the linearized equations and use the Laplace transform to convert the equations of motion to the s-domain.

- (b) Find the transfer functions from the input  $\tilde{F}(s)$  to the outputs  $\tilde{Z}(s)$  and  $\tilde{\Theta}(s)$ .  
 Find the transfer function from the input  $\tilde{\Theta}(s)$  to the output  $\tilde{Z}(s)$ .
- (c) Assume that deviations in the gravity torque away from the equilibrium torque due to motions of the ball along the beam are small compared to the torque required to hold up the beam. This will allow you to ignore the  $m_1 g \tilde{z}$  term in the second equation of motion. How does this simplify your transfer functions?
- (d) Draw a block diagram of your simplified system transfer functions in a cascade from the input  $\tilde{F}(s)$  to the intermediate state  $\tilde{\Theta}(s)$  and then to the output  $\tilde{Z}(s)$ .



## Homework E.6

Defining the states as  $\tilde{x} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$ , the input as  $\tilde{u} = \tilde{F}$ , and the measured output as  $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$ , find the linear state space equations in the form

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y} &= C\tilde{x} + D\tilde{u}.\end{aligned}$$



## Homework E.8

For the ball and beam problem, do the following:

- (a) Using the principle of successive loop closure, draw a block diagram that uses PD control for both inner loop control and outer loop control. For design purposes, let the equilibrium position for the ball be  $z_e = \frac{\ell}{2}$ . The input to the outer loop controller is the desired ball position  $z_r$  and the output of the controller is the desired beam angle  $\tilde{\theta}_r$ . The input to the inner loop controller is the desired beam angle  $\tilde{\theta}_r$  and the output is the force  $\tilde{F}$  on the end of the beam.
- (b) Focusing on the inner loop, find the PD gains  $k_{P_\theta}$  and  $k_{D_\theta}$  so that the rise time of the inner loop is  $t_{r_\theta} = 1$  second, and the damping ratio is  $\zeta_\theta = 0.707$ .
- (c) Find the DC gain  $k_{DC_\theta}$  of the inner loop.
- (d) Replacing the inner loop by its DC-gain, find the PD gains  $k_{P_z}$  and  $k_{D_z}$  so that the rise time of the outer loop is  $t_{r_z} = 10t_{r_\theta}$  and the damping ratio is  $\zeta_z = 0.707$ .

- (e) Implement the successive loop closure design for the ball and beam in simulation where the commanded ball position is given by a square wave with magnitude  $0.25 \pm 0.15$  meters and frequency 0.01 Hz. Use the actual position in the ball as a feedback linearizing term for the equilibrium force.
- (f) Suppose that the size of the input force on the beam is limited to  $F_{\max} = 15$  N. Modify the simulation to include a saturation block on the force  $F$ . Using the rise time of the outer loop, tune the PD control law to get the fastest possible response without input saturation when a step of size 0.25 meter is placed on  $\tilde{z}^r$ .



## Homework E.9

- (a) When the inner loop controller for the ballbeam system is PD control, what is the system type of the inner loop? Characterize the steady state error when  $\tilde{\theta}^d$  is a step, a ramp, and a parabola. What is the system type with respect to an input disturbance?
- (b) When the outer loop controller for the ballbeam is PD control, what is the system type of the outer loop? Characterize the steady state error when  $z^d$  is a step, a ramp, and a parabola. How does this change if you add an integrator? What is the system type with respect to an input disturbance for both PD and PID control?



## Homework E.P.6

Adding an integrator to obtain PID control for the outer loop of the ballbeam system, put the characteristic equation in Evan's form and use the Matlab `rlocus` command to plot the root locus verses the integrator gain  $k_I$ . Note that since the integrator gain will be negative, the transfer function in Evan's form must be negated since the Matlab `rlocus` command only plots the return for  $k_I > 0$ . Select a value for  $k_I$  that does not significantly change the other locations of the closed loop poles.



## Homework E.10

The objective of this problem is to implement the PID controller using only measured outputs of the system.

- (a) Modify the system dynamics file so that the parameters  $m_1$ ,  $m_2$ , and  $\ell$  vary by up to 20% of their nominal value each time they are run (uncertainty parameter = 0.2).

- (b) Change the simulation files so that the input to the controller is the output and not the state. Implement the nested PID loops in Problems E.8 in a separate class. Use the dirty derivative gain of  $\sigma = 0.05$ . Tune the integrators so that there is no steady state error. The controller should only assume knowledge of the position  $z$ , the angle  $\theta$ , and the reference position  $z_r$ .
- (c) Note that the integrator gain will need to be negative which will cause problems for the anti-windup scheme that we have been implementing. Remove the old anti-windup scheme and implement a new scheme where the integrator only winds up when  $|\dot{z}|$  is small.



## Homework E.11

The objective of this problem is to implement a state feedback controller for the ball and beam problem. Start with the simulation files developed in Homework E.10.

- (a) Using the values for  $\omega_{n_z}$  and  $\zeta_z$  from Homework E.8, choose the desired locations of the four closed-loop poles so that their natural frequency is greater than  $\omega_{n_z}$  and damping ratio is greater than  $\zeta_z$ .
- (b) Add the state space matrices  $A$ ,  $B$ ,  $C$ ,  $D$  derived in Homework E.6 to your param file.
- (c) Verify that the state space system is controllable by checking that  $\text{rank}(\mathcal{C}_{A,B}) = n$ .
- (d) Find the feedback gain  $K$  such that the eigenvalues of  $(A - BK)$  are equal to the desired closed loop poles. Find the reference gain  $k_r$  so that the DC-gain from  $z_r$  to  $z$  is equal to one.
- (e) Implement the state feedback scheme in simulation and tune the closed-loop poles to vary the response of the system. Notice the effect of pole locations on the speed of response and the control effort required.



## Homework E.12

- (a) Modify the state feedback solution developed in Homework E.11 to add an integrator with anti-windup to the feedback loop for  $z$ .
- (b) Add a constant input disturbance of 1 Newtons to the input of the plant and allow the plant parameters to vary up to 20%.
- (c) Tune the integrator pole (and other gains if necessary) to get good tracking performance.



## Homework E.13

The objective of this problem is to design an observer that estimates the state of the system and to use the estimated state in the controller designed in Homework E.12.

- (a) For the sake of understanding the function of the observer, for this problem we will use exact parameters, without an input disturbance. Modify the ball beam dynamics so that the parameters known to the controller are the actual plant parameters (uncertainty parameter  $\alpha = 0$ ).
- (b) Verify that the state space system is observable by checking that  $\text{rank}(\mathcal{O}_{A,C}) = n$ .
- (c) In the control block, add an observer to estimate the state  $\hat{x}$ , and use the estimate of the state in your feedback controller. Tune the poles of the controller and observer to obtain good performance.
- (d) Modify the simulation files so that the controller outputs both  $u$  and  $\hat{x}$ . Add a plotting routine to plot both the state and the estimated state of the system on the same graph.
- (e) As motivation for the next chapter, add an input disturbance to the system of 0.5 and observe that there is steady state error in the response even though there is an integrator. This is caused by a steady state error in the observation error. In the next chapter we will show how to remove the steady state error in the observation error.



## Homework E.14

- (a) Modify your solution from HW E.13 so that the uncertainty parameter is  $\alpha = 0.2$ , representing 20% inaccuracy in the knowledge of the system parameters, and so that the input disturbance is 0.5. Also, add noise to the output channels  $z_m$  and  $\theta_m$  with standard deviation of 0.001.
- (b) Add a disturbance observer to the controller, and verify that the steady state error in the estimator has been removed. Tune the system to get good response.



## Homework E.15

- (a) Draw by hand the Bode plot of the inner loop transfer function from force  $\tilde{F}$  to angle  $\tilde{\theta}$  for the ball beam system. Use the `bode` command from Matlab or Python and compare your results.

- (b) Draw by hand the Bode plot of the outer loop transfer function from angle  $\tilde{\theta}$  to position  $\tilde{z}(t)$  for the ball beam system. Use the `bode` command and compare your results.



## Homework E.16

For the inner loop of the ball & beam system, use the Matlab or Python `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework E.8. Use the dirty derivative when implementing the derivative portion of the PD control.

- (a) To what percent error can the closed loop system under PD control track the desired input if all of the frequency content of  $\theta_r(t)$  is below  $\omega_r = 1.0$  radians per second?
- (b) If the frequency content of the input disturbance is all contained below  $\omega_{d_{in}} = 0.8$  radians per second, what percentage of the input disturbance shows up in the output?
- (c) If all of the frequency content of the noise  $n(t)$  is greater than  $\omega_{no} = 300$  radians per second, what percentage of the noise shows up in the output signal  $\theta$ ?

For the outer loop of the ball & beam, use the Matlab or Python `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PID control using the control gains calculated in Homework E.10. Use the dirty derivative when implementing the derivative portion of the PID control.

- (d) If the frequency content of an output disturbance is contained below  $\omega_{d_{out}} = 0.1$  radian/sec, what percentage of the output disturbance will be contained in the output under PID control?
- (e) If the reference signal is  $y_r(t) = 2 \sin(0.6t)$  what is the output error under PID control?



## Homework E.17

For this problem, use the gains found in HW E.10.

- (a) For the inner loop of the ball & beam system, use the `bode` and `margin` commands (from Matlab or Python) to find the phase and gain margin for the inner loop system under PD control. On the same graph, plot the open

loop Bode plot and the closed loop Bode plot. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency?

- (b) For the outer loop of the ball beam system, use the `bode` and `margin` commands to find the phase and gain margin for the outer loop system under PID control. Plot the open and closed loop Bode plots for the outer loop on the same plot as the open and closed loop for the inner loop. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency?
- (c) What is the bandwidth separation between the inner (fast) loop, and the outer (slow) loop? For this design, is successive loop closure justified?



## Homework E.18

For this homework assignment we will use the loopshaping design technique to design a successive loop closure controller for the ball and beam problem.

- (a) First consider the inner loop, where  $P_{in}(s)$  is the transfer function of the inner loop derived in HW E.5. Design the inner control system  $C_{in}(s)$  to stabilize the system with a phase margin close to 60 degrees, and to ensure that reference inputs with frequency below  $\omega_r = 1$  radians/sec have tracking error  $\gamma_r = 0.0032$ , and to ensure that noise above  $\omega_n = 1000$  radians/second is rejected by  $\gamma_n = 0.0032$ .
- (b) Now consider the design of the controller for the outer loop system. The 'plant' for the design of the outer loop controller is

$$P = P_{out} \frac{P_{in}C_{in}}{1 + P_{in}C_{in}}.$$

Design the outer loop controller  $C_{out}$  so that the system is stable with phase margin close to  $PM = 60$  degrees, and so that constant input disturbances are rejected, reference signals with frequency below  $\omega_r = 0.1$  radians/sec are tracked with error  $\gamma_r = 0.01$ , and noise with frequency content above  $\omega_n = 100$  radians/sec are rejected with  $\gamma_n = 0.001$ . Add a prefilter  $F(s)$  to reduce peaking in the closed loop transfer function.

- (c) Implement the developed control strategy in simulation using the state space equivalents of  $C(s)$  and  $F(s)$ .

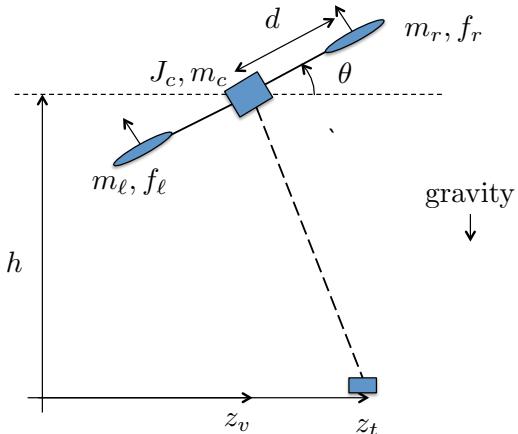
# F

## Design Study: Planar VTOL

In this design study we will explore the control design for a simplified planar version of a quadrotor following a ground target. In particular, we will constrain the dynamics to be in two dimension plane comprising vertical and one dimension of horizontal, as shown in [Fig. 21-1](#). The planar vertical take-off and landing (VTOL) system is comprised of a center pod of mass  $m_c$  and inertia  $J_c$ , a right motor/rotor that is modeled as a point mass  $m_r$  that exerts a force  $f_r$  at a distance  $d$  from the center of mass, and a left motor/rotor that is modeled as a point mass  $m_\ell$  that exerts a force  $f_\ell$  at a distance  $-d$  from the center of mass.

The position of the center of mass of the planar VTOL system is given by horizontal position  $z_v$  and altitude  $h$ . The airflow through the rotor creates a change in the direction of flow of air and causes what is called “momentum drag.” Momentum drag can be modeled as a viscous drag force that is proportional to the horizontal velocity  $\dot{z}_v$ . In other words, the drag force is  $F_{\text{drag}} = -\mu \dot{z}_v$ . The target on the ground will be modeled as an object with position  $z_t$  and altitude  $h = 0$ . We will not explicitly model the dynamics of the target.

Use the following physical parameters:  $m_c = 1 \text{ kg}$ ,  $J_c = 0.0042 \text{ kg m}^2$ ,  $m_r = 0.25 \text{ kg}$ ,  $m_\ell = 0.25 \text{ kg}$ ,  $d = 0.3 \text{ m}$ ,  $\mu = 0.1 \text{ kg/s}$ ,  $g = 9.81 \text{ m/s}^2$ .



[Figure 21-1](#): Planar vertical take-off and landing (VTOL)

**Homework Problems:****F.2** Kinetic energy.**F.3** Equations of Motion.**F.4** Linearize equations of motion.**F.5** Transfer function model.**F.6** State space model.**F.7** Pole Placement using PD.**F.8** Successive loop closure.**F.9** Integrators and system type.**F.P.6** Root locus.**F.10** Digital PID.**F.11** Full state feedback.**F.12** Full state with integrator.**F.13** Observer based control.**F.14** Disturbance observer.**F.15** Frequency Response.**F.16** Loop gain.**F.17** Stability margins.**F.18** Loopshaping design.



## Homework F.2

- (a) Using the configuration variables  $z_v$ ,  $h$ , and  $\theta$ , write an expression for the kinetic energy of the system.
- (b) Create an animation of the planar VTOL system. The inputs to the animation should be  $z_v$ ,  $z_t$ ,  $h$ , and  $\theta$ . Turn in a screen capture of the animation.



## Homework F.3

- (a) Find the potential energy for the system.
- (b) Define the generalized coordinates and damping forces.
- (c) Find the generalized forces. Note that the right and left forces are more easily modeled as a total force on the center of mass, and a torque about the center of mass.
- (d) Derive the equations of motion for the planar VTOL system using the Euler-Lagrange equations.
- (e) Referring to Appendices P.1, P.2, and P.3, write a class or s-function that implements the equations of motion. Simulate the system using a variable force inputs  $f_r$  and  $f_\ell$ . The output should connect to the animation function developed in homework F.2.



## Homework F.4

Since  $f_r$  and  $f_\ell$  appear in the equations as either  $f_r + f_\ell$  or  $d(f_r - f_\ell)$ , it is easier to think of the inputs as the total force  $F \triangleq f_r + f_\ell$ , and the torque  $\tau = d(f_r - f_\ell)$ . Note that since

$$\begin{pmatrix} F \\ \tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ d & -d \end{pmatrix} \begin{pmatrix} f_r \\ f_\ell \end{pmatrix},$$

that

$$\begin{pmatrix} f_r \\ f_\ell \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ d & -d \end{pmatrix}^{-1} \begin{pmatrix} F \\ \tau \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2d} \\ \frac{1}{2} & -\frac{1}{2d} \end{pmatrix} \begin{pmatrix} F \\ \tau \end{pmatrix}.$$

Therefore, in subsequent exercises, if we determine  $F$  and  $\tau$ , then the right and left forces are given by

$$\begin{aligned} f_r &= \frac{1}{2}F + \frac{1}{2d}\tau \\ f_\ell &= \frac{1}{2}F - \frac{1}{2d}\tau. \end{aligned}$$

- (a) Find the equilibria of the system.
- (b) Linearize the equations about the equilibria using Jacobian linearization.
- (c) If possible, linearize the system using feedback linearization.



## Homework F.5

For the planar VTOL system, note that the system naturally divides into so-called longitudinal dynamics (up-down motion) with total force  $\tilde{F}$  as the input and altitude  $\tilde{h}$  as the output, and lateral dynamics (side-to-side motion) with  $\tilde{\tau}$  as the input and  $\tilde{z}$  as the output, with  $\tilde{\theta}$  as an intermediate variable.

- (a) Start with the linearized equations of motion and use the Laplace transform to convert the equations of motion to the s-domain.
- (b) For the longitudinal dynamics, find the transfer function from the input  $\tilde{F}(s)$  to the output  $\tilde{H}(s)$ .
- (c) For the lateral dynamics, find the transfer function from the input  $\tilde{\tau}(s)$  to the intermediate state  $\tilde{\Theta}(s)$  and the output  $\tilde{Z}(s)$ . Find the transfer function from  $\tilde{\Theta}(s)$  to  $\tilde{Z}(s)$ .
- (d) Draw a block diagram of the open loop longitudinal and lateral systems.



## Homework F.6

- (a) Defining the longitudinal states as  $\tilde{x}_{lon} = (\tilde{h}, \dot{\tilde{h}})^\top$  and the longitudinal input as  $\tilde{u}_{lon} = \tilde{F}$  and the longitudinal output as  $\tilde{y}_{lon} = \tilde{h}$ , find the linear state space equations in the form

$$\begin{aligned}\dot{\tilde{x}}_{lon} &= A\tilde{x}_{lon} + B\tilde{u}_{lon} \\ \tilde{y}_{lon} &= C\tilde{x}_{lon} + D\tilde{u}_{lon}.\end{aligned}$$

- (b) Defining the lateral states as  $\tilde{x}_{lat} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$  and the lateral input as  $\tilde{u}_{lat} = \tilde{\tau}$  and the lateral output as  $\tilde{y}_{lat} = (\tilde{z}, \tilde{\theta})^\top$ , find the linear state space equations in the form

$$\begin{aligned}\dot{\tilde{x}}_{lat} &= A\tilde{x}_{lat} + B\tilde{u}_{lat} \\ \tilde{y}_{lat} &= C\tilde{x}_{lat} + D\tilde{u}_{lat}.\end{aligned}$$



## Homework F.7

For this problem we will only consider the longitudinal (up-down) dynamics of the VTOL system.

- (a) Given the open-loop transfer function from total force  $\tilde{F}$  to altitude  $\tilde{h}$  found in homework F.5, find the open-loop poles of the system.
- (b) Using PD control architecture shown in Figure 7-2, find the closed-loop transfer function from  $\tilde{h}_r$  to  $\tilde{h}$  and find the closed-loop poles as a function of  $k_P$  and  $k_D$ .
- (c) Select  $k_P$  and  $k_D$  to place the closed-loop poles at  $p_1 = -0.2$  and  $p_2 = -0.3$ .
- (d) Using the gains from part (c), implement the PD control for the altitude control in simulation and plot the step response.



## Homework F.8

For the VTOL system, do the following:

- (a) For the longitudinal (altitude) dynamics of the VTOL, suppose that the design requirements are that the rise time is  $t_r \approx 8$  seconds, with a damping ratio of  $\zeta = 0.707$ . Find the desired closed-loop characteristic polynomial  $\Delta_{cl}^d(s)$ , and the associated pole locations. Find the proportional and derivative gains  $k_{P_h}$  and  $k_{D_h}$  to achieve these specifications, and modify the computer simulation from HW F.8 to verify that the step response satisfies the requirements.
- (b) The lateral dynamics (side-to-side) will be designed using successive loop closure with the inner-loop using torque to regulate the roll angle, and the outer-loop using roll angle to regulate the position  $z$ . Focusing on the inner loop, find the PD gains  $k_{P_\theta}$  and  $k_{D_\theta}$  so that the rise time of the inner loop is  $t_{r_\theta} = 0.8$  seconds, and the damping ratio is  $\zeta_\theta = 0.707$ .
- (c) Find the DC gain  $k_{DC_\theta}$  of the inner loop.
- (d) Replacing the inner loop by its DC-gain, find the PD gains  $k_{P_z}$  and  $k_{D_z}$  so that the rise time of the outer loop is  $t_{r_z} = 10t_{r_\theta}$  and the damping ratio is  $\zeta_z = 0.707$ .
- (e) Implement the successive loop closure design for the lateral control of the VTOL system where the commanded lateral position is given by a square wave with magnitude  $3 \pm 2.5$  meters and frequency 0.08 Hz.

- (f) Suppose that the size of the force on each rotor is constrained by  $0 \leq f_\ell \leq f_{\max} = 10$  N, and  $0 \leq f_r \leq f_{\max} = 10$  N. Modify the simulation to include saturations on the right and left forces  $f_\ell$  and  $f_r$ . Using the rise time for the altitude controller, and the rise time for the outer loop of the lateral controller as tuning parameters, tune the controller to achieve the fastest possible response without input saturation.



## Homework F.9

- (a) When the longitudinal controller for the VTOL system is PD control, what is the system type? Characterize the steady state error when  $h^d$  is a step, a ramp, and a parabola. How does this change if you add an integrator? What is the system type with respect to an input disturbance for both PD and PID control?
- (b) When the inner loop of the lateral controller for the VTOL system is PD control, what is the system type of the inner loop? Characterize the steady state error when  $\tilde{\theta}^d$  is a step, a ramp, and a parabola. What is the system type with respect to an input disturbance?
- (c) When the outer loop of the lateral controller for the VTOL system is PD control, what is the system type of the outer loop? Characterize the steady state error when  $z^d$  is a step, a ramp, and a parabola. How does this change if you add an integrator? What is the system type with respect to an input disturbance for both PD and PID control?



## Homework F.P.6

- (a) Adding an integrator to obtain PID control for the longitudinal controller for the VTOL system, put the characteristic equation in Evan's form and use the Matlab/Python `rlocus` command to plot the root locus versus the integrator gain  $k_{I_h}$ . Select a value for  $k_{I_h}$  that does not significantly change the other locations of the closed loop poles.
- (b) Adding an integrator to obtain PID control for the outer loop of the lateral controller for the VTOL system, put the characteristic equation of the outer loop in Evan's form and use the Matlab/Python `rlocus` command to plot the root locus versus the integrator gain  $k_{I_z}$ . Select a value for  $k_{I_z}$  that does not significantly change the other locations of the closed loop poles.



## Homework F.10

The objective of this problem is to implement the PID controller using only measured outputs of the system.

- (a) Modify the system dynamics file so that the parameters  $m_c$ ,  $J_c$ ,  $d$ , and  $\mu$  vary by up to 20% of their nominal value each time they are run (uncertainty parameter = 0.2).
- (b) Change the simulation files so that the input to the controller is the output and not the state. Implement the nested PID loops in Problems F.8. Use the dirty derivative gain of  $\sigma = 0.05$ . Tune the integrator so that there is no steady state error. The controller should only assume knowledge of the position  $z$ , the angle  $\theta$ , the altitude  $h$ , the reference position  $z_r$ , and the reference altitude  $h_r$ .



## Homework F.11

The objective of this problem is to implement a state feedback controller for the VTOL system. Start with the simulation files developed in Homework F.10.

- (a) Using the values for  $\omega_{n_h}$ ,  $\zeta_h$ ,  $\omega_{n_z}$ , and  $\zeta_z$  from Homework F.8, choose the closed-loop pole locations to ensure that the longitudinal poles have damping ratios greater than  $\zeta_h$  and natural frequencies greater than  $\omega_{n_h}$ . Similarly choose the closed-loop poles for the lateral dynamics to ensure that their damping ratios are greater than  $\zeta_z$  and natural frequencies greater than  $\omega_{n_z}$ .
- (b) Add the state space matrices  $A$ ,  $B$ ,  $C$ ,  $D$  derived in Homework F.6 to your param file.
- (c) Verify that the state space system is controllable by checking that  $\text{rank}(\mathcal{C}_{A,B}) = n$ .
- (d) Find the feedback gain  $K$  such that the eigenvalues of  $(A - BK)$  are equal to desired closed loop poles. Find the reference gains  $k_{r_h}$  and  $k_{r_z}$  so that the DC-gain from  $h_r$  to  $h$  is equal to one, and the DC-gain from  $z_r$  to  $z$  is equal to one.
- (e) Implement the state feedback scheme and tune the closed-loop poles to get acceptable response. What changes would you make to your closed-loop pole locations to decrease the rise time of the system? How would you change them to lower the overshoot in response to a step command?



## Homework F.12

- (a) Modify the state feedback solution developed in Homework F.11 to add an integrator with anti-windup to the altitude feedback loop and to the position feedback loop.
- (b) Allow the plant parameters to vary up to 20% and add a constant input disturbance of 0.1 Newtons to the input of position dynamics simulating wind.  
*Hint: The best place to add the wind force is in the class that implements the dynamics. For example, one possibility is to modify the z dynamics as:*  

$$\ddot{z} = (-(f_r + f_l)\sin(\theta) - \mu\dot{z} + F_{wind}) / (m_c + 2m_r)$$
- (c) Tune the integrator poles on both loops (and other gains if necessary) to get good tracking performance.



## Homework F.13

The objective of this problem is to design an observer that estimates the state of the system and to use the estimated state in the controller designed in Homework F.12.

- (a) For the sake of understanding the function of the observer, for this problem we will use exact parameters, without an input disturbance. Modify the VTOL dynamics so that the parameters known to the controller are the actual plant parameters (uncertainty parameter  $\alpha = 0$ ).
- (b) Verify that the state space system is observable by checking that  $\text{rank}(\mathcal{O}_{A,C}) = n$ .
- (c) In the control block, add an observer to estimate the state  $\hat{x}$ , and use the estimate of the state in your feedback controller. Tune the poles of the controller and observer to obtain good performance.
- (d) Modify the simulation files so that the controller outputs both  $u$  and  $\hat{x}$ . Add a plotting routine to plot both the state and the estimated state of the system on the same graph.
- (e) As motivation for the next chapter, add an input force disturbance to the system of 1.0 and an input torque disturbance of 0.1 and observe that there is steady state error in the response even though there is an integrator. This is caused by a steady state error in the observation error. In the next chapter we will show how to remove the steady state error in the observation error.



## Homework F.14

- (a) Modify your solution from HW F.13 so that the uncertainty parameter is  $\alpha = 0.2$ , representing 20% inaccuracy in the knowledge of the system parameters. Modify VTOL\_dynamics to add an altitude disturbance of 1.0 and a wind disturbance of 1.0 m/s. A Python code snippet that implements these disturbances is given by

```

1  from math import sin
2  # disturbances
3  wind = 1.0
4  altitude_dist = 1.0
5
6  # add wind disturbance
7  zdot = zdot + wind
8
9  zddot = -(fr+f1)*sin(theta) / (mc+2*mr)
10 hddot = (-(mc+2*mr)*g \
11         + (fr+f1)*cos(theta)) / (mc+2*mr) \
12         + altitude_dist
13 thetaddot = d*(fr-f1) / (Jc+2*mr*d**2)

```

Before adding the disturbance observer, run the simulation and note that the controller is not robust to the input disturbance.

- (b) Add a disturbance observer to both controllers, and verify that the steady state error in the estimator has been removed and that the disturbances have been adequately compensated. Tune the system to get good response.



## Homework F.15

- (a) Draw by hand the Bode plot of the altitude transfer function from force  $\tilde{F}$  to altitude  $\tilde{h}$ . Use the `bode` command (from Matlab or Python) and compare your results.
- (b) Draw by hand the Bode plot of the inner loop transfer function for the lateral dynamics from torque  $\tau$  to angle  $\theta$ . Use the `bode` command and compare your results.
- (c) Draw by hand the Bode plot of the outer loop transfer function for the lateral dynamics from angle  $\theta$  to position  $z(t)$ . Use the `bode` command and compare your results.



## Homework F.16

For the altitude hold loop of the VTOL system, use the `bode` command (from Matlab or Python) to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PID control using the control gains calculated in Homework F.10. Use the dirty derivative when implementing the derivative portion of the PID control.

- (a) What is the tracking error if the reference input is a parabola with curvature 5?
- (b) If all of the frequency content of the noise  $n(t)$  is greater than  $\omega_{no} = 30$  radians per second, what percentage of the noise shows up in the output signal  $h$ ?

For the inner loop of the lateral controller for the VTOL system, use the `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework F.8. Use the dirty derivative when implementing the derivative portion of the PD control.

- (c) If the frequency content of the input disturbance is all contained below  $\omega_{d_{in}} = 2$  radians per second, what percentage of the input disturbance shows up in the output?
- (d) If a sensor for  $\theta$  is to be selected, what are the characteristics of the sensor (frequency band and size) that would result in measurement noise for  $\theta$  that is less than 0.1 degrees?

For the outer loop of the lateral controller for the VTOL system, use the `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PID control, using the control gains calculated in Homework F.10.

- (e) To what percent error can the closed loop system track the desired input if all of the frequency content of  $z_r(t)$  is below  $\omega_r = 0.1$  radians per second?
- (f) If the frequency content of an output disturbance is contained below  $\omega_{d_{out}} = 0.01$  radian/sec, what percentage of the output disturbance will be contained in the output?



## Homework F.17

For this problem, use the gains found in Homework F.10.

- (a) For the altitude hold loop of the VTOL system, use the `bode` and `margin` commands (from Matlab or Python) to find the phase and gain margin for the closed loop system under PID control. On the same graph, plot the open loop Bode plot and the closed loop Bode plot. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency?
- (b) For the inner loop of the lateral control loop, use the `bode` and `margin` commands to find the phase and gain margin for the inner loop system under PD control. On the same graph, plot the open loop Bode plot and the closed loop Bode plot. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency?
- (c) For the outer loop of the lateral control loop, use the `bode` and `margin` commands to find the phase and gain margin for the outer loop system under PD control. Plot the open and closed loop Bode plots for the outer loop on the same plot as the open and closed loop for the inner loop. What is the bandwidth of the closed loop system, and how does this relate to the crossover frequency?
- (d) What is the bandwidth separation between the inner (fast) loop, and the outer (slow) loop? For this design, is successive loop closure justified?



## Homework F.18

For this homework assignment we will use the loopshaping design technique to design a controller for the VTOL flight control problem.

- (a)** First consider the longitudinal or altitude control problem. For this part, the force command will be

$$F = F_e + C_{lon}(s)E_h(s),$$

where  $F_e$  is the equilibrium force as found in other homework problems, and the error signal is  $e_h = h_f^r - h_m$ , where  $h_f^r$  is the prefiltered reference altitude command, and  $h_m$  is the measured altitude. Find the longitudinal controller  $C_{lon}(s)$  so that the closed loop system meets the following specifications.

- Reject constant input disturbances.
- Track reference signals with frequency content below  $\omega_r = 0.1$  rad/s to within  $\gamma_r = 0.01$ .
- Attenuate noise on the measurement of  $h$  for all frequencies above  $\omega_n = 200$  rad/s by  $\gamma_n = 10^{-4}$ .
- The phase margin is close to  $PM = 60$  deg.

- Use a prefilter to improve the closed loop response.
- (b) Now consider the inner loop of the lateral, or position, dynamics. Find the inner loop controller  $C_{lat_{in}}(s)$  so that the inner loop is stable with phase margin approximately  $PM = 60$  deg, and with closed loop bandwidth approximately  $\omega_{co} = 10$  rad/s. Add a low pass filter to enhance noise rejection.
- (c) For the outer loop of the lateral system, find the controller  $C_{lat_{out}}(s)$  to meet the following specifications:
- Cross over frequency is around  $\omega_{co} = 1$  rad/s.
  - Reject constant input disturbances.
  - Attenuate noise on the measurement of  $z$  for all frequencies above  $\omega_{no} = 100$  rad/s by  $\gamma_{no} = 10^{-5}$ .
  - The phase margin is close to  $PM = 60$  deg.
  - Use a prefilter to reduce overshoot in the step response.
- (c) Implement the developed control strategy in simulation using the state space or digital filter equivalents of  $C(s)$  and  $F(s)$ .

# Appendices



# P.1

## Simulating Control Systems Using Python

In this appendix we will show how to simulate the feedback control of a dynamic system using Python. Consider the general feedback control system shown in Figure 1-1. The output of the physical system is the  $y(t)$ , the input to the physical system is  $u(t)$  plus an external disturbance signal  $d(t)$ . The output of the system is corrupted by zero-mean noise  $n(t)$ . The input to the controller is the reference signal  $r(t)$  as well as the output corrupted by noise. In Python, each block shown in Figure 1-1 is implemented using a Python Class.

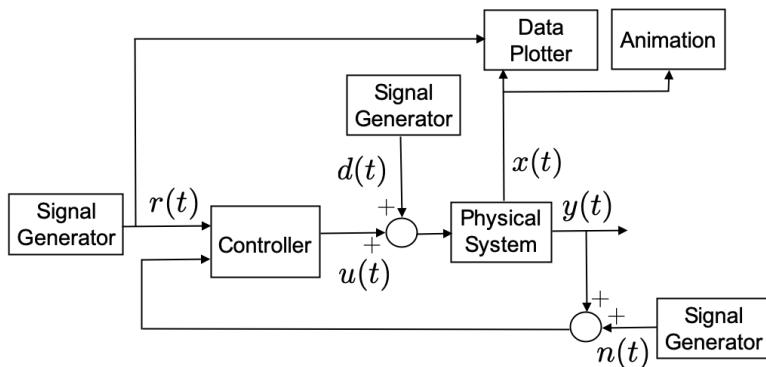


Figure 1-1: Block diagram of a general feedback system.

We will illustrate how to implement feedback control systems in Python by simulating the physical system described by the differential equation

$$\ddot{y} + 2\dot{y} + 3y = 4u, \quad (\text{P.1.1})$$

and using the PID controller

$$u(t) = k_p(r(t) - y(t)) + k_i \int_{-\infty}^t (r(\tau) - y(\tau)) d\tau - k_d \dot{y}. \quad (\text{P.1.2})$$

The feedback control system shown in Figure 1-1 is simulated in `systemSim.py` which is listed below.

```

1 import matplotlib.pyplot as plt
2 from systemDynamics import systemDynamics
3 from systemController import systemController
4 from signalGenerator import signalGenerator
5 from systemAnimation import systemAnimation
6 from dataPlotter import dataPlotter
7
8 # simulation parameters
9 t_start = 0.0 # start time
10 t_end = 20.0 # end time
11 t_plot = 0.1 # sample rate for plotter and animation
12 Ts = 0.01 # sample rate for dynamics and controller
13
14 # instantiate system, controller, and reference classes
15 system = systemDynamics(sample_rate=Ts)
16 controller = pendulumController(sample_rate=Ts)
17 reference = signalGenerator(amplitude=0.5, frequency=0.05)
18 disturbance = signalGenerator(amplitude=1.0, frequency = 0.0)
19 noise = signalGenerator(amplitude=0.01)
20
21 # instantiate the simulation plots and animation
22 dataPlot = dataPlotter()
23 animation = systemAnimation()
24
25 t = t_start # time starts at t_start
26 y = system.h() # output of system at start of simulation
27 while t < t_end: # main simulation loop
28     # set time for next plot
29     t_next_plot = t + t_plot
30     # Propagate dynamics and controller at fast rate Ts
31     while t < t_next_plot:
32         r = reference.square(t) # assign reference
33         d = disturbance.step(t) # simulate input disturbance
34         n = noise.random(t) # simulate sensor noise
35         u = controller.update(r, y + n) # update controller
36         y = system.update(u + d) # Propagate the dynamics
37         t = t + Ts # advance time by Ts
38     # update animation and data plots
39     animation.update(system.state)
40     dataPlot.update(t, ref_input, pendulum.state, u)
41     # pause causes the figure to be displayed during the simulation
42     plt.pause(0.0001)
43
44 # Keeps the program from closing until the user presses a button.
45 print('Press key to close')
46 plt.waitforbuttonpress()
47 plt.close()
```

**Listing P.1.1:** `systemSim.py`

The physical system is implemented in a class contained in `systemDynamic.py`, which will be discussed below, and which is imported on line 2, and instantiated on line 15. The system is updated at sample rate  $T_s$  given

the control input  $u$  and the input disturbance  $d$  on line 36.

The controller is implemented in a class contained in `systemController.py`, which will be discussed below, and which is imported on line 3, and instantiated on line 16. Given the reference input  $r$ , and the system output  $y$ , the controller is updated at the fast sample rate  $T_s$  on line 35. At the initial time, the controller needs the initial output at time  $t = 0$ , which is computed on line 26 before the main loop begins.

The reference input is implemented using the `signalGenerator` class that is contained in `signalGenerator.py`, which is described below, and which is imported on line 4 and instantiated on line 17. The reference input at time  $t$ , i.e.,  $r(t)$  is assigned a value on line 32.

The external disturbance  $d(t)$  is also implemented using the `signalGenerator` class and is instantiated on line 17, and assigned a value on line 33. For this example, the external disturbance will be a constant value of  $d(t) = 1.0$ . The disturbance is assigned on line 33, and is input to the system on line 36.

In addition to the feedback loop as implemented above, the system will be visualized through an animation, and also by plotting important variables like the reference input, the system states, and the control input. The animation and data plotting functions are implemented in `systemAnimation.py` and `dataPlotter.py`, are imported on lines 5 and 6, instantiated on lines 22 and 23, and updated on lines 39 and 40. The `pause` command on line 42 forces the animation and data plotting functions to update their displays each time through the loop.

## Appendix P.1.1 System Dynamics

This section will describe how to implement the system dynamics given in Equation (P.1.1). The first step is to convert the dynamics to the state space form

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u).\end{aligned}$$

For the differential equation in Equation (P.1.1) let  $x = (y, \dot{y})^\top$ , which gives

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ -2\dot{y} - 3y + 4u \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_2 - 3x_1 + 4u \end{pmatrix} \triangleq f(x, u) \quad (\text{P.1.3})$$

$$y = x_1 \triangleq h(x). \quad (\text{P.1.4})$$

The dynamics of the system are implemented in `systemDynamics.py` which is listed below.

```
1 import numpy as np
2
```

```

3 class systemDynamics:
4     def __init__(self, sample_rate):
5         # Initial state conditions
6         y0 = 0.0
7         ydot0 = 0.0
8         self.state = np.array([
9             [y0], # initial condition for y
10            [ydot0], # initial condition for ydot
11        ])
12        self.Ts = sample_rate # sample rate of system
13        self.limit = 1.0 # input saturation limit
14        # system parameters
15        self.a0 = 3.0
16        self.a1 = 2.0
17        self.b0 = 4.0
18        # modify the system parameters by random value
19        alpha = 0.2 # Uncertainty parameter
20        self.a1 = self.a1 * (1.+alpha*(2.*np.random.rand()-1.))
21        self.a0 = self.a0 * (1.+alpha*(2.*np.random.rand()-1.))
22        self.b0 = self.b0 * (1.+alpha*(2.*np.random.rand()-1.))
23
24    def f(self, state, u):
25        # for system xdot = f(x,u), return f(x,u)
26        y = state.item(0)
27        ydot = state.item(1)
28        # The equations of motion.
29        yddot = -self.a1 * ydot - self.a0 * y + self.b0 * u
30        # build xdot and return
31        xdot = np.array([[ydot], [yddot]])
32        return xdot
33
34    def h(self):
35        # Returns the measured output y = h(x)
36        y = self.state.item(0)
37        # return output
38        return y
39
40    def update(self, u):
41        # This is the external method that takes the input u(t)
42        # and returns the output y(t).
43        u = self.saturate(u, self.limit) # saturate the input
44        self.rk4_step(u) # propagate the state by one time step
45        y = self.h() # compute the output at the current state
46        return y
47
48    def rk4_step(self, u):
49        # Integrate ODE using Runge-Kutta RK4 algorithm
50        F1 = self.f(self.state, u)
51        F2 = self.f(self.state + self.Ts / 2 * F1, u)
52        F3 = self.f(self.state + self.Ts / 2 * F2, u)
53        F4 = self.f(self.state + self.Ts * F3, u)
54        self.state += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
55
56    def saturate(self, u, limit):
57        if abs(u) > limit:
58            u = limit*np.sign(u)
59        return u

```

---

 Listing P.1.2: systemDynamics.py

The system is initialized in lines 4–22. The state is initialized in lines 6–11 based on the initial conditions for  $y(0)$  and  $\dot{y}(0)$ . The system parameters are initialized in lines 15–22. The parameters for any physical system are never known exactly. Therefore, feedback systems need to be designed to be robust to this uncertainty. In the simulation, we model uncertainty in our knowledge of the physical parameters by changing the nominal value by a uniform random variable. Let  $\alpha$  be the amount of change allowed for each parameter, i.e.,  $\text{alpha} = 0.2$ , means that the physical parameters may change by up to 20%. The function `np.random.rand()` returns a uniform random variable in the interval  $[0, 1]$ . Therefore  $(1 + \alpha(2 * \text{np.random.rand()} - 1))$  is a random variable in the interval  $[-\alpha, \alpha]$ . The nominal value for the physical parameters that are known to the designer, are specified in lines 15–17. The actual values used by the simulation, are randomly selected in lines 19–22.

The system dynamics

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u).\end{aligned}$$

are specified in lines 24–38. Lines 24–32 define  $f(x, u)$ , and lines 34–38 define  $h(x, u)$ , as given by Equations (P.1.3) and (P.1.4) respectively.

The main feedback loop calls the `update` routine in lines 40–46. The `update` function first saturates the input in line 43, where the `saturation` function is defined in lines 56–59. The system dynamics are then propagated forward by one sample time  $T_s$  using the Runge-Kutta 4 algorithm in line 44, where the RK4 algorithm is specified in lines 48–54, as described in Appendix P.4. The `update` function finally computes and returns the output  $y$  in lines 45–46.

## Appendix P.1.2 Controller

This section will describe how to implement the PID controller given in Equation (P.3.2). The controller is implemented in `systemController.py`, which is listed below.

```

1 import numpy as np
2
3 class systemController:
4     def __init__(self, sample_rate):
5         # specify gains
6         self.kp = 1.0
7         self.kd = 1.0
8         self.ki = 1.0
9         self.Ts = sample_rate
10        self.sigma = 2.0 * self.Ts
11        self.beta = (2.0 * self.sigma - self.Ts) / (2.0 * self.sigma + self.Ts)

```

```

12     self.limit = 1.0
13     # internal storage for differentiator
14     self.y_dot = 0.0
15     self.y_d1 = 0.0
16     # internal storage for integrator
17     self.integrator = 0
18     self.error_d1 = 0
19
20     def update(self, r, y):
21         e = r - y # define the error
22         I = self.integrate_e(e) # integrate error
23         ydot = self.differentiate_y(y) # differentiate y
24         # form PID controller, saturate, and return
25         u = self.kp * e + self.ki * I - self.kd * ydot
26         u = self.saturate(u)
27         return u
28
29     def integrate_e(self, e):
30         self.integrator = self.integrator \
31             + (self.Ts / 2) * (error + self.error_d1)
32         self.error_d1 = error
33         return self.integrator
34
35     def differentiate_y(self, y):
36         self.y_dot = self.betta*self.y_dot \
37             + (1-self.betta)/self.Ts * (y - self.y_d1)
38         self.y_d1 = y
39         return self.y_dot
40
41     def saturate(self, u):
42         if abs(u) > self.limit:
43             u = self.limit*np.sign(u)
44         return u

```

Listing P.1.3: systemController.py

The controller is instantiated in lines 4–18. The PID gains are specified in lines 6–8. The sample rate of the controller is passed into the system at the time of instantiation and is assigned an internal variable  $T_s$  in line 9. The dirty derivative gain  $\sigma$  described in Chapter 10 is specified in line 10, and the intermediate gain  $\beta = \frac{2\sigma-T_s}{2\sigma+T_s}$  is specified on line 11. Internal storage variables for the differentiator and the integrator, as explained in Chapter 10 are given in lines 13–18. The PID control update routine is specified in lines 20–27. The error  $e = r - y$  as shown in Figure 1-1 is defined in line 21. The PID controller in Equations (P.3.2) uses the integral of  $e$  and the derivative of  $y$ . The integral of  $e$  is computed on line 22 using the routine specified in lines 29–33. A derivation of the integral equations is given in Chapter 10. The derivative of  $y$  is computed in line 23 using the routine specified in lines 35–39. A derivation of the dirty derivative equations is given in Chapter 10. Finally, the PID controller computed, the value is saturated, and returned in lines 25, 26, and 27, respectively.

### Appendix P.1.3 Signal Generator

Python code for the signal generator class is listed below.

```

1 import numpy as np
2
3 class signalGenerator:
4     def __init__(self, amplitude=1.0, frequency=0.001, y_offset=0):
5         self.amplitude = amplitude # signal amplitude
6         self.frequency = frequency # signal frequency
7         self.y_offset = y_offset # signal y-offset
8
9     def square(self, t):
10        if t % (1.0/self.frequency) <= 0.5/self.frequency:
11            out = self.amplitude + self.y_offset
12        else:
13            out = - self.amplitude + self.y_offset
14        return out
15
16    def sawtooth(self, t):
17        tmp = t % (0.5/self.frequency)
18        out = 4 * self.amplitude * self.frequency*tmp \
19              - self.amplitude + self.y_offset
20        return out
21
22    def step(self, t):
23        if t >= 0.0:
24            out = self.amplitude + self.y_offset
25        else:
26            out = self.y_offset
27        return out
28
29    def random(self, t):
30        out = np.random.normal(self.y_offset, self.amplitude)
31        return out
32
33    def sin(self, t):
34        out = self.amplitude * np.sin(2*np.pi*self.frequency*t) \
35              + self.y_offset
36        return out

```

Listing P.1.4: signalGenerator.py

The class is instantiated by passing in the desired amplitude of the signal, the frequency in Hertz, and a y-offset value. For random signals, the amplitude is interpreted to be the standard deviation of the signal. The signal generator can then be used to produce a square wave, a sawtooth signal, a step, a random signal, or a sinusoid, as specified in lines 9–36.

### Appendix P.1.4 Data Plotter

The objective of the data plotter is to plot the relevant signals during the simulation. For this simple example, we will plot the output  $y$  and the reference  $r$  on

one plot, the velocity  $\dot{y}$  on another plot, and the control signal  $u$  on a third plot. For plotting and animation, we will use the Python Matplotlib library. Python code for the data plotter class is listed below.

```

1 import matplotlib.pyplot as plt
2 from matplotlib.lines import Line2D
3 import numpy as np
4
5 plt.ion() # enable interactive drawing
6
7 class dataPlotter:
8     def __init__(self):
9         # Number of subplots = num_of_rows*num_of_cols
10        self.num_rows = 3      # Number of subplot rows
11        self.num_cols = 1      # Number of subplot columns
12        # Create figure and axes handles
13        self.fig, self.ax = plt.subplots(self.num_rows,
14                                         self.num_cols,
15                                         sharex=True)
16        # Instantiate lists to hold the time and data histories
17        self.time_history = [] # time
18        self.r_history = [] # reference r
19        self.y_history = [] # output y
20        self.ydot_history = [] # velocity ydot
21        self.u_history = [] # input u
22        # create a handle for every subplot.
23        self.handle = []
24        self.handle.append(subplotWindow(self.ax[0],
25                                         ylabel='y',
26                                         title='Simple System'))
27        self.handle.append(subplotWindow(self.ax[1],
28                                         ylabel='ydot'))
29        self.handle.append(subplotWindow(self.ax[2],
30                                         xlabel='t(s)',
31                                         ylabel='u'))
32
33    def update(self, time, reference, state, control):
34        # update the time history of all plot variables
35        self.time_history.append(time)
36        self.r_history.append(reference)
37        self.y_history.append(state.item(0))
38        self.ydot_history.append(state.item(1))
39        self.u_history.append(control)
40
41        # update the plots with associated histories
42        self.handle[0].update(self.time_history,
43                             [self.r_history, self.y_history])
44        self.handle[1].update(self.time_history,
45                             [self.ydot_history])
46        self.handle[2].update(self.time_history,
47                             [self.u_history])
48
49    class subplotWindow:
50        # Create each individual subplot.
51        def __init__(self, ax, xlabel='', ylabel='', title='',
52                     legend=None):

```

```

53     """
54         ax - This is a handle to the axes of the figure
55         xlabel - Label of the x-axis
56         ylabel - Label of the y-axis
57         title - Plot title
58         legend - A tuple of strings that identify the data.
59             EX: ("data1","data2", ... , "dataN")
60     """
61     self.legend = legend
62     self.ax = ax # Axes handle
63     self.colors = ['b', 'g', 'r', 'c', 'm', 'y', 'b']
64     # A list of colors. The first color in the list corresponds
65     # to the first line object, etc.
66     # 'b' - blue, 'g' - green, 'r' - red, 'c' - cyan,
67     # 'm' - magenta, 'y' - yellow, 'k' - black
68     self.line_styles = ['-', '--', '---', '-.', ':']
69     # A list of line styles. The first line style in the list
70     # corresponds to the first line object.
71     # '-' solid, '--' dashed, '-.' dash_dot, ':' dotted
72     self.line = []
73     # Configure the axes
74     self.ax.set_ylabel(ylabel)
75     self.ax.set_xlabel(xlabel)
76     self.ax.set_title(title)
77     self.ax.grid(True)
78     # Keeps track of initialization
79     self.init = True
80
81 def update(self, time, data):
82     # Adds data to the plot.
83     # Initialize the plot the first time routine is called
84     if self.init==True:
85         for i in range(len(data)):
86             # Instantiate line object and add it to the axes
87             self.line.append(Line2D(time, data[i],
88                                     color=self.colors[np.mod(i,len(self.colors))-1]),
89                                     ls=self.line_styles
90                                     [np.mod(i, len(self.line_styles) - 1)],
91                                     label=self.legend
92                                     if self.legend!=None else None))
93             self.ax.add_line(self.line[i])
94         self.init = False
95         # add legend if one is specified
96         if self.legend!=None:
97             plt.legend(handles=self.line)
98         else: # Add new data to the plot
99             # Updates the x and y data of each line.
100            for i in range(len(self.line)):
101                self.line[i].set_xdata(time)
102                self.line[i].set_ydata(data[i])
103
104        # Adjusts the axis to fit all of the data
105        self.ax.relim()
106        self.ax.autoscale()

```

Listing P.1.5: dataPlotter.py

The plotter is initialized in lines 9–31. Lines 10–11, specify that the plot window will contain three plots arranged as a  $3 \times 1$  array. The routine keeps track of the time histories of all variables to be plotted. Lines 17–21 initialize time histories for the time  $t$ , the reference signal  $r$ , the output  $y$ , and the velocity  $\dot{y}$ , and the input  $u$ . Lines 23–31 set up the individual plots with labels, by instantiating objects from the `subplotWindow` class, which will be described below.

The `update` method receives as input the current `time`, `reference`, `state`, and `control` values. Lines 35–39 appends the current values to the history lists maintained in memory for each of those values. The plots are then updated with the new histories in lines 42–47 by calling the `subplotWindow.update` method, which will be described below.

The `subplotWindow` class is instantiated in lines 50–79. If multiple data streams are on a single plot, the color of the line correspond to the order specified on line 63, and the style of the lines will be in the order specified on line 68. The `subplotWindow.update` method is specified in lines 81–102. The first time the `subplotWindow.update` method is called, the line objects are initialized and plotted in lines 85–97. In subsequent calls to `subplotWindow.update` the data lines are modified in lines 100–102.

## Appendix P.1.5 Animation

In this section we will show how to create an animation for the inverted pendulum case study. Consider the image of the inverted pendulum shown in [Fig. 1-2](#), where the configuration is completely specified by the position of the cart  $z$ , and the angle of the rod from vertical  $\theta$ . The physical parameters of the system are the rod length  $\ell$ , the base width  $w$ , the base height  $h$ , and the gap between the base and the track  $g$ .

The first step in developing the animation is to determine the position of points that define the animation. The pendulum will require that we draw three objects: the rectangular cart, the circular bob at the end of the rod, and the rod. We will draw the cart as a rectangle, the bob as a circle, and the rod as a line. In Python, rectangles are specified by their lower-left corner, their width, and their height. The lower left corner of the rectangle is  $(z - w/2, g)$ . In Python, circles are specified by their center and radius. The center of the bob is  $(z + (\ell + R) \sin \theta, g + h + (\ell + R) \cos \theta)$ . In Python, lines are specified by the  $x$  and  $y$  locations of their endpoints. The  $x$  locations of the endpoints of the rod are  $X = (z, z + \ell \sin \theta)$ , and the  $y$  locations of the endpoints are  $Y = (g + h, g + h + \ell \cos \theta)$ .

Python code for the animation class is listed below.

```

1 import matplotlib.pyplot as plt
2 import matplotlib.patches as mpatches
3 import numpy as np
4 import pendulumParam as P
5
6 class pendulumAnimation:
7     def __init__(self):

```

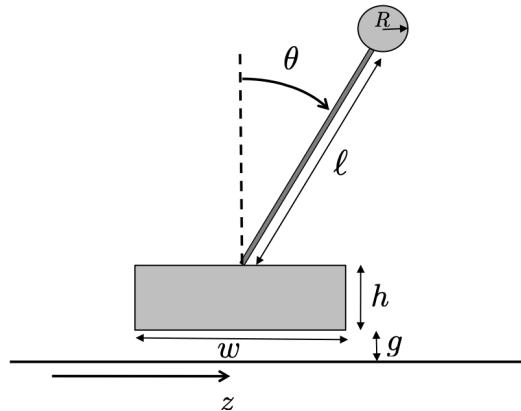


Figure 1-2: Drawing for inverted pendulum. The first step in developing an animation is to draw a figure of the object to be animated and identify all of the physical parameters.

```

8     self.flag_init = True # Used to indicate initialization
9     # Initialize a figure and axes object
10    self.fig, self.ax = plt.subplots()
11    # Initializes a list of objects (patches and lines)
12    self.handle = []
13    # Specify the x,y axis limits
14    plt.axis([-3*P.ell, 3*P.ell, -0.1, 3*P.ell])
15    # Draw line for the ground
16    plt.plot([-2*P.ell, 2*P.ell], [0, 0], 'b--')
17    # label axes
18    plt.xlabel('z')
19
20    def update(self, state):
21        z = state.item(0) # Horizontal position of cart, m
22        theta = state.item(1) # Angle of pendulum, rads
23        # draw plot elements: cart, bob, rod
24        self.draw_cart(z)
25        self.draw_bob(z, theta)
26        self.draw_rod(z, theta)
27        self.ax.axis('equal')
28        # Set initialization flag to False after first call
29        if self.flag_init == True:
30            self.flag_init = False
31
32    def draw_cart(self, z):
33        # specify bottom left corner of rectangle
34        x = z-P.w/2.0
35        y = P.gap
36        corner = (x, y)
37        # create rectangle on first call, update on subsequent calls
38        if self.flag_init is True:
39            # Create the Rectangle patch and append its handle
40            # to the handle list

```

```

41         self.handle.append(
42             mpatches.Rectangle(corners, P.w, P.h,
43                                 fc='blue', ec='black'))
44             # Add the patch to the axes
45             self.ax.add_patch(self.handle[0])
46     else:
47         self.handle[0].set_xy(corners) # Update patch
48
49     def draw_bob(self, z, theta):
50         # specify center of circle
51         x = z+(P.ell+P.radius)*np.sin(theta)
52         y = P.gap+P.h+(P.ell+P.radius)*np.cos(theta)
53         center = (x, y)
54         # create circle on first call, update on subsequent calls
55         if self.flag_init is True:
56             # Create the CirclePolygon patch and append its handle
57             # to the handle list
58             self.handle.append(
59                 mpatches.CirclePolygon(center, radius=P.radius,
60                                         resolution=15, fc='limegreen', ec='black'))
61             # Add the patch to the axes
62             self.ax.add_patch(self.handle[1])
63         else:
64             self.handle[1].xy = center
65
66     def draw_rod(self, z, theta):
67         # specify x-y points of the rod
68         X = [z, z+P.ell*np.sin(theta)]
69         Y = [P.gap+P.h, P.gap+P.h+P.ell*np.cos(theta)]
70         # create rod on first call, update on subsequent calls
71         if self.flag_init is True:
72             # Create the line object and append its handle
73             # to the handle list.
74             line, = self.ax.plot(X, Y, lw=1, c='black')
75             self.handle.append(line)
76         else:
77             self.handle[2].set_xdata(X)
78             self.handle[2].set_ydata(Y)

```

Listing P.1.6: pendulumAnimation.py

The pendulumAnimation class is initialized in lines 7–18. The flag `self.flag_init` is used to indicate the first time the animation function is called. On the first call when `self.flag_init==True`, all of the plot elements are initialized and configured (see lines 38–45, lines 55–62, and lines 71–75). After the first call, when when `self.flag_init==False`, each plot element is simply updated (see lines 47, lines 64, and lines 77–78). Line 10 initializes the plot window and axes, line 12 initializes an empty list of handles to plot objects, line 14 sets the size of the axes to be  $\pm 3\ell$  along the  $x$ -axes and  $-0.1$  to  $3\ell$  along the  $y$ -axis, line 16 draws the line representing the ground, and line 18 labels the  $x$ -axis as the configuration variable  $z$ .

The animation function is updated each time step using the `update` method shown in lines 20–30. The input to the `update` method is the state of the pendulum, of which the first element is  $z$  and the second element is  $\theta$  (lines 21–

22). Lines 24–26 then call methods, that will be described below, that draw the cart, bob, and rod. Line 27 ensures that the axes are not distorted. Lines 29–30 set the initialization flag `self.flat_init` to `False` after the first call to update.

The `draw_cart` method is listed in lines 32–47. The cart is drawn as a rectangle patch, which is specified by its lower-left corner, and the width and height of the rectangle. The lower-left corner or the rectangle are specified in lines 34–35, and the width and height are specified in line 42. The rectangle patch is created in lines 42–43 and appended to the list of handles in line 41. The patch is added to the graph in line 45. After initialization, the only element of the cart rectangle that changes with time is the lower-left corner, which is updated in line 47.

The `draw_bob` method is listed in lines 49–64. The bob is represented as a circle patch, which in Python is specified by the  $x - y$  coordinates of the center of the circle, and by its radius. The  $x - y$  center coordinates are given in lines 51–53. The circle patch is initialized in lines 59–60, where `resolution` is the angular resolution of the circle patch, `fc` is the color of the bob, and `ec` is the color of the outline. The circle patch is added to the handle list in line 58, and the handle is added to figure in line 62. After initialization, the only element of the bob that changes with time is it center, which is updated on line 64.

The `draw_rod` method is listed in lines 66–78. The rod is draw as a line, which in Python is specified by the  $x$  and  $y$ -coordinates of the start and end of the line, which are given in lines 68–69. The line object is created in line 74, and added to the handle list in line 75. After initialization, both ends of the line change with time and are updated on lines 77–78.



# P.2

## Simulating Control Systems Using Matlab

In this appendix we will show how to simulate the feedback control of a dynamic system using Matlab's object oriented class structure. Consider the general feedback control system shown in Figure 2-1. The output of the physical system is the  $y(t)$ , the inputs to the physical system are  $u(t)$  and an external disturbance signal  $d(t)$ . The output of the system  $y(t)$  is corrupted by zero-mean noise  $n(t)$ . The input to the controller is the reference signal  $r(t)$  as well as the output corrupted by noise. In Matlab, each block shown in Figure 2-1 can be implemented using a Matlab Class.

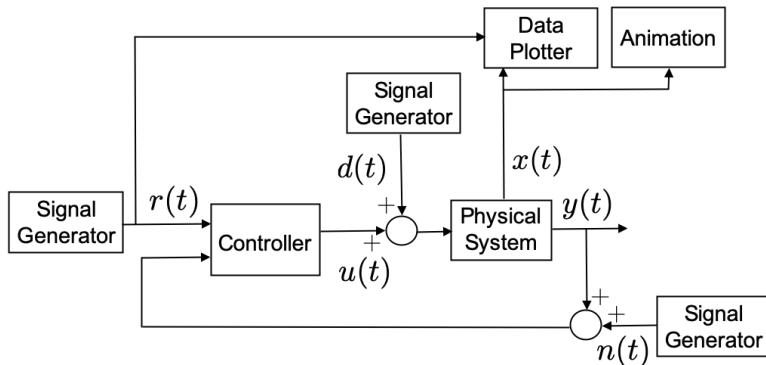


Figure 2-1: Block diagram of a general feedback system.

We will illustrate how to implement feedback control systems in Matlab by simulating the physical system described by the differential equation

$$\ddot{y} + 2\dot{y} + 3y = 4u, \quad (\text{P.2.1})$$

and using the PID controller

$$u(t) = k_p(r(t) - y(t)) + k_i \int_{-\infty}^t (r(\tau) - y(\tau)) d\tau - k_d \dot{y}. \quad (\text{P.2.2})$$

The feedback control system shown in Figure 2-1 is simulated in `systemSim.m` which is listed below.

```

1 t_start = 0; % start time
2 t_end = 20; % end time
3 t_plot = 0.1; % sample rate for plotter and animation
4 Ts = 0.01; % sample rate for dynamics and controller
5
6 % instantiate system, controller, and reference classes
7 system = systemDynamics(Ts);
8 controller = systemController(Ts);
9 reference = signalGenerator(0.5, 0.05);
10 disturbance = signalGenerator(1.0, 0.0);
11 noise = signalGenerator(0.01);
12
13 % instantiate the data plots and animation
14 dataPlot = dataPlotter();
15 animation = systemAnimation();
16
17 % main simulation loop
18 t = t_start; % time starts at t_start
19 y = system.h(); % system output at start of simulation
20 while t < t_end
21     % set time for next plot
22     t_next_plot = t + P.t_plot;
23     % updates control and dynamics at faster simulation rate
24     while t < t_next_plot
25         r = reference.square(t); % assign reference
26         d = disturbance.step(t); % simulate input disturbance
27         n = noise.random(t); % simulate noise
28         u = controller.update(r, y + n); % update controller
29         y = system.update(u+d); % Propagate the dynamics
30         t = t + Ts; % advance time by Ts
31     end
32     % update animation and data plots
33     animation.update(system.state);
34     dataPlot.update(t, r, system.state, u);
35 end

```

Listing P.2.1: `systemSim.m`

The physical system is implemented in a class contained in `systemDynamic.m`, which will be discussed below, and which is instantiated on line 7. The system is updated at sample rate  $T_s$  given the control input  $u$  and the input disturbance  $d$  on line 29.

The controller is implemented in a class contained in `systemController.m`, which will be discussed below, and which is instantiated on line 8. Given the reference input  $r$ , and the system output  $y$ , the controller is updated at the fast sample rate  $T_s$  on line 28. At the initial time, the controller

needs the initial output at time  $t = 0$ , which is computed on line 19 before the main loop begins.

The reference input is implemented using the `signalGenerator` class that is contained in `signalGenerator.m`, which is described below, and which is instantiated on line 9. The reference input at time  $t$ , i.e.,  $r(t)$  is assigned a value on line 25.

The external disturbance  $d(t)$  is also implemented using the `signalGenerator` class and is instantiated on line 10 and assigned a value on line 26. For this example, the external disturbance will be a constant value of  $d(t) = 1.0$ . The noise  $n(t)$  is also implemented using the `signalGenerator` class and is instantiated on line 11 and assigned a value on line 27.

In addition to the feedback loop as implemented above, the system will be visualized through an animation, and also by plotting important variables like the reference input, the system states, and the control input. The animation and data plotting functions are implemented in `systemAnimation.m` and `dataPlotter.m`, which are instantiated on lines 14–15, and updated on lines 33–34.

## Appendix P.2.1 System Dynamics

This section will describe how to implement the system dynamics given in Equation (P.2.1). The first step is to convert the dynamics to the state space form

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u).\end{aligned}$$

For the differential equation in Equation (P.2.1) let  $x = (y, \dot{y})^\top$ , which gives

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ -2\dot{y} - 3y + 4u \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_2 - 3x_1 + 4u \end{pmatrix} \triangleq f(x, u) \quad (\text{P.2.3})$$

$$y = x_1 \triangleq h(x). \quad (\text{P.2.4})$$

The dynamics of the system are implemented in `systemDynamics.m` which is listed below.

```

1 classdef systemDynamics < handle
2     properties
3         state
4         Ts
5         limit
6         a0
7         a1
8         b0
9     end
10    methods
11        %---constructor-----
```

```

12 function self = systemDynamics(sample_rate)
13     y0 = 0; % initial condition for y
14     ydot0 = 0; % initial condition for ydot
15     self.state = [... % initial state
16         y0;...
17         ydot0;...
18     ];
19     self.Ts = sample_rate;
20     self.limit = 1.0; % saturation limit
21     % system parameters
22     self.a0 = 3.0;
23     self.a1 = 2.0;
24     self.b0 = 4.0;
25     % modify the system parameters by random value
26     alpha = 0.2; % Uncertainty parameter
27     self.a0 = self.a0 * (1+2*alpha*rand-alpha);
28     self.a1 = self.a1 * (1+2*alpha*rand-alpha);
29     self.b0 = self.b0 * (1+2*alpha*rand-alpha);
30 end
31
32 function xdot = f(self, state, u)
33     % Return xdot = f(x,u)
34     y = state(1);
35     ydot = state(2);
36     % The equations of motion.
37     yddot = -self.a1 * ydot - self.a0 * y + self.b0 * u;
38     % build xdot and return
39     xdot = [ydot; yddot];
40 end
41
42 function y = h(self)
43     % y = h(x)
44     y = self.state(1);
45 end
46
47 function y = update(self, u)
48     % This is the external method that takes input u
49     % and returns output y.
50     u = self.saturate(u, self.limit); % saturate input
51     self.rk4_step(u); % propagate state by one time step
52     y = self.h(); % compute the output at current time
53 end
54
55 function self = rk4_step(self, u)
56     % Integrate ODE using Runge-Kutta-4 algorithm
57     F1 = self.f(self.state, u);
58     F2 = self.f(self.state + self.Ts/2*F1, u);
59     F3 = self.f(self.state + self.Ts/2*F2, u);
60     F4 = self.f(self.state + self.Ts*F3, u);
61     self.state = self.state...
62         + self.Ts/6 * (F1 + 2*F2 + 2*F3 + F4);
63 end
64
65 function out = saturate(self, in, limit)
66     if abs(in) > limit
67         out = limit * sign(in);
68     else

```

```

69         out = in;
70     end
71 end
72
73 end
74 end

```

Listing P.2.2: systemDynamics.m

In Matlab, the class definition `classdef systemDynamics < handle` must contain the `< handle` to ensure that internal methods can modify the class properties. The internal properties are specified in lines 2–9. The class methods are specified in lines 10–66. The class constructor is given in lines 12–30 and is called when a class object is instantiated. The state is initialized on lines 13–18 based on the initial conditions for  $y(0)$  and  $\dot{y}(0)$ . The system parameters are initialized in lines 21–29. The parameters for any physical system are never known exactly. Therefore, feedback systems need to be designed to be robust to this uncertainty. In the simulation, we model uncertainty in our knowledge of the physical parameters by changing the nominal value by a uniform random variable. Let  $\alpha$  be the amount of change allowed for each parameter, i.e., `alpha = 0.2`, means that the physical parameters may change by up to 20%. The function `rand()` returns a uniform random variable in the interval  $[0, 1]$ . Therefore  $(1 + \alpha(2 * np.random.rand() - 1))$  is a random variable in the interval  $[-\alpha, \alpha]$ . The nominal value for the physical parameters that are known to the designer, are specified in lines 22–24. The actual values used by the simulation, are randomly selected in lines 27–29.

The system dynamics

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u).\end{aligned}$$

are specified in lines 32–45. Lines 32–40 define  $f(x, u)$ , and lines 42–45 define  $h(x, u)$ , as given by Equations (P.1.3) and (P.1.4) respectively.

The main feedback loop calls the `update` method on lines 47–53. The `update` function first saturates the input on line 50, where the saturation function is defined in lines 64–70. The system dynamics are then propagated forward by one sample time  $T_s$  using the Runge-Kutta 4 algorithm on line 51, where the RK4 algorithm is specified on lines 55–62, as described in Appendix P.4. The `update` function finally computes and returns the output  $y$  on line 52.

## Appendix P.2.2 Controller

This section will describe how to implement the PID controller given in Equation (P.2.2). The controller is implemented in `systemController.m`, which is listed below.

```
| classdef systemController
```

```

2 properties
3     kp
4     kd
5     ki
6     Ts
7     sigma
8     beta
9     limit
10    ydot
11    y_d1
12    integrator
13    error_d1
14 end
15 methods
16     %-----constructor-----
17     function self = systemController(sample_rate)
18         % specify gains
19         self.kp = 1.0;
20         self.kd = 1.0;
21         self.ki = 1.0;
22         self.Ts = sample_rate;
23         self.sigma = 2.0 * self.Ts;
24         self.beta = (2.0*sigma-self.Ts)/(2.0*sigma+self.Ts);
25         self.limit = 1.0;
26         % internal storage for differentiator
27         self.ydot = 0.0;
28         self.y_d1 = 0.0;
29         % internal storage for integrator
30         self.integrator = 0;
31         self.error_d1 = 0;
32     end
33
34     function u = update(self, r, y)
35         e = r - y; % define the error
36         I = self.integrate_e(e); % integrate error
37         ydot = self.differentiate_y(y); %differentiate y
38         % form PID controller, saturate, and return
39         u = self.kp * e + self.ki * I - self.kd * ydot;
40         u = self.saturate(u);
41     end
42
43     function I = integrate_e(self, error)
44         self.integrator = self.integrator...
45             + (self.Ts / 2) * (error + self.error_d1);
46         self.error_d1 = error;
47         I = self.integrator;
48     end
49
50     function ydot = differentiate_y(self, y)
51         self.ydot = self.beta*self.ydot...
52             + (1-self.beta)/self.Ts * (y - self.y_d1);
53         self.y_d1 = y;
54         ydot = self.ydot;
55     end
56
57     function out = saturate(self, in):
58         if abs(in) > self.limit

```

```

59         out = self.limit * sign(u);
60     else
61         out = in;
62     end
63 end
64
65 end
66 end

```

Listing P.2.3: systemController.m

The controller properties are specified on lines 2–14, and instantiated in the constructor on lines 17–32. The PID gains are specified on lines 19–21. The sample rate of the controller is passed into the system at the time of instantiation and is assigned an internal variable  $T_s$  in line 9. The dirty derivative gain  $\sigma$  described in Chapter 10 is specified on line 22, and the intermediate gain  $\beta = \frac{2\sigma-T_s}{2\sigma+T_s}$  is specified on line 24. Internal storage variables for the differentiator and the integrator, as explained in Chapter 10 are given in lines 27–31. The PID control update routine is specified in lines 34–41. The error  $e = r - y$  as shown in Figure 1-1 is defined on line 35. The PID controller in Equations (P.2.2) uses the integral of  $e$  and the derivative of  $y$ . The integral of  $e$  is computed on line 36 using the routine specified on lines 43–48. A derivation of the integral equations is given in Chapter 10. The derivative of  $y$  is computed in line 37 using the routine specified in lines 50–53. A derivation of the dirty derivative equations is given in Chapter 10. Finally, the PID controller computed, the value is saturated, and returned on lines 39, and 40, respectively.

## Appendix P.2.3 Signal Generator

Matlab code for the signal generator class is listed below.

```

1 classdef signalGenerator
2 % produces: square_wave, step, sawtooth_wave,
3 %             random_wave, sin_wave.
4 properties
5     amplitude
6     frequency
7     y_offset
8 end
9 methods
10    %---constructor-----
11    function self = signalGenerator...
12        amplitude, frequency, y_offset)
13        self.amplitude = amplitude;
14        if nargin>1
15            self.frequency = frequency;
16        else
17            self.frequency = 1;
18        end
19        if nargin>2
20            self.y_offset = y_offset;

```

```

21         else
22             self.y_offset = 0;
23         end
24     end
25
26     function out = square(self, t)
27         if mod(t, 1/self.frequency)<=0.5/self.frequency
28             out = self.amplitude + self.y_offset;
29         else
30             out = -self.amplitude + self.y_offset;
31         end
32     end
33
34     function out = step(self, t)
35         if t>=0
36             out = self.amplitude + self.y_offset;
37         else
38             out = 0;
39         end
40     end
41
42     function out = sawtooth(self, t)
43         out = 4*self.amplitude*self.frequency...
44             * mod(t, 0.5/self.frequency)...
45             - self.amplitude + self.y_offset;
46     end
47
48     function out = random(self, t)
49         out = sqrt(self.amplitude)*randn + self.y_offset;
50     end
51
52     function out = sin(self, t)
53         out = self.amplitude*sin(2*pi*self.frequency*t)...
54             + self.y_offset;
55     end
56 end
57 end

```

Listing P.2.4: signalGenerator.m

The class is instantiated by passing in the desired amplitude of the signal, the frequency in Hertz, and a y-offset value. For random signals, the amplitude is interpreted to be the standard deviation of the signal. The signal generator can then be used to produce a square wave, a sawtooth signal, a step, a random signal, or a sinusoid, as specified in lines 26–55.

## Appendix P.2.4 Data Plotter

The objective of the data plotter is to plot the relevant signals during the simulation. For this simple example, we will plot the output  $y$  and the reference  $r$  on one plot, the velocity  $\dot{y}$  on another plot, and the control signal  $u$  on a third plot. Matlab code for the data plotter class is listed below.

```

1 classdef dataPlotter < handle
2     properties
3         % data histories
4         time_history
5         r_history
6         y_history
7         ydot_history
8         u_history
9         index
10        % figure handles
11        r_handle
12        y_handle
13        ydot_handle
14        u_handle
15    end
16    methods
17        %--constructor-----
18        function self = dataPlotter(P)
19            % Instantiate lists to hold the time and data histories
20            self.time_history...
21                = NaN*ones(1, (P.t_end-P.t_start)/P.t_plot);
22            self.r_history...
23                = NaN*ones(1, (P.t_end-P.t_start)/P.t_plot);
24            self.y_history...
25                = NaN*ones(1, (P.t_end-P.t_start)/P.t_plot);
26            self.ydot_history...
27                = NaN*ones(1, (P.t_end-P.t_start)/P.t_plot);
28            self.u_history...
29                = NaN*ones(1, (P.t_end-P.t_start)/P.t_plot);
30            self.index = 1;
31            % Create figure and axes handles
32            figure(2), clf
33            subplot(311)
34                hold on
35                self.r_handle...
36                    = plot(self.time_history, ...
37                        self.r_history, 'g');
38                self.y_handle...
39                    = plot(self.time_history, ...
40                        self.y_history, 'b');
41                ylabel('y')
42                title('System Data')
43            subplot(312)
44                hold on
45                self.ydot_handle...
46                    = plot(self.time_history, ...
47                        self.ydot_history, 'b');
48                ylabel('ydot')
49            subplot(313)
50                hold on
51                self.u_handle...
52                    = plot(self.time_history, ...
53                        self.u_history, 'b');
54                ylabel('u')
55            end
56
57        function self=update(self, time, reference, states, control)

```

```

58    % update the time history of all plot variables
59    self.time_history(self.index) = time;
60    self.r_history(self.index) = reference;
61    self.y_history(self.index) = states(1);
62    self.ydot_history(self.index) = states(2);
63    self.u_history(self.index) = control;
64    self.index = self.index + 1;
65    % update the plots with associated histories
66    set(self.r_handle, ...
67        'Xdata', self.time_history, ...
68        'Ydata', self.r_history)
69    set(self.y_handle, ...
70        'Xdata', self.time_history, ...
71        'Ydata', self.y_history)
72    set(self.ydot_handle, ...
73        'Xdata', self.time_history, ...
74        'Ydata', self.ydot_history)
75    set(self.u_handle, ...
76        'Xdata', self.time_history, ...
77        'Ydata', self.u_history)
78
79    end
80
81 end

```

Listing P.2.5: dataPlotter.m

The plotter constructor is listed on lines 18–55. Lines 20–29 initialize arrays that will hold the data histories. Allocating memory for the arrays upon initialization speeds up the run-time execution of the data plotter. Lines 32–54 set up the plot window with two subplots and plot the initial data. The Matlab `plot` routine returns a Matlab `handle` to the line, which is a data structure the contains all data associated with each line. The first subplot (311) will plot  $r(t)$  in green ('`g`'), and  $y(t)$  in blue ('`b`'). The second subplot (312) will plot  $\dot{y}$  in blue, and the third subplot (313) will plot  $u(t)$  in blue.

The `update` method receives as input the current `time`, `reference`, `state`, and `control` values. Lines 59–63 appends the current values to their histories. The plots are then updated with the new histories in lines 66–77 by calling the `set` routine and passing in the associated `handle`.

## Appendix P.2.5 Animation

In this section we will show how to create an animation for the inverted pendulum case study. Consider the image of the inverted pendulum shown in [Fig. 2-2](#), where the configuration is completely specified by the position of the cart  $z$ , and the angle of the rod from vertical  $\theta$ . The physical parameters of the system are the rod length  $\ell$ , the base width  $w$ , the base height  $h$ , and the gap between the base and the track  $g$ .

The first step in developing the animation is to determine the position of points that define the animation. The pendulum will require that we draw three objects: the rectangular cart, the circular bob at the end of the rod, and the rod. We will

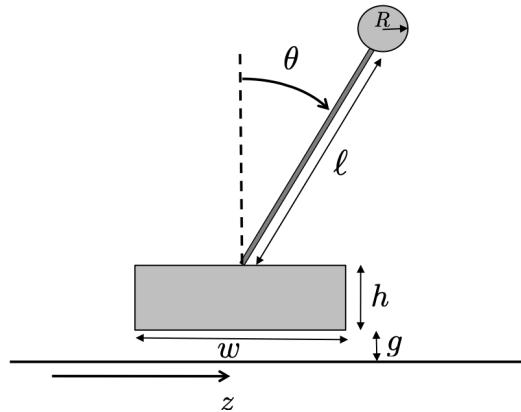


Figure 2-2: Drawing for inverted pendulum. The first step in developing an animation is to draw a figure of the object to be animated and identify all of the physical parameters.

draw the cart as a rectangle, the bob as a circle, and the rod as a line. In Matlab, rectangles are specified by their corners. The four corners of the rectangle are given by

$$C = \begin{bmatrix} z - w/2, & g \\ z_w/2, & g \\ z + w/2, & g + h \\ z - w/2, & g + h \end{bmatrix}.$$

In Matlab, lines are specified by the  $x$  and  $y$  locations of their endpoints. The  $x$  locations of the endpoints of the rod are  $X = [z, z + \ell \sin \theta]$ , and the  $y$  locations of the endpoints are  $Y = [g + h, g + h + \ell \cos \theta]$ . In Matlab, circles are drawn as a sequence of small line segments, that must be specified by the  $x$  and  $y$  locations. The center of the bob is given by  $C = (z + (\ell + R) \sin \theta, g + h + (\ell + R) \cos \theta)$ , and so points on the edge of the bob will be given by  $X = [C_x + R \cos \theta]$  and  $Y = [C_y + R \sin \theta]$ . Matlab code for the animation class is listed below.

```

1 classdef pendulumAnimation
2 properties
3     cart_handle
4     rod_handle
5     bob_handle
6     ell
7     width
8     height
9     gap
10    radius
11 end
12 methods
13     %-----constructor-----
14     function self = pendulumAnimation(P)
15         self.ell = P.ell;

```

```

16     self.width = P.w;
17     self.height = P.h;
18     self.gap = P.gap;
19     self.radius = P.radius;
20
21     figure(1), clf
22     % draw ground track
23     plot([-2*self.ell, 2*self.ell], [0,0],'k');
24     hold on
25     % initialize the cart, rod, and bob
26     self=self.draw_cart(P.z0);
27     self=self.draw_rod(P.z0, P.theta0);
28     self=self.draw_bob(P.z0, P.theta0);
29     % Set the x,y axis limits
30     axis([-3*self.ell, 3*self.ell, -0.1, 3*self.ell]);
31     xlabel('z'); % label x-axis
32 end
33
34 function self=update(self, state)
35     % external method to update the animation
36     z= state(1); % Horizontal position of cart, m
37     theta = state(2); % Angle of pendulum, rads
38     self=self.draw_cart(z);
39     self=self.draw_rod(z, theta);
40     self=self.draw_bob(z, theta);
41     drawnow
42 end
43
44 function self=draw_cart(self, z)
45     % specify X-Y locations of corners of the cart
46     X = [z-self.width/2, z+self.width/2, ...
47           z+self.width/2, z-self.width/2];
48     Y = [self.gap, self.gap, ...
49           self.gap+self.height, self.gap+self.height];
50     % plot or update cart
51     if isempty(self.cart_handle)
52         self.cart_handle = fill(X,Y,'b');
53     else
54         set(self.cart_handle,'XData',X);
55     end
56 end
57
58 function self=draw_rod(self, z, theta)
59     % specify X-Y locations of ends or rod
60     X = [z, z+self.ell*sin(theta)];
61     Y = [...
62           self.gap+self.height, ...
63           self.gap + self.height + self.ell*cos(theta)...
64           ];
65     % plot or update rod
66     if isempty(self.rod_handle)
67         self.rod_handle = plot(X, Y, 'k');
68     else
69         set(self.rod_handle,'XData', X, 'YData', Y);
70     end
71 end
72

```

```

73 function self=draw_bob(self, z, theta)
74     th = 0:2*pi/10:2*pi;
75     % center of bob
76     center = [z + (self.ell+self.radius)*sin(theta) ...
77                 self.gap+self.height + ...
78                 (self.ell+self.radius)*cos(theta)];
79     % points that define the bob
80     X = center(1) + self.radius*cos(th);
81     Y = center(2) + self.radius*sin(th);
82     % plot or update bob
83     if isempty(self.bob_handle)
84         self.bob_handle = fill(X,Y,'g');
85     else
86         set(self.bob_handle,'XData',X);
87         set(self.bob_handle,'YData',Y);
88     end
89 end
90 end
91 end

```

Listing P.2.6: pendulumAnimation.m

The `pendulumAnimation` class is initialized on lines 14–32. Lines 15–19 sets all of the plot parameters based on data passed through the structure `P`. Line 21 initializes the plot window and line 23 draws the ground track, which will not change during the animation. Lines 26–26 draws the base, bob, and line for the first time, where a Matlab handle is returned to each object in the animation. Line 30 sets the plot axes, and line 31 labels the plot.

The animation function is updated each time step using the `update` method on lines 34–42. The input to the `update` method is the state of the pendulum, of which the first element is  $z$  and the second element is  $\theta$  (lines 36–37). Lines 38–40 then call methods, that will be described below, that draw the cart, rod, and bob. Line 41 forces Matlab to update the animation at each time step.

The `draw_cart` method is listed on lines 44–56. The cart is drawn using the `fill` routine, where the  $x$  and  $y$  locations of each corner must be specified, as shown on lines 46–49. The first time `draw_cart` is called, the `cart_handle` will not exist, and so it is created through a call to `fill` on line 52. Upon subsequent calls to `draw_cart`, the handle will exist and so the cart location is updated through the `set` command on line 54. Since the  $y$  data of the cart doesn't change with time, only the  $x$  data is updated.

The `draw_rod` method is listed on lines 58–71. The rod is drawn using the `plot` routine, where the  $x$  and  $y$  locations of the ends of the rod must be specified, as shown on lines 60–64. The first time `draw_rod` is called, the `rod_handle` will not exist, and so it is created through a call to `plot` on line 67. Upon subsequent calls to `draw_rod`, the handle will exist and so the rod points are updated through the `set` command on line 69. Note that both the  $x$  and the  $y$  data of the rod must be updated at each time step.

The `draw_bob` method is listed on lines 73–90. The bob is drawn using the `fill` routine, where the  $x$  and  $y$  locations of each point on the bob must be specified. The points are parametrized by  $\vartheta$  which ranges from zero to  $2\pi$ . Line 74

establishes an array of  $N = 10$  angle points. The center of the bob is specified on lines 76–79, and the  $x$  and  $y$  points along the outline of the bob are given on lines 81–82. The first time `draw_bob` is called, the `bob_handle` will not exist, and so it is created through a call to `fill` on line 85. Upon subsequent calls to `draw_bob`, the handle will exist and so the cart location is updated through the `set` command on lines 87–88. Note that both the  $x$  and the  $y$  data of the bob must be updated at each time step.

# P.3

## Simulating Control Systems Using Simulink

In this appendix we will show how to simulate the feedback control of a dynamic system using Simulink. Consider the general feedback control system shown in Figure 3-1. The output of the physical system is  $y(t)$ , the inputs to the physical system are  $u(t)$  and an external disturbance signal  $d(t)$ . The output of the system  $y(t)$  is corrupted by zero-mean noise  $n(t)$ . The input to the controller is the reference signal  $r(t)$  as well as the output corrupted by noise.

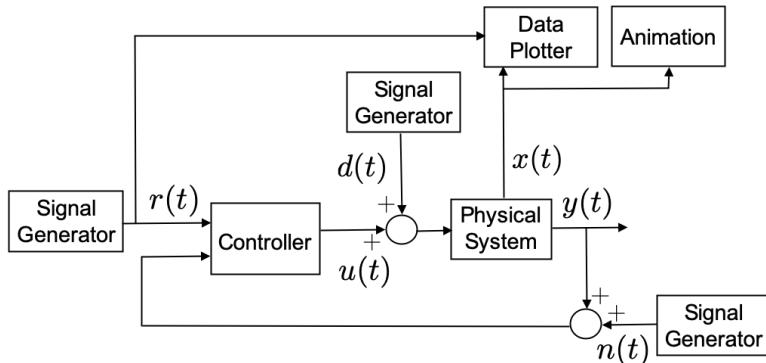


Figure 3-1: Block diagram of a general feedback system.

We will illustrate how to implement feedback control systems in Simulink by simulating the physical system described by the differential equation

$$\ddot{y} + 2\dot{y} + 3y = 4u, \quad (\text{P.3.1})$$

and using the PID controller

$$u(t) = k_p(r(t) - y(t)) + k_i \int_{-\infty}^t (r(\tau) - y(\tau)) d\tau - k_d \dot{y}. \quad (\text{P.3.2})$$

The Simulink diagram that corresponds to Figure 3-1 is shown in Figure 3-2. The system dynamics are implemented in the `system` block, the controller is

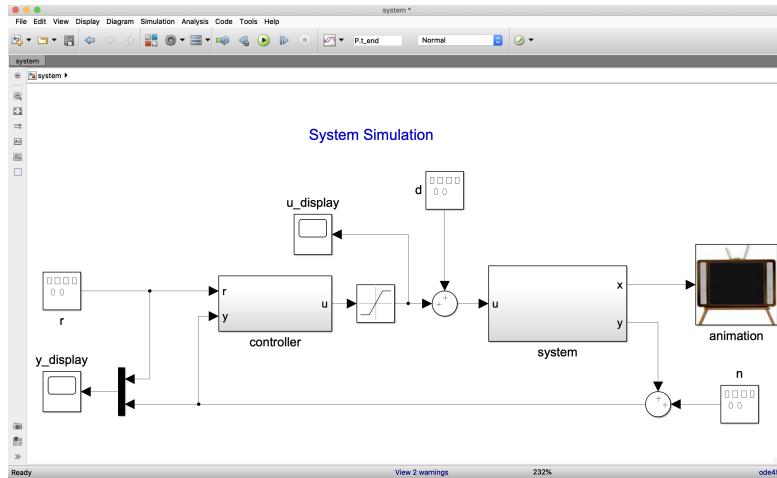


Figure 3-2: Simulink diagram implementing the feedback control loop in Figure 3-1.

implemented in the `controller` block, and the animation is implemented in the `animation` block. The reference  $r(t)$ , the disturbance  $d(t)$ , and the noise  $n(t)$  are all implemented using the Simulink built in Signal Generator block. The built in Simulink Scope blocks are used to display the output  $y$  and the reference  $r$ , as well as the input  $u$ .

System parameters can be loaded into Simulink at the beginning of a simulation by clicking on the Model Properties menu, as shown in Figure 3-3, and then selecting Callbacks/InitFcn, as shown in Figure 3-4. The commands listed under Model initialization function: are executed at the beginning of each Simulink simulation.

### Appendix P.3.1 System Dynamics

In this section we will describe how to implement the system dynamics. Double clicking on the `system` block shown in Figure 3-2 results in the Simulink sub-diagram shown in Figure 3-5. Double clicking on the System Dynamics block results in Figure 3-6, which shows an s-function dialog block. In Simulink, system dynamics are implemented using s-functions. Simulink is essentially a sophisticated tool for solving interconnected hybrid ordinary differential and

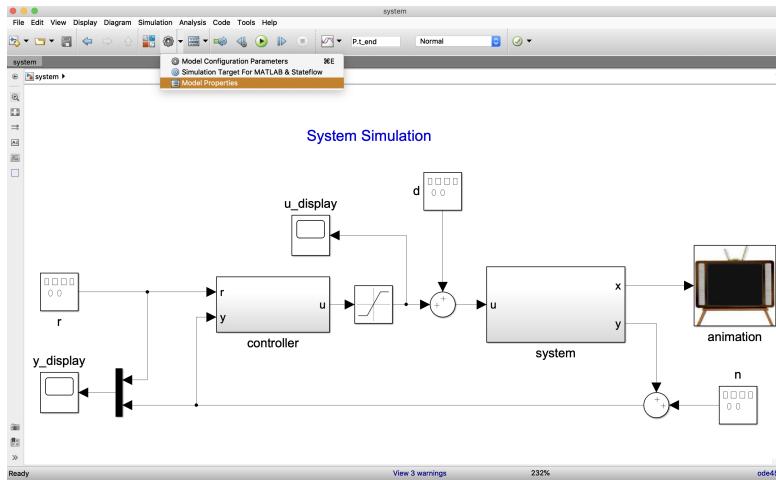


Figure 3-3: Simulink Model Properties menu.

difference equations. Each block in Simulink is assumed to have the structure

$$\dot{x}_c = f(t, x_c, x_d, u); \quad x_c(0) = x_{c0} \quad (\text{P.3.3})$$

$$x_d[k+1] = g(t, x_c, x_d, u); \quad x_d[0] = x_{d0} \quad (\text{P.3.4})$$

$$y = h(t, x_c, x_d, u), \quad (\text{P.3.5})$$

where  $x_c \in \mathbb{R}^{n_c}$  is a continuous state with initial condition  $x_{c0}$ ,  $x_d \in \mathbb{R}^{n_d}$  is a discrete state with initial condition  $x_{d0}$ ,  $u \in \mathbb{R}^m$  is the input to the block,  $y \in \mathbb{R}^p$  is the output of the block, and  $t$  is the elapsed simulation time. An s-function is a Simulink tool for explicitly defining the functions  $f$ ,  $g$ , and  $h$  and the initial conditions  $x_{c0}$  and  $x_{d0}$ . As explained in the Matlab/Simulink documentation, there are a number of methods for specifying an s-function. In this section, we will overview the method of using a level-1 m-file s-function.

The system given in Equation (P.3.1) is implemented using the s-function shown below.

```

1 function [sys,x0,str,ts,simStateCompliance]...
2     =system_dynamics(t,x,u,flag)
3 switch flag
4     case 0 % Initialize block
5         [sys,x0,str,ts,simStateCompliance]=mdlInitializeSizes;
6     case 1 % define xdot = f(t,x,u)
7         sys=mdlDerivatives(t,x,u);
8     case 3 % define y = h(t,x,u)
9         sys=mdlOutputs(t,x,u);
10    % GetTimeOfNextVarHit
11 otherwise
12     DASstudio.error('...
13         'Simulink:blocks:unhandledFlag',num2str(flag));
14 end

```

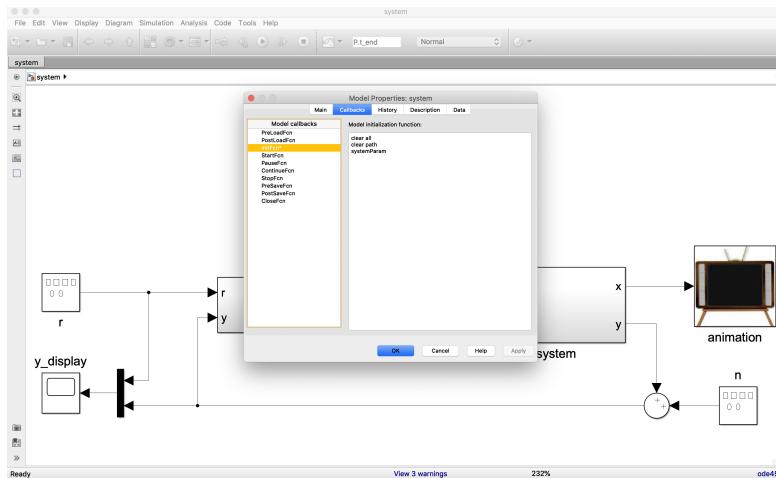


Figure 3-4: Simulink Callback/InitFcn script.

```

15 %=====
16 function [sys,x0,str,ts,simStateCompliance]=mdlInitializeSizes
17 sizes = simsizes;
18 sizes.NumContStates = 2;
19 sizes.NumDiscStates = 0;
20 sizes.NumOutputs = 1+2;
21 sizes.NumInputs = 1;
22 sizes.DirFeedthrough = 0;
23 sizes.NumSampleTimes = 1;
24 sys = simsizes(sizes);
25
26 % define initial conditions
27 y0 = 0; % initial condition for y
28 ydot0 = 0; % initial condition for ydot
29 x0 = [y0; ydot0];
30 str = []; % str is always an empty matrix
31 % initialize the array of sample times
32 ts = [0 0];
33 simStateCompliance = 'UnknownSimState';
34
35 %=====
36 function sys=mdlDerivatives(t,x,u)
37 % Return xdot = f(x,u)
38 y = x(1);
39 ydot = x(2);
40
41 % system parameters
42 persistent a0
43 persistent a1
44 persistent b0
45 if t==0
46     a0 = 3.0;
47     a1 = 2.0;

```

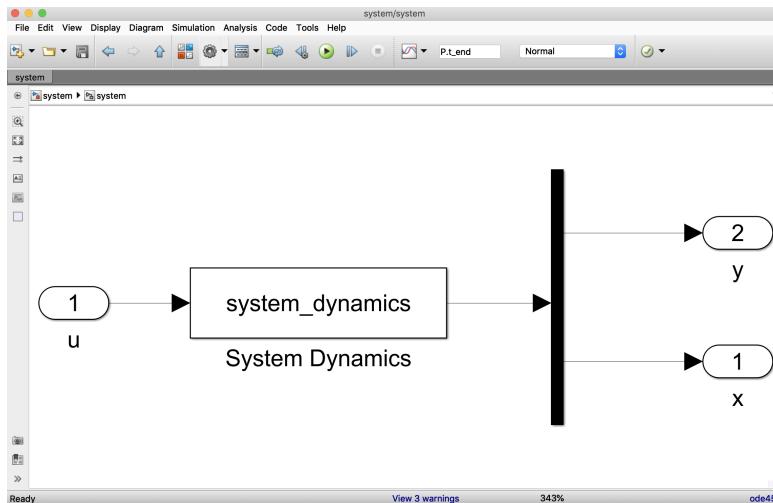


Figure 3-5: Simulink system dynamics sub-diagram.

```

48 b0 = 4.0;
49 % modify the system parameters by random value
50 alpha = 0.2; % Uncertainty parameter
51 a0 = a0 * (1+2*alpha*rand-alpha);
52 a1 = a1 * (1+2*alpha*rand-alpha);
53 b0 = b0 * (1+2*alpha*rand-alpha);
54 end
55 % The equations of motion.
56 yddot = -a1 * ydot - a0 * y + b0 * u;
57 % build xdot and return
58 sys = [ydot; yddot];
59
60 =====
61 function sys=mdlOutputs(t, x, u)
62 y = x(1);
63 sys = [y; x]; % return output and state

```

Listing P.3.1: system\_dynamics.m

Lines 1-2 defines the main s-function file. The inputs to this function are always the elapsed time  $t$ ; the state  $x$ , which is a concatenation of the continuous state and discrete state; the input  $u$ ; and a flag, followed by user defined input parameters, which are absent in this case. The Simulink engine calls the s-function and passes the parameters  $t$ ,  $x$ ,  $u$ , and flag. When  $\text{flag}==0$ , the Simulink engine expects the s-function to return the structure  $\text{sys}$ , which defines the block; initial conditions  $x_0$ ; an empty string  $\text{str}$ ; and an array  $\text{ts}$  that defines the sample times of the block. When  $\text{flag}==1$ , the Simulink engine expects the s-function to return the function  $f(t, x, u)$ ; and when  $\text{flag}==3$  the Simulink engine expects the s-function to return  $h(t, x, u)$ . The switch statement that calls the proper functions based on the value of flag is shown in Lines 3–13. The block

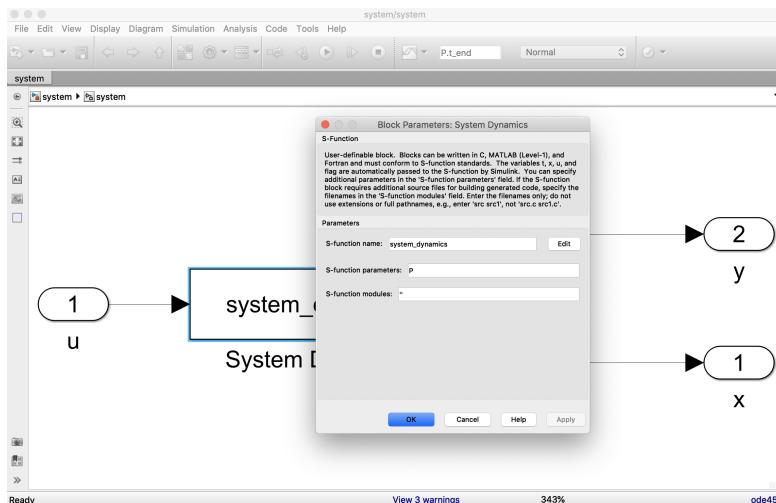


Figure 3-6: Simulinks–function dialog.

setup and the definition of the initial conditions are shown in Lines 16–33. The number of continuous states, discrete states, outputs, and inputs are defined on Lines 18–21, respectively. The direct feedthrough term on Line 22 is set to one if the output depends explicitly on the input  $u$ . For example, if  $D \neq 0$  in the linear state-space output equation  $y = Cx + Du$ . The initial conditions are defined on Lines 26–29. The sample times are defined on Line 32. The format for this line is  $ts = [\text{period } \text{offset}]$ , where `period` defines the sample period and is 0 for continuous time or  $-1$  for inherited, and where `offset` is the sample time offset, which is typically 0. The function  $f(t, x, u)$  is defined in Lines 36–58, and the output function  $h(t, x, u)$  is defined in Lines 61–63.

## Appendix P.3.2 Controller

In this section we will describe how to implement the system dynamics. Double clicking on the controller block shown in Figure 3-2 results in the Simulink sub-diagram shown in Figure 3-7, which also shows the pop-up window that results from clicking on the Interpreted Matlab Function block. This block calls the Matlab function `system_controller.m` with inputs  $u$ , coming into the Simulink block, and the parameter structure  $P$ , which must be defined in the Matlab workspace. The m-file that defines the controller is listed below.

```

1 function u=system_controller(in, Ts)
2     r    = in(1);
3     y    = in(2);
4     t    = in(3);
5     % controller gains
6     kp = 1;

```

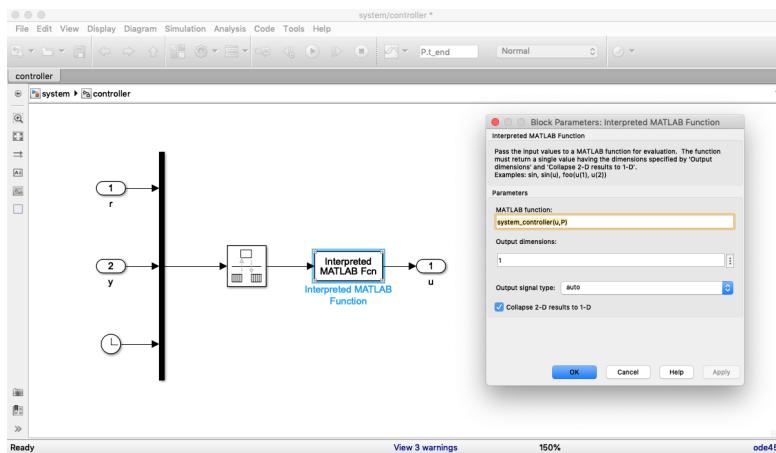


Figure 3-7: Simulink controller sub-diagram, with pop-up window.

```

7 kd = 1;
8 ki = 1;
9 sigma = 2*Ts;
10 beta = (2*sigma-Ts) / (2*sigma+Ts);
11 limit = 1;
12
13 % define the error
14 error = r - y;
15
16 % set persistent variables at start of simulation
17 persistent integrator
18 persistent ydot
19 persistent error_d1
20 persistent y_d1
21 if t<Ts
22     integrator = 0;
23     ydot = 0;
24     error_d1 = error;
25     y_d1 = y;
26 end
27
28 % update the integrator
29 integrator = integrator + (Ts/2)*(error+error_d1);
30 % update the derivative of y
31 ydot = beta*ydot + (1-beta)*(y-y_d1)/Ts;
32 % form PID controller, saturate, and return
33 u_unsat = kp*error + ki*integrator - kd*ydot;
34 u = saturate(u_unsat, limit);
35 % update persistent delay variables
36 error_d1 = error;
37 y_d1 = y;
38 end
39
40 %-----

```

```

41 % saturation function
42 function out = sat(in,limit)
43     if      in > limit,      out = limit;
44     elseif in < -limit,    out = -limit;
45     else                  out = in;
46     end
47 end

```

Listing P.3.2: system\_controller.m

The input to `system_controller.m` are `in` and the sample rate `Ts`. The input array is decomposed into the reference `r`, the system output `y`, and the time `t` on lines 2–4. The controller gains are set on lines 5–11, and the error is calculated on line 14. The persistent variables that must persist between function calls, are defined and initialized on lines 16–26. The integrator is updated on line 29, the differentiator is updated on line 31, and the PID controller is specified on line 33–34. The delay variables are updated on lines 36–37.

### Appendix P.3.3 Animation

In this section we will show how to create an animation for the inverted pendulum case study in Simulink.

When a graphics function like `plot` is called in Matlab, the function returns a *handle* to the plot. A graphics handle is similar to a pointer in C/C++ in the sense that all of the properties of the plot can be accessed through the handle. For example, the Matlab command

```
>> plot_handle=plot(t,sin(t))
```

returns a pointer, or handle, to the plot of `sin(t)`. Properties of the plot can be changed by using the handle, rather than reissuing the `plot` command. For example, the Matlab command

```
>> set(plot_handle, 'YData', cos(t))
```

changes the plot to `cos(t)` without redrawing the axes, title, label, or other objects that may be associated with the plot. If the plot contains drawings of several objects, a handle can be associated with each object. For example,

```

1 >> plot_handle1 = plot(t,sin(t))
2 >> hold on
3 >> plot_handle2 = plot(t,cos(t))

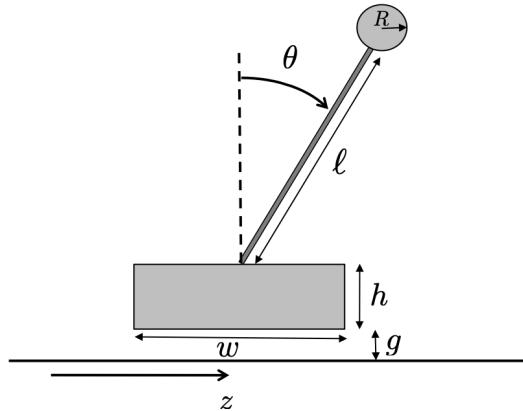
```

draws both `sin(t)` and `cos(t)` on the same plot, with a handle associated with each object. The objects can be manipulated separately without redrawing the other object. For example, to change `cos(t)` to `cos(2t)`, issue the command

```
>> set(plot_handle2, 'YData', cos(2*t))
```

We can exploit this property to animate simulations in Simulink by redrawing only the parts of the animation that change in time, and thereby significantly reducing the simulation time. To show how handle graphics can be used to produce animations in Simulink, we will show how to draw the inverted pendulum in a 2-D animation.

Consider the image of the inverted pendulum shown in **Fig. 3-8**, where the configuration is completely specified by the position of the cart  $z$ , and the angle of the rod from vertical  $\theta$ . The physical parameters of the system are the rod length  $\ell$ , the base width  $w$ , the base height  $h$ , and the gap between the base and the track  $g$ .



**Figure 3-8:** Drawing for inverted pendulum. The first step in developing an animation is to draw a figure of the object to be animated and identify all of the physical parameters.

The first step in developing the animation is to determine the position of points that define the animation. For example, for the inverted pendulum in **Fig. 3-8**, the four corners of the base are

$$(z + w/2, g), (z + w/2, g + h), (z - w/2, g + h), \text{ and } (z - w/2, g),$$

and the two ends of the rod are given by

$$(z, g + h) \text{ and } (z + L \sin \theta, g + L \cos \theta).$$

Since the base and the rod can move independently, each will need its own figure handle. The `drawBase` command can be implemented with the following Matlab code:

```

1 function handle...
2     = drawBase(z, width, height, gap, handle)
3     X = [z-width/2,z+width/2,z+width/2,z-width/2];
4     Y = [gap, gap, gap+height, gap+height];
5     if isempty(handle),

```

```

6     handle = fill(X,Y,'m');
7 else
8 set(handle,'XData',X,'YData',Y);
9 end

```

Lines 2 and 3 define the X and Y locations of the corners of the base. Note that in Line 1, handle is both an input and an output. If an empty array is passed into the function, then the fill command is used to plot the base in Line 5. On the other hand, if a valid handle is passed into the function, then the base is redrawn using the set command in Line 7.

The Matlab code for drawing the rod is similar and is listed below.

```

1 function handle...
2     = drawRod(z,theta,L,gap,height,handle)
3 X = [z, z+L*sin(theta)];
4 Y = [gap+height, gap + height + L*cos(theta)];
5 if isempty(handle),
6     handle = plot(X, Y, 'g');
7 else
8     set(handle,'XData',X,'YData',Y);
9 end

```

The main routine for the pendulum animation is listed below.

```

1 function drawPendulum(u)
2 % process inputs to function
3 z        = u(1);
4 theta    = u(2);
5 t        = u(3);
6
7 % drawing parameters
8 L = 1;
9 gap = 0.01;
10 width = 1.0;
11 height = 0.1;
12
13 % define persistent variables
14 persistent base_handle
15 persistent rod_handle
16
17 % first time function is called, initialize plot
18 % and persistent vars
19 if t==0,
20     figure(1), clf
21     track_width=3;
22     % plot track
23     plot([-track_width,track_width],[0,0],'k');
24     hold on
25     base_handle...
26         =drawBase(z, width, height, gap, []);
27     rod_handle...
28         =drawRod(z, theta, L, gap, height, []);
29     axis([-track_width,track_width, ...
30             -L,2*track_width-L]);
31 % at every other time step, redraw base and rod
32 else

```

```
33 drawBase(z, width, height, gap, base_handle);  
34 drawRod(z, theta, L, gap, height, rod_handle);  
35 end
```

The routine `drawPendulum` is called from the Simulink file shown above, where there are three inputs: the position  $z$ , the angle  $\theta$ , and the time  $t$ . Lines 3-5 rename the inputs to  $z$ ,  $\theta$ , and  $t$ . Lines 8-11 define the drawing parameters. We require that the handle graphics persist between function calls to `drawPendulum`. Since a handle is needed for both the base and the rod, we define two persistent variables in Lines 14 and 15. The `if` statement in Lines 19-32 is used to produce the animation. Lines 20–27 are called once at the beginning of the simulation, and draw the initial animation. Line 20 brings the figure 1 window to the front and clears it. Lines 21 and 23 draw the ground along which the pendulum will move. Line 25 calls the `drawBase` routine with an empty handle as input, and returns the handle `base_handle` to the base. Line 26 calls the `drawRod` routine, and Line 25 sets the axes of the figure. After the initial time step, all that needs to be changed are the locations of the base and rod. Therefore, in Lines 30 and 31, the `drawBase` and `drawRod` routines are called with the figure handles as inputs.



# P.4

## Numerical Solutions to Differential Equations

This appendix provides a brief overview of techniques for numerically solving ordinary differential equations, when the initial value is specified. In particular, we are interested in solving the differential equation

$$\dot{x} = \frac{dx}{dt}(t) = f(x(t), u(t)), \quad x(t_0), \quad (\text{P.4.1})$$

over the time interval  $[t_0, t_f]$ , where  $x(t) \in \mathbf{R}^n$ , and  $u(t) \in \mathbf{R}^m$  is assumed to be known, and where we assume that  $f(\cdot, \cdot)$  is sufficiently smooth and that unique solutions to the differential equation exist for every  $x_0$ .

Integrating both sides of Equation (P.4.1) from  $t_0$  to  $t < t_f$  gives

$$x(t) = x(t_0) + \int_{t_0}^t f(x(\tau), u(\tau)) d\tau. \quad (\text{P.4.2})$$

The difficulty with solving Equation (P.4.2) is that  $x(t)$  appears on both sides of the equation, and cannot therefore be solved explicitly for  $x(t)$ . All numerical solution techniques attempt to approximate the solution to Equation (P.4.2). Most techniques break the solution into small time increments, usually equal to the sample rate of the computer, which we denote by  $T_s$ . Accordingly we write Equation (P.4.2) as

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t-T_s} f(x(\tau), u(\tau)) d\tau + \int_{t-T_s}^t f(x(\tau), u(\tau)) d\tau \\ &= x(t - T_s) + \int_{t-T_s}^t f(x(\tau), u(\tau)) d\tau. \end{aligned} \quad (\text{P.4.3})$$

The most simple form of numerical approximation for the integral in Equation (P.4.3) is to use the left endpoint method where

$$\int_{t-T_s}^t f(x(\tau), u(\tau)) d\tau \approx T_s f(x(t - T_s), u(t - T_s)).$$

Using the notation  $x_k \triangleq x(t_0 + kT_s)$  we get the numerical approximation to the differential equation in Equation (P.4.1) as

$$x_k = x_{k-1} + T_s f(x_k, u_k), \quad x_0 = x(t_0). \quad (\text{P.4.4})$$

Equation (P.4.4) is called the Euler method, or a Runge-Kutta first order method (RK1).

While the RK1 method is often effective for numerically solving ODEs, it typically requires a small sample size  $T_s$ . The advantage of the RK1 method is that it only requires one evaluation of  $f(\cdot, \cdot)$ .

To improve the numerical accuracy of the solution, a second order method can be derived by approximating the integral in Equation (P.4.3) using the trapezoidal rule shown in Figure 10-1, where

$$\int_a^b f(\xi) d\xi \approx (b-a) \left( \frac{f(a) + f(b)}{2} \right).$$

Accordingly,

$$\int_{t-T_s}^t f(x(\tau), u(\tau)) d\tau \approx \frac{T_s}{2} (f(x(t-T_s), u(t-T_s)) + f(x(t), u(t))).$$

Since  $x(t)$  is unknown, we use the RK1 method to approximate it as

$$x(t) \approx x(t-T_s) + T_s f(x(t-T_s), u(t-T_s)).$$

In addition, we typically assume that  $u(t)$  is constant over the interval  $[t-T_s, t]$  to get

$$\begin{aligned} \int_{t-T_s}^t f(x(\tau), u(\tau)) d\tau &\approx \frac{T_s}{2} (f(x(t-T_s), u(t-T_s)) \\ &\quad + f(x(t-T_s) + T_s f(x(t-T_s), u(t-T_s)), u(t-T_s))). \end{aligned}$$

Accordingly, the numerical integration routine can be written as

$$\begin{aligned} x_0 &= x(t_0) \\ F_1 &= f(x_{k-1}, u_{k-1}) \\ F_2 &= f(x_{k-1} + T_s F_1, u_{k-1}) \\ x_k &= x_{k-1} + \frac{T_s}{2} (F_1 + F_2). \end{aligned} \quad (\text{P.4.5})$$

Equations (P.4.5) is the Runge-Kutta second order method or RK2. It requires two function calls to  $f(\cdot, \cdot)$  for every time sample.

The most common numerical method for solving ODEs is the Runge-Kutta forth order method RK4. The RK4 method is derived using Simpson's Rule

$$\int_a^b f(\xi) d\xi \approx \left( \frac{b-a}{6} \right) \left( f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right),$$

which is found by approximating the area under the integral using a parabola. The standard strategy is to write the approximation as

$$\int_a^b f(\xi) \xi \approx \left( \frac{b-a}{6} \right) \left( f(a) + 2f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+b}{2}\right) + f(b) \right),$$

and then to use

$$F_1 = f(a) \quad (\text{P.4.6})$$

$$F_2 = f\left(a + \frac{T_s}{2} F_1\right) \approx f\left(\frac{a+b}{2}\right) \quad (\text{P.4.7})$$

$$F_3 = f\left(a + \frac{T_s}{2} F_2\right) \approx f\left(\frac{a+b}{2}\right) \quad (\text{P.4.8})$$

$$F_4 = f(a + T_s F_3) \approx f(b), \quad (\text{P.4.9})$$

resulting in the RK4 numerical integration rule

$$\begin{aligned} x_0 &= x(t_0) \\ F_1 &= f(x_{k-1}, u_{k-1}) \\ F_2 &= f\left(x_{k-1} + \frac{T_s}{2} F_1, u_{k-1}\right) \\ F_3 &= f\left(x_{k-1} + \frac{T_s}{2} F_2, u_{k-1}\right) \\ F_4 &= f\left(x_{k-1} + T_s F_3, u_{k-1}\right) \\ x_k &= x_{k-1} + \frac{T_s}{6} (F_1 + 2F_2 + 2F_3 + F_4). \end{aligned} \quad (\text{P.4.10})$$

It can be shown that the RK4 method matches the Taylor series approximation of the integral in Equation (P.4.3) up to the fourth order term. Higher order methods can be derived, but generally do not result in significantly better numerical approximations to the ODE. The step size  $T_s$  must still be chosen to be sufficiently small to result in good solutions. It is also possible to develop adaptive step size solvers. For example, the ODE45 algorithm in Matlab solves the ODE using both an RK4 and an RK5 algorithm. The two solutions are then compared and if they are sufficiently different, then the step size is reduced. If the two solutions are close, then the step size can be increased.

## Appendix P.4.1 Python Implementation of Numerical ODE Solvers

In the following code example, we show how to implement a system that is described by the second order differential equation

$$\ddot{y} + a_1 \dot{y} + a_2 \sin y = bu,$$

where  $a_1$ ,  $a_2$  and  $b$  are constant coefficients and  $u$  is the system input. The first step is to put the system into state space form. Selecting the states as

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \triangleq \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

and differentiating gives the state space equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -a_1x_2 - a_2 \sin x_1 + bu \end{pmatrix}$$

$$y = x_1,$$

which is in the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= f(\boldsymbol{x}, u) \\ y &= h(\boldsymbol{x}, u). \end{aligned}$$

Python code that implements a class for this system is given below. The class has two external properties: the sample rate `self.Ts` and the state `self.state`. The constant coefficients are treated as internal properties known only to the system and are therefore labeled as `self.a1`, `self.a2`, and `self.b`. When the class is instantiated, the `__init__` function is called, which is where the states are initialized with their initial conditions, and the system coefficients  $a_1$ ,  $a_2$ , and  $b$  are initialized. The external method is `update`, and takes as its input the system input  $u$ , and returns the system output  $y$ . The `update` function first propagates the system dynamics  $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, u)$  using one of the Runge-Kutta methods. In the code below, the RK4 algorithm is used, but internal methods that implement the RK1, RK2, and RK4 methods are all shown for educational purposes. The RK methods each use the internal function `self.f`, which is where the system dynamics  $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, u)$  are specified. This function is system dependent. The `update` function then computes the output  $y = h(\boldsymbol{x})$  by calling the internal function `self.h`, which is also system dependent.

```

1 import numpy as np
2
3 class systemDynamics:
4     # Model the system yddot + a1*ydot + a2*sin(y) = b*u
5     def __init__(self, Ts=0.01):
6         self.Ts = Ts    # sample rate
7         # Specify the initial condition
8         y0 = 0.0        # y at time zero
9         ydot0 = 0.0      # ydot at time zero
10        self.state = np.array([
11            [y0],
12            [ydot0],
13        ])    # state at time zero
14        self.a1 = 2.0
15        self.a2 = 2.0
16        self.b = 4.0
17

```

```

18     def update(self, u):
19         # This is the external method that takes the input u and
20         # returns the output y.
21         self.rk4_step(u) # propagate the state by one time sample
22         y = self.h() # return the corresponding output
23         return y
24
25     def rk1_step(self, u):
26         # Integrate ODE using Runge-Kutta RK1 algorithm
27         F1 = self.f(self.state, u)
28         self.state += self.Ts * F1
29
30     def rk2_step(self, u):
31         # Integrate ODE using Runge-Kutta RK2 algorithm
32         F1 = self.f(self.state, u)
33         F2 = self.f(self.state + self.Ts / 2 * F1, u)
34         self.state += self.Ts / 2 * (F1 + F2)
35
36     def _rk4_step(self, u):
37         # Integrate ODE using Runge-Kutta RK4 algorithm
38         F1 = self.f(self.state, u)
39         F2 = self.f(self.state + self.Ts / 2 * F1, u)
40         F3 = self.f(self.state + self.Ts / 2 * F2, u)
41         F4 = self.f(self.state + self.Ts * F3, u)
42         self.state += self.Ts / 6 * (F1 + 2 * F2 + 2 * F3 + F4)
43
44     def f(self, state, u):
45         # Return xdot = f(x,u)
46         x1 = state.item(0)
47         x2 = state.item(1)
48         xdot = np.array([
49             [x2],
50             [-self._a1 * x2 - self._a2 * np.sin(x1) + self._b * u],
51             ])
52         return xdot
53
54     def h(self):
55         # return the output equations y=h(x)
56         x1 = self.state.item(0)
57         y = np.array([
58             [x1],
59             ])
60         return y

```

Listing P.4.1: systemDynamics.py

The system is simulated using the following Python code. The system is instantiated on line 13. The while loop in lines 20–28 update the internal system at the sample rate  $T\_s$  rather than at the plot rate  $T\_plot$ . The input is specified as a square wave on line 23. The system is propagated one step forward using the input  $u$  on line 24.

```

1 import matplotlib.pyplot as plt
2 from signalGenerator import signalGenerator
3 from plotData import plotData
4 from systemDynamics import systemDynamics

```

```

5 # simulation parameters
6 Ts = 0.01 # sample rate of simulation
7 T_plot = 0.1 # plot update rate
8 t_start = 0.0 # start time of simulation
9 t_end = 20.0 # end time of simulation
10
11
12 # instantiate system and input classes
13 system = systemDynamics(Ts)
14 input = signalGenerator(amplitude=0.5, frequency=0.1)
15
16 # instantiate the simulation plots and animation
17 dataPlot = plotData()
18
19 t = t_start # time starts at t0=0
20 while t < t_end: # main simulation loop
21     t_next_plot = t + T_plot
22     while t < t_next_plot:
23         u = input.square(t)
24         y = system.update(u)
25         t = t + Ts # advance time by Ts
26         # update animation and data plots
27         dataPlot.update(t, system.state, u)
28         plt.pause(0.0001)
29
30 # Keeps the program from closing until the user presses a button.
31 print('Press key to close')
32 plt.waitforbuttonpress()
33 plt.close()

```

Listing P.4.2: systemSim.py

## Appendix P.4.2 Matlab Implementation of Numerical ODE Solvers

Matlab code that implements a class for the system defined in the previous section is given below. The class has five properties: the sample rate `self.Ts`, the state `self.state`, and the constant coefficients `self.a1`, `self.a2`, and `self.b`. When the class is instantiated, constructor `systemDynamics(Ts)` is called, which is where the states are initialized with their initial conditions, and the system coefficients  $a_1$ ,  $a_2$ , and  $b$  are initialized. The class method `update` takes as its input the system input  $u$  and returns the system output  $y$ . The `update` function first propagates the system dynamics  $\dot{x} = f(x, u)$  using one of the Runge-Kutta methods. In the code below, the RK4 algorithm is used, but internal methods that implement the RK1, RK2, and RK4 methods are all shown for educational purposes. The RK methods each use the internal function `self.f`, which is where the system dynamics  $\dot{x} = f(x, u)$  are specified. This function is system dependent. The `update` function then computes the output  $y = h(x)$  by calling the internal function `self.h`, which is also system dependent.

```

1 classdef systemDynamics < handle
2 properties
3     Ts
4     state
5     a1
6     a2
7     b
8 end
9 methods
10    %---constructor-----
11    function self = systemDynamics(Ts)
12        self.Ts = Ts; % sample rate
13        y0 = 0; % initial y
14        ydot0 = 0; % initial y_dot
15        self.state = [y0; ydot0]; % initial state
16        self.a1 = 2; % system parameters
17        self.a2 = 2;
18        self.b = 4;
19    end
20    %-----
21    function y = update(self, u)
22        % external method that takes u as input
23        % and returns y
24        self.rk4_step(u);
25        y = self.h();
26    end
27    function self = rk1_step(self, u)
28        % Integrate ODE using Runge-Kutta RK1 algorithm
29        F1 = self.f(self.state, u);
30        self.state = self.state + self.Ts * F1;
31    end
32    function self = rk2_step(self, u)
33        % Integrate ODE using Runge-Kutta RK4 algorithm
34        F1 = self.f(self.state, u);
35        F2 = self.f(self.state + self.Ts/2*F1, u);
36        self.state = self.state + self.Ts/2 * (F1 + F2);
37    end
38    function self = rk4_step(self, u)
39        % Integrate ODE using Runge-Kutta RK4 algorithm
40        F1 = self.f(self.state, u);
41        F2 = self.f(self.state + self.Ts/2*F1, u);
42        F3 = self.f(self.state + self.Ts/2*F2, u);
43        F4 = self.f(self.state + self.Ts*F3, u);
44        self.state = self.state...
45            + self.Ts/6 * (F1 + 2*F2 + 2*F3 + F4);
46    end
47    function xdot = f(self, state, u)
48        % Return xdot = f(x, u),
49        x1 = state(1);
50        x2 = state(2);
51        xdot = [x2;...
52            -self.a1*x2 - self.a2*sin(x2) + self.b*u];
53    end
54    function y = h(self)
55        % return y = h(x)
56        x1 = self.state(1);

```

```

57         y = x1;
58     end
59 end
60 end

```

Listing P.4.3: systemDynamics.m

The system is simulated using the following Matlab code. The system is instantiated on line 7. The while loop in lines 13–23 update the internal system at the sample rate  $T_s$  rather than at the plot rate  $T_{plot}$ . The input is specified as a square wave on line 17. The system is propagated one step forward using the input  $u$  on line 18.

```

1 % simulation parameters
2 Ts = 0.01; % sample rate
3 T_plot = 0.1; % plot update rate
4 t_start = 0.0; % simulation start time
5 t_end = 10.0; % simulation end timm
6 % instantiate dynamics and plotter
7 system = systemDynamics(Ts);
8 input = signalGenerator(0.5, 0.2);
9 dataPlot = dataPlotter();
10
11 % main simulation loop
12 t = t_start;
13 while t < t_end
14     % Propagate dynamics in between plot samples
15     t_next_plot = t + P.t_plot;
16     while t < t_next_plot
17         u = input.square(t); % input is a square wave
18         y = system.update(u); % Propagate the dynamics
19         t = t + P.Ts; % advance time by Ts
20     end
21     % update data plots
22     dataPlot.update(t, system.state, u);
23 end

```

Listing P.4.4: systemSim.m

### Appendix P.4.3 Simulink Implementation of Numerical ODE Solvers

The Simulink engine implements a variety of different ODE solvers. The system dynamics are specified in an s-function. An s-function that implements the system described above is shown below.

```

1 function [sys,x0,str,ts,simStateCompliance]...
2     = system_dynamics(t,x,u,flag)
3 switch flag
4     % Initialization
5     case 0
6         [sys,x0,str,ts,simStateCompliance]...

```

```
7      =mdlInitializeSizes();
8 % Derivatives
9 case 1
10    sys=mdlDerivatives(t,x,u);
11 % Update
12 case 2
13    sys=mdlUpdate(t,x,u);
14 % Outputs
15 case 3
16    sys=mdlOutputs(t,x,u);
17 % GetTimeOfNextVarHit
18 case 4
19    sys=mdlGetTimeOfNextVarHit(t,x,u);
20 % Terminate %
21 case 9
22    sys=mdlTerminate(t,x,u);
23 % Unexpected flags %
24 otherwise
25    DAStudio.error(...,
26      'Simulink:blocks:unhandledFlag',...
27      num2str(flag));
28
29 end
30 %
31 %=====
32 % mdlInitializeSizes
33 % Return the sizes, initial conditions, and
34 % sample times for the S-function.
35 %=====
36 %
37 function [sys,x0,str,ts,simStateCompliance]...
38     =mdlInitializeSizes()
39
40 sizes = simsizes;
41
42 sizes.NumContStates = 2;
43 sizes.NumDiscStates = 0;
44 sizes.NumOutputs = 1;
45 sizes.NumInputs = 1;
46 sizes.DirFeedthrough = 0;
47 sizes.NumSampleTimes = 1;
48
49 sys = simsizes(sizes);
50
51 % initialize the initial conditions%
52 y0 = 0;
53 ydot0 = 0;
54 x0 = [y0; ydot0];
55 % str is always an empty matrix
56 str = [];
57
58 % initialize the array of sample times
59 ts = [0 0];
60 simStateCompliance = 'UnknownSimState';
61
62 %
63 %
```

```
64 %=====
65 % mdlDerivatives
66 % Return the derivatives for the continuous
67 % states.
68 %=====
69 %
70 function sys=mdlDerivatives(t,x,u,P)
71     x1 = x(1);
72     x2 = x(2);
73     xdot = [x2;...
74             -self.a1*x2 - self.a2*sin(x2) + self.b*u];
75 sys = xdot;
76 % end mdlDerivatives
77 %
78 %
79 %=====
80 % mdlUpdate
81 % Handle discrete state updates, sample time
82 % hits, and major time step requirements.
83 %=====
84 %
85 function sys=mdlUpdate(t,x,u)
86
87 sys = [];
88
89 % end mdlUpdate
90 %
91 %
92 %=====
93 % mdlOutputs
94 % Return the block outputs.
95 %=====
96 %
97 function sys=mdlOutputs(t,x,u)
98     x1 = x(1);
99     y = x1;
100 sys = y;
101
102 % end mdlOutputs
103 %
104 %
105 %=====
106 % mdlGetTimeOfNextVarHit
107 %=====
108 %
109 function sys=mdlGetTimeOfNextVarHit(t,x,u)
110
111 sampleTime = 1;
112 sys = t + sampleTime;
113
114 % end mdlGetTimeOfNextVarHit
115 %
116 %
117 %=====
118 % mdlTerminate
119 % Perform any end of simulation tasks.
120 %=====
```

```

121 %
122 function sys=mdlTerminate(t,x,u)
123
124 sys = [];
125
126 % end mdlTerminate

```

Listing P.4.5: systemDynamics.m

The state equation  $\dot{x} = f(x, u)$  are implemented in the function `mdlDerivatives` on lines 70–75, and the output equation  $y = h(x)$  is implemented in the function `mdlOutputs` on lines 97–102. The system is initialized in `mdlInitializeSizes` on lines 38–61.

The numerical ODE solver can be specified by using the menu `Simulation > Model Configuration Parameters`. Simulink has two types of solvers: fixed-step solvers and variable-step solvers. The dialog menu for fixed-step solvers is shown in Figure 4-1, where “ode4(Runge-Kutta)” corresponds to an RK4 method, “ode2(Heun)” corresponds to an RK2 method, and “ode1(Euler)” corresponds to an RK1 method. Simulink also has variable step size ODE solvers. The options for variable step size are shown in Figure 4-2. The “ode45 (Dormand-Prince)” solver works by solving the ODE over the next time step using both an `ode4` and an `ode5` method, comparing the two, and then reducing the step size if the difference is too large, and increasing the step size if the difference is within a tolerance.

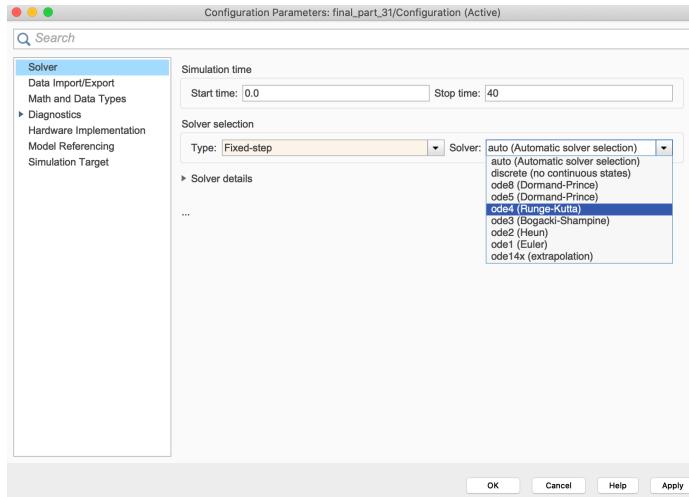


Figure 4-1: Dialog window for Simulink fixed-step size solver.

## Notes and References

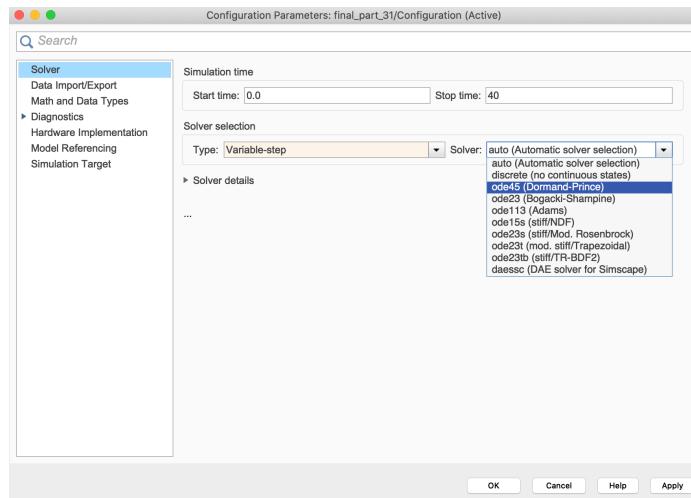


Figure 4-2: Dialog window for Simulink variable-step size solver.

# P.5

## Review of Ordinary Differential Equations

This appendix provides a refresher on how to solve linear ordinary differential equations with constant coefficients. In particular, we focus on the step response of such system.

### Appendix P.5.1 First order ODE's

This section provides a review of how to use an integrating factor to find the solution to a scalar first order differential equations.

Consider the first order linear constant coefficient ordinary differential equation

$$\frac{dy}{dt}(t) = ay(t) + bu(t), \quad (\text{P.5.1})$$

with initial condition  $y(0) = y_0$ , where  $a$  and  $b$  are constants. Equation (P.5.1) can be rearranged as

$$\frac{dy}{dt}(t) - ay(t) = bu(t). \quad (\text{P.5.2})$$

Multiplying both sides of Equation (P.5.2) by the integrating factor  $e^{-at}$  gives

$$e^{-at} \left( \frac{dy}{dt}(t) - ay(t) \right) = e^{-at}bu(t). \quad (\text{P.5.3})$$

Noting that the left hand side of Equation (P.5.3) is the derivative of  $e^{-at}y(t)$  gives

$$\frac{d}{dt} (e^{-at}y(t)) = e^{-at}bu(t). \quad (\text{P.5.4})$$

Integrating both sides of Equation (P.5.4) from 0 to  $t$  yields

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau}y(\tau)) d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau,$$

which implies that

$$e^{-at}y(t) - e^{-a \cdot 0}y(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau.$$

Solving for  $y(t)$  gives

$$y(t) = \underbrace{e^{at}y_0}_{\text{homogeneous response}} + \underbrace{\int_0^t e^{a(t-\tau)}bu(\tau)d\tau}_{\text{forced response}}, \quad (\text{P.5.5})$$

where the first term on the right is the homogenous response, or the response to initial conditions, and the second term on the right is the forced response, or the response to the input  $u(t)$ , which should be noted is the convolution of  $u(t)$  with the impulse response  $e^{at}$ .

For the discussion to follow, it is important to note that if  $a$  is a complex number, then Equation (P.5.5) still holds, but that  $y(t)$  will be a complex function of time. To see this, assume that  $a = \sigma + j\omega$ , then

$$\begin{aligned} e^{at} &= e^{(\sigma+j\omega)t} \\ &= e^{\sigma t}e^{j\omega t} \\ &= e^{\sigma t}(\cos(\omega t) + j \sin(\omega t)) \\ &= e^{\sigma t} \cos(\omega t) + j e^{\sigma t} \sin(\omega t), \end{aligned}$$

substituting into Equation (P.5.5) yields the complex function

$$\begin{aligned} y(t) &= \left( e^{\sigma t} \cos(\omega t) y_0 + \int_0^t e^{\sigma(t-\tau)} \cos(\omega\tau) bu(\tau)d\tau \right) \\ &\quad + j \left( e^{\sigma t} \sin(\omega t) y_0 + \int_0^t e^{\sigma(t-\tau)} \sin(\omega(t-\tau)) bu(\tau)d\tau \right). \end{aligned}$$

Note that if the real part of  $a$  is negative, or in other words if  $a$  is in the left half of the complex plane, then the homogeneous response decays to zero and the forced response is bounded if  $u(t)$  is bounded.

## Appendix P.5.2 Inverse Laplace Transform

Taking the Laplace transform of Equation (P.5.1) gives

$$(sY(s) - y_0) = aY(s) + bU(s).$$

Solving for  $Y(s)$  yields

$$Y(s) = \left( \frac{1}{s-a} \right) y_0 + \left( \frac{b}{s-a} \right) U(s).$$

Taking the inverse Laplace transform gives

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)y_0 + \mathcal{L}^{-1}\left(\frac{b}{s-a}\right) * \mathcal{L}^{-1}(U(s)), \quad (\text{P.5.6})$$

where  $*$  denotes the convolution operator. Comparing Equation (P.5.5) with Equation (P.5.6) implies that

$$\mathcal{L}^{-1}\left(\frac{b}{s-a}\right) = be^{at}, \quad t \geq 0. \quad (\text{P.5.7})$$

In the limit, as  $a \rightarrow 0$  we get

$$\mathcal{L}^{-1}\left(\frac{b}{s}\right) = b, \quad t \geq 0. \quad (\text{P.5.8})$$

Again we note that Equations (P.5.7) and (P.5.8) are true even when  $a$  and  $b$  are complex numbers.

### Appendix P.5.3 Partial Fraction Expansion

The inverse Laplace transform for first order systems can be used to find the response to step inputs, for first order and for higher order systems, through the use of partial fraction expansion. For example, consider the task of finding the step response for the first order system

$$\dot{y} = -ay + bu, \quad (\text{P.5.9})$$

where  $a$  and  $b$  are real positive constants, where  $u$  is a step input of size  $A$ , i.e.,

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The Laplace transform of  $u(t)$  is  $U(s) = \frac{A}{s}$ . Similarly, taking the Laplace transform of Equation (P.5.9) and solving for  $Y(s)$  gives

$$Y(s) = \left(\frac{b}{s+a}\right)U(s) = \frac{bA}{s(s+a)}. \quad (\text{P.5.10})$$

The partial fraction expansion of  $Y(s)$  expresses  $Y(s)$  as a linear combination of each factor in the denominator, i.e.,

$$Y(s) = \frac{c_1}{s} + \frac{c_2}{s+a}. \quad (\text{P.5.11})$$

To find the coefficients  $c_1$  and  $c_2$ , set Equation (P.5.11) equal to Equation (P.5.10) to obtain

$$\frac{c_1}{s} + \frac{c_2}{s+a} = \frac{bA}{s(s+a)}. \quad (\text{P.5.12})$$

To find  $c_1$  multiply both sides of Equation (P.5.12) by  $s$  to obtain

$$c_1 + \frac{sc_2}{s+a} = \frac{bA}{s+a}. \quad (\text{P.5.13})$$

To remove the term involving  $c_2$  in Equation (P.5.13), set  $s = 0$  to obtain

$$c_1 = \frac{bA}{a}.$$

To find  $c_2$  multiply both sides of Equation (P.5.12) by  $s+a$  to obtain

$$\frac{c_1(s+a)}{s} + c_2 = \frac{bA}{s}. \quad (\text{P.5.14})$$

To remove the term involving  $c_1$  in Equation (P.5.14), set  $s = -a$  to obtain

$$c_2 = -\frac{bA}{a}.$$

Therefore,

$$Y(s) = \frac{\frac{bA}{a}}{s} - \frac{\frac{bA}{a}}{s+a}. \quad (\text{P.5.15})$$

Since the inverse Laplace transform is a linear operator, we get

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{\frac{bA}{a}}{s}\right) - \mathcal{L}^{-1}\left(\frac{\frac{bA}{a}}{s+a}\right) \\ &= \frac{bA}{a} - \frac{bA}{a}e^{-at}, \quad t \geq 0 \\ &= \frac{bA}{a}(1 - e^{-at}), \quad t \geq 0. \end{aligned}$$

The same technique works for finding the step response for higher order systems. As an example, find the unit step response of the system described by

$$y^{(4)} + 10y^{(3)} + 35\ddot{y} + 50\dot{y} + 24y = u.$$

Taking the Laplace transform and solving for  $Y(s)$ , and using the fact that  $U(s) = \frac{1}{s}$  gives

$$\begin{aligned} Y(s) &= \left(\frac{1}{s^4 + 10s^3 + 35s^2 + 50s + 24}\right) U(s) \\ &= \left(\frac{1}{s^4 + 10s^3 + 35s^2 + 50s + 24}\right) \frac{1}{s} \\ &= \frac{1}{s} \left(\frac{1}{s+1}\right) \left(\frac{1}{s+2}\right) \left(\frac{1}{s+3}\right) \left(\frac{1}{s+4}\right). \end{aligned} \quad (\text{P.5.16})$$

Using partial fraction expansion we have that

$$Y(s) = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+2} + \frac{c_4}{s+3} + \frac{c_5}{s+4}. \quad (\text{P.5.17})$$

Setting Equation (P.5.17) equal to Equation (P.5.16) gives

$$\begin{aligned} \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+2} + \frac{c_4}{s+3} + \frac{c_5}{s+4} \\ = \frac{1}{s} \left( \frac{1}{s+1} \right) \left( \frac{1}{s+2} \right) \left( \frac{1}{s+3} \right) \left( \frac{1}{s+4} \right). \end{aligned} \quad (\text{P.5.18})$$

The coefficient  $c_1$  is found by multiplying both sides of Equation (P.5.18) by  $s$  and then setting  $s = 0$  to remove the coefficients  $c_2-c_5$  to obtain  $c_1 = 1/24$ . Following a similar procedure for each coefficient yields  $c_2 = -1/6$ ,  $c_3 = 1/4$ ,  $c_4 = -1/6$ , and  $c_5 = 1/24$ . The step response is therefore given by

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left( \frac{1/24}{s} \right) + \mathcal{L}^{-1} \left( \frac{-1/6}{s+1} \right) + \mathcal{L}^{-1} \left( \frac{1/4}{s+2} \right) \\ &\quad + \mathcal{L}^{-1} \left( \frac{-1/6}{s+3} \right) + \mathcal{L}^{-1} \left( \frac{1/24}{s+4} \right) \\ &= \frac{1}{24} - \frac{1}{6}e^{-t} + \frac{1}{4}e^{-2t} - \frac{1}{6}e^{-3t} + \frac{1}{24}e^{-4t}, \quad t \geq 0 \\ &= \frac{1}{24} (1 - 4e^{-t} + 6e^{-2t} - 4e^{-3t} + e^{-4t}), \quad t \geq 0. \end{aligned}$$

In Matlab, the coefficients can be found by using the `residue` command. For the problem above we could use the matlab commands

```

1 num = 1; % numerator polynomial
2 % denominator polynomial
3 den = poly([0,-1,-2,-3,-4]);
4 % compute partial fraction expansion
5 [R,P,K] = residue(num,den);

```

The vector `R` will contain the coefficients  $c_1-c_5$ , the vector `P` contains the associated roots, and `K` will be the remainder polynomial if the numerator is a higher order polynomial than the denominator.

The method of partial fraction expansion is also valid if the poles of the system are complex. For example, consider finding the response of the second order differential equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u,$$

to as step of size  $A$ . Taking the Laplace transform and letting  $U(s) = A/s$  gives

$$\begin{aligned} Y(s) &= \left( \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + 2\omega_n^2} \right) \frac{A}{s} \\ &= \frac{A\omega_n^2}{s} \left( \frac{1}{s + \zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}} \right) \end{aligned} \quad (\text{P.5.19})$$

$$\cdot \left( \frac{1}{s + \zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}} \right). \quad (\text{P.5.20})$$

Expanding  $Y(s)$  as a sum of partial fractions yields

$$\begin{aligned} Y(s) &= \frac{c_1}{s} + \left( \frac{c_2}{s + \zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}} \right) \\ &\quad + \left( \frac{c_3}{s + \zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}} \right). \end{aligned} \quad (\text{P.5.21})$$

Setting Equation (P.5.21) equal to Equation (P.5.20) gives

$$\begin{aligned} \frac{c_1}{s} + \left( \frac{c_2}{s + \zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}} \right) + \left( \frac{c_3}{s + \zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}} \right) \\ = \frac{A\omega_n^2}{s} \left( \frac{1}{s + \zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}} \right) \left( \frac{1}{s + \zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}} \right). \end{aligned}$$

Solving for  $c_1$ ,  $c_2$ , and  $c_3$  gives

$$\begin{aligned} c_1 &= A \\ c_2 &= -\frac{A}{2} \left( 1 + j \frac{\zeta}{\sqrt{1 + \zeta^2}} \right) \\ c_3 &= -\frac{A}{2} \left( 1 - j \frac{\zeta}{\sqrt{1 - \zeta^2}} \right). \end{aligned}$$

Therefore,

$$Y(s) = \frac{A}{s} + \frac{-\frac{A}{2} \left( 1 + j \frac{\zeta}{\sqrt{1 + \zeta^2}} \right)}{s + \zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}} + \frac{-\frac{A}{2} \left( 1 - j \frac{\zeta}{\sqrt{1 - \zeta^2}} \right)}{s + \zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}}.$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
 y(t) &= A - \frac{A}{2} \left( 1 + j \frac{\zeta}{\sqrt{1+\zeta^2}} \right) e^{-\zeta\omega_n t - j\omega_n \sqrt{1-\zeta^2}t} \\
 &\quad - \frac{A}{2} \left( 1 - j \frac{\zeta}{\sqrt{1-\zeta^2}} \right) e^{-\zeta\omega_n t + j\omega_n \sqrt{1-\zeta^2}t} \\
 &= A \left( 1 - \frac{1}{2} \left| 1 + j \frac{\zeta}{\sqrt{1+\zeta^2}} \right| e^{-\zeta\omega_n t} \right. \\
 &\quad \cdot \left. \left( e^{-j\omega_n \sqrt{1-\zeta^2}t} e^{j\angle\left(1+j\frac{\zeta}{\sqrt{1+\zeta^2}}\right)} + e^{+j\omega_n \sqrt{1-\zeta^2}t} e^{j\angle\left(1-j\frac{\zeta}{\sqrt{1+\zeta^2}}\right)} \right) \right) \\
 &= A \left( 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \right. \\
 &\quad \cdot \left. \left( \frac{e^{-j\left(\omega_n \sqrt{1-\zeta^2}t - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)}}{2} + e^{j\left(\omega_n \sqrt{1-\zeta^2}t - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)} \right) \right) \\
 &= A \left( 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}t - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right) \right),
 \end{aligned}$$

for  $t \geq 0$ . In general, complex conjugate poles will always lead to complex conjugate coefficients that end up canceling to produce a decaying exponential multiplied by a cosine term with a phase shift. If the coefficients of the original ordinary differential equation are real, then the response will always be real.



# P.6

## Root Locus

The *root locus* has been an important pedagogical tool in control systems since the 1940s. It was important in the design of feedback amplifiers for communication circuits where the classic design involved the tuning of a small number of parameters. In this appendix we introduce the root locus and show how it can be used to understand the effect of adding an integrator to a second order system with PD control. However, we should note that root locus can be used in many other settings. We are not including this appendix as a main chapter in the book because in our experience root locus is rarely used for actual design.

### Appendix P.6.1 Theory

We begin by noting that for the closed loop system shown in Figure 6-1, the transfer function from  $Y_r(s)$  to  $Y(s)$  can be computed as

$$\begin{aligned} Y(s) &= P(s)C(s)(Y_r(s) - Y(s)) \\ \implies Y(s) &= \frac{P(s)C(s)}{1 + P(s)C(s)}Y_r(s). \end{aligned}$$

Therefore, the characteristic equation of the closed loop system is given by

$$\Delta_{cl} = 1 + P(s)C(s) = 0.$$

We have seen in Chapter 7 that by using PD control for a second order system, it is possible to precisely select the locations of the closed loop poles. When an integrator is added for PID control, it is also possible to precisely select the location of the three closed-loop poles. In this chapter we will introduce a simple and convenient tool for visualizing how the closed loop poles change as a function of a single parameter. As motivation, consider a first order plant under proportional control, as shown in Fig. 6-2. The closed-loop transfer function is given by

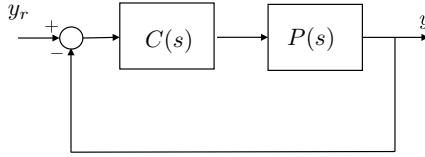


Figure 6-1: General feedback system with plant  $P(s)$  and controller  $C(s)$ .

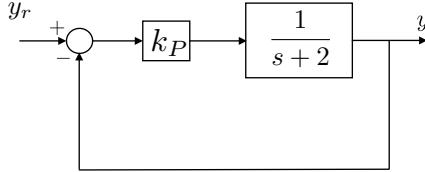


Figure 6-2: A first order plant under proportional control.

$$Y(s) = \frac{k_P}{s + (2 + k_P)} Y_r(s)$$

and the closed loop characteristic polynomial is given by

$$\Delta_{cl}(s) = s + (2 + k_P).$$

Note that the location of the closed loop pole is

$$p_{cl} = -(2 + k_P).$$

We can create a plot of the location of the poles in the complex plane as shown in Fig. 6-3. When  $k_P = 0$ , the closed loop pole is located at the open loop pole  $-2$ , but as  $k_P$  is increased greater than zero, the closed loop pole moves from  $-2$  into the left half plane in the direction of the arrow shown in Fig. 6-3. The plot of the closed loop pole as a function of  $k_P$  is an example of a *root locus*.

As a second example, consider the second order system under proportional control shown in Fig. 6-4. In this case the closed loop transfer function is given by

$$Y(s) = \frac{k_P}{s^2 + 2s + k_P} Y_r(s)$$

Note that the open loop poles are located at  $0$  and  $-2$ . The closed loop characteristic polynomial is given by

$$\Delta_{cl}(s) = s^2 + 2s + k_P \quad (\text{P.6.1})$$

and the closed loop poles are located at

$$p_{cl} = -1 \pm \sqrt{1 - k_P}.$$

The position of the closed loop poles as a function of  $k_P > 0$  are shown in Fig. 6-5. When  $k_P = 0$ , the closed loop poles are equal to the open loop poles

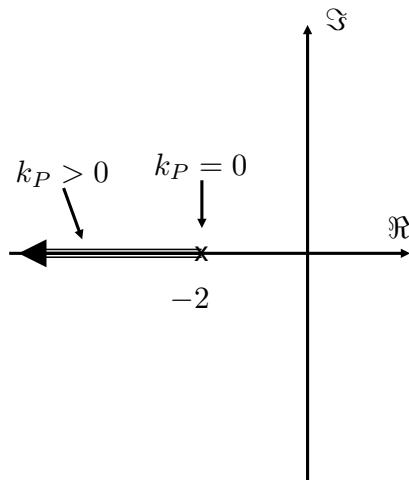


Figure 6-3: A plot of the closed loop pole associated with [Fig. 6-2](#) as a function of  $k_P > 0$ .

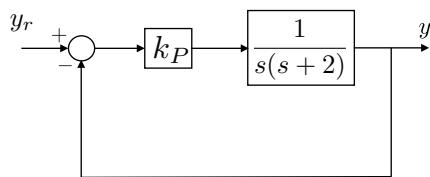


Figure 6-4: A second order plant under proportional control.

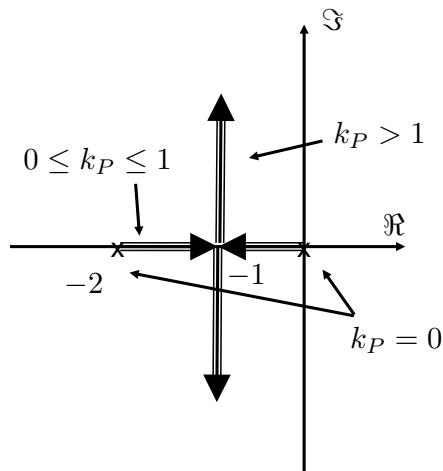


Figure 6-5: A plot of the closed loop pole associated with [Fig. 6-4](#) as a function of  $k_P > 0$ .

at 0 and  $-2$ . When  $0 < k_P \leq 1$ , the closed loop poles are both real and move toward  $-1$  with increasing  $k_P$ . When  $k_P = 1$  both poles are located at  $-1$ . When  $k_P > 1$  the two poles are complex with real part equal to  $-1$  and increasing complex part as  $k_P$  increases. The poles move in the direction of the arrows shown in **Fig. 6-5**.

The Matlab command `rlocus` will plot the root locus and can be used interactively to find the gain associated with specific pole locations. To use the `rlocus` command in Matlab, the characteristic equation must be put in *Evan's form*, which is

$$1 + kL(s) = 0,$$

where  $k$  is the gain of interest. For example, the characteristic equation associated with the characteristic polynomial in Equation (P.6.1) is

$$\Delta_{cl}(s) = s^2 + 2s + k_P = 0.$$

The characteristic equation can be put in Evan's form by dividing both sides by  $s^2 + 2s$  and rearranging to obtain

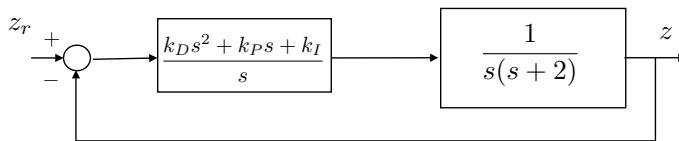
$$1 + k_P \left( \frac{1}{s^2 + 2s} \right) = 0,$$

where  $L(s) = \frac{1}{s^2 + 2s}$ . The Matlab command that draws the root locus for this equation is

```
1 >> L = tf([1], [1, 2, 0]);
2 >> figure(1), clf, rlocus(L)
```

where the first line defines the transfer function  $L(s) = \frac{1}{s^2 + 2s}$  and the second line invokes the root locus command using  $L$ .

The root locus tool is particularly useful for selecting the integrator gains after PD gains have been selected. For example, suppose that a PD controller has been designed for the second order plant shown in **Fig. 6-6** so that the closed loop poles are located at  $-2 \pm j2$ , where  $k_P = 8$  and  $k_D = 2$ . Now suppose that an



**Figure 6-6:** Closed loop system where the proportional gains have been selected as  $k_P = 8$  and  $k_D = 2$ .

integrator is to be added to reject disturbances and eliminate steady state error, but that the integrator gain  $k_I$  is to be selected so that the closed loop poles deviate from  $-2 \pm j2$  by less than 10%. The characteristic equation for the closed loop

system including the integrator is given by

$$\begin{aligned}
 & 1 + \left( \frac{1}{s^2 + 2s} \right) \left( \frac{k_D s^2 + k_P s + k_I}{s} \right) = 0 \\
 \implies & \frac{(s^3 + 2s^2) + (k_D s^2 + k_P s + k_I)}{s^3 + 2s^2} = 0 \\
 \implies & s^3 + (2 + k_D)s^2 + k_P s + k_I = 0 \\
 \implies & \frac{(s^3 + (2 + k_D)s^2 + k_P s) + k_I}{s^3 + (2 + k_D)s^2 + k_P s} = 0 \\
 \implies & 1 + k_I \left( \frac{1}{s^3 + (2 + k_D)s^2 + k_P s} \right) = 0 \\
 \implies & 1 + k_I \left( \frac{1}{s^3 + 4s^2 + 8s} \right) = 0
 \end{aligned}$$

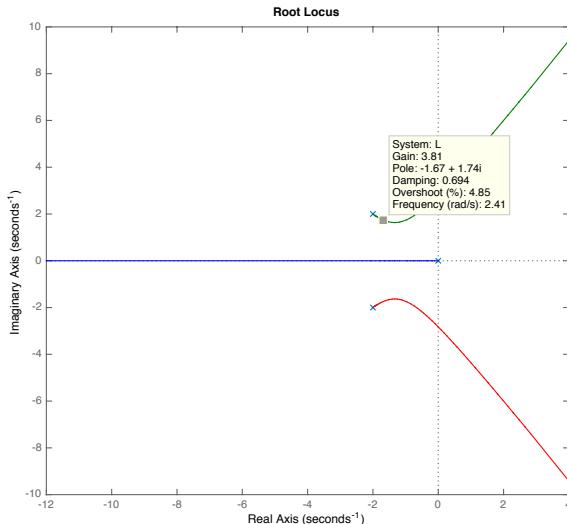
where the later equation is in Evan's form. Therefore the Matlab command for drawing the root locus is

```

1 >> L = tf([1], [1, 4, 8, 0]);
2 >> figure(1), clf, rlocus(L)

```

and the resulting plot is shown in **Fig. 6-7**. Note from **Fig. 6-7** that the poles



**Figure 6-7:** The result of Matlab's `rlocus` command, and by interactively selecting pole locations close to  $-2 \pm j2$ .

when  $k_I = 0$  are at  $0, -2 \pm j2$  and are denoted by 'x'. The root locus shows what

happens to the closed loop poles as  $k_I$  is increased from zero. Note from [Fig. 6-7](#) that for large values of  $k_I$  the closed-loop poles are in the right half plane and the system will be unstable. However, for a gain of  $k_I = 3.81$  the closed loop poles are located at  $-1.67 \pm j1.74$ . This technique can be used to select reasonable values for  $k_I$ .

## Appendix P.6.2 Design Study A. Single Link Robot Arm



### Example Problem A.P.6

For the single link robot arm, use the PD gains derived in [HW A.8](#). Add an integrator to the controller to get PID control. Put the resulting closed loop characteristic equation in Evan's form and use the Matlab `rlocus` command to plot the root locus versus the integrator gain  $k_I$ . Select a value for  $k_I$  that does not significantly change the other locations of the closed loop poles.

### Solution

The closed loop block diagram including an integrator is shown in [Fig. 6-8](#). The

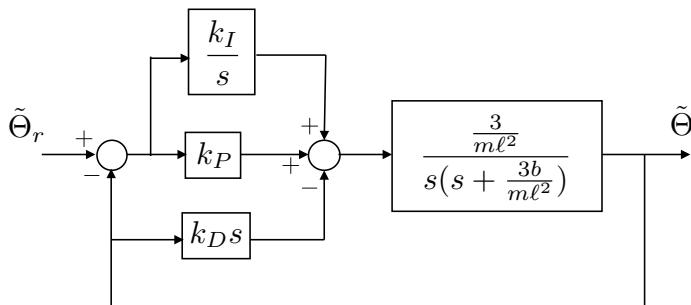


Figure 6-8: PID control for the single link robot arm.

closed loop transfer function is given by

$$\tilde{\Theta}(s) = \frac{\frac{3k_P}{m\ell^2}s + \frac{3k_I}{m\ell^2}}{s^3 + \left(\frac{3b+3k_D}{m\ell^2}\right)s^2 + \frac{3k_P}{m\ell^2}s + \frac{3k_I}{m\ell^2}} \tilde{\Theta}^d(s).$$

The characteristic equation is therefore

$$s^3 + \left(\frac{3b+3k_D}{m\ell^2}\right)s^2 + \frac{3k_P}{m\ell^2}s + \frac{3k_I}{m\ell^2} = 0.$$

In Evan's form we have

$$1 + k_I \left( \frac{\frac{3}{m\ell^2}}{s^3 + \left(\frac{3b+3k_D}{m\ell^2}\right)s^2 + \frac{3k_P}{m\ell^2}s} \right) = 0.$$

The appropriate Matlab command is therefore

```

1 >> L = tf([3/m/L^2], ...
2           [1, (3*b+3*kd)/m/L^2, 3*kp/m/L^2, 0]);
3 >> figure(1), clf, rlocus(L);

```

### Appendix P.6.3 Design Study B. Inverted Pendulum

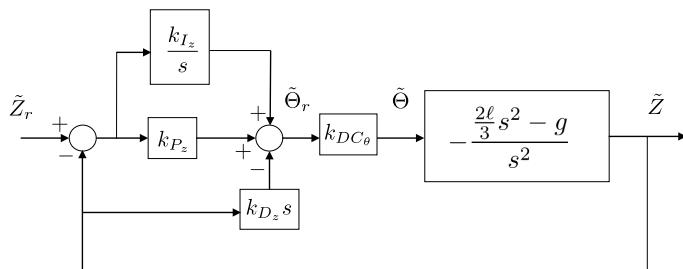


#### Example Problem B.P.6

Adding an integrator to obtain PID control for the outer loop of the inverted pendulum system, put the characteristic equation in Evan's form and use the Matlab `rlocus` command to plot the root locus verses the integrator gain  $k_I$ . Select a value for  $k_I$  that does not significantly change the other locations of the closed loop poles.

#### Solution

The closed loop block diagram for the outer loop of the inverted pendulum including an integrator is shown in [Fig. 6-9](#). The closed loop transfer function is given



[Figure 6-9:](#) PID control for the outer loop of the inverted pendulum.

by

$$\tilde{Z}(s) = \frac{(g - \frac{2}{3}\ell s^2)(k_{Pz}s + k_{Iz})}{-\frac{2}{3}\ell k_{Dz}s^4 + \left(\frac{1}{k_{DC_\theta}} - \frac{2}{3}\ell k_{Pz}\right)s^3 + (gk_{Dz} - \frac{2}{3}\ell k_{Iz})s^2 + gk_{Pz}s + gk_{Iz}} \tilde{Z}_r(s).$$

The characteristic equation is therefore

$$-\frac{2}{3}\ell k_{Dz}s^4 + \left(\frac{1}{k_{DC_\theta}} - \frac{2}{3}\ell k_{Pz}\right)s^3 + (gk_{Dz} - \frac{2}{3}\ell k_{Iz})s^2 + gk_{Pz}s + gk_{Iz} = 0.$$

In Evan's form we have

$$1 + k_{I_z} \left( \frac{-\frac{2}{3}\ell s^2 + g}{-\frac{2}{3}\ell k_{Dz} s^4 + \left(\frac{1}{k_{DC_\theta}} - \frac{2}{3}\ell k_{Pz}\right) s^3 + g k_{Dz} s^2 + g k_{Pz} s} \right) = 0.$$

The appropriate Matlab command is therefore

```

1 >> L = tf([-2/3*ell,0,g],...
2      [-2/3*ell*kd,1/kDCth-2/3*ell*kp,g*kd,g*kp,0]);
3 >> figure(1), clf, rlocus(L);

```

## Appendix P.6.4 Design Study C. Satellite Attitude Control



### Example Problem C.P.6

Adding an integrator to obtain PID control for the outer loop of the satellite system, put the characteristic equation in Evan's form and use the Matlab `rlocus` command to plot the root locus verses the integrator gain  $k_I$ . Select a value for  $k_I$  that does not significantly change the other locations of the closed loop poles.

### Solution

The closed loop block diagram for the outer loop including an integrator is shown in **Fig. 6-10**. The closed loop transfer function is given by

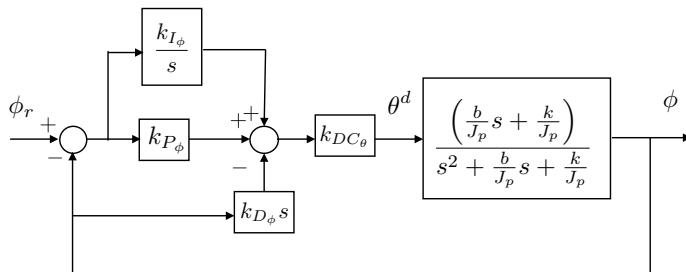


Figure 6-10: PID control for the outer loop of the satellite system.

$$\Phi(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \Phi_r(s),$$

The appropriate Matlab command is therefore

```
1 >> L = tf([b*kDC/Jp, k*kDC/Jp], ...
2           [1+b*kDC*kD/Jp, ...
3            b/Jp+k*kDC*kD/Jp+b*kDC*kP/Jp, ...
4            k/Jp+k*kDC*kP/Jp, 0]);
5 >> figure(1), clf, rlocus(L);
```

## Notes and References

[TOC](#)



# P. 7

## Review of Linear Algebra

Let  $A$  be an  $m \times n$  matrix of real numbers, denoted as  $A \in \mathbb{R}^{m \times n}$ . Then

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Matrix addition is defined element wise for matrices of compatible dimensions as

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_A + \underbrace{\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}}_B = \underbrace{\begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}}_C.$$

To multiply a matrix times a vector, the inner dimensions must match:

$$\underbrace{A}_{m \times n} \underbrace{x}_{n \times 1} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_{1i}x_i \\ \vdots \\ \sum_{i=1}^n a_{mi}x_i \end{pmatrix} = \underbrace{b}_{m \times 1}.$$

For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}.$$

Multiplying two matrices also requires matching dimensions:

$$\underbrace{AB}_{m \times n \quad n \times p} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \dots & \sum_{i=1}^n a_{1i}b_{ip} \\ \vdots & & \vdots \\ \sum_{i=1}^n a_{mi}b_{i1} & \dots & \sum_{i=1}^n a_{mi}b_{ip} \end{pmatrix} = \underbrace{C}_{m \times p}.$$

Note that if  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times p}$ , then  $AB$  is a valid product, but  $BA$  is not a valid product. Even if the dimensions are compatible, i.e., if  $A$  and  $B$  are square, in general

$$AB \neq BA.$$

## Appendix P.7 .1 Transpose

If  $A \in \mathbb{R}^{m \times n}$ , then the transpose of  $A$ , denoted as  $A^\top \in \mathbb{R}^{n \times m}$  is obtained by interchanging the rows and the columns. For example

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\top = (x_1 \quad x_2 \quad x_3),$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^\top = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

A square matrix  $A$  is said to be *symmetric* if  $A^\top = A$ . Similarly,  $A$  is said to be *skew-symmetric* if  $A^\top = -A$ . Every square matrix  $A \in \mathbb{R}^{n \times n}$  can be decomposed into the sum of a symmetric and a skew-symmetric matrix as

$$\begin{aligned} A &= \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^\top - \frac{1}{2}A^\top \\ &= \left(\frac{1}{2}A + \frac{1}{2}A^\top\right) + \left(\frac{1}{2}A - \frac{1}{2}A^\top\right) \\ &= A_s + A_{ss}, \end{aligned}$$

where  $A_s = \frac{1}{2}(A + A^\top)$  is symmetric, and  $A_{ss} = \frac{1}{2}(A - A^\top)$  is skew-symmetric.

The following properties are easily verified by direct manipulation:

$$(AB)^\top = B^\top A^\top$$

$$(A + B)^\top = A^\top + B^\top.$$

## Appendix P.7 .2 Trace

The trace operator is defined for square matrices. If  $A \in \mathbb{R}^{n \times n}$ , then

$$\text{tr}(A) = \sum_{i=1}^n a_{ii},$$

or in other words, the sum of the diagonals of  $A$ . For example

$$\text{tr} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 + 4 = 5.$$

## Appendix P.7 .3 Determinant

The *determinant* operator is defined for square matrices  $A \in \mathbb{R}^{n \times n}$  as

$$\det(A) = \sum_{j=1}^n a_{ij} \gamma_{ij},$$

for any  $i = 1, \dots, n$ , where the *co-factor*

$$\gamma_{ij} = (-1)^{(i+j)} \det(M_{ij})$$

and where the minor  $M_{ij}$  is the  $(n - 1) \times (n - 1)$  matrix obtained by removing the  $i^{th}$  row and the  $j^{th}$  column of  $A$ . As an example,

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 7 & 8 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} 5 & 0 \\ 7 & 8 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 4 & 0 \\ 6 & 8 \end{pmatrix} \\ &\quad + 3 \cdot \det \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} \\ &= 1 \cdot (5 \cdot 8 - 7 \cdot 0) - 2 \cdot (4 \cdot 8 - 6 \cdot 0) \\ &\quad + 3 \cdot (4 \cdot 7 - 6 \cdot 5) \\ &= -30. \end{aligned}$$

The *adjugate* of  $A$  is defined as the transpose of the co-factors of  $A$ :

$$\text{adj}(A) = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{n1} \\ \vdots & & \vdots \\ \gamma_{1n} & \cdots & \gamma_{nn} \end{pmatrix}.$$

We have the following properties for the determinant:

$$\begin{aligned} \det(A) &= \det(A^\top) \\ \det(AB) &= \det(A) \det(B) \\ \det(A^{-1}) &= \frac{1}{\det(A)}. \end{aligned}$$

There are also well defined rules for finding the determinant of a block matrix. Let

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times p}$ ,  $C \in \mathbb{R}^{m \times n}$  and  $D \in \mathbb{R}^{m \times m}$ , then it can be shown that if  $A$  is invertible, then

$$\det(E) = \det(A) \det(D - CA^{-1}B).$$

As a special case, if either  $B = 0$  or  $C = 0$ , then

$$\det(E) = \det(A) \det(D).$$

For example,

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 9 & 10 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \cdot \det \begin{pmatrix} 8 & 9 \\ 9 & 10 \end{pmatrix}.$$

## Appendix P.7 .4 Matrix Inverses

**Definition.** If  $A \in \mathbb{R}^{n \times n}$ , and there exists a matrix  $B \in \mathbb{R}^{n \times n}$  such that

$$AB = BA = I,$$

then  $B$  is said to be the inverse of  $A$  and is denoted as  $B = A^{-1}$ .

The inverse of any square matrix can be expressed in terms of its adjugate and its determinant as

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}.$$

For a  $2 \times 2$  matrix we have the simple formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{\text{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad - bc}.$$

If  $\det(A) \neq 0$ , then  $A$  is called *non-singular*.

If  $A$  and  $B$  are nonsingular, then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

For block matrices it can be verified by direct substitution that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix} \quad (\text{P.7.1})$$

$$= \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}, \quad (\text{P.7.2})$$

from which we get the famous matrix inversion lemma

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}. \quad (\text{P.7.3})$$

When  $C = 0$  we get the simple formula

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

## Appendix P.7 .5 Eigenvalues and Eigenvectors

The pair  $(\lambda, v)$  where  $\lambda \in \mathbb{C}$  and  $0 \neq v \in \mathbb{C}^{n \times 1}$  is said to be an eigenvalue-eigenvector pair of the matrix  $A \in \mathbb{R}^{n \times n}$  if

$$Av = \lambda v.$$

The following statements are equivalent

1.  $(\lambda, v)$  are an eigenvalue-eigenvector pair of  $A$ ,
2.  $Av = \lambda v$ ,
3.  $(\lambda I - A)v = 0$ , where  $I$  is the  $n \times n$  identity matrix,
4.  $v$  is in the null space of  $(\lambda I - A)$ ,
5.  $\det(\lambda I - A) = 0$ .

For example, to find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix},$$

first compute

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{pmatrix} \\ &= \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1). \end{aligned}$$

Setting  $\det(\lambda I - A) = 0$  implies that the eigenvalues are

$$\lambda_1 = -2, \quad \lambda_2 = -1.$$

To find the eigenvector associated with  $\lambda_1$  we need to find a vector  $v_1$  that is in the null space of  $(\lambda_1 I - A)$ . Since

$$(\lambda_1 I - A)v_1 = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we have that  $2v_{11} = -v_{12}$ . Therefore  $v_1 = (1, -2)^\top$  is in the null space of  $(\lambda_1 I - A)$  and is an eigenvector of  $A$  associated with eigenvalue  $\lambda_1$ .

To find the eigenvector associated with  $\lambda_2$  we need to find a vector  $v_2$  that is in the null space of  $(\lambda_2 I - A)$ . Since

$$(\lambda_2 I - A)v_2 = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we have that  $v_{21} = -v_{22}$ . Therefore  $v_2 = (-1, 1)^\top$  is in the null space of  $(\lambda_2 I - A)$  and is an eigenvector of  $A$  associated with eigenvalue  $\lambda_2$ .

Since  $Av_1 = \lambda_1 v_1$  and  $Av_2 = \lambda_2 v_2$  we can write

$$A(v_1 \ v_2) = (v_1 \ v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Defining  $M = (v_1 \ v_2)$  and  $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$  we have

$$AM = M\Lambda.$$

When  $M$  is invertible, we have

$$A = M\Lambda M^{-1}.$$

$M$  will always be invertible when the eigenvalues are distinct. For example, it can be verified that

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 11 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}^{-1}.$$

The formula  $A = M\Lambda M^{-1}$  is called the eigen-decomposition of  $A$ .

The trace and the determinant of  $A$  can be written in terms of the eigenvalues of  $A$  as

$$\text{tr}(A) = \sum_{i=1}^n \lambda_i(A),$$

$$\det(A) = \prod_{i=1}^n \lambda_i(A).$$

where  $\lambda_i(A)$  is the  $i^{th}$  eigenvalue of  $A$ .

## Appendix P.7 .6 Linear Independence and Rank

The vectors  $x_1, x_2, \dots, x_p \in \mathbb{R}^{n \times 1}$  are said to be *linearly independent* if

$$c_1 x_1 + c_2 x_2 + \cdots + c_p x_p = 0,$$

implies that  $c_1 = c_2 = \cdots = c_p = 0$ . For example, the vectors  $(1, 0)^\top$  and  $(0, 1)^\top$  are linearly independent since

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

implies that  $c_1 = c_2 = 0$ . Alternatively, the vectors  $(1, 0)^\top$  and  $(3, 0)^\top$  are not linearly independent since

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is true if  $c_1 = -3c_2$  for any  $c_2$ .

The *rank* of a matrix is defined to be the number of linearly independent rows or columns. For example

$$\text{rank} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\text{rank} \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} = 1.$$

## Appendix P.7 .7 Special Matrices

The matrix  $A \in \mathbb{R}^{n \times n}$  is said to be *diagonal* if

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

For example,

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

is a diagonal matrix.

The matrix  $A \in \mathbb{R}^{n \times n}$  is said to be *upper triangular* if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

For example,

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 0 & 11 \end{pmatrix}$$

is an upper triangular matrix.

The matrix  $A \in \mathbb{R}^{n \times n}$  is said to be in *upper companion form* if

$$A = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

For example,

$$A = \begin{pmatrix} -2 & -3 & -4 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is in upper companion form.

The matrix  $A \in \mathbb{R}^{n \times n}$  is said to be a *Vandermonde matrix* if

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \dots & a_3^{n-1} \\ 1 & a_4 & a_4^2 & \dots & a_4^{n-1} \end{pmatrix}.$$

For example,

$$A = \begin{pmatrix} 1 & 2 & 4 & 9 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{pmatrix}$$

is a Vandermonde matrix.

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