Chapter 1

Transmission Lines and S-Parameters

1.1 Review of Transmission Lines

In our study of microwave circuits, we will be extensively using our knowledge of transmission lines. We therefore need to review a few concepts from transmission line theory. We will only need the sinusoidal steady state, so line voltages and currents will be in phasor form. The phasor voltage $V(z)$ and current $I(z)$ are defined in terms of the time-dependent voltage according to

$$v(z,t) = \text{Re}\left\{V(z)e^{j\omega t}\right\} \tag{1.1}$$

$$i(z,t) = \text{Re}\left\{I(z)e^{j\omega t}\right\} \tag{1.2}$$

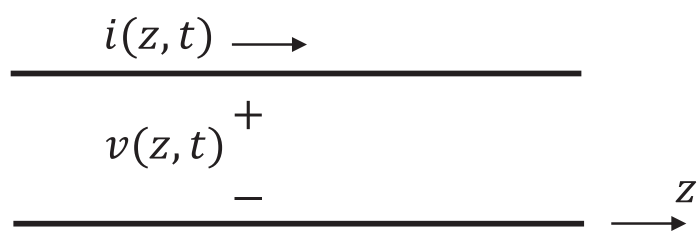

Figure 1.1: A transmission line. The two wires are only a schematic representation, because many different types of structures can be modeled as transmission lines.

1. **Differential (Telegrapher's) Equations:**

   The governing equations for the line current and voltage at a point z on a lossless line are

$$\frac{\partial v(z,t)}{\partial z} = -L\frac{\partial i(z,t)}{\partial t} \tag{1.3}$$

$$\frac{\partial i(z,t)}{\partial z} = -C\frac{\partial v(z,t)}{\partial t} \tag{1.4}$$

where L (H/m) and C (F/m) are the distributed inductance and capacitance, respectively. In the sinusoidal steady state,

$$\begin{aligned} \frac{dV(z)}{dz} &= -j\omega LI(z) && (1.5) \\ \frac{dI(z)}{dz} &= -j\omega CV(z) && (1.6) \end{aligned}$$

Substituting the second equation into the first gives

$$\frac{d^2V(z)}{dz^2} = -j\omega L\frac{dI(z)}{dz} = -\omega^2 LCV(z) \tag{1.7}$$

which leads to the Helmholtz equation,

$$\frac{d^2V(z)}{dz^2} + \underbrace{\omega^2 LC}_{\beta^2} V(z) = 0 \tag{1.8}$$

where $\beta = \omega\sqrt{LC}$ is the wavenumber of the sinusoidal waves.

The general solution to this differential equation is

$$V(z) = Ae^{-j\beta z} + Be^{j\beta z} \tag{1.9}$$

where A and B are unknown coefficients. The corresponding current is

$$\begin{aligned} I(z) &= -\frac{1}{j\omega L}\frac{dV(z)}{dz} = \frac{\beta}{\omega L}\left[Ae^{-j\beta z} - Be^{j\beta z}\right] \\ &= \frac{1}{\sqrt{L/C}}\left[Ae^{-j\beta z} - Be^{j\beta z}\right] && (1.10) \end{aligned}$$

We define

$$Z_0 = \sqrt{\frac{L}{C}} \tag{1.11}$$

as the *characteristic impedance* of the transmission line, because it gives the ratio between the voltage and current waves at a point on the line. Note that this quantity is nonzero even though the line is lossless. What is the difference between characteristic impedance of a transmission line and the impedance of a circuit element?

Using the characteristic impedance, we can write the current on the line as

$$I(z) = \frac{A}{Z_0}e^{-j\beta z} - \frac{B}{Z_0}e^{j\beta z} \tag{1.12}$$

In the time domain,

$$v(z,t) = \text{Re}\left\{Ae^{j(\omega t-\beta z)} + Be^{j(\omega t+\beta z)}\right\} = A\cos(\omega t - \beta z) + B\cos(\omega t + \beta z) \tag{1.13}$$

so that the A term represents a forward traveling ($+z$) wave and the B term is the reverse wave.

2. **Wavelength:** This is the distance between two peaks of the forward or reverse wave:

$$\beta z = \beta\lambda = 2\pi \qquad \rightarrow \qquad \lambda = \frac{2\pi}{\beta} \tag{1.14}$$

3. **Phase Velocity:** In order to stay on a peak of the wave, the argument of one of the cosine functions in (1.13) must be constant. From this idea, we can determine the phase velocity of the wave:

$$\omega t - \beta z = \phi = \text{constant} \tag{1.15}$$

$$z = \frac{\omega t - \phi}{\beta} \tag{1.16}$$

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{1.17}$$

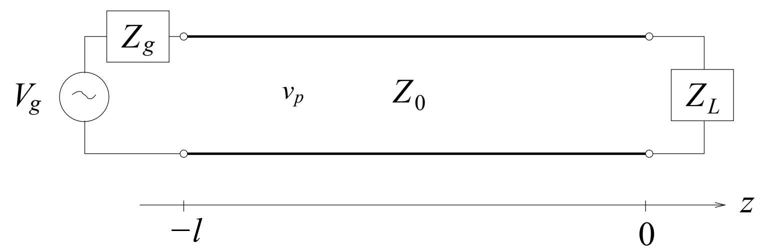

Figure 1.2: A transmission line with generator at $z = -l$ and load at $z = 0$.

4. **Reflection Coefficient:** Consider a line of characteristic impedance Z_0 and length ℓ terminated with a load impedance of Z_L, as shown in Fig. 1.2. Let $z = 0$ be defined at the load, such that the input is at $z = -\ell$. From the general solutions derived above, the voltage and current on the line are

$$V(z) = Ae^{-j\beta z} + Be^{j\beta z} \tag{1.18}$$

$$I(z) = \frac{A}{Z_0}e^{-j\beta z} - \frac{B}{Z_0}e^{j\beta z} \tag{1.19}$$

The coefficients A and B are determined by the boundary conditions at the load and generator ends of the transmission line. At the load end, the current and voltage on the line must be equal to the voltage across the load and the current through the load, which are related by Ohm's law, so that

$$Z_L = \frac{V(0)}{I(0)} = Z_0\frac{A+B}{A-B} \tag{1.20}$$

At the generator end, Kirchhoff's voltage law gives

$$\begin{aligned} V_g &= V(-\ell) + Z_g I(-\ell) \\ &= Ae^{j\beta\ell} + Be^{-j\beta\ell} + Z_g\left[\frac{A}{Z_0}e^{j\beta\ell} - \frac{B}{Z_0}e^{-j\beta\ell}\right] \end{aligned} \tag{1.21}$$

Equations (1.20) and (1.21) provide two simultaneous equations that can be solved for the two unknowns in the problem, A and B. Once we have these two coefficients, we know the current and voltage everywhere on the line, and the problem is solved, because we can get any derived quantity that we want from the current and voltage.

We could solve Eqs. (1.20) and (1.21) using any method for solving systems of equations. It is convenient to solve these equations in a way that gives values for two very useful quantities, the load reflection coefficient at $z = 0$ and the input impedance at $z = -\ell$. To do this, we first solve the load equation (1.20) for B/A, which is the load reflection coefficient:

$$\Gamma_0 = \frac{B}{A} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{1.22}$$

We can also define a generalized reflection coefficient at other points on the transmision line as

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{Be^{j\beta z}}{Ae^{-j\beta z}} = \Gamma_0 e^{j2\beta z} \tag{1.23}$$

5. **Input Impedance:** At the generator end of the line ($z = -\ell$),

$$Z_{\text{in}} = \frac{V(-\ell)}{I(-\ell)} = Z_0 \frac{Ae^{j\beta\ell} + Be^{-j\beta\ell}}{Ae^{j\beta\ell} - Be^{-j\beta\ell}} = Z_0 \frac{1 + \Gamma_0 e^{-j2\beta\ell}}{1 - \Gamma_0 e^{-j2\beta\ell}} \tag{1.24}$$

Using (1.22) and (1.24),

$$Z_{\text{in}} = Z_0 \frac{(Z_L + Z_0)e^{j\beta\ell} + (Z_L - Z_0)e^{-j\beta\ell}}{(Z_L + Z_0)e^{j\beta\ell} - (Z_L - Z_0)e^{-j\beta\ell}} = Z_0 \frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell} \tag{1.25}$$

This can be used with the generator voltage and impedance to solve for $V(-\ell)$ using circuit theory formulas:

$$V(-\ell) = \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_g} V_g \tag{1.26}$$

Using $V(-\ell)$ together with Eq. (1.18) written in the form

$$V(-\ell) = A(e^{j\beta\ell} + \Gamma_0 e^{-j\beta\ell}) \tag{1.27}$$

allows us to find A, which completes the solution of the transmission line problem in Fig. 1.2.

6. **VSWR:** The voltage standing wave ratio is the ratio of the maximum magnitude of the voltage along the line to the minimum:

$$V(z) = Ae^{-j\beta z}\left[1 + \Gamma_0 e^{j2\beta z}\right] \tag{1.28}$$

$$|V(z)|_{\text{max}} = |A|\left[1 + |\Gamma_0|\right] \tag{1.29}$$

$$|V(z)|_{\text{min}} = |A|\left[1 - |\Gamma_0|\right] \tag{1.30}$$

$$\text{VSWR} = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|} \tag{1.31}$$

7. **Matched Line:**

$$Z_L = Z_0 \qquad \Gamma_0 = 0$$

$$Z_{\text{in}}(-\ell) = Z_0 \qquad \text{VSWR} = 1$$

8. **Short-Circuited Line:**

$$Z_{\text{in}}(-\ell) = Z_0 \frac{jZ_0 \tan\beta\ell}{Z_0} = jZ_0 \tan\beta\ell \tag{1.32}$$

9. **Open-Circuited Line:**

$$Z_{\text{in}}(-\ell) = \frac{Z_0}{j\tan\beta\ell} = -jZ_0 \cot\beta\ell \tag{1.33}$$

10. **Quarter-Wave Line:**

$$Z_{\text{in}}(-\ell = -\lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan\pi/2}{Z_0 + jZ_L \tan\pi/2} = \frac{Z_0^2}{Z_L} \tag{1.34}$$

1.2 Microwave Networks and S-Parameters

Previously in our study of transmission line theory, we have focused mainly on the transmission line itself. Now, we would like to view transmission lines merely as connections between devices, and shift our emphasis to the properties of the devices themselves.

A microwave network is a device or structure to which one or more transmission lines are connected. Each transmission line connection to the device is a port. An N-port network is shown in Fig. 1.3. Power splitters, lumped elements, amplifiers, antennas, a section of transmission line, and many other structures can be modeled as networks.

As with a circuit device, a network can be characterized in terms of the voltage/current relationship at each transmission line port. For a simple parallel wire transmission line, defining the voltage at a point on the line is straightforward. But for waveguides, voltages are more difficult to define for non-TEM modes. Even for a TEM mode, voltages and currents are difficult to measure directly at microwave frequencies, because of the rapid oscillation with time. Consider, for example, how fast a digital sampling oscilloscope would have to operate to accomplish this measurement.

Instead of using voltages and currents to characterize the signals on the transmission line ports, we use a representation that is based on the relative amplitudes of incident, reflected, and transmitted waves for a structure. Regardless of what the network may be physically, on each transmission line port there can only be a forward wave (incident) and a reverse wave (reflected or transmitted by the network) with some steady state amplitude. This means that instead of characterizing the signal on a transmission line using $V(z)$ or $I(z)$, we use the ratios between the forward and reverse waves on the transmission lines. For a network connected at each port by transmission lines to other devices, each transmission line will have a given value for this ratio. If the other devices are all matched and a unit amplitude forward wave is excited as an input into one of the ports, then the values of the reverse waves coming out the ports are completely determined by the device and can be used to characterize the microwave properties of the device. These values are called the *scattering matrix* or *S-matrix*. We will generally use the term *S-parameters* to describe the elements of the S-matrix.

The S-matrix is defined to relate the incident and reflected wave amplitudes at the ports of the network:

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & & & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix} \tag{1.35}$$

or

$$\left[V^-\right] = [S]\left[V^+\right] \tag{1.36}$$

From this expression, it can be seen that in general, the output wave at a given port is a function of the input waves at every port. If there is no input wave on any port except for the jth port, then all of the elements of the vector V^+ are zero except for V_j^+, and the S-parameters can be expressed as

$$S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0 \text{ for } k\neq j} \tag{1.37}$$

This formula shows that an S-parameter is similar to a generalized reflection coefficient, since it is defined the same way as $\Gamma(z)$ in Eq. (1.23). The only difference between (1.37) and (1.23) is that the forward wave is on the jth port and the reverse wave is on the ith port.

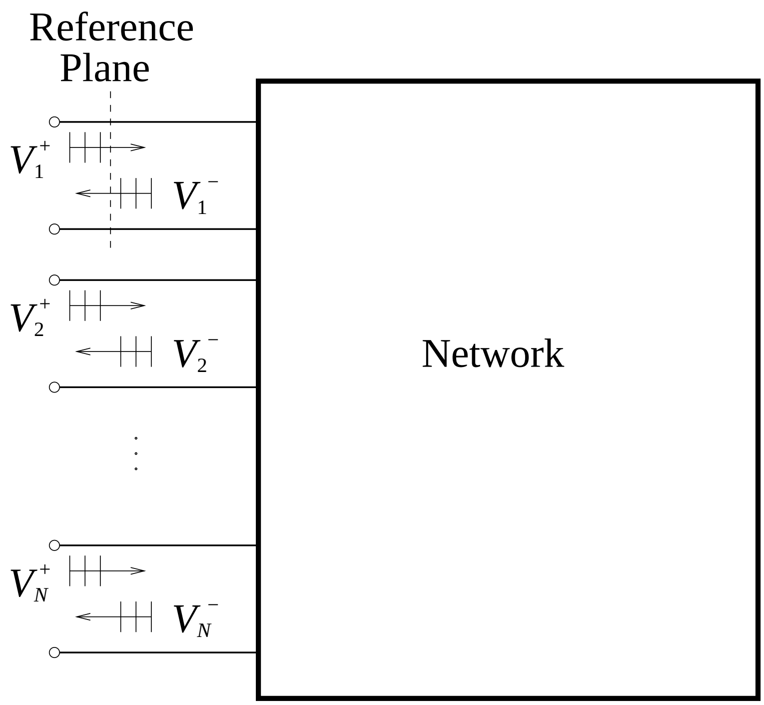

Figure 1.3: N-port network. Each port is a transmission line, leading to a device or structure that connects the ports. At a fixed reference plane for each port, the voltage amplitudes of the forward and reverse traveling waves are V_n^+ and V_n^-, respectively.

1.2.1 Measuring or Computing S-parameters

Equation (1.37) shows how to measure the S-parameters S_{ij}: inject a known signal into port j and measure the signal at port i with all other ports terminated with a matched load. We can use this same idea to calculate S-parameters for a given network. To find the S-parameters for a network, we use these steps:

1. Choose one of the ports (often we start with port 1) to excite with an input wave.
2. Terminate all other ports with a matched load.
3. Write down the amplitudes of the forward and reverse waves on the transmission lines connected to each port. Matched ports will only have an input wave. To avoid confusion about the direction of forward and reverse waves, we often use the a and b notation from Sec. 1.2.3 on generalized S-parameters.
4. Find the voltages or currents at each port and relate them using the properties of the network.
5. Solve for ratios of forward and reverse waves to calculate S-parameters.
6. Choose a different port to excite and repeat the process to find remaining S-parameters. For many networks, additional excitations aren't needed, because reciprocity and port symmetry discussed in Sec. 1.2.5 mean that pairs the S-parameters are equal.

1.2.2 Reference Planes

The S-parameter S_{ij} is a complex number that can be represented by a magnitude and phase. The phase of the ratio in Eq. (1.37) depends on the location along the transmission line at which we measure the S-parameter. The location at which the measurement is made is called the reference plane.

We can derive an expression for how the S-matrix changes if the reference planes at each port are changed. For a N-port network, $[V^-] = [S][V^+]$. If we change the reference position of the nth port a distance ℓ_n away from the device, the new input and output waves are related to the new S-matrix by $[V^{-'}] = [S'][V^{+'}]$. From transmission line theory,

$$\begin{aligned} V_n^{+'} &= V_n^+ e^{j\beta_n \ell_n} = V_n^+ e^{j\theta_n} && \rightarrow && V_n^+ = V_n^{+'} e^{-j\theta_n} \\ V_n^{-'} &= V_n^- e^{-j\beta_n \ell_n} = V_n^- e^{-j\theta_n} && \rightarrow && V_n^- = V_n^{-'} e^{j\theta_n} \end{aligned} \tag{1.38}$$

where $\theta_n = \beta \ell_n$. Placing these equations into matrix form gives

$$\begin{bmatrix} e^{j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{j\theta_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{j\theta_N} \end{bmatrix} \left[V^{-'}\right] = [S] \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{-j\theta_N} \end{bmatrix} \left[V^{+'}\right] \tag{1.39}$$

or

$$[V^{-'}] = \underbrace{[\Theta][S][\Theta]}_{[S']} [V^{+'}] = [S'][V^{+'}] \tag{1.40}$$

where Θ is the matrix of phase shifts on the right hand side of Eq. (1.39). This result shows how the scattering matrices for the same network but with a different choice of reference planes are related.

If we change the reference plane on port 1 of a network by adding a length ℓ of matched transmission, for example, then S_{11} changes according to

$$S'_{11} = e^{-2j\beta\ell} S_{11} \tag{1.41}$$

The factor of two arises because the added length of line changes the phase of both the forward and the reverse wave that enter into the S-parameter.

1.2.3 Generalized S-Parameters

So far we have assumed that the characteristic impedances of all of the transmssion line port are the same. This is often the case. But in many practical situations, the impedances are different. In this case, for convenience we redefine the S-parameters so that they are related in a simple way to the power flowing through each of the ports.

Recall from transmission line analysis that the average (real) power associated with a voltage wave is given as $|V|^2/2Z_0$. Therefore, we can write for a two-port device where the reference impedance is different for the two ports (and the device is matched)

$$P_{\text{in}} = \frac{|V_1^+|^2}{2Z_{01}} \tag{1.42}$$

$$P_{\text{out}} = \frac{|V_2^-|^2}{2Z_{02}} \tag{1.43}$$

so that

$$\frac{P_{\text{out}}}{P_{\text{in}}} = \frac{|V_2^-|^2}{|V_1^+|^2}\frac{Z_{01}}{Z_{02}} \tag{1.44}$$

If we define generalized S-parameters by

$$a_n = \frac{V_n^+}{\sqrt{Z_{0n}}} \qquad b_n = \frac{V_n^-}{\sqrt{Z_{0n}}} \qquad S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k=0 \text{ for } k\neq j} \tag{1.45}$$

then

$$P_{\text{in}} = \frac{1}{2}|a_1|^2 \tag{1.46}$$

$$P_{\text{out}} = \frac{1}{2}|b_2|^2 \tag{1.47}$$

The ratio of input and output power is

$$\frac{P_{\text{out}}}{P_{\text{in}}} = \frac{|b_2|^2}{|a_1|^2} = |S_{21}|^2 \tag{1.48}$$

which is more convenient than (1.44). Notice that generalized S-parameters reduce to the same definition as in (1.35) if the impedances at each port are the same.

When solving network problems, we often use a and b to denote input and output waves, even if the port transmission line impedances are the same. This avoids confusion with the direction of input and output waves at the ports, since $+$ and $-$ waves aren't always moving to the right and left, respectively, as in earlier course material.

1.2.4 Lossless Networks

For lossless networks, no real power is delivered to the network. The time average power delivered to each port is

$$\begin{aligned} P_{\text{av}} &= \frac{1}{2}\text{Re}\left\{[V]^T[I]^*\right\} \\ &= \frac{1}{2}\text{Re}\left\{([a]^T + [b]^T)[\sqrt{Z_0}][\sqrt{Z_0}]^{-1}([a]^* - [b]^*)\right\} \\ &= \frac{1}{2}\text{Re}\left\{([a]^T[a]^* - [a]^T[b]^* + [b]^T[a]^* - [b]^T[b]^*)\right\} \end{aligned} \tag{1.49}$$

where $[\sqrt{Z_0}]$ is a diagonal matrix with nth diagonal entry $\sqrt{Z_{0n}}$ and the superscript ^{T} represents the matrix transpose operation. The middle two terms of this expression give an imaginary result. The other two terms are purely real. But since P_{av} is the average power delivered to the network, the real part must be zero:

$$P_{\text{av}} = \frac{1}{2}([a]^T[a]^* - [b]^T[b]^*) = 0 \tag{1.50}$$

The first and second terms represent the power entering and exiting the ports, respectively. This equation implies that

$$\begin{aligned} [a]^T[a]^* = [b]^T[b]^* &= (S[a])^T(S[a])^* \\ &= [a]^T S^T S^*[a]^* \end{aligned} \tag{1.51}$$

which in turn implies that

$$S^T S^* = SS^H = I = \text{Identity Matrix} \tag{1.52}$$

A matrix that satisfies this relationship is *unitary*. We can also write this condition as

$$\sum_{k=1}^{N} S_{ik}^T S_{kj}^* = \delta_{ij} \tag{1.53}$$

or

$$\sum_{k=1}^{N} S_{ki} S_{kj}^* = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.54}$$

Equations (1.52)-(1.54) all state the same condition, which if satisfied by the scattering matrix means that the network must be lossless.

1.2.5 Reciprocity

A network is reciprocal if

$$S = S^T \tag{1.55}$$

where the superscript denotes the matrix transpose operation. For a reciprocal network, the S-matrix is symmetric.

Consider the S-parameters of an isolator, which only allows a one-way flow of energy through a two-port network. Would the matrix satisfy (1.55)?

Reciprocity should not be confused with network symmetry. For a 2-port device, network symmetry means that $S_{11} = S_{22}$, so that port 1 behaves like port 2 if a signal is input to one of the ports and the other port is terminated with a matched load. (as well as the $S_{12} = S_{21}$ imposed by (1.55) if the network is reciprocal).

Chapter 2

Passive Structures

2.1 Lumped Element Matching

Matching is a key part of any RF/Microwave circuit design. The basic concept of matching is to couple energy from a transmission line into a load with as little reflection as possible. But in practice, matching is used for much more than simply minimizing reflection coefficients. Using appropriate matching techniques we can control gain, noise figure, and stability for a device.

There are a variety of techniques available for matching. We will first consider the simple but very useful case of using two lumped elements to accomplish a match at a single frequency. There are eight potential topologies available of this form (not all of which will be suitable for a given problem). Two possible matching configurations are shown in Fig. 2.1.

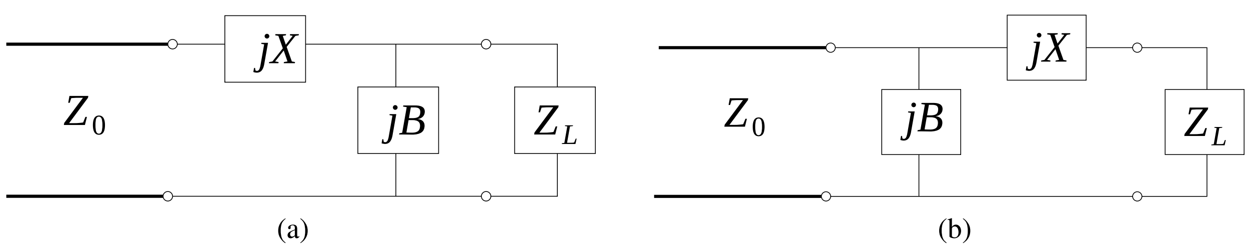

Figure 2.1: Matching network topologies. (a) Series reactance followed by a shunt susceptance. (b) Shunt susceptance followed by a series reactance.

2.1.1 Analytic Solution

We begin by formulating an analytic solution for this matching problem. Consider the case of a series reactance followed by a shunt susceptance. Let $Z_L = R_L + jX_L$ with admittance $Y_L = G_L + jB_L$. In order to match the input to Z_0, we must have

$$Z_0 = jX + \frac{1}{jB + G_L + jB_L} \tag{2.1}$$

If we cross multiply and separate into real and imaginary parts, we obtain the two equations

$$1 - (B_L + B)X = G_L Z_0 \tag{2.2}$$
$$(B_L + B)Z_0 = G_L X \tag{2.3}$$

If we solve the second equation for X and put this into the first equation, we obtain the solutions

$$B = \pm\sqrt{G_L(Y_0 - G_L)} - B_L \tag{2.4}$$
$$X = \pm\frac{\sqrt{(Y_0 - G_L)/G_L}}{Y_0} \tag{2.5}$$

where if you choose the top (bottom) sign for the first equation you must use the top (bottom) sign for the second. So, this topology provides two possible solutions depending on the sign used. Note that in order for the term in the radical to remain positive, we must have $Y_0 > G_L$. This condition holds if $R_L > Z_0$.

If we repeat the analysis for the topology in Fig. 2.1b, a shunt susceptance followed by a series reactance, we obtain the solutions

$$X = \pm\sqrt{R_L(Z_0 - R_L)} - X_L \tag{2.6}$$
$$B = \pm\frac{\sqrt{(Z_0 - R_L)/R_L}}{Z_0} \tag{2.7}$$

where if you choose the top (bottom) sign for the first equation, you must use the top (bottom) sign for the second equation. In this case, we must have $R_L < Z_0$.

Note that in these derivations, we are matching a complex load to a real impedance. We could also match a complex load to a complex impedance using these networks.

2.1.2 Smith Chart Solution

It is perhaps most instructive to demonstrate lumped-element matching on the Smith Chart using an example. Suppose we want to match $Z_L = 25\Omega$ to a 50Ω line at 1 GHz..

1. Normalize Z_L:
$$z_L = \frac{Z_L}{Z_0} = 0.5$$
Pick a topology: Since $R_L < Z_0$, we must use the topology with a shunt susceptance followed by a series reactance.

2. Work from the load to the source impedance. We want to use the series reactance to get a normalized admittance of the form $1 + jb$, so we can use the shunt susceptance to cancel the jb part and get a match (normalized impedance = 1). We do this by adding enough reactance to move to the circle with $g = 1$, so from the Smith chart, there are two points we can use, $x = 0.5$, and $x = -0.5$.

3. Now, add enough susceptance to move to the center of the Smith Chart: $b = 1.0$, or $b = -1.0$ for the other solution.

4. Convert the values to components:
$$X = Z_0 x = \begin{cases} 2\pi f L_s & X > 0 \\ -\frac{1}{2\pi f C_s} & X < 0 \end{cases}$$

$$B = Y_0 b = \frac{b}{Z_0} = \begin{cases} 2\pi f C_p & B > 0 \\ -\frac{1}{2\pi f L_p} & B < 0 \end{cases}$$

For this example, we get

$$\begin{aligned} jX &= +j25 = j\omega L \Rightarrow L \simeq 4\,\text{nH} \\ jB &= +j/50 = j\omega C \Rightarrow C \simeq 3.2\,\text{pF} \end{aligned}$$

or $C \simeq 6.4\,$pF, $L \simeq 8\,$nH for the other solution.

5. At higher frequencies, you might have to realize these components using transmission line stubs instead of lumped elements. In this case:

   Open-Circuit Stubs

$$\begin{aligned} x &= -\frac{Z_0 \cot \beta\ell}{Z_0} = -\cot \beta\ell \\ b &= \frac{Y_0 \tan \beta\ell}{Y_0} = \tan \beta\ell \end{aligned}$$

   Short-Circuit Stubs

$$\begin{aligned} x &= \frac{Z_0 \tan \beta\ell}{Z_0} = \tan \beta\ell \\ b &= -\frac{Y_0 \cot \beta\ell}{Y_0} = -\cot \beta\ell \end{aligned}$$

   You can then solve for the appropriate length.

Note that we can very easily use combinations of transmission lines and lumped elements when we are using the Smith Chart to match.

2.1.3 Stub Matching

Instead of using two reactances to match a load, we can use a length of transmission line to move the impedance to the $1 + jb$ circle. Then, we can use a shunt stub at that location with input impedance $y_{\text{stub}} = -jb$. The resulting normalized admittance is $y = 1$, which corresponds to an impedance matched to Z_0. The stub can either be open circuited or short circuited.

2.2 Multisection Quarter-Wave Transformers

Recall that a quarter-wave section of transmission line can be used as a matching network. From Eq. (1.25), the input impedance looking into a quarter-wave line ($\beta\ell = \pi/2$) is

$$Z_{\text{in}} = Z_1 \frac{Z_L + jZ_1 \tan\beta\ell}{Z_1 + jZ_L \tan\beta\ell} = \frac{Z_1^2}{Z_L} \tag{2.8}$$

where Z_1 is the characteristic impedance of the quarter-wavelength line. If we want to match a line of characteristic impedance Z_0 to the load, we must have

$$Z_0 = \frac{Z_1^2}{Z_L} \;\Rightarrow\; Z_1 = \sqrt{Z_0 Z_L} \tag{2.9}$$

What is the bandwidth of this matching network? If we change the frequency, then $\beta\ell$ is no longer equal to $\pi/2$. The reflection coefficient is

$$\Gamma = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} \tag{2.10}$$

where Z_{in} is given by (1.25). Figure 2.2 shows the magnitude of the reflection coefficient as a function of frequency relative to f_0, where f_0 is the frequency at which we have a perfect match. How can we get a broadband match?

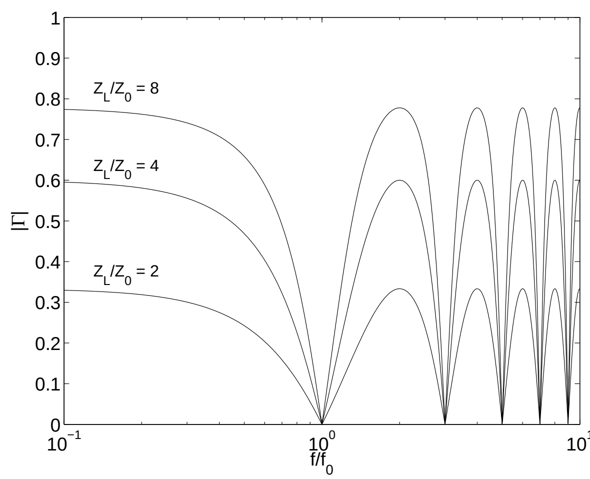

Figure 2.2: Magnitude of reflection coefficient as a function of frequency for a quarter-wavelength transformer.

2.2.1 Small Reflection Theory.

Suppose that Z_0, Z_1, and Z_L are close in value. If this is the case, then the reflection coefficient can be approximated by the first reflections from the two discontinuities in the transmission line:

$$\Gamma \simeq \Gamma_0 + \Gamma_1 e^{-2j\theta} \tag{2.11}$$

where

$$\begin{aligned}\Gamma_0 &= \frac{Z_1 - Z_0}{Z_1 + Z_0}\\ \Gamma_1 &= \frac{Z_L - Z_1}{Z_L + Z_1}\\ \theta &= \beta\ell\end{aligned}$$

For a multisection transformer as shown in Fig. 2.3 with sections of impedance $Z_0, Z_1, Z_2, \ldots, Z_N$ followed by a load Z_L, the reflection coefficient is approximately

$$\Gamma \simeq \Gamma_0 + \Gamma_1 e^{-2j\theta} + \cdots + \Gamma_N e^{-2jN\theta} \tag{2.12}$$

If we assume that the sections are symmetrical, so that $\Gamma_0 = \Gamma_N$, $\Gamma_1 = \Gamma_{N-1}$, and so on, then we can combine pairs of terms to get

$$\Gamma = 2e^{-jN\theta}\left(\Gamma_0 \cos N\theta + \Gamma_1 \cos{(N-2)\theta} + \cdots + \Gamma_{N/2}/2\right) \tag{2.13}$$

for N even, with a similar expression for N odd. Notice that this is a Fourier cosine series! This means that we can get any reflection coefficient as a function of frequency we want, with enough sections and by choosing the individual reflection coefficients Γ_n properly. There are a number of different ways to accomplish this.

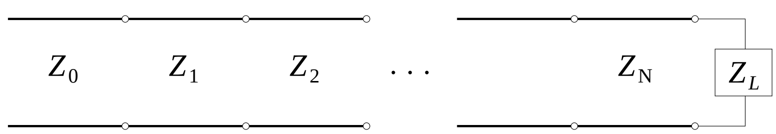

Figure 2.3: Multisection quarter wavelength transformer.

Binomial Multisection Matching. One possibility is to choose the reflection coefficient to be of the form

$$\Gamma = A(1 + e^{-2j\theta})^N \tag{2.14}$$

Since $1 + e^{-2j\theta} = 0$ for $\theta = \pi/2$ (at the center frequency), near the center frequency this quantity is small, and raising it to the Nth power makes it even smaller. This means that the reflection coefficient is closer to zero over a broader band around the center frequency for large N. More rigorously, this form means that $N - 1$ derivatives of Γ at $\theta = \pi/2$ are zero.

The constant A can be found by letting the frequency go to zero in (2.14) and using the fact that at very low frequencies the sections are electrically short and have negligible effect on the reflection coefficient:

$$\Gamma(0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.15}$$

If we expand Γ using the binomial theorem, we get

$$\Gamma = A\binom{N}{0} + A\binom{N}{1}e^{-2j\theta} + \cdots A\binom{N}{N}e^{-2jN\theta} \tag{2.16}$$

where the binomial coefficient is

$$\binom{N}{n} = \frac{N!}{(N-n)!n!} \tag{2.17}$$

By comparing this equation to (2.12), we can read off the values of Γ_n and determine the impedances $Z_1, Z_1, \ldots, Z_N$.

One way to get the impedances of each section from the values of Γ_n would be to use the expression for the reflection coefficient in terms of Z_n and Z_{n+1}, but if we do this for Z_2, Z_3, and so on then Z_{N+1} will be slightly different from Z_L. If we use

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \simeq \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n} \tag{2.18}$$

instead, we avoid this problem and get a formula for Z_n that is easy to use. From Eqs. (2.12) and (2.16), we find that

$$\begin{aligned} \ln \frac{Z_{n+1}}{Z_n} &\simeq 2A \binom{N}{n} \\ &= 2^{-N} 2 \frac{Z_L - Z_0}{Z_L + Z_0} \binom{N}{n} \\ &\simeq 2^{-N} \binom{N}{n} \ln \frac{Z_L}{Z_0} \end{aligned} \tag{2.19}$$

from which each of the impedances Z_n can be found. We begin with $n = 0$, which allows Z_1 to be found in terms of Z_0, and proceed to find $Z_2, \ldots, Z_N$. To check the results, Z_{N+1} should be equal to Z_L.

2.3 Power Dividers

Power dividers are important in signal splitting and combining. An ideal power divider would be matched at all ports, lossless, and reciprocal (so expensive nonreciprocal materials would not be needed in its construction). Is this possible? We can use S-parameters to answer this question.

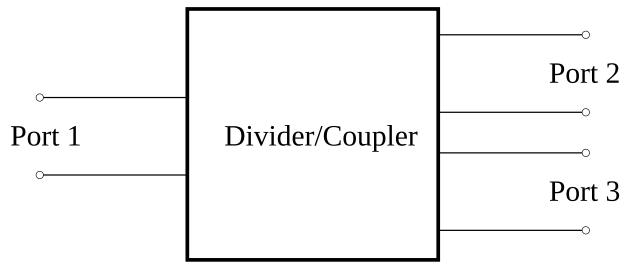

Figure 2.4: Three port microwave network for used as a power divider or coupler (combiner).

Consider a 3-port device with S-matrix

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \tag{2.20}$$

Suppose all three ports are matched so that $S_{ii} = 0$, and that the network is reciprocal so that $S_{ij} = S_{ji}$. The most general possible S-matrix for a device with these properties is

$$[S] = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{12} & 0 & S_{23} \\ S_{13} & S_{23} & 0 \end{bmatrix} \tag{2.21}$$

If the network is lossless, then this matrix must be unitary. Using one of the relationships in Section 1.2.4 derived for lossless networks leads to

$$|S_{12}|^2 + |S_{13}|^2 = 1 \qquad S_{13}S_{23}^* = 0 \tag{2.22}$$
$$|S_{12}|^2 + |S_{23}|^2 = 1 \qquad S_{12}S_{13}^* = 0 \tag{2.23}$$
$$|S_{13}|^2 + |S_{23}|^2 = 1 \qquad S_{12}S_{23}^* = 0 \tag{2.24}$$

The second column shows that at least two of the three unique S-parameters must be zero. But if two of them were zero, then one of the equations in the first column would be violated.

From this we can conclude that it is impossible to have a lossless, matched, and reciprocal three port device. Relaxing any one of these constraints makes it possible for the other two constraints to be satisfied. The possibilities are a network that is

1. Lossless and reciprocal, but not matched;
2. Reciprocal and matched but lossy;
3. Matched and lossless using a nonreciprocal material.

2.3.1 Tee Power Divider

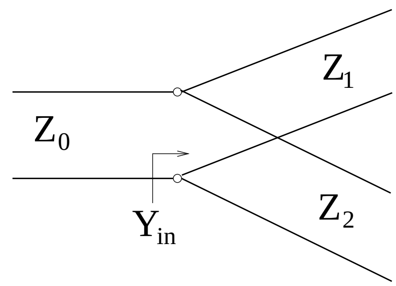

Figure 2.5: Transmission line tee.

The transmission line tee is a three port device that is lossless and reciprocal, but not matched on all ports. From Fig. 2.5, the input admittance is

$$Y_{\text{in}} = \frac{1}{Z_1} + \frac{1}{Z_2} \tag{2.25}$$

If we require an input match at port 1, then we must have

$$\frac{1}{Z_1} + \frac{1}{Z_2} = \frac{1}{Z_0} \tag{2.26}$$

If $Z_0 = 50\,\Omega$, and we choose $Z_1 = Z_2 = 100\Omega$, $Z_0 = 50\Omega$, then we will have an equal power split (3 dB) and the input will be matched. Because the lines of impedance Z_1 and Z_2 at the other two ports both see an input impedance of 50Ω in parallel with 100Ω, the tee is not matched at the output ports. If we want the impedances of the output lines to be the same as Z_0, we can replace them with impedance matching networks that transform Z_0 to the impedances required by (2.26) at the junction. (Does this mean that the tee with matching networks is matched at all ports?)

We can also choose Z_1 and Z_2 for an unequal power split. For example, if we want 5/6 of the power to go down one line and the remaining 1/6 down the other, then we set the two output line impedances to obtain the desired power ratio:

$$\begin{aligned}
P_{\text{in}} &= \frac{1}{2}\frac{|V_0|^2}{Z_0} \\
P_1 &= \frac{1}{2}\frac{|V_0|^2}{Z_1} = \frac{5}{6}P_{\text{in}} = \frac{5}{6}\frac{1}{2}\frac{|V_0|^2}{Z_0} \\
P_2 &= \frac{1}{2}\frac{|V_0|^2}{Z_2} = \frac{1}{6}P_{\text{in}} = \frac{1}{6}\frac{1}{2}\frac{|V_0|^2}{Z_0} \\
Z_1 &= \frac{6}{5}Z_0 \\
Z_2 &= 6Z_0
\end{aligned} \tag{2.27}$$

where we have assumed that the input is matched so that the incident voltage V_0 is the voltage at the junction. This will result in

$$Z_{\text{in}} = \frac{1}{5/6Z_0 + 1/6Z_0} = Z_0 \tag{2.28}$$

which is matched. The input impedance seen looking in from the other lines is

$$Z_{\text{in},1} = \frac{1}{1/Z_0 + 1/6Z_0} = \frac{6}{7}Z_0 \tag{2.29}$$

$$Z_{\text{in},2} = \frac{1}{1/Z_0 + 5/6Z_0} = \frac{6}{11}Z_0 \tag{2.30}$$

which still represents a mismatch.

2.3.2 Wilkinson Power Divider

The Wilkinson power divider shown in Fig. 2.6 is a three port network that is matched on all ports, but is lossy. The Wilkinson divider also provides output port isolation ($S_{23} = S_{32} = 0$).

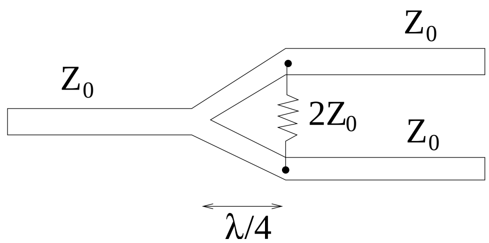

Figure 2.6: Wilkinson power divider in microstrip form. The characteristic impedance of the two quarter-wavelength branches is $\sqrt{2}Z_0$.

Analyzing this network directly to find its S-parameters would require the solution of many simultaneous equations. The amount of work required to find the S-parameters of this and other similar networks is greatly simplified by the use of the even/odd mode analysis technique.

Finding the S-parameters S_{i1} with a source at port 1 is relatively straightforward. It is more difficult to find the S-parameters with an input wave on port 2 or port 3. We will first consider the case of a source at port 2. As we will have to redraw the network several times, we will use a single wire to represent transmission lines to make the picture simpler, as shown in Fig. 2.7. This can be thought of as a microstrip circuit viewed from the top, with the other conductor for the transmission line being the ground plane.

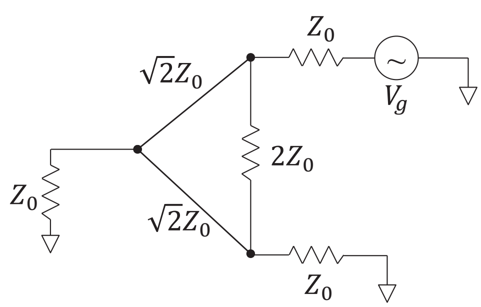

Figure 2.7: Equivalent circuit for the Wilkinson power divider.

To begin the even/odd mode analysis, we make the circuit symmetric vertically and break up the source into a sum of even and odd sources on ports 2 and 3, as in Fig. 2.8. Using superposition, we can analyze the circuit with the first pair of sources and then with the second pair. If we add the results, we have the voltages for the original single source. Because of the symmetry of the new network with even or odd sources, the analysis is much easier than is the case for the original source on port 2.

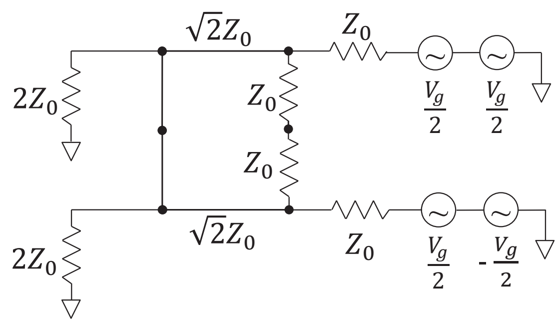

Figure 2.8: Symmetric equivalent circuit for the Wilkinson power divider.

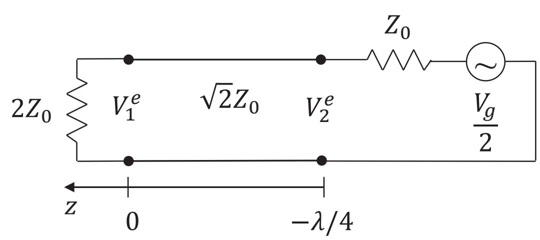

Figure 2.9: Even mode equivalent circuit for the Wilkinson power divider.

Even Mode

For equal excitations on the output ports (2 and 3), the symmetry of the system makes it such that we have no current flowing between the upper and lower portions of the circuit. Therefore, we can treat only the top portion, recognizing that the resistor is open-circuited.

We now analyze the simplified even mode network as follows:

1. Find Z_{in}^e: For $\ell = \lambda/4$,

$$Z_{\text{in}}^e = \frac{(\sqrt{2}Z_0)^2}{2Z_0} = \frac{2Z_0^2}{2Z_0} = Z_0 \tag{2.31}$$

   so that port 2 is matched.

2. Find V_2^e: By voltage division,

$$V_2^e = \frac{V_g}{2}\frac{Z_{\text{in}}^e}{Z_0 + Z_{\text{in}}^e} = \frac{V_g}{2}\frac{Z_0}{2Z_0} = \frac{V_g}{4} \tag{2.32}$$

3. The voltage on the $\lambda/4$ length of line is

$$V(z) = V^+e^{-j\beta z} + V^-e^{j\beta z} \tag{2.33}$$

4. Applying the reflection coefficient at the load (left end) leads to

$$\Gamma = \frac{V^-}{V^+} = \frac{2Z_0 - \sqrt{2}Z_0}{2Z_0 + \sqrt{2}Z_0} = \frac{2-\sqrt{2}}{2+\sqrt{2}} \tag{2.34}$$

$$V(z) = V^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) \tag{2.35}$$

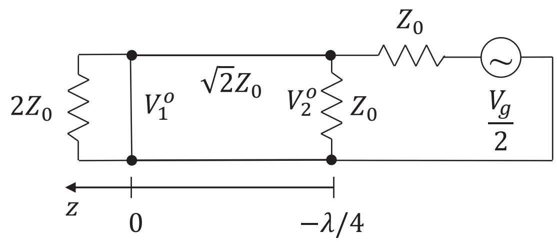

Figure 2.10: Odd mode equivalent circuit for the Wilkinson power divider.

5. This allows us to find V^+ in terms of V_g, using

$$V(-\lambda/4) = V^+\left[e^{j\pi/2} + \Gamma e^{-j\pi/2}\right] = V^+[j - j\Gamma] = jV^+(1-\Gamma) \tag{2.36}$$

$$V_2^e = \frac{V_g}{4} = V(-\lambda/4) = jV^+(1-\Gamma) \tag{2.37}$$

$$V^+ = -j\frac{V_g}{4}\frac{1}{1-\Gamma} \tag{2.38}$$

6. Now we can get the voltage V_1^e at port 1 in terms of the source voltage:

$$V_1^e = V(0) = V^+[1+\Gamma] = -j\frac{V_g}{4}\frac{1+\Gamma}{1-\Gamma} \tag{2.39}$$

$$= -j\frac{V_g}{4}\frac{2+\sqrt{2}+2-\sqrt{2}}{2+\sqrt{2}-2+\sqrt{2}} = -j\frac{V_g}{4}\frac{4}{2\sqrt{2}} = -j\frac{V_g}{2\sqrt{2}} \tag{2.40}$$

Summarizing these results, we have for the even mode

$$V_1^e = -j\frac{V_g}{2\sqrt{2}} \tag{2.41}$$

$$V_2^e = \frac{V_g}{4} \tag{2.42}$$

$$V_3^e = \frac{V_g}{4} \tag{2.43}$$

where the value for V_3^e results from the symmetry of the network.

Odd Mode

For equal but opposite excitations on ports 2 and 3, the line of symmetry through the middle of the network must be an equipotential at 0 V, so we can replace the nodes at the equipotential with grounds:

We can now analyze the odd mode network:

1. Find Z_{in}^o: For $\ell = \lambda/4$

$$Z_{\text{in}}^o = \frac{(\sqrt{2}Z_0)^2}{0} = \infty \tag{2.44}$$

which is an open circuit.

2. The voltage at port 2 is
$$V_2^o = \frac{V_g}{2}\frac{Z_0}{Z_0 + Z_0} = \frac{V_g}{4} \tag{2.45}$$

3. Because of the short to ground at port 1, the voltage at port 1 is zero. Because the sources at ports 2 and 3 are opposite in polarity, the voltage at port 3 is the negative of the voltage at port 2. This leads to the port voltages
$$V_1^o = 0 \tag{2.46}$$
$$V_2^o = \frac{V_g}{4} \tag{2.47}$$
$$V_3^o = -\frac{V_g}{4} \tag{2.48}$$

Superposition

We can now combine the voltages for the even and odd excitations using superposition:

$$V_1 = V_1^e + V_1^o = -j\frac{V_g}{2\sqrt{2}} \tag{2.49}$$
$$V_2 = V_2^e + V_2^o = \frac{V_g}{2} \tag{2.50}$$
$$V_3 = V_3^e + V_3^o = 0 \tag{2.51}$$

These are the voltages at each port with a source at port 2.

The input impedance at port 2 (and 3) is Z_0 for both even and odd excitations. Therefore, the reflected voltage at port 2 is
$$V_2^- = V_2^{e-} + V_2^{o-} = 0 \tag{2.52}$$
This means that ports 2 and 3 are matched.

Input Match

When excited from port 1, power splits equally between branches. Therefore, no current flows in the $2Z_0$ resistor. On one branch:
$$Z_{\text{in},1}^1 = \frac{(\sqrt{2}Z_0)^2}{Z_0} = 2Z_0 \tag{2.53}$$
Since we see two such branches in parallel, $Z_{\text{in}}^1 = Z_0$, indicating an input match. Note that when excited from port 1, there is no power loss since no current flows through the resistor.

S-Parameter Matrix

We now use the results of the even/odd mode analysis to determine the S-paramet ers of the Wilkinson divider:

$$\begin{aligned}
S_{11} &= S_{22} = S_{33} = 0 \\
S_{32} &= \tfrac{V_3^-}{V_2^+} = \tfrac{V_3}{V_2} = 0 \qquad \text{since } V_3 = 0 \\
S_{23} &= 0 \qquad \text{by symmetry} \\
S_{12} &= \tfrac{V_1^-}{V_2^+} = \tfrac{V_1}{V_2} = -\tfrac{j}{\sqrt{2}} \\
S_{21} &= S_{31} = S_{13} = -\tfrac{j}{\sqrt{2}}
\end{aligned} \tag{2.54}$$

In matrix form,

$$[S] = -\frac{j}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.55}$$

If we send in signals V_2^+ and V_3^+ on ports 2 and 3, the output wave at port 1 is

$$V_1^- = -\frac{j}{\sqrt{2}} V_2^+ - \frac{j}{\sqrt{2}} V_3^+ = -\frac{j}{\sqrt{2}} (V_2^+ + V_3^+) \tag{2.56}$$

Note that the sum is scaled in magnitude by $1/\sqrt{2}$. The circuit will therefore present the sum of the two voltage signals at port 1. If the input signals at ports 2 and 3 are in phase, then the signals are combined with no power loss. This is an even mode excitation and no current flows through the resistor. If the signals are not in phase, then power is dissipated in the resistor.

2.4 Passive Filters

Filters are an important part of microwave engineering. A filter is a two-port microwave network which attenuates signal components at some frequencies and passes others. The basic filter types are low-pass, high-pass, bandpass, and band-reject or notch filters. One approach to microwave filter design is to first come up with a low frequency lumped element design, and then map the design to transmission line sections.

2.4.1 Insertion Loss Design

A common method for specifying a filter characteristic is through the insertion loss versus frequency, or power loss ratio:

$$P_{LR}(\omega) = \frac{\text{Power available from source}}{\text{Power delivered to load}} = \frac{1}{1 - |\Gamma(\omega)|^2} \tag{2.57}$$

where Γ is the reflection coefficient looking into the input of the filter network.

For a low pass filter, one standard form for this quantity is

$$P_{LR} = 1 + k^2(\omega/\omega_c)^{2N} \quad \text{Maximally Flat/Butterworth/Binomial} \tag{2.58}$$

The loss ratio grows as frequency increases, so that high frequency components are attenuated. At the band edge ($\omega = \omega_c$), the power loss is $1 + k^2$. This filter characteristic is called maximally flat because derivatives of the response vanish at $\omega = 0$ up to order $2N$, so for a given value of N this function is as flat as possible near $\omega = 0$.

Another form for the insertion loss is

$$P_{LR} = 1 + k^2 T_N(\omega/\omega_c) \quad \text{Equal Ripple/Chebychev} \tag{2.59}$$

where $T_N(x) = \cos(N\cos^{-1} x)$ is a Chebychev polynomial. In the range $-1 \le x \le 1$, Chebychev polynomials oscillate between ± 1, which is where the name “equal ripple” comes from. The height of the ripples for a Chebychev filter is $1 + k^2$. Both of these filter types have an integer order N, which is determined by the number of stages in the filter. The larger the order, the faster the rolloff of the frequency response, but the filter also becomes more expensive to implement.

Another property of a filter characteristic is linear phase, where the phase response of a filter is specified as well as the magnitude. Why would linear phase be desirable?

2.4.2 Low Pass Filter Prototypes and Transformations

For various types of filter characteristics, such as those given above for the maximally flat and equal ripple cases, lumped element values have been computed and tabulated. To save space, this is done for low pass filters only, with a corner frequency of $\omega_c = 1$ and a source impedance of $R_s = 1\,\Omega$ (see Pozar, Section 8.3). If a different corner frequency or a high-pass or bandpass filter is desired, simple transformations can be applied to the low pass prototype design to get the desired filter type.

The tabulated lumped element values give capacitances and inductances for LC sections. For an Nth order filter, N of these sections are cascaded to give the desired response. The response as a function of frequency is also computed (Pozar, Section 8.3), so the required value of N can be obtained from the desired attenuation at the corner frequency ω_c.

Impedance scaling. In order to scale the source impedance to R_0, we use the transformations

$$L' = R_0 L \tag{2.60}$$

$$C' = C/R_0 \tag{2.61}$$

Frequency scaling. To change the corner frequency of a low-pass filter from unity to ω_c, we replace ω in the impedances of the lumped elements with $\omega \to \omega/\omega_c$. This leads to the transformations

$$L'' = L'/\omega_c \tag{2.62}$$

$$C'' = C'/\omega_c \tag{2.63}$$

Low-pass to high-pass transformation. To change a low-pass filter prototype into a high-pass filter, we make the replacement $\omega \to -\omega_c/\omega$. This leads to

$$L'' = \frac{1}{\omega_c C'} \tag{2.64}$$

$$C'' = \frac{1}{\omega_c L'} \tag{2.65}$$

Low-pass to bandpass transformation. To change a low-pass filter prototype into a band-pass filter, we use

$$\omega \to \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \tag{2.66}$$

where $\Delta = (\omega_2 - \omega_1)/\omega_0$ and the corners of the passband are ω_1 and ω_2. The center frequency is often chosen to be $\omega_0 = \sqrt{\omega_1\omega_2}$. For the bandpass filter, the element transformations are a little more complicated. A series inductance L' is transformed into a series LC circuit with

$$L'' = \frac{L'}{\Delta\omega_0} \tag{2.67}$$

$$C'' = \frac{\Delta}{\omega_0 L'} \tag{2.68}$$

and a shunt capacitance C' is transformed into a shunt LC circuit with

$$L'' = \frac{\Delta}{\omega_0 C'} \tag{2.69}$$

$$C'' = \frac{C'}{\Delta\omega_0} \tag{2.70}$$

This should be intuitive, because an inductor is a low-pass filter, and an LC circuit is a bandpass filter with the band center at the resonance frequency.

2.4.3 Implementation

The capacitors and inductors resulting from the low-pass prototype approach can be realized using either lumped elements or transmission line sections.

Other methods for implementing microstrip filters are stepped-impedance filters, consisting of alternating sections of low impedance and high impedance lines, and coupled line filters, which are sections of transmission lines placed nearby with frequency dependent coupling between the lines.

Chapter 3

Amplifiers

For low frequency amplifier design, a transistor can be modeled using an equivalent circuit. At microwave frequencies, reflections from the input and output ports become important, so a network description characterized by S-parameters as a function of frequency is required. The key parameters of a transistor for microwave amplification are

- Gain as a function of frequency
- $f_T =$ frequency at which gain drops to unity
- S-parameters as a function of frequency
- Noise figure: characterizes noise added to signal by the device ($\mathrm{SNR}_{\mathrm{out}} < \mathrm{SNR}_{\mathrm{in}}$).

f_T determines the usable bandwidth of the transistor, the S-parameters are used to design matching networks to match to the transistor and determine the gain and stability of the amplifier, and the noise figure is used to determine the amount of noise produced by the transistor.

3.1 Gain

The most important quantity for a microwave amplifier is power gain. In order to analyze amplifier gain, we first need to determine the gain in terms of the S-parameters of the transistor. Because there are forward and reverse waves at the input and output ports, unlike in circuit theory, there several different power measures that can be used to define gain:

- $P_{\mathrm{in}} =$ power delivered to network
- $P_{\mathrm{avs}} =$power available from source
- $P_L =$ power delivered to load
- $P_{\mathrm{avn}} =$ power available from network

Available power is the maximum power that can be supplied with over all possible load impedances. The actual power delivered may be smaller than the available power due to reflections and is less than or equal to the available power.

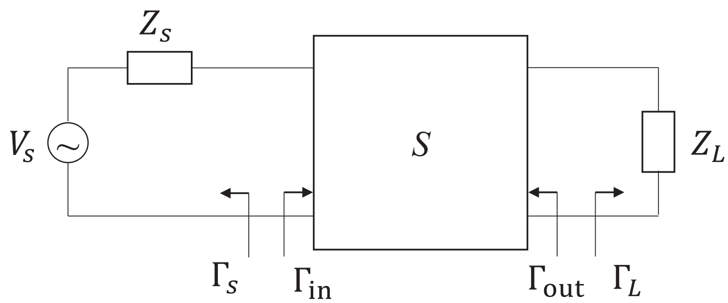

Figure 3.1: A microwave amplifier connected to a source at the input (port 1) and a load impedance at the output (port 2).

We first need to find the reflection coefficients looking into ports 1 and 2, as shown in Fig. 3.1. The source and load reflection coefficients are

$$\Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{3.1}$$

Given these definitions, we can write the amplitudes b_1 and b_2 of the waves exiting ports 1 and 2 as

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 + S_{12}\Gamma_L b_2 \tag{3.2}$$
$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 + S_{22}\underbrace{\Gamma_L b_2}_{a_2} \tag{3.3}$$

Solving the second equation for b_2 gives

$$b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L}a_1 \tag{3.4}$$

Using this expression in the first equation gives

$$b_1 = S_{11}a_1 + S_{12}\Gamma_L\frac{S_{21}}{1 - S_{22}\Gamma_L}a_1 \tag{3.5}$$

The ratio b_1/a_1 gives the reflection coefficient looking into the input port:

$$\Gamma_{\text{in}} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}\Gamma_L S_{21}}{1 - S_{22}\Gamma_L} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} \tag{3.6}$$

Repeating this procedure for a source at port 2, we find that

$$\Gamma_{\text{out}} = S_{22} + \frac{S_{12}\Gamma_s S_{21}}{1 - S_{11}\Gamma_s} \tag{3.7}$$

This expression can be understood in an intuitive way. The first term accounts for reflections at port 2, and would be the entire reflection coefficient if port 1 were matched. In the second term, S_{12} takes the signal from port 2 to port 1, Γ_s reflects the signal from the source impedance, S_{21} returns the signal to port 2, and the denominator takes into account multiple reflections between the ports.

We can compute P_{in} and P_L in terms of these reflection coefficients using

$$P_{\text{in}} = \frac{|a_1|^2}{2} - \frac{|b_1|^2}{2} = \frac{|a_1|^2}{2}\left(1 - |\Gamma_{\text{in}}|^2\right) \tag{3.8}$$

$$P_L = \frac{|b_2|^2}{2} - \frac{|a_2|^2}{2} = \frac{|b_2|^2}{2}\left(1 - |\Gamma_L|^2\right) \tag{3.9}$$

Now, we need to find a_1 and b_2. We can express the voltage at port 1 in terms of the source voltage V_s using a voltage divider as well as in terms of the generalized voltage waves a_1 and b_1. Equating these two representations gives

$$V_1 = V_s \frac{Z_{\text{in}}}{Z_s + Z_{\text{in}}} = \sqrt{Z_0}(a_1 + b_1) \tag{3.10}$$

$$= \sqrt{Z_0}a_1(1 + \Gamma_{\text{in}}) \tag{3.11}$$

Substituting $Z_{\text{in}} = Z_0(1 + \Gamma_{\text{in}})/(1 - \Gamma_{\text{in}})$ into this expression gives

$$V_s \frac{(1 + \Gamma_{\text{in}})Z_0}{Z_s(1 - \Gamma_{\text{in}}) + Z_0(1 + \Gamma_{\text{in}})} = \sqrt{Z_0}a_1(1 + \Gamma_{\text{in}}) \tag{3.12}$$

Solving for a_1 and using that $Z_s = Z_0(1 + \Gamma_s)/(1 - \Gamma_s)$ gives

$$\begin{aligned} a_1 &= \frac{Z_0}{Z_s(1 - \Gamma_{\text{in}}) + Z_0(1 + \Gamma_{\text{in}})} \frac{V_s}{\sqrt{Z_0}} \\ &= \frac{Z_0}{Z_0 \frac{1+\Gamma_s}{1-\Gamma_s}(1 - \Gamma_{\text{in}}) + Z_0(1 + \Gamma_{\text{in}})} \frac{V_s}{\sqrt{Z_0}} \\ &= \frac{1 - \Gamma_s}{(1 + \Gamma_s)(1 - \Gamma_{\text{in}}) + (1 - \Gamma_s)(1 + \Gamma_{\text{in}})} \frac{V_s}{\sqrt{Z_0}} \\ &= \frac{1}{2} \frac{1 - \Gamma_s}{1 - \Gamma_{\text{in}}\Gamma_s} \frac{V_s}{\sqrt{Z_0}} \end{aligned} \tag{3.13}$$

If we put this into the expression for P_{in}, we find that

$$\begin{aligned} P_{\text{in}} &= \frac{1}{2}\left|\frac{1}{2} \frac{1 - \Gamma_s}{1 - \Gamma_{\text{in}}\Gamma_s} \frac{V_s}{\sqrt{Z_0}}\right|^2 (1 - |\Gamma_{\text{in}}|^2) \\ &= \frac{|V_s|^2}{8Z_0} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_{\text{in}}\Gamma_s|^2}(1 - |\Gamma_{\text{in}}|^2) \end{aligned} \tag{3.14}$$

Using (3.4) together with (3.13) gives the amplitude of the wave propagating out of port 2,

$$b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L}a_1 = \frac{V_s}{2\sqrt{Z_0}} \frac{S_{21}(1 - \Gamma_s)}{(1 - \Gamma_{\text{in}}\Gamma_s)(1 - S_{22}\Gamma_L)} \tag{3.15}$$

This can be used to obtain the power delivered to the load,

$$P_L = \frac{|V_s|^2}{8Z_0}|S_{21}|^2 \frac{(1 - |\Gamma_L|^2)|1 - \Gamma_s|^2}{|1 - S_{22}\Gamma_L|^2|1 - \Gamma_s\Gamma_{\text{in}}|^2} \tag{3.16}$$

Power Gain

Using these results, the power gain of the amplifier is

$$\begin{aligned} G_P = \frac{P_L}{P_{\text{in}}} &= |S_{21}|^2 \frac{(1-|\Gamma_L|^2)|1-\Gamma_s|^2}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s\Gamma_{\text{in}}|^2} \frac{|1-\Gamma_s\Gamma_{\text{in}}|^2}{|1-\Gamma_s|^2(1-|\Gamma_{\text{in}}|^2)} \\ &= |S_{21}|^2 \frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2(1-|\Gamma_{\text{in}}|^2)} \end{aligned} \tag{3.17}$$

This is the gain of the amplifier in terms of power delivered to the load relative to power coming into the amplifier input port.

In this expression, if the load reflection coefficient is one, then the gain vanishes, as expected. If the load reflection coefficient is zero, then

$$G_P = |S_{21}|^2 \frac{1}{1-|\Gamma_{\text{in}}|^2} = |S_{21}|^2 \frac{1}{1-|S_{11}|^2} \tag{3.18}$$

Note, however, that $\Gamma_L = 0$ does not mean the power gain is maximized, since it simply means $Z_L = Z_0$ whereas maximum power gain occurs for a conjugate match condition (we will see this more clearly later). More generally, the power gain is equal to $|S_{21}|^2$ scaled by a factor which takes into account reflections at the load (which means G_P can be larger than $|S_{21}|^2$).

Transducer Power Gain

The problem with power gain as defined in Eq. (3.18) is that it does not consider the match between the source and the input impedance to the network. In some cases, therefore, a more meaningful measure of gain is the power dissipated by the load relative to the maximum power that the source can supply. This is the transducer power gain,

$$G_T = \frac{P_L}{P_{\text{avs}}} \tag{3.19}$$

If we consider a source with impedance Z_s driving a line with input impedance Z_{in}, the maximum power transfer occurs when Z_{in} is the complex conjugate of Z_s, so that

$$Z_{\text{in}} = Z_s^* \tag{3.20}$$

This can be proved by taking the derivative of the power delivered to the line with respect to the real and imaginary parts of Z_{in} and setting the derivatives to zero. The imaginary parts of Z_{in} and Z_s are equal in magnitude and opposite in sign, which is what happens with the impedances of the inductor and capacitor in an LCR circuit at resonance. This is called a **conjugate match**.

With a conjugate match to the source impedance, the power available from the source is

$$\begin{aligned} P_{\text{avs}} &= P_{\text{in}}\big|_{\Gamma_{\text{in}}=\Gamma_s^*} \quad \text{(conjugate match)} \\ &= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{|1-\underbrace{\Gamma_s^*}_{\text{cm}}\Gamma_s|^2}(1-|\Gamma_s|^2) \\ &= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{1-|\Gamma_s|^2} \end{aligned} \tag{3.21}$$

The transducer gain is then

$$\begin{aligned} G_T &= |S_{21}|^2 \frac{(1-|\Gamma_L|^2)|1-\Gamma_s|^2}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s\Gamma_{\text{in}}|^2} \frac{1-|\Gamma_s|^2}{|1-\Gamma_s|^2} \\ &= |S_{21}|^2 \frac{(1-|\Gamma_s|^2)(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s \underbrace{\Gamma_{\text{in}}}_{\neq \text{ cm}}|^2} \end{aligned} \tag{3.22}$$

Notice that the input reflection coefficient in the power delivered to the load does not assume a conjugate match at the input. This is because transducer gain does not mean that we have actually conjugate matched to the input impedance. Instead, we just want to compute the gain relative to the input power we would have if the source were conjugate matched.

Available Power Gain

Another difficulty with power gain in Eq. (3.18) is that if the load is not well matched to the network output impedance, then the power gain is small, even though the amplifier is capable of delivering more power to a better matched load. If we want to characterize the amplifier independently of the load impedance, we can define a measure of gain in terms of the power delivered to a conjugate matched load. This is the available power gain,

$$G_A = \frac{P_{\text{avn}}}{P_{\text{avs}}} \tag{3.23}$$

which is the gain if both the source and load were conjugate matched. The power available from the amplifier network at port 2 is

$$\begin{aligned} P_{\text{avn}} &= P_L|_{\Gamma_L=\Gamma^*_{\text{out}}} \\ &= \frac{|V_s|^2}{8Z_0}|S_{21}|^2 \frac{(1-|\Gamma_{\text{out}}|^2)|1-\Gamma_s|^2}{|1-S_{22}\Gamma^*_{\text{out}}|^2|1-\Gamma_s\Gamma_{\text{in}}|^2} \end{aligned} \tag{3.24}$$

The available gain is

$$G_A = |S_{21}|^2 \frac{(1-|\Gamma_{\text{out}}|^2)(1-|\Gamma_s|^2)}{|1-S_{22}\Gamma^*_{\text{out}}|^2|1-\Gamma_s\Gamma_{\text{in}}|^2} \tag{3.25}$$

In order to simplify the expression for available power gain by eliminating Γ_{in}, we use

$$\begin{aligned} 1-\Gamma_s\Gamma_{\text{in}} &= 1-\Gamma_s\left(S_{11}+\frac{S_{12}\Gamma_L S_{21}}{1-S_{22}\Gamma_L}\right) \\ &= \frac{1-S_{22}\Gamma_L - S_{11}\Gamma_s + S_{11}S_{22}\Gamma_s\Gamma_L - S_{12}S_{21}\Gamma_s\Gamma_L}{1-S_{22}\Gamma_L} \\ &= \frac{1-S_{11}\Gamma_s}{1-S_{22}\Gamma_L}\left[1-\Gamma_L \underbrace{\left(S_{22}+\frac{S_{12}\Gamma_s S_{21}}{1-S_{11}\Gamma_s}\right)}_{\Gamma_{\text{out}}}\right] \\ &= \frac{1-S_{11}\Gamma_s}{1-S_{22}\Gamma_L}(1-\Gamma_L\Gamma_{\text{out}}) \end{aligned} \tag{3.26}$$

With this result, the available gain becomes

$$G_A = |S_{21}|^2 \frac{1-|\Gamma_s|^2}{|1-S_{11}\Gamma_s|^2(1-|\Gamma_{\text{out}}|^2)} \tag{3.27}$$

Special Cases

1. Both source and load impedances are equal to the Z_0 ($\Gamma_s = \Gamma_L = 0$). In this case,

$$G_T = |S_{21}|^2 \tag{3.28}$$

   It is important to be aware that if we instead choose a conjugate match at the source and load ($\Gamma_s = \Gamma_{\text{in}}^*, \Gamma_L = \Gamma_{\text{out}}^*$), the gain can be larger than $|S_{21}|^2$.

2. Unilateral device ($S_{12} = 0$ or very small). In this case, $\Gamma_{\text{in}} = S_{11}$ and $\Gamma_{\text{out}} = S_{22}$, and the transducer gain becomes

$$\begin{aligned} G_{TU} &= |S_{21}|^2 \frac{(1-|\Gamma_s|^2)(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s S_{11}|^2} \\ &= \underbrace{\frac{1-|\Gamma_s|^2}{|1-\Gamma_s S_{11}|^2}}_{\substack{\text{source} \\ G_s}} \underbrace{|S_{21}|^2}_{\substack{\text{device} \\ G_0}} \underbrace{\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}}_{\substack{\text{load} \\ G_L}} \end{aligned} \tag{3.29}$$

   which can be broken up into a product of source, device, and load gain factors. Again, it is possible for G_s and G_L to be greater than one, depending on the source and load matches.

   Using the expression for transducer gain for a bilateral device, it can be shown that the error made in assuming that a transistor is unilateral is bounded by

$$\frac{1}{(1+U)^2} \leq \frac{G_T}{G_{TU}} < \frac{1}{(1-U)^2} \tag{3.30}$$

   where

$$U = \frac{|S_{12}S_{21}S_{11}S_{22}|}{(1-|S_{11}|^2)(1-|S_{22}|^2)} \tag{3.31}$$

   is the unilateral figure of merit. The smaller U, the closer the device is to unilateral.

Summary

For a two-port amplifier, we define three gain quantities:

$$G_P = \frac{P_L}{P_{\text{in}}} \quad \text{(Power Gain)} \tag{3.32}$$

$$G_T = \frac{P_L}{P_{\text{avs}}} \quad \text{(Transducer Gain)} \tag{3.33}$$

$$G_A = \frac{P_{\text{avn}}}{P_{\text{avs}}} \quad \text{(Available Power Gain)} \tag{3.34}$$

In developing amplifier design procedures, we will use whichever type of gain is most convenient for a given problem.

3.2 Unilateral Amplifier Design - Gain Circles

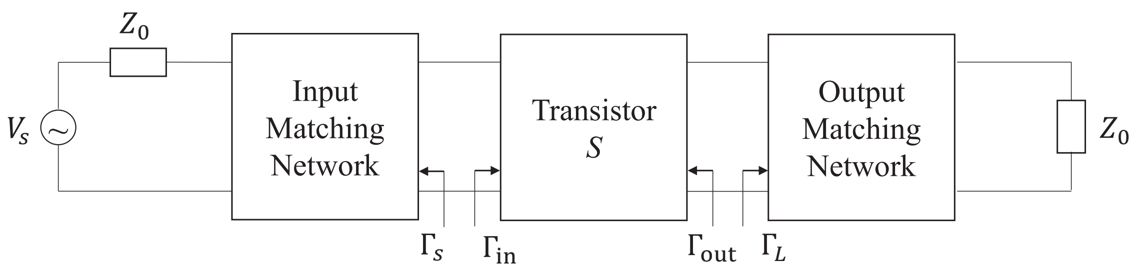

Figure 3.2: Amplifier with device and source and load matching networks.

When designing an amplifier, we typically design matching networks at the source and load as shown in Fig. 3.2 with values of Γ_s and Γ_L such that a given set of design criteria for the amplifier are met (gain, stability, noise performance, bandwidth, etc.).

In order to obtain a specified gain, we can use the method of constant gain circles, which are circles of values of Γ_s and Γ_L that give constant gain. For a unilateral device,

$$G_{TU} = G_s G_0 G_L \tag{3.35}$$

where

$$G_s = \frac{1-|\Gamma_s|^2}{|1-\Gamma_s S_{11}|^2} \qquad G_L = \frac{1-|\Gamma_L|^2}{|1-\Gamma_L S_{22}|^2} \tag{3.36}$$

The maximum values of the source and load gain factors are

$$\begin{aligned} G_{s,\max} &= G_s|_{\Gamma_s = S_{11}^*} \quad \text{(conjugate match)} \\ &= \frac{1-|S_{11}|^2}{|1-|S_{11}|^2|^2} = \frac{1}{1-|S_{11}|^2} \end{aligned} \tag{3.37}$$

$$\begin{aligned} G_{L,\max} &= G_s|_{\Gamma_L = S_{22}^*} \quad \text{(conjugate match)} \\ &= \frac{1}{1-|S_{22}|^2} \end{aligned} \tag{3.38}$$

In terms of the maximum values, we define normalized gains,

$$g_s = \frac{G_s}{G_{s,\max}} = \frac{(1-|\Gamma_s|^2)(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s|^2} \tag{3.39}$$

$$g_L = \frac{G_L}{G_{L,\max}} = \frac{(1-|\Gamma_L|^2)(1-|S_{22}|^2)}{|1-S_{22}\Gamma_L|^2} \tag{3.40}$$

so that $0 \le g_s \le 1$ and $0 \le g_L \le 1$.

If we want to design for a specified value of g_s, then rearranging the expression for normalized gain gives

$$\begin{aligned} g_s|1-S_{11}\Gamma_s|^2 &= (1-|\Gamma_s|^2)(1-|S_{11}|^2) \\ g_s(1-S_{11}\Gamma_s - S_{11}^*\Gamma_s^* + |S_{11}|^2|\Gamma_s|^2) &= 1-|S_{11}|^2 - |\Gamma_s|^2 + |S_{11}\Gamma_s|^2 \\ (g_s|S_{11}|^2 + 1 - |S_{11}|^2)|\Gamma_s|^2 - g_s(S_{11}\Gamma_s + S_{11}^*\Gamma_s^*) &= 1-|S_{11}|^2 - g_s \end{aligned} \tag{3.41}$$

This can be placed in the form

$$\Gamma_s\Gamma_s^* - \frac{g_s(S_{11}\Gamma_s + S_{11}^*\Gamma_s^*)}{1-(1-g_s)|S_{11}|^2} = \frac{1-|S_{11}|^2 - g_s}{1-(1-g_s)|S_{11}|^2} \tag{3.42}$$

We will now show that this is an equation for a circle in the Γ_s plane. The equation for a circle with center C_s and radius r_s in the complex plane is

$$\begin{aligned} |\Gamma_s - C_s|^2 &= r_s^2 \\ (\Gamma_s - C_s)(\Gamma_s^* - C_s^*) &= r_s^2 \\ \Gamma_s\Gamma_s^* - (C_s^*\Gamma_s + C_s\Gamma_s^*) &= r_s^2 - |C_s|^2 \end{aligned} \tag{3.43}$$

By comparing Eqs. (3.42) and (3.43), we can see that Γ_s lies on a circle with center at

$$C_s = \frac{g_s S_{11}^*}{1-(1-g_s)|S_{11}|^2} \tag{3.44}$$

The radius of the circle is given by

$$\begin{aligned} r_s^2 &= |C_s|^2 + \frac{1-|S_{11}|^2 - g_s}{1-(1-g_s)|S_{11}|^2} \\ &= \frac{(1-|S_{11}|^2 - g_s)[1-(1-g_s)|S_{11}^2|] + g_s^2|S_{11}|^2}{[1-(1-g_s)|S_{11}|^2]^2} \\ &= \frac{(1-g_s)(|S_{11}|^4 - 2|S_{11}|^2 + 1)}{[1-(1-g_s)|S_{11}|^2]^2} \\ &= \frac{(1-g_s)(1-|S_{11}|^2)^2}{[1-(1-g_s)|S_{11}|^2]^2} \end{aligned} \tag{3.45}$$

so that

$$r_s = \frac{\sqrt{1-g_s}(1-|S_{11}|^2)}{1-(1-g_s)|S_{11}|^2} \tag{3.46}$$

The center C_L and radius r_L of the Γ_L circle for a constant value of g_L can be obtained from Eqs. (3.44) and (3.46) by replacing g_s with g_L and S_{11} with S_{22}.

These results lead to a design procedure for a desired value of the gain:

1. From $G_{TU} = G_s|S_{21}|^2 G_L$, determine desired values for G_s and G_L. One approach is to conjugate match the input port so that $G_s = G_{s,\max}$, and then choose G_L to obtain the desired gain.

2. Compute the normalized gains g_s and g_L.

3. Compute the load gain circle center and radius, C_L and r_L. If the source is not conjugate matched, compute the source gain circle center and radius C_s and r_s also.

4. Choose convenient values of Γ_s and Γ_L on these circles. Any value on the circle will meet the gain target, so we need some way to choose a particular value. One possibility is to choose the reflection coefficient with smallest magnitude ($|\Gamma|$ closest to zero). Later, we will consider other performance metrics such as noise figure that will dictate the choices of Γ_s and Γ_L.

5. Find the source network impedance and load network impedance using

$$Z_s = Z_0 \frac{1+\Gamma_s}{1-\Gamma_s} \tag{3.47}$$

$$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L} \tag{3.48}$$

If $g_s = 1$, then $C_s = S_{11}^*$ and $r_s = 0$, so that $\Gamma_s = S_{11}^*$, which is what we expect, since this is a conjugate match.

If we conjugate match both the input and output, then we can have both $G_s > 1$ and $G_L > 1$. How is this possible with passive matching networks?

3.3 Stability

After gain, the next key amplifier concept is stability. If an amplifier is unstable, noise feedback will lead to oscillation. Stability can be determined from the reflection coefficients Γ_{in} and Γ_{out} looking into the input and output of the transistor. If the magnitude of one or both of these reflection coefficients is greater than unity, then the amplifier is unstable.

Since Γ_{in} and Γ_{out} depend on the reflection coefficients Γ_s and Γ_L looking from the device into the source and load, the matching networks determine the stability of the amplifier. There are two possible situations:

1. Unconditional stability: $|\Gamma_{\text{in}}| < 1$ and $|\Gamma_{\text{out}}| < 1$ for all passive source and load impedances ($|\Gamma_s| < 1$, $|\Gamma_L| < 1$).
2. Conditional stability: $|\Gamma_{\text{in}}| < 1$ and $|\Gamma_{\text{out}}| < 1$ only for a certain range of source and load impedances. This is also called potentially unstable. For this case, we design the source and load matching networks to be such that the amplifier is in the stable region.

In order to determine the stability of an amplifier, we need to examine the reflection coefficients looking into the two device ports. The stability conditions are

$$|\Gamma_{\text{in}}| = \left| S_{11} + \frac{S_{12}\Gamma_L S_{21}}{1 - S_{22}\Gamma_L} \right| < 1 \tag{3.49}$$

$$|\Gamma_{\text{out}}| = \left| S_{22} + \frac{S_{12}\Gamma_s S_{21}}{1 - S_{11}\Gamma_s} \right| < 1 \tag{3.50}$$

We will use these to find out what values Γ_s and Γ_L may take on in order to have stability.

Unilateral device. For a unilateral device, $S_{12} = 0$, and the stability conditions become

$$|S_{11}| < 1 \tag{3.51}$$

$$|S_{22}| < 1 \tag{3.52}$$

which are a function of the device only, and not the source and load matching networks.

Bilateral device. The bilateral case is more complicated, because stability depends on the source and load matching networks. The boundary of the region of stability is defined by

$$|\Gamma_{\text{in}}| = |\Gamma_{\text{out}}| = 1 \tag{3.53}$$

The first condition becomes

$$|S_{11}(1 - S_{22}\Gamma_L) + S_{12}\Gamma_L S_{21}| = |1 - S_{22}\Gamma_L| \tag{3.54}$$

$$|S_{11} - \Delta\Gamma_L| = |1 - S_{22}\Gamma_L| \tag{3.55}$$

where $\Delta = S_{11}S_{22} - S_{12}S_{21}$ is the determinant of the S-parameter matrix of the device. Squaring both sides of the last expression leads to

$$|S_{11}|^2 + |\Delta|^2|\Gamma_L|^2 - \Delta\Gamma_L S_{11}^* + \Delta^*\Gamma_L^* S_{11} = 1 + |S_{22}|^2|\Gamma_L|^2 - (S_{22}^*\Gamma_L^* + S_{22}\Gamma_L) \tag{3.56}$$

Combining terms containing Γ_L gives

$$|\Gamma_L|^2 - \frac{(S_{22} - \Delta S_{11}^*)\Gamma_L + (S_{22}^* - \Delta^* S_{11})\Gamma_L^*}{|S_{22}|^2 - |\Delta|^2} = \frac{|S_{11}|^2 - 1}{|S_{22}|^2 - |\Delta|^2} \tag{3.57}$$

If we compare this to Eq. (3.43) for a circle in the complex plane, we find that the center and radius of the circle in the Γ_L plane is

$$C_L = \frac{S_{22}^* - \Delta^* S_{11}}{|S_{22}|^2 - |\Delta|^2} \tag{3.58}$$

$$r_L = \left| \frac{S_{12}S_{21}}{|S_{22}|^2 - |\Delta|^2} \right| \tag{3.59}$$

For the condition on Γ_{out}, we get similar expressions for the center C_s and radius r_s of the stability circle in the Γ_s plane, with S_{11} and S_{22} interchanged. Why does the input stability condition lead to a circle for Γ_L, which is on the output side, and the output stability condition lead to a condition on Γ_s, the reflection coefficient looking into the source network?

If we have a matched load, then $Z_L = Z_0$ and $\Gamma_L = 0$, so that

$$|\Gamma_{\text{in}}| = |S_{11}| \tag{3.60}$$

If $|S_{11}| < 1$, the center of the Smith chart represents a stable value of Γ_L. Otherwise, the center of the Smith chart is in the unstable region. This can be used to determine whether the inside or the outside of a stability circle represents the stable region for a device.

In practice, it is good to be well inside the stable region, and to be sure that Γ_s and Γ_L are inside the stable region over a range of frequencies near the design frequency.

Unconditional stability. If a device is unconditionally stable, then the entire Smith chart ($|\Gamma| < 1$) is inside the stability circles. It can be shown that a device is unconditionally stable if

$$\frac{1 - |S_{11}|^2}{|S_{22} - S_{11}^*\Delta| + |S_{12}S_{21}|} > 1 \tag{3.61}$$

and the greater this quantity is, the more stable the device.

3.4 Bilateral Design

If $S_{12} \neq 0$, the constant gain circle design approach used above needs some changes. For a bilateral device, the transducer gain cannot be separated into independent factors for the source and load ports, so to achieve a given value of the gain we would have to adjust both Γ_s and Γ_L at the same time. To avoid this difficulty, we will work with either the power gain G_P or the available power gain G_A since they are independent of the source or load, respectively.

The power gain is

$$G_P = \frac{P_L}{P_{\text{in}}} = \frac{1}{1-|\Gamma_{\text{in}}|^2}|S_{21}|^2\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2} \tag{3.62}$$

The goal here is a way to find Γ_L given a specified power gain. First, we need to write Γ_{in} in terms of Γ_L, using

$$|\Gamma_{\text{in}}| = \left|S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L}\right| = \left|\frac{S_{11} - S_{11}S_{22}\Gamma_L + S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L}\right| = \left|\frac{S_{11}-\Delta\Gamma_L}{1-S_{22}\Gamma_L}\right| \tag{3.63}$$

With this result, the power gain can be expressed as

$$G_P = \frac{1}{1-\left|\frac{S_{11}-\Delta\Gamma_L}{1-S_{22}\Gamma_L}\right|^2}|S_{21}|^2\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2} \tag{3.64}$$

which is a function of Γ_L and the device parameters only. We will use this expression to design the value of Γ_L to achieve a specified power gain, and then use a conjugate match for the source network.

The power gain normalized by the intrinsic transistor gain $|S_{21}|^2$ is

$$\begin{aligned} g_P &= \frac{G_P}{|S_{21}|^2} \\ &= \frac{1-|\Gamma_L|^2}{1-S_{22}\Gamma_L - S_{22}^*\Gamma_L^* + |S_{22}\Gamma_L|^2 - |S_{11}|^2 + S_{11}^*\Delta\Gamma_L + S_{11}\Delta^*\Gamma_L^* - |\Delta\Gamma_L|^2} &(3.65) \\ &= \frac{1-|\Gamma_L|^2}{1-|S_{11}|^2 + |\Gamma_L|^2(|S_{22}|^2-|\Delta|^2) - \Gamma_L(S_{22}-\Delta S_{11}^*) - \Gamma_L^*(S_{22}^*-\Delta^* S_{11})} &(3.66) \end{aligned}$$

Cross multiplying and rearranging leads to

$$|\Gamma_L|^2 - \Gamma_L\frac{g_P(S_{22}-\Delta S_{11}^*)}{D} - \Gamma_L^*\frac{g_P(S_{22}^*-\Delta^* S_{11})}{D} + \frac{g_P(1-|S_{11}|^2)}{D} = \frac{1}{D} \tag{3.67}$$

where $D = 1 + g_P(|S_{22}|^2 - |\Delta|^2)$. We recognize this as a circle in the Γ_L plane. The center and radius are

$$\begin{aligned} C_P &= \frac{g_P(S_{22}^*-\Delta^* S_{11})}{1+g_P(|S_{22}|^2-|\Delta|^2)} &(3.68) \\ r_P &= \frac{[1-2k|S_{12}S_{21}|g_P + |S_{12}S_{21}|^2 g_P^2]^{1/2}}{|1+g_P(|S_{22}|^2-|\Delta|^2)|} &(3.69) \\ k &= \frac{1-|S_{11}|^2-|S_{22}|^2+|\Delta|^2}{2|S_{12}S_{21}|} &(3.70) \end{aligned}$$

This is similar to the unilateral design procedure we developed earlier, except that we do not know the range of possible values for g_P.

In order to understand the range of attainable gains in the bilateral case, we need to find out the maximum value of g_P. Let the quantity inside the square brackets in the expression for r_P be denoted by $f(g_P)$, so that

$$f(g_P) = 1 - 2k|S_{12}S_{21}|g_P + |S_{12}S_{21}|^2 g_P^2 \tag{3.71}$$

Since we must have $r_P > 0$, then $f(g_P) > 0$ as well. We can also see that $f(0) = 1$. Using the quadratic formula, the zeros of $f(g_P)$ are

$$g_{1,2} = \frac{1}{|S_{12}S_{21}|}[k \mp \sqrt{k^2 - 1}] \tag{3.72}$$

where g_1 corresponds to the upper sign and g_2 to the lower. Using the zeros, we can factor the polynomial into the product form

$$\begin{aligned} f(g_P) &= |S_{12}S_{21}|^2 \left[g_P - \frac{1}{|S_{12}S_{21}|}(k - \sqrt{k^2-1})\right]\left[g_P - \frac{1}{|S_{12}S_{21}|}(k + \sqrt{k^2-1})\right] && (3.73) \\ &= |S_{12}S_{21}|^2 (g_P - g_1)(g_P - g_2) && (3.74) \end{aligned}$$

If we assume that $k > 1$ (which is true for a transistor that is unconditionally stable), then both zeros are positive.

Putting all of this together, we can see that $f(g_P)$ is a parabola that is one at $g_P = 0$, goes negative at g_1, and becomes positive again at g_2. Since $f(g_P)$ must be positive, we can see that the interval for physically meaningful values of the normalized gain is $0 \le g_P \le g_1$. We therefore have

$$\begin{aligned} g_{P,\max} &= \frac{1}{|S_{12}S_{21}|}[k - \sqrt{k^2-1}] && (3.75) \\ G_{P,\max} &= \left|\frac{S_{21}}{S_{12}}\right|[k - \sqrt{k^2-1}] && (3.76) \end{aligned}$$

Also, since $r_P = 0$ at $g_P = g_{P,\max}$, C_P becomes

$$\Gamma_{ML} = \frac{g_{P,\max}(S_{22}^* - \Delta^* S_{11})}{1 + g_{P,\max}(|S_{22}|^2 - |\Delta|^2)} \tag{3.77}$$

which is the value of Γ_L that maximizes G_P. These results provide a design approach for a bilateral transistor amplifier.

Design Procedure

1. For the desired power gain G_P, compute the normalized gain g_p and plot the resulting gain circle on the Γ_L plane.

2. *Load reflection coefficient:* Choose a value of Γ_L on the gain circle.

3. *Source reflection coefficient:* Compute Γ_{in} using the selected value for Γ_L. Conjugate match the source, so that $\Gamma_s = \Gamma_{\text{in}}^*$. For this matching condition, $P_{\text{in}} = P_{\text{avs}}$ and therefore $G_T = G_P$.

Our choice of Γ_L produces a value for Γ_{in} which in turn determines Γ_s. This value of Γ_s results in a value of Γ_{out} which determines the output VSWR. If we don't like the output VSWR that we obtain for some reason, we can always choose a different value of Γ_L.

We won't take the time to prove this, but it can be shown that if we pick $\Gamma_L = \Gamma_{ML}$, then $\Gamma_{\text{out}}^* = \Gamma_L$. In other words, maximizing the gain is equivalent to conjugate matching the input and output.

3.5 Noise in Communications Systems

There are several types of noise that are included in communication systems.

1. Thermal Noise (Johnson or Nyquist noise): Created by thermal vibration of bound charges.
2. Shot Noise: Random fluctuations of charge carriers in a solid-state device.
3. Flicker Noise ($1/f$ noise): Occurs in solid-state components. The noise power varies as $1/f$.
4. Plasma Noise: Random motion of charges in an ionized gas.

Thermal noise tends to be dominant in most systems, so we will concentrate on this.

Consider a resistor with resistance R at a temperature T (in Kelvin). The kinetic energy of the electrons is proportional to T. The random motion of the electrons creates voltage fluctuations at the resistor terminals. The voltage has zero average, but the RMS value is given by Planck's blackbody radiation equation

$$\overline{v}_n = \sqrt{\frac{4hfBR}{e^{hf/k_BT} - 1}} \tag{3.78}$$

where

B Bandwidth in Hertz
h Planck's constant = 6.546×10^{-34} J·sec
k_B Boltzmann's constant = 1.380×10^{-23} J/K
f frequency (Hz)

If the frequency is large, say $f = 100$ GHz, and the temperature is low, say $T = 100$ K, then

$$hf = 6.5 \times 10^{-23} \ll k_BT = 1.38 \times 10^{-21} \tag{3.79}$$

This means that the exponent hf/k_BT is very small. The inequality gets even larger for microwave frequencies at room temperature ($T = 290$ K). Because of this, at microwave frequencies the exponential can be approximated by the first two terms of the Taylor series,

$$e^{hf/k_BT} \approx 1 + \frac{hf}{k_BT} \tag{3.80}$$

This simplifies the RMS voltage to

$$\overline{v}_n \approx \sqrt{\frac{4hfBR}{1 + hf/k_BT - 1}} = \sqrt{4k_BTBR} \tag{3.81}$$

In this approximation, $\overline{v}_n$ is independent of frequency. For this reason, the thermal noise signal is called "white noise". We generally model the noise voltage as a random variable with a zero mean Gaussian distribution and variance $\overline{v}_n^2$. Given multiple noise sources, the distributions are independent. Mathematically, this means that if you combine multiple noise sources, the variance of the sum is equal to the sum of the variances (we add the noise powers, not the voltages).

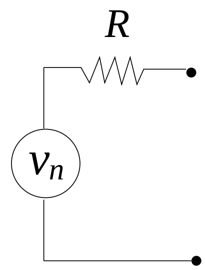

Figure 3.3: Equivalent circuit of a noise source.

We can replace any noisy (warm) resistor with a Thevenin equivalent of a noise source and an ideal, noiseless resistor (Fig. 3.3). If we connect this equivalent circuit to a bandpass filter with bandwidth B Hz and then to a second ideal resistor R (where the resistance of the load is chosen for maximum power transfer), the noise power delivered to the load, which is the noise power available from the network, is

$$P_n = \left(\frac{\overline{v}_n}{2R}\right)^2 R = \frac{\overline{v}_n^2}{4R} \tag{3.82}$$

Note that we do not have another factor of two in the denominator as we would for phasor voltages since $\overline{v}_n$ is already an RMS quantity. Using our expression for $\overline{v}_n$ leads to

$$P_n = \frac{4k_B TBR}{4R} = k_B TB \tag{3.83}$$

This result is very often used for other noise sources than resistors. The noise source may not even be at a physical temperature equal to T, in which case T in (3.83) becomes an equivalent noise temperature.

When working with microwave signals, it is often convenient to use units of dBm, which means power expressed in decibels relative to 1 milliwatt (dBm is $10\log_{10}$[Power(mW)]). For a resistor at room temperature (approximately $T = 290$ K), $10\log_{10} k_B T = -174$ dBm/Hz. In order to go from this quantity, which measures the amount of noise power in a 1 Hz bandwidth, we multiply by the system bandwidth, or add $10\log_{10} B$ in dB to find the total in-band noise power.

3.5.1 Noise Figure

A key measure of system performance is signal-to-noise ratio (SNR):

$$\text{SNR} = \frac{S}{N} = \frac{\text{Signal Power}}{\text{Noise Power}} \tag{3.84}$$

A high SNR means that it is easy to recognize the signal, and a low SNR means that the signal is obscured by noise. The signal and noise powers can in general be defined in a number of ways, but to be consistent we use available powers for all quantities.

Amplifiers, lossy transmission lines, mixers, and almost any other component of a microwave system add noise to the signal. An ideal component does not add any noise, so the SNR at the output is the same as the SNR at the input. But for a non-ideal component, the output SNR is always less than the input SNR.

Noise figure is a measure of the degradation in signal-to-noise ratio (SNR) as a signal passes through a system component. The definition of noise figure (F) is the ratio of the total available noise power at the

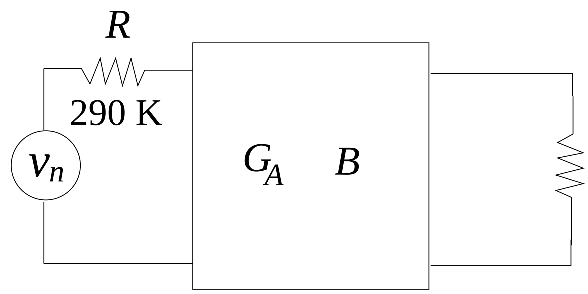

Figure 3.4: Noisy amplifier.

amplifier output to the available noise power at the output due to the input noise only:

$$F = \frac{\text{Output noise power}}{\text{Ideal output noise power = Gain} \times \text{input noise power}} \geq 1 \tag{3.85}$$

For an ideal component, $F = 1$. The gain used in this expression is the available gain

$$G_A = \frac{P_{\text{avn}}}{P_{\text{avs}}} = \frac{S_o}{S_i} \tag{3.86}$$

It can be seen that noise figure is also equal to the ratio of the input SNR to the output SNR:

$$F = \frac{N_o}{N_i G_A} = \frac{N_o}{N_i S_o/S_i} = \frac{S_i/N_i}{S_o/N_o} = \frac{\text{SNR}_{\text{in}}}{\text{SNR}_{\text{out}}} \tag{3.87}$$

We can also write

$$F = \frac{G_A N_i + P_n}{G_A N_i} = 1 + \frac{P_n}{G_A N_i} \tag{3.88}$$

where P_n is the extra noise power at the output introduced by the component. As a convention, we assume that the input noise corresponds to room temperature, so that $N_i = k_B T_0 B$ with $T_0 = 290\,\text{K}$. Since noise figure is a dimensionless quantity, it is often expressed in dB.

3.5.2 Equivalent Noise Temperature

We can also express the "noisiness" of a component in terms of an equivalent noise temperature using $P = k_B T B$. If we consider an ideal, noiseless component with a warm resistor at the input, then the equivalent temperature T_e is defined to be the temperature of the resistor such that it supplies the same noise as the non-ideal component, so that

$$P_n = G_A k_B T_e B \tag{3.89}$$

Using this in Eq. (3.88) together with $N_i = k_B T_0 B$ leads to

$$F = 1 + \frac{T_e}{T_0} \tag{3.90}$$

Equivalent temperature is often used for very low noise figure devices.

3.5.3 Lossy Components

A lossy system component such as a length of lossy transmission line leads to a degradation in SNR. The basic principle for determining the noise figure of a lossy component is to realize that the noise power at the output of the component must be the same as the noise power at the input (thermal equilibrium), so that

$$G_A N_i + P_n = N_i \tag{3.91}$$

Solving for the equivalent additional power at the input gives $P_n = N_i(1 - G_A)$. The noise figure is then

$$F = 1 + \frac{P_n}{G_A N_i} = 1 + \frac{N_i(1 - G_A)}{G_A N_i} = \frac{1}{G_A} = L \tag{3.92}$$

where L is the power loss of the device. Thus, the noise figure is the same as the loss.

3.5.4 Cascaded Networks

If we have two stages in a system,

$$N_o = G_{A2} N_{o1} + P_{n2} = G_{A2}(G_{A1} N_i + P_{n1}) + P_{n2} \tag{3.93}$$

$$F = \frac{G_{A2}(G_{A1} N_i + P_{n1}) + P_{n2}}{N_i G_{A1} G_{A2}} = 1 + \frac{P_{n1}}{N_i G_{A1}} + \frac{P_{n2}}{N_i G_{A1} G_{A2}} \tag{3.94}$$

In terms of the noise figures of the two stages,

$$F_1 = 1 + \frac{P_{n1}}{N_i G_{A1}} \tag{3.95}$$

$$F_2 = 1 + \frac{P_{n2}}{N_i G_{A2}} \tag{3.96}$$

the noise figure of the system is

$$F = F_1 + \frac{F_2 - 1}{G_{A1}} \tag{3.97}$$

The noise figure of the second state is divided by the gain of the first stage. We can see that the first stage is most critical in determining the noise figure of the system. The idea is that we want to boost the signal as much as possible early in the system while adding as little possible noise so that the signal is larger than noise added by subsequent components in the system. For a receive antenna, for example, we want to have an amplifier with high gain and a noise figure close to unity before a long length of lossy coaxial cable.

3.6 Low Noise Amplifiers

For an amplifier, it can be shown that

$$F = F_{\min} + \frac{R_N}{G_s}|Y_s - Y_{\text{opt}}|^2 \tag{3.98}$$

where

$$\begin{aligned}
Y_s &= G_s + jB_s = \text{source admittance} \\
Y_{\text{opt}} &= \text{optimum source admittance resulting in minimum noise figure} \\
F_{\min} &= \text{minimum noise figure} \\
R_N &= \text{equivalent noise resistance of the transistor}
\end{aligned}$$

Y_{opt}, $F_{\min}$, and R_N are noise parameters for the transistor, and would typically be measured or included in a spec sheet for the transistor.

We want to put Eq. (3.98) in terms of reflection coefficients rather than admittances. Using

$$Y_s = \frac{1}{Z_0}\frac{1-\Gamma_s}{1+\Gamma_s}, \qquad Y_{\text{opt}} = \frac{1}{Z_0}\frac{1-\Gamma_{\text{opt}}}{1+\Gamma_{\text{opt}}} \tag{3.99}$$

the magnitude squared term in Eq. (3.98) becomes

$$\begin{aligned}
|Y_s - Y_{\text{opt}}|^2 &= \frac{1}{Z_0^2}\left|\frac{1-\Gamma_s}{1+\Gamma_s} - \frac{1-\Gamma_{\text{opt}}}{1+\Gamma_{\text{opt}}}\right|^2 \\
&= \frac{1}{Z_0^2}\left|\frac{1-\Gamma_s+\Gamma_{\text{opt}}-\Gamma_s\Gamma_{\text{opt}} - 1 - \Gamma_s + \Gamma_{\text{opt}} + \Gamma_s\Gamma_{\text{opt}}}{(1+\Gamma_s)(1+\Gamma_{\text{opt}})}\right|^2 \\
&= \frac{1}{Z_0^2}\left|\frac{-2\Gamma_s + 2\Gamma_{\text{opt}}}{(1+\Gamma_s)(1+\Gamma_{\text{opt}})}\right|^2 \\
&= \frac{4}{Z_0^2}\left|\frac{\Gamma_s - \Gamma_{\text{opt}}}{(1+\Gamma_s)(1+\Gamma_{\text{opt}})}\right|^2
\end{aligned} \tag{3.100}$$

The source conductance is

$$\begin{aligned}
G_s &= \text{Re}\{Y_s\} = \frac{1}{2}(Y_s + Y_s^*) \\
&= \frac{1}{2Z_0}\left[\frac{1-\Gamma_s}{1+\Gamma_s} + \frac{1-\Gamma_s^*}{1+\Gamma_s^*}\right] \\
&= \frac{1}{2Z_0}\left[\frac{1-\Gamma_s+\Gamma_s^* - |\Gamma_s|^2 + 1 + \Gamma_s - \Gamma_s^* - |\Gamma_s|^2}{|1+\Gamma_s|^2}\right] \\
&= \frac{1}{Z_0}\frac{1-|\Gamma_s|^2}{|1+\Gamma_s|^2}
\end{aligned} \tag{3.101}$$

Using these expressions, the amplifier noise figure becomes

$$\begin{aligned}
F &= F_{\min} + R_N Z_0 \frac{|1+\Gamma_s|^2}{1-|\Gamma_s|^2}\frac{4}{Z_0^2}\left|\frac{\Gamma_s - \Gamma_{\text{opt}}}{(1+\Gamma_s)(1+\Gamma_{\text{opt}})}\right|^2 \\
&= F_{\min} + \frac{4R_N}{Z_0}\frac{|\Gamma_s - \Gamma_{\text{opt}}|^2}{(1-|\Gamma_s|^2)|1+\Gamma_{\text{opt}}|^2}
\end{aligned} \tag{3.102}$$

Now, what we would like is to know the values of Γ_s that give a fixed noise figure. To do this, we first define a noise figure parameter N, which consists of all the factors in (3.102) that do not depend on Γ_s:

$$N = \frac{F - F_{\text{min}}}{4R_N/Z_0}|1 + \Gamma_{\text{opt}}|^2 \tag{3.103}$$

We do this to isolate the terms containing Γ_s, and lump the rest into N. Therefore,

$$\begin{aligned}
(\Gamma_s - \Gamma_{\text{opt}})(\Gamma_s^* - \Gamma_{\text{opt}}^*) &= N(1 - \Gamma_s \Gamma_s^*) \\
|\Gamma_s|^2 - \Gamma_s \Gamma_{\text{opt}}^* - \Gamma_s^* \Gamma_{\text{opt}} + |\Gamma_{\text{opt}}|^2 &= N(1 - |\Gamma_s|^2) \\
|\Gamma_s|^2 - \Gamma_s \frac{\Gamma_{\text{opt}}^*}{1+N} - \Gamma_s^* \frac{\Gamma_{\text{opt}}}{1+N} &= \frac{N - |\Gamma_{\text{opt}}|^2}{1+N} \qquad (3.104)
\end{aligned}$$

Once again, we see this is a circle in the complex plane, with center and radius given by

$$C_F = \frac{\Gamma_{\text{opt}}}{N+1} \tag{3.105}$$

$$r_F = \frac{\sqrt{N(N+1-|\Gamma_{\text{opt}}|^2)}}{N+1} \tag{3.106}$$

Using these expressions, we can draw gain, stability, and noise figure circles on the Γ_s Smith chart and pick a value of Γ_s to achieve multiple specifications.

3.7 Dynamic Range

There are a few things we need to understand about amplifiers and other components with a limited operating range.

1. *1 dB compression point:* When a component is operating within its linear region, the slope of the output versus input power curve is 1 dB/dB. When the amplifier or other component is overdriven, as the input power is increased, the output power no longer increases linearly. This causes spurious harmonic distortion and sets a limit on the dynamic range of the system. The output power at which the gain has dropped 1 dB from the ideal linear value is the 1 dB compression point. We often denote this point as P_{1dB}.

2. *System noise floor:* The system noise floor is the sum of the input noise and the noise added by components in the receiver chain. The system noise can be represented as an equivalent temperature at the receiver input,
$$T_{\text{sys}} = T_i + T_{\text{rec}} \tag{3.107}$$
where T_i is the noise level at the receiver input represented in Kelvin and $T_{\text{rec}} = (F_{\text{rec}} - 1)T_0$ is the equivalent noise temperature of the receiver. If the receiver input is an antenna, then we use T_A for the input antenna noise temperature. The output noise power is
$$N_o = k_B T_{\text{sys}} B G_A \tag{3.108}$$
where G_A is the overall receiver available gain and B is the system noise bandwidth. If the input noise is at the reference temperature $T_0 = 290\,\text{K}$, then $N_o = F k_B T_0 B G_A$.

3. *Dynamic range:* Range of input signal levels that can be detected by the receiver without appreciable distortion. The system dynamic range is

$$\mathrm{DR} = P_{1\mathrm{dB}} - S_{o,\mathrm{mds}} \tag{3.109}$$

where $S_{o,\mathrm{mds}}$ is the minimum detectable signal level. For reliable detection, the signal at the output of a receiver normally must be above the noise power by some level X in dB. The minimum detectable signal level is

$$\begin{aligned} S_{o,\mathrm{mds}} &= N_{o,\mathrm{dBm}} + X \\ &= -174 \text{ dBm/Hz } + 10\log_{10} B + 10\log_{10}(T_{\mathrm{sys}}/T_0) + X + G_{A,\mathrm{dB}} \end{aligned} \tag{3.110}$$

The dynamic range is then

$$\mathrm{DR} = P_{1\mathrm{dB}} + 174 \text{ dBm/Hz } - 10\log_{10} B - 10\log_{10}(T_{\mathrm{sys}}/T_0) - X - G_{A,\mathrm{dB}} \tag{3.111}$$

This is the range of signal levels that can be reliably detected by the receiver. For typical wireless communications systems, the dynamic range may be 40–50 dB or more.

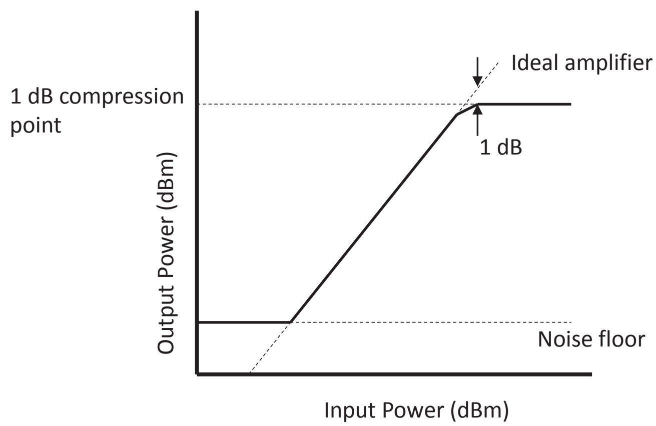

Figure 3.5: Dynamic range of an amplifier.

4. *Third Order Intercept (TOI, TOIP, IP$_3$):* Consider a two-tone test where the input signal is

$$v(t) = A\cos(2\pi f_1 t) + A\cos(2\pi f_2 t) \tag{3.112}$$

where $|f_1 - f_2|$ 5 to 10 MHz. The output frequencies will be of the form

$$f_0 = mf_1 + nf_2 \tag{3.113}$$

where m and n are integers. The order of the intermodulation product (IP) is given by $|m| + |n|$.

Note that $2f_1 - f_2$ and $2f_2 - f_1$ will be inside the communication band. The third order intercept point P_{IP} is defined as the output power at which the third order IP power intersects the linear power (assuming no gain compression or saturation occurs). The slope of the third order intermodulation product output power versus input power is 3 dB/dB.

5. *Spurious Free Dynamic Range:* To compute this dynamic range, we continue to use $S_{o,\mathrm{mds}}$ as the lower bound. However, for the upper bound, we take the output power (in the fundamental signal) at which the third order intermodulation product output power reaches $S_{o,\mathrm{mds}}$.

Chapter 4

Oscillators

There are several types of sources of microwave signals:

- *Black-body radiation.* All materials give off a small amount of microwave radiation due to black-body or thermal radiation. This effect is used in passive remote sensing and receiver calibration.
- *Microwave tubes*. These sources are typically used for very high power applications.
- *Diodes.* A diode source converts DC into microwave energy by making use of a negative resistance voltage-current characteristic (one-port oscillators).
- *Transistors.* Transistor microwave oscillators are similar in principle to low frequency oscillators—amplifiers with feedback (two-port oscillators).

4.1 Oscillator Basics

In studying the stability of an amplifier design, we recognized that we can have a transistor with an input or output reflection coefficients have a magnitude greater than unity. The question is: how do we use this fact to form an oscillator?

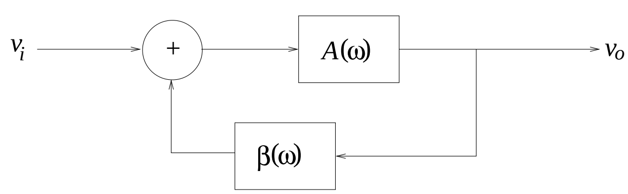

Figure 4.1: Block diagram of a simple feedback network.

By analyzing a simple feedback network, we can find the conditions necessary for oscillation. From the block diagram in Fig. 4.1, the output signal is

$$v_o = A(\omega)[v_i + \beta(\omega)v_o] \tag{4.1}$$

Rearranging this expression gives the transfer function of the system:

$$\frac{v_o}{v_i} = \frac{A(\omega)}{1 - \beta(\omega)A(\omega)} \tag{4.2}$$

For oscillation to occur, we want an output v_o with no input v_i ($v_o/v_i \to \infty$). This leads to the condition

$$\beta(\omega)A(\omega) = 1 \tag{4.3}$$

We see that the loop gain must have: 1) unity magnitude and 2) $2\pi n$ phase, where n is an integer.

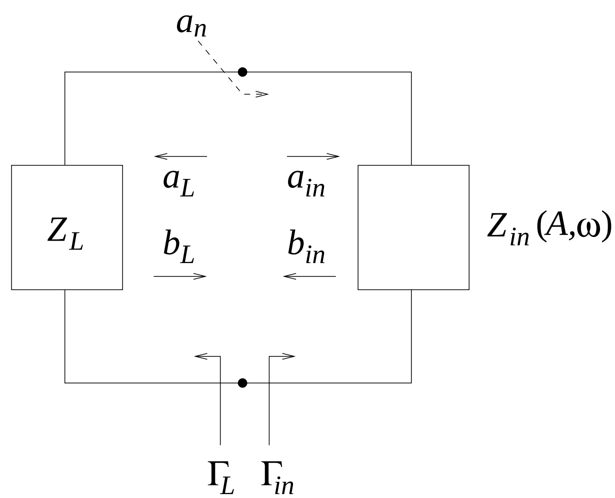

Figure 4.2: Oscillator network with load on the left and active (nonlinear) device on the right. Because the device is nonlinear, its input impedance Z_{in} is a function of the current amplitude A and frequency ω.

Now, let us look at the same concept from a microwave network point of view, as shown in Fig. 4.2. We assume that some noise signal a_n due to thermal or another type of noise is input as an additional forward wave into the active device. The total forward wave into the device is

$$\begin{aligned} a_{\text{in}} &= a_n + \Gamma_{\text{in}}\Gamma_L a_{\text{in}} \\ &= \frac{a_n}{1 - \Gamma_{\text{in}}\Gamma_L} \end{aligned} \tag{4.4}$$

The forward wave into the load is

$$a_L = \Gamma_{\text{in}} a_{\text{in}} = \frac{a_n \Gamma_{\text{in}}}{1 - \Gamma_{\text{in}}\Gamma_L} \tag{4.5}$$

For oscillation, we must have

$$\Gamma_{\text{in}}\Gamma_L = 1 \tag{4.6}$$

We now want to change this to a condition on the load and device impedances:

$$\begin{gathered} \Gamma_{\text{in}}(A,\omega)\Gamma_L(\omega) = 1 \\ \left[\frac{Z_{\text{in}}(A,\omega) - Z_0}{Z_{\text{in}}(A,\omega) + Z_0}\right]\left[\frac{Z_L(\omega) - Z_0}{Z_L(\omega) + Z_0}\right] = 1 \\ Z_{\text{in}}Z_L - Z_0(Z_{\text{in}} + Z_L) + Z_0^2 = Z_{\text{in}}Z_L + Z_0(Z_{\text{in}} + Z_L) + Z_0^2 \\ Z_{\text{in}} + Z_L = 0 \end{gathered} \tag{4.7}$$

where A is the signal amplitude. Breaking up the impedances into real and reactive parts leads to the conditions

$$R_{\text{in}}(A,\omega) + R_L(\omega) = 0 \tag{4.8}$$

$$X_{\text{in}}(A,\omega) + X_L(\omega) = 0 \tag{4.9}$$

For the reactive part, we choose $X_L(\omega) = -X_{\text{in}}(A,\omega)$. Now, the question is how to pick $R_L(\omega)$ so that oscillation starts up properly and to provide high output power.

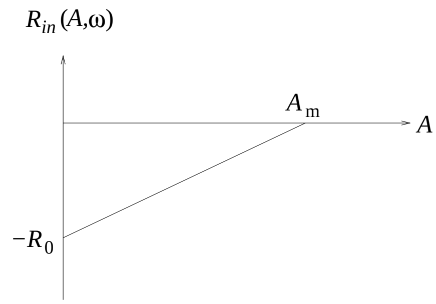

Figure 4.3: Device resistance as a function of current amplitude A.

Let us assume that $R_{\text{in}}(A,\omega)$ is linear in A. (Does this mean that the device is linear?) Let us denote the $A = 0$ intercept as $R_{\text{in}}(0,\omega) = -R_0$, as shown in Fig. 4.3. The input resistance can be expressed as

$$R_{\text{in}}(A,\omega) = -R_0\left[1 - A/A_m\right] \tag{4.10}$$

where A_m is the output signal level at which $R_{\text{in}}(A_m,\omega) = 0$. The power available to the load is

$$\begin{aligned} P_{\text{avn}} &= \frac{1}{2}\text{Re}\{VI^*\} && (4.11)\\ &= \frac{1}{2}|I|^2\text{Re}\{Z_{\text{in}}(A,\omega)\} && (4.12)\\ &= \frac{1}{2}|I|^2 R_{\text{in}}(A,\omega) && (4.13)\\ &= -\frac{1}{2}A^2 R_0\left[1 - A/A_m\right] && (4.14) \end{aligned}$$

Note that P_{avn} is negative, indicating that the network is supplying power to the load rather than dissipating power. We can find the maximum of the power delivered to the load by setting the derivative of $-P_{\text{avn}}$ with respect to A to zero:

$$\left|\frac{d(-P_{\text{avn}})}{dA}\right|_{A=A_0} = \frac{1}{2}R_0[2A_0 - 3A_0^2/A_m] = 0 \tag{4.15}$$

The solution is

$$A_0 = \frac{2}{3}A_m \tag{4.16}$$

At $A = A_0$, the device input resistance is

$$R_{\text{in}}(A_0,\omega) = -R_0\left[1 - \frac{2}{3}\right] = -\frac{1}{3}R_0 \tag{4.17}$$

so that in steady state oscillation we want $R_{\text{in}} = -R_0/3$ to maximize the output power from the oscillation. Intuitively speaking, as the current amplitude A through the device increases, the device saturates and $|R_{\text{in}}|$ will decrease until $R_{\text{in}} + R_L = 0$. So, we back off from the saturation point to a smaller value of A. The resulting design rule is that if $R_{\text{in}}(0,\omega) = -R_0$ for small signal conditions, then we choose $R_L = R_0/3$.

4.2 Negative Resistance Oscillator Design

We will now look at a design procedure for a transistor oscillator. The goal is to take a transistor and add networks to one of the ports to produce a one port device that is unstable, or in other words, exhibits negative resistance. The one-port procedure developed above can then be used to design a tuning network to complete the oscillator.

If the transistor is not unstable enough for use in an oscillator, an additional step is required to increase the device instability. Generally, we consider a transistor as a two-port device, but if the transistor is too stable, we can treat it as a three-port device and add reactance to one of the ports, to produce a new two-port device which is more unstable.

The design procedure is as follows:

1. Select a transistor and a DC bias point. Generally, you are given the two-port common emitter or common source S-parameters from the manufacturer.

2. *Increasing the instability of the transistor.* If the transistor is unconditionally stable or it is potentially unstable but the unstable region is not large enough, the stability characteristics of the transistor need to be changed. This can generally be done by adding a reactance to one of the device terminals. Commonly, a reactance in the base/gate will work well.

   (a) Compute the 3-port S-parameters for the device from the common emitter parameters (see the section Appendix). Port designations are: 1 = base/gate, 2 = collector/drain, 3 = emitter/source.

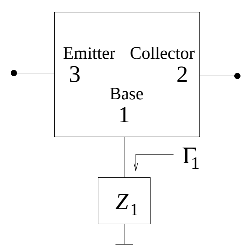

Figure 4.4: Terminating the gate of a transistor with an impedance in order to change its stability characteristics.

   (b) Decide which port will receive the destabilizing reactance. The base/gate is typically used. Assuming that port 1 sees a reflection coefficient of Γ_1 due to an impedance terminating the base (that we are trying to design), we want the new two-port S-parameters for ports 2 and 3 to be such that the reflection coefficients are as large as possible. For the three-port S-parameters,

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{4.18}$$

   Since $a_1 = \Gamma_1 b_1 \rightarrow b_1 = a_1/\Gamma_1$, the first equation in the linear system becomes

$$\frac{a_1}{\Gamma_1} = S_{11}a_1 + S_{12}a_2 + S_{13}a_3 \tag{4.19}$$

$$a_1 = \frac{S_{12}\Gamma_1}{1 - S_{11}\Gamma_1}a_2 + \frac{S_{13}\Gamma_1}{1 - S_{11}\Gamma_1}a_3 \tag{4.20}$$

The remaining equations become

$$b_2 = \left[S_{22} + \frac{S_{21}S_{12}\Gamma_1}{1 - S_{11}\Gamma_1}\right] a_2 + \left[S_{23} + \frac{S_{21}S_{13}\Gamma_1}{1 - S_{11}\Gamma_1}\right] a_3 \tag{4.21}$$

$$b_3 = \left[S_{32} + \frac{S_{31}S_{12}\Gamma_1}{1 - S_{11}\Gamma_1}\right] a_2 + \left[S_{33} + \frac{S_{31}S_{13}\Gamma_1}{1 - S_{11}\Gamma_1}\right] a_3 \tag{4.22}$$

This procedure can of course be performed with the terminating reactance at one of the other ports if needed.

We now have a new common base two-port network with S-parameters S^T, or

$$S^T_{11} = S_{22} + \frac{S_{21}S_{12}\Gamma_1}{1 - S_{11}\Gamma_1} \tag{4.23}$$

$$S^T_{12} = S_{23} + \frac{S_{21}S_{13}\Gamma_1}{1 - S_{11}\Gamma_1} \tag{4.24}$$

$$S^T_{21} = S_{32} + \frac{S_{31}S_{12}\Gamma_1}{1 - S_{11}\Gamma_1} \tag{4.25}$$

$$S^T_{22} = S_{33} + \frac{S_{31}S_{13}\Gamma_1}{1 - S_{11}\Gamma_1} \tag{4.26}$$

(c) We need to find a good value for the reflection coefficient Γ_1 of the base reactance. To make the new network unstable, our ultimate goal is to make $|\Gamma_{\text{in}}| > 1$, where Γ_{in} is the reflection coefficient looking into port 1 of the new 2 port network:

$$\Gamma_{\text{in}} = S^T_{11} + \frac{S^T_{12}S^T_{21}\Gamma_T}{1 - S^T_{22}\Gamma_T} \tag{4.27}$$

To make this reflection coefficient large in magnitude, we will pick a value of Γ_1 that makes $|S^T_{11}|$ large. One approach is to consider the Smith Chart as the S^T_{11} plane and plot the $|\Gamma_1| = 1$ circle (corresponding to a reactive element at the base) on the Smith Chart. Solving the S^T_{11} equation for Γ_1 leads to

$$(1 - S_{11}\Gamma_1)S^T_{11} = S_{22}(1 - S_{11}\Gamma_1) + S_{21}S_{12}\Gamma_1 = S_{22} - \Delta\Gamma_1 \tag{4.28}$$

$$(\Delta - S_{11}S^T_{11})\Gamma_1 = S_{22} - S^T_{11} \tag{4.29}$$

$$\Gamma_1 = \frac{S_{22} - S^T_{11}}{\Delta - S_{11}S^T_{11}} \tag{4.30}$$

where $\Delta = S_{11}S_{22} - S_{12}S_{21}$. Setting the magnitude of this expression to one leads to a circle on the S^T_{11} plane with center and radius given by

$$C_1 = \frac{S_{22} - \Delta S^*_{11}}{1 - |S_{11}|^2} \tag{4.31}$$

$$r_1 = \left|\frac{S_{12}S_{21}}{1 - |S_{11}|^2}\right| \tag{4.32}$$

We choose the point $S^T_{11,\max}$ on this circle for which $|S^T_{11}|$ is a maximum. Using Eq. (4.30) with $S^T_{11} = S^T_{11,\max}$, we can compute the corresponding value of Γ_1 that maximimizes $|S^T_{11}|$. The required base reactance can be found from the reflection coefficient:

$$jX_1 = Z_0\frac{1 + \Gamma_1}{1 - \Gamma_1} \tag{4.33}$$

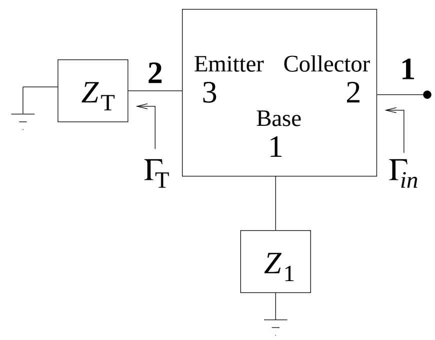

Figure 4.5: Terminating the emitter of the transistor with an impedance in order to achieve a reflection coefficient looking into the collector with magnitude greater than one.

3. *Terminate the 2 port device to make an unstable 1 port device.* We need to design a termination on one of the remaining ports to make the system unstable at the desired frequency.

   This can be done in two ways. We can either draw stability circles and choose a reflection coefficient that makes the device unstable.

   Or, we can find the center and radius of the circle traced by the input reflection coefficient as the terminating impedance moves around the unit circle on the Smith chart. We then find the terminating impedance that makes the input reflection coefficient as large as possible.

   If we choose to put a termination on the emitter (port 2 of the two-port network), then the reflection coefficient at the collector (port 1) is

$$\Gamma_{\text{in}} = S_{11}^T + \frac{S_{12}^T S_{21}^T \Gamma_T}{1 - S_{22}^T \Gamma_T} \tag{4.34}$$

   Solving for the reflection coefficient of the port 2 termination gives

$$\Gamma_T = \frac{S_{11}^T - \Gamma_{\text{in}}}{\Delta^T - S_{22}^T \Gamma_{\text{in}}} \tag{4.35}$$

   where $\Delta^T = S_{11}^T S_{22}^T - S_{12}^T S_{21}^T$. Assuming a purely reactive termination, the $|\Gamma_T| = 1$ circle on the Γ_{in} plane has center and radius

$$C_{\text{in}} = \frac{S_{11}^T - \Delta^T S_{22}^{T*}}{1 - |S_{22}^T|^2} \tag{4.36}$$

$$r_{\text{in}} = \left| \frac{S_{12}^T S_{21}^T}{1 - |S_{22}^T|^2} \right| \tag{4.37}$$

   The maximum achievable magnitude for Γ_{in} is

$$\Gamma_{\text{in,max}} = (|C_{\text{in}}| + r_{\text{in}}) \angle C_{\text{in}} \tag{4.38}$$

   and we compute the corresponding value of Γ_T from Eq. (4.35). The termination reactance is

$$jX_T = Z_0 \frac{1 + \Gamma_T}{1 - \Gamma_T} \tag{4.39}$$

4. Since $|\Gamma_{\rm in}| > 1$, the real part of the impedance looking into port 1 is negative: $\text{Re}\{Z_{\rm in}\} < 1$. We now have a one-port negative resistance device, and can use the one-port negative resistance design procedure to determine the load impedance that the collector port should see. We design a tuning network on the collector port to make

$$X_L = -X_{\rm in} \tag{4.40}$$

$$R_L = |R_{\rm in}|/3 \tag{4.41}$$

   Since the load that the oscillator is driving is a part of the tuning network on the collector side of the transistor, we will take the signal output power from the collector, which is port 1 as a two-port device (or port 2 as a three-port device).

   It is also possible to take the power from the emitter (port 2), so that the load is part of the termination network at the emitter instead of the tuning network at the collector. The termination and tuning networks can also be swapped between the emitter and collector.

If the transistor is already potentially unstable and it is not necessary to destabilize it further, step 2 can be skipped.

Section Appendix: Three Port S-Parameters

Suppose we have a three port device which has terminal three connected to ground. The two-port S-parameters for this configuration are specified as

$$[S^E] = \begin{bmatrix} S^E_{11} & S^E_{12} \\ S^E_{21} & S^E_{22} \end{bmatrix}.$$

The corresponding three-port S-parameters may be computed using

$$S_{\rm sum} = \sum_{i=1,2}\sum_{j=1,2} S^E_{ij} \tag{4.42}$$

$$S_{33} = \frac{S_{\rm sum}}{4 - S_{\rm sum}} \tag{4.43}$$

$$S_{32} = \frac{1+S_{33}}{2}\left(1 - S^E_{12} - S^E_{22}\right) \tag{4.44}$$

$$S_{23} = \frac{1+S_{33}}{2}\left(1 - S^E_{21} - S^E_{22}\right) \tag{4.45}$$

$$S_{22} = S^E_{22} + \frac{S_{23}S_{32}}{1+S_{33}} \tag{4.46}$$

$$S_{13} = 1 - S_{23} - S_{33} \tag{4.47}$$

$$S_{31} = 1 - S_{33} - S_{32} \tag{4.48}$$

$$S_{12} = 1 - S_{22} - S_{32} \tag{4.49}$$

$$S_{21} = 1 - S_{22} - S_{23} \tag{4.50}$$

$$S_{11} = 1 - S_{21} - S_{31} \tag{4.51}$$

Chapter 5

Mixers

A mixer is a device that multiplies two signals. In RF and microwave design, mixers are used to upconvert or downconvert a signal in frequency. If a narrowband signal centered around a carrier frequency is mixed with a sinusoidal signal at the same frequency as the carrier, one of the mixing products that appears at the output is the narrowband signal centered at DC. Often, instead of mixing a signal all the way to DC, it is common to mix to an intermediate frequency (IF) such as 70 MHz or 140 MHz that is much smaller than the carrier frequency. Further processing and detection of the information carried in the signal can then be done at the lower frequency with much simpler circuits than would be required at the original carrier frequency.

RF mixers can also be used for modulation/demodulation, although this is uncommon since most modern communication signals use complex modulations which would require two mixers with nearly identical phase performance. Since this is difficult to construct at high frequencies, modulation is usually done at a lower IF frequency and then the signal is upconverted to a center frequency in the microwave band.

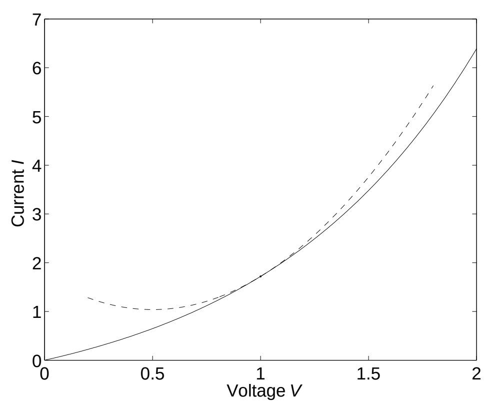

Figure 5.1: Nonlinear diode current/voltage characteristic. Solid line: Eq. (5.1) with $\alpha = I_s = 1$. Dashed line: second order approximation using the first three terms of Eq. (5.2).

One way to perform mixing is take two signals of similar strength, add them together, and then run them through a diode. To see how this works mathematically, consider that a diode has a voltage-current relationship of

$$I(V) = I_s(e^{\alpha V} - 1) \tag{5.1}$$

If the voltage is written as $V = V_0 + v$, where V_0 is a DC bias voltage and v is a small AC term, then we can write a Taylor series of the current as

$$I(V_0 + v) = I(V_0) + v\, \frac{dI}{dV}\bigg|_{V=V_0} + \frac{v^2}{2}\, \frac{d^2 I}{dV^2}\bigg|_{V=V_0} + \cdots \tag{5.2}$$

where we can neglect higher order terms as long as v is small. Defining $dI/dV = 1/R_j$ at $V = V_0$, where R_j is the small-signal junction resistance, then $d^2I/dV^2 = \alpha/R_j$ at $V = V_0$. If $v = v_1 + v_2$, then the second-order term in the Taylor series produces

$$\frac{v^2}{2}\, \frac{d^2 I}{dV^2}\bigg|_{V=V_0} = \frac{\alpha}{2R_j}(v_1^2 + 2v_1v_2 + v_2^2) \tag{5.3}$$

The middle term performs the multiplication we desire. So, all we have to do is match the diode to the feedline in order to maximize the voltage across the diode junction and thereby maximize the signal strength of the multiplied signal.

This nonlinear diode relationship also indicates how to do power detection. We often need to do power sampling in a wireless communication system, for example, when the gain of the receiver must be be automatically varied depending on the received signal strength. If the input voltage is $v(t) = A\cos\omega t$, then the squaring operation leads to

$$\frac{v^2}{2}\, \frac{d^2 I}{dV^2}\bigg|_{V=V_0} = \frac{\alpha A^2}{4R_j}\,[1 + \cos(2\omega t)] \tag{5.4}$$

Filtering out the term at $2\omega t$ leaves a DC term that is proportional to the received signal power (voltage squared).

5.1 Switching (Sampling) Mixers

The diode mixer described above is relatively straightforward to design and build, and is used in some cases for special purpose designs, but most commercial mixers use an different approach based on switching. Switching mixers are not as intuitive as nonlinear mixing, but can be readily analyzed using Fourier analysis.

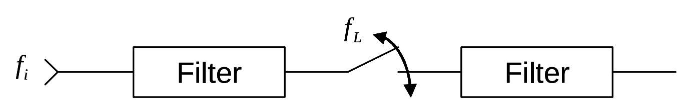

Figure 5.2: A simple sampling mixer configuration.

Consider a system with a high-speed switch that samples an input signal at frequency f_i at a sample rate of f_L. The system is shown in Fig. 5.2. Assume that the switch is an ideal sampling device for now, which means that it is closed only instantaneously. The sampling function is therefore a sequence of delta functions. The spectrum of the sampling function will also be a sequence of delta functions, as shown in Fig. 5.3.

Since we are multiplying the signal by $v_L(t)$, we are convolving in the frequency domain. This will replicate the signal spectrum at f_L intervals in the frequency domain. We can then filter out images of the spectrum that we do not want. Note that we call v_L the *Local Oscillator (LO)* signal. The high frequency signal is typically called the *Radio Frequency (RF)* signal, and the low frequency signal the *Intermediate Frequency*

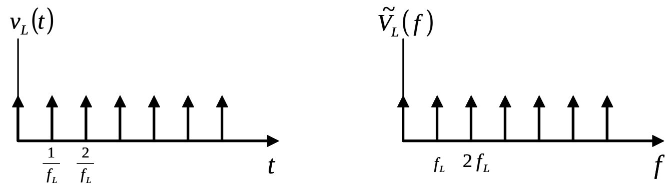

Figure 5.3: Sampling voltage and spectrum.

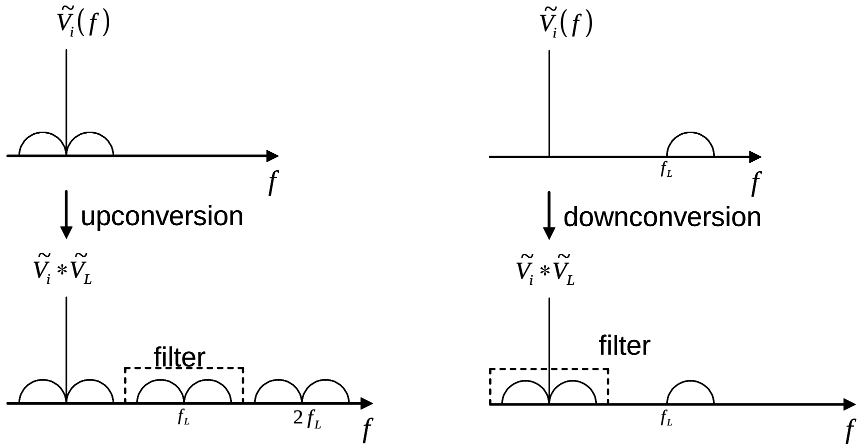

Figure 5.4: Sampling mixer up- and downconversion of a signal.

(IF) signal. If the RF is the input, IF is the output (downconversion). If the IF is the input, the RF is the output (upconversion). Upconversion and downconversion are represented in the spectral domain in Fig. 5.4.

Now, suppose $v_L(t)$ is not a train of impulses, but rather some periodic function with a period of $T_L = 1/f_L$. $\tilde{V}_L(f)$ will still be impulses at nf_L. However, they will not be equal amplitude, since the spectrum of the sampling function decays with frequency. This is what we obtain using a diode as the switching device (Fig. 5.5).

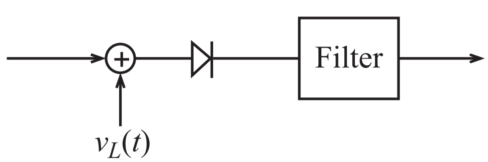

Figure 5.5: Diode switching mixer.

The local oscillator is a strong signal, which when positive causes the diode to conduct (switch closed), when negative causes the diode to be reverse biased (switch open). If the required turn-on voltage of the diode is v_d, then the diode is on when the switching signal is greater than v_d, as shown in Fig. 5.6.

For simplicity, we will assume that the switching voltage is strong enough that we can neglect the tiny turn-on voltage V_d, which means that when $v_L > 0$ the diode conducts. Mathematically, we can write

$$v_o(t) = [v_L(t) + v_i(t)]\, S_s(t) \tag{5.5}$$

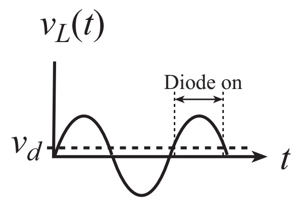

Figure 5.6: Diode voltage over one cycle of the switching signal.

where

$$\begin{aligned}
S_s(t) &= \begin{cases} 1 & v_L > 0 \\ 0 & v_L < 0 \end{cases} \\
&= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos(n\omega_L t) \qquad (5.6)
\end{aligned}$$

Therefore, if we assume that $v_i(t) = V_i \cos\omega_i t$ and $v_L(t) = V_L \cos\omega_L t$, then

$$\begin{aligned}
v_o(t) &= \frac{1}{2}v_L(t) + \frac{1}{2}v_i(t) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\{V_L\cos(n\omega_L t)\cos(\omega_L t) + V_i\cos(n\omega_L t)\cos(\omega_i t)\} \\
&= \frac{1}{2}v_L(t) + \frac{1}{2}v_i(t) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\{V_L\cos[(n-1)\omega_L t] + V_L\cos[(n+1)\omega_L t] + \\
&\qquad V_i\cos[(n\omega_L - \omega_i)t] + V_i\cos[(n\omega_L + \omega_i)t]\} \qquad (5.7)
\end{aligned}$$

Note that the $\sin(n\pi/2)$ term is zero for n even. Therefore, we will have signals at

1. ω_i
2. ω_L
3. $m\omega_L$ for m even
4. $n\omega_L \pm \omega_i$ for n odd

The desired term will be either $\omega_L + \omega_i$ for upconversion or $\omega_L - \omega_i$ for downconversion. The remaining undesired components will need to be filtered out.

5.2 Single Balanced Mixers

The simple mixer introduced above is a single-ended mixer. We have shown that this mixer produces a large variety of undesired signals. If we use a more balanced configuration, then some of these undesired signals can be suppressed. For example, consider the single-balanced mixer shown in Fig. 5.7.

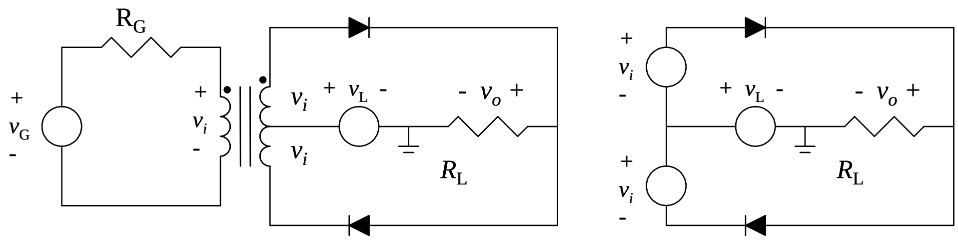

Figure 5.7: Single-balanced diode switching-type mixer.

For this circuit,

$$\begin{aligned} v_o &= \begin{cases} v_L + v_i & v_L > 0 \\ v_L - v_i & v_L < 0 \end{cases} \\ &= v_L + v_i S_b(t) \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} S_b(t) &= \begin{cases} +1 & v_L > 0 \\ -1 & v_L < 0 \end{cases} \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos(n\omega_L t) \end{aligned} \tag{5.9}$$

So,

$$v_o(t) = v_L(t) + \frac{2V_i}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\{\cos[(n\omega_L - \omega_i)t] + \cos[(n\omega_L + \omega_i)t]\} \tag{5.10}$$

We now have signals at

1. ω_L
2. $n\omega_L \pm \omega_i$ for n odd

The balanced configuration has removed many of our undesired components. This tends to be a good choice for downconversion, since $\omega_L >> \omega_L - \omega_i$, so it is easy to filter out the undesirable signals at ω_L and $n\omega_L \pm \omega_i$ for $n > 1$.

If we change the single-balanced mixer design as in Fig. 5.8 we can instead suppress the LO. In this case,

$$v_o(t) = v_i(t)S_s(t) = \frac{1}{2}v_i(t) + \frac{V_i}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\{\cos[(n\omega_L - \omega_i)t] + \cos[(n\omega_L + \omega_i)t]\} \tag{5.11}$$

This is a good choice for upconversion since $\omega_L >> \omega_i$.

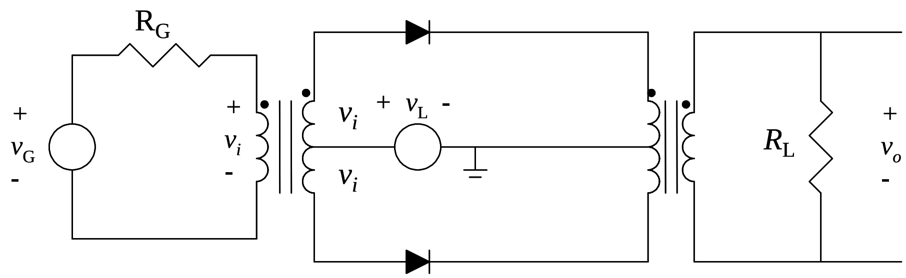

Figure 5.8: Alternate single-balanced diode switching-type mixer.

5.3 Double Balanced Mixer

To suppress both the signal and LO frequencies, we must go to a double balanced design:

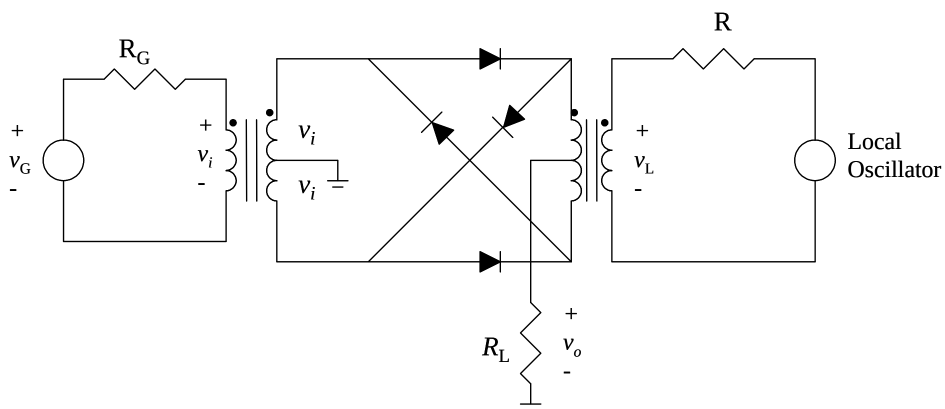

Figure 5.9: Double-balanced diode switching-type mixer.

For $v_L > 0$, we can simplify the circuit by removing the diodes that are off:

For the simplified circuit ($v_L > 0$):

$$v_i - (i_1 - i_2)R_L + v_L - r_d i_1 = 0 \tag{5.12}$$
$$v_i - (i_1 - i_2)R_L - v_L + r_d i_2 = 0 \tag{5.13}$$

Adding these equations leads to

$$2v_i - 2R_L(i_1 - i_2) - r_d(i_1 - i_2) = 0 \tag{5.14}$$

$$i_1 - i_2 = \frac{v_i}{R_L + r_d/2} = -\frac{v_o}{R_L} \tag{5.15}$$

For $v_L > 0$, we have

$$\frac{v_o}{v_i} = -\frac{R_L}{R_L + r_d/2} \tag{5.16}$$

A similar analysis for $v_L < 0$ leads to

$$\frac{v_o}{v_i} = \frac{R_L}{R_L + r_d/2} \tag{5.17}$$

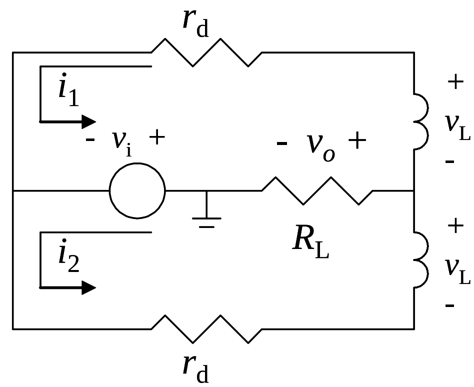

Figure 5.10: Simplified double-balanced mixer circuit for $v_L > 0$.

Combining these two results leads to

$$\begin{aligned} v_o &= \frac{R_L}{R_L + r_d/2} v_i S_b(t) \\ &= \frac{R_L}{R_L + r_d/2} \frac{2V_i}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \{\cos[(n\omega_L - \omega_i)t] + \cos[(n\omega_L + \omega_i)t]\} \end{aligned} \tag{5.18}$$

This shows that the double balanced mixer eliminates both the LO and input signals at the output.

5.3.1 Conversion Loss

For the double-balanced mixer,

$$R_{\text{in}} = \frac{v_i}{i_1 - i_2} = R_L + r_d/2 \approx R_L \tag{5.19}$$

The maximum available power from the source is

$$P_i = \frac{V_p^2}{8R_L} \tag{5.20}$$

where V_p is the peak value of the source sinusoidal signal. The peak of the output voltage in a single sideband is then

$$V_o = \frac{2V_i}{\pi} = \frac{V_p}{\pi} \tag{5.21}$$

due to the voltage division of V_p. The output power is

$$P_o = \frac{V_p^2}{2\pi^2 R_L} \tag{5.22}$$

and the conversion loss is

$$\begin{aligned} L &= \frac{P_i}{P_o} = \frac{\pi^2}{4} && (5.23) \\ L_{\text{dB}} &= 10 \log\left(\frac{\pi^2}{4}\right) = 3.92 \text{ dB} \approx 4 \text{ dB} && (5.24) \end{aligned}$$

For the single-balanced mixer,

$$L_{\text{dB}} = 9.94 \text{ dB} \approx 10 \text{ dB} \tag{5.25}$$