4-4 Conditional Distributions

Definition

- The *conditional distribution* $F_{\mathbf{x}|M}(x|M)$ of a random variable $\mathbf{x}$ assuming the event M is defined as the conditional probability of the event $\mathbf{x} \leq x$:

$$F_{\mathbf{x}|M}(x|M) = P(\mathbf{x} \leq x|M) = \frac{P(\mathbf{x} \leq x, M)}{P(M)}$$

  where the notation $(\mathbf{x} \leq x, M)$ means

$$\{\zeta \in \mathcal{S} \colon \mathbf{x}(\zeta) \leq x\} \cap \{\zeta \in M\}.$$

- To find $F_{\mathbf{x}|M}(x|M)$ one must, in general, know the underlying experiment.
- However, if the event M is an event that can be expressed in terms of the raon variable $\mathbf{x}$, then, for the determination of $F_{\mathbf{x}|M}(x|M)$, knowledge of $F_{\mathbf{x}}(x)$ is sufficient.

1. $M = \{\zeta \in \mathcal{S} \colon \mathbf{x}(\zeta) \leq a\}$: $F_{\mathbf{x}|\mathbf{x}\leq a}(x|\mathbf{x} \leq a) = \begin{cases} 1 & x \geq a \\ \dfrac{F_{\mathbf{x}}(x)}{F_{\mathbf{x}}(a)} & x < a \end{cases}$
2. $M = \{\zeta \in \mathcal{S} \colon b < \mathbf{x}(\zeta) \leq a\}$: $F_{\mathbf{x}|b<\mathbf{x}\leq a}(x|b < \mathbf{x} \leq a) = \begin{cases} 1 & x \geq a \\ \dfrac{F_{\mathbf{x}}(x) - F_{\mathbf{x}}(b)}{F_{\mathbf{x}}(a) - F_{\mathbf{x}}(b)} & b \leq x < a \\ 0 & x < b \end{cases}$

Properties

- The *conditional distribution* $F_{\mathbf{x}|M}(x|M)$ is a distribution function.
- $F_{\mathbf{x}|M}(x|M)$ possesses all 8 properties of a distribution.

Definition

The *conditional density* $f_{\mathbf{x}|M}(x|M)$ of a random variable $\mathbf{x}$ assuming the event M is the derivative of the conditional distribution:

$$f_{\mathbf{x}|M}(x|M) = \frac{d}{dx} F_{\mathbf{x}|M}(x|M) = \lim_{\Delta x \to 0} \frac{P(x < \mathbf{x} \leq x + \Delta x|M)}{\Delta x}$$

Total Probability and Bayes' Theorem

- Bayes' Rule for a random variable $\mathbf{x}$:

$$f_{\mathbf{x}|A}(x|A) = \frac{P(A|\mathbf{x}=x)}{P(A)} f_{\mathbf{x}}(x)$$
$$P(A|\mathbf{x}=x) = \frac{f_{\mathbf{x}|A}(x|A)P(A)}{f_{\mathbf{x}}(x)}.$$

- Total Probability Theorem for a random variable $\mathbf{x}$:

$$P(A) = \int_{-\infty}^{\infty} P(A|\mathbf{x}=x) f_{\mathbf{x}}(x) dx.$$

- Bayes' Theorem for a random variable $\mathbf{x}$:

$$f_{\mathbf{x}|A}(x|A) = \frac{P(A|\mathbf{x}=x) f_{\mathbf{x}}(x)}{\int_{-\infty}^{\infty} P(A|\mathbf{x}=x) f_{\mathbf{x}}(x) dx}$$