**6-1 Bivariate Distributions**

Definitions

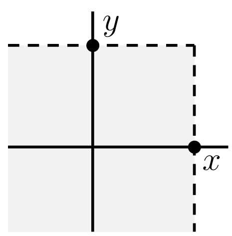
- The *joint* (*bivariate*) *distribution* of two random variables $\mathbf{x}$ and $\mathbf{y}$ is

$$F_{\mathbf{xy}}(x,y) = P(\{\zeta \in \mathcal{S} \colon \mathbf{x}(\zeta) \le x\} \cap \{\zeta \in \mathcal{S} \colon \mathbf{y}(\zeta) \le y\})$$

- Shorthand notation: $F_{\mathbf{xy}}(x,y) = P(\mathbf{x} \le x, \mathbf{y} \le y)$

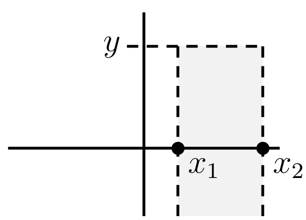
Properties

0 $F\mathbf{xy}(x,y)$ is the probability $\mathbf{x}$ and $\mathbf{y}$ are in an open rectangle.

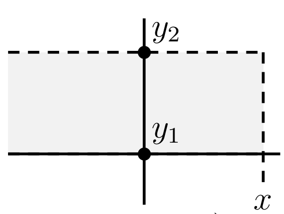



1 $F\mathbf{xy}(-\infty,y) = 0$, $F\mathbf{xy}(x,-\infty) = 0$, $F\mathbf{xy}(\infty,\infty) = 1$.

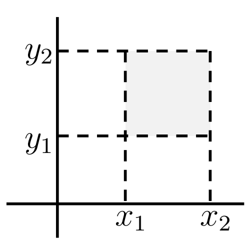
2a $P(x_1 < \mathbf{x} \le x_2, y_1 < \mathbf{y} \le y_2) = F_{\mathbf{xy}}(x_2,y) - F_{\mathbf{xy}}(x_1,y)$



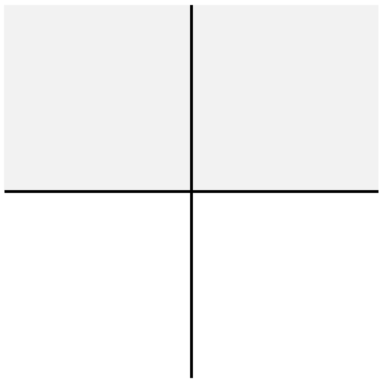

2b $P(\mathbf{x} \le x, y_1 < \mathbf{y} \le y_2) = F_{\mathbf{xy}}(x,y_2) - F_{\mathbf{xy}}(x,y_1)$

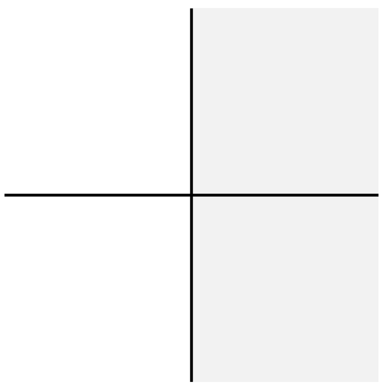


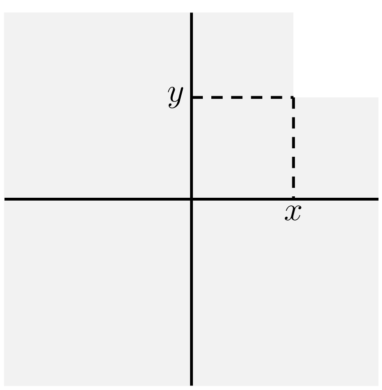

3 $P(x_1 < \mathbf{x} \le x_2, y_1 < \mathbf{y} \le y_2)$
$= F_{\mathbf{xy}}(x_2,y_2) - F_{\mathbf{xy}}(x_1,y_2) - F_{\mathbf{xy}}(x_2,y_1) + F_{\mathbf{xy}}(x_1,y_1)$.



Can you work out these?







Definitions

- The *joint density* of $\mathbf{x}$ and $\mathbf{y}$ is defined by the function

$$f_{\mathbf{xy}}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{\mathbf{xy}}(x, y).$$

- If $F_{\mathbf{xy}}(x, y)$ has step discontinuities, then the pdf $f_{\mathbf{xy}}(x, y)$ contains impulses (Dirac deltas). Alternatively, the joint *probability mass function* can be used:

$$P(\mathbf{x} = x_i, \mathbf{y} = y_k)$$

Properties

1. $F_{\mathbf{xy}}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{\mathbf{xy}}(u, v)\, dv\, du$

2. Joint Statistics

$$P((\mathbf{x}, \mathbf{y}) \in D) = \iint_D f_{\mathbf{xy}}(x, y)\, dx\, dy$$

3. Marginal Statistics

   (a) Marginal distribution and density/pmf of $\mathbf{x}$

$$\begin{aligned} F_{\mathbf{x}}(x) &= F_{\mathbf{xy}}(x, \infty) & \\ f_{\mathbf{x}}(x) &= \int_{-\infty}^{\infty} f_{\mathbf{xy}}(x, y)\, dy & \text{jointly continuous RVs} \\ P(\mathbf{x} = x_i) &= \sum_k P(\mathbf{x} = x_i, \mathbf{y} = y_k) & \text{jointly discrete RVs} \end{aligned}$$

   (b) Marginal distribution and density/pmf of $\mathbf{y}$

$$\begin{aligned} F_{\mathbf{y}}(y) &= F_{\mathbf{xy}}(\infty, y) & \\ f_{\mathbf{y}}(y) &= \int_{-\infty}^{\infty} f_{\mathbf{xy}}(x, y)\, dx & \text{jointly continuous RVs} \\ P(\mathbf{y} = y_k) &= \sum_i P(\mathbf{x} = x_i, \mathbf{y} = y_k) & \text{jointly discrete RVs} \end{aligned}$$

Definition

Two random variables $\mathbf{x}$ and $\mathbf{y}$ are called (*statistically*) *independent* if the events $\{\zeta \in \mathcal{S}\colon \mathbf{x}(\zeta) \in A\}$ and $\{\zeta \in \mathcal{S}\colon \mathbf{y}(\zeta) \in B\}$ are independent, that is, if (using shorthand notation)

$$P(\mathbf{x} \in A, \mathbf{y} \in B) = P(\mathbf{x} \in A)\, P(\mathbf{y} \in B)$$

Properties

1. If $A = \{\zeta \in \mathcal{S}\colon \mathbf{x}(\zeta) \leq x\}$ and $B = \{\zeta \in \mathcal{S}\colon \mathbf{y}(\zeta) \leq y\}$ then independence means
$$F_{\mathbf{xy}}(x, y) = F_{\mathbf{x}}(x) F_{\mathbf{y}}(y)$$

2. From property 1 for jointly continuous RVs
$$f_{\mathbf{xy}}(x, y) = f_{\mathbf{x}}(x) f_{\mathbf{y}}(y)$$

3. From property 1 (or the definition) for jointly discrete RVs
$$P(\mathbf{x} = x_i, \mathbf{y} = y_k) = P(\mathbf{x} = x_i)\, P(\mathbf{y} = y_k)$$

4. Theorem 6-1: If the random variables $\mathbf{x}$ and $\mathbf{y}$ are independent then the random variables
$$\mathbf{z} = g(\mathbf{x}) \qquad \mathbf{w} = h(\mathbf{y})$$
are also independent.

5. Theorem 6-2: Let the random variable $\mathbf{x}$ be defined by experiment $\mathcal{S}_1$ and the random variable $\mathbf{y}$ by experiment $\mathcal{S}_2$. If the experiments $\mathcal{S}_1$ and $\mathcal{S}_2$ are independent, then the random variables $\mathbf{x}$ and $\mathbf{y}$ are independent.

Definition

Joint Normality: Two random variables $\mathbf{x}$ and $\mathbf{y}$ are called *jointly normal* if the joint density is given by [(6-23)–(6-24)]

$$f_{\mathbf{xy}}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2r\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right\}$$

for $|r| < 1$.

Properties

1. The form (6-23)–(6-24) is horrific: one cannot tell what is going on. MDR prefers the form

$$f_{\mathbf{xy}}(x,y) = \frac{1}{2\pi\sqrt{\det(C)}} \exp\left\{-\frac{1}{2}\begin{bmatrix}(x-\mu_1) & (y-\mu_2)\end{bmatrix} C^{-1} \begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}\right\}$$

   where

$$C = \begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

   is the *covariance matrix.*

2. The *marginal density* of $\mathbf{x}$ is

$$f_{\mathbf{x}}(x) = \int_{-\infty}^{\infty} f_{\mathbf{xy}}(x,y)\,dy = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}$$

3. The *marginal density* of $\mathbf{y}$ is

$$f_{\mathbf{y}}(y) = \int_{-\infty}^{\infty} f_{\mathbf{xy}}(x,y)\,dxy = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}$$

4. if $r = 0$ in (6-23)–(6-24) then $f_{\mathbf{xy}}(x,y) = f_{\mathbf{x}}(x)\,f_{\mathbf{y}}(y)$

Definition

We say that the joint density of two random variables $\mathbf{x}$ and $\mathbf{y}$ is *circularly symmetric* (or *symmetrical*) if it depends only on its distance from the origin, that is if

$$f_{\mathbf{xy}}(x,y) = g(r) \qquad r = \sqrt{x^2+y^2}.$$

Note: this $r$ is not the same $r$ used in (6-23)–(6-24)!