7-2 Conditional Densities, Characteristic Functions, and Normality

Definitions

- The *joint conditional density* of the random variables $\mathbf{x}_n, \ldots, \mathbf{x}_{k+1}$ assuming $\mathbf{x}_k, \ldots, \mathbf{x}_1$ is

$$f(x_n, \ldots, x_{k+1} | x_k, \ldots x_1) = \frac{f(x_1, \ldots, x_n)}{f(x_1, \ldots, x_k)}$$

- The *joint conditional distribution* of the random variables $\mathbf{x}_n, \ldots, \mathbf{x}_{k+1}$ assuming $\mathbf{x}_k, \ldots, \mathbf{x}_1$ is

$$\begin{aligned} &F(x_n, \ldots, x_{k+1} | x_k, \ldots x_1) \\ &\quad = \int_{u_n=-\infty}^{x_n} \cdots \int_{u_{k+1}=-\infty}^{x_{k+1}} f(u_n, \ldots, u_{k+1} | x_k, \ldots x_1) \, du_{k+1} \cdots du_n \end{aligned}$$

Properties

1. Chain Rule

$$f(x_1, \ldots, x_n) = f(x_n | x_{n-1}, \ldots, x_1) \cdots f(x_2 | x_1) f(x_1)$$

2. Removing variables on the left or on the right of the conditional line:

   (a) Some example equations

$$\begin{aligned} f(x_1 | x_3) &= \int_{-\infty}^{\infty} f(x_1, x_2 | x_3) \, dx_2 \\ f(x_1 | x_4) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1 | x_2, x_3, x_4) f(x_2, x_3 | x_4) \, dx_2 \, dx_3 \end{aligned}$$

   (b) To remove any number of variables on the left of the conditional line, integrate with respect to them.

   (c) To remove any number of variables to the right of the conditional line, multiply by their conditional density with respect to the remaining variables on the right (of the conditional line) and integrate the product.

   (d) The Chapman-Kolmogorov equation is a special case:

$$f(x_1 | x_3) = \int_{-\infty}^{\infty} f(x_1 | x_2, x_3) f(x_2 | x_3) \, dx_2.$$

The definitions and properties hold for discrete random variables if the pdfs are replaced by pmfs and the integrals by summations.

Definitions

- The conditional mean of $\mathbf{y}_1, \ldots, \mathbf{y}_k = g(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ given the event M is

$$E\{g(\mathbf{x}_1, \ldots, \mathbf{x}_n)|M\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f(x_1, \ldots x_n|M)\, dx_1 \cdots dx_n$$

Properties

1. Special case

$$E\{\mathbf{x}_1|x_2, \ldots, x_n\} = \int_{-\infty}^{\infty} x_1 f(x_1|x_2, \ldots, x_n)\, dx_1$$

   $E\{\mathbf{x}_1|x_2, \ldots, x_n\}$ is a function of $x_2, \ldots, x_n$.

2. $E\{\mathbf{x}_1|\mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a *random variable*. The expected value of $E\{\mathbf{x}_1|\mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is

$$\begin{aligned} E\{E\{\mathbf{x}_1|\mathbf{x}_2, \ldots, \mathbf{x}_n\}\} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E\{\mathbf{x}_1|x_2, \ldots, x_n\} f(x_2, \ldots, x_n)\, dx_2 \cdots dx_n \\ &= E\{\mathbf{x}_1\} \end{aligned}$$

3. $E\{\mathbf{x}_1|x_2, x_3, x_4\}$ is a function of x_2, x_3, x_4. $E\{\mathbf{x}_1|x_2, x_3, \mathbf{x}_4\}$ is a random variable. The conditional expected value of $E\{\mathbf{x}_1|x_2, x_3, \mathbf{x}_4\}$ is

$$\begin{aligned} E\{E\{\mathbf{x}_1|x_1, x_2, \mathbf{x}_4\}\} &= \int_{-\infty}^{\infty} E\{\mathbf{x}_1|x_2, x_3, x_4\} f(x_4|x_2, x_3)\, dx_4 \\ &= E\{\mathbf{x}_1|x_2, x_3\} \end{aligned}$$

4. Generalization: To remove any number of variables on the right of the conditional expected value line, multiply by their conditional density with respect to the remaining variables on the right (of the conditional expected value line) and integrate.

The definitions and properties hold for discrete random variables if the pdfs are replaced by pmfs and the integrals by summations.

Law of Large Numbers

- $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are an IID sequence with $E\{\mathbf{x}_i\} = \mu$.
- Sample mean $\overline{\mathbf{x}} = \dfrac{\mathbf{x}_1 + \cdots + \mathbf{x}_n}{n}$
- Law of Large Numbers: $\dfrac{\mathbf{x}_1 + \cdots + \mathbf{x}_n}{n} \underset{n\to\infty}{\longrightarrow} \mu$

Central Limit Theorem

- $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are a sequence of *independent* random variables with

$$E\{\mathbf{x}_i\} = \mu_i \qquad E\{(\mathbf{x}_i - \mu_i)^2\} = \sigma_i^2$$

- The sum $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_n$

$$E\{\mathbf{x}\} = \mu = \mu_1 + \cdots + \mu_n$$
$$E\{(\mathbf{x} - \mu)^2\} = \sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$$

- Central Limit Theorem: under certain general conditions

$$\mathbf{z} = \frac{\mathbf{x} - \mu}{\sigma} \quad F_{\mathbf{z}}(z) \underset{n\to\infty}{\longrightarrow} \text{standard normal CDF}$$

- If the random variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are jointly continuous then, under the same general conditions

$$\mathbf{z} = \frac{\mathbf{x} - \mu}{\sigma} \quad f_{\mathbf{z}}(z) \underset{n\to\infty}{\longrightarrow} \text{standard normal PDF}$$