5-1 The Random Variable *g*(*x*)

Definition

- $g(x)$ is a real-valued function of the real-valued variable x.
- $\mathbf{x} = \mathbf{x}(\zeta)$ is a random variable.
- $g(\mathbf{x})$ is a new random variable.
  - $\mathbf{x}(\zeta)$ is a map $\mathcal{S} \to \mathbb{R}$.
  - For each $\zeta \in \mathcal{S}$, $\mathbf{x}(\zeta)$ is a real number.
  - $g(\mathbf{x}(\zeta))$ is another real number.
  - $g(\mathbf{x}(\zeta))$ is a composite map $\mathcal{S} \to \mathbb{R}$.
  - This composite map is called $\mathbf{y}(\zeta)$: the random variable $\mathbf{y}$.
- $\mathbf{y} = g(\mathbf{x})$ is a random variable defined by the event $\{\zeta\colon \mathbf{y}(\zeta) \leq y\}$ and its probability (the distribution function)

$$F_{\mathbf{y}}(y) = P(\{\zeta\colon \mathbf{y}(\zeta) \leq y\}) = P(\mathbf{y} \leq y).$$

- The event $\{\zeta\colon \mathbf{y}(\zeta) \leq y\}$ may also be written $\{\zeta \in \mathcal{S}\colon g(\mathbf{x}(\zeta)) \leq y\}$ so that the distribution function may also be written

$$F_{\mathbf{y}}(y) = P(\{\zeta \in \mathcal{S}\colon g(\mathbf{x}(\zeta)) \leq y\}) = P(g(\mathbf{x}) \leq y).$$

- Let $D_y = \{x \in \mathbb{R}\colon g(x) \leq y\}$. Then

$$\{\zeta \in \mathcal{S}\colon g(\mathbf{x}(\zeta)) \leq y\} = \{\zeta \in \mathcal{S}\colon \mathbf{x}(\zeta) \in D_y\}$$

  The probability of this event is

$$\begin{aligned} F_{\mathbf{y}}(y) &= P(\{\zeta \in \mathcal{S}\colon g(\mathbf{x}(\zeta)) \leq y\}) \\ &= P(\{\zeta \in \mathcal{S}\colon \mathbf{x}(\zeta) \in D_y\}) \\ &= \int_{D_y} f_{\mathbf{x}}(x)dx \end{aligned}$$

- Probability density function of $\mathbf{y}$

$$f_{\mathbf{y}}(y) = \frac{d}{dy}F_{\mathbf{y}}(y) = \frac{d}{dy}\int_{D_y} f_{\mathbf{x}}(x)dx$$

5-2 The Distribution of *g*(*x*)

Examples

1. $\mathbf{y} = a\mathbf{x} + b$

2. $\mathbf{y} = \mathbf{x}^2$

3. $\mathbf{y} = e^{\mathbf{x}}$

4. $\mathbf{y} = \begin{cases} 1 & \mathbf{x} > 0 \\ -1 & \mathbf{x} \leq 0 \end{cases}$

5. $\mathbf{y} = \begin{cases} \sqrt{\mathbf{x}} & \mathbf{x} > 0 \\ 0 & \mathbf{x} \leq 0 \end{cases}$

6. $\mathbf{y} = \begin{cases} 1 & \mathbf{x} > 1 \\ \mathbf{x} & -1 \leq \mathbf{x} < 1 \\ -1 & \mathbf{x} \leq -1 \end{cases}$