Complex-Valued Random Variables

Statistical Vectors and Matrices for the complex-valued random vector

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 + j\mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n + j\mathbf{y}_n \end{bmatrix}$$

- Mean Vector

$$\mu_{\mathbf{Z}} = E\{\mathbf{Z}\} = \begin{bmatrix} E\{\mathbf{z}_1\} \\ \vdots \\ E\{\mathbf{z}_n\} \end{bmatrix} = \begin{bmatrix} \mu_{\mathbf{z}_1} \\ \vdots \\ \mu_{\mathbf{z}_n} \end{bmatrix}$$

- Covariance Matrix

$$C_{\mathbf{ZZ}} = E\{(\mathbf{Z} - \mu_{\mathbf{Z}})(\mathbf{Z} - \mu_{\mathbf{Z}})^H\}$$

$$= E\left\{ \begin{bmatrix} \mathbf{z}_1 - \mu_{\mathbf{z}_1} \\ \vdots \\ \mathbf{z}_n - \mu_{\mathbf{z}_n} \end{bmatrix} \begin{bmatrix} (\mathbf{z}_1^* - \mu_{\mathbf{z}_1}^*) & \cdots & (\mathbf{z}_n^* - \mu_{\mathbf{z}_n}^*) \end{bmatrix} \right\}$$

$$= E\left\{ \begin{bmatrix} (\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{z}_1^* - \mu_{\mathbf{z}_1}^*) & \cdots & (\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{z}_n^* - \mu_{\mathbf{z}_n}^*) \\ \vdots & & \vdots \\ (\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{z}_1^* - \mu_{\mathbf{z}_1}^*) & \cdots & (\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{z}_n^* - \mu_{\mathbf{z}_n}^*) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E\{(\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{z}_1^* - \mu_{\mathbf{z}_1}^*)\} & \cdots & E\{(\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{z}_n^* - \mu_{\mathbf{z}_n})^*\} \\ \vdots & & \vdots \\ E\{(\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{z}_1^* - \mu_{\mathbf{z}_1})^*\} & \cdots & E\{(\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{z}_n^* - \mu_{\mathbf{z}_n}^*)\} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$$

- Correlation Matrix

$$R_{\mathbf{ZZ}} = E\{\mathbf{Z}\mathbf{Z}^H\} = E\left\{ \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} \begin{bmatrix} \mathbf{z}_1^* & \cdots & \mathbf{z}_n^* \end{bmatrix} \right\}$$

$$= E\left\{ \begin{bmatrix} \mathbf{z}_1\mathbf{z}_1^* & \cdots & \mathbf{z}_1\mathbf{z}_n^* \\ \vdots & & \vdots \\ \mathbf{z}_n\mathbf{z}_1^* & \cdots & \mathbf{z}_n\mathbf{z}_n^* \end{bmatrix} \right\} = \begin{bmatrix} E\{\mathbf{z}_1\mathbf{z}_1^*\} & \cdots & E\{\mathbf{z}_1\mathbf{z}_1^*\} \\ \vdots & & \vdots \\ E\{\mathbf{z}_n\mathbf{z}_1^*\} & \cdots & E\{\mathbf{z}_n\mathbf{z}_n^*\} \end{bmatrix}$$

$$= \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & & \vdots \\ R_{n1} & \cdots & R_{nn} \end{bmatrix}$$

Statistical Matrices for the complex-valued random vectors

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}$$

- Cross-Covariance Matrix

$$C_{\mathbf{ZW}} = E\{(\mathbf{Z} - \mu_{\mathbf{Z}})(\mathbf{W} - \mu_{\mathbf{W}})^H\}$$

$$= E\left\{ \begin{bmatrix} \mathbf{z}_1 - \mu_{\mathbf{z}_1} \\ \vdots \\ \mathbf{z}_n - \mu_{\mathbf{z}_n} \end{bmatrix} \begin{bmatrix} (\mathbf{w}_1^* - \mu_{\mathbf{w}_1}^*) & \cdots & (\mathbf{w}_n^* - \mu_{\mathbf{w}_n}^*) \end{bmatrix} \right\}$$

$$= E\left\{ \begin{bmatrix} (\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{w}_1^* - \mu_{\mathbf{w}_1}^*) & \cdots & (\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{w}_n^* - \mu_{\mathbf{w}_n}^*) \\ \vdots & & \vdots \\ (\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{w}_1^* - \mu_{\mathbf{w}_1}^*) & \cdots & (\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{w}_n^* - \mu_{\mathbf{w}_n}^*) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} E\{(\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{w}_1^* - \mu_{\mathbf{w}_1}^*)\} & \cdots & E\{(\mathbf{z}_1 - \mu_{\mathbf{z}_1})(\mathbf{w}_n^* - \mu_{\mathbf{w}_n})^*\} \\ \vdots & & \vdots \\ E\{(\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{w}_1^* - \mu_{\mathbf{w}_1})^*\} & \cdots & E\{(\mathbf{z}_n - \mu_{\mathbf{z}_n})(\mathbf{w}_n^* - \mu_{\mathbf{w}_n}^*)\} \end{bmatrix}$$

$$= \begin{bmatrix} C_{\mathbf{z}_1\mathbf{w}_1} & \cdots & C_{\mathbf{z}_1\mathbf{w}_n} \\ \vdots & & \vdots \\ C_{\mathbf{z}_n\mathbf{w}_1} & \cdots & C_{\mathbf{z}_n\mathbf{w}_n} \end{bmatrix}$$

- Correlation Matrix

$$R_{\mathbf{ZW}} = E\{\mathbf{ZW}^H\} = E\left\{ \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} \begin{bmatrix} \mathbf{w}_1^* & \cdots & \mathbf{w}_n^* \end{bmatrix} \right\}$$

$$= E\left\{ \begin{bmatrix} \mathbf{z}_1\mathbf{w}_1^* & \cdots & \mathbf{z}_1\mathbf{w}_n^* \\ \vdots & & \vdots \\ \mathbf{z}_n\mathbf{w}_1^* & \cdots & \mathbf{z}_n\mathbf{w}_n^* \end{bmatrix} \right\} = \begin{bmatrix} E\{\mathbf{z}_1\mathbf{w}_1^*\} & \cdots & E\{\mathbf{z}_1\mathbf{w}_1^*\} \\ \vdots & & \vdots \\ E\{\mathbf{z}_n\mathbf{w}_1^*\} & \cdots & E\{\mathbf{z}_n\mathbf{w}_n^*\} \end{bmatrix}$$

$$= \begin{bmatrix} R_{\mathbf{z}_1\mathbf{w}_1} & \cdots & R_{\mathbf{z}_1\mathbf{w}_n} \\ \vdots & & \vdots \\ R_{\mathbf{z}_n\mathbf{w}_1} & \cdots & R_{\mathbf{z}_n\mathbf{w}_n} \end{bmatrix}$$

Complex-Valued Multivariate Normal 1:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 + j\mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n + j\mathbf{y}_n \end{bmatrix} \quad \Rightarrow \quad \mathbf{V} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{matrix}}\right\} \mathbf{X} \\ \left.\vphantom{\begin{matrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{matrix}}\right\} \mathbf{Y} \end{matrix}$$

- PDF of $\mathbf{Z}$ is the joint PDF of $\mathbf{V}$:

$$f_{\mathbf{V}}(V) = \frac{1}{(2\pi)^n \sqrt{\det(C_{\mathbf{VV}})}} \exp\left\{ -\frac{1}{2}(V - \mu_{\mathbf{V}})^t C_{\mathbf{VV}}^{-1}(V - \mu_{\mathbf{V}}) \right\}$$

The mean vector and covariance matrix of V have special forms

$$\mu_{\mathbf{V}} = E\{\mathbf{V}\} = E\left\{ \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \right\} = \begin{bmatrix} \mu_{\mathbf{X}} \\ \mu_{\mathbf{Y}} \end{bmatrix}$$

$$\begin{aligned}
C_{\mathbf{VV}} &= E\left\{ (\mathbf{V} - \mu_{\mathbf{V}})(\mathbf{V} - \mu_{\mathbf{V}})^T \right\} \\
&= E\left\{ \begin{bmatrix} \mathbf{X} - \mu_{\mathbf{X}} \\ \mathbf{Y} - \mu_{\mathbf{Y}} \end{bmatrix} \begin{bmatrix} (\mathbf{X} - \mu_{\mathbf{X}})^t & (\mathbf{Y} - \mu_{\mathbf{Y}})^t \end{bmatrix} \right\} \\
&= E\left\{ \begin{bmatrix} (\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^t & (\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Y} - \mu_{\mathbf{Y}})^t \\ \\ (\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{X} - \mu_{\mathbf{X}})^t & (\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Y} - \mu_{\mathbf{Y}})^t \end{bmatrix} \right\} \\
&= \begin{bmatrix} C_{\mathbf{XX}} & C_{\mathbf{XY}} \\ C_{\mathbf{YX}} & C_{\mathbf{YY}} \end{bmatrix}
\end{aligned}$$

Complex-Valued Multivariate Normal 2:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 + j\mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n + j\mathbf{y}_n \end{bmatrix}$$

- PDF of $\mathbf{Z}$ in terms of Z:

$$f_{\mathbf{Z}}(Z) = \frac{1}{\pi^n \sqrt{\det(C_{\mathbf{ZZ}})\det(\Gamma)}} \times$$
$$\exp\left\{ -\frac{1}{2} \begin{bmatrix} (Z - \mu_{\mathbf{Z}})^H & (Z - \mu_{\mathbf{Z}})^t \end{bmatrix} \begin{bmatrix} C_{\mathbf{ZZ}} & P_{\mathbf{ZZ}} \\ P_{\mathbf{ZZ}}^H & C_{\mathbf{ZZ}}^* \end{bmatrix}^{-1} \begin{bmatrix} (Z - \mu_{\mathbf{Z}}) \\ (Z - \mu_{\mathbf{Z}})^* \end{bmatrix} \right\}$$

  where

$$\begin{aligned} \mu_{\mathbf{Z}} &= E\{\mathbf{Z}\} && \text{mean vector} \\ C_{\mathbf{ZZ}} &= E\left\{(\mathbf{Z} - \mu_{\mathbf{Z}})(\mathbf{Z} - \mu_{\mathbf{Z}})^H\right\} && \text{covariance matrix} \\ P_{\mathbf{ZZ}} &= E\left\{(\mathbf{Z} - \mu_{\mathbf{Z}})(\mathbf{Z} - \mu_{\mathbf{Z}})^t\right\} && \text{pseudo-covariance matrix} \\ \Gamma &= C_{\mathbf{ZZ}}^* - P_{\mathbf{ZZ}}^H \mathbf{C}_{ZZ}^{-1} P_{\mathbf{ZZ}} \end{aligned}$$

- A *Proper* complex-valued multivariate normal random vector is one for which the pseudo-covariance matrix is the all-zeros matrix.

- The pdf for a *proper* complex-valued multivariate normal random vector is

$$f_{\mathbf{Z}}(Z) = \frac{1}{\pi^n \sqrt{\det(C_{\mathbf{ZZ}})}} \exp\left\{ -(\mathbf{Z} - \mu_{\mathbf{Z}})^H C_{\mathbf{ZZ}}^{-1} (\mathbf{Z} - \mu_{\mathbf{Z}}) \right\}$$