# Single Dimensional Linear Least Squares

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Given: Point pairs $\{(x_1, y_1), \ldots, (x_n, y_n)\}$

Find: A model of form $y = ax + b$ that maps from $x$ to $y$ and best matches the passed in data.
(we need to find $a$ and $b$).

## Fully Determined Case → Same # eqns and variables

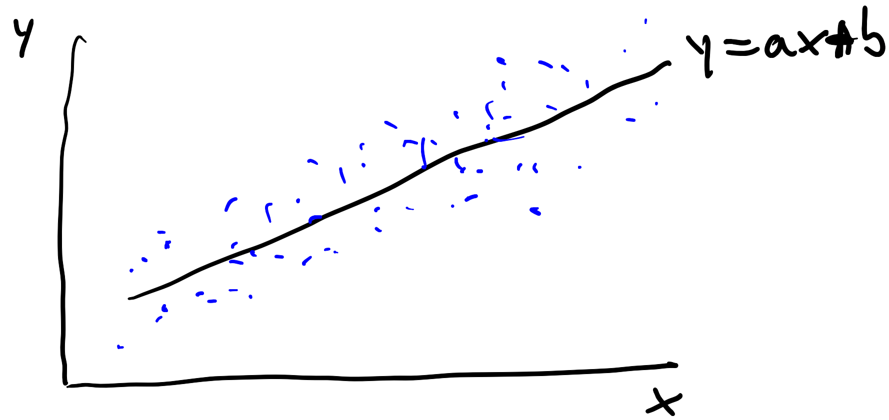
$$y_1 = ax_1 + b, \quad y_2 = ax_2 + b, \ldots$$

Exact Solution can be found with **2** points

## Over determined System → More eqns than variable

If Noisy sample points have error

$$y_1 = ax_1 + b + \epsilon_1, \quad y_2 = ax_2 + b + \epsilon_2, \ldots$$




No perfect solution to set of equations

Need to find the best line to minimize error.

## Least Squared Error

If we select $\ell 2$-norm as our error metric,
we want to minimize

$$f(a,b) = \sum_i ||\epsilon_i||^2$$

$||\underline{x}||^2$ - $\ell 2$ norm
$= (\underline{x})^2$ for scalars
and
$\underline{x}^T\underline{x}$ for vectors

$$\underset{a,b}{\operatorname{argmin}} \sum_i ||(ax_i + b) - y_i||^2$$

$$= \underset{\underline{x}}{\operatorname{argmin}} ||(A\underline{x} - \underline{y}||^2 \text{ with}$$

$$\Bigg|\Bigg| \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{\substack{n \times 2 \\ A}} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\substack{2 \times 1 \\ \underline{x}}} - \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\substack{n \times 1 \\ \underline{b}'}} \Bigg|\Bigg|^2$$

$$= \underset{\underline{x}}{\operatorname{arg\,min}} \; f(\underline{x})$$

$$f(\underline{x}) = (A\underline{x} - \underline{b}')^T(A\underline{x} - \underline{b}')$$

$$= \underline{x}^T A^T A \underline{x} - \underline{x}^T A^T \underline{b}' - \underline{b}'^T A \underline{x} + \underline{b}'^T \underline{b}'$$

$$= \underline{x}^T A^T A \underline{x} - 2\underline{x}^T A^T \underline{b}' + \underline{b}'^T \underline{b}'$$

At the minimum of a cost function, the derivative or gradient must be zero ($\underline{0}$)

$$J(f) = \nabla f = 2A^T A \underline{x} - 2A^T \underline{b}' = 0$$

w/ respect to $\underline{x}$

$$\Rightarrow \boxed{A^T A \underline{x} = A^T \underline{b}'}$$

$$\underline{x}^* = \underbrace{(A^T A)^{-1} A^T}_{\text{Pseudo Inverse}} \underline{b}'$$