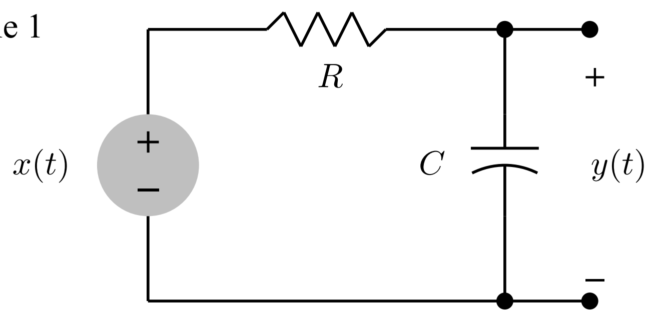
A Review of Continuous Time Linear Systems

- Description of Linear Systems
- Frequency Domain Analysis

Example 1



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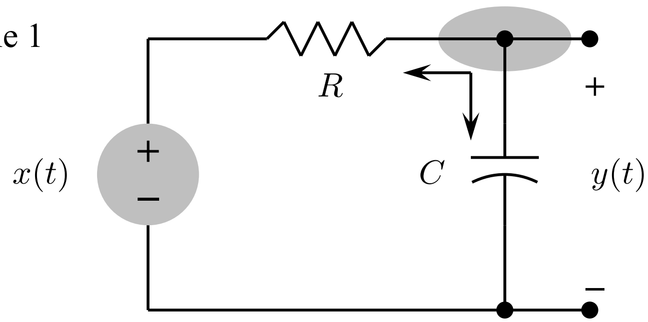

sum of the currents leaving the node $= 0$

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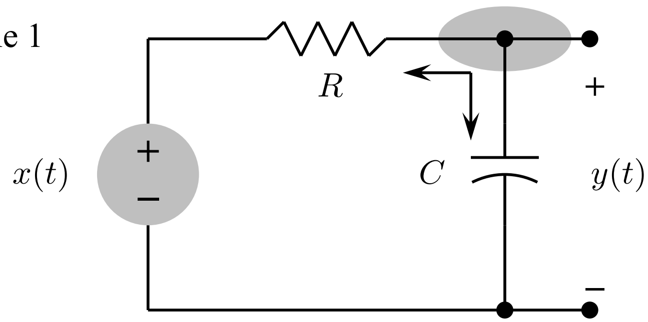

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$$\frac{y(t) - x(t)}{R} + C\frac{d}{dt}y(t) = 0$$

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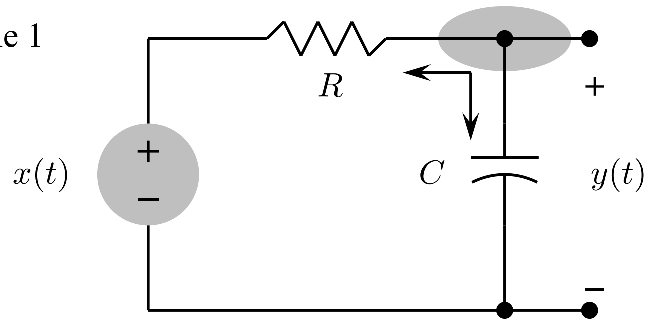

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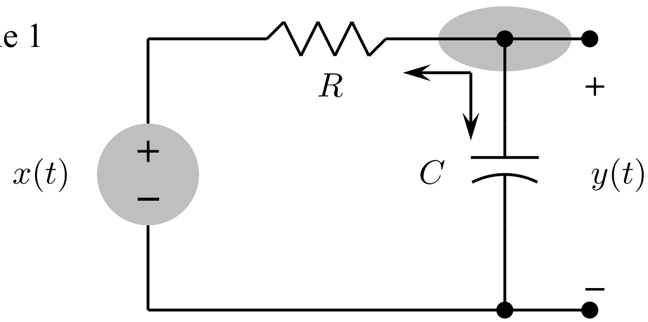

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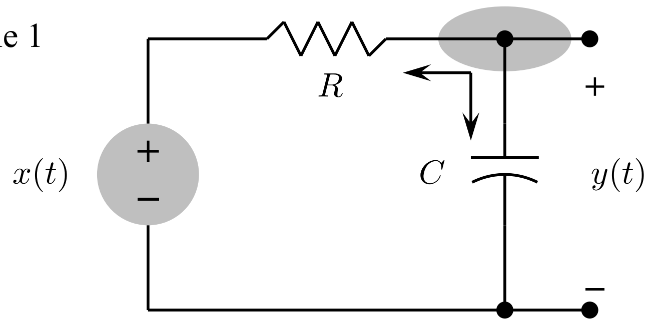

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LCCDE

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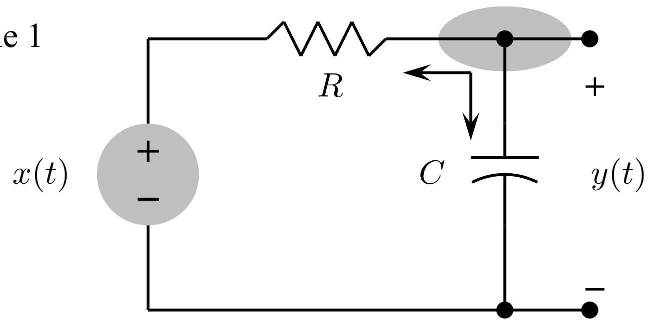

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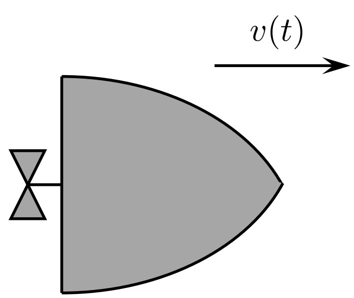

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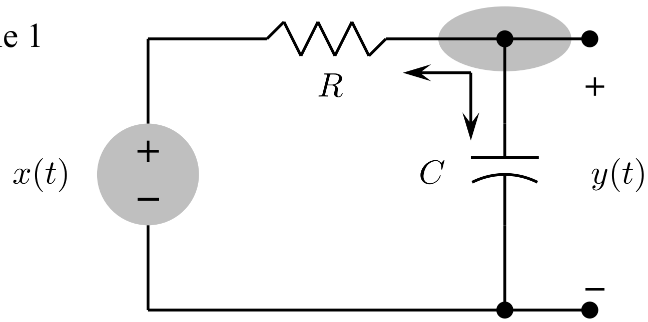

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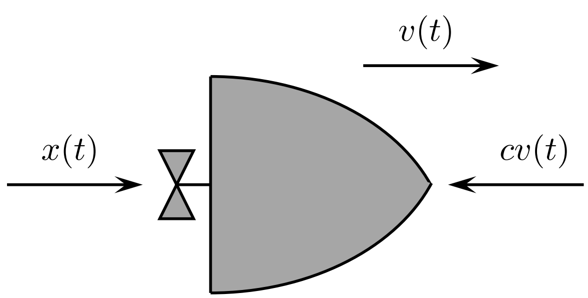

force $=$ mass $\times$ acceleration

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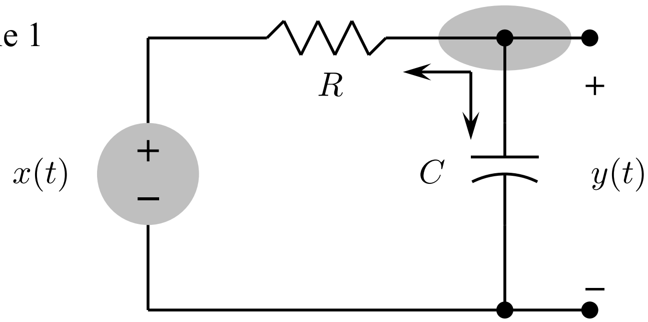

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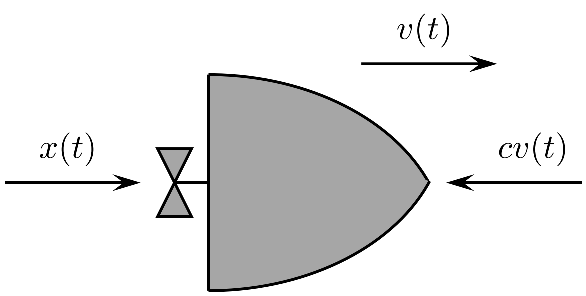

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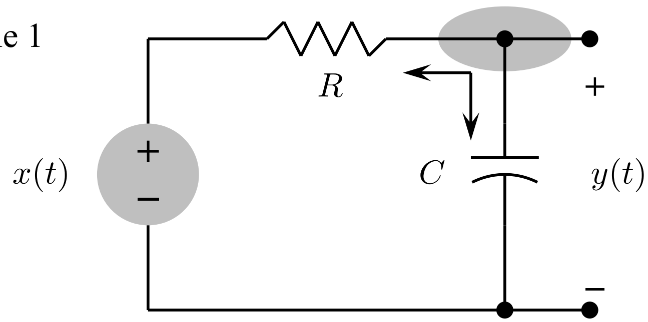

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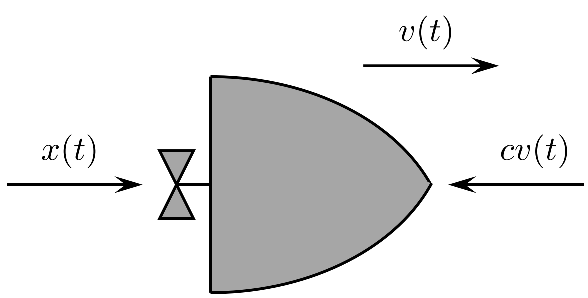

$$\text{force} = \text{mass} \times \text{acceleration}$$

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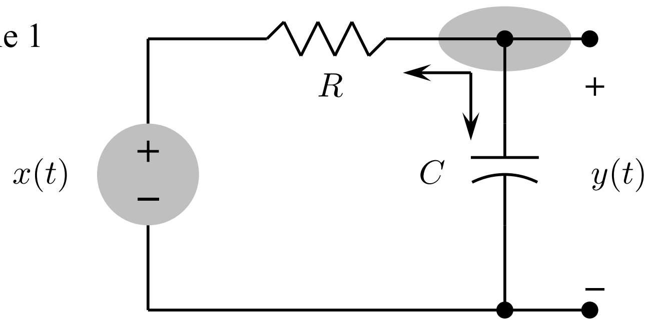

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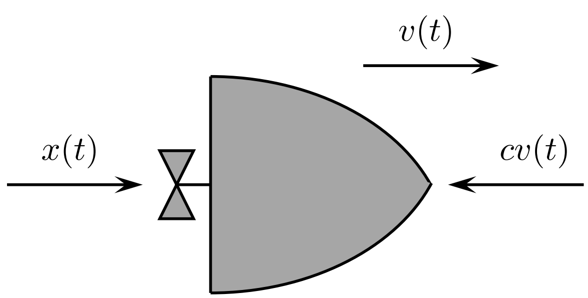

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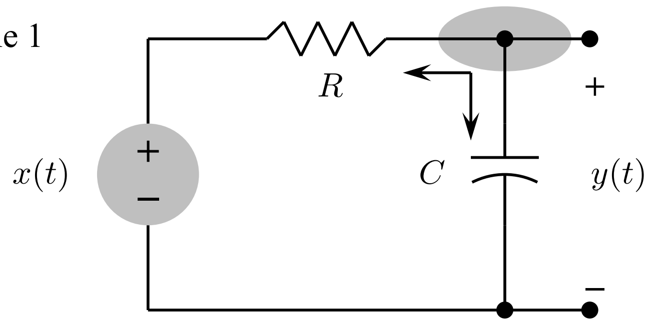

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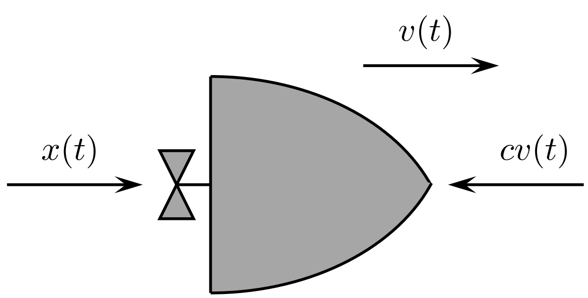

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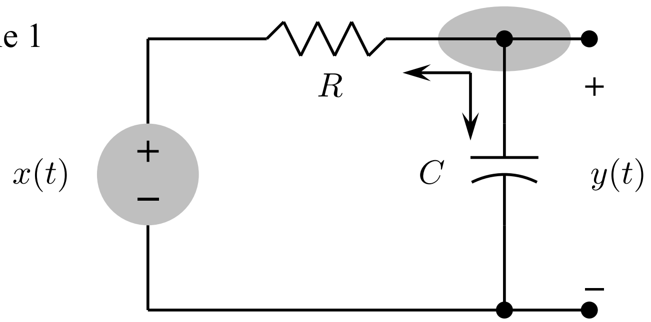

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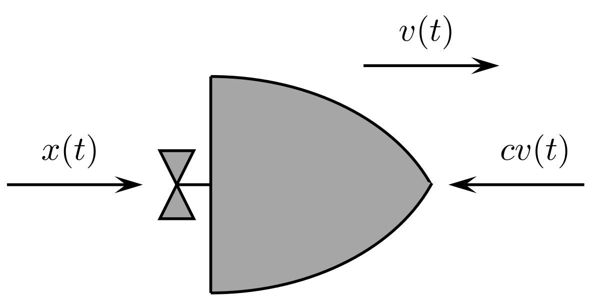

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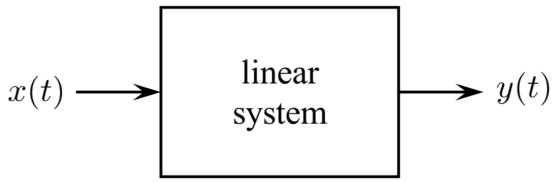
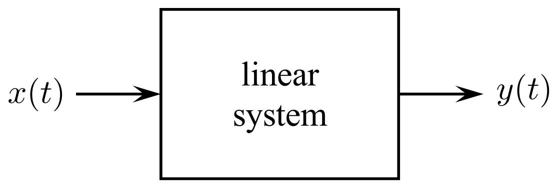


The linear systems of interest to us have an input/output relationship defined by LCCDE

e.g., $y'(t) + ay(t) = x(t)$



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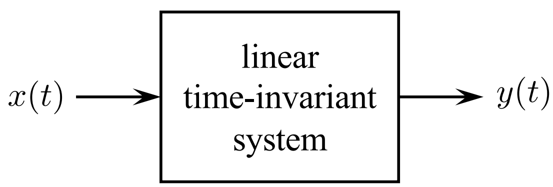
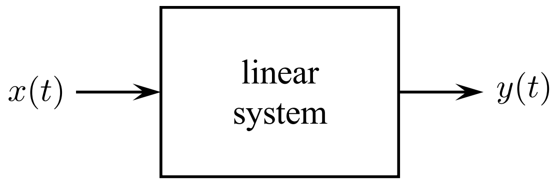


Systems defined by an LCCDE are also time-invariant



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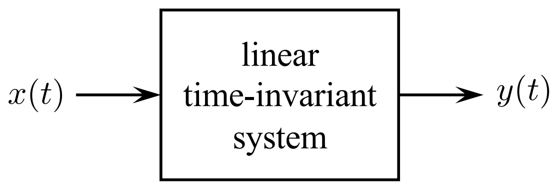

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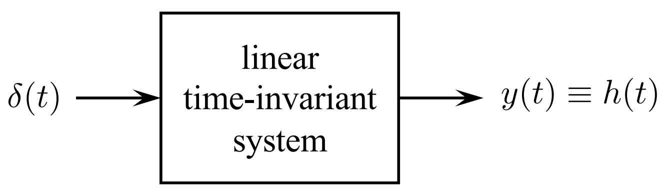
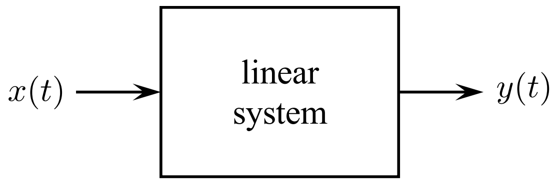


The input/output relationship of a linear time-invariant system may also be described using the impulse response and convolution.



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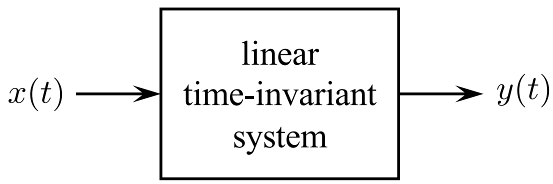

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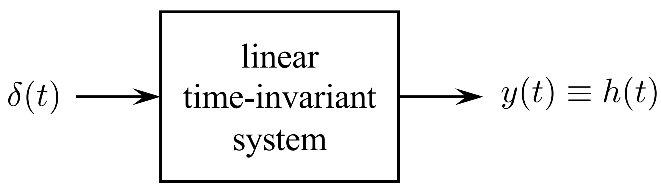

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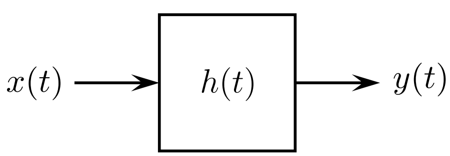

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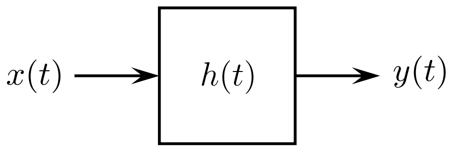
Convolution Example



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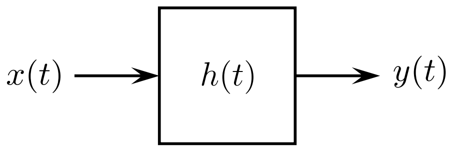

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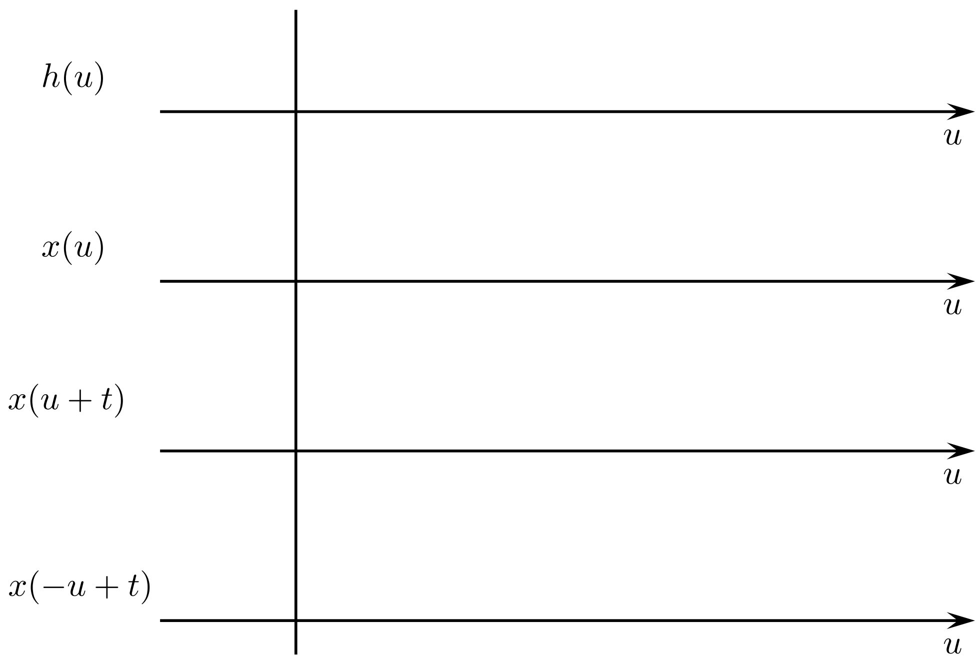

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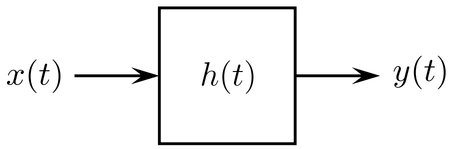

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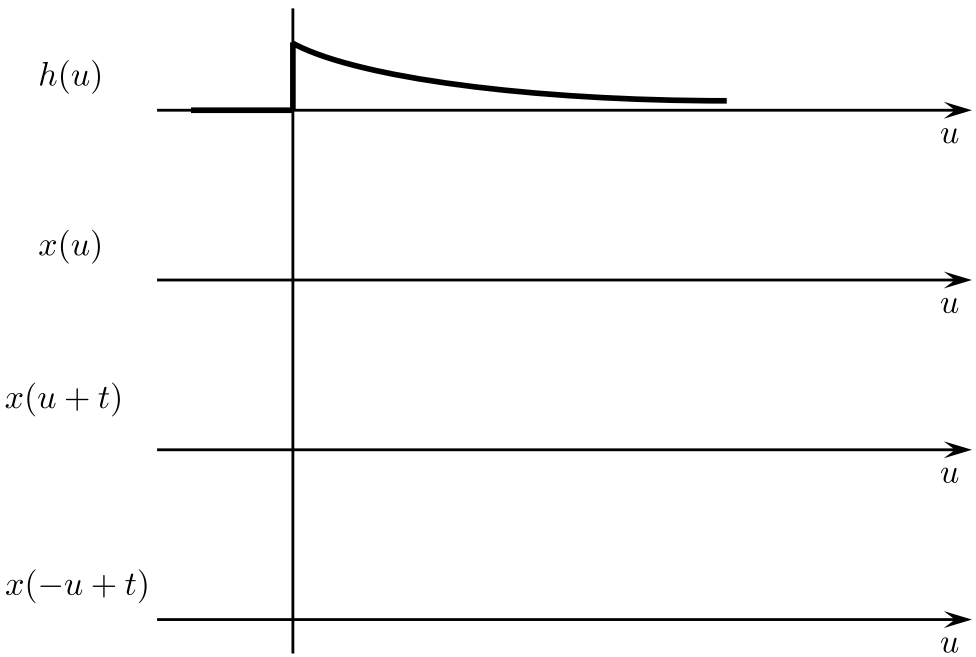

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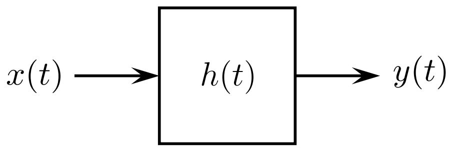

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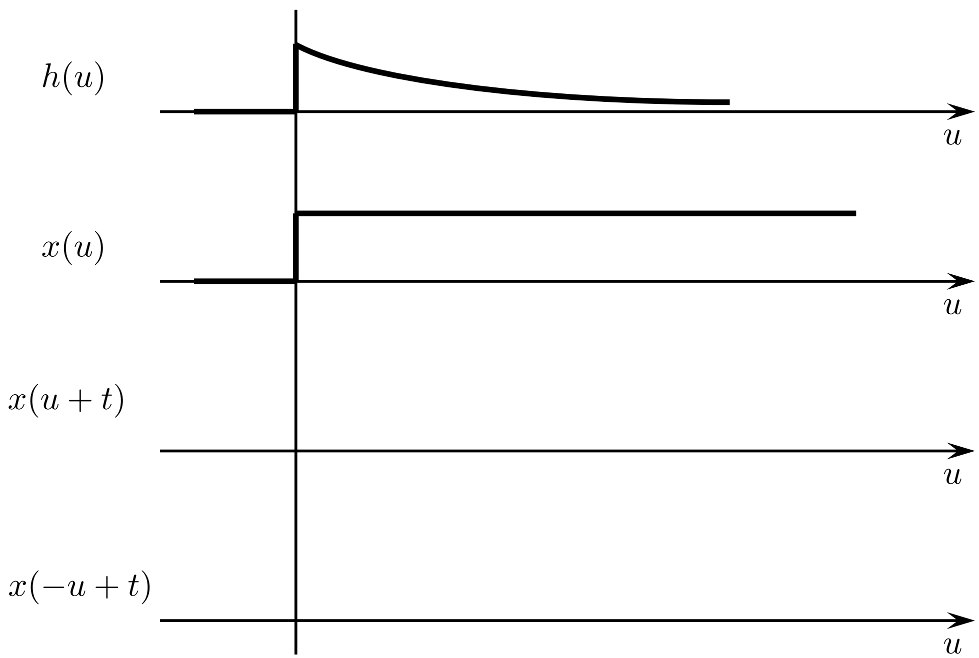

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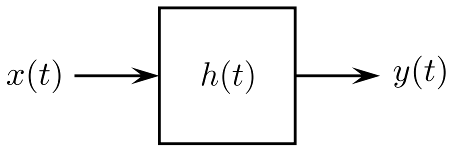

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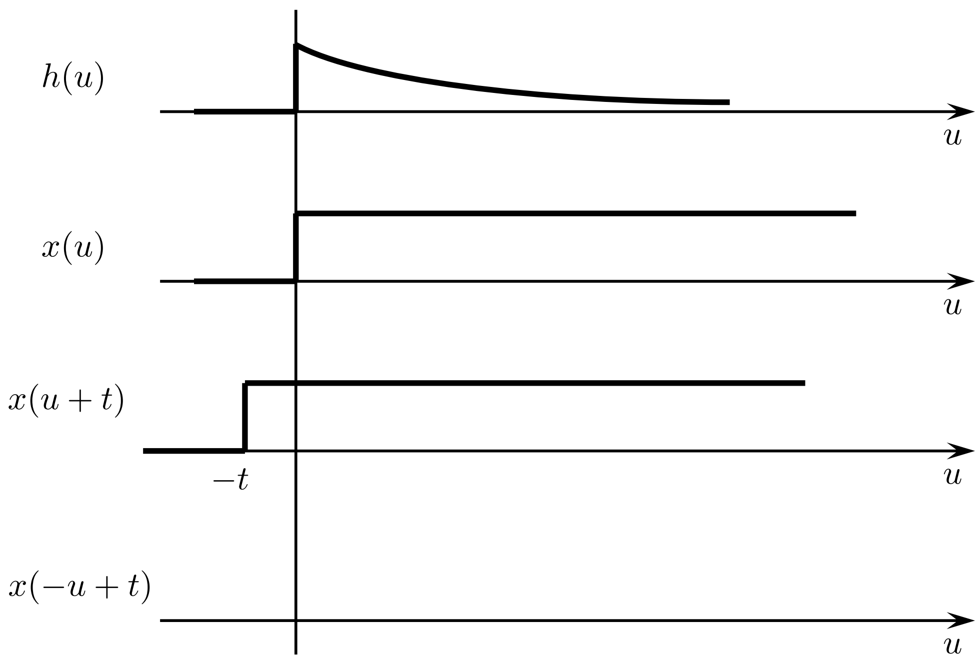

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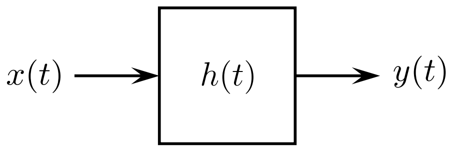

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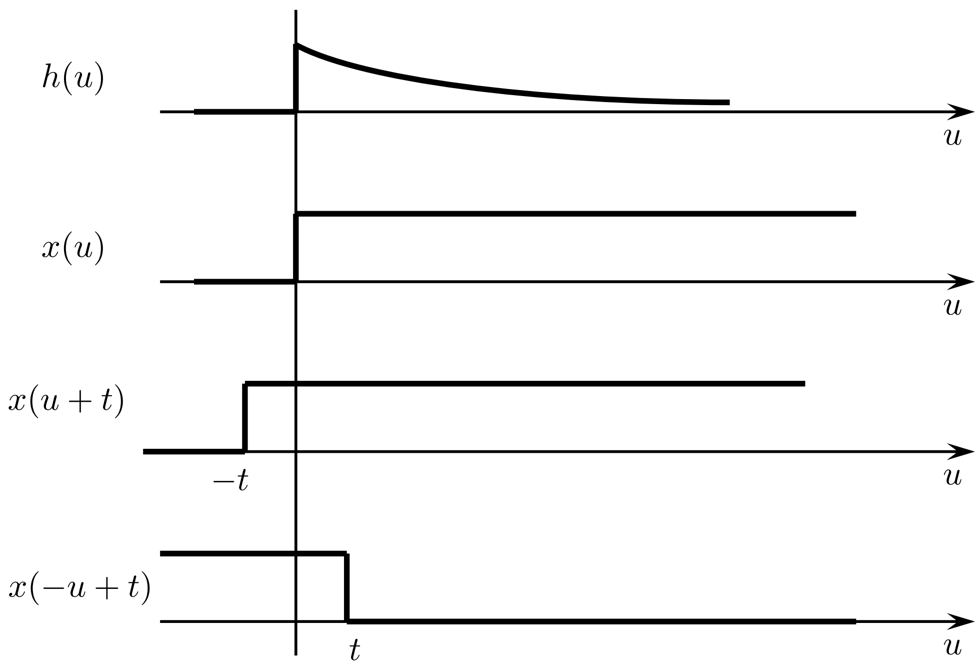

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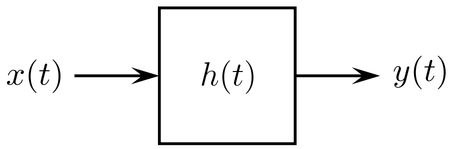

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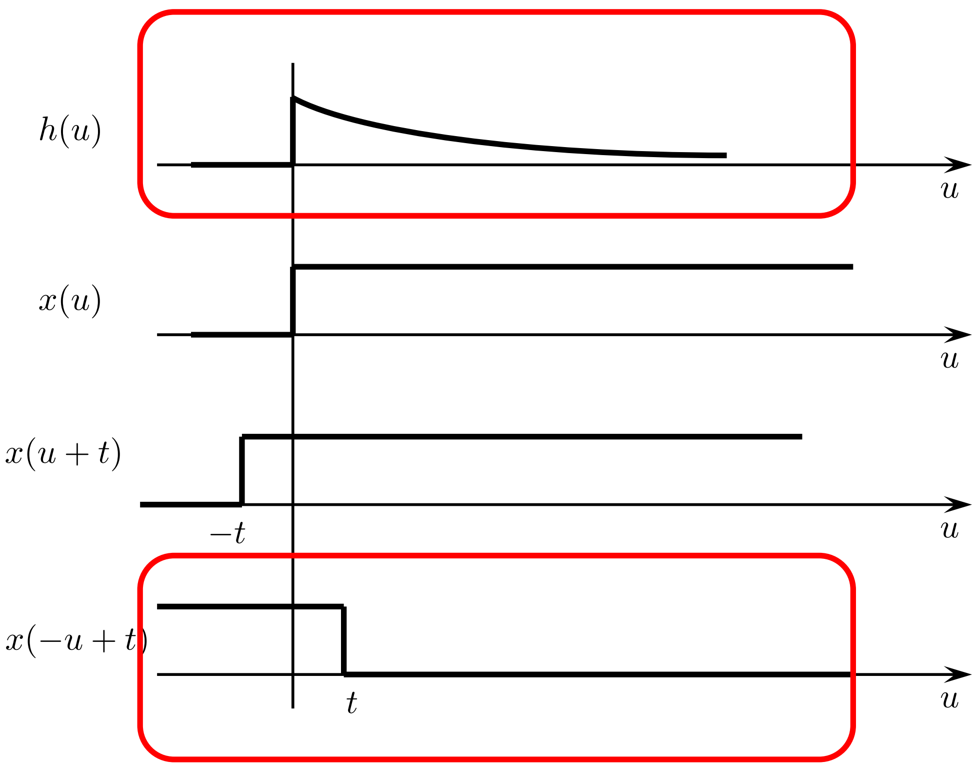

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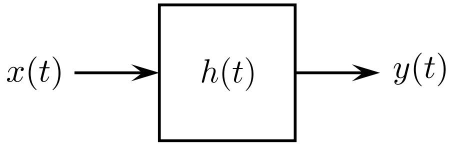

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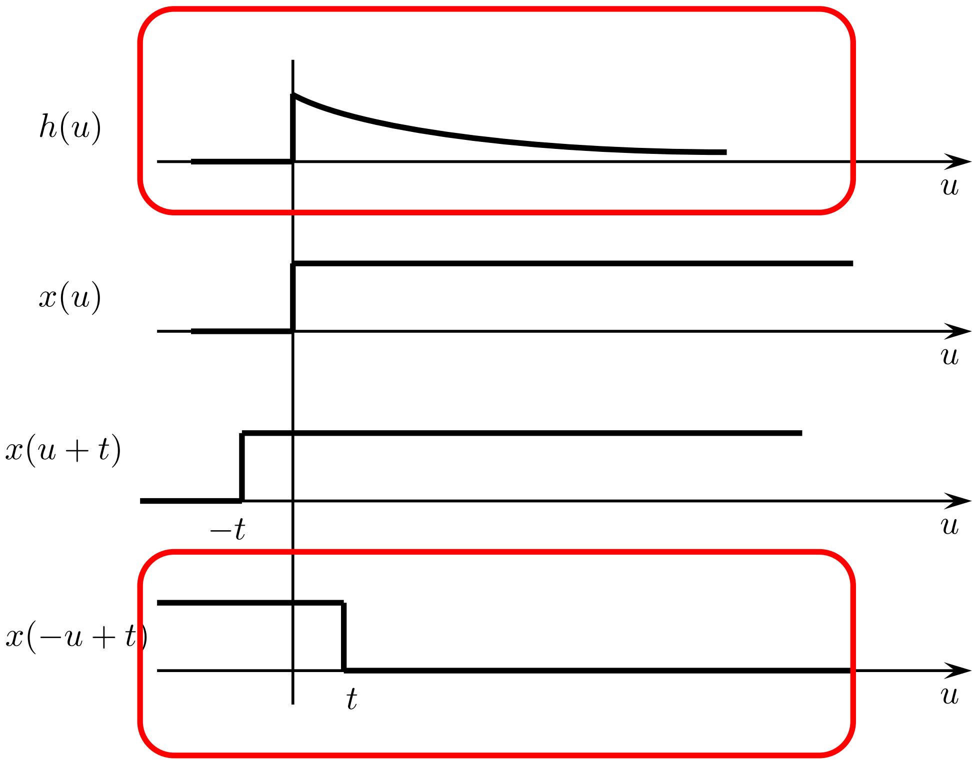

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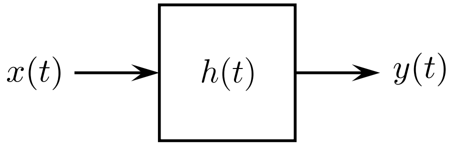

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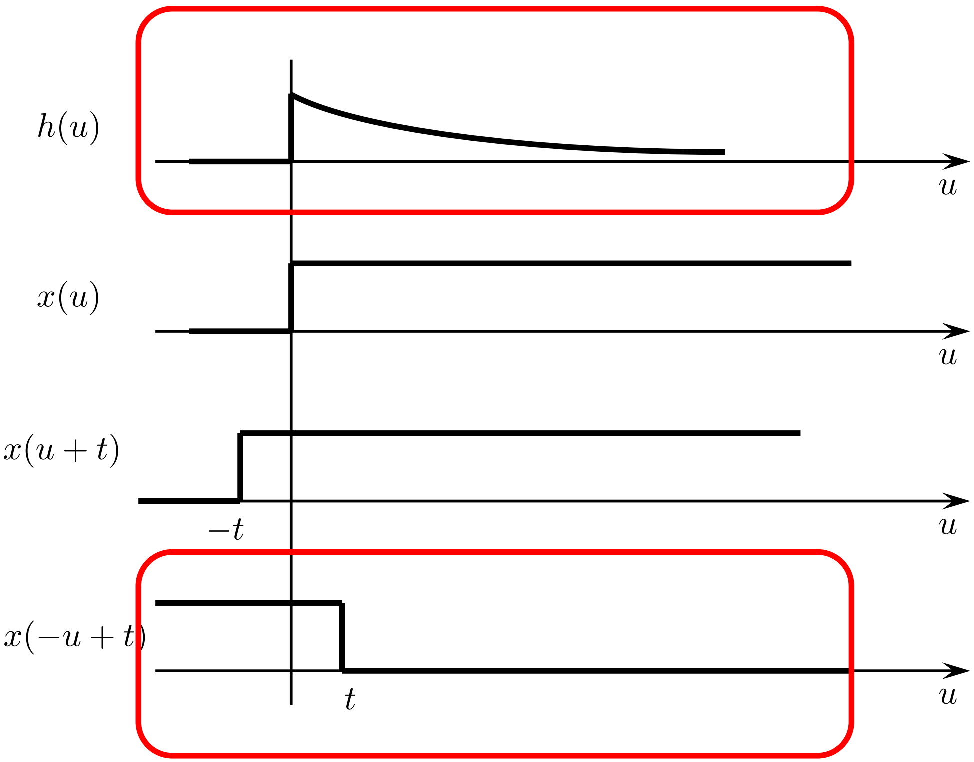

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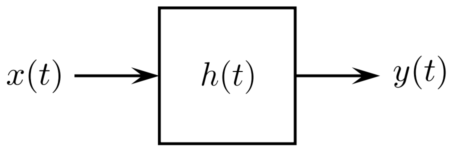

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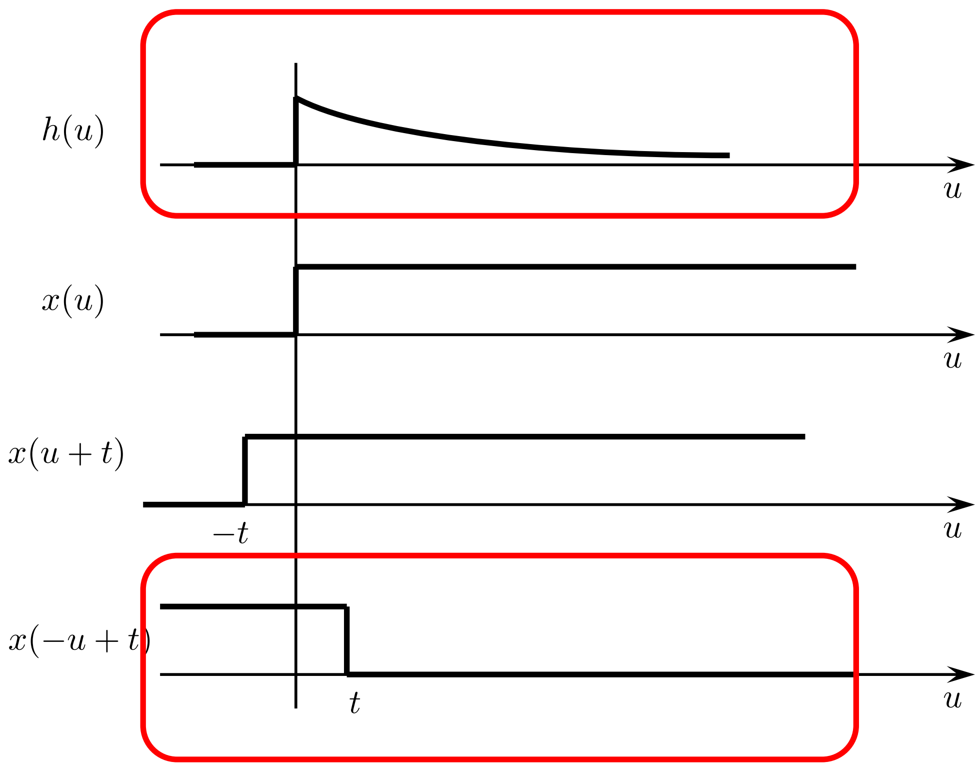

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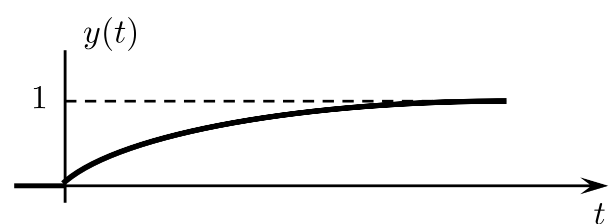
$t \geq 0, \qquad y(t) = \int_0^t \frac{1}{a}e^{-au}du = 1 - e^{-at}$



Frequency Domain Analysis

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{j2\pi} \oint X(s)e^{st}ds$$

Frequency Domain Analysis

Laplace Transform

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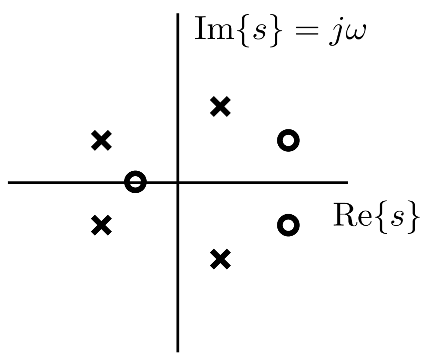
- s is a complex variable and $X(s)$ is a complex-valued function of the complex variable s.



Frequency Domain Analysis

Laplace Transform

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Comments:

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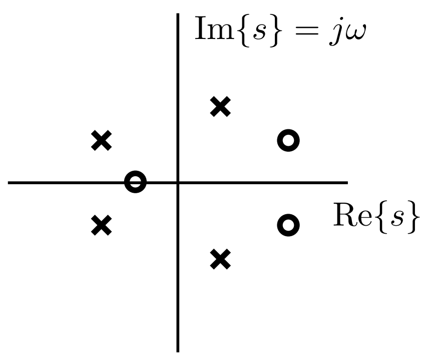
- LCCDEs always produce a ratio of polynomials in s:
$$X(s) = \frac{B(s)}{A(s)}$$



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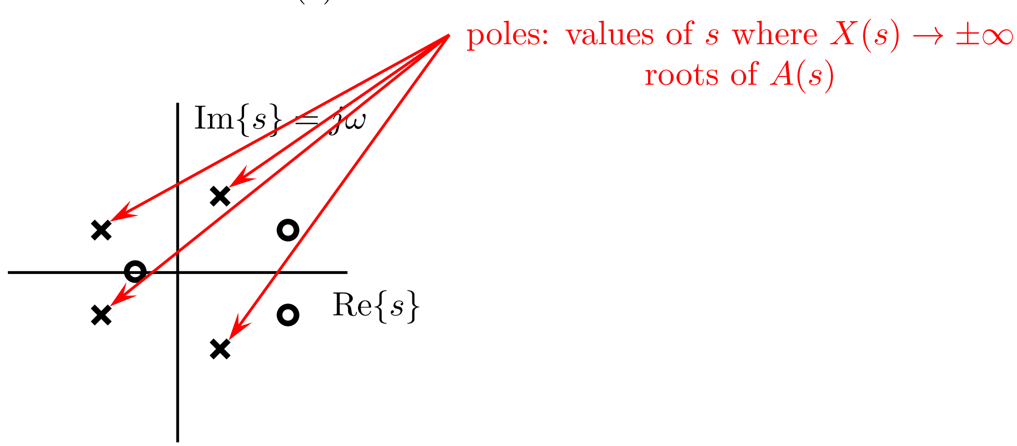

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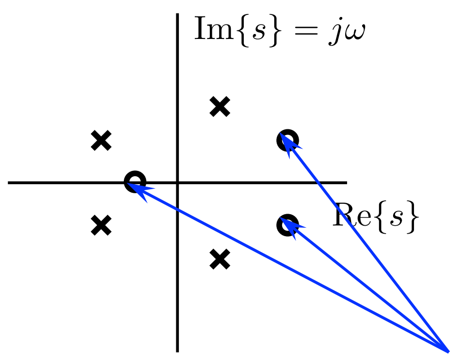
poles: values of s where $X(s) \rightarrow \pm\infty$
roots of $A(s)$



zeros: values of s where $X(s) = 0$
roots of $B(s)$

Frequency Domain Analysis

Laplace Transform

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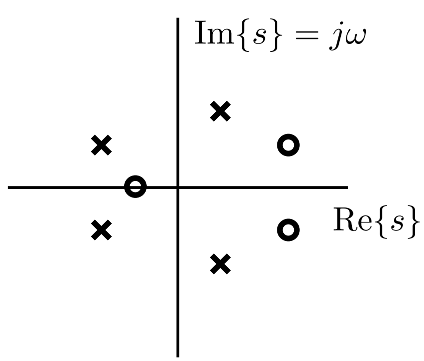

The Laplace transform integral

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

only converges for certain values of s. These values are called *the region of convergence* (ROC).

- The ROC cannot contain any poles.
- The ROC for the Laplace transform of a stable (bounded) time-domain signal contains the $\mathrm{Im}\{s\} = j\omega$ axis.

Frequency Domain Analysis

Laplace Transform

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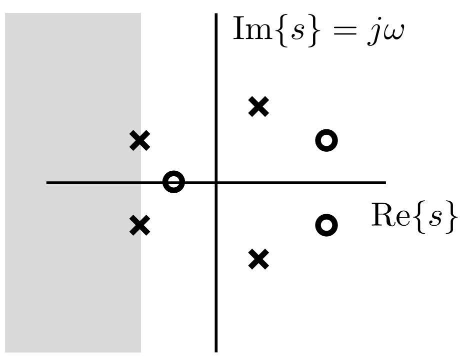
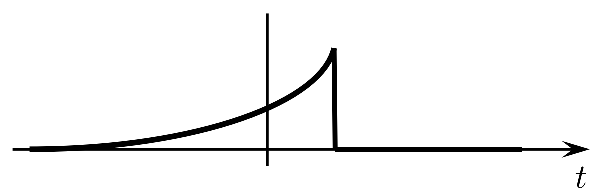


ROC for left-sided signals



Frequency Domain Analysis

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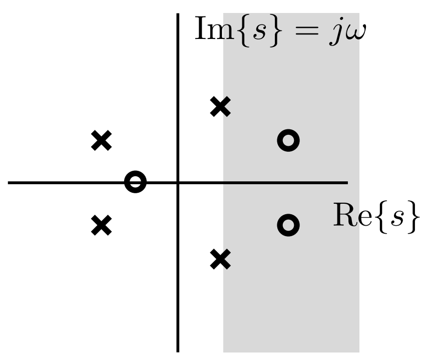
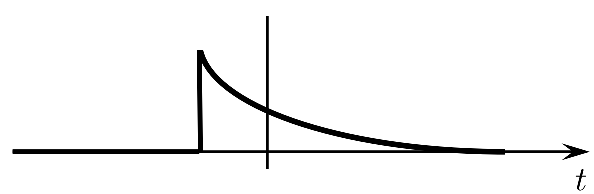


ROC for right-sided signals



Frequency Domain Analysis

Laplace Transform

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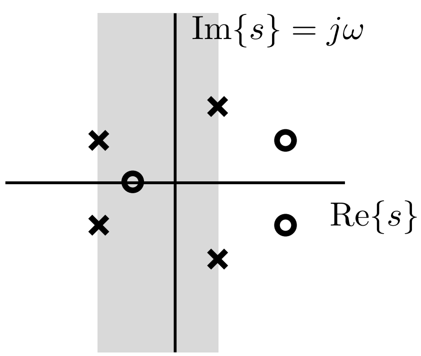
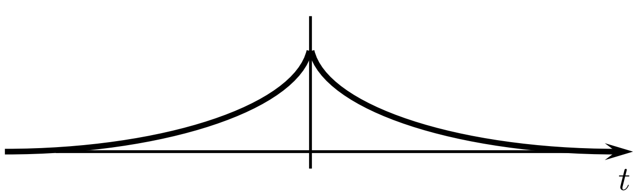


ROC for two-sided signals



Frequency Domain Analysis

Laplace Transform

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$$X(s) = \frac{B(s)}{A(s)}$$
- Because s is a complex variable, s as a *variable of integration* defines a *contour integral* in the complex plane.

Frequency Domain Analysis

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Frequency Domain Analysis

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$x(t)$	$X(s)$	ROC
$\delta(t)$	1	all s
$U(t)$	$\frac{1}{s}$	Real $\{s\} > 0$
$\frac{t^{n-1}}{(n-1)!}U(t)$	$\frac{1}{s^n}$	Real $\{s\} > 0$
$e^{-at}U(t)$	$\frac{1}{s+a}$	Real $\{s\} > -a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}U(t)$	$\frac{1}{(s+a)^n}$	Real $\{s\} > 0$
$\delta(t-T)$	e^{-sT}	all s
$\cos(\omega_0 t)\,U(t)$	$\frac{s}{s^2+\omega_0^2}$	Real $\{s\} > 0$
$\sin(\omega_0 t)\,U(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	Real $\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\,U(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	Real $\{s\} > -a$
$e^{-at}\sin(\omega_0 t)\,U(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	Real $\{s\} > -a$
$\frac{d^n\delta(t)}{dt}$	s^n	all s
$\underbrace{U(t)*\cdots*U(t)}_{n\text{ times}}$	$\frac{1}{s^n}$	Real $\{s\} > 0$

Frequency Domain Analysis

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

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valued function of the comp
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of integration defines a *co*
complex plane.
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verse Laplace transforn
  - Everyone else uses pa
sion to construct table
bles.

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R_x
	$y(t)$	$Y(s)$	R_y
Linearity	$ax(t) + by(t)$	$aX(s) + bY(s)$	at least $R_x \cap R_y$
Time Shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R_x
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	all s for which s/a is in R_x
Conjugation	$x^*(t)$	$X^*(s^*)$	R_x
Convolution	$x(t) * y(t)$	$X(s)Y(s)$	at least $R_x \cap R_y$
Differentiation	$\frac{d}{dt}x(t)$	$sX(s) - x(0^-)$	at least R_x
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	at least $R_x \cap \{\text{Real}\{s\} > 0\}$
Initial Value Theorem[a]		$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)$	
Final Value Theorem[b]		$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$	

[a] The Initial Value Theorem is valid for signals $x(t)$ that satisfy the following conditions: $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$.

[b] The Final Value Theorem is valid for signals $x(t)$ that satisfy the following conditions: $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$.

Frequency Domain Analysis

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{j2\pi} \oint X(s)e^{st}ds$$

Comments:

- s is a complex variable and $X(s)$ is a complex-valued function of the complex variable s.
- LCCDEs always produce a ratio of polynomials in s:
$$X(s) = \frac{B(s)}{A(s)}$$
- Because s is a complex variable, s as a *variable of integration* defines a *contour integral* in the complex plane.
  - Contour integral wizards compute the inverse Laplace transform from the definition.
  - Everyone else uses partial fraction expansion to construct tables, then uses the tables.

Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Frequency Domain Analysis

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Comments:

- w is a real variable and $X(\omega)$ is a complex-valued function of the real variable ω.

Frequency Domain Analysis

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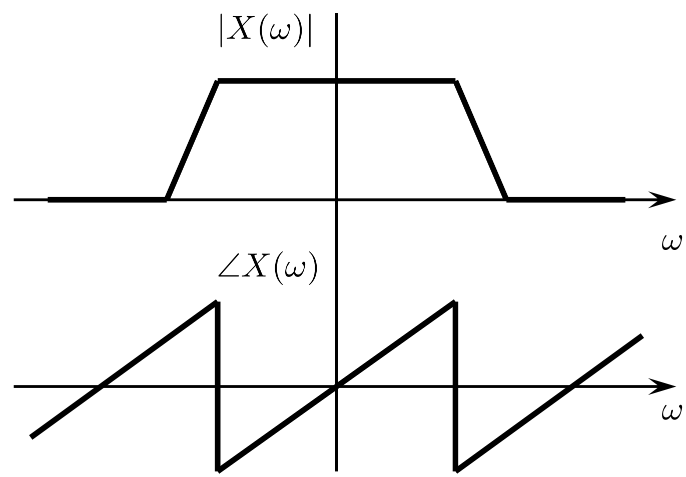

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$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

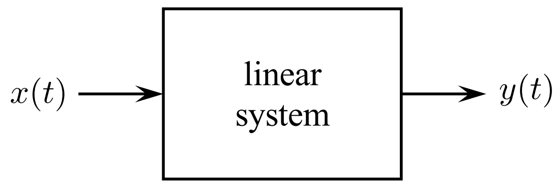
Comments:

- w is a real variable and $X(\omega)$ is a complex-valued function of the real variable ω.
- Because ω is a real variable, the integral that defines the inverse transform is the familiar integral introduced in your first and second calculus courses.
  - Integral wizards compute the inverse Fourier transform from the definition.
  - Everyone else uses uses tables.

$x(t)$	$X(\omega)$	$X(f)$
$\delta(t)$	1	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$	$e^{-j2\pi f t_0}$
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	$\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(a+j\omega)^n}$	$\frac{1}{(a+j2\pi f)^n}$
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $(\omega_0 = 2\pi f_0)$	$2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega-k\omega_0)$	$\sum_{k=-\infty}^{\infty} a_k\delta(f-kf_0)$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T}\right)$	$\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right)$
1	$2\pi\delta(\omega)$	$\delta(f)$
$\begin{cases} 1 & \lvert t\rvert < T \\ 0 & \lvert t\rvert > T \end{cases}$	$2T\frac{\sin(\omega T)}{\omega T}$	$2T\frac{\sin(2\pi f T)}{2\pi f T}$
$2B\frac{\sin(2\pi B t)}{2\pi B t}$	$\begin{cases} 1 & -2\pi B \le \omega \le 2\pi B \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$
$e^{j\omega_0 t}$ $(\omega_0 = 2\pi f_0)$	$2\pi\delta(\omega-\omega_0)$	$\delta(f-f_0)$
$\cos(\omega_0 t)$ $(\omega_0 = 2\pi f_0)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(\omega_0 t)$ $(\omega_0 = 2\pi f_0)$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$	$\frac{1}{j2}\delta(f-f_0)-\frac{1}{j2}\delta(f+f_0)$
$\exp\{-a\lvert t\rvert\}$	$\frac{2a}{a^2+\omega^2}$	$\frac{2a}{a^2+(2\pi f)^2}$
$\exp\{-\pi t^2\}$	$\exp\left\{-\pi\left(\frac{\omega}{2\pi}\right)^2\right\}$	$\exp\left\{-\pi f^2\right\}$

Property	Signal	Fourier Transform in ω	Fourier Transform in f
	$x(t)$	$X(\omega)$	$X(f)$
	$y(t)$	$Y(\omega)$	$Y(f)$
Linearity	$ax(t)+by(t)$	$aX(\omega)+bY(\omega)$	$aX(f)+bY(f)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$	$e^{-j2\pi f t_0}X(f)$
Time Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$	$\frac{1}{\|a\|}X\left(\frac{f}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$	$X^*(-f)$
Convolution	$x(t)*y(t)$	$X(\omega)Y(\omega)$	$X(f)Y(f)$
Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$	$j2\pi f X(f)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
Frequency Shifting ($\omega_0 = 2\pi f_0$)	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$	$X(f-f_0)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(\omega)*Y(\omega)$	$X(f)*Y(f)$
Parseval's Theorem	$\int_{-\infty}^{\infty}\|x(t)\|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\|X(\omega)\|^2d\omega = \int_{-\infty}^{\infty}\|X(f)\|^2df$		

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



The linear systems of interest to us have an input/output relationship defined by LCCDE

e.g., $y'(t) + ay(t) = x(t)$

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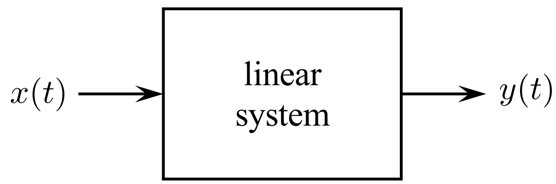

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$$sY(s) - y(0^-) + aY(s) = X(s)$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{linear}\\ \text{system}\end{array}} \longrightarrow y(t)$$

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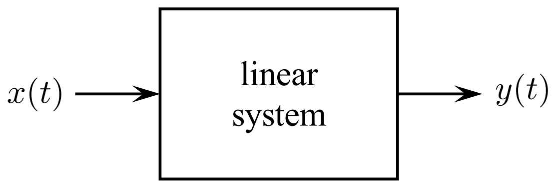
Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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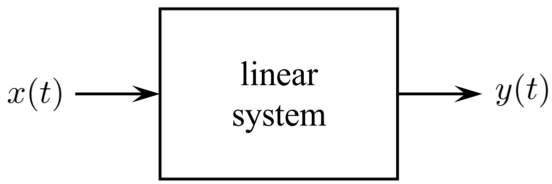

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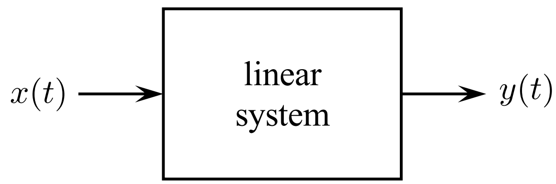
$$sY(s) + aY(s) = X(s)$$

$$(s+a)Y(s) = X(s)$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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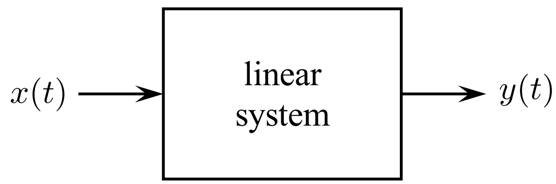
Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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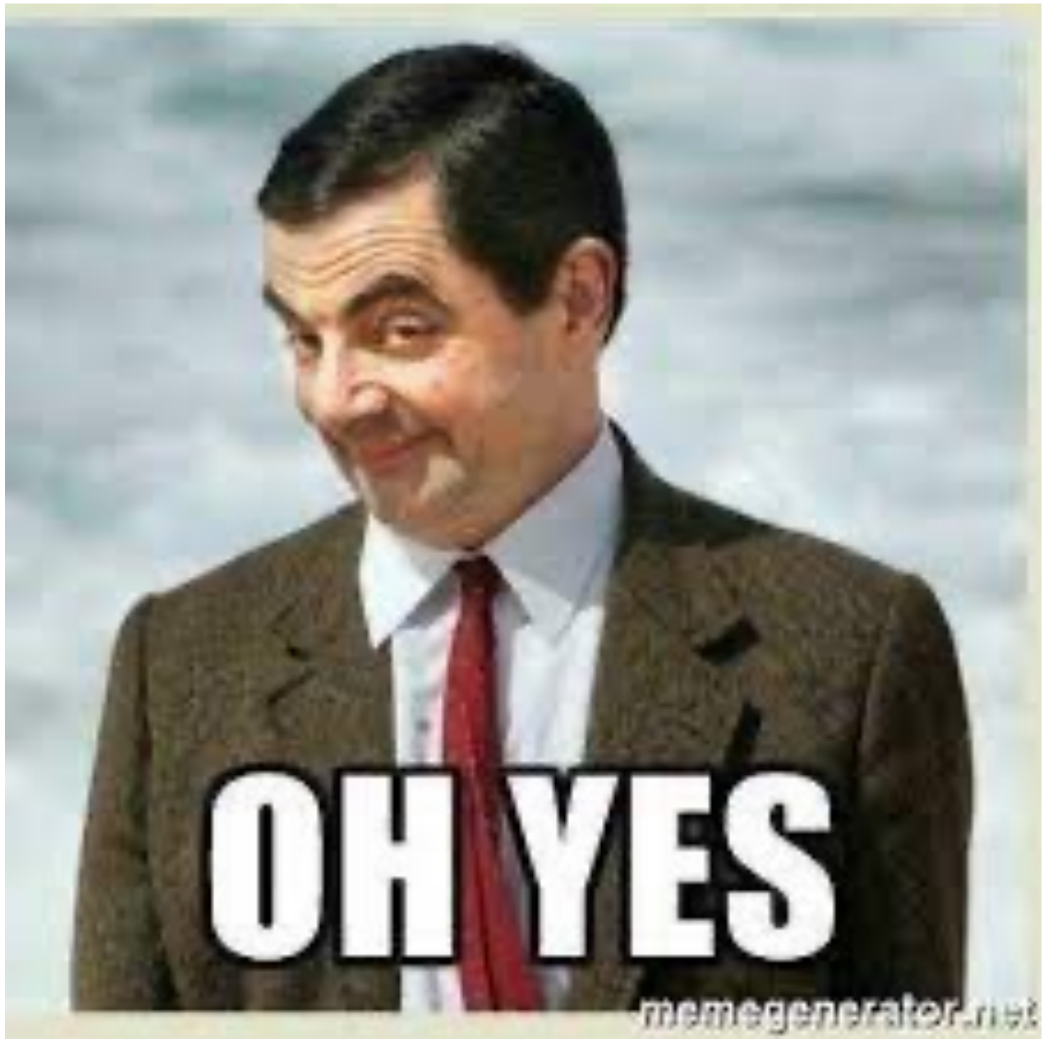
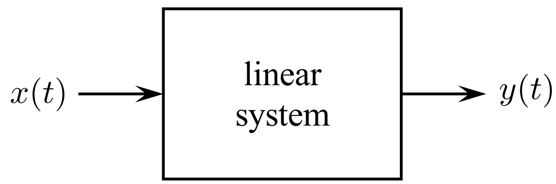
Is this really the impulse response?


OH YES
memegenerator.net

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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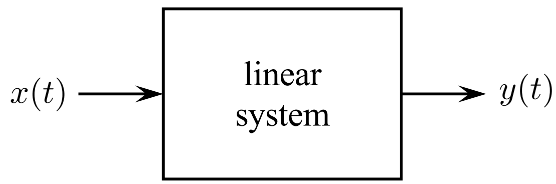

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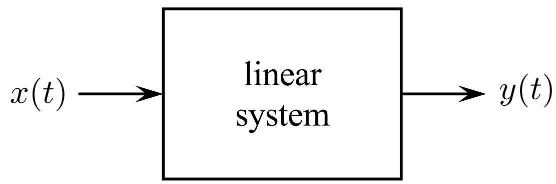

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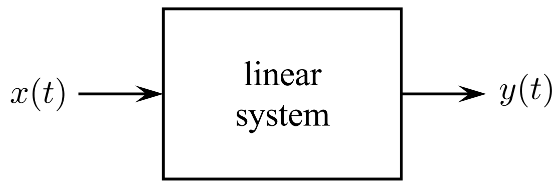

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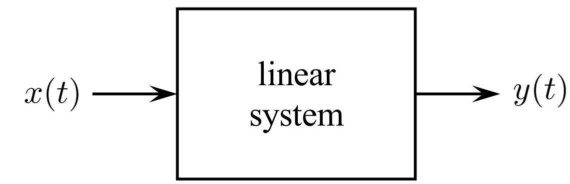

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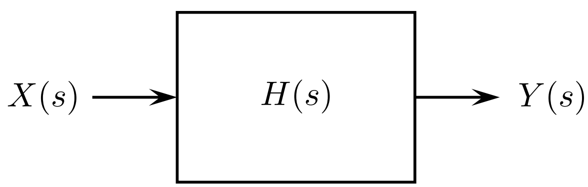
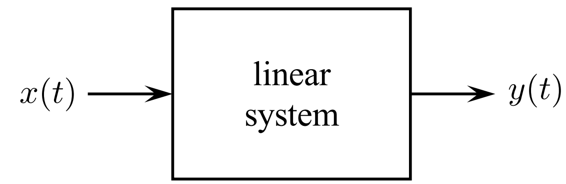


Laplace domain transfer function

$$Y(s) = H(s)X(s)$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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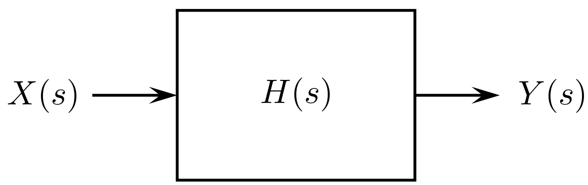

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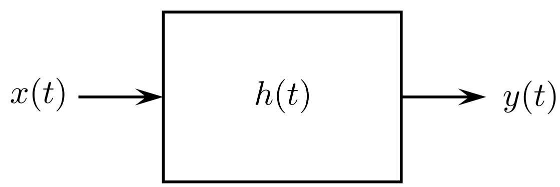
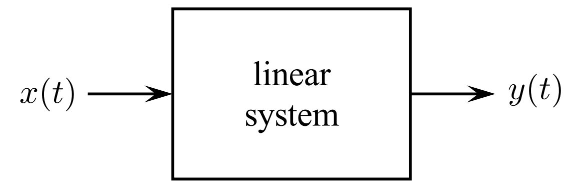


Time domain convolution

$$y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)du = \int_{-\infty}^{\infty} h(u)x(t-u)du$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain



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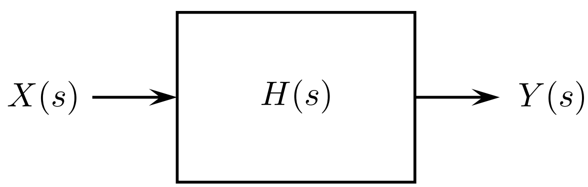

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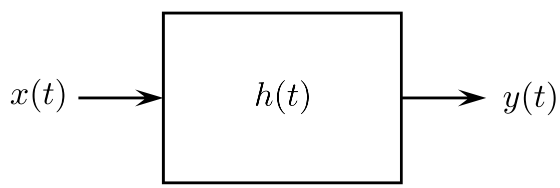

Time domain convolution

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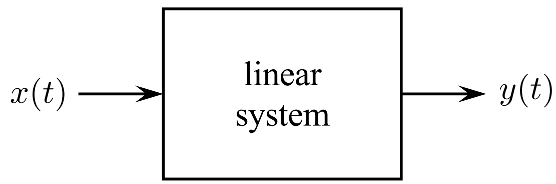
To find the impulse response of an LTI system described by an LCCDE

1. Solve the LCCDE using the Laplace transform with all-zeros initial conditions.
2. Write the Laplace domain solution as $Y(s) = H(s)X(s)$.
3. Identify $H(s)$ in the solution.
4. The impulse response $h(t)$ is the inverse Laplace transform of $H(s)$.

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain + Fourier Transform



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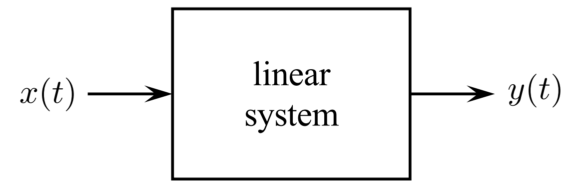

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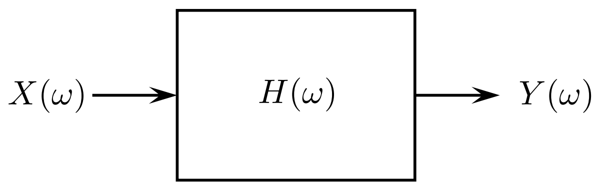
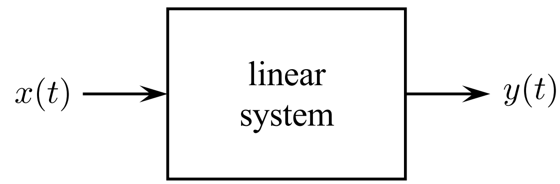


Fourier domain frequency response

$$Y(\omega) = H(\omega)X(\omega)$$

Relationships: LCCDE, Impulse Response, Convolution, Frequency Domain + Fourier Transform



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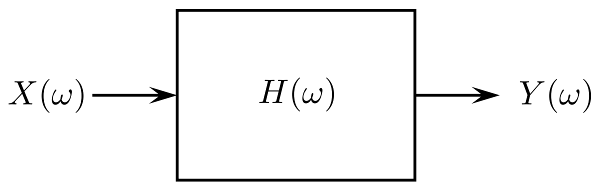

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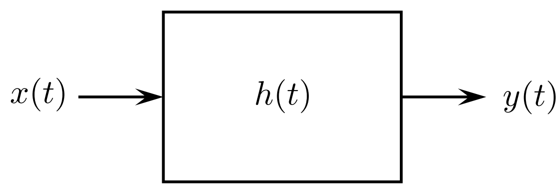

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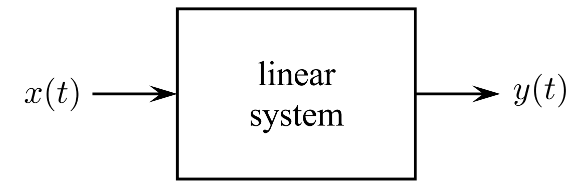

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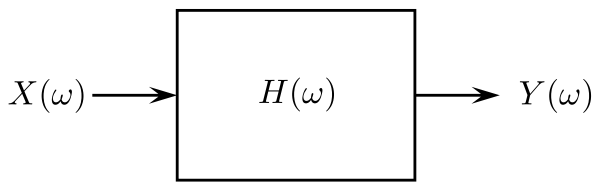

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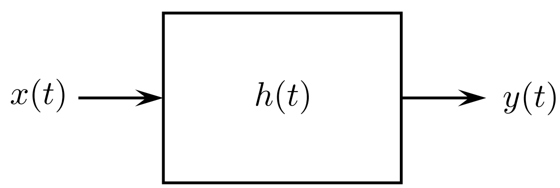

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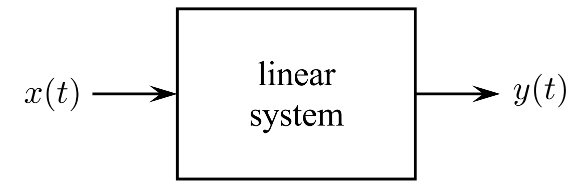

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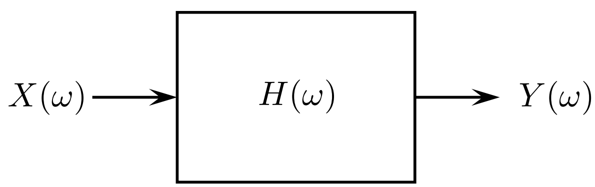

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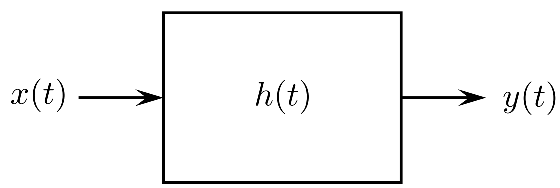

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- After identifying the Laplace domain transfer function $H(s)$, the Fourier domain frequency response is

$$H(\omega) = H(s)\Big|_{s=j\omega}$$

This works only as long as the ROC of $H(s)$ contains the $s = j\omega$ axis.

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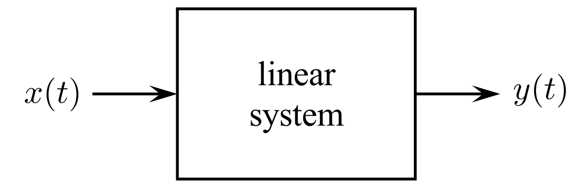

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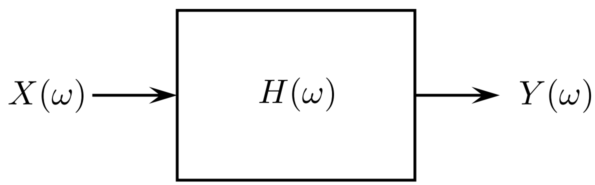

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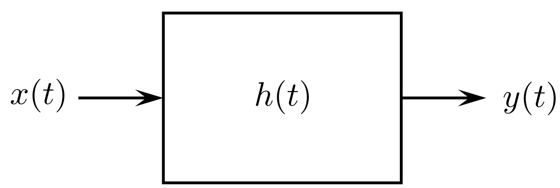

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- For stable systems, this is always true.