6-2 One Function of Two Random Variables

Let $\mathbf{x}$ and $\mathbf{y}$ be two random variables with joint pdf $f_{\mathbf{xy}}(x, y)$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function denoted $z = g(x, y)$. The meaning of $\mathbf{z} = g(\mathbf{x}, \mathbf{y})$.

- $\zeta \in \mathcal{S}$ is an outcome.
- $\mathbf{x}(\zeta)$ is a real number.
- $\mathbf{y}(\zeta)$ is a real number.
- $g(\mathbf{x}(\zeta), \mathbf{y}(\zeta))$ is a real number.
- $g(\mathbf{x}(\zeta), \mathbf{y}(\zeta))$ is a composite map $\mathcal{S} \to \mathbb{R}$.
- Call this composite map $\mathbf{z}(\zeta)$.
- $\mathbf{z}(\zeta) = g(\mathbf{x}(\zeta), \mathbf{y}(\zeta))$ is a random variable.
- The events of $\mathbf{z}(\zeta)$ are described by the set

$$\{\zeta \in \mathcal{S} \colon \mathbf{z}(\zeta) \le z\} = \{\zeta \in \mathcal{S} \colon g(\mathbf{x}(\zeta), \mathbf{y}(\zeta)) \le z\}$$

- Let $D_z = \{(x, y) \in \mathbb{R}^2 \colon g(x, y) \le z\}$. Then

$$\{\zeta \in \mathcal{S} \colon \mathbf{z}(\zeta) \le z\} = \{\zeta \in \mathcal{S} \colon g(\mathbf{x}(\zeta), \mathbf{y}(\zeta)) \le z\} = \Big\{\zeta \in \mathcal{S} \colon \Big(\mathbf{x}(\zeta), \mathbf{y}(\zeta)\Big) \in D_z\Big\}$$

- Cumulative distribution function of $\mathbf{z}$:

$$\begin{aligned} F_{\mathbf{z}}(z) &= P(\{\zeta \in \mathcal{S} \colon \mathbf{z}(\zeta) \le z\}) \\ &= P(\{\zeta \in \mathcal{S} \colon g(\mathbf{x}(\zeta), \mathbf{y}(\zeta)) \le z\}) \\ &= P\left(\Big\{\zeta \in \mathcal{S} \colon \Big(\mathbf{x}(\zeta), \mathbf{y}(\zeta)\Big) \in D_z\Big\}\right) \\ &= \iint_{D_z} f_{\mathbf{xy}}(x, y)\, dx\, dy \end{aligned}$$

- Probability density function of $\mathbf{z}$:

$$f_{\mathbf{z}}(z) = \frac{d}{dz} F_{\mathbf{z}}(z) = \frac{d}{dz} \iint_{D_z} f_{\mathbf{xy}}(x, y)\, dx\, dy$$