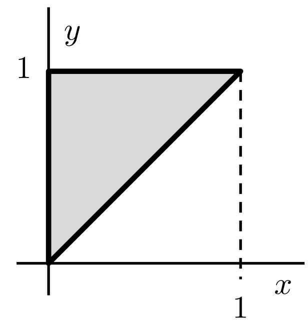
Example 6-40: The Extended Version

ECEn 670: Stochastic Processes

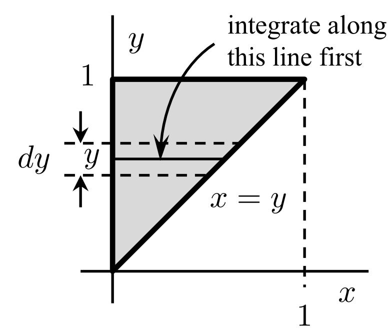
The joint density of the random variables $\mathbf{x}$ and $\mathbf{y}$ is

$$f_{\mathbf{xy}}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$



The region of support for the joint density $f_{\mathbf{xy}}(x,y)$ is shown by the gray area in the figure to the right. The joint density function $f_{\mathbf{xy}}(x,y)$ is the constant 2 over the gray region of support and zero everywhere else in the (x,y) plane. To see that this is a valid joint density function, the double integral must be one. The double integral can be formulated two ways: $dx\,dy$ or $dy\,dx$.

The "$dx\,dy$" order is developed with the aid of the figure to the right.



1. Pick a value of y. For $0 < y < 1$, the value of y defines a horizontal line that passes through the region of support of $f_{\mathbf{xy}}(x,y)$.

2. The *area* under the slice defined by the horizontal line $y = y$ is

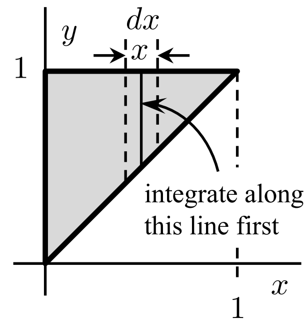
$$\text{area of the “slice” } y = y = \int_{x=0}^{y} 2\,dx$$

3. Multiplying the area under the slice $y = y$ by the incremental width dy creates the incremental *volume* for the slice.

4. The *volume* of the joint density $f_{\mathbf{xy}}(x,y)$ is obtained by summing the incremental volumes. In the limit $dy \to 0$, the sum becomes the integral:

$$\text{volume} = \int_{y=0}^{1} \int_{x=0}^{y} 2\,dx\,dy = \int_{y=0}^{1} 2y\,dy = 1.$$

The "$dy\,dx$" order is developed with the aid of the figure to the right.



1. Pick a value of x. For $0 < x < 1$, the value of x defines a vertical line that passes through the region of support of $f_{\mathbf{xy}}(x, y)$.

2. The *area* under the slice defined by the vertical line $x = x$ is

$$\text{area of the "slice" } x = x = \int_{y=x}^{1} 2\,dy$$

3. Multiplying the area under the slice $y = y$ by the incremental width dx creates the incremental *volume* for the slice.

4. The *volume* of the joint density $f_{\mathbf{xy}}(x, y)$ is obtained by summing the incremental volumes. In the limit $dx \to 0$, the sum becomes the integral:

$$\text{volume} = \int_{x=0}^{1} \int_{y=x}^{1} 2\,dy\,dx = \int_{x=0}^{1} 2(1-x)\,dx = 1.$$

The marginal density $f_{\mathbf{x}}(x)$: The marginal density $f_{\mathbf{x}}(x)$ is obtained from the joint density by integrating with respect to the unwanted variable y:

$$f_{\mathbf{x}}(x) = \int_{-\infty}^{\infty} f_{\mathbf{xy}}(x, y)\,dy.$$

For a given x, the integral with respect to y is along the line $x = x$:

$$\begin{aligned} f_{\mathbf{x}}(x) &= \begin{cases} \displaystyle\int_{y=x}^{1} 2\,dy & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The marginal density function is plotted in Figure 1.

The marginal density $f_{\mathbf{y}}(y)$: The marginal density $f_{\mathbf{y}}(y)$ is obtained from the joint density by

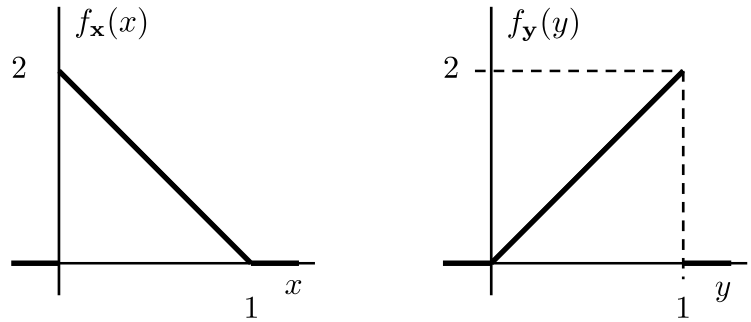


Figure 1: The marginal density functions.

integrating with respect to the unwanted variable x:

$$f_{\mathbf{y}}(y) = \int_{-\infty}^{\infty} f_{\mathbf{xy}}(x, y)\, dx.$$

For a given y, the integral with respect to x is along the line $y = y$:

$$\begin{aligned} f_{\mathbf{y}}(y) &= \begin{cases} \displaystyle\int_{x=0}^{x} 2\, dx & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The marginal density function is plotted in Figure 1.

Marginal statistics:

$$\begin{aligned} \mu_{\mathbf{x}} &= \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x)\, dx = \int_0^1 x 2(1-x)\, dx = \frac{1}{3} \\ \sigma_{\mathbf{x}}^2 &= \int_{-\infty}^{\infty} (x - \mu_{\mathbf{x}})^2 f_{\mathbf{x}}(x)\, dx = \int_0^1 \left(x - \frac{1}{3}\right)^2 2(1-x)\, dx = \frac{1}{18} \\ \mu_{\mathbf{y}} &= \int_{-\infty}^{\infty} y f_{\mathbf{y}}(y)\, dy = \int_0^1 y 2y\, dy = \frac{2}{3} \\ \sigma_{\mathbf{y}}^2 &= \int_{\infty}^{\infty} (y - \mu_{\mathbf{y}})^2 f_{\mathbf{y}}(y)\, dy = \int_0^1 \left(y - \frac{2}{3}\right)^2 2y\, dy = \frac{1}{18}. \end{aligned}$$

Joint statistics:

$$\begin{aligned}
C_{\mathbf{xy}} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_{\mathbf{x}})(y-\mu_{\mathbf{y}})f_{\mathbf{xy}}(x,y)\,dx\,dy \\
&= \int_{y=0}^{1}\int_{x=0}^{y}\left(x-\frac{1}{3}\right)\left(y-\frac{2}{3}\right)2\,dx\,dy \\
&= 2\int_{y=0}^{1}\left(y-\frac{2}{3}\right)\left(\frac{1}{2}y^2-\frac{1}{3}y\right)\,dy \\
&= \frac{1}{36} \\
\rho_{\mathbf{xy}} &= \frac{C_{\mathbf{xy}}}{\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}} = \frac{\frac{1}{36}}{\sqrt{\frac{1}{18}\frac{1}{18}}} = \frac{1}{2} \\
R_{\mathbf{xy}} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{\mathbf{xy}}(x,y)\,dx\,dy \\
&= \int_{y=0}^{1}\int_{x=0}^{y} xy2\,dx\,dy \\
&= \int_{y=0}^{1} y\,y^2\,dy \\
&= \frac{1}{4}.
\end{aligned}$$

Note that the covariance may also be computed using

$$C_{\mathbf{xy}} = R_{\mathbf{xy}} - \mu_{\mathbf{x}}\mu_{\mathbf{y}} = \frac{1}{4} - \frac{1}{3}\times\frac{2}{3} = \frac{1}{36}.$$

Conditional densities and expectations:

$$f_{\mathbf{x}|\mathbf{y}}(x|y) = \frac{f_{\mathbf{xy}}(x,y)}{f_{\mathbf{y}}(y)} = \begin{cases} \dfrac{2}{2y} = \dfrac{1}{y} & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\mathbf{y}|\mathbf{x}}(y|x) = \frac{f_{\mathbf{xy}}(x,y)}{f_{\mathbf{x}}(x)} = \begin{cases} \dfrac{2}{2(1-x)} = \dfrac{1}{1-x} & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The behavior of $f_{\mathbf{x}|\mathbf{y}}(x|y)$ is illustrated by examining $f_{\mathbf{x}|\mathbf{y}}(x|y)$ at a few trial values of y:

$$f_{\mathbf{x}|\mathbf{y}}(x|\mathbf{y}=0.2) = \begin{cases} 5 & 0 < x < 0.2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\mathbf{x}|\mathbf{y}}(x|\mathbf{y}=0.5) = \begin{cases} 2 & 0 < x < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\mathbf{x}|\mathbf{y}}(x|\mathbf{y}=0.8) = \begin{cases} 1.25 & 0 < x < 0.8 \\ 0 & \text{otherwise} \end{cases}$$

These cases are plotted in Figure 2 (a).

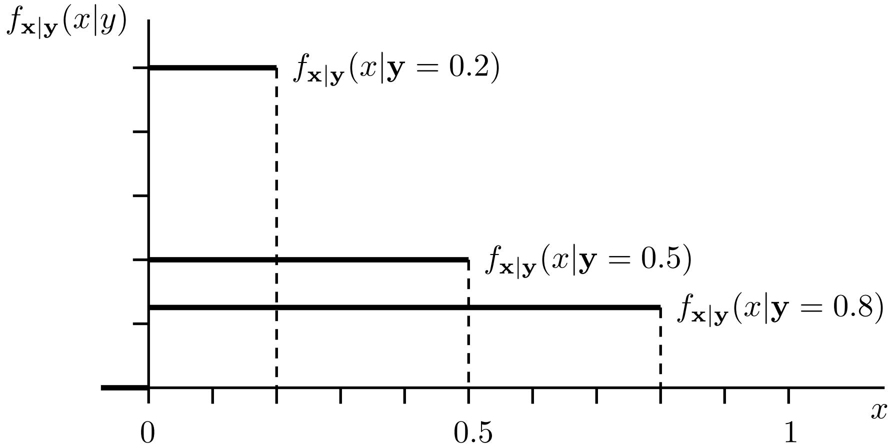
The behavior of $f_{\mathbf{y}|\mathbf{x}}(y|x)$ is illustrated by examining $f_{\mathbf{y}|\mathbf{x}}(y|x)$ at a few trial values of x:

$$f_{\mathbf{y}|\mathbf{x}}(y|\mathbf{x}=0.2) = \begin{cases} 1.25 & 0.2 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

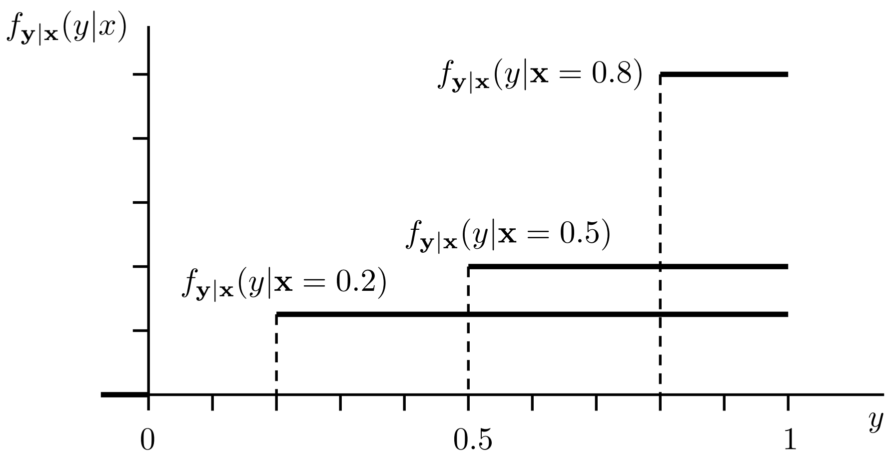
$$f_{\mathbf{y}|\mathbf{x}}(y|\mathbf{x}=0.5) = \begin{cases} 2 & 0.5 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\mathbf{y}|\mathbf{x}}(y|\mathbf{x}=0.8) = \begin{cases} 5 & 0.5 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

These cases are plotted in Figure 2 (b).



(a)



(b)

Figure 2: The conditional density functions.

The conditional expectation $E\{\mathbf{x}|\mathbf{y}=y\}$ is computed as follows:

$$\begin{aligned}
\mu_{\mathbf{x}|y} &= E\{\mathbf{x}|\mathbf{y}=y\} \\
&= \int_{-\infty}^{\infty} x f_{\mathbf{x}|\mathbf{y}}(x|\mathbf{y}=y)\,dx \\
&= \begin{cases} \displaystyle\int_{x=0}^{y} x\frac{1}{y}\,dx & 0<y<1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \dfrac{y}{2} & 0<y<1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$E\{\mathbf{x}|\mathbf{y}=y\}$ is a function of y. (This is true, in general.) A plot of $E\{\mathbf{x}|\mathbf{y}=y\}$ vs. y is shown in Figure 3 (a). $E\{\mathbf{x}|\mathbf{y}=y\}$ as a function of y is called a "regression line" even though in general it is not a "line."

Replacing the y in the conditional expectation with $\mathbf{y}$ produces the random variable

$$g(\mathbf{y}) = \begin{cases} \dfrac{y}{2} & 0<\mathbf{y}<1 \\ 0 & \text{otherwise.} \end{cases}$$

The expected value of this random variable is

$$E\{g(\mathbf{y})\} = \int_{-\infty}^{\infty} g(y) f_{\mathbf{y}}(y) dy = \int_0^1 \frac{y}{2}\,2y\,dy = \frac{1}{3} = \mu_{\mathbf{x}}.$$

This demonstrates the property

$$E\{\,E\{\,\mathbf{x}|\mathbf{y}\,\}\,\} = E\{\mathbf{x}\}.$$

The conditional expectation $E\{\mathbf{y}|\mathbf{x}=x\}$ is computed as follows:

$$\begin{aligned}
\mu_{\mathbf{y}|x} &= E\{\mathbf{y}|\mathbf{x}=x\} \\
&= \int_{-\infty}^{\infty} y f_{\mathbf{y}|\mathbf{x}}(y|\mathbf{x}=x)\,dy \\
&= \begin{cases} \displaystyle\int_{y=x}^{1} y\frac{1}{1-x}\,dy & 0<x<1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \dfrac{x+1}{2} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$E\{\mathbf{y}|\mathbf{x}=x\}$ is a function of x. A plot of $E\{\mathbf{y}|\mathbf{x}=x\}$ vs. x is shown in Figure 3 (b). $E\{\mathbf{y}|\mathbf{x}=x\}$ as a function of x is called a “regression line” even though in general it is not a “line.”

Replacing x in the conditional expectation with $\mathbf{x}$ produces the random variable

$$g(\mathbf{x}) = \begin{cases} \displaystyle\int_{y=x}^{1} y\frac{1}{1-\mathbf{x}}\,dy & 0<\mathbf{x}<1 \\ 0 & \text{otherwise} \end{cases}$$

The expected value of this random variable is

$$E\{g(\mathbf{x})\} = \int_{-\infty}^{\infty} g(x) f_{\mathbf{x}}(x)\,dx = \int_{0}^{1} \frac{x+1}{2}\,2(1-x)\,dx = \frac{2}{3} = \mu_{\mathbf{y}}.$$

This demonstrates the property

$$E\{\,E\{\,\mathbf{y}|\mathbf{x}\,\}\,\} = E\{\mathbf{y}\}.$$

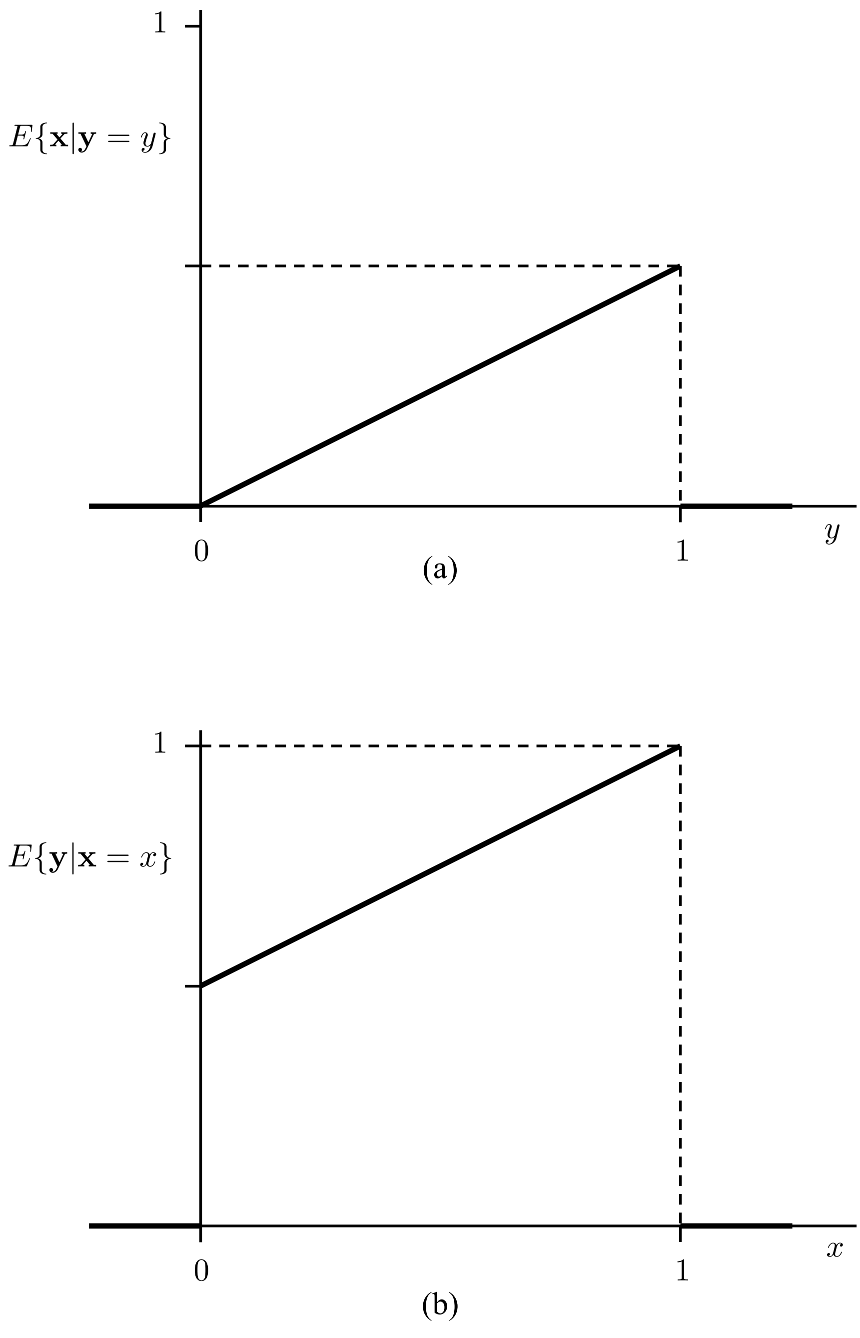


Figure 3: The regression lines (a) $E\{\mathbf{x}|\mathbf{y} = y\}$ and (b) $E\{\mathbf{y}|\mathbf{x} = x\}$.