Complex Random Variables

Chapters 2- 7

Page 75: A *complex* random variable $\mathbf{z}$ is a sum

$$\mathbf{z} = \mathbf{x} + j\mathbf{y}$$

where $\mathbf{x}$ and $\mathbf{y}$ are real random variables.

Page 77: **Note** A complex random variable $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ has no distribution function because the inequality $\mathbf{x} + j\mathbf{y} \le x + jy$ has no meaning.* The statistical properties of $\mathbf{z}$ are specified in terms of the joint distribution of the random variables $\mathbf{x}$ and $\mathbf{y}$ (see Chap. 6).

Note to students: *This statement has no meaning for two reasons. First, following the definition for real random variables, the distribution function of the complex variable $\mathbf{z}$ must be

$$F_{\mathbf{z}}(z) = \mathrm{P}\{\ \mathbf{z} \le z\ \}.$$

Second, anyone from the math department will tell you that complex numbers are not *ordered*. Ordered means that from the natural definition of complex numbers one cannot say one complex number is greater than another. Sure, we can conceive of all sorts of possible measures such as "greater means greater in magnitude" or $z = x + jy$ is greater than $w = u + jv$ is both $x > u$ and $y > v$." But in both cases, we are imposing a definition after the fact.

Page 143: ***Complex random variables:*** If $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ is a complex random variable, then its expected value is by definition

$$E\{\mathbf{z}\} = E\{\mathbf{x}\} + jE\{\mathbf{y}\}$$

Note to students: The variance of a complex random variable is not defined here (in Chap. 5) because joint distributions have not yet been introduced: the variance of a complex random variable is defined in terms of the covariance – so we have to wait.

Section 6-1, Pages 169– 179:

Here, joint distributions and densities are introduced. This is the natural place to define the distribution and density functions of the complex random variable $\mathbf{z} = \mathbf{x} + j\mathbf{y}$. But I cannot find the definitions. So here they are

$$z = x + jy$$

$$\downarrow$$

Distribution function of $\mathbf{z} = \mathbf{x} + j\mathbf{y}$: $F_{\mathbf{z}}(z) = F_{\mathbf{xy}}(x,\mathrm{y}) = \mathrm{P}\{\mathbf{x} \leq \mathrm{x},\ \mathbf{y} \leq y\}$

Density function of $\mathbf{z} = \mathbf{x} + j\mathbf{y}$: $f_{\mathbf{z}}(z) = f_{\mathbf{xy}}(x, y) = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y} F_{\mathbf{xy}}(x, y)$

Example 6-15 examines the (complex to complex) transformation $\mathbf{x} + j\mathbf{y} \to \mathbf{r}e^{j\boldsymbol{\theta}}$

1. From the definition of a complex random variable, $\mathbf{x} + j\mathbf{y}$ is described by the joint density function $f_{\mathbf{xy}}(\mathrm{x},y)$.
2. The complex random variable $\mathbf{r}e^{j\boldsymbol{\theta}}$ can also be described a joint density function. Here the joint density function is $f_{\mathbf{r\theta}}(r,\theta)$.
3. The transformation $\mathbf{x} + j\mathbf{y} \to \mathbf{r}e^{j\boldsymbol{\theta}}$ can be thought of as the 2×2 transformation

$$\mathbf{r} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$$
$$\boldsymbol{\theta} = \tan^{-1}\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$$

This transformation is examined in Example 6-22 (pp. 202-203).

For the special case where $\mathbf{x}$ and $\mathbf{y}$ are independent normal random variables with zero mean and common variance σ^2, the joint density function of $\mathbf{r}$ and $\boldsymbol{\theta}$ is

$$f_{\mathbf{r}\boldsymbol{\theta}}(r, \theta) = \frac{r}{2\pi\sigma^2} \exp\left\{-\frac{r^2}{2\sigma^2}\right\}$$

This joint density function describes the complex random variable $\mathbf{r}e^{j\boldsymbol{\theta}}$.

Page 249: **CORRELATION AND COVARIANCE MATRICES**. The covariance C_{ij} of two real random variables $\mathbf{x}_i$ and $\mathbf{x}_j$ is defined as in (6-163). For complex random variables

$$C_{ij} = E\left\{(\mathbf{x}_i - \eta_i)\left(\mathbf{x}_j^* - \eta_j^*\right)\right\} = E\left\{\mathbf{x}_i \mathbf{x}_j^*\right\} - \eta_i \eta_j^*$$

by definition. The variance of $\mathbf{x}_i$ is given by

$$\sigma_i^2 = E\left\{|\mathbf{x}_i - \eta_i|^2\right\} = E\{|\mathbf{x}_i|^2\} - |E\{\mathbf{x}_i\}|^2$$

COVARIANCE. The covariance C or C_{xy} of two [real] random variables $\mathbf{x}$ and $\mathbf{y}$ is by definition the number

$$C_{xy} = E\{(\mathbf{x} - \eta_x)(\mathbf{y} - \eta_y)\} \tag{6-123}$$

where E{$\mathbf{x}$} = η_x and E{$\mathbf{y}$} = η_y. Expanding the product in (6-123) and using (6-121) we obtain

$$C_{xy} = E\{\mathbf{xy}\} - E\{\mathbf{x}\}E\{\mathbf{y}\}$$

Notes to class:

1. The definition for the covariance of two complex random variables (at the top of this page) is first given in Chapter 7, Sequences of Random Variables. Clearly, the definition could have been given earlier. I do not know why it was not.
2. A sequence of random variables is simply a list of random variables, say $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. That is $\mathbf{x}_i$ and $\mathbf{x}_j$ from the list are two different random variables for $i \neq j$. One could call these two different random variables $\mathbf{x}$ and $\mathbf{y}$. Then the definition would have been the more familiar
$$C_{xy} = E\left\{(\mathbf{x} - \eta_x)(\mathbf{y} - \eta_y)^*\right\} = E\{\mathbf{xy}^*\} - E\{\mathbf{x}\}E\{\mathbf{y}^*\}$$
3. The term $E\{\mathbf{xy}^*\}$ or $E\{\mathbf{x}_i\mathbf{x}_j^*\}$ above is the *correlation*. I am not sure why the definition is not made explicit.