5-3 Mean and Variance

Definitions

- The *expected value* or *mean* of the random variable $\mathbf{x}$ is

$$E\{\mathbf{x}\} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} x f_{\mathbf{x}}(x)dx & \text{continuous RV} \\ \displaystyle\sum_i x_i P(\mathbf{x} = x_i) & \text{discrete RV} \end{cases}$$

- The *conditional mean* of the random variable $\mathbf{x}$ assuming an event M is

$$E\{\mathbf{x}|M\} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} x f_{\mathbf{x}|M}(x|M)dx & \text{continuous RV} \\ \displaystyle\sum_i x_i P(\mathbf{x} = x_i|M) & \text{discrete RV} \end{cases}$$

- The *mean of* $g(\mathbf{x})$ can be computed two ways.

  1. Write $\mathbf{y} = g(\mathbf{x})$ and find the pdf $f_{\mathbf{y}}(y)$ or pmf $P(\mathbf{y} = y_i)$. The mean is

$$E\{g(\mathbf{x})\} = E\{\mathbf{y}\} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} y f_{\mathbf{y}}(y)dy & \text{continuous RV} \\ \displaystyle\sum_i y_i P(\mathbf{y} = y_i) & \text{discrete RV} \end{cases}$$

  2. Apply the *Law of the Unconscious Statistician*

$$E\{g(\mathbf{x})\} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} g(x) f_{\mathbf{x}}(x)dx & \text{continuous RV} \\ \displaystyle\sum_i g(x_i) P(\mathbf{x} = x_i) & \text{discrete RV} \end{cases}$$

- The *variance* of the random variable $\mathbf{x}$ with mean $\mu_{\mathbf{x}} = E\{\mathbf{x}\}$ is

$$E\{(\mathbf{x} - \mu_{\mathbf{x}})^2\} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} (x - \mu_{\mathbf{x}})^2 f_{\mathbf{x}}(x)dx & \text{continuous RV} \\ \displaystyle\sum_i (x_i - \mu_{\mathbf{x}})^2 P(\mathbf{x} = x_i) & \text{discrete RV} \end{cases}$$

5-4 Moments

Definitions

- The n-th *moment* of the random variable $\mathbf{x}$ is

$$m_n = E\{\mathbf{x}^n\} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} x^n f_{\mathbf{x}}(x)dx & \text{continuous RV} \\ \displaystyle\sum_i x_i^n P(\mathbf{x} = x_i) & \text{discrete RV} \end{cases}$$

- The n-th *central moment* of the random variable $\mathbf{x}$ is

$$\mu_n = E\{(\mathbf{x} - m_1)^n\} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} (x - m_1)^n f_{\mathbf{x}}(x)dx & \text{continuous RV} \\ \displaystyle\sum_i (x_i - m_1)^n P(\mathbf{x} = x_i) & \text{discrete RV} \end{cases}$$

- The n-th *absolute moment* of the random variable $\mathbf{x}$ is $E\{|\mathbf{x}|^n\}$.
- The n-th *absolute central moment* of the random variable $\mathbf{x}$ is $E\{|\mathbf{x} - m_1|^n\}$.
- The n-th *generalized moment* of the random variable $\mathbf{x}$ is $E\{(\mathbf{x} - a)^n\}$.
- The n-th *generalized absolute moment* of the random variable $\mathbf{x}$ is $E\{|\mathbf{x} - a|^n\}$.

5-5 Characteristic Functions

Definitions

- The *characteristic function* of the random variable $\mathbf{x}$ is

$$\Phi_{\mathbf{x}}(\omega) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} f_{\mathbf{x}}(x) e^{j\omega x} dx & \text{continuous RV} \\ \displaystyle\sum_i P(\mathbf{x} = x_i) e^{j\omega x_i} & \text{discrete RV} \end{cases}$$

- The inversion formula are

$$f_{\mathbf{x}}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{\mathbf{x}}(\omega) e^{-j\omega x} d\omega \qquad \text{continuous RV}$$

$$P(\mathbf{x} = x_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{\mathbf{x}}(\omega) e^{-j\omega x_i} d\omega \qquad \text{discrete RV}$$

Properties

- From the Law of the Unconscious Statistician, the characteristic function may be defined as an expectation:

$$\Phi_{\mathbf{x}}(\omega) = E\{e^{j\omega \mathbf{x}}\}.$$

- Characteristic functions are used to analyze sums of random variables. Sums of random variables are examined in Chapter 6.