9-3 The Power Spectrum

Definitions

- The *power spectrum* (or *spectral density*) of a WSS random process $\mathbf{x}(t)$, real or complex, is the Fourier transform of its autocorrelation function $R_{\mathbf{xx}}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}(t)\}$:

$$S_{\mathbf{xx}}(\omega) = \int_{-\infty}^{\infty} R_{\mathbf{xx}}(\tau) e^{-j\omega\tau}\, d\tau$$

- From the Fourier inversion formula

$$R_{\mathbf{xx}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\mathbf{xx}}(\omega) e^{j\omega\tau}\, d\omega$$

- The *cross power spectrum* of two random processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is the Fourier transform of their cross correlation $R_{\mathbf{xy}}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{y}^*(t)\}$:

$$S_{\mathbf{xy}}(\omega) = \int_{-\infty}^{\infty} R_{\mathbf{xy}}(\tau) e^{-j\omega\tau}\, d\tau$$

- From the Fourier inversion formula

$$R_{\mathbf{xy}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\mathbf{xy}}(\omega) e^{j\omega\tau}\, d\omega$$

Properties

1. $S_{\mathbf{xx}}(\omega)$ is a real-valued function of ω.
2. If $\mathbf{x}(t)$ is real, then $S_{\mathbf{xx}}(\omega)$ is real and even.
3. $S_{\mathbf{xx}}(\omega) \geq 0$ for all ω.
4. $S_{\mathbf{xy}}(\omega)$ is, in general, complex valued, even when both processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real-valued processes.
5. $S_{\mathbf{xy}}(\omega) = S^*_{\mathbf{yx}}(\omega)$

TABLE 9-1

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \quad \leftrightarrow \quad S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$$

$$\delta(\tau) \leftrightarrow 1 \qquad\qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega-\beta) \qquad \cos(\beta\tau) \leftrightarrow \pi\delta(\omega-\beta) + \pi\delta(\omega+\beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2} \qquad\qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos(\beta\tau) \leftrightarrow \frac{\alpha}{\alpha^2+(\omega-\beta)^2} + \frac{\alpha}{\alpha^2+(\omega+\beta)^2}$$

$$2e^{-\alpha\tau^2}\cos(\beta\tau) \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\left[e^{-(\omega-\beta)^2/4\alpha} + e^{-(\omega+\beta)^2/4\alpha}\right]$$

$$\begin{cases} 1-\dfrac{|\tau|}{T} & |\tau|<T \\ 0 & |\tau|>T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

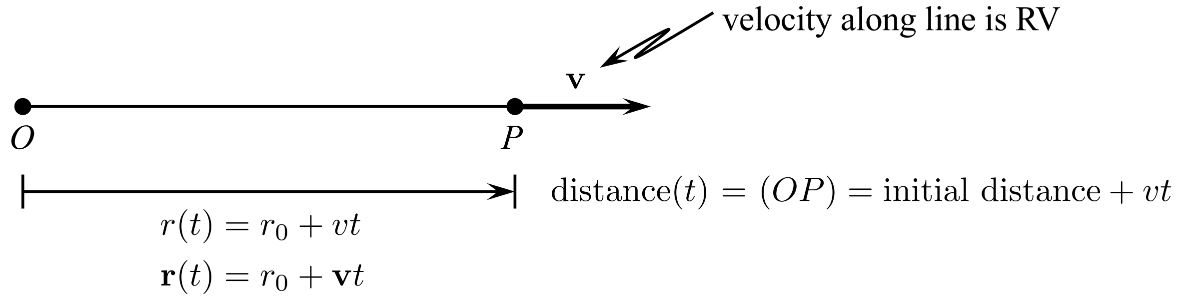
$$\frac{\sin(\sigma\tau)}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega|<\sigma \\ 0 & |\omega|>\sigma \end{cases}$$

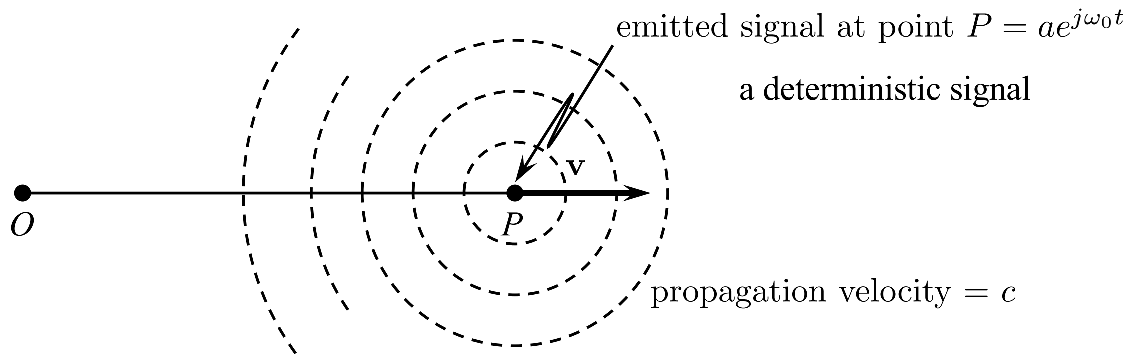
$$e^{-\alpha|\tau|}\sin(\beta|\tau|) \leftrightarrow \frac{\omega+\beta}{\alpha^2+(\omega+\beta)^2} - \frac{\omega-\beta}{\alpha^2+(\omega-\beta)^2}$$

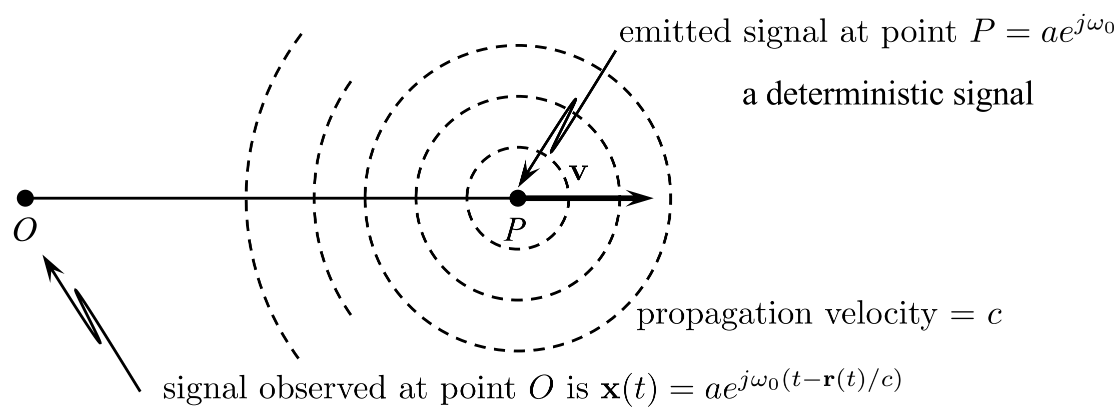
$$|\tau|e^{-\alpha|\tau|} \leftrightarrow 2\frac{\alpha^2-\omega^2}{(\alpha^2+\omega^2)^2}$$

These are valid Fourier transform pairs, but the left-hand sides by themselves are not valid auto-correlation functions. (See Property 6.)

Example 9-24: Doppler Effect







Observed signal is a random process because $\mathbf{r}(t)$ is a random process.
$\mathbf{r}(t)$ is a random process because $\mathbf{v}$ is a random variable.

If $\mathbf{x}(t)$ is WSS, we can ask, "What is the power spectral density of $\mathbf{x}(t)$?"

LTI System with WSS Input

$$\mathbf{x}(t) \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{y}(t)$$

$$\mu_{\mathbf{y}} = \mu_{\mathbf{x}} \int_{-\infty}^{\infty} h(u) du \qquad \mu_{\mathbf{y}} = \mu_{\mathbf{x}} H(0)$$

$$R_{\mathbf{xy}}(\tau) = \int_{-\infty}^{\infty} R_{\mathbf{xx}}(\tau + u) h^*(u) du \qquad S_{\mathbf{xy}}(\omega) = S_{\mathbf{xx}}(\omega) H^*(\omega)$$

$$R_{\mathbf{yy}}(\tau) = \int_{-\infty}^{\infty} R_{\mathbf{xy}}(\tau - u) h(u) du \qquad S_{\mathbf{yy}}(\omega) = S_{\mathbf{xy}}(\omega) H(\omega) = S_{\mathbf{xx}}(\omega) |H(\omega)|^2$$

Example 9-27

$$\mathbf{x}(t) \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{y}(t)$$

(a) $\quad \mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t) \quad \text{all } t \quad \mu_{\mathbf{x}} = 0 \quad R_{\mathbf{xx}}(\tau) = q\delta(\tau)$

$$H(s) = \frac{1}{s+c}$$

$$H(\omega) = \frac{1}{c+j\omega}$$

$$|H(\omega)|^2 = \frac{1}{c^2+\omega^2}$$

$$S_{\mathbf{yy}}(\omega) = S_{\mathbf{xx}}(\omega)|H(\omega)|^2 = \frac{q}{c^2+\omega^2}$$

$$R_{\mathbf{yy}}(\tau) = \frac{q}{2c}e^{-c|\tau|}$$

$$H(s) = \frac{1}{s+c}$$

$$h(t) = e^{-ct}U(t)$$

$$R_{\mathbf{xy}}(\tau) = \begin{cases} qe^{c\tau} & \tau < 0 \\ 0 & \tau > 0 \end{cases}$$

$$R_{\mathbf{yy}}(\tau) = \begin{cases} \displaystyle\int_0^\infty qe^{c(\tau-u)}e^{-cu}du & \tau < 0 \\ \displaystyle\int_\tau^\infty qe^{c(\tau-u)}e^{-cu}du & \tau > 0 \end{cases}$$

$$= \begin{cases} \dfrac{q}{2c}e^{c\tau} & \tau < 0 \\ \dfrac{q}{2c}e^{-c\tau} & \tau > 0 \end{cases}$$

$$= \frac{q}{2c}e^{-c|\tau|}$$

$$\rho(\tau) = \begin{cases} \displaystyle\int_0^\infty e^{-c(u-\tau)}e^{-cu}du & \tau < 0 \\ \displaystyle\int_\tau^\infty e^{-c(u-\tau)}e^{-cu}du & \tau > 0 \end{cases} = \begin{cases} \dfrac{1}{2c}e^{c\tau} & \tau < 0 \\ \dfrac{1}{2c}e^{-c\tau} & \tau > 0 \end{cases} = \frac{1}{2c}e^{-c|\tau|}$$

$$R_{\mathbf{yy}}(\tau) = q\rho(\tau) = \frac{q}{2c}e^{-c|\tau|}$$

$$E\{\mathbf{y}^2(t)\} = R_{\mathbf{yy}}(0) = \frac{q}{2c}$$

Example 9-27

$$\mathbf{x}(t) \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{y}(t)$$

(b) $\mathbf{y}''(t) + b\mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t)$ all t $\mu_{\mathbf{x}} = 0$ $R_{\mathbf{xx}}(\tau) = q\delta(\tau)$

$$H(s) = \frac{1}{s^2 + bs + c}$$

$$H(\omega) = \frac{1}{c - \omega^2 + jb\omega}$$

$$|H(\omega)|^2 = \frac{1}{(c-\omega^2)^2 + b^2\omega^2}$$

$$S_{\mathbf{yy}}(\omega) = S_{\mathbf{xx}}(\omega)|H(\omega)|^2 = \frac{q}{(c-\omega^2)^2 + b^2\omega^2}$$

$\underline{b^2 < 4c}$

$$R_{\mathbf{yy}}(\tau) = \frac{q}{2bc} e^{-\alpha|\tau|}\left(\cos(\beta\tau) + \frac{\alpha}{\beta}\sin(\beta|\tau|)\right) \qquad \alpha = \frac{b}{2} \qquad \alpha^2 + \beta^2 = c$$

$\underline{b^2 = 4c}$

$$R_{\mathbf{yy}}(\tau) = \frac{q}{2bc} e^{-\alpha|\tau|}\left(1 + \alpha|\tau|\right) \qquad \alpha = \frac{b}{2}$$

$\underline{b^2 > 4c}$

$$R_{\mathbf{yy}}(\tau) = \frac{q}{4\gamma bc}\left[(\alpha+\gamma)e^{-(\alpha-\gamma)|\tau|} - (\alpha-\gamma)e^{-(\alpha+\gamma)|\tau|}\right]$$

$$\alpha = \frac{b}{2} \qquad \alpha^2 - \gamma^2 = c$$

In all cases, $E\{\mathbf{y}^2(t)\} = \dfrac{q}{2bc}$.