CENTRAL ACTIONS AND $\mathcal{W}$-CATEGORIES

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Abstract. We prove a categorical analog of Kostant's identification of the center $\mathcal{Z} = \mathcal{Z}(U\mathfrak{g})$ of the enveloping algebra of a reductive Lie algebra with the Whittaker Hecke algebra, i.e., the finite $\mathcal{W}$-algebra. Namely we show that the Whittaker Hecke category, or $\mathcal{W}$-category, associated to a reductive group is symmetric monoidal (answering a question of Arinkin, Drinfeld and Gaitsgory), and acts centrally on the monoidal category of Harish Chandra bimodules HC_G (or equivalently of $\mathcal{D}$-modules on G), lifting the action of $\mathcal{Z}$ (quantum characteristic polynomial map). This action provides a notion of Langlands parameters for categorical representations of G, a categorical integration of quantum Hamiltonian systems arising from Harish-Chandra Laplacians (the action of $\mathcal{Z}$), and a new commutative symmetry of homology of character varieties of surfaces.

We deduce the result from a general symmetry principle for convolution algebras: the category of modules for a convolution algebra carries a symmetric monoidal structure, and acts centrally on the convolution category of sheaves. In particular modules for the nil-Hecke algebra for any Kac-Moody algebra act centrally on the corresponding Hecke category. We apply this principle to the spherical quotient of the affine Grassmannian for the Langlands dual group $G^\vee$, where the renormalized Satake theorem of Bezrukavnikov-Finkelberg identifies the respective categories with the Whittaker Hecke category and Harish Chandra bimodules.

1. Introduction

Theorem 1.1 (Informal). *Let X denote a stack and $\mathcal{G} \circlearrowright X$ a groupoid acting on X. Let $H = (\omega(\mathcal{G}), *)$ denote the associated convolution algebra, considered as a monad on $\mathcal{C} = Shv(X)$. Let $\mathcal{H} = (Shv(\mathcal{G}), *)$ denote the associated convolution category, equipped with the diagonal action of $(Shv(X), \otimes)$. Then there is a symmetric monoidal structure on $\mathcal{W} = \mathbf{Mod}_H$ compatible with the forgetful functor to $\mathcal{C}$, and the diagonal action $\mathcal{C} \to \mathcal{H}$ lifts to a central action of $\mathcal{W} \to \mathcal{Z}(\mathcal{H})$ with a monoidal left inverse:*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "E_2"] \arrow[d, "E_\infty"'] & \mathcal{Z}(\mathcal{H}) \arrow[l, bend right, "E_1"'] \arrow[d, "E_1"] \\ \mathcal{C} \arrow[r, "E_1"'] & \mathcal{H} \end{tikzcd}$$

1.1. **Toy examples.** Let us first illustrate the result with two toy examples.

- Let $\pi : X \to Y$ denote a map of finite sets, and $\mathcal{G} = X \times_Y X$. In this case the convolution algebra $(H = \mathbb{C}[\mathcal{G}], *)$ is the algebra of $|X|$ by $|X|$ block-diagonal matrices (with blocks labeled by Y), which is Morita equivalent to the commutative algebra $\mathbb{C}[Y]$. We also consider the convolution category $(\mathcal{H} = Vect(X \times_Y X), *)$. In this case the inclusion of block-scalar matrices $Vect(Y) \hookrightarrow Vect(X \times_Y X)$ identifies

$$\mathbf{Mod}_H \simeq Vect(Y) \xrightarrow{\sim} \mathcal{Z}(Vect(X \times_Y X))$$

with the Drinfeld center of $(Vect(X \times_Y X), *)$, categorifying the familiar identification of block-scalar matrices $\mathbb{C}[Y]$ as the center of block-diagonal matrices $\mathbb{C}[X \times_Y X]$.

• Let G denote a finite group, and $X = pt \to Y = BG$, so that $G \simeq X \times_Y X$. In this case the convolution algebra $H = (\mathbb{C}[G], *)$ is the group algebra, and $\mathbf{Mod}_H = Rep(G)$ is the symmetric monoidal category of representations. The Drinfeld center of the monoidal category $(Vect(G), *)$ is now the braided tensor category $Vect(G/G)$, which contains $Rep(G) \simeq Vect(pt/G)$ as the tensor subcategory of equivariant vector bundles supported on the identity. The latter is in fact a Lagrangian subcategory of $Vect(G/G)$ in the sense of [DGNO]. We expect our general construction provides (derived analogues of) Lagrangian subcategories as well. The action of $Vect(G)$ on a $Vect(pt) = Vect$ induces an action of its center

$$\mathcal{Z}(Vect(G)) = Vect(G/G) \longrightarrow End_{Vect(G)}(Vect) \simeq Rep(G)$$

which provides the desired left inverse.

1.2. **Kac-Moody groups.** Let $\underline{G}$ denote a Kac-Moody group, with Borel and Cartan subgroups $\underline{B}$, $\underline{H}$ and Cartan Lie algebra $\mathfrak{h}$. The flag variety $\underline{G}/\underline{B}$ is an ind-projective ind-scheme of ind-finite type. We let $\mathcal{G} = \underline{B}\backslash\underline{G}/\underline{B}$ denote the corresponding "Hecke" groupoid acting on $pt/\underline{B}$. Let $H_{\underline{G}} = H_*(\underline{B}\backslash\underline{G}/\underline{B})$ denote the nil-Hecke algebra associated to $\underline{G}$, and $\mathcal{H}_{\underline{G}} = \breve{\mathcal{D}}_{hol}(\underline{B}\backslash\underline{G}/\underline{B})$ the Iwahori-Hecke category.

Theorem 1.2. *There is a natural symmetric monoidal structure on modules $\mathbf{Mod}_{H_{\underline{G}}}$ for the nil-Hecke algebra of $\underline{G}$, compatible with the forgetful functor to $\mathbf{Mod}_{H^*_{\underline{B}}(pt)=H^*_{\underline{H}}(pt)} \simeq \mathbb{C}[\mathfrak{h}^*]$, and the action $\mathbb{C}[\mathfrak{h}^*] \to \mathcal{H}_{\underline{G}}$ lifts to a central action $\mathbf{Mod}_{H_{\underline{G}}} \to \mathcal{Z}(\mathcal{H}_{\underline{G}})$ with an E_2 section:*

$$\begin{tikzcd}
\mathbf{Mod}_{H_{\underline{G}}} \arrow[r] \arrow[d] & \mathcal{Z}(\mathcal{H}_{\underline{G}}) \arrow[d] \arrow[l, bend right] \\
\mathbb{C}[\mathfrak{h}^*] \arrow[r] & \mathcal{H}_{\underline{G}}
\end{tikzcd}$$

Example 1.3. G reductive, $X = pt/B$, $Y = pt/G$, $\mathcal{G} = B\backslash G/B$. In this case $H = C_*(\mathcal{G})$ is the nil-Hecke algebra, acting on $\breve{\mathcal{D}}_{hol}(X) \simeq \mathbf{Mod}_{C*(BT)}$ by Demazure operators, with the category of H-modules identified with $\breve{\mathcal{D}}_{hol}(Y) = \mathbf{Mod}_{C*(pt/G)}$. The result is the linearity of the finite Hecke category $\mathcal{H} = \breve{\mathcal{D}}_{hol}(B\backslash G/B)$ over the G-equivariant cohomology ring.

1.3. **Coxeter groups.** Let W denote a Coxeter group and $\mathfrak{h}$ its reflection representation. For $w \in W$ we let $\Gamma_w \subset \mathfrak{h} \times \mathfrak{h}$ denote the graph of the corresponding reflection. Let

$$\Gamma_W = \coprod_{w \in W} \Gamma_w.$$

Then $\mathcal{G} = \Gamma_W$ is an ind-proper groupoid acting on the scheme $\mathfrak{h}$.

Let $H_W = \Gamma(\omega_{\mathcal{G}})$ denote the corresponding (ind-coherent) Hecke algebra. It is a variant of the nil-Hecke algebra [KK] associated to W. David BZ: Do we know if H_W is the nil-Hecke algebra outside of the finite case?? Let $\mathcal{H}_W = \mathcal{Q}^!(\Gamma_W)$ denote the ind-coherent Hecke category. It is closely related to the category of Soergel bimodules associated to W.

Theorem 1.4. *There is a natural symmetric monoidal structure on modules $\mathbf{Mod}_{H_W}$ for the Coxeter Hecke algebra compatible with the forgetful functor to $\mathbb{C}[\mathfrak{h}^*]$, and the action $\mathbb{C}[\mathfrak{h}^*] \to \mathcal{H}_W$ on the Coxeter Hecke category lifts to a central action $\mathbf{Mod}_{H_W} \to \mathcal{Z}(\mathcal{H}_W)$ with an E_2 section:*

$$\begin{tikzcd}
\mathbf{Mod}_{H_W} \arrow[r] \arrow[d] & \mathcal{Z}(\mathcal{H}_W) \arrow[l, bend right] \arrow[d] \\
\mathbb{C}[\mathfrak{h}^*] \arrow[r] & \mathcal{H}_W
\end{tikzcd}$$

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2. Sheaf Theory Formalism

We will study monoidal properties of categories of sheaves on stacks. The geometric spaces that appear are ind-algebraic stacks and groupoids (Section 2.0.1). We require a theory of sheaves that attaches to a stack X a presentable DG category $Shv(X)$ with continuous pull-back and pushforward functors $p_*, p^!$ for maps $p : X \to Y$ of ind-finite type, satisfying base change and an adjunction $(p_*, p^!)$ in the case that p is ind-proper.

Two important examples of such a theory of sheaves, developed in [GR3], are the theory of ind-coherent sheaves $IndCoh(X)$ and the theory of $\mathcal{D}$-modules $\mathcal{D}(X)$. Their properties are summarized in the following:

Theorem 2.1. *[GR3, Theorem III.3.5.4.3, III.3.6.3] There is a uniquely defined right-lax symmetric monoidal functor $IndCoh$ from the $(\infty, 2)$-category whose objects are* laft *prestacks, morphisms are correspondences with vertical arrow ind-inf-schematic, and 2-morphisms are ind-proper and ind-inf-schematic, to the $(\infty, 2)$ category of DG categories with continuous morphisms.*

The theorem encodes a tremendous amount of structure. Let us highlight some salient features useful in practice. The theorem assigns a symmetric monoidal dg category $IndCoh(X)$ to any reasonable (locally almost of finite type) stack. The symmetric monoidal structure, the !-tensor product, is induced by !-pullback along diagonal maps. For an arbitrary morphism $p : X \to Y$ there is a continuous symmetric monoidal pullback functor $p^! : IndCoh(Y) \to IndCoh(X)$, while for p schematic or ind-schematic there is a continuous pushforward $p_* : IndCoh(X) \to IndCoh(Y)$, which satisfies base change with respect to !-pullbacks. Moreover for p ind-proper, $(p_*, p^!)$ form an adjoint pair. Furthermore, the formalism of *inf-schemes* greatly extends the validity of the construction. In particular the same formal properties holds for the theory of $\mathcal{D}$-modules, defined by the assignment $X \mapsto \mathcal{D}(X) = IndCoh(X_{dR})$, ind-coherent sheaves on the de Rham space of X.

For our applications we require a minor variation, the theory of ind-holonomic $\mathcal{D}$-modules $\breve{\mathcal{D}}_{hol}(X)$, the main instance of which is the renormalized Satake category $\breve{\mathcal{D}}_{hol}(\underline{Gr})$ studied in [AG] (and, implicitly, [BeF]). We will explain the appropriate modifications of the formalism of [GR3] needed to establish the minimal functoriality of ind-holonomic $\mathcal{D}$-modules we will require.

2.0.1. *Geometric context.* We adopt the following geometric conventions: all schemes will be of almost finite type, and all algebraic stacks will be *laft* QCA stacks, as studied in particular in [DG1]. In other words, an algebraic stack X is a prestack whose diagonal is affine and which admits a smooth and surjective map from an affine scheme of almost finite type.

By an *ind-algebraic stack* we refer to a prestack X which is equivalent to a filtered colimit $X = \lim_{\rightarrow} X_i$ of algebraic stacks under closed embeddings.

In our applications X will be realized as the quotient of an ind-scheme of ind-finite type by an affine algebraic group. The main example of interest is the equivariant affine Grassmannian

$$X = \underline{Gr} = G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})$$

of a reductive group G.

2.1. **Motivating Ind-Holonomic $\mathcal{D}$-modules.** First recall (see e.g. [DG1]) that for a scheme of finite type we have an equivalence $\mathcal{D}(X) \simeq \operatorname{Ind}\mathcal{D}_{coh}(X)$, and that we have a full stable subcategory $\mathcal{D}_{coh,hol} \subset \mathcal{D}_{coh}(X)$. Thus we have a fully faithful embedding

$$\breve{\mathcal{D}}_{hol}(X) := \operatorname{Ind}\mathcal{D}_{coh,hol}(X) \subset \mathcal{D}(X)$$

of ind-holonomic $\mathcal{D}$-modules into all $\mathcal{D}$-modules. Holonomic $\mathcal{D}$-modules are preserved by !-pullback and $*$-pushforward for finite type morphisms, and carry a symmetric monoidal structure through !-tensor product for which !-pullback is naturally symmetric monoidal.

This picture persists for X an ind-scheme of ind-finite type $X = \lim_{\rightarrow} X_i$, for example the affine Grassmannian $Gr = G(\mathcal{K})/G(\mathcal{O})$. The $(i_*, i^!)$ adjunction for a closed embeddings provides the alternative descriptions

$$\mathcal{D}(X) \simeq \lim_{\leftarrow,(-)^!} \mathcal{D}(X_i) \simeq \lim_{\rightarrow,(-)_*} \mathcal{D}(X_i).$$

As a result (by a general lemma of [DG2]) $\mathcal{D}(X)$ is compactly generated by coherent $\mathcal{D}$-modules, which by definition are the pushforwards of coherent $\mathcal{D}$-modules on the finite type closed subschemes X_i, and include the similarly defined holonomic $\mathcal{D}$-modules. Note that with this definition the pullback of a holonomic $\mathcal{D}$-module by an ind-finite type morphism (for example, the dualizing complex of an ind-scheme) is ind-holonomic but not necessarily holonomic (i.e. compact).

For X an algebraic stack, the situation (as studied in detail in [DG1]) changes: coherent (and in particular holonomic) $\mathcal{D}$-modules, defined by descent using a smooth atlas, are no longer compact in general. The category $\mathcal{D}(X)$ is compactly generated by *safe* objects, which are coherent objects satisfying a restriction on the action of stabilizers (in the case of quotient stacks). One can thus measure the lack of safety of X by the difference between $\mathcal{D}(X)$ and the category $\breve{\mathcal{D}}(X) := \operatorname{Ind}\mathcal{D}_{coh}(X)$ of *ind-coherent* or *renormalized* $\mathcal{D}$-modules. This is analogous to the difference between quasicoherent and ind-coherent sheaves on a derived stack measuring its singularities, with safe (respectively, coherent) $\mathcal{D}$-modules taking on the role of perfect (respectively, coherent) complexes of $\mathcal{O}$-modules.

Example 2.2. Suppose $X = pt/G$ is the classifying stack of a reductive group. Let $\Lambda = C_*G \simeq \mathbb{C}[\mathfrak{g}^*[-1]]^G$ and $S = C^*X \simeq \mathbb{C}[\mathfrak{g}[2]]^G$ be the corresponding Koszul dual exterior and symmetric algebras. Then

$$\mathcal{D}(X) \simeq \mathbf{Mod}_\Lambda \simeq QC(\mathfrak{g}[2]//G)_0$$

is the completion of sheaves on the graded version of the adjoint quotient $\mathfrak{g}//G \simeq \mathfrak{h}//W$ at the origin. On the other hand we have

$$\breve{\mathcal{D}}_{hol}(X) = \breve{\mathcal{D}}_{hol}(X) \simeq \operatorname{Ind}(Coh\Lambda) \simeq \mathbf{Mod}_S \simeq QC(\mathfrak{g}[2]//G)$$

is the "anticompleted" version of the same category.

This can also be described in terms of the corresponding homotopy type X_{top} (as a constant prestack) and $\mathfrak{X} = \operatorname{Spec} C^*(X)$ the corresponding coaffine stack. We then have equivalences

$$\mathcal{D}(X) \simeq QC(X_{top}) \simeq QC(\mathfrak{X}).$$

Sam: Is QC of the coaffine stack $\mathfrak{X} = \operatorname{Spec}(S)$ really Λ-mod, not S-mod? David BZ: I'm pretty sure the answer is yes.. $\mathbf{Mod}_\Lambda$ is a t-complete category, like QC of any stack, while its anticompletion is shown by Jacob to be modules for the corresponding cosimplicial ring. Though maybe bringing in coaffine stacks is just a distraction in the current version? On the other hand we have the following description of renormalized sheaves:

$$\breve{\mathcal{D}}_{hol}(X) \simeq \mathbf{Mod}_{C^*(X)}.$$

In particular $\mathcal{D}(X)$ is the completion of $\breve{\mathcal{D}}_{hol}(X)$.

We will be interested in a combined setting of ind-algebraic stacks. In this setting the category $\breve{\mathcal{D}}_{hol}(X)$ (defined formally in the next section) is identified with the Ind-category of (coherent) holonomic $\mathcal{D}$-modules, which are pushforwards of holonomic $\mathcal{D}$-modules on algebraic substacks. Thus ind-holonomic $\mathcal{D}$-modules form a full subcategory of *ind-coherent* (or renormalized) $\mathcal{D}$-modules $\breve{\mathcal{D}}_{hol}(X) = \operatorname{Ind} \mathcal{D}_{coh}(X)$.

Example 2.3. Our main motivating example is the equivariant affine Grassmannian $X = \underline{Gr}$. The *renormalized Satake category* $\breve{\mathcal{D}}_{hol}(\underline{Gr})$ of [AG] is a variant of the usual Satake category $\mathcal{D}(\underline{Gr})$ which appears (implicitly) in the derived Satake correspondence of [BeF]. It can be defined as the ind-category $\operatorname{Ind}(Shv_{lc}(\underline{Gr}))$ of the category of *locally compact* sheaves on $\underline{Gr}$, i.e., equivariant sheaves on the affine Grassmannian for which the underlying sheaves are constructible (hence compact). In the language of $\mathcal{D}$-modules, it is the Ind-category of the category of holonomic $\mathcal{D}$-modules on $\underline{Gr}$ - note that (as in the previous example) all coherent $\mathcal{D}$-modules on $\underline{Gr}$ are holonomic, in fact regular holonomic, hence identified with constructible sheaves. The renormalized Satake theorem [BeF,AG] is an equivalence of monoidal categories

$$\breve{\mathcal{D}}_{hol}(\underline{Gr}) = \breve{\mathcal{D}}_{hol}(\underline{Gr}) \simeq IndCoh(\mathfrak{g}^\vee[2]/G^\vee).$$

Dropping the renormalization of $\mathcal{D}$-modules corresponds to imposing finiteness conditions on the right hand side.

Remark 2.4 (Ind-constructible sheaves)**.** The notion of ind-holonomic $\mathcal{D}$-modules has a natural analog in the setting of l-adic sheaves or constructible sheaves in the analytic topology. Namely on a scheme X the compact objects in $Shv(X)$ are the constructible sheaves, but this is no longer the case on a stack. A *locally compact* sheaf on a stack X is a sheaf whose stalks are perfect complexes – i.e., whose pullback under any map $pt \to X$ is compact. We denote $Shv(X)_{lc} \subset Shv(X)$ the full subcategory of locally compact sheaves, and define the category $\breve{Shv}(X)$ of renormalized, or ind-constructible, sheaves as $\operatorname{Ind} Shv(X)_{lc}$. It has $Shv(X)$ as a colocalization:

$$\Xi : Shv(X) \rightleftarrows \breve{Shv}(X) : \Psi$$

For example for $X = Y/G$ a quotient stack, $\breve{Shv}(X)$ can be identified with the Ind category of G-equivariant constructible complexes on Y in the sense of Bernstein–Lunts [**?**].

The !-tensor structure on $Shv(X)$ respects locally compact objects, hence extends by continuity to define a symmetric monoidal structure on $\breve{Shv}(X)$, for which the functors Ξ, Ψ upgrade to symmetric monoidal functors.

When X is a finite orbit stack (for example, a quotient stack Y/G where G acts on Y with finitely many orbits) or an ind-finite orbit stack such as $\underline{Gr}$, every coherent complex on X is regular holonomic. Thus, via the Riemann-Hilbert correspondence, $\breve{\mathcal{D}}_{hol}(X) = \breve{\mathcal{D}}_{hol}(X) \simeq S\breve{h}v(X)$.

2.2. **Formalism of ind-holonomic $\mathcal{D}$-modules.** Recall [GR1, GR3, Be, R] the construction of the contravariant functor of $\mathcal{D}$-modules $\mathcal{D}^!$ on ind-schemes. Namely we start with the functor

$$\mathcal{D}^! : AffSch^{f.t.,op} \to DGCat$$

of $\mathcal{D}$-modules with !-pullback on schemes of finite type as constructed e.g. in [GR1, GR3]. We then right Kan extend to ind-schemes of ind-finite type (or more generally to *laft* prestacks).

Definition 2.5. The right-lax symmetric monoidal functor $\breve{\mathcal{D}}^!_{hol} : QCA^{op} \to DGCat$ is defined as the (symmetric monoidal) ind-construction

$$QCA^{op} \xrightarrow{\mathcal{D}^!_{coh,hol}} DGCat^{sm} \xrightarrow{Ind} DGCat$$

applied to the subfunctor of $\mathcal{D}^!$ defined by coherent holonomic $\mathcal{D}$-modules.

Lemma 2.6. *For $p : X \to Y$ a finite type morphism of QCA stacks, we have continuous pullback and pushforward functors*

$$p_* : \breve{\mathcal{D}}_{hol}(X) \rightleftarrows \breve{\mathcal{D}}_{hol}(Y) : p^!$$

satisfying base change. Moreover for $p : X \to Y$ a proper morphism, $(p_*, p^!)$ form an adjoint pair.

Proof. Pullback and pushforward of holonomic $\mathcal{D}$-modules on stacks under finite type morphisms remain holonomic. Hence the functors

$$p_* : \mathcal{D}_{coh,hol}(X) \rightleftarrows \mathcal{D}_{coh,hol}(Y) : p^!$$

extend by continuity to the ind-categories. The property of base-change can likewise be checked on the compact objects. $\square$

If we need to consider schemes beyond finite type, we first perform a left Kan extension to extend from affine schemes to all affines and then right Kan extend extend $\mathcal{D}^!$ to all ind-schemes [R]. Another formulation [Be] is to consider schemes of pro-finite type or simply *pro-schemes*, schemes that can be written as filtered limits of schemes of finite type along affine smooth surjective maps. Again $\mathcal{D}^!$ is extended from finite type schemes to pro-schemes as a left Kan extension, and then to ind-pro-schemes by a right Kan extension.

We are interested in objects such as the equivariant affine Grassmannian $\underline{Gr}$, which is nearly but not quite an ind-finite type algebraic stack. Namely $\underline{Gr}$ is the inductive limit (under closed embeddings) of stacks of the form X/K where X is a scheme of finite type and K ($G(\mathcal{O})$ in our setting) is an algebraic group acting on X through a finite type quotient $K_i = K/K^i$ with pro-unipotent kernel. Thus

$$X/K = \varprojlim X/K_i$$

is a projective limit of finite type algebraic stacks under morphisms which are gerbes for unipotent group schemes. In particular the category of $\mathcal{D}$-modules on X_i/K is equivalent to that of any of the finite type quotients X/K_i:

Thus we make the following more modest variant of the constructions in [Be, R]:

Definition 2.7. (1) By a stack nearly of finite type we refer to an algebraic stack expressible as a projective limit of QCA stacks under morphisms which are gerbes for unipotent group schemes.
(2) By an ind-nearly finite type stack, or simply *ind-stack*, we denote a prestack equivalent to an inductive limit of stacks nearly of finite type under closed embeddings. The symmetric monoidal category of ind-stacks is denoted $IndSt$.

Definition 2.8. The functor $\breve{\mathcal{D}}^!_{hol} : IndSt^{op} \to DGCat$ on ind-stacks is defined by first left Kan extending $\breve{\mathcal{D}}^!_{hol}$ from QCA stacks to stacks nearly of finite type, and then right Kan extending to ind-nearly finite type stacks.

Proposition 2.9. *The functor $\breve{\mathcal{D}}^!_{hol}$ admits a right-lax symmetric monoidal structure extending that previously defined on QCA stacks.*

Lemma 2.10. (1) *For $\mathcal{X} = \lim_{\leftarrow} X_n$ an inverse limit of stacks of finite type under unipotent gerbes, the functor*

$$\lim_{\leftarrow} \breve{\mathcal{D}}_{hol}(X_n) \to \breve{\mathcal{D}}_{hol}(X_i)$$

is an equivalence for any i.
(2) *The assertions of Lemma 2.6 extend to morphisms of nearly finite type stacks.*

To calculate the abstractly defined functor $\breve{\mathcal{D}}_{hol}$ on ind-stacks, we follow the strategy of [GR3] (see also [GR2, Section 2]):

Lemma 2.11. *For X an ind-stack, expressed as a filtered colimit of closed embeddings $i_n : X_n \hookrightarrow X$ with X_n nearly of finite type, we have identifications*

$$\breve{\mathcal{D}}_{hol}(X) \simeq \lim_{\leftarrow, i_n^!} \breve{\mathcal{D}}_{hol}(X_n) \simeq \lim_{\rightarrow, i_{n,*}} \breve{\mathcal{D}}_{hol}(X_n).$$

Proof. The functor $\breve{\mathcal{D}}_{hol}$ takes colimits in $IndSt$ to limits in $DGCat$. Hence for an ind-stack $X = \lim_{\leftarrow, i_n} X_n$ written as a colimit of nearly finite type stacks under closed embeddings, we have an identification $\breve{\mathcal{D}}_{hol}(X) \simeq \lim_{\rightarrow, i_n^!} \breve{\mathcal{D}}_{hol}(X_n)$. Since the X_n are nearly finite type stacks and i_n are proper morphisms, we may apply proper adjunction to further identify the limit over the pullbacks with the colimit over their left adjoints, $\breve{\mathcal{D}}_{hol}(X) \simeq \lim_{\leftarrow, i_{n*}} \breve{\mathcal{D}}_{hol}(X_n)$ as desired. □

Proposition 2.12. *For $p : X \to Y$ an ind-finite type morphism in $IndSch$, we have continuous pushforward and pullback functors*

$$p_* : \breve{\mathcal{D}}_{hol}(X) \rightleftarrows \breve{\mathcal{D}}_{hol}(Y) : p^!$$

satisfying base change. For $p : X \to Y$ ind-proper, $(p_*, p^!)$ form an adjoint pair.

Proof. Let us write Y as the filtered colimit of closed embeddings of nearly finite type substacks $t_n : Y_n \hookrightarrow Y$, and $s_n : X_n = X \times_Y Y_n \hookrightarrow X$. Then by hypothesis we can further decompose X_n as the colimit of substacks $i_{m,n} : X_{m,n} \hookrightarrow X_n$ with $p_{m,n} : X_{m,n} \to Y_n$ finite type.

A holonomic $\mathcal{D}$-modules $\mathcal{F}$ on X can be represented as the pushforward of a holonomic $\mathcal{D}$-module $\mathcal{F}_{m,n}$ on some $X_{m,n}$. Hence $p_*\mathcal{F} = p_{m,n*}\mathcal{F}_{m,n}$ is holonomic. Thus pushforward on all $\mathcal{D}$-modules restricts to a functor

$$p_* : \mathcal{D}_{coh,hol}(X) \to \mathcal{D}_{coh,hol}(Y)$$

which thus extends by continuity to the ind-categories $\breve{\mathcal{D}}_{hol}$.

Pullback defines a functor

$$p^!_{m,n} : \mathcal{D}_{coh,hol}(Y_n) \to \mathcal{D}_{coh,hol}(X_{m,n}),$$

and thus passing to ind-categories by continuity

$$p^!_{m,n} : \breve{\mathcal{D}}_{hol}(Y_n) \to \breve{\mathcal{D}}_{hol}(X_{m,n}).$$

By Lemma 2.11, these functors assemble to a continuous functor to the inverse limit category and on to the target,

$$\breve{\mathcal{D}}_{hol}(Y_n) \xrightarrow{p^!_n} \breve{\mathcal{D}}_{hol}(X_n) \xrightarrow{s_{n,*}} \breve{\mathcal{D}}_{hol}(X)$$

Finally by (finite type) base change the functors $s_{n,*}p^!_n \simeq p^! t_{n,*}$ assemble to a functor from the direct limit category

$$\varinjlim \breve{\mathcal{D}}_{hol}(Y_n) = \breve{\mathcal{D}}_{hol}(Y)$$

to $\breve{\mathcal{D}}_{hol}(X)$. The resulting functors inherit the base change property from their finite type constituents.

□

Remark 2.13 (Bivariant functoriality)**.** The key 2-categorical extension theorem of Gaitsgory-Rozenblyum, [GR3, Theorem V.1.3.2.2], allows one to define functors out of correspondence 2-categories given 1-categorical data, namely a functor (in our case $\breve{\mathcal{D}}^!_{hol}$) satisfying an adjunction and base change property for a particular class of morphisms (in our case ind-proper morphisms). Thus we find that the functor $\breve{\mathcal{D}}^!_{hol} : IndSt^{op} \to DGCat$ extends to a functor of $(\infty, 2)$-categories

$$\breve{\mathcal{D}}_{hol} : Corr^{ind-prop}_{ind-f.t,ind-prop}(IndSt) \to DGCat^{(\infty,2)}.$$

3. Hecke algebras and Hecke categories

In this section we describe a general formalism for constructing symmetric monoidal categories acting centrally on convolution categories. We work in the setting of ind-holonomic $\mathcal{D}$-modules on ind-stacks described above, since our main example is the renormalized Satake category $\breve{\mathcal{D}}_{hol}(\underline{Gr})$ and more generally Hecke categories for Kac-Moody groups $\breve{\mathcal{D}}_{hol}(\underline{P}\backslash\underline{G}/\underline{P})$. However the discussion of this section works identically when applied to the sheaf theories of ind-coherent sheaves $QC^!$ or $\mathcal{D}$-modules $\mathcal{D}$ when restricted to *laft* prestacks, as in [GR3].

3.1. **Groupoids.**

Definition 3.1. By an *ind-proper groupoid* we refer to a groupoid object $\mathcal{G} \circlearrowright X$ in ind-stacks, with ind-proper source and target maps $\pi_1, \pi_2 : \mathcal{G} \to X$.

More precisely, the groupoid object is given by a simplicial object $\mathcal{G}_\bullet$ satisfying a Segal condition resulting in an identification of the simplices with iterated fiber products:

$$\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \mathcal{G} \times_X \mathcal{G} \times_X \mathcal{G} \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \mathcal{G} \times_X \mathcal{G} \rightrightarrows \mathcal{G} \rightrightarrows X \tag{1}$$

See [GR3, Sections II.2.5.1, III.3.6.3] for a discussion of ind-proper groupoid objects, and more generally Segal (or monoid) objects (to which all of our constructions apply equally). David BZ: Not sure about what level of generality to take here - we never use the inverse map of the groupoid, so things apply equally well to monoid / category / Segal objects instead of groupoids, which is the generality that [GR3] take for convolution constructions. I have been sticking to groupoid since the word is more familiar and can say everything works in general - and also can speak of the mythical quotient Y as a crutch then..

It will be psychologically convenient (but technically irrelevant) to think in terms of the (potentially very poorly behaved) quotient prestack $Y = |\mathcal{G}_\bullet| = X/\mathcal{G}$, so that $\mathcal{G}_\bullet$ is identified with the Čech simplicial object $\{X \times_Y X \times_Y \cdots \times_Y X\}$. We denote $i = (\pi_1, \pi_2) : \mathcal{G} \to X \times X$.

Our main example of an ind-proper groupoid will be the equivariant Grassmannian $\mathcal{G} = \underline{Gr}$ acting on $X = pt/G(\mathcal{O})$, i.e., the Čech construction for the ind-proper, ind-schematic morphism $X = pt/G(\mathcal{O}) \to Y = pt/G(\mathcal{K})$.

3.2. **Hecke algebras.** We start with an ind-stack X and the corresponding base category $\mathcal{C} = \breve{\mathcal{D}}_{hol}(X)$ of sheaves on X.

Let $\mathcal{G} \circlearrowright X$ denote an ind-proper groupoid as above, and $\pi_1, \pi_2 : \mathcal{G} \to X$ the ind-proper source and target maps (with $i = (\pi_1, \pi_2) : \mathcal{G} \to X \times X$), $\delta : X \to \mathcal{G}$ the diagonal and $\pi : X \to Y = X/\mathcal{G}$ the quotient stack, so $\mathcal{G} \simeq X \times_Y X$. Note that Y may not be well behaved, so we avoid working directly with sheaves on Y.

The groupoid $\mathcal{G}$ defines a monad acting on $\mathcal{C} = \breve{\mathcal{D}}_{hol}(X)$ following the general mechanism discussed in [GR3, II.2.5.1, V.3.4] which we call the Hecke algebra $\underline{H}$. The Hecke algebra is an algebra object structure on the functor $\pi_{2,*}\pi_1^! \simeq p^! p_* \in End(\mathcal{C})$.

Definition 3.2. The $\mathcal{W}$-category associated to the groupoid $\mathcal{G}$ is the category $\mathcal{W} = \mathbf{Mod}_{\underline{H}}$ of $\underline{H}$-modules in $\mathcal{C} = \breve{\mathcal{D}}_{hol}(X)$.

Since Diagram 1 is a diagram of ind-stacks and ind-finite type maps, we can pass to $\breve{\mathcal{D}}_{hol}$ and !-pullbacks to find the cosimplicial symmetric monoidal category $\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)$:

$$\cdots \Lleftarrow \breve{\mathcal{D}}_{hol}(\mathcal{G} \times_X \mathcal{G} \times_X \mathcal{G}) \Lleftarrow \breve{\mathcal{D}}_{hol}(\mathcal{G} \times_X \mathcal{G}) \Lleftarrow \breve{\mathcal{D}}_{hol}(\mathcal{G}) \Longleftarrow \breve{\mathcal{D}}_{hol}(X)$$

Definition 3.3. The symmetric monoidal category $\breve{\mathcal{D}}_{hol}(X)^{\mathcal{G}}$ of $\mathcal{G}$-equivariant sheaves on X is the totalization $Tot(\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet))$.

Proposition 3.4. *The cosimplicial symmetric monoidal category $\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)$ satisfies the monadic Beck-Chevalley conditions. Moreover the associated monad on $\breve{\mathcal{D}}_{hol}(X)$ is identified with the Hecke algebra $\underline{H}$ as an algebra in $End(\breve{\mathcal{D}}_{hol}(X))$. Thus we have an identification $\mathcal{W} \simeq \breve{\mathcal{D}}_{hol}(X)^{\mathcal{G}}$, and hence a symmetric monoidal structure on the $\mathcal{W}$-category for which the forgetful functor $\mathcal{W} \to \mathcal{C}$ is symmetric monoidal.*

David BZ: needs proof - should just be reminding that base change holds..

Remark 3.5 (Tensor structure on $\mathcal{W}$)**.** One can explain the symmetric monoidal structure on $\mathcal{W} = \mathbf{Mod}_{\underline{H}}$ in alternative ways.

The symmetric monoidal structure on !-pullback and oplax monoidal structure on ind-proper $*$-pushforward endow $\underline{H} = \pi_{2,*}\pi_1^! \simeq \pi^!\pi_*$ with a canonical oplax symmetric monoidal structure. Explicitly, given $\underline{H}$-modules M, N with structure maps $\underline{H}M \to M$, $\underline{H}N \to N$, we give $M \otimes N$ a $\underline{H}$-module structure with structure map

$$\underline{H}(M \otimes N) \to \underline{H}M \otimes \underline{H}N \to M \otimes N.$$

We also have a natural transformation

$$\underline{H}(\omega_X) = \pi_{2,*}\omega_{\mathcal{G}} \simeq \pi_{2,*}\pi_2^!\omega_X \longrightarrow \omega_X.$$

In other words $\underline{H}$ forms a (derived analog of) a cocommutative *bimonad* in the sense of Moerdijk and Bruguières-Virelizier, see [Bö] (in fact it's naturally a Hopf monad). Hence its modules form a symmetric monoidal category. David BZ: Don't know if this is helpful? should be clear we're doing E_∞ not discrete stuff

More formally, any stack canonically upgrades to a cocommutative coalgebra object Sam: should this be *co*algebra object? Although I guess it doesn't matter in the correspondence category...David BZ: definitely, fixed in the category of stacks using the diagonal maps. Hence any groupoid, considered as an algebra object in the correspondence category of stacks, canonically upgrades to an algebra object in cocommutative algebra objects in the correspondence category. Then the functor $\breve{\mathcal{D}}_{hol}$ recovers the structure of cocommutative bimonad on $\underline{H}$.

3.3. **Monads vs. algebras.** In the generality we're working, the Hecke algebra $\underline{H}$ is only a monad, i.e., algebra object in endofunctors of $\breve{\mathcal{D}}_{hol}(X)$. In the cases of practical interest however this reduces to an ordinary algebra, thanks to "affineness" (or rather coaffineness). For example, in the case of a classifying space $X = pt/G$, we have equivalences

$$\breve{\mathcal{D}}_{hol}(X) \simeq \mathbf{Mod}_{C^*(X)}$$

and

$$End(\breve{\mathcal{D}}_{hol}(X)) \simeq \mathbf{Mod}_{C^*(X)\otimes C^*(X)}.$$

Thus a monad on $\breve{\mathcal{D}}_{hol}(X)$ is identified as an algebra object in $C^*(X)$-bimodules. The monad $\underline{H}$ corresponding to an ind-proper groupoid $\mathcal{G}$ over X is

$$\underline{H} \leftrightarrow i_*\omega_{\mathcal{G}} \in \breve{\mathcal{D}}_{hol}(X \times X) \simeq \mathbf{Mod}_{C^*(X)\otimes C^*(X)},$$

which can be identified with $C_*(\mathcal{G}) = \Gamma(\omega_G)$, with bimodule structure given by the diagonal morphism

$$C^*(X) \longrightarrow C_*(\mathcal{G}).$$

Proposition 3.6. *For $X = pt/K$ a classifying space, the category $\mathcal{W} = \mathbf{Mod}_{\underline{H}}$ is equivalent to the category $\mathbf{Mod}_H$ of modules for the k-algebra $H = \Gamma(\underline{H}) \in Alg(Vect_k)$.*

Proof. We use the equivalence $\breve{\mathcal{D}}_{hol}(X) \simeq \mathbf{Mod}_{C^*(X)}$, identifying $\underline{H}$ with a monad on $\mathbf{Mod}_{C^*(X)}$, i.e., algebra object in $C^*(X)$-bimodules. The forgetful functor from H-modules in $\mathbf{Mod}_{C^*(X)}$ has a quasi-inverse, defined by using the unit map $C^*(X) \to H$ to endow any H-module in $Vect_k$ with a $C^*(X)$-module structure. □

3.3.1. *Pulling back Hecke algebras.* We include an observation about base-changing Hecke algebras and the corresponding $\mathcal{W}$-categories, that is needed for Proposition 5.11 below

Proposition 3.7. *Let $\mathcal{G} \to X \times X$ denote an ind-proper groupoid, and $\underline{H}_X$ the Hecke algebra. Let $p : Z \to X$ denote an ind-proper morphism,*

$$\mathcal{G}_Z = \mathcal{G} \times_{X\times X} Z \times Z \to Z \times Z$$

the pullback groupoid and $\underline{H}_Z$ the resulting Hecke algebra. Then $p^!$ lifts to a symmetric monoidal functor

$$\mathbf{Mod}_{\underline{H}_X} \longrightarrow \mathbf{Mod}_{\underline{H}_Z}.$$

David BZ: This is a little out of place - just stuck in so as to make the Webster Morita equivalence argument work.

Proof. This is immediate from the description of modules for the Hecke algebras as totalizations of cosimplicial symmetric monoidal categories - the functor in question is given by pullback of sheaves on simplices, hence is symmetric monoidal. ☐

3.4. **Hecke Categories.** We now consider a categorical analog of the above discussion. The *Hecke category*

$$\mathcal{H} := \breve{\mathcal{D}}_{hol}(\mathcal{G})$$

carries a canonical monoidal structure, the convolution product, following the general mechanism discussed in [GR3, II.2.5.1, V.3.4] — it is inherited on applying $\breve{\mathcal{D}}_{hol}$ to the structure on $\mathcal{G}$ of algebra object in correspondences. The diagonal embedding (unit map) $i : X \to \mathcal{G}$ induces a monoidal functor

$$Char : \mathcal{C} \to \mathcal{H}$$

making $\mathcal{H}$ into a $\mathcal{C}$*-ring*, i.e., algebra object in $\mathcal{C}$-bimodules.

Consider the cosimplicial symmetric monoidal category $\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)$. We may pass to module categories, obtaining a cosimplicial symmetric monoidal category $\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)}$.

Definition 3.8. The symmetric monoidal category $\mathbf{Mod}^{\mathcal{G}}_{\breve{\mathcal{D}}_{hol}(X)}$ of $\mathcal{G}$-equivariant module categories on X is the totalization $Tot(\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)})$.

Remark 3.9 (Algebra vs Monad, revisited)**.** As we noted in Remark 3.3, we treat the Hecke algebra in general as a monad on $\breve{\mathcal{D}}_{hol}(X)$, but in situations of interest this reduces to an algebra object in $C^*(X)$-bimodules. Here we chose to treat the Hecke category directly as an algebra in $\breve{\mathcal{D}}_{hol}(X)$-bimodules. One could instead consider the monad on sheaves of categories on X obtained by push-pull along $\mathcal{G}$. Likewise the category $\mathbf{Mod}^{\mathcal{G}}_{\breve{\mathcal{D}}_{hol}(X)}$ is an avatar for the category of $\mathcal{G}$-equivariant sheaves of categories on X, with which it is connected by the localization-global sections adjunction, and which it would recover if we were in a 1-affine situation. Thus we can also consider it as an avatar of sheaves of categories on the quotient stack $Y = X/\mathcal{G}$, which is the source of its symmetric monoidal structure.

Proposition 3.10. *The cosimplicial category* $\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)}$ *satisfies the monadic Beck-Chevalley conditions. Moreover the associated monad on* $\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(X)}$ *is identified with the* Hecke category $\mathcal{H} = \breve{\mathcal{D}}_{hol}(\mathcal{G})$ *as an algebra in* $\breve{\mathcal{D}}_{hol}(X)$*-bimodules via the diagonal map* $\delta_* : \breve{\mathcal{D}}_{hol}(X) \to \mathcal{H}$*. Thus we have an identification* $\mathbf{Mod}^{\mathcal{G}}_{\breve{\mathcal{D}}_{hol}(X)} \simeq \mathbf{Mod}_{\mathcal{H}}$*.*

Proof. The Beck-Chevalley conditions for $\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)}$ follow from those for $\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)$ upon applying the functor **Mod**.

☐

It follows that the category $\mathbf{Mod}_{\mathcal{H}}$ of $\mathcal{G}$-equivariant $\breve{\mathcal{D}}_{hol}(X)$-modules inherits a symmetric monoidal structure, such that the forgetful functor $\mathbf{Mod}_{\mathcal{H}} \to \mathbf{Mod}_{\mathcal{C}}$ is symmetric monoidal. The unit object is the $\mathcal{H}$-module $\mathcal{C}$ itself, which corresponds to the cosimplicial category $\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)$.

3.5. **Hecke algebras vs. Hecke categories.** We now compare descent for module categories with descent for sheaves. Given a $\mathcal{H}$-module $\mathcal{M}$, or equivalently $\mathcal{M}^\bullet \in Tot(\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)})$, we define the $\mathcal{G}$*-equivariant objects* $\mathcal{M}^{\mathcal{G}}$ to be

$$\mathcal{M}^{\mathcal{G}} := Hom_{\mathcal{H}}(\breve{\mathcal{D}}_{hol}(X), \mathcal{M}).$$

Thus we have
$$\mathcal{M}^{\mathcal{G}} \simeq Tot(Hom(\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet), \mathcal{M}^\bullet)).$$

Proposition 3.11. (1) *The $\mathcal{G}$-equivariant objects in the $\mathcal{H}$-module $\mathcal{C}$ recover the category of $\mathcal{G}$-equivariant sheaves on X, i.e.,*
$$\mathcal{C}^{\mathcal{G}} \simeq \mathcal{W}.$$

(2) *The resulting equivalence of $\mathcal{C}^{\mathcal{G}}$ with the endomorphisms of the unit $\mathcal{C}$ of the symmetric monoidal category $\mathbf{Mod}_{\mathcal{H}}$ lifts to a symmetric monoidal equivalence.*

Proof. We apply the above definition in the case $\mathcal{C} = \breve{\mathcal{D}}_{hol}(X)$, which corresponds to $\mathcal{C}^\bullet = \breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)$:
$$\begin{aligned}
[\breve{\mathcal{D}}_{hol}(X)]^{\mathcal{G}} &:= Hom_{\mathcal{H}}(\breve{\mathcal{D}}_{hol}(X), \breve{\mathcal{D}}_{hol}(X)) \\
&\simeq Hom_{Tot(\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)})}(\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet), \breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)) \\
&\simeq Tot(\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)) \\
&\simeq \breve{\mathcal{D}}_{hol}(X)^{\mathcal{G}}.
\end{aligned}$$

Tracing through the identifications above, we see that the symmetric monoidal structure on $Tot(\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet))$ coming from tensor product of sheaves is identified with the symmetric monoidal structure on endomorphisms of the unit in $Tot(\mathbf{Mod}_{\breve{\mathcal{D}}_{hol}(\mathcal{G}_\bullet)})$, as claimed. □

3.6. **Abstract nonsense.** The following result is a standard feature of monoidal categories.

Proposition 3.12. *Let us fix a presentable symmetric monoidal category $\mathcal{P}$ and a commutative algebra object $\mathcal{O} \in Alg_{E_\infty}(\mathcal{P})$.*

For any monoidal category $\mathcal{A}$ the identification of functors $1_{\mathcal{A}} \otimes -$ and $Id_{\mathcal{A}}$ induces an E_2-monoidal morphism
$$1_{\mathcal{A}} \otimes - : End(1_{\mathcal{A}}) \to End(Id_{\mathcal{A}}).$$
This morphism, considered as an E_1-morphism, admits a left inverse
$$act_{1_{\mathcal{A}}} : End(Id_{\mathcal{A}}) \to End(1_{\mathcal{A}}),$$
given by the action of $End(Id_{\mathcal{A}})$ on the object $1_{\mathcal{A}}$.

Proof. We follow the outline of [AG, Section E.2] in the stable setting of dg categories, though stability is not needed for the result. Namely we work

2 category means

Let $\mathfrak{a}$ denote the E_2-algebra $End(1_{\mathcal{A}})$. We have monoidal functors $-\mathrm{mod}_{\mathfrak{a}} \longrightarrow \mathcal{A} \longrightarrow End(\mathcal{A})$ □

3.7. **Central actions.** Our main result asserts that $\mathcal{G}$-equivariant sheaves give central objects in the groupoid category $\mathcal{H}$. This central action can be thought of as expressing the linearity of convolution on $\mathcal{G} = X \times_Y X$ over sheaves on the (possibly ill-behaved) quotient $Y = X/\mathcal{G}$.

Theorem 3.13. *Let $\mathcal{G}$ denote an ind-proper groupoid acting on an ind-stack X, $\underline{H}$ the corresponding monad on $\mathcal{C}$ and $\mathcal{H} = \breve{\mathcal{D}}_{hol}(\mathcal{G})$ the groupoid category. Then there is a canonical E_2-morphism $Ng\hat{o}$ with a monoidal left inverse section $Kost$ ($Kost \circ Ng\hat{o} \simeq Id$),*
$$\begin{tikzcd} \mathbf{Mod}_{\underline{H}} \arrow[r, "Ng\hat{o}"] & \mathcal{Z}(\mathcal{H}) \arrow[l, bend right, "Kost"'] \end{tikzcd}$$
lifting the diagonal map $Char : \mathcal{C} \to \mathcal{H}$:

$$\begin{tikzcd}
\mathbf{Mod}_{\underline{H}} \arrow[r, "E_2"'] \arrow[d, "E_\infty"'] & \mathcal{Z}(\mathcal{H}) \arrow[l, bend right, "E_2"'] \arrow[d, "E_1"] \\
\breve{\mathcal{D}}_{hol}(X) \arrow[r, "E_1"] & \mathcal{H}
\end{tikzcd}$$

Proof. We take

$$\mathcal{A} = \mathbf{Mod}_{\mathcal{H}} \simeq \mathbf{Mod}_{\mathcal{C}}^{\mathcal{G}}$$

to be the category of modules for the Hecke category, i.e., $\mathcal{G}$-equivariant $\mathcal{C}$-modules. We have identified

$$End(1_{\mathcal{A}}) \simeq \mathbf{Mod}_{\underline{H}} \simeq \breve{\mathcal{D}}_{hol}(X)^{\mathcal{G}}$$

as categories. We need to show that this identification can be upgraded to an E_2 identification, hence obtaining the desired E_2-morphism from $\mathcal{W} = \mathbf{Mod}_{\underline{H}}$ to $End(Id_{\mathcal{A}}) = \mathcal{Z}(\mathcal{H})$. However we have seen in Proposition 3.11 that the identification is in fact naturally E_∞.

□

3.8. **Digression: The averaging idempotent.** The following is not strictly necessary, but gives a useful picture of equivariance as modules for an categorical form of the averaging idempotent in a finite group algebra:

Proposition 3.14. (1) *The object $e = \omega_{\mathcal{G}} \in \mathcal{H}$ has a canonical structure of algebra object.*
(2) *Under the monoidal action map $\mathcal{H} \to End(\breve{\mathcal{D}}_{hol}(X))$, e is taken to the Hecke algebra $\underline{H}$. Thus we have an equivalence $\breve{\mathcal{D}}_{hol}(X)^{\mathcal{G}} \simeq \breve{\mathcal{D}}_{hol}(X)^e$ of equivariant sheaves with e-modules in $\breve{\mathcal{D}}_{hol}(X)$.*
(3) *More generally, for any $\mathcal{H}$-module $\mathcal{C}$ [assuming $\mathcal{H}$ rigid and $\breve{\mathcal{D}}_{hol}(X)$ dualizable] we have an identification $\mathcal{C}^{\mathcal{G}} \simeq \mathcal{C}^e$ of equivariant objects with e-modules in $\mathcal{C}$.*

Proof. For the third part, we assume that $\mathcal{H}$ is rigid (which follows from the assumption that $X \to Y$ is ind-proper) and that $\breve{\mathcal{D}}_{hol}(X)$ is self-dual over $\mathcal{G}$ (which follows from rigidity and the self-duality of $\breve{\mathcal{D}}_{hol}(X)$ over k, which holds as long as $\breve{\mathcal{D}}_{hol}(X)$ is dualizable, i.e., in great generality.) It follows that for any $\mathcal{H}$-module $\mathcal{C}$ we have

$$\begin{aligned}
\mathcal{C}^{\mathcal{G}} &= Hom_{\mathcal{H}}(\breve{\mathcal{D}}_{hol}(X), \mathcal{C}) \\
&\simeq \breve{\mathcal{D}}_{hol}(X) \otimes_{\mathcal{H}} \mathcal{C} \\
&\simeq \mathcal{H}^e \otimes_{\mathcal{H}} \mathcal{C} \\
&\simeq \mathcal{C}^e
\end{aligned}$$

where the last step is [**?**][Proposition 4.1] (or rather its extension from symmetric monoidal to monoidal categories, as cited somewhere else..)

□

4. Kac-Moody groups

Let us recall some constructions from the theory of Kac-Moody flag varieties following [M, K] and nil-Hecke algebras following [KK, A, LLMSSZ, Gi1].

Let A be a generalized Cartan matrix, together with a realization $(\underline{\mathfrak{h}}, R, R^\vee)$ on a k-vector space $\underline{\mathfrak{h}}$. (We assume k has characteristic zero.) We denote by $\underline{\mathfrak{g}}$ the associated Kac-Moody algebra, with root decomposition

$$\underline{\mathfrak{g}} = \underline{\mathfrak{h}} \oplus \bigoplus_{\alpha \in R} \underline{\mathfrak{g}}_\alpha.$$

Let $\underline{W}$ denote the associated Weyl group, which is a Coxeter group acting on $\underline{\mathfrak{h}}$.

To this data is associated a group ind-scheme $\underline{G}$ over k, the Kac-Moody group, together with a Borel subgroup (an affine group scheme) and pro-unipotent radical $\underline{U} \subset \underline{B} \subset \underline{G}$ and Cartan subgroup H (a finite dimensional split torus over k) with $\underline{B} = \underline{UH}$. The flag variety $\underline{G}/\underline{B}$ has the structure of an ind-proper ind-scheme of ind-finite type, and this structure is compatible with the pro-structure on $\underline{B}$ as described in [K, Section 7.1] so that the flag stack

$$\mathcal{G} = \underline{B}\backslash\underline{G}/\underline{B}$$

defines an ind-proper groupoid over $X = pt/\underline{B}$.

The nil-Hecke algebra of Kostant-Kumar [KK] associated to $\underline{G}$ is by definition the subalgebra $H_{\underline{G}}$ of endomorphisms of $\mathbb{C}[\underline{\mathfrak{h}}^*]$ generated by multiplication operators $\mathbb{C}[\underline{\mathfrak{h}}^*]$ and the Demazure, or divided-difference, operators associated to simple reflections $\sigma_i \in \underline{W}$ with associated simple roots α_i

$$A_i = (1 - \sigma_i)/\alpha_i.$$

The A_i generate the nil-Coxeter algebra $Nil_{\underline{G}}$, with basis A_w ($w \in \underline{W}$) and relations $A_v A_w = A_{vw}$ if $l(vw) = l(v) + l(w)$ and 0 otherwise (it is the associated graded of $\mathbb{C}\underline{W}$ with respect to a natural filtration [Gi1]).

Theorem 4.1. *[A] The nil-Hecke algebra $H_{\underline{G}}$ is isomorphic to the convolution algebra $H_*(\underline{B}\backslash\underline{G}/\underline{B})$.*

It follows that $H_{\underline{G}}$ is a Hopf algebroid over $H^*(pt/\underline{B}) \simeq \mathbb{C}[\underline{\mathfrak{h}}^*]$. One can also write the cocommutative coproduct on the nil-Hecke algebra explicitly: it is determined by the assignment

$$\begin{aligned}\Delta(A_i) &= A_i \otimes 1 + \sigma_i \otimes A_i \\ &= \frac{1}{\alpha_i}(1 \otimes 1 - \sigma_i \otimes \sigma_i)\end{aligned}$$

In other words, the coproduct is compatible with the standard coproduct on the groupoid algebra $\mathbb{C}W \otimes \mathbb{C}[\underline{\mathfrak{h}}^*]$.

Remark 4.2 (Parabolic versions)**.** One can replace the Borel $\underline{B}$ by other parabolic subgroups, resulting in parabolic Hecke categories $\breve{\mathcal{D}}_{hol}(\underline{P}\backslash\underline{G}/\underline{P})$ and parabolic nil-Hecke algebras $H_*(\underline{P}\backslash\underline{G}/\underline{P})$. See [Gi2] for a study of the spherical affine nil-Hecke algebra, i.e. equivariant homology of the affine Grassmannian, which we return to below.

4.1. **Coarse quotients of Coxeter groups.** We continue with the Coxeter group $\underline{W} \circlearrowright \underline{\mathfrak{h}}^*$ associated to the Kac-Moody group $\underline{G}$.

Let $\Gamma \subset \underline{\mathfrak{h}}^* \times \underline{\mathfrak{h}}^*$ denote the union of the graphs of the elements of $\underline{W}$ acting on $\underline{\mathfrak{h}}^*$. In other words, Γ is the graph of the equivalence relation on $\underline{\mathfrak{h}}^*$ determined by $\underline{W}$. We denote by $\underline{\mathfrak{h}}^*//W$ the quotient functor of $\underline{\mathfrak{h}}^*$ by the equivalence relation Γ. In other words, $\underline{\mathfrak{h}}^*//W$ is the coarse moduli space of the stack quotient $\underline{\mathfrak{h}}^*/W$. For $\underline{G}$ a reductive group, this agrees with the usual terminology $\mathfrak{h}^*//W = Spec\mathbb{C}[\mathfrak{h}^*]^W$. Note that $\Gamma \to \underline{\mathfrak{h}}^* \times \underline{\mathfrak{h}}^*$ is ind-proper, so that we may identify Γ-equivariant ind-coherent sheaves on $\underline{\mathfrak{h}}^*$, i.e., ind-coherent sheaves on $\underline{\mathfrak{h}}^*//W$, with modules for the groupoid algebra $\omega(\Gamma)$.

Proposition 4.3. *$H_{\underline{G}}$ is naturally identified as Hopf algebroid over $\underline{\mathfrak{h}}^*$ with $\omega(\Gamma)$, the groupoid algebra of the equivalence relation Γ on $\underline{\mathfrak{h}}^*$ determined by the $\underline{W}$-action. Thus we have an equivalence of symmetric monoidal categories*

$$\mathbf{Mod}_{H_{\underline{G}}} \simeq IndCoh(\underline{\mathfrak{h}}^*//\underline{W}).$$

Let W denote a Coxeter group and $\mathfrak{h}$ its reflection representation. For $w \in W$ we let $\Gamma_w \subset \mathfrak{h} \times \mathfrak{h}$ denote the graph of the corresponding reflection. Let

$$\Gamma_W = \coprod_{w \in W} \Gamma_w.$$

Then $\mathcal{G} = \Gamma_W$ is an ind-proper groupoid acting on the scheme $\mathfrak{h}$.

Let $H_W = \Gamma(\omega_{\mathcal{G}})$ denote the corresponding (ind-coherent) Hecke algebra. It is a variant of the nil-Hecke algebra [KK] associated to W. David BZ: Do we know if H_W is the nil-Hecke algebra outside of the finite case?? Let $\mathcal{H}_W = \mathcal{Q}^!(\Gamma_W)$ denote the ind-coherent Hecke category. It is closely related to the Iwahori-Hecke category associated to W.

5. The Kostant Category

5.1. **Whittaker reduction.** In this section we recall some results about Whittaker reductions of differential operators on G.

Caution: In this section we discuss associative algebras and abelian categories rather than their dg or ∞-versions.

Let $G \supset B \supset N$ denote a reductive group with Borel subgroup and unipotent radical, $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{n}$ their Lie algebras. Fix a nondegenerate character $\psi \in \mathfrak{n}^*$. A seminal result of Kostant **??** identifies the center of $U\mathfrak{g}$ with $U\mathfrak{g}//_\psi U\mathfrak{n}$, the quantum Hamiltonian reduction of $U\mathfrak{g}$ by $U\mathfrak{n}$ twisted by the character ψ. The latter, the Whittaker Hecke algebra, is the endomorphism algebra of the functor $(-)^{\mathfrak{n},\psi}$ of Whittaker vectors in $\mathfrak{g}$-modules.

If we replace $U\mathfrak{g}$ by the full algebra $\mathcal{D}_G$ of differential operators on G, the analogous Hamiltonian reduction produces the quantized (and partially completed) phase space of the Toda lattice associated to G.

Theorem 5.1. *[BeF] The following associative algebras are canonically isomorphic:*

1. *the Whittaker Hamiltonian reduction $Toda(G) = N_\psi \backslash\backslash \mathcal{D}_G //_\psi N$ of the algebra of differential operators on G (the partially completed quantized Toda lattice)*
2. *$H = H_*(\mathcal{G}r^\vee/\mathbb{G}_m)$, the equivariant homology of the Grassmannian of the dual group*

The theorem is a quantization of the result of [BFM] describing the equivariant homology of the Grassmannian (without loop rotation) as functions on the group-scheme $J \to \mathfrak{c}$ of regular centralizers.

The results of Kostant-Kumar [KK, K] (see [Gi2]) identify the algebra H with an explicitly defined algebra, namely the spherical subalgebra H_{sph} of the nil Hecke algebra associated to the affine Weyl group.

Theorem 5.2. *[Gi2, Lo1, Lo2] The following abelian categories are canonically equivalent:*

1. *the bi-Whittaker Hecke category of G*
2. *the category $\mathbf{Mod}_H$ of modules for the algebra $H \simeq H_{sph} \simeq Toda(G)$ described in Theorem 5.1*
3. *The category $QCoh(\mathfrak{h}^*//W^{aff})$ of quasicoherent sheaves on the coarse quotient of $\mathfrak{h}^*//W^{aff}$.*

Remark 5.3. The last assertion in Theorem 5.2 is a paraphrase of the results of [Gi2, Lo1, Lo2]. In [Lo1] the category is described as quasi-coherent sheaves equivariant for the "adjancency groupoid" of W^{aff}, namely the groupoid Γ defining the coarse quotient. The category is also described in *loc. cit.* as the full subcategory of W^{aff}-equivariant quasicoherent sheaves on $\mathfrak{h}^*$ which carry a trivial action of derived inertia, or equivalently descend to the categorical quotient by the finite Weyl group or by every finite parabolic subgroup of W^{aff} the finite Weyl group.

5.2. **Asymptotic Harish Chandra modules.** Given a variety X we let $\mathcal{R}_X$ denote the graded Rees algebra of differential operators on X, i.e.,

$$\mathcal{R}_X = \bigoplus \mathcal{D}_{\leqslant i, X}[-2i].$$

We let $\mathcal{R}(X) = \mathbf{Mod}_{\mathcal{R}_X}$ denote the category of asymptotic $\mathcal{D}$-modules on X.

Let HC denote the category of asymptotic Harish Chandra bimodules, $HC = \mathcal{R}_G(G)_G$, the category of G-weakly bi-equivariant asymptotic $\mathcal{D}$-modules on G.

The following result is a consequence of Gaitsgory's 1-affineness theorem [G] due to Beraldo [Be]:

Proposition 5.4. *There is a Morita equivalence of monoidal categories* $\mathbf{Mod}_{\mathcal{R}(G)} \simeq \mathbf{Mod}_{HC}$ *sending a* $\mathcal{R}(G)$*-module category* M *to its weak* G*-invariants* $M^{QC(G)}$.

Under the Morita equivalence, the $\mathcal{R}(G)$-module $\mathcal{R}(G/N)_\psi$ is exchanged with its weak G-invariants,

$$(\mathcal{D}_\hbar(G/N)_\psi)^{w,G} = \mathcal{D}_{\hbar,G}(G/N)_\psi = Whit,$$

the category of Whittaker $U\mathfrak{g}$-modules, which by Skryabin's theorem is identified as

$$Whit \simeq \mathbf{Mod}_{\mathcal{Z}}.$$

Sam: Should the $\mathcal{D}_{\hbar,G}$ etc. be $\mathcal{R}_G$ to be consistent? David BZ: Should definitely be consistent. I was trying to eliminate all the floating $\hbar$'s from the notation, especially if in this version we don't do Ngo, i.e. set $\hbar = 0$.. Not a huge fan of the $\mathcal{R}_G$ notation though, maybe something else with a $\mathcal{D}$?

Theorem 5.5 (Renormalized Satake Equivalence)**.** *[BeF] (see also [AG]) There is an equivalence of monoidal categories*

$$\breve{\mathcal{D}}_{hol}(\mathcal{G}r^\vee/\mathbb{G}_m) \simeq HC_\hbar.$$

The equivalence exchanges the global cohomology functor

$$\breve{\mathcal{D}}_{hol}(\mathcal{G}r^\vee/\mathbb{G}_m) \to \breve{\mathcal{D}}_{hol}(X \times_{B\mathbb{G}_m} X) \simeq \mathbf{Mod}_{C^*(BG^\vee \times BG^\vee \times B\mathbb{G}_m)}$$

with the Whittaker functor

$$HC_\hbar \to \mathbf{Mod}_{\mathcal{Z}\otimes\mathcal{Z}\otimes\mathbb{C}[\hbar]}.$$

Definition 5.6. The *Kostant category* $\mathcal{K}$ is the Whittaker Hecke category, i.e., the monoidal category

$$\mathcal{D}_{\hbar,\psi}(N\backslash G/N)_\psi \simeq End_{\mathcal{D}_\hbar(G)}(\mathcal{D}_\hbar(G/N)_\psi) \simeq End_{HC_\hbar}(Whit)$$

In particular the classical Kostant category

$$\mathcal{K}_0 = \mathcal{K}_\hbar \otimes_{\mathbb{C}[\hbar]} \mathbb{C} \simeq \mathcal{Q}(N_\psi\backslash\backslash G//_\psi N)$$

is identified as a monoidal category with $(\mathcal{Q}(J), *)$, the category of sheaves on the groupscheme $J \to \mathfrak{c}$ of regular centralizers equipped with convolution.

5.3. **Langlands duality.**

Proposition 5.7. *There is an equivalence of monoidal categories*

$$\mathcal{K}_\hbar \simeq \mathbf{Mod}_H.$$

Proof. By Proposition 3.11 we have a monoidal identification of $\mathbf{Mod}_{\underline{H}}$ with $\mathcal{H}$-endomorphisms of $\breve{\mathcal{D}}_{hol}(X = pt/G^\vee(\mathcal{O}) \rtimes \mathbb{G}_m) \simeq \mathbf{Mod}_{C*(BG^\vee \times \mathbb{G}_m)}$. By Theorem 5.5 on the other hand we have a monoidal identification of the latter category with $HC_\hbar$-endofunctors of $\mathbf{Mod}_{\mathcal{Z}\otimes\mathbb{C}[\hbar]}$, in other words with the Whittaker Hecke category $\mathcal{K}_\hbar \simeq End_{HC_\hbar}(Whit)$. $\square$

Theorem 5.8. *The monoidal structure of $\mathcal{K}_\hbar$ can be extended to a symmetric monoidal structure. Moreover there is an E_2 functor defining a central action*

$$\mathcal{K}_\hbar \to \mathcal{Z}(HC_\hbar) \simeq \mathcal{D}_\hbar(G/G).$$

5.4. **The equivariant Satake category.** We take $X = pt/G^\vee(\mathcal{O}) \rtimes \mathbb{G}_m$, $Y = pt/G^\vee(\mathcal{K}) \rtimes \mathbb{G}_m$, $\mathcal{G}r^\vee = G^\vee(\mathcal{O})\backslash G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ and $\mathcal{G} = \mathcal{G}r^\vee/\mathbb{G}_m \simeq G^\vee(\mathcal{O})\backslash G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$. Note that X is an ind-stack and $\mathcal{G}r^\vee$ an ind-proper groupoid acting on X. Thus Theorem 3.13 applies:

Corollary 5.9. *There is a symmetric monoidal structure on modules for $H = C_*(\mathcal{G}r^\vee/\mathbb{G}_m)$, and a central functor $\mathbf{Mod}_H \to \mathcal{Z}(\breve{\mathcal{D}}_{hol}(\mathcal{G}r^\vee/\mathbb{G}_m))$.*

5.5. **Homology version.**

Proposition 5.10. *The equivariant chains on the Grassmannian $C_*(\mathcal{G}r^\vee/\mathbb{G}_m)$ are formal both as algebra and as coalgebra.*

In this section we denote by $H_{sph} = H_*^{G^\vee \times \mathbb{G}_m}(Gr_{G^\vee})$ the equivariant homology of the affine Grassmannian, and $H_{aff} = H_*^{H\times\mathbb{G}_m}(Fl_{G^\vee})$ the equivariant homology of the affine flag variety. We consider both as cocommutative Hopf algebroids under convolution. Their categories of modules are equivalent by a result of [**?**], in fact we have the following:

Proposition 5.11. *There is a natural equivalence of symmetric monoidal categories $\mathbf{Mod}_{H_{sph}} \simeq \mathbf{Mod}_{H_{aff}}$.*

Sam: What is the status of this section? Do we have a reference for this proposition? Horel? David BZ: This section is somewhat cobbled. Webster's proof gives the Morita equivalence on homology level. Somewhere we still need to/can do the Horel formality argument to relate the chains to homology, but I've left that be for now.

Proof. We apply Proposition 3.7 to $X = pt/G^\vee(\mathcal{O})$, $Z = pt/I^\vee$, and $\mathcal{G}_X = \mathcal{G}r^\vee = X \times_{pt/G^\vee(\mathcal{K})} X$. The pullback groupoid is

$$\mathcal{G}_Z = Z \times_{pt/G^\vee(\mathcal{K})} Z \simeq I^\vee\backslash G^\vee(\mathcal{K})/I^\vee,$$

the equivariant affine flag variety. It follows that we have a symmetric monoidal functor from $\mathbf{Mod}_{H_{sph}}$ to $\mathbf{Mod}_{H_{aff}}$[1]. By [**?**, Theorem 3.3] the underlying functor is an equivalence (in fact identifying H_{aff} as an $|W| \times |W|$-matrix algebra over H_{sph}). $\square$

[1]or rather the chain level version.

5.6. **Affine Nil-Hecke Algebras.** David BZ: This section can probably be cut. Just hanging around for now, from an earlier version We now specialize to the affine Kac-Moody case, where $\underline{G}$ is the semidirect product of a central extension of the loop group $G^\vee(\mathcal{K})$ by $\mathbb{G}_m$ acting by loop rotation. In this case the flag variety is a line bundle over $G^\vee(\mathcal{K})/I^\vee$,

$$\underline{\mathfrak{h}} \simeq \mathfrak{h}^\vee \oplus \mathbb{C}d \oplus \mathbb{C}K$$

and

$$H^*(pt/\underline{G}) \simeq H^*(pt/G^\vee)[\epsilon, k] = \mathbb{C}[\mathfrak{h}^*]^W[\epsilon, k],$$
$$H^*(pt/\underline{B}) \simeq H^*(pt/B^\vee)[\epsilon, k] = \mathbb{C}[\mathfrak{h}^*][\epsilon, k],$$

The affine Weyl group $\underline{W} = W^{aff} = \Lambda \rtimes W$ with W the finite Weyl group. Let $\underline{\mathfrak{h}}_0 = \underline{\mathfrak{h}}/\mathbb{C}K \simeq \mathfrak{h}^* \oplus \mathbb{C}d$. An element $\widetilde{w} = w\lambda \in \underline{W}$ acts on $\underline{\mathfrak{h}}^0$ by

$$\widetilde{w} \cdot (\xi, t) = (w \cdot \xi + \epsilon\lambda, \epsilon).$$

In other words, at $\epsilon = 1$ we recover the standard action of W^{aff} on $\mathfrak{h}^*$. However we will treat ϵ as a graded parameter, so it will only make sense to invert ϵ, not set it to 1.

Let $H^{aff}_{G^\vee} = H_{\underline{G}}/(k)$ be the level zero specialization of the nil-Hecke algebra of $\underline{G}$, which we consider as a $\mathbb{C}[\epsilon]$-algebra.

Proposition 5.12. *We have an equivalence of symmetric monoidal categories*

$$\mathbf{Mod}_{H^{aff}_{G^\vee}} \simeq IndCoh(\mathfrak{h}^* \times \mathbb{A}^1_\epsilon//W^{aff}),$$

where the W^{aff} action is the standard action on $\mathfrak{h}^*$ with the lattice action rescaled by ϵ.

References

[A] A. Arabia, Cohomologie T-équivariante de la variété de drapeaux d'un groupe de Kac-Moody. Bull. Soc. Math. France 117 (1989), no. 2, 129-165.

[AG] D. Arinkin and D. Gaitsgory, Singular support of coherent sheaves, and the geometric Langlands conjecture, arXiv:1201.6343. Selecta Math. (N.S.) 21 (2015), no. 1, 1-199.

[BD] A. Beilinson and V. Drinfeld, Quantization of Hitchin Hamiltonians and Hecke Eigensheaves. Preprint, available at math.uchicago.edu/~mitya.

[BGO] D. Ben-Zvi, S. Gunningham and H. Orem, Highest weights for categorical representations. arXiv:1608.08273.

[BGN] D. Ben-Zvi, S. Gunningham and D. Nadler, The character field theory and homology of character varieties. arXiv:1705.04266.

[BN1] D. Ben-Zvi and D. Nadler, The Character Theory of a Complex Group. arXiv:0904.1247.

[Be] D. Beraldo, Loop group actions on categories and Whittaker invariants. Berkeley PhD Thesis, 2013.

[BeF] R. Bezrukavnikov and M. Finkelberg, Equivariant Satake category and Kostant-Whittaker reduction. Mosc. Math. J. 8 (2008), no. 1, 39-72, 183.

[BFM] R. Bezrukavnikov, M. Finkelberg and I. Mirković, Equivariant homology and K-theory of affine Grassmannians and Toda lattices. Compos. Math. 141 (2005), no. 3, 746-768.

[Bö] G. Böhm, Hopf algebroids. arXiv:0805.3806. Chapter in Handbook of Algebra.

[dCHM] M. de Cataldo, T. Hausel and L. Migliorini,Topology of Hitchin systems and Hodge theory of character varieties: the case A1. Ann. of Math. (2) 175 (2012), no. 3, 1329-1407.

[CH] J. Cirici and G. Horel, Mixed Hodge structures and formality of symmetric monoidal functors. e-print arXiv:1703.06816.

[DG1] V. Drinfeld and D. Gaitsgory, On some finiteness questions for algebraic stacks. Geom. Funct. Anal. 23 (2013), no. 1, 149-294. e-print arXiv:1108.5351.

[DG2] V. Drinfeld and D. Gaitsgory, Compact generation of $\mathcal{D}$-modules on Bun_G. Available at http://www.math.harvard.edu/~gaitsgde/GL/

[DGNO] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik, On Braided Fusion Categories I. eprint arXiv:0906.0620.

[G1] D. Gaitsgory, Generalities on DG categories. Available at http://www.math.harvard.edu/~gaitsgde/GL/

[G] D. Gaitsgory, Sheaves of categories and the notion of 1-affineness. Stacks and categories in geometry, topology, and algebra, 127-225, Contemp. Math., 643, Amer. Math. Soc., Providence, RI, 2015.

[GR1] D. Gaitsgory and N. Rozenblyum, Crystals and $\mathcal{D}$-modules, available at http://www.math.harvard.edu/~gaitsgde/GL/

[GR2] D. Gaitsgory and N. Rozenblyum, DG ind-schemes, available at http://www.math.harvard.edu/~gaitsgde/GL/

[GR3] D. Gaitsgory and N. Rozenblyum, A study in derived algebraic geometry. Preliminary version (May 2016), available at http://www.math.harvard.edu/~gaitsgde/GL/

[Gi1] V. Ginzburg, Geometric methods in the representation theory of Hecke algebras and quantum groups. Notes by Vladimir Baranovsky., NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 127-183, Kluwer Acad. Publ., Dordrecht, 1998.

[Gi2] V. Ginzburg, Nil Hecke algebras and Whittaker D-modules. e-print arXiv:1706.06751.

[Ha] T. Hausel, Mirror symmetry and Langlands duality in the non-abelian Hodge theory of a curve. Geometric methods in algebra and number theory, 193–217, Progr. Math., 235, Birkhäuser Boston, Boston, MA, 2005.

[HRV] T. Hausel and F. Rodriguez-Villegas, Mixed Hodge polynomials of character varieties. With an appendix by Nicholas M. Katz. Invent. Math. 174 (2008), no. 3, 555–624.

[HLRV] T. Hausel, E. Letellier and F. Rodriguez-Villegas Arithmetic harmonic analysis on character and quiver varieties. e-print arXiv:0810.2076

[HK] R. Hotta and M. Kashiwara, The invariant holonomic system on a semisimple Lie algebra. Invent. Math. 75 (1984), no. 2, 327–358.

[KW] A. Kapustin and E. Witten, "Electric-Magnetic Duality And The Geometric Langlands Program," Commun. Number Theory Phys. 1 (2007), no. 1, 1-236. e-print hep-th/0604151.

[Ko1] B. Kostant, On Whittaker vectors and representation theory. Invent. Math. 48 (1978), no. 2, 101-184.

[Ko2] B. Kostant, The solution to a generalized Toda lattice and representation theory. Adv. in Math. 34 (1979), no. 3, 195-338.

[KK] B. Kostant and S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. Adv. in Math. 62 (1986), no. 3, 187-237.

[K] S. Kumar, Kac-Moody groups, their flag varieties and representation theory. Progress in Mathematics, 204. Birkhäuser Boston, Inc., Boston, MA, 2002. xvi+606 pp.

[LLMSSZ] T. Lam, L. Lapointe, J. Morse, A. Schilling, M. Shimozono and M. Zabrocki, k-Schur functions and affine Schubert calculus. Fields Institute Monographs, 33. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2014.

[L1] J. Lurie, Higher topos theory. arXiv:math.CT/0608040. Annals of Mathematics Studies, 170. Princeton University Press, Princeton, NJ, 2009.

[L2] J. Lurie, Higher Algebra. Available at http://www.math.harvard.edu/~lurie/

[L3] J. Lurie, On the classification of topological field theories. Available at http://www.math.harvard.edu/~lurie/ Current developments in mathematics, 2008, 129–280, Int. Press, Somerville, MA, 2009.

[Lo1] G. Lonergan, A Fourier transform for the quantum Toda lattice. e-print arXiv:1706.05344.

[Lo2] G. Lonergan, A remark on descent for Coxeter groups. arXiv:1707.01156.

[Lu] G. Lusztig, Character sheaves I. Adv. Math 56 (1985) no. 3, 193-237.

[M] O. Mathieu, Construction d'un groupe de Kac-Moody et applications. Compositio Math. 69 (1989), no. 1, 37–60.

[Ngô] Ngô Bao-Châu, Le lemme fondamental pour les algèbres de Lie. Pub. Math. IHES No. 111 (2010), 1–169.

[R] S. Raskin, $\mathcal{D}$-modules on infinite dimensional varieties. Available at http://math.mit.edu/~sraskin/dmod.pdf

[T] C. Teleman, Gauge theory and mirror symmetry. arXiv:1404.6305. To appear, Proc. of the 2014 ICM.

[W] B. Webster, Koszul duality between Higgs and Coulomb categories $\mathcal{O}$. e-print arXiv:1611.06541.

[Y] Z. Yun, Goresky-MacPherson calculus for the affine flag varieties. e-print arXiv:0712.4395.

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