Stochastic Processes

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Rolando Rebolledo
Universidad de Valparaíso (Chile)

STOCHASTIC PROCESSES

Rolando Rebolledo [1]
Professor, and Head of the Center for Stochastic Analysis
Pontificia Universidad Católica de Chile, Chile

The word "stochastic process" is derived from the Greek noun "stokhos" which means "aim". Another related Greek word "stokhastikos", "the dart game", provides an alternative image for randomness or chance. Although the concept of Probability is often associated with dice games, the dart game seems to be more adapted to the modern approach to both Probability Theory and Stochastic Processes. Indeed, the fundamental difference between a dice game and darts is that while in the first, one cannot control the issue of the game, in the dart game, one tries to attain an objective with different degrees of success, thus, the player increases his knowledge of the game at each trial. As a result, time is crucial in the dart game, the longer you play, the better you increase your skills.

1. Definition of a Stochastic Process

The mathematical definition of a stochastic process, in the Kolmogorov model of Probability Theory, is given as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, that is, Ω is a non empty set called *sample space*, $\mathcal{F}$ is a sigma field of subsets of Ω, which represents the family of *events*, and $\mathbb{P}$ is a *probability measure* defined on $\mathcal{F}$. T is another non empty set, and $(E, \mathcal{E})$ a measurable space to represent all possible *states*. Then, a *stochastic process with states in* E is a map $X : T \times \Omega \to E$ such that for all $t \in T$, $\omega \mapsto X(t, \omega)$ is a measurable function. In other words, a primary interpretation of a stochastic process X is as a collection of random variables, and as such, notations like $(X_t)_{t \in T}$ are used to refer to X, that is $X_t(\omega) = X(t, \omega)$, for all $(t, \omega) \in T \times \Omega$. If T is an ordered number set, (eg. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$, $\mathbb{R}$), it is often referred as the set of *time variables* and taken as a subset of integers or real numbers.For each $\omega \in \Omega$, the map $X(\cdot, \omega) : t \mapsto X(t, \omega)$ is called the *trajectory* of the process. Thus, each trajectory is an element of E^T, the set of all E-valued functions defined

[1] Rolando Rebolledo obtained his "Doctorat d'État" at the Université Pierre et Marie Curie (Paris VI), France, in 1979. He is Professor at the Faculty of Mathematics, and Head of the Center for Stochastic Analysis, Pontificia Universidad Católica de Chile. He was President of the Sociedad de Matemática de Chile during five periods (1982–1985, 1994–1995, 1995–1998). He was Chairman of the Latin American Regional Committee of the Bernoulli Society (1989–1993), member of the Council of that Society and Scientific Secretary of the Committee for the Year 2000 of the Bernoulli Society. He chaired the Commission on Development and Exchanges of the International Mathematical Union (1994–1998, 1998–2002). Professor Rebolledo is a member of the American Mathematical Society, Bernoulli Society, Fellow of the International Statistics Institute since 1994. He has been twice awarded with the "Presidential Chair" in Chile (1995–1998, 1999–2002), and with the Medal of the Catholic University for outstanding research achievements (1996 and 1999). Dr. Rebolledo has published over 80 research papers, and edited five Proceedings of the International ANESTOC Workshops. Rolando Rebolledo has been Visiting Professor at many universities all over the world, including Denmark, Germany, Brazil, Venezuela, Italy, France, Russia, Australia, US, and Portugal.

on T. Particularly, if T is a countable set, the process is said to be indexed by *discrete times* (the expression *Time Series* is also in use in this case). Discrete time stochastic processes were the first studied in Probability Theory under the name of *chains* (see *Markov Chains*).

Examples

1. Consider a sequence $(\xi_n)_{n\geq 1}$ of real random variables. According to the definition, this is a stochastic process. New stochastic processes can be defined on this basis. For instance, take $(S_n)_{n\geq 1}$, defined as, $S_n = \xi_1 + \ldots + \xi_n$, for each $n \geq 1$.

   Suppose now that the random variables $(\xi_n)_{n\geq 1}$ are independent and identically distributed on $\{-1, 1\}$ with $\mathbb{P}(\xi = \pm 1) = 1/2$. Then, $(S_n)_{n\geq 1}$ becomes a *Simple Symmetric Random Walk*.

2. Consider a real function $x : [0, \infty[\to \mathbb{R}$, this is also a stochastic process. It suffices to consider any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define $X(\omega, t) = x(t)$, for all $\omega \in \Omega$, $t \geq 0$. This is a trivial stochastic process.

3. Consider an initial value problem given by

$$\begin{cases} x' = f(t, x); \\ x(0) = x, \end{cases} \tag{1}$$

   where f is a continuous function on the two variables (t, x). Newtonian Mechanics can be written within this framework, which is usually referred as a mathematical model for a *closed dynamical system* in Physics. That is, the system has no interaction with the environment, and time is reversible. Now define Ω as the set of all continuous functions from $[0, \infty[$ into $\mathbb{R}$. Endow Ω with the topology of uniform convergence on compact subsets of the positive real line and call $\mathcal{F}$ the corresponding Borel σ-field. Thus, any $\omega \in \Omega$ is a function $\omega = (\omega(t); t \geq 0)$. Define the stochastic process $X(\omega, t) = \omega(t)$, known as the *canonical process*. The initial value problem is then written as

$$X(\omega, t) = x + \int_0^t f(s, X(\omega, s))ds. \tag{2}$$

   This can be phrased as an example of a *Stochastic Differential Equation*, without noise term. The solution is a deterministic process which provides a description of the given closed dynamical system. Apparently, there is no great novelty and one can wonder whether the introduction of Ω is useful. However, this framework includes processes describing open dynamical systems too, embracing the interaction of the main system with the environment, and that is an important merit of the stochastic approach. Typically, the interaction of a given system with the environment is described through the action of so-called noises interfering with the main dynamics. Let us complete our example adding a noise term to the closed dynamics.

To consider the action of a *noise*, take a sequence $(\xi_n)_{n\geq 1}$ of random variables defined on Ω, such that $\xi_n(\omega) \in \{-1, 1\}$. Let be given a probability $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$ such that $\mathbb{P}(\xi_n = \pm 1) = 1/2$. Call $S_n = \xi_1 + \ldots + \xi_n$ and denote $[t]$ the greatest integer $\leq t$. The equation

$$X(\omega, t) = x + \int_0^t f(s, X(\omega, s))ds + S_{[t]}(\omega), \tag{3}$$

is an example of a stochastic differential equation driven by a random walk. The stochastic process obtained as a solution is no longer deterministic and describes an open system dynamics.

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2. Distributions

The space of trajectories E^T is usually endowed with the product σ-field $\mathcal{E}^{\otimes T}$ generated by all projections $\pi_t : E^T \to E$, which associate to each function $x \in E^T$ its value $x(t) \in E$, $t \in T$. Thus, a stochastic process is, equivalently, a random variable $X : \Omega \to E^T$, $\omega \mapsto X(\cdot, \omega)$. The *Law* or *Probability Distribution* P_X of a stochastic process X is the image of the probability $\mathbb{P}$ on the measurable space $(E^T, \mathcal{E}^{\otimes T})$ of all trajectories. Given a probability measure P on the space $(E^T, \mathcal{E}^{\otimes T})$, one may construct a *Canonical Process* X whose distribution P_X coincides with P. Indeed, it suffices to consider $\Omega = E^T$, $\mathcal{F} = \mathcal{E}^{\otimes T}$, $\mathbb{P} = P$, $X(t, \omega) = \omega(t)$, for each $\omega = (\omega(s);\ s \in T) \in E^T$, $t \in T$.

Let a finite set $I = \{t_1, \ldots, t_n\} \subset T$ be given, and denote π_I the canonical projection defined on E^T with values in E^I, such that $x \mapsto (x(t_1), \ldots, x(t_n))$. Call $\mathcal{P}_f(T)$ the family of all finite subsets of T. The *Finite Dimensional Distributions* or *Marginal Probability Distributions* of an E-valued stochastic process is the family $(P_{X,I})_{I\in\mathcal{P}_f(T)}$ of distributions, where $P_{X,I}$ is defined as

$$P_{X,I}(A) = P_X(\pi_i^{-1}(A)) = \mathbb{P}\left((X(t_1, \cdot), \ldots, X(t_n, \cdot)) \in A\right), \tag{4}$$

for all $A \in \mathcal{E}^{\otimes I}$.

Examples

1. A *Poisson Process* $(N_t)_{t\geq 0}$ is defined as a stochastic process with values in $\mathbb{N}$ such that

   (a) $N_0(\omega) = 0$ and $t \mapsto N_t(\omega)$ is increasing, for all $\omega \in \mathbb{N}$,

   (b) for all $0 \leq s \leq t < \infty$, $N_t - N_s$ is independent of $(N_u;\ u \leq s)$,

(c) for all $0 \leq s \leq t < \infty$, the distribution of $N_t - N_s$ is Poisson with parameter $t-s$, that is

$$\mathbb{P}(N_t - N_s = k) = \frac{(t-s)^k}{k!} e^{-(t-s)}.$$

2. A d-dimensional *Brownian Motion* is a stochastic process $(B_t)_{t\geq 0}$, taking values in $\mathbb{R}^d$ such that:

   (a) if $0 \leq s < t < \infty$, then $B_t - B_s$ is independent of $(B_u;\ u \leq s)$,

   (b) if $0 \leq s < t < \infty$, then

   $$\mathbb{P}(B_t - B_s \in A) = (2\pi(t-s))^{-d/2} \int_A e^{-|x|^2/2(t-s)} dx,$$

   where dx represents the Lebesgue measure on $\mathbb{R}^d$ and $|x|$ is the euclidian norm in that space.

   The Brownian Motion starts at x if $\mathbb{P}(B_0 = x) = 1$.

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3. Construction of canonical processes

An important problem in the construction of a canonical stochastic process given the family of its finite dimensional distributions was solved by Kolmogorov in the case of a countable set T and extended to continuous time later by several authors. At present, a particular case, general enough for applications, is the following version of the Daniell-Kolmogorov Theorem. Suppose that E is a Polish space (complete separable metric space) and let $\mathcal{E}$ be its Borel σ-field. Let T be a subset of $\mathbb{R}^+$. Suppose that for each $I \in \mathcal{P}_f(T)$ a probability P_I is given on the space $(E, \mathcal{E}^{\otimes I})$. Then, there exists a probability P on $(E^T, \mathcal{E}^{\otimes T})$ such that for all $I \in \mathcal{P}_f(T)$,

$$P_I(A) = P \circ \pi_I^{-1}(A) = P(\pi_I^{-1}(A)), \tag{5}$$

for all $A \in \mathcal{E}^{\otimes I}$, if and only if the following *Consistency Condition* is satisfied:

$$P_I = P_J \circ \pi_{J,I}^{-1}, \tag{6}$$

for all $I, J \in \mathcal{P}_f(T)$ such that $I \subset J$, where $\pi_{J,I}$ denotes the canonical projection from the space E^J onto E^I.

Example Consider $J = \{t_1, \dots, t_n\}$ and let Φ_t be the normal distribution of mean zero and variance $t \geq 0$, that is,

$$\Phi_t(A) = (2\pi t)^{-1/2} \int_A e^{-x^2/2t} dx.$$

Let $P_J = \Phi_{t_1} \otimes \Phi_{t_2-t_1} \otimes \ldots \Phi_{t_n-t_{n-1}}$, that is for all Borel sets $A_1, \ldots, A_n$,

$$P_J(A_1 \times A_2 \times \ldots \times A_n) = \Phi_{t_1}(A_1)\Phi_{t_2-t_1}(A_2) \ldots \Phi_{t_n-t_{n-1}}(A_n).$$

This is a probability on $\mathbb{R}^n$. Take $I = \{t_1, \ldots, t_{n-1}\}$. Notice that $\pi_{J,I}^{-1}(A_1 \times \ldots \times A_{n-1}) = A_1 \times \ldots \times A_{n-1} \times \mathbb{R}$, thus

$$\begin{aligned} P_I(A_1 \times A_2 \times \ldots \times A_{n-1}) &= \Phi_{t_1}(A_1)\Phi_{t_2-t_1}(A_2) \ldots \Phi_{t_{n-1}-t_{n-2}}(A_{n-1}) \\ &= P_J(A_1 \times A_2 \times \ldots \times \mathbb{R}). \end{aligned}$$

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4. Regularity of trajectories

Another interpretation of a stochastic process is based on regularity properties of trajectories. Indeed, if one knows that each trajectory belongs almost surely to a function space $S \subset E^T$, endowed with a σ-field $\mathcal{S}$, one may provide another characterization of the stochastic process X as an S-valued random variable, $\omega \mapsto X(\cdot, \omega)$ defined on Ω.

Regarding the regularity, Kolmogorov first proved one of the most useful criteria on continuity of trajectories. Suppose that $X = (X(t,\omega);\ t \in [0,1],\ \omega \in \Omega)$ is a real-valued stochastic process and assume that there exist $\alpha, \delta > 0$ and $0 < C < \infty$ such that

$$\mathbb{E}\left(|X(t+h) - X(t)|^{\alpha}\right) < C\,|h|^{1+\delta}\,, \tag{7}$$

for all $t \in [0,1]$ and all sufficiently small $h > 0$, then X has continuous trajectories with probability 1. Therefore, if X satisfies (**??**), then there exists a random variable $\widetilde{X} : \Omega \to C[0,1]$, where $C[0,1]$ is the metric space of real continuous functions defined on $[0,1]$, endowed with the metric of uniform distance, such that $\mathbb{P}\left(\left\{\omega \in \Omega :\ X(\cdot,\omega) = \widetilde{X}(\omega)\right\}\right) = 1$.

5. Wiener Measure, Brownian Motion

The above result is crucial to construct the *Wiener Measure* on the space $C[0,1]$ or, more generally, on $C(\mathbb{R}^+)$, which is the law of the *Brownian Motion*. Indeed, by means of Kolmogorv's Consistency Theorem, one first constructs a probability measure P on the product space $(\mathbb{R}^{\mathbb{R}^+}, \mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}^+})$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field of $\mathbb{R}$, considering the consistent family of probability distributions

$$P_I = \Phi_{t_1} \otimes \Phi_{t_2-t_1} \otimes \ldots \otimes \Phi_{t_n-t_{n-1}}, \tag{8}$$

where $I = \{t_1, \ldots, t_n\}$, and Φ_t denotes the normal distribution with mean 0 and variance t. Since the family $(P_I)_{I \in \mathcal{P}_f(\mathbb{R}^+)}$ is consistent, there exists a unique P probability measure on $(\mathbb{R}^{\mathbb{R}^+}, \mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}^+})$ such that $P_I = P \circ \pi_I^{-1}$. One can construct the canonical process with law P which should correspond to the Brownian

Motion. Unfortunately, the set of real-valued continuous functions defined on $\mathbb{R}^+$ is not an element of $\mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}^+}$. However, thanks to (**??**) one proves that the exterior probability measure P^* defined by P is concentrated on the subset $C(\mathbb{R}^+)$ of $\mathbb{R}^{\mathbb{R}^+}$ thus, the restriction P_W of P^* to $C(\mathbb{R}^+)$ gives the good definition of Wiener Measure. Thus, a canonical version of the Brownian Motion is given by the canonical process on the space $C(\mathbb{R}^+)$.

6. Series expansion in L^2

In the early years of the Theory of Stochastic Processes, a number of authors, among them Karhunen and Loève, explored other regularity properties of trajectories, deriving some useful representations by means of series expansions in an L^2 space. More precisely, let $T \in \mathcal{B}(\mathbb{R}^+)$ be given and call $\mathfrak{h} = L^2(T)$ the Hilbert space of all real-valued Lebesgue-square integrable functions defined on T. Suppose that all trajectories $X(\cdot,\omega)$ belong to $\mathfrak{h}$ for all $\omega \in \Omega$, and denote $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis of $\mathfrak{h}$. Therefore, $x_n(\omega) = \langle X(\cdot,\omega), e_n\rangle$ satisfies $\sum_{n\in\mathbb{N}} |x_n(\omega)|^2 < \infty$, for all $\omega \in \Omega$. And the series

$$\sum_{n\in\mathbb{N}} x_n(\omega) e_n, \tag{9}$$

converges in $\mathfrak{h}$, providing a representation of $X(\cdot,\omega)$. So that, by an abuse of language one can represent $X(t,\omega)$ by $\sum_{n\in\mathbb{N}} x_n(\omega)e_n(t)$.

Example Consider $T = [0,1]$ and the Haar orthonormal basis on the space $\mathfrak{h} = L^2([0,1])$ constructed by induction as follows: $e_1(t) = 1$ for all $t \in [0,1]$;

$$e_{2^m+1} = \begin{cases} 2^{m/2}, \text{ if } 0 \le t < 2^{-m-1} \ , \\ -2^{m/2}, \text{ if } 2^{-m-1} \le t < 2^{-m}, \\ 0, \text{ otherwise.} \end{cases}$$

And finally, define $e_{2^m+j}(t) = e_{2^m+1}\left(t - 2^{-m}(j-1)\right)$, for $j = 1,\ldots,2^m$, $m = 0,1,\ldots$. Given a sequence $(b_n)_{n\ge 1}$ of independent standard normal random variables (that is, with distribution $\mathcal{N}(0,1)$), the $L^2(\Omega\times[0,1])$-convergent series $\sum_{n\ge1} b_n(\omega)e_n(t)$ provides a representation of the Brownian Motion $(B_t)_{t\in[0,1]}$. $\triangledown$

7. The General Theory of Processes

The General Theory of Processes emerged in the seventies as a contribution of the Strasbourg School initiated by Paul André Meyer. This Theory uses the concept of a *History* or *Filtration*, which consists of an increasing family of σ-fields $\mathbb{F} = (F_t)_{t\in T}$, where T is an ordered set, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s \le t$. Thus, a stochastic process X is *adapted* to $\mathbb{F}$ if for all $t \in T$, the variable $X(t,\cdot)$ is $\mathcal{F}_t/\mathcal{E}$–measurable. Stronger measurability conditions mixing regularity conditions have been introduced motivated by the construction of stochastic integrals and the modern theory of Stochastic Differential Equations. Let $T = \mathbb{R}^+$ and assume E to be a Polish space endowed with the σ-field of its Borel sets. Denote $C_E = C(\mathbb{R}^+, E)$ (respectively $D_E = D(\mathbb{R}^+, E)$) the space of all E-valued continuous functions defined on $\mathbb{R}^+$ to E (resp. the space of all E-valued functions which have left hand limit at each point $t > 0$ and are right-continuous at $t > 0$, endowed with the Skorokhod's topology). Consider now the family $\mathcal{C}_E$ (resp. $\mathcal{D}_E$) of all $\mathbb{F}$-adapted stochastic processes $X : \mathbb{R}^+ \times \Omega \to E$ such that their trajectories belong to C_E (resp. to D_E). The *Predictable* (resp. *Optional*) σ-field on the product set $\mathbb{R}^+ \times \Omega$ is the one generated by $\mathcal{C}_E$ (resp. $\mathcal{D}_E$), that is $\mathcal{P} = \sigma(\mathcal{C}_E)$, (resp. $\mathcal{O} = \sigma(\mathcal{D}_E)$). Then, a process X is *predictable* (resp. *optional*) if $(t, \omega) \mapsto X(t, \omega)$ is measurable with respect to $\mathcal{P}$, (resp. $\mathcal{O}$). A crucial notion in the development of this theory is that of *Stopping Time*: a function $\tau : \Omega \to [0, \infty]$ is a stopping time if for all $t > 0$, $\{\omega \in \Omega : \tau(\omega) \leq t\}$ is an element of the σ-field $\mathcal{F}_t$. This definition is equivalent to say that τ is a stopping time if and only if $(t, \omega) \mapsto 1_{[0,\tau(\omega)[}(t)$ is an optional process, where the notation 1_A is used for the indicator or characteristic function of a set A.

The development of the General Theory of Processes encountered at least two serious difficulties which could not be solved in the framework of Measure Theory and required a use of Capacity Theory. They are the *Section Theorem* and the *Projection Theorem*. The Section Theorem asserts that if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete (that is $\mathcal{F}$ contains all $\mathbb{P}$-null sets) and $A \in \mathcal{O}$, then there exists a stopping time τ such that its graph is included in A. And the Projection Theorem states that given an optional set $A \subset \mathbb{R}^+ \times \Omega$, the projection $\pi(A)$ on Ω belongs to the complete σ-field $\mathcal{F}$. For instance, this result allows to prove that given a Borel set B of the real line, the random variable $\tau_B(\omega) = \inf \{t \geq 0 : X(t, \omega) \in B\}$ ($\inf \emptyset = \infty$), defines a stopping time for an $\mathbb{F}$-adapted process X with trajectories in D almost surely, provided the filtration $\mathbb{F}$ is right-continuous, that is, for all $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$, and in addition each σ-field contains all $\mathbb{P}$-null sets. Within this theory, the system $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})$ is usually called a *Stochastic Basis* and a system $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, E, \mathcal{E}, \mathbb{P}, (X_t)_{t \in T})$ provides the whole structure needed to define an E-valued adapted stochastic process.

Attending to measurability properties only, stochastic processes may be classified as optional or predictable, as mentioned before, for which no probability is needed. However, richer properties of processes strongly depend on the probability considered in the stochastic basis. For instance, the definitions of *martingales*, *submartingales*, *supermartingales*, *semimartingales* depend on a specific probability measure, through the concept of *conditional expectation*. Let us mention that *semimartingales* form the most general class of possible integrands to give a rigorous meaning to *Stochastic Integrals* and *Stochastic Differential Equations*.

Probability is moreover fundamental for introducing concepts as *Markov Process*, *Gaussian Process*, *Stationary Sequence and Stationary Process*.

8. Extensions of the Theory

Extensions to the theory have included changing either the nature of T to consider *Random Fields*, where $t \in T$ may have the meaning of a space label (T is no more a subset of the real line), or the state space E, to deal for instance with measure-valued processes, or random distributions.

Example. Let $(T, \mathcal{T}, \nu)$ be a σ-finite measure space, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Call $\mathcal{T}_\nu$ the family of all sets $A \in \mathcal{T}$ such that $\nu(A) < \infty$. A *Gaussian white noise* based on ν is a random set function W defined on $\mathcal{T}_\nu$ and values in $\mathbb{R}$ such that

(a) $W(A)$ is centered Gaussian and $\mathbb{E}\left(W(A)^2\right) = \nu(A)$, for all $A \in \mathcal{T}_\nu$;

(b) If $A \cap B = \emptyset$, then $W(A)$ and $W(B)$ are independent.

In particular, if $T = \mathbb{R}^{+2}$, $\mathcal{T}$ the corresponding Borel σ-field, and $\nu = \lambda$ the product Lebesgue measure, define $B_{t_1,t_2} = W(]0,t_1]\times]0,t_2])$, for all $(t_1, t_2) \in T$. The process $(B_{t_1,t_2})_{(t_1,t_2)\in T}$ is called the *Brownian sheet.* $\triangledown$

Going further, on the state space E consider the algebra $\mathfrak{E}$ of all bounded $\mathcal{E}$-measurable complex-valued functions. Then, to each E-valued stochastic process X one associates a family of maps $j_t : \mathfrak{E} \to L^\infty_{\mathbb{C}}(\Omega, \mathcal{F}, \mathbb{P})$, where $j_t(f)(\omega) = f(X(t,\omega))$, for all $t \geq 0$, $\omega \in \Omega$. The family $(j_t)_{t\in\mathbb{R}^+}$, known as the *Algebraic Flow* can be viewed as a family of complex random measures (each j_t is a Dirac measure supported by $X(t,\omega)$) or, better, as a *-homomorphism between the two *-algebras $\mathfrak{E}$, $L^\infty_{\mathbb{C}}(\Omega, \mathcal{F}, \mathbb{P})$, the * operation being here the complex conjugation. The stochastic process is completely determined by the algebraic flow $(j_t)_{t\in\mathbb{R}^+}$.

Example. Consider a Brownian motion B defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$, with states in $\mathbb{R}$, and call $\mathfrak{B}$ the algebra of bounded complex valued Borel function defined on the real line. $\mathfrak{B}$ is a *-algebra of functions, that is, there exists an involution * (the conjugation), such that $f \mapsto f^*$ is antilinear and $(fg)^* = g^* f^*$, for all $f, g \in \mathfrak{B}$. The algebraic flow associated to B is given by $j_t(f) = f(B_t)$, for all $t \geq 0$, and any $f \in \mathfrak{B}$, that is $j_t : \mathfrak{B} \to L^\infty_{\mathbb{C}}(\Omega, \mathcal{F}, \mathbb{P})$. If $\mathbb{P}(B_0 = x) = 1$, then $j_0(f) = f(x)$ almost surely. Moreover, notice that Itô's formula implies that for all bounded f of class C^2, it holds

$$j_t(f) = f(x) + \int_0^t j_s\left(\frac{d}{dx}f\right) dB_s + \int_0^t j_s\left(\frac{1}{2}\frac{d^2}{dx^2}f\right) ds.$$

$\triangledown$

Algebraic flows provide a suitable framework to deal with more generalized evolutions, like those arising in the description of *Open Quantum System Dynamics*, where the algebras are non commutative. Thus, given two unital *-algebras (possibly non commutative) $\mathfrak{A}, \mathfrak{B}$, a notion of *Algebraic Stochastic Process* is given by a flow $(j_t)_{t\in\mathbb{R}^+}$, where $j_t : \mathfrak{B} \to \mathfrak{A}$ is a *-homomorphisms, for all $t \geq 0$. That is, each j_t is a linear map, which satisfies $(j_t(b))^* = j_t(b^*)$, $j_t(a^*b) = j_t(a)^* j_t(b)$, for all $a, b \in \mathfrak{B}$, and $j_t(\mathbf{1}_{\mathfrak{B}}) = \mathbf{1}_{\mathfrak{A}}$, where $\mathbf{1}_{\mathfrak{A}}$ (resp. $\mathbf{1}_{\mathfrak{B}}$) is the unit of $\mathfrak{A}$ (resp. $\mathfrak{B}$).

9. The dawning of Stochastic Analysis as a pillar of Modern Mathematics

These days, Stochastic Processes provide the better description of complex evolutionary phenomena in Nature. Coming from our understanding of the macro world, through our everyday life, exploring matter at its smallest component, stochastic modeling has become fundamental. In other words, stochastic processes have become influential in all sciences, namely, in biology (population dynamics, ecology, neurosciences), computer science, engineering (especially electric and operation research), economics (via finance), physics, among others. The new branch of Mathematics, known as Stochastic Analysis, is founded on stochastic processes. Stochastics is invading all branches of Mathematics: Combinatorics, Graph Theory, Partial and Ordinary Differential Equations, Group Theory, Dynamical Systems, Geometry, Functional Analysis, among many other specific subjects. The dawning of Stochastic Analysis era is a fundamental step in the evolution of human understanding of Chance as a natural interconnection and interaction of matter in Nature. This has been a long historical process which started centuries ago with the dart game.

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The following bibliography is nothing but a very small sample of references on stochastics, which could be termed classic, as well as more recent textbooks. General references as well as specialized books on the field are fast increasing, following the success of stochastic modeling, and one can be involuntarily and easily unfair by omitting outstanding authors.

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