Deepayan Bhadra - hmwk1 Solutions

Solution to Q1

Hilbert matrices are known for being ill-conditioned. Thus, due to the high condition number, the noise amplifies during reconstruction. The amplification added to the trailing eigen-vector is $1/\lambda_{min}$

Solution to Q2

The dual norm of $\|\cdot\|$ is defined as

$$\|x\|_* = \max_{\|z\|\leq 1} z^T x.$$

We need to prove that the dual norm is indeed a norm. Thus, we need to verify if the dual norm satisfies the properties of a norm as follows:

1. $||x|| \geq 0 \ \forall$ x$\in R^n$, and $||x|| = 0$ iff x = 0 (Non-negativity)
2. $||\alpha x|| = |\alpha| ||x|| \quad \forall x \in R^n \quad \forall \alpha \in R$ (Homogeneity)
3. Triangle inequality: $||x + y|| \leq ||x|| + ||y|| \quad \forall x, y \in R^n$

The first two properties are easy to verify as follows:

1. z can always be chosen such $||x||_* \geq 0$. Since z can also be > 0, $||x||_* = 0$ iff $x = 0$. Hence.
2. $\|\alpha x\|_* = \max_{\|z\|\leq 1} z^T(\alpha x) = |\alpha| \max_{\|z\|\leq 1} z^T x = |\alpha|\|x\|_*$
3. $||x + y||_* = \max_{\|z\|\leq 1}(z^T x + z^T y) \leq \max_{\|z\|\leq 1} z^T x + \max_{\|z\|\leq 1} z^T y = \|x\|_* + \|y\|_*$

Solution to Q3

We have $||\hat{x} - x|| = ||A^{-1}\hat{b} - A^{-1}b|| \leq ||A^{-1}||||\hat{b} - b||$

Also, $||Ax|| = ||b|| \leq ||A||||x||$

Dividing the first inequality by the latter one yields

$\frac{||\hat{x}-x||}{||A||||x||} \leq \frac{||A^{-1}||||\hat{b}-b||}{||b||}$

Condition number of a square nonsingular matrix A is defined as $||A||.||A||^{-1}$
Thus re-arranging the inequality, we get our desired result.

Solution to Q4

We have

$p(y|x) = \frac{1}{\sqrt{(2\pi)^m|\Sigma|}}exp\big(- \frac{1}{2}(y - Dx)^T\Sigma^{-1}(y - Dx)\big)$ $p(x) = \Pi_i \frac{1}{2b}exp(-\frac{|x_i|}{b}) = \frac{1}{(2b)^n}exp(-\frac{||x||_1}{b})$ From Baye's rule, we have

$p(x|y) \sim p(y|x)p(x) = \frac{1}{\sqrt{(2\pi)^m|\Sigma|}}exp\big(-\frac{1}{2}(y-Dx)^T\Sigma^{-1}(y-Dx)\big)\frac{1}{(2b)^n}exp(-\frac{||x||_1}{b})$
The log-likelihood (LL) follows by taking log

$-\frac{m}{2}log(2\pi) - \frac{1}{2}log|\Sigma| - \frac{1}{2}(y - Dx)^T\Sigma^{-1}(y - Dx) - nlog(2b) - \frac{1}{b}||x||_1$

The negative log-likelihood (NLL) occurs by multiplying with -1 and re-arranging
:

$\frac{1}{b}||x||_1 + \frac{1}{2}(y - Dx)^T\Sigma^{-1}(y - Dx) + \frac{1}{2}(mlog(2\pi) + log|\Sigma|) + nlog(2b)$

For minimization, we can ignore the constant terms and simply :

Minimize $\frac{1}{b}||x||_1 + \frac{1}{2}(y - Dx)^T\Sigma^{-1}(y - Dx)$.

Solution to Q5: Code from hmwk1.py

```
import numpy as np
def buildmat(m,n,condNumber):
    A = np.random.randn(m, n)
    np.linalg.cond(A)
    U, S, V = np.linalg.svd(A)
    S = np.array([[S[j] if i==j else 0 for j in range(n)] for i in range(m)])
    S[S!=0]= np.linspace(condNumber,1,min(m,n))
    A=U.dot(S).dot(V)
    return A
# For a 3x5 matrix

m,n,condNumber = 3,5,2
print("The 3x5 matrix A and the condition no. are\n")
A = buildmat(m,n,condNumber)
print(np.matrix(A),"\n")
print(np.linalg.cond(A),"\n")

# For a 5x4 matrix

m,n,condNumber = 5,4,4
print("The 5x4 matrix A and the condition no. are\n")
A = buildmat(m,n,condNumber)
print(np.matrix(A),"\n")
print(np.linalg.cond(A),"\n")
```