

# Linear Algebra -25MT103 - Module Bank

Section: 7, 14, 21

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## Question 1 (10 Marks)

- a) Define the characteristic equation of a matrix. Explain how eigenvalues are obtained from the characteristic equation. **(2 marks)**
- b) Find the characteristic equation and eigenvalues of the matrix  $A = \begin{bmatrix} 3 & -1 & 1 & 1 \end{bmatrix}$ . Verify your eigenvalues by substituting back into the characteristic equation. **(4 marks)**
- c) Check that the sum of eigenvalues of a matrix equals its trace, and the product of eigenvalues equals its determinant. Demonstrate this property with a  $3 \times 3$  matrix of your choice. **(4 marks)**

## Question 2 (10 Marks)

- a) State the Cayley-Hamilton Theorem. Indicate whether the theorem holds for every  $n \times n$  matrix over the complex numbers. **(2 marks)**
- b) Verify the Cayley-Hamilton Theorem for the matrix  $A = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ . Use this result to compute  $A^3$ . **(4 marks)**
- c) Outline the Cayley-Hamilton recipe that rewrites  $A^n$  as a linear combination of  $I$  and integer powers of  $A$ . Demonstrate with a specific example for  $n = 3$ . **(4 marks)**

## Question 3 (10 Marks)

- a) Define a diagonalizable matrix. State the necessary and sufficient condition for a matrix to be diagonalizable. **(2 marks)**
- b) Determine whether the matrix  $A = \begin{pmatrix} 3 & 1 & 0 & 3 \end{pmatrix}$  is diagonalizable. If yes, find matrices  $P$  and  $D$  such that  $A = PDP^{-1}$ . If no, justify your answer. **(4 marks)**
- c) Give one concrete  $3 \times 3$  matrix that has three distinct eigenvalues and show that it is diagonalizable. Construct a counterexample showing that a matrix can be diagonalizable even without  $n$  distinct eigenvalues. **(4 marks)**

## Question 4 (10 Marks)

- a) Define a real vector space and list the eight axioms that must be satisfied by the addition and scalar multiplication operations. **(3 marks)**
- b) Verify whether the set  $V = \{(x, y) : x, y \in \mathbb{R}, x \geq 0\}$  with standard addition and scalar multiplication forms a vector space. **(3 marks)**
- c) Can you give examples of vector space each with different operations but the same underlying set, where one is vector space and the other is not. **(4 marks)**

### Question 5 (10 Marks)

- a) Explain the concepts of linear combination and linear span of a set of vectors. **(2 marks)**
- b) Express the vector  $v = (2, -1, 5)$  as a linear combination of  $u_1 = (1, 0, 1)$ ,  $u_2 = (0, 1, 2)$ , and  $u_3 = (1, 1, 0)$ . Also, determine  $\text{span}\{u_1, u_2, u_3\}$ . **(4 marks)**
- c) Show that  $\text{span}\{v_1, v_2, \dots, v_n\}$  is the smallest subspace containing all the vectors  $v_1, v_2, \dots, v_n$ . Create a geometric interpretation for  $n = 2$  in  $\mathbb{R}^3$ . **(4 marks)**

### Question 6 (10 Marks)

- a) Define the row space and column space of an  $m \times n$  matrix  $A$ . How are they related to the matrix operations? **(3 marks)**
- b) For the matrix  $A = \begin{pmatrix} 1 & 2 & 3 & 2 & 4 & 6 & 1 & 1 & 2 \end{pmatrix}$ , find: (i) A basis for the row space (ii) A basis for the column space (iii) Verify that  $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$  **(4 marks)**
- c) Explain why elementary row operations preserve the row space but may change the column space. **(3 marks)**

### Question 7 (10 Marks)

- a) What is the difference between an inner product and a norm? How are they related? **(3 marks)**
- b) Let  $u = (1, 2)$  and  $v = (2, 3)$ . Find the inner product  $u$  and  $v$ . Also, find angle between  $u$  and  $v$  using the inner product. **(3 marks)**
- c) Analyze whether the following function is an inner product on  $\mathbb{R}^2$ :  $\langle (x_1, x_2), (y_1, y_2) \rangle = 3x_1y_1 + x_2y_2$ . Justify using axioms. **(4 marks)**

### Question 8 (10 Marks)

- Define the norm of a vector in an inner product space and express it in terms of the inner product. **(2 marks)**
- For the vectors  $u = (1, 2, 3)$  and  $v = (4, 5, 6)$  in  $R^3$  with the standard inner product, calculate  $\|u\|$ ,  $\|v\|$ , and  $\|u + v\|$ . **(4 marks)**
- Prove that for any vector  $v$  in an inner product space,  $\|\alpha v\| = |\alpha| \|v\|$  where  $\alpha$  is a scalar. Discuss the geometric significance of this property. **(4 marks)**

### Question 9 (10 Marks)

- Define linear dependence and linear independence of vectors. Provide one example of each. **(3 marks)**
- Test whether the vectors  $v_1 = (1, 2, 3)$ ,  $v_2 = (2, 3, 4)$ , and  $v_3 = (3, 5, 7)$  are linearly dependent or independent. If dependent, express one vector as a linear combination of others. **(4 marks)**
- Show that if a set  $S = \{v_1, v_2, \dots, v_n\}$  contains the zero vector, then  $S$  is linearly dependent. Extend this to prove that any set containing a linearly dependent subset is itself linearly dependent. **(3 marks)**

### Question 10 (10 Marks)

- List the steps involved in the Gram-Schmidt orthogonalization process. **(2 marks)**
- Apply the Gram-Schmidt process to orthogonalize the vectors  $v_1 = (1, 1, 0)$ ,  $v_2 = (1, 0, 1)$ , and  $v_3 = (0, 1, 1)$  in  $R^3$ . **(5 marks)**
- Design a modified Gram-Schmidt algorithm that produces an orthonormal set directly (without a separate normalization step at the end). Explain why this might be computationally advantageous. **(3 marks)**

### Question 11 ( 10 Marks)

- For the matrix  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ , find: (i) All eigenvalues (ii) Corresponding eigenvectors for each eigenvalue (iii) Verify that  $Av = \lambda v$  for each eigenvalue-eigenvector pair **(3 marks)**
- Determine the rank of the matrix  $A = \begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 3 & 2 & 2 & 4 & 2 & 5 & 7 \end{pmatrix}$  by: (i) Finding the dimension of row space (ii) Finding the dimension of column space (iii) Verifying both methods yield the same result **(5 marks)**
- Verify whether the function  $\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$  defines an inner product on  $R^2$ , where  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$ . **(2 marks)**