📘 About This Document

This document contains clear and detailed solutions to the questions included in the module bank 📚. It is designed to help students understand concepts better and revise effectively.

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Sec : 07

Q1.

a) Define the charecteristic equation of a matrix. Explain how eigenvalues are obtained from the characteristic equation.

Characteristic Equation:

For square $A \in \mathbb{R}^{n \times n}$, the charecteristic polynomial is $P_A(\lambda) = \det(A - \lambda I)$.

* The scalar roots of $P_A(\lambda) = 0$ are the eigen values of A.

Example:

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = P_A(\lambda)$$

$$= \det\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = P_A(\lambda)$$

$$= \det\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = P_A(\lambda)$$

$$= \det\left(\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\right) = P_A(\lambda)$$

$$= [(5-\lambda)(2-\lambda) - 4] = P_A(\lambda)$$

$$= \boxed{\lambda^2 - 7\lambda + 6 = 0} \rightarrow \text{charecteristic equation}$$

$$\boxed{\lambda = 1, 6} \rightarrow \text{eigen values}$$

b) Find the charecteristic equation and eigen values of the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Verify your eigenvalues by substituting back into the charecteristic equation.

Given,

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$\det(A - \lambda I) = P_A(\lambda)$

$$\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = P_A(\lambda)$$

$$\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) = P_A(\lambda)$$

$= [(3-\lambda)(1-\lambda)+1] = P_A(\lambda)$

$= 3 - 3\lambda - \lambda + \lambda^2 + 1$

$= \boxed{\lambda^2 - 4\lambda + 4 = 0} \rightarrow$ charecteristic equation

$\boxed{\lambda = 2, 2} \rightarrow$ eigen values

Substituting eigen values back into charecteristic equation

$= \lambda^2 - 4\lambda + 4 = 0$

$= (2)^2 - 4(2) + 4 = 0$

$= 4 - 8 + 4 = 0$

$= 0$

c) check that the sum of eigenvalues of a matrix equals its trace, and the product of eigen values equals its determinant. Demonstrate this property with a 3X3 matrix of your choice.

* $tr(ABCD) = tr(DABC) = tr(CDAB) = tr(BCDA)$

* $\det(ABCD) = \det A \cdot \det B \cdot \det C \cdot \det D$

$\Rightarrow$

$$tr(A) = tr(PDP^{-1}) = tr(D\underbrace{P^{-1}P}_{I})$$
$$= tr(DI)$$
$$= tr(D)$$

$\because A = PDP^{-1}$

$P^{-1}P = I$

$\therefore \boxed{tr(A) = tr(D)}$

where, $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$Tr(D) = \lambda_1 + \lambda_2 + \lambda_3$

= sum of eigen Values

$\Rightarrow \det(A) = \det(PDP^{-1})$

$\det(AB) = \det(A) \cdot \det(B)$

$$= \det(P) \cdot \det(D) \cdot \underbrace{\det(P^{-1})}_{I}$$
$$= \det(D)$$

$\therefore \boxed{\det(A) = \det(D)}$

$\because \det(P) \cdot \det(P^{-1}) = \det(PP^{-1})$

$= \det(I)$

$= 1$

where, $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\det(D) = \lambda_1[\lambda_2\lambda_3 - 0]$

$= \lambda_1 \lambda_2 \lambda_3$

= product of eigen values

$\therefore$ Sum of eigen values of a matrix equals its trace and the product of eigen values equals its det.

Example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$\det(A - \lambda I) = P_A(\lambda)$

$$\det\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = P_A(\lambda)$$

$$\det\left(\begin{bmatrix} (2-\lambda) & 1 & 0 \\ 0 & (3-\lambda) & 4 \\ 0 & 0 & (5-\lambda) \end{bmatrix}\right) = P_A(\lambda)$$

$= (2-\lambda)\left[(3-\lambda)(5-\lambda) - 0\right] - 1\left[0 - 0\right]$

$= (2-\lambda)(3-\lambda)(5-\lambda)$

$\therefore \lambda = 2, 3, 5$

* sum of eigen values of a matrix equals its trace

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$tr(A) = 2+3+5$
$= 10$

sum of $\lambda = 2+3+5$
$= 10$

$\therefore tr(A) =$ sum of eigen values (A)

* product of eigen values of a matrix equals its det

$\det(A) = 2(15) - 1(0)$
$= 30$

Product of $\lambda = 2 \times 3 \times 5$
$= 30$

$\therefore \det(A) =$ product of eigen values(A)

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Section - 07

a) State the Cayley-Hamilton theorem. Indicates whether the theorem holds for every nxn matrix over the complex numbers.

Soln

They cayley-Hamilton theorem states that every square matrix satisfies its own charactristic equation.

yes, the theorem holds for every nxn matrix over the complex numbers.

b) Verify the cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ use the result to compute A^3

Step 1

$A - \lambda I$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

Step 2

$\det(A - \lambda I)$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$$

$$(1-\lambda)(4-\lambda) - 6 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

Verify cayley hamilton theorm,

$$A^2 - 5A - 2I = 0$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} - \left[\begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ theorem verified

using result find A^3

$$A^2 - 5A - 2I = 0$$

$$A^2 = 5A + 2I \quad - (1)$$

multiply with A

$$A^3 = 5A^2 + 2A$$

then substitute A^2

$$A^3 = 5(5A + 2I) + 2A$$

$$= 25A + \cancel{10}10I + 2A$$

$$= 27A + 10I$$

$$= 27\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 54 \\ 81 & 108 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix}$$

c) outline the Cayley Hamilton recipe that rewrites A^n as a linear Combination of I and integer powers of A.
Demostrate with a specific example for $n=3$.

Sol^n Here using Cayley Hamilton theorem, any power A^n can be expressed as the linear combination of I and lower powers of A.

For $n=3$,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det(A - \lambda I)$$

$$= (1-\lambda)(4-\lambda) - 6 = 0$$

$$4 - 4\lambda - \lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

$$A^2 - 5A - 2I = 0 \Rightarrow A^2 = 5A + 2I$$

$$\boxed{A^3 = 5A^2 + 2A}$$

$\therefore$ Hence proved

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sec :- 21 (ECE)

3) a) Define a diagonalizable matrix. State the necessary and sufficient Condition for a matrix to be diagonalizable

b) Determine whether the matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is diagonalizable if yes, find matrices P and D such that $A = PDP^{-1}$ if no, Justify your answer

c) Give one Concrete 3x3 matrix that has three distinct eigen values and show that it is diagonalizable. Construct a counterexample showing that a matrix can be diagonalizable even without n distinct eigen values.

A is diagonalizable if there exists invertible P with $P^{-1}AP = D$ (diagonal) Then $A^k = PD^kP^{-1}$ which is cheap to Compute.

(or)

A square matrix A is said to be diagonalizable if it is similar to a diagonal matrix D. It means there exists an invertible matrix P and diagonal matrix D

$$A = PDP^{-1}$$

Necessary and Sufficient Condition for Diagonalizability.

* A $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigen vectors.
* An equivalent, more practical Condition relating the eigen values to the eigen vectors is:

A $n \times n$ matrix A is diagonalizable if and only if for every eigenvalue λ of A:

$$\boxed{\text{Algebraic Multiplicity of } \lambda = \text{Geometric multiplicity of } \lambda}$$

- Algebraic Multiplicity (Am): is the number of times λ is repeated as a root of the charecteristic polynomial $\det(A - \lambda I) = 0$
- Geometric Multiplicity (GM): is the dimension of the eigenspace Corresponding to λ, which is $\dim(\text{Null}(A - \lambda I))$ This is also the maximum number of linearly independent eigenvectors associated with λ.

b) Determine whether the matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is diagonalizable if yes, find matrices P and D such that $A = PDP^{-1}$ if no, Justify your answer.

The matrix A is an upper triangular matrix, so its eigen values are the entries on the main diagonal

The characteristic equation is $\boxed{\det(A - \lambda I) = 0}$

$$\det\left(\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}$$

$$\Rightarrow (3-\lambda)(3-\lambda) - 1(0) = (3-\lambda)^2 = 0$$

Eigen values are $\lambda = 3, 3$ $\xrightarrow{}$ ② $\rightarrow$ AM

The "Algebraic Multiplicity" (Am) of $\lambda = 3$ is $\boxed{AM(3) = 2}$ (since the factor $(3-\lambda)$ is repeated twice)

The Geometric Multiplicity (GM) and Eigen Vectors

For $\lambda = 3$

$$(A - 3I)X = 0 \quad \text{where } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(A - 3I) = \begin{pmatrix} 3-3 & 1 \\ 0 & 3-3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The system of equations is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$0x_1 + 1x_2 = 0 \quad \Rightarrow x_2 = 0$

$0x_1 + 0x_2 = 0 \quad \Rightarrow 0$

The eigen vectors are of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x_1 \neq 0$$

The Geometric Multiplicity (GM) of $\lambda = 3$ is the dimension of the eigen space, $GM(3) = 1$

Since $AM(3) = 2$ and $GM(3) = 1$

$AM \neq GM$

Therefore, the matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is not diagonalizable.

c) Give one concrete 3x3 matrix that has three distinct eigen values and show that it is diagonalizable. Construct a counter example showing that a matrix can be diagonalizable even without n distict eigen-values let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Eigenvalues

since A is diagonal matrix, its eigen values are the diagonal entries: $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

Diagonalizability

if an $n \times n$ matrix has n distinct eigenvalues then it is diagonalizable.

- P can be any invertible matrix whose columns are the eigen vectors. The eigen vectors for a diagonal matrix are the standard basis vectors.

For $\lambda_1 = 1$, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Equations from $(A - 1I)x = 0$

$0x + 1x_2 + 0x_3 = 0 \Rightarrow x_2 = 0$

$0x_1 + 0x_2 + 2x_3 = 0 \Rightarrow x_3 = 0$

$\therefore x_1 \neq 0 \Leftarrow 0x_1 + 0x_2 + 0x_3 = 0$

For $\lambda_2 = 2$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Equations from $(A - 2I)x = 0$

$-1x_1 + 0x_2 + 0x_3 = 0 \Rightarrow x_1 = 0$ $\quad 0x_1 + 0x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + 1x_3 = 0 \Rightarrow x_3 = 0$; $x_2 \neq 0$

For $\lambda_3 = 3$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

equations from $(A - 3I)x = 0$

$-2x_1 + 0x_2 + 0x_3 = 0 \Rightarrow x_1 = 0$

$0x_1 - 1x_2 + 0x_3 = 0 \Rightarrow x_2 = 0$

$x_3 \neq 0 \Leftarrow 0x_1 + 0x_2 + 0x_3 = 0$

The matrix P formed by these eigen vectors is

$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

D is Diagonal matrix with eigen values on the diagonal

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Since $A = D$ and $P = I$ we have $P^{-1} = I^{-1} = I$

$$PDP^{-1} = IDI = D = A$$

Thus. A is diagonalizable

Counter example:-

let B be the diagonal matrix

We need a 3×3 matrix that is diagonalizable but has at least one repeated eigenvalue (i.e. not three distinct ones) The easiest way is to choose a diagonal matrix with a repeated entry.

$$B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Eigen values

Since B is diagonal matrix, its eigen values are $\lambda_1 = 5$ and $\lambda_2 = 7$ It has only two distinct eigenvalues not three.

* $\lambda = 5$: $AM(5) = 2$
* $\lambda = 7$ $AM(7) = 1$

Diagonalizability

B is diagonalizable. $B = IBI^{-1}$

The geometric multiplicity

For $\lambda = 5$; $B - 5I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ | $0x_1 + 0x_2 + 2x_3 = 0$; $x_3 = 0$; $x_2 \neq 0$; $x_1 \neq 0$

The null space has dimension $3 - \text{rank}(B - 5I) = 3 - 1 = 2$

$GM(5) = 2 = AM(5)$ | $x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ free variables

Eigenvectors are $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\boxed{\text{Number of } \cancel{\text{free}} \text{ variables} = \text{Rank} + \text{No. of free variables}}$

For $\lambda = 7$

$B - 7I = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ | $-2x_1 = 0 \Rightarrow x_1 = 0$; $-2x_2 = 0 \Rightarrow x_2 = 0$; $0(x_3) = 0 \Rightarrow x_3$ can be real number.

Null space has dimension $3 - \text{rank}(B - 7I)$

$\Rightarrow 3 - 2 = 1$

$GM(7) = 1 = AM(7)$

The eigen vector is $x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$AM = GM$ for both eigen values, the matrix B is diagonalizable, even though it does not have three distinct eigen values.

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4 a) Define a real vector space and list the eight axioms that must be satisfied by the addition and scalar multiplication operations.

A real vector space V is a set together with two operations vector addition $+ : V \times V \to V$ and scalar multiplication : $R \times V \to V$. Such that the following axioms hold for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

1 $u + v = v + u$ (commutativity)

2 $(u+v)+w = u+(v+w)$ (associativity)

3 There exists $0 \in V$ with $v+0 = v$ for all v (additive identity)

4 each v has an additive inverse $-v$

5 $\alpha(v+w) = \alpha v + \alpha w$ (distributivity)

6 $(\alpha+\beta)v = \alpha v + \beta v$ (distributivity)

7 $(\alpha\beta)v = \alpha(\beta v)$

8 $1 \cdot v = v$

9 closure of vector addition :

for every $v, w \in V$, then $v+w \in V$

10 closure of scalar multiplication :

for every $k \in \mathbb{R}$, $v \in V$, then $kv \in V$

4 b) Verify whether the set $V = \{(x,y) : x, y \in \mathbb{R},\ x \geq 0\}$ with standard addition & scalar multiplication forms a vector space.

Additive Identity:

$\exists\ 0 \in V,\quad v + 0 = v \quad \forall\ v \in V$

Let $v = (x,y) \in \mathbb{R}^2,\ 0 = (a,b) \in \mathbb{R}^2$

$v + 0 = v$

$(x+a,\ y+b) = (x,y)$

$x + a = x \quad | \quad y + b = y$

$a = 0 \quad | \quad b = 0$

So, $0 = (0,0)$

Additive Inverse:

$v + (-v) = 0$

Let $v = (x,y) \in \mathbb{R}^2$

$v + (-v) = 0$

$(x,y) + (a,b) = (0,0)$

$(x+a,\ y+b) = (0,0)$

$x + a = 0 \quad | \quad y + b = 0$

$a = -x \quad | \quad b = -y$

So, $-v = (-x,-y) \notin V$

because $x \geq 0$ condition not satisfied

Thus, the given set V is not a vector space.

4c) Can you give examples of vector space each with different operations but the same underlying set, where one is vector space and other is not ?

<u>example for a vector space</u>

Set $V = \mathbb{R}^2$ $\quad (\mathbb{R}^2, +, \cdot)$

with standard addition,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$$

& standard scalar multiplication,

$$\alpha(x, y) = (\alpha x, \alpha y) \in \mathbb{R}^2$$

example for not a vector space

Set $V = \mathbb{R}^2$ $(\mathbb{R}^2, \oplus, \cdot)$

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 1, \; y_1 + y_2 + 1) \in \mathbb{R}^2$

$\alpha \cdot (x, y) = (\alpha x, \alpha y)$

Distributivity:

$\alpha (u + v) = \alpha \cdot u + \alpha \cdot v$

Let $\alpha \in \mathbb{R}$ $\quad u = (x_1, y_1) \quad v = (x_2, y_2)$

LHS

$\alpha \cdot (u \oplus v)$

$= \alpha \cdot (x_1 + x_2 + 1, \; y_1 + y_2 + 1)$

$= (\alpha x_1 + \alpha x_2 + \alpha, \alpha y_1 + \alpha y_2 + \alpha)$

RHS

$\alpha \cdot u \oplus \alpha \cdot v$

$= \alpha \cdot (x_1, y_1) \oplus \alpha \cdot (x_2, y_2)$

$= (\alpha x_1, \alpha y_1) \oplus (\alpha x_2, \alpha y_2)$

$= (\alpha x_1 + \alpha x_2 + 1, \alpha y_1 + \alpha y_2 + 1)$

LHS $\neq$ RHS

So, its not a vector space.

5) a) Explain the concepts of linear combination and linear span of a set of vectors

Linear combination of vector:

"Let $v_1, v_2, \ldots v_k$ in a vector space V is any vector of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$. where $\alpha \in \mathbb{R}$ are scalars" is known as linear combination of vector.

Linear span :

"Set of linear combination of vector"

for a subset $S \subset V$, the span of S is

$$\text{Span}(S) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \mid k \in \mathbb{N}, v \in S, \alpha \in \mathbb{R} \}$$

where, It is the smallest subspace containing S.

b) Express the vector $v = (2,-1,5)$ as a linear combination of $u_1 = (1,0,1)$, $u_2 = (0,1,2)$ and $u_3 = (1,1,0)$. Also determine Span $\{u_1, u_2, u_3\}$

Given vector $(v) = (2,-1,5)$

To Express these vector in linear combination, we Scalars, let a, b, c be scalars of u_1, u_2 and u_3

Linear combination: $\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_K V_K$

i.e $K=3$; $a \in \alpha_1$; $b \in \alpha_2$; $c \in \alpha_3$; $u_1 \in V_1$; $u_2 \in V_2$; $u_3 \in V_3$

$\therefore a(1,0,1) + b(0,1,2) + c(1,1,0) = (2,-1,5)$

The above system result in the form of linear combination.

let solve it;

$$1 \cdot a + 0 \cdot b + 1 \cdot c = 2$$
$$0 \cdot a + 1 \cdot b + 1 \cdot c = -1$$
$$1 \cdot a + 2 \cdot b + 0 \cdot c = 5$$

This result in the form of

$$a + c = 2 \quad \text{--- (1)}$$
$$b + c = -1 \quad \text{--- (2)}$$
$$a + 2b = 5 \quad \text{--- (3)}$$

From Eq (1) & (2)

$a + \cancel{c} = 2$

$b + \cancel{c} = -1$

$a - b = 3$ --- (4)

From Eq (3) & (4)

$\cancel{a} + 2b = 5$

$\cancel{a} - b = 3$

$3b = 2 \Rightarrow \boxed{b = \frac{2}{3}}$

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From Eq ②

$b + c = -1 \quad \because b = \frac{2}{3}$

$\frac{2}{3} + c = -1$

$c = -1 - \frac{2}{3} = \frac{-3-2}{3} = \frac{-5}{3}$

$\boxed{c = -\frac{5}{3}}$

From Eq ①

$a + c = 2$

$a + \left(\frac{-5}{3}\right) = 2$

$a = 2 + \frac{5}{3} = \frac{6+5}{3} = \frac{11}{3}$

$\boxed{a = \frac{11}{3}}$

Since; $a = \frac{11}{3}$; $b = \frac{2}{3}$; $c = \frac{-5}{3}$

$\text{Span}(S) : \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \mid k \in \mathbb{N}, v \in S, \alpha \in \mathbb{R}\}$

$S = \{a(1,0,1) + b(0,1,2) + c(1,1,0) \mid a,b,c \in \mathbb{R}\}$

This can be expressed as

$S = \{\frac{11}{3}(1,0,1) + \frac{2}{3}(0,1,2) + \left(\frac{-5}{3}\right)(1,1,0) \mid \frac{11}{3}, \frac{2}{3}, \frac{-5}{3} \in \mathbb{R}\}$

i.e $\text{Span}\{v_1, v_2, v_3\} = \{av_1 + bv_2 + cv_3 \mid a,b,c \in \mathbb{R}\}$

$a(1,0,1) + b(0,1,2) + c(1,1,0) = (a+c, b+c, a+2b)$

$\text{Span}\{v_1, v_2, v_3\} = \{(x,y,z) \in \mathbb{R}^3 \mid x = a+c, y = b+c, z = a+2b, a,b,c \in \mathbb{R}\}$

Since the three vectors are linearly independent, their span is the entire space

$\text{Span}\{v_1, v_2, v_3\} = \mathbb{R}^3$

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c) span as the smallest subspace containing all the vectors $v_1, v_2, \ldots, v_n$. create a geometric interpretation for $n=2$ in $\mathbb{R}^3$.

1. The span is a subspace containing $v_1, \ldots, v_n$

By definition, the span of $v1, \ldots, v_n$ is the set of all linear combinations $a1v1 + a2v2 + \ldots + anvn$, where the coefficients are real numbers. this set contains the zero vector, is closed under addition, and is closed under scalar multiplication. Hence, it is a subspace. Each vector v_i is in the span by choosing it's coefficient to be 1 and others 0.

2. Minimality of the span.

let w be any subspace that contains $v_1, \ldots v_n$. since w is closed under scalar multiplication and addition. Every linear combination of $v_1, \ldots v_n$ lies in w. Therefore, span $\{v_1, \ldots, v_n\}$ is contained in w.

since, the span is contained in every subspace that contains $v_1, \ldots v_n$, it is the smallest such subspace.

Geometric Interpretation ($n = 2$ in $\mathbb{R}^3$).

If v_1 and v_2 are linearly independent, their span is a plane through the origin in $\mathbb{R}^3$. This plane is the smallest subspace containing both vectors.

If v_1 and v_2 are linearly dependent, their span is a line through the origin. This line is the smallest subspace containing both vectors.

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Question. 6 (10 Marks)

a) Define the row space and column space of an $m \times n$ matrix A. How are they related to the matrix operations? (3 marks)

Ans:- Let A be an $m \times n$ matrix.

Row Space:

* The row space of A is the set of all linear combinations of the rows of A.
* It is a subspace of $\mathbb{R}^n$.
* The row space represents all possible vectors that can be formed by combining the rows.

Column Space:

* The column space of A is the set of all linear combinations of the columns of A.
* It is a subspace of $\mathbb{R}^n$.
* The column space represents all vectors that can be expressed as Ax for some vector $x \in \mathbb{R}^n$.

Relation to Matrix Operations:

* Row operations do not change the row space but may change the column space.
* Column operations do not change the column space but may change the row space.
* The rank of matrix A equals the dimension of both the row space and the column space.

$$\boxed{\dim R(A) = \dim C(A)}$$

* Row space is related to solving $Ax=0$, while column space is related to solving $Ax=b$.

(P.T.O)

b) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{pmatrix}$, find: i, A basis for the row space ii, A basis for the column space iii, Verify that $\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A))$

(4 marks)

Sol:- Given matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{bmatrix}.$$

i,

Now, To find a basis for the row space, we reduce matrix A to row-echelon form using elementary row operations.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-1}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, the non-zero rows form a basis for the row space.

$\therefore$ Basis for Row space $= \{(1,2,3), (0,1,1)\}$.

ii, To find column space, column space is spanned by the linearly independent columns of the "original matrix".

Now,

Column-1 : $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; column-2 : $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ ($\because$ Here, by identifying Pivot colums of REF, we should take the corresponding colums)

(P.T.O)

Here, by identifying the Pivot columns, we should take the corresponding Columns from the original matrix.

$\therefore$ Basis for Column space $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$.

iii) Verify that dim(Row(A)) = dim (col(A))

* Number of basis vectors in row space = 2.
* Number of basis vectors in column space = 2.

$$\boxed{\therefore \dim(\text{Row}(A)) = \dim(\text{Col}(A)) = 2}$$

Also, this common value is called the rank of matrix A.

c) Explain why elementary row operations Preserve the row space but may change the column space. (3 marks)

Ans:- Row Space :

* Elementary row operations replace rows by linear combination of rows.
* Since row space consists of all linear combination of rows, it remains unchanged.
* Hence, row operations Preserve the row space.

Column Space:

* Row operations change the entries of columns.
* This alters the set of vectors spanned by the columns.
* Therefore, Column space may change under row operations.

Now, we can determine that :

* Row operations Preserve row space.
* Row operations may change column space.
* However, the dimension (rank) remains unchanged. (i.e rank does not change).

O.Sahithi
Section-14

7)

a) what is the difference between an Inner product and a norm? How are they related?

Sol

An inner product is a function that takes two vectors and produces a scalar measuring their Geomentric relationships like angle and orthogonality.

for vectors an inner product $\langle u,v \rangle$ satisfies

1. Linearity : $\langle a_1 u_1 + a_2 u_2 , v \rangle = a_1 \cdot \langle u_1 , v \rangle + a_2 \langle u_2 , v \rangle$

2. Symmetry: $\langle u,v \rangle = \langle v,u \rangle$

3. Positive definite: $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0$ iff $u=0$

where Norm is a function that takes one vector and gives its length.

For a vector, a norm $\|u\|$ satisfies:

1. $\|u\| \geq 0$ 2. $\|u\| = 0$ 3. $\|\alpha u\| = |\alpha| \cdot \|u\|$

Relation b/w Inner product and Norm

$$\boxed{\|u\| = \sqrt{\langle u,u \rangle}}$$

The norm of a vector u is the square root of its inner product with itself.

b) Let $u=(1,2)$ and $v=(2,3)$. find the innerproduct u and v

Also, find angle between u and v using the inner product.

Sol angle using Innerproduct $\cos\theta = \dfrac{\langle u,v\rangle}{\|u\|\cdot\|v\|}$

Given $u=\begin{pmatrix}1\\2\end{pmatrix}$ $v=\begin{pmatrix}2\\3\end{pmatrix}$

$\langle u,v\rangle = u^T \cdot v$ → format

$u^T = (1,2)$ $v=\begin{pmatrix}2\\3\end{pmatrix}$

$u^T \Rightarrow \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)_{(1,2)}$ $v \Rightarrow \cancel{(2,3)}\begin{pmatrix}2\\3\end{pmatrix}$ $u^T \cdot v = (1)(2)+(2)(3) = 8$

$\|u\| = \sqrt{1^2+2^2} = \sqrt{5}$ $\|v\| = \sqrt{2^2+3^2} = \sqrt{13}$

So by substituting $\cos\theta = \dfrac{8}{\sqrt{5}\sqrt{13}}$

$\theta = \cos^{-1}\left(\dfrac{8}{\sqrt{5}.\sqrt{13}}\right)$ The angle b/w 'u' and 'v' is acute angle because $\cos\theta$ is positive.

c) Analyze whether the following is an inner function is an inner product on $\mathbb{R}^2$ of $\langle (x_1 x_2),(y_1,y_2)\rangle = 3x_1y_1+x_2y_2$.

Justify using axioms.

Sol $\langle (x_1 x_2)(y_1 y_2)\rangle = 3x_1y_1+x_2y_2$

there are three axioms

1) Linearity 2) Symmetry 3) positive definite

1) Linearity $\langle (x_1,x_2)(y_1,y_2)\rangle = 3x_1y_1 + x_2y_2$

Let $u=(u_1,u_2)$ $v=(v_1,v_2)$ $y=(y_1,y_2)$

$\langle a(u_1,u_2)+b(v_1,v_2), (y_1,y_2)\rangle = \langle (au_1+bv_1, au_2+bv_2),(y_1,y_2)\rangle$

$\Rightarrow 3(au_1+bv_1)y_1 + (au_2+bv_2)y_2$

$\Rightarrow a(3u_1y_1+u_2y_2)+b(3v_1y_1+v_2y_2)$

$\Rightarrow a\langle (u_1,u_2),(y_1,y_2)\rangle + b\langle (v_1,v_2),(y_1,y_2)\rangle$

$\therefore \langle a(u_1u_2)+b(v_1v_2),(y_1,y_2)\rangle$

$= a\langle (u_1u_2),(y_1y_2)\rangle + b\langle (v_1v_2),(y_1y_2)\rangle$

$\therefore$ Linearity satisfied

2) Symmetry $\langle u,v\rangle = \langle v,u\rangle$

$u=(x_1y_1)$ $v=(x_2y_2)$ $\langle u,v\rangle = 3x_1y_1 + x_2y_2$

$\langle v,u\rangle = 3y_1x_1 + y_2x_2$

$\therefore \langle u,v\rangle = \langle v,u\rangle$ $\therefore$ symmetry satisfied

3) Positive Definite : The innerproduct must $\langle u,u\rangle \geq 0$

and $\langle u,u\rangle = 0$ iff $u=0$

$\langle u,u\rangle = \langle (x_1x_2),(x_1x_2)\rangle = 3x_1x_1 + 3x_2x_2$

$\langle u,u\rangle = 3x_1^2 + x_2^2$ Positive definite satisfied

$3x_1^2 + x_2^2 \geq 0$

The given functions is an inner product on $\mathbb{R}^2$.

8)

a) Define the norm of vector in an inner product Space and express it in terms of the inner product.

A) The norm of a Vector v in an inner product space V is a function that assign a strictly positive length or Magnitude to the vector.

$$\|V\| = \sqrt{\langle v, v \rangle}$$

This means the norm is the square root of the inner Product of the vector with itself.

b) Given,

$U = (1,2,3)$ $\|U\| = ?$

$V = (4,5,6)$ $\|V\| = ?$ $\|U+V\| = ?$

norm of vector in an inner Product space

$\|U\| = \sqrt{\langle U, U \rangle}$

Already we know that $\langle U, U \rangle = U^T U$

Sub in $\|U\|$

$\|U\| = \sqrt{U^T U}$

R^n Show always be as a column vector

So $U = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$U^T = (1 \ 2 \ 3)$

$U^T \cdot U = [1 \ 2 \ 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1+4+9] = 14$

$\|U\| = \sqrt{14}$

→ To calculate $\|V\|$

norm of vector in inner product space

$\|V\| = \sqrt{\langle V, V \rangle}$

Already we know that $\langle v, v \rangle = V^T V$

Given,

$$V = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$V^T = (4\ 5\ 6)$$

$$V^T . V = \left[(4\ 5\ 6) \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right] = 16 + 25 + 36 = 77$$

$$\|V\| = \sqrt{V^T . V} = \sqrt{77}$$

$\therefore \|V\| = \sqrt{77}$

→ To calculate $\|U+V\|$

Given,

$U = (1, 2, 3)$

$V = (4, 5, 6)$

$U+V = (1+4, 2+5, 3+6) = (5, 7, 9)$

norm of vector in inner product space

$\|U+V\| = \sqrt{\langle U+V, U+V \rangle}$

Alredy we know that $\langle U+V, U+V \rangle = (U+V)^T \cdot U+V$

R^n should always be as a column vector

$$U+V = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$(U+V)^T = (5 \quad 7 \quad 9)$

$(U+V)^T \cdot U+V = \cancel{85}\,(5 \quad 7 \quad 9)\begin{bmatrix}5\\7\\9\end{bmatrix} = 25+49+81$

$= 155$

$\|U+V\| = \sqrt{(U+V)^T \cdot (U+V)} = \sqrt{155}$

$\therefore \|U+V\| = \sqrt{155}$

c) Prove that for any vector v in an inner product space, $|\alpha v| = |\alpha|\,|v|$ where α is a Scalar. Discuss the geometric significance.

A) norm of vector in inner product Space

$\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v \rangle}$

α — is Scalar

The inner product has properties of linearity in the first argument and Conjugate linearity in the Second argument.

For Understanding

1) Scalars Can Come out of the linear Product

$\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$

2) When the Scalar is in the Second position, it comes out as its Conjugate

$\langle v, \alpha w \rangle = \bar{\alpha} \langle v, w \rangle$

So in the problem

$\langle \alpha v, \alpha v \rangle = \alpha \langle v, \alpha v \rangle$

$\varepsilon / \cancel{K} \alpha$

$= \alpha \bar{\alpha} \langle v, v \rangle$

Since $\alpha \bar{\alpha} = |\alpha|^2$

$= |\alpha|^2 \langle v, v \rangle$

$\langle \alpha v, \alpha v \rangle = |\alpha|^2 \langle v, v \rangle$

$\|\alpha v\| = \sqrt{|\alpha|^2 \langle v, v \rangle}$

$= |\alpha| \sqrt{\langle v, v \rangle}$

We know that $\sqrt{\langle v, v \rangle} = \|v\|$

$= |\alpha| \, \|v\|$

$\therefore \|\alpha v\| = |\alpha| \, \|v\|$

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Section-14
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Q 9

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(a) • Linear dependence

A finite set $\{v_1, v_2, \dots, v_k\}$ in V is linearly dependent if there exist scalars $\alpha_1, \alpha_2 \dots \alpha_k$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0$$

Ex : In $\mathbb{R}^2$, the vectors $(1,2)$ and $(2,4)$ are linearly dependent because

$$a(1,2) + b(2,4) = (0,0)$$
$$-2(1,2) + 1(2,4) = (-2+2, -4+4) = (0,0)$$

$a = -2 \quad b = 1$

• Linear independence

A finite set $\{v_1, v_2 \dots v_k\}$ in V is linearly independent if the only solution is $\alpha_1 = \dots = \alpha_k = 0$

Ex: In $\mathbb{R}^2$ the vectors $(1,0)$ and $(0,1)$ are linearly independent because

$$a(1,0) + b(0,1) = (0,0)$$

forces $a = 0$ & $b = 0$ such that it satisfies above

$$0(1,0) + 0(0,1) = (0,0)$$

b. given $V_1 = (1, 2, 3)$

$V_2 = (2, 3, 4)$

$V_3 = (3, 5, 7)$

$\Rightarrow \quad aV_1 + bV_2 + cV_3 = 0$

$\Rightarrow \quad a(1, 2, 3) + b(2, 3, 4) + c(3, 5, 7) = 0$

$$\Rightarrow \quad \begin{aligned} a + 2b + 3c &= 0 \longrightarrow (1) \\ 2a + 3b + 5c &= 0 \longrightarrow (2) \\ 3a + 4b + 7c &= 0 \longrightarrow (3) \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 2R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank (A) $= 2 <$ No. of unknowns

$\therefore$ has non-trivial Solution

$\approx$ linearly dependent.

So, therefore

$$a + 2b + 3c = 0 \longrightarrow (1)$$

$$-b - c = 0 \longrightarrow (2)$$

From eq^n (2) $\quad b = -c$

Substitute $b = -c$ in (1)

$$a + 2(-c) + 3c = 0$$

$$a + c = 0$$

$$a = -c$$

Let $c = 1$, Then $a = -1$, $b = -1$. So,

$$-1 \cdot V_1 - 1 V_2 + 1 \cdot V_3 = 0$$

$$V_3 = V_1 + V_2.$$

(c) let $S = \{v_1, v_2, \dots v_n\}$ be a subset of a Vector space & suppose that for some k, $v_k = 0$, the zero vector

consider the linear combination

$$1 \cdot v_k + 0 \cdot v_1 + \dots + 0 \cdot v_{k-1} + 0 \cdot v_{k+1} + \dots + 0 \cdot v_n = 0$$

Here not all coefficients are zero (since co-efficient of v_k is 1)

So, by defination this is a non-trivial linear dependence relation & hence S is linearly dependent.

Now suppose a set S contains a linearly dependent subset U by definition of linear dependence, there exist vectors $v_1, v_2 \dots v_m \in U$ & Scalars $a_1, \dots, a_m$, not all zero, such that $a_1 v_1 + \dots a_m v_m = 0$

this shows same linear combination shows a linearly dependence among vectors in S as well, because all v_i are also in S, so S is linearly dependent

Question - 10: A N V pragna (25IFA04E22) Sec-07

Q) List the steps involved in the Gram-Schmidt orthogonalisation process (2Marks).

A) **Step-by-step procedure:**

This process accepts a set of linearly independent vectors and converts them into orthonormal set (unit + orthogonal).

Assume we have a set of vectors:

$$v_1, v_2, v_3, \ldots, v_n.$$

We need to convert them into a new orthonormal set of vectors:

$$u_1, u_2, u_3, \ldots, u_n$$

Step-1: Select the first vector.

The first orthogonal vector is just the first vector of original set.

$$u_1 = v_1$$

To make it normalised, divide it by it's Length.

$$e_1 = \frac{u_1}{\|u_1\|} \longrightarrow 1^{st} \text{ orthonormal vector.}$$

Step-2: To find second orthogonal vector, remove component of v_2 that is in the direction of e_1

$$u_2 = v_2 - \text{proj}_{e_1}(v_2)$$

$$= v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1$$

Projection of e_1 on v_2

$$\text{proj}_{e_1}(v_2) = \frac{\langle v_2 \cdot e_1 \rangle}{\langle e_1, \underset{u_1}{e_1} \rangle} e_1$$

After obtaining u_2, normalise it,

$$e_2 = \frac{u_2}{\|u_2\|}.$$

Step-3: For 3^{rd} vector, remove components of v_3 in the direction of e_1 and e_2:

$$u_3 = v_3 - \text{Proj}_{e_1}(v_3) - \text{Proj}_{e_2}(v_3) \Rightarrow v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2}$$

After obtaining u_3, normalize it:

$$e_3 = \frac{u_3}{\|u_3\|}.$$

Step-4: Repeat process for all vectors in original set

$$u_k = v_k - \sum_{i=1}^{k-1} \text{Proj}_{e_i}(v_k) = v_k - \sum_{i=1}^{k-1} \frac{\langle v_k, u_i \rangle}{\|u_i\|^2} u_i$$

After obtaining u_k, normalise it:

$$e_k = \frac{u_k}{\|u_k\|}.$$

This ensures all vectors are orthonormal.

b) Apply the Gram-Schmidt process to orthogonalise the vectors $v_1 = (1,1,0)$, $v_2 = (1,0,1)$ and $v_3 = (0,1,1)$ in $\mathbb{R}^3$ (5 marks).

$$v_1 = (1,1,0) \; ; \; v_2 = (1,0,1) \; ; \; v_3 = (0,1,1)$$

Step-1: First orthogonal vector, u_1

$$u_1 = (1,1,0)$$

Normalising $u_1 \Rightarrow \frac{u_1}{\|u_1\|} = \frac{(1,1,0)}{\sqrt{1^2+1^2+0^2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

Step-2: Second orthogonal vector, u_2.

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1$$

$$= (1,0,1) - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1$$

$\langle v_2, u_1 \rangle = \langle (1,0,1), (1,1,0) \rangle = 1 \times 1 + 0 \times 1 + 1 \times 0$

$= 1$

$||u_1||^2 = 1^2 + 1^2 + 0^2 = 2$

$$u_2 = (1, 0, 1) - \frac{1}{2}(1, 1, 0)$$

$$u_2 = \left(1 - \frac{1}{2}, 0 - \frac{1}{2}, 1 - 0\right)$$

$$u_2 = \left(\frac{1}{2}, \frac{-1}{2}, 1\right) \longrightarrow \text{orthogonal vector - 2nd}$$

$$e_2 = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 1^2}} \left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$

$$= \sqrt{\frac{2}{3}}\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$

$$= \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right) \longrightarrow \text{orthonormal vector - 2nd}$$

<u>step-3</u>: Third orthogonal vector, u_3:

$$u_3 = v_3 - \frac{\langle v_3, u_2 \rangle}{||u_2||^2} \cdot u_2 - \frac{\langle v_3, u_1 \rangle}{||u_1||^2} \cdot u_1$$

$$\langle v_3, u_2 \rangle = 0 \times \frac{1}{2} + 1 \times \frac{-1}{2} + 1 \times 1$$

$$= 1/2$$

$$\langle v_3, u_1 \rangle = 0 \times 1 + 1 \times 1 + 1 \times 0$$

$$= 1$$

$$||u_2||^2 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{6}{4} = \frac{3}{2}$$

$$||u_1||^2 = 2$$

$$u_3 = (0,1,1) - \frac{1/2}{3/2}\left(\frac{1}{2}, \frac{-1}{2}, 1\right) - \frac{1}{2}(1,1,0)$$

$$= (0,1,1) - \left(\frac{1}{6}, \frac{-1}{6}, \frac{1}{3}\right) - \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$= \left(-\frac{1}{6} - \frac{1}{2}, 1 + \frac{1}{6} - \frac{1}{2}, 1 - \frac{1}{3}\right)$$

$$= \left(\frac{-1-3}{6}, \frac{6+1-3}{6}, \frac{2}{3}\right) = \left(\frac{-4}{6}, \frac{7-3}{6}, \frac{2}{3}\right)$$

$$= \left(\frac{-2}{3}, \frac{7-3}{6}, \frac{2}{3}\right)$$

$$= \left(\frac{-2}{3}, \frac{4}{6}, \frac{2}{3}\right) = \left(\frac{-2}{3}, \frac{2}{3}, \frac{2}{3}\right) \rightarrow \text{orthogonal vector } 3^{rd}$$

$$e_3 = \frac{1}{\sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}}}\left(\frac{-2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{1}{\sqrt{\frac{12}{9}}}\left(\frac{-2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{-2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \rightarrow \text{orthonormal vector, } 3^{rd}$$

Orthogonal vectors $\Rightarrow$ $u_1 = (1,1,0)$

$u_2 = (1/2, -1/2, 1)$

$u_3 = \left(\frac{-2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Orthonormal vectors $\Rightarrow$ $e_2 = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$

$e_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$e_3 = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

C) Design a modified Gram-Schmidt algorithm that produces an orthonormal set directly (without a seperate normalisation step at the end). Explain why this might be computationally Advantageous.

A) In Gram-schmidt orthogonalisation process, the orthogonal vectors are calculated by removing the components that are in the direction of vec. After this, these orthogonal vectors are divided by norm to get orthonormal set.

In the modified process, orthonormal vectors are calculated immediately after calculating the orthogonal vector.

Computational Advantages:

1. Avoids rounding errors: The vectors are rounded off to nearly numbers which results in loss of significant points.

2. Reduces floating point errors: As computer doesnot have infinite precision, calculating norm immediately reduces errors.

3. Available for Immediate use: The use of orthonormal vectors at every step of QR and other methods will speed up Calculations.

M. Thrividha

sec - 14

Q11)

a) Given matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(i) eigenvalues find eigenvalues from $\det(A-\lambda I)=0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$$

$$\lambda = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-2)}}{2(1)}$$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{33}}{2}$$

eigen values :- $\boxed{\lambda_1 = \frac{5+\sqrt{33}}{2}, \quad \lambda_2 = \frac{5-\sqrt{33}}{2}}$

(ii) eigen vectors

for λ_1, $(A - \lambda_1 I)x = 0$

$$= \begin{pmatrix} 1 - \left(\frac{5+\sqrt{33}}{2}\right) & 2 \\ 3 & 4 - \left(\frac{5+\sqrt{33}}{2}\right) \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\frac{-3-\sqrt{33}}{2}\right)x + 2y = 0$$

$$\Rightarrow x(-\sqrt{33} - 3) + 4y = 0 \quad - (1)$$

$$\Rightarrow 3x + \left(\frac{3+\sqrt{33}}{2}\right)y = 0$$

$$\Rightarrow 6x + (3 \mp \sqrt{33})y = 0 \quad - (2)$$

from ① $y = \dfrac{3x + \sqrt{33}x}{4}$

substitute in ②

$$6x + (3-\sqrt{33})\left(\frac{3+\sqrt{33}}{4}\right)x = 0$$

$$6x + \left(\frac{-24}{4}\right)x = 0 \Rightarrow 6x - 6x = 0$$

$$\therefore x = K$$

then

$$y = \left(\frac{3+\sqrt{33}}{4}\right)K = \left(\frac{\sqrt{33}+3}{4}\right)K$$

$$\therefore V_1 = \begin{pmatrix} K \\ \left(\frac{\sqrt{33}+3}{4}\right)K \end{pmatrix} = \boxed{K \begin{pmatrix} 1 \\ \left(\frac{\sqrt{33}+3}{4}\right) \end{pmatrix} ; K \in R}$$

for λ_2, $(A - \lambda_2 I)X = 0$

$$\Rightarrow \begin{pmatrix} 1 - \left(\frac{5-\sqrt{33}}{2}\right) & 2 \\ 3 & 4 - \left(\frac{5-\sqrt{33}}{2}\right) \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{-3+\sqrt{33}}{2}\right)x + 2y = 0 \Rightarrow (-3+\sqrt{33})x + 4y = 0 - ①$$

$$3x + \left(\frac{3+\sqrt{33}}{2}\right)y = 0 \Rightarrow 6x + (3+\sqrt{33})y = 0 - ②$$

from ① $y = \dfrac{3x - \sqrt{33}x}{4}$

substitute in ②

$$6x + (3+\sqrt{33})\left(\frac{3-\sqrt{33}}{4}\right)x = 0$$

$$6x + \left(\frac{-24}{4}\right)x = 0 \Rightarrow 6x - 6x = 0 \Rightarrow x = K$$

$$\Rightarrow y = \left(\frac{3-\sqrt{33}}{4}\right)K \Rightarrow V_1 = \begin{pmatrix} K \\ \left(\frac{3-\sqrt{33}}{4}\right)K \end{pmatrix} = \boxed{K \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{4} \end{pmatrix} ; K \in R}$$

(iii) verify $\boxed{Av = \lambda v}$

$$AV_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{4} \end{pmatrix} = \begin{pmatrix} 1(1) + 2\left(\frac{3+\sqrt{33}}{4}\right) \\ 3(1) + 4\left(\frac{3+\sqrt{33}}{4}\right) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 + \frac{3+\sqrt{33}}{2} \\ 3 + 3 + \sqrt{33} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ 6 + \sqrt{33} \end{pmatrix}$$

$$\lambda_1 V_1 = \frac{5+\sqrt{33}}{2}\begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{4} \end{pmatrix} = \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ \frac{(5+\sqrt{33})(3+\sqrt{33})}{8} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ \frac{48+8\sqrt{33}}{8} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{5+\sqrt{33}}{2} \\ 6 + \sqrt{33} \end{pmatrix}$$

$\boxed{\therefore AV_1 = \lambda_1 V_1 \text{ (verified)}}$

$$AV_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{4} \end{pmatrix} = \begin{pmatrix} 1 + \frac{3-\sqrt{33}}{2} \\ 3 + 3 - \sqrt{33} \end{pmatrix} = \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ 6 - \sqrt{33} \end{pmatrix}$$

$$\lambda_2 V_2 = \frac{5-\sqrt{33}}{2}\begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{4} \end{pmatrix} = \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ \frac{(5-\sqrt{33})(3-\sqrt{33})}{8} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ \frac{48-8\sqrt{33}}{8} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{5-\sqrt{33}}{2} \\ 6 - \sqrt{33} \end{pmatrix}$$

$\boxed{\therefore AV_2 = \lambda_2 V_2 \text{ (verified)}}$

b) Given matrix, $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 3 & 2 & 2 \\ 4 & 2 & 5 & 7 \end{pmatrix}$

Apply elementary row operations

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & 2 & 5 & -1 \end{pmatrix}$$

$R_2 \to \frac{1}{3} R_2$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2/3 & 0 \\ 0 & 2 & 5 & -1 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 11/3 & -1 \end{pmatrix}$$

$R_3 \to \frac{3}{11} R_3$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & -3/11 \end{pmatrix} \to R.R.E.F$$

(i)

$\to$ Here as we applied row operations,

Row space (A) = Row space (R.R.E.F)

$\therefore$ Basis of Row space = $\boxed{\{(1,0,0,2),(0,1,2/3,0),(0,0,11/3,-1)\}}$

- Dimension of Row space = no. of vectors in the basis

$$\boxed{\therefore \text{Dim(Row space)} = 3}$$

(ii) → now identify pivot columns in R.E.F and write corresponding columns in matrix A.

$\therefore$ Basis of column space = $\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \right\}}$.

dimension of column space = no. of vectors in basis

$\boxed{\therefore \text{Dim}(\text{column space}) = 3}$

(iii) For any matrix A,

dim (Row space of A) = dim (column space of A)

This common value is called Rank of the matrix.

$\boxed{\text{Here, Rank}(A) = \text{dim}(\text{row space}) = \text{dim}(\text{column space}) = 3}$

<u>Hence verified</u>

c) Given function $\langle u, v\rangle = 2u_1v_1 + 3u_2v_2$

$u = (u_1, u_2)$ and $v = (v_1, v_2)$

1) Linearity ÷ $\langle(x_1, x_2), (y_1, y_2)\rangle = 2x_1y_1 + 3x_2y_2$

let, $u = (u_1, u_2)$, $v = (v_1, v_2)$, $y = (y_1, y_2)$

$$\langle a(u_1, u_2) + b(v_1, v_2), (y_1, y_2)\rangle = \langle(au_1 + bv_1, au_2 + bv_2), (y_1, y_2)\rangle$$

$$= 2(au_1 + bv_1)y_1 + 3(au_2 + bv_2)y_2$$

$$= a(2u_1y_1 + 3u_2y_2) + b(2v_1y_1 + 3v_2y_2)$$

$$= a\langle(u_1, u_2), (y_1, y_2)\rangle + b\langle(v_1, v_2), (y_1, y_2)\rangle$$

$$\therefore \langle a(u_1, u_2) + b(v_1, v_2), (y_1, y_2)\rangle = a\langle(u_1, u_2), (y_1, y_2)\rangle + b\langle(v_1, v_2), (y_1, y_2)\rangle$$

∴ linearity satisfied.

2) symmetry ÷ $\langle u, v\rangle = 2u_1v_1 + 3u_2v_2$

$\langle v, u\rangle = 2v_1u_1 + 3v_2u_2$

since multiplication of real numbers is commutative

$2u_1v_1 = 2v_1u_1$, $3u_2v_2 = 3v_2u_2$

$\Rightarrow \langle u, v\rangle = \langle v, u\rangle$

∴ symmetry satisfied

3) Positive definite:-

$$\langle u,u\rangle = 2(u_1)u_1 + 3(u_2)u_2$$

$$= 2u_1^2 + 3u_2^2$$

since $u_1^2 \geq 0$ and $u_2^2 \geq 0$

$$2u_1^2 + 3u_2^2 \geq 0$$

Hence, function is positive definete

$\therefore$ given function is an inner product on $\mathbb{R}^2$