SCHOOL OF APPLIED SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

Subject: Linear Algebra — Subject Code : 25MT103

Sem. : I — Academic Year: 2025-2026

Section: 7, 14, 21 — Regulation: R25

Problem Set – Unit 5

Question 1 (10 Marks)

a) Define a real inner product space and state any four axioms that the inner product must satisfy. **(3 marks)**

b) Verify whether the function $\langle u, v\rangle = u_1v_1 + 2u_2v_2$ defines an inner product on R^2, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. **(3 marks)**

c) Construct an example of a function on R^2 that satisfies three axioms of inner product but fails one axiom. Justify your answer with appropriate mathematical reasoning. **(4 marks)**

Question 2 (10 Marks)

a) Define the norm of a vector in an inner product space and express it in terms of the inner product. **(2 marks)**

b) For the vectors $u = (1, 2, 3)$ and $v = (4, 5, 6)$ in R^3 with the standard inner product, calculate $||u||$, $||v||$, and $||u + v||$. **(4 marks)**

c) Prove that for any vector v in an inner product space, $||\alpha v|| = |\alpha|\ ||v||$ where α is a scalar. Discuss the geometric significance of this property. **(4 marks)**

Question 3 (10 Marks)

a) State the Cauchy-Schwarz inequality for real inner product spaces. **(2 marks)**

b) Apply the Cauchy-Schwarz inequality to prove that for any vectors u and v in an inner product space, $|\langle u, v\rangle| \leq ||u||\ ||v||$. Verify this inequality for $u = (1, -2, 3)$ and $v = (2, 1, -1)$ in R^3. **(4 marks)**

c) Analyze the conditions under which equality holds in the Cauchy-Schwarz inequality. Provide a geometric interpretation and create an example demonstrating this case. **(4 marks)**

Question 4 (10 Marks)

a) Define orthogonal vectors and orthogonal set in an inner product space. **(2 marks)**

b) Determine whether the set of vectors {(1, 1, 0), (1, -1, 0), (0, 0, 1)} is orthogonal in R^3 with the standard inner product. **(3 marks)**

c) Prove that if S = {v_1, v_2, ..., v_k} is an orthogonal set of non-zero vectors in an inner product space, then S is linearly independent. **(5 marks)**

Question 5 (10 Marks)

a) Distinguish between an orthogonal set and an orthonormal set. Provide one example of each. **(3 marks)**

b) Convert the orthogonal set {(1, 2, 3), (-6, 3, 0), (1, -2, 1)} into an orthonormal set. **(4 marks)**

c) Evaluate the advantages of using an orthonormal basis over a standard basis for computations in vector spaces. Support your answer with specific examples. **(3 marks)**

Question 6 (10 Marks)

a) List the steps involved in the Gram-Schmidt orthogonalization process. **(2 marks)**

b) Apply the Gram-Schmidt process to orthogonalize the vectors v_1 = (1, 1, 0), v_2 = (1, 0, 1), and v_3 = (0, 1, 1) in R^3. **(5 marks)**

c) Design a modified Gram-Schmidt algorithm that produces an orthonormal set directly (without a separate normalization step at the end). Explain why this might be computationally advantageous. **(3 marks)**

Question 7 (10 Marks)

a) Explain what is meant by projecting a vector onto another vector in an inner product space. Write the formula for projection. **(3 marks)**

b) For vectors u = (2, 3, 1) and v = (1, 0, -1) in R^3, compute the projection of u onto v and find the component of u orthogonal to v. **(4 marks)**

c) Analyze how the Gram-Schmidt process uses the concept of projection iteratively. Create a geometric illustration (describe it) for the case of three vectors in R^3. **(3 marks)**

Question 8 (10 Marks)

a) Define QR decomposition of a matrix and state the relationship between Q and R. **(3 marks)**

b) Find the QR decomposition of the matrix A = $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ using the Gram-Schmidt process. **(4 marks)**

c) Evaluate the role of QR decomposition in solving least squares problems. Compare its numerical stability with the normal equations method. **(3 marks)**

Question 9 (10 Marks)

a) State the theorem regarding the uniqueness of QR decomposition. Under what conditions is the decomposition unique? **(2 marks)**

b) Given the matrix A = $\begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 1 \end{pmatrix}$, perform QR decomposition and verify your answer by computing QR. **(5 marks)**

c) Construct an algorithm to use QR decomposition iteratively to find eigenvalues of a matrix (QR algorithm). Explain the convergence properties. **(3 marks)**

Question 10 (10 Marks)

a) Define Singular Value Decomposition (SVD) of a matrix. What are the properties of the matrices U, Σ, and V in the decomposition $A = U\Sigma V^T$? **(3 marks)**

b) For a 2×2 matrix A = $\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$, calculate A^TA and find its eigenvalues. Use this to determine the singular values of A. **(4 marks)**

c) Prove that for any m×n matrix A, the singular values are the non-negative square roots of the eigenvalues of A^TA. **(3 marks)**

Question 11 (10 Marks)

a) Recall the relationship between SVD and the four fundamental subspaces of a matrix. **(2 marks)**

b) Given a matrix A with SVD $A = U\Sigma V^T$ where $\Sigma = \text{diag}(5, 3, 0)$, determine the rank of A, and identify the dimensions of its null space and column space. **(4 marks)**

c) Design a procedure to use SVD for image compression. Analyze how the number of singular values retained affects image quality and storage requirements. Provide a mathematical framework for your analysis. **(4 marks)**

Question 12 (10 Marks)

a) Describe the concept of the Moore-Penrose pseudoinverse and its relation to SVD. **(2 marks)**

b) Apply SVD to find the pseudoinverse of the matrix A = $\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$. Use your result to find the least squares solution to Ax = b where b = $(1, 2, 2)^T$. **(5 marks)**

c) Compare and contrast the methods of solving overdetermined systems: (i) Normal equations, (ii) QR decomposition, and (iii) SVD. Evaluate each method in terms of computational cost, numerical stability, and applicability to rank-deficient problems. **(3 marks)**