25MT103: Linear Algebra

Unit 1: Matrices

Dr. D Bhanu Prakash
Course Page: dbhanuprakash233.github.io/LA

Assistant Professor,
Department of Mathematics and Statistics.
Contact: db_maths@vignan.ac.in.
dbhanuprakash233.github.io.

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Matrices - Lecture Slides

Syllabus

- Elementary row and column operations
- Elementary Matrices
- Similar Matrices
- Echelon form
- Row reduced echelon form
- Rank of a matrix
- Inverse of a matrix by Gauss-Jordan Method
- LU decomposition

Outline

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Elementary Row Operations

Definition

Elementary row operations are the three basic manipulations one may perform on rows of a matrix (they are reversible):

1. Row swap: interchange two rows, $R_i \leftrightarrow R_j$.
2. Row scaling: multiply a row by a nonzero scalar, $R_i \to kR_i$ with $k \neq 0$.
3. Row replacement: add a multiple of one row to another, $R_i \to R_i + kR_j$ for $i \neq j$.

Elementary Matrices

Definition

An *elementary matrix* is a square matrix obtained by performing a single elementary row operation on the identity matrix I_n.

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Types of elementary matrices:

1. **Row swap:** Obtained by interchanging two rows of I_n.
2. **Row scaling:** Obtained by multiplying one row of I_n by a nonzero scalar k.
3. **Row replacement:** Obtained by adding k times one row of I_n to another.

Elementary Matrix 1: Row Swap

If S denotes the matrix obtained from I_n by swapping rows i and j, then for any A we have SA equals A with rows i and j swapped.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \qquad R_1 \leftrightarrow R_3, \qquad \begin{bmatrix} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} = S_{1\leftrightarrow 3}A.$$

Then elementary matrix

$$S_{1\leftrightarrow 3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Elementary Matrix 2: Row Scaling

The elementary matrix is obtained from I_n by replacing the (i,i) entry by k.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad \text{scale } R_2 \text{ by } 3, \qquad \begin{bmatrix} 2 & 1 \\ 12 & 9 \end{bmatrix} = EA$$

Then elementary matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

which is the matrix obtained by multiplying the second row of A by 3.

Elementary Matrix 3: Row Replacement

For adding k times row j to row i, the elementary matrix is I_n with an extra entry k in position (i,j).

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{here } R_2 \to R_2 - 4R_1 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix} = EA.$$

Then elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Elementary Matrices: Quick Reference

- Swap rows i and j: take I and swap rows i,j to get E. Then EA swaps rows of A.
- Scale row i by k: replace I_{ii} by k to get E. Then EA scales that row.
- Add k times row j to row i: put k in position (i,j) of I to get E. Then EA performs the replacement.

Note: Every elementary matrix is invertible and its inverse is also elementary (the inverse operation).

Row Echelon Form (REF)

Definition

A matrix is in *row echelon form* if:

1. All nonzero rows are above any zero rows.
2. The leading entry (pivot) of each nonzero row is strictly to the right of the leading entry of the row above it.
3. All entries below each leading entry are zero.

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Example:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in REF.

Practice: Row Echelon Form

Problems:

1. Put the following matrix into REF:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

2. True or False: Every matrix has a unique row echelon form.

Reduced Row Echelon Form (RREF)

Definition

A matrix is in *reduced row echelon form* (RREF) if:

1. It is in REF.
2. The leading entry (pivot) in each nonzero row is 1.
3. Each pivot is the only nonzero entry in its column.

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Definition

A matrix is in *reduced row echelon form* (RREF) if:

1. It is in REF.
2. The leading entry (pivot) in each nonzero row is 1.
3. Each pivot is the only nonzero entry in its column.

Example:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

is in RREF.

Practice: RREF

Problems:

1. Reduce to RREF:
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$
2. Explain why the RREF of a matrix is always unique.

Rank of a Matrix

Definition

The *rank* of a matrix A, denoted $\text{rank}(A)$, is:

- The number of leading 1's (pivot columns) in its RREF, OR
- The number of non-zero rows in REF of A.

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Example:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}.$$

Row reducing gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\text{rank}(A) = 2$.

Practice: Rank

Problems:

1. Find the rank of
$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Similarity of Matrices

Definition

Two $n \times n$ matrices A and B are *similar* if

$$B = P^{-1}AP$$

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Two $n \times n$ matrices A and B are *similar* if

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Properties:

- Similar matrices represent the same linear transformation under different bases.
- They have the same determinant, trace, rank, and eigenvalues.

Similarity Example

Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

So A is similar to a diagonal matrix.

Practice: Similarity

Problems:

1. Determine whether the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

are similar. Justify your answer.

Outline

1. Definitions
   - Elementary Row Operations
   - Elementary Matrices
   - Row Echelon Form
   - Reduced Row Echelon Form
   - Rank of a Matrix
   - Similarity of Matrices
2. Matrix Operations
   - Method 1: Upper triangular matrix
   - Method 2: LU Decomposition
   - Method 3: Inverse via Gauss-Jordan

Problem Statement

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}.$$

We will:

1. Convert A to upper triangular form using elementary row operations.
2. Compute an LU decomposition (without pivoting).
3. Find A^{-1} using the Gauss–Jordan method.

Method 1: Upper triangular matrix

Start with

$$A_0 = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}.$$

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$$A_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}.$$

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$$A_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}.$$

Eliminate (3,1): $R_3 \to R_3 + R_1$.

$$A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}.$$

Method 1: Upper triangular matrix

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Eliminate (3,2): $R_3 \to R_3 + R_2$.

$$U = A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Method 2: LU Decomposition

LU decomposition was introduced by mathematician *Tadeusz Banachiewicz* in 1938. We decompose the matrix A into the product of lower triangular (L) and upper triangular (U) matrices.

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$$\text{Let } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

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Check: $LU = A$.

Method 3: Inverse via Gauss-Jordan

Begin with

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right].$$

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$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right].$$

Step 1: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$.

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Step 1: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right].$$

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Step 2: $R_3 \to R_3 + R_2$.

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$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

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Step 2: $R_3 \to R_3 + R_2$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Step 3: Clear above column 3. $R_1 \to R_1 - R_3$, $R_2 \to R_2 + 2R_3$.

Method 3: Inverse via Gauss-Jordan

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$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

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Step 3: Clear above column 3. $R_1 \to R_1 - R_3$, $R_2 \to R_2 + 2R_3$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Step 4: Scale R_2 then eliminate above.

Method 3: Inverse via Gauss-Jordan

Step 3: Clear above column 3. $R_1 \to R_1 - R_3$, $R_2 \to R_2 + 2R_3$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Step 4: Scale R_2 then eliminate above.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -5/16 & -3/8 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Method 3: Inverse via Gauss-Jordan

Step 3: Clear above column 3. $R_1 \to R_1 - R_3$, $R_2 \to R_2 + 2R_3$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

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So

$$A^{-1} = \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix}.$$

Summary of Example

- Upper triangular (U): $\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.
- Lower factor (L): $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.
- Inverse: $\begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix}$.

Thank You!

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