SCHOOL OF APPLIED SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

Subject: Linear Algebra | Subject Code : 25MT103

Sem. : I | Academic Year: 2025-2026

Section: 7, 14, 21 | Regulation: R25

Problem Set – Unit 4

Question 1 (10 Marks)

a) Define a real vector space and list the eight axioms that must be satisfied by the addition and scalar multiplication operations. **(3 marks)**

b) Verify whether the set $V = \{(x, y) : x, y \in \mathbb{R}, x \geq 0\}$ with standard addition and scalar multiplication forms a vector space. **(3 marks)**

c) Construct two different examples: one where a set with non-standard operations forms a vector space, and another where standard operations fail to produce a vector space. Justify each case by examining the axioms. **(4 marks)**

Question 2 (10 Marks)

a) Define a subspace of a vector space and state the subspace test (three conditions). **(2 marks)**

b) Determine whether the following subsets of $\mathbb{R}^3$ are subspaces: (i) $W_1 = \{(x, y, z) : x + y + z = 0\}$ (ii) $W_2 = \{(x, y, z) : x + y + z = 1\}$ (iii) $W_3 = \{(x, y, z) : xy = 0\}$ **(5 marks)**

c) Is the intersection of two subspaces of a vector space is always a subspace. Analyze whether the union of two subspaces is necessarily a subspace, providing justification with examples. **(3 marks)**

Question 3 (10 Marks)

a) Explain the concepts of linear combination and linear span of a set of vectors. **(2 marks)**

b) Express the vector $v = (2, -1, 5)$ as a linear combination of $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 2)$, and $u_3 = (1, 1, 0)$. Also, determine $\text{span}\{u_1, u_2, u_3\}$. **(4 marks)**

c) Show that $\text{span}\{v_1, v_2, ..., v_n\}$ is the smallest subspace containing all the vectors $v_1, v_2, ..., v_n$. Create a geometric interpretation for $n = 2$ in $\mathbb{R}^3$. **(4 marks)**

Question 4 (10 Marks)

a) Define linear dependence and linear independence of vectors. Provide one example of each. **(3 marks)**

b) Test whether the vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 3, 4)$, and $v_3 = (3, 5, 7)$ are linearly dependent or independent. If dependent, express one vector as a linear combination of others. **(4 marks)**

c) Show that if a set $S = \{v_1, v_2, ..., v_n\}$ contains the zero vector, then S is linearly dependent. Extend this to prove that any set containing a linearly dependent subset is itself linearly dependent. **(3 marks)**

Question 5 (10 Marks)

a) State the relationship between the number of vectors in a linearly independent set and the dimension of the space they span. **(2 marks)**

b) Given the vectors $v_1 = (1, -1, 0, 2)$, $v_2 = (2, 0, 1, -1)$, $v_3 = (0, 2, 1, -5)$, and $v_4 = (3, -1, 2, 1)$ in $\mathbb{R}^4$, determine the maximum number of linearly independent vectors from this set. **(4 marks)**

c) Analyze and prove: "If $\{v_1, v_2, ..., v_n\}$ is linearly independent and v is a vector not in span$\{v_1, v_2, ..., v_n\}$, then $\{v_1, v_2, ..., v_n, v\}$ is linearly independent." Discuss the significance of this result in building bases. **(4 marks)**

Question 6 (10 Marks)

a) Define a basis of a vector space and explain what is meant by the dimension of a vector space. **(2 marks)**

b) Show that the set $B = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$ forms a basis for $\mathbb{R}^3$. Express the vector (3, -2, 1) as a linear combination of basis vectors. **(5 marks)**

c) Prove that all bases of a finite-dimensional vector space have the same number of elements. Evaluate the consequences if this property were not true. **(3 marks)**

Question 7 (10 Marks)

a) Describe the standard basis for $\mathbb{R}^n$ and explain its significance. **(2 marks)**

b) Find a basis and determine the dimension of the subspace W of $\mathbb{R}^4$ defined by: $W = \{(x_1, x_2, x_3, x_4) : x_1 - x_2 + x_3 = 0 \text{ and } x_2 + x_4 = 0\}$ **(5 marks)**

c) Extend the set $\{(1, 0, 1), (0, 1, 2)\}$ to basis in $\mathbb{R}^3$. **(3 marks)**

Question 8 (10 Marks)

a) Define the row space and column space of an m×n matrix A. How are they related to the matrix operations? **(3 marks)**

b) For the matrix A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{pmatrix}$, find: (i) A basis for the row space (ii) A basis for the column space (iii) Verify that dim(Row(A)) = dim(Col(A)) **(4 marks)**

c) Explain why elementary row operations preserve the row space but may change the column space. **(3 marks)**

Question 9 (10 Marks)

a) State the definition of the rank of a matrix and explain its relationship to row space and column space. **(2 marks)**

b) Determine the rank of the matrix A = $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & 0 \\ 1 & 2 & 2 & -3 \end{pmatrix}$ by: (i) Finding the dimension of row space (ii) Finding the dimension of column space (iii) Verifying both methods yield the same result **(5 marks)**

c) Analyze the relationship between rank(A), rank(A^T), rank(A^TA), and rank(AA^T) for any matrix A. Provide rigorous justification for each relationship. **(3 marks)**

Question 10 (10 Marks)

a) Explain the Rank-Nullity Theorem and define the null space of a matrix. **(2 marks)**

b) For the matrix A = $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 11 \end{pmatrix}$, find: (i) The rank of A (ii) A basis for the null space of A (iii) Verify the Rank-Nullity Theorem **(5 marks)**

c) Prove the Rank-Nullity Theorem: For an m×n matrix A, rank(A) + nullity(A) = n. Discuss its applications in solving systems of linear equations. **(3 marks)**

Question 11 (10 Marks)

a) Define similar matrices and state the condition for two matrices to be similar. **(2 marks)**

b) Show that if A and B are similar matrices, then they have the same rank. Given A = $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, verify their ranks and determine if they could be similar. **(4 marks)**

c) Prove that if matrices A and B are similar, then: (i) rank(A) = rank(B) (ii) dim(Row(A)) = dim(Row(B)) (iii) dim(Col(A)) = dim(Col(B)) Evaluate whether similar matrices must have the same row spaces. **(4 marks)**

Question 12 (10 Marks)

a) Recall the definitions of row rank, column rank, and rank of a matrix. State the fundamental theorem connecting them. **(2 marks)**

b) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 1 \\ 1 & 2 & 2 & 1 \\ 3 & 6 & 4 & 1 \end{pmatrix}$. Apply row reduction to: (i) Find a basis for Row(A) (ii) Identify pivot columns to find a basis for Col(A) (iii) Determine the rank and compare with nullity **(5 marks)**

c) Design a comprehensive procedure to analyze any m×n matrix by determining:

- Rank using both row and column space methods
- Bases for all four fundamental subspaces (row space, column space, null space, left null space)
- Verification using the Rank-Nullity Theorem Demonstrate your procedure on a 3×4 matrix of your choice. **(3 marks)**