

SCHOOL OF APPLIED SCIENCE & HUMANITIES
DEPARTMENT OF MATHEMATICS

Subject: Linear Algebra

Sem. : I

Section: 21

Subject Code : 25MT103

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Regulation: R25

Module Bank

Question 1 (10 Marks)

- a) Define the characteristic equation of a matrix. Explain how eigenvalues are obtained from the characteristic equation. **(2 marks)**
- b) Find the characteristic equation and eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Verify your eigenvalues by substituting back into the characteristic equation. **(4 marks)**
- c) Check that the sum of eigenvalues of a matrix equals its trace, and the product of eigenvalues equals its determinant. Demonstrate this property with a 3×3 matrix of your choice. **(4 marks)**

Question 2 (10 Marks)

- a) State the Cayley-Hamilton Theorem. Indicate whether the theorem holds for every $n \times n$ matrix over the complex numbers. **(2 marks)**
- b) Verify the Cayley-Hamilton Theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Use this result to compute A^3 . **(4 marks)**
- c) Outline the Cayley-Hamilton recipe that rewrites A^n as a linear combination of I and integer powers of A. Demonstrate with a specific example for $n = 3$. **(4 marks)**

Question 3 (10 Marks)

- a) Define a diagonalizable matrix. State the necessary and sufficient condition for a matrix to be diagonalizable. **(2 marks)**
- b) Determine whether the matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is diagonalizable. If yes, find matrices P and D such that $A = PDP^{-1}$. If no, justify your answer. **(4 marks)**

c) Give one concrete 3×3 matrix that has three distinct eigenvalues and show that it is diagonalizable. Construct a counterexample showing that a matrix can be diagonalizable even without n distinct eigenvalues. **(4 marks)**

Question 4 (10 Marks)

a) Define a real vector space and list the eight axioms that must be satisfied by the addition and scalar multiplication operations. **(3 marks)**

b) Verify whether the set $V = \{(x, y) : x, y \in \mathbb{R}, x \geq 0\}$ with standard addition and scalar multiplication forms a vector space. **(3 marks)**

c) Can you give examples of vector space each with different operations but the same underlying set, where one is vector space and the other is not. **(4 marks)**

Question 5 (10 Marks)

a) Explain the concepts of linear combination and linear span of a set of vectors. **(2 marks)**

b) Express the vector $v = (2, -1, 5)$ as a linear combination of $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 2)$, and $u_3 = (1, 1, 0)$. Also, determine $\text{span}\{u_1, u_2, u_3\}$. **(4 marks)**

c) Show that $\text{span}\{v_1, v_2, \dots, v_n\}$ is the smallest subspace containing all the vectors v_1, v_2, \dots, v_n . Create a geometric interpretation for $n = 2$ in \mathbb{R}^3 . **(4 marks)**

Question 6 (10 Marks)

a) Define the row space and column space of an $m \times n$ matrix A . How are they related to the matrix operations? **(3 marks)**

b) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{pmatrix}$, find: (i) A basis for the row space (ii) A basis for the column space (iii) Verify that $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$ **(4 marks)**

c) Explain why elementary row operations preserve the row space but may change the column space. **(3 marks)**

Question 7 (10 Marks)

a) What is the difference between an inner product and a norm? How are they related? **(3 marks)**

- b) Let $u = (1,2)$ and $v = (2,3)$. Find the inner product u and v . Also, find angle between u and v using the inner product. **(3 marks)**
- c) Analyze whether the following function is an inner product on \mathbb{R}^2 : $\langle (x_1, x_2), (y_1, y_2) \rangle = 3x_1y_1 + x_2y_2$. Justify using axioms. **(4 marks)**

Question 8 (10 Marks)

- a) Define the norm of a vector in an inner product space and express it in terms of the inner product. **(2 marks)**
- b) For the vectors $u = (1, 2, 3)$ and $v = (4, 5, 6)$ in R^3 with the standard inner product, calculate $\|u\|$, $\|v\|$, and $\|u + v\|$. **(4 marks)**
- c) Prove that for any vector v in an inner product space, $\|\alpha v\| = |\alpha| \|v\|$ where α is a scalar. Discuss the geometric significance of this property. **(4 marks)**

Question 9 (10 Marks)

- a) Recall the relationship between *SVD* and the four fundamental subspaces of a matrix. **(2 marks)**
- b) Given a matrix A with SVD $A = U\Sigma V^T$ where $\Sigma = \text{diag}(5, 3, 0)$, determine the rank of A , and identify the dimensions of its null space and column space. **(4 marks)**
- c) Design a procedure to use *SVD* for image compression. Analyze how the number of singular values retained affects image quality and storage requirements. Provide a mathematical framework for your analysis. **(4 marks)**

Question 10 (10 Marks)

- a) State the theorem regarding the uniqueness of *QR* decomposition. Under what conditions is the decomposition unique? **(2 marks)**
- b) Given the matrix $A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 1 \end{pmatrix}$, perform *QR* decomposition and verify your answer by computing *QR*. **(5 marks)**
- c) Construct an algorithm to use *QR* decomposition iteratively to find eigenvalues of a matrix (*QR* algorithm). Explain the convergence properties. **(3 marks)**