Notes on Transformations and the Laplace Transform

1 Transformations

A transformation T operates on a function $f(x)$ to produce another function. T is **linear** if $T[\alpha f(x) + \beta g(x)] = \alpha T[f(x)] + \beta T[g(x)]$.

Common examples include Differentiation, $D[f(x)] = f'(x)$, and Integration, $I[f(x)] = \int_0^x f(t)dt$.

2 The Laplace Transformation

A **general linear integral transformation** is defined by a kernel $K(p, x)$:

$$T[f(x)] = \int_a^b K(p, x)f(x)dx = F(p) \tag{1}$$

The **Laplace transformation** $\mathcal{L}$ is a special case where $a = 0$, $b = \infty$, and the kernel is $K(p, x) = e^{-px}$.

2.1 Definition

The Laplace transform $\mathcal{L}$ acts on $f(x)$ to produce $F(p)$:

$$\mathcal{L}[f(x)] = F(p) = \int_0^\infty e^{-px}f(x)dx \tag{2}$$

This improper integral must **converge**, meaning the limit $\lim_{b\to\infty} \int_0^b e^{-px}f(x)dx$ must exist.

3 Examples of Basic Laplace Transforms

$$\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{p} && (p > 0)\\
\mathcal{L}\{x\} &= \frac{1}{p^2} && (p > 0)\\
\mathcal{L}\{x^n\} &= \frac{n!}{p^{n+1}} && (n \in \mathbb{Z}^+, p > 0)\\
\mathcal{L}\{e^{ax}\} &= \frac{1}{p-a} && (p > a)\\
\mathcal{L}\{\sin(ax)\} &= \frac{a}{p^2+a^2} && (p > 0)\\
\mathcal{L}\{\cos(ax)\} &= \frac{p}{p^2+a^2} && (p > 0)
\end{aligned}$$

4 Proof of $\mathcal{L}\{t^n\}$ (using the Gamma Function)

This proof derives the transform for $f(t) = t^n$. By definition:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

We use the substitution $x = st$. This implies:

- $t = \frac{x}{s}$
- $dt = \frac{1}{s} \, dx$
- Limits: $t = 0 \implies x = 0$ and $t = \infty \implies x = \infty$

Substitute these into the integral:

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \left(\frac{1}{s} \, dx\right) \\
&= \int_0^\infty e^{-x} \frac{x^n}{s^n} \frac{1}{s} \, dx \\
&= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}} \, dx \\
&= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx
\end{aligned}$$

This integral is a form of the **Gamma function**, $\Gamma(z)$.

Gamma Function Definition

The Gamma function is defined as:

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} \, dx$$

By comparing this definition to our integral $\int_0^\infty e^{-x} x^n \, dx$, we see that $z - 1 = n$, which means $z = n + 1$. Therefore, the integral is equal to $\Gamma(n + 1)$.

$$\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} \Gamma(n + 1)$$

A key property of the Gamma function is that for any positive integer n, $\Gamma(n + 1) = n!$.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$