Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

ODE notes

- Gauss's Hypergeometric Equation
- Special functions:
    - Legendre Polynomials
    - Bessel functions
    - Gamma functions

Gauss's Hypergeometric
ooooooooooooooooooooooooooo

Legendre's
oooooooooooooooooooooo

Bessel's
ooooooooooooooooooooooooo

Solving a Differential Equation using Hypergeometric Functions

Gauss's Hypergeometric Equation (GHE)

The differential equation

$$x(1-x)\,y'' + \big[c - (a+b+1)x\big]\,y' - ab\,y = 0 \tag{1}$$

where a, b, and c are constants, is called Gauss's Hypergeometric equation (GHE).

Learnings:?

- We will find the general solution of GHE at its regular points.
- As an application, in particular, we find the general solutions of the equation

$$(x - A)(x - B)\,y'' + (C + Dx)\,y' + Ey = 0$$

where $A \neq B$ by using the general solution of GHE.

Singularities

If we simplify GHE in the form of a standard second-order linear equation, we obtain

$$P(x) = \frac{c - [a + b + 1]x}{x(1 - x)} \quad \text{and} \quad Q(x) = \frac{-abx}{x(1 - x)}.$$

What are the singular points:

Singularities

If we simplify GHE in the form of a standard second-order linear equation, we obtain

$$P(x) = \frac{c - [a + b + 1]x}{x(1-x)} \quad \text{and} \quad Q(x) = \frac{-abx}{x(1-x)}.$$

What are the singular points: $x = 0$ and $x = 1$ are only singular points.
Are these regular points?

Singularities

If we simplify GHE in the form of a standard second-order linear equation, we obtain

$$P(x) = \frac{c - [a+b+1]x}{x(1-x)} \quad \text{and} \quad Q(x) = \frac{-abx}{x(1-x)}.$$

What are the singular points: $x = 0$ and $x = 1$ are only singular points.
Are these regular points? Yes, $x = 0$ and $x = 1$ are regular points, since

$$xP(x) = \frac{c - [a+b+1]x}{(1-x)} = [c - (a+b+1)x]\left(1 + x + x^2 + \cdots\right)$$

$$= c + [c - (a+b+1)]x + \cdots$$

and

$$x^2 Q(x) = \frac{-abx}{1-x} = -abx(1 + x + x^2 + \cdots) = -abx - abx^2 - \cdots.$$

Similarly, we can show that $x = 1$ is a regular point. Aim to find the general solutions of GHE at $x = 0$ and $x = 1$.

Gauss's Hypergeometric
○○○●○○○○○○○○○○○○○○○○○○○○○○○○○○○
Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○○○○○○○○○○○○○○○○○○○○○○

General solution at the regular point $x = 0$

- From the above expansions of $xP(x)$ and $x^2Q(x)$, $p_0 = c$ and $q_0 = 0$ so the indicial equation is

$$m(m-1) + mp_0 + q_0 = 0 \quad \Rightarrow \quad m[m - (1 - c)] = 0$$

and the exponents are $m = 0$ and $m = 1 - c$.

- **Solution corresponding to the exponent m = 0:** The exponent $m = 0$ corresponds to a Frobenius series solution by **Theorem A** if $1 - c < 0$ or the difference $(1 - c) - 0 = 1 - c$ is not a positive integer.

  Note that the second condition implies the first one, so if $1 - c$ is not a positive integer, equivalently c is neither zero nor a negative integer, then $m = 0$ corresponds to a Frobenius series solution of the form

$$y = x^0 \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (2)$$

  where a_0 is nonzero.

On substituting the above into (1) and equating the coefficients of x^n to zero, we obtain the following relations:

$$a_1 = \frac{ab}{c} a_0; \quad a_2 = \frac{(a+1)(b+1)}{2(c+1)} a_1 = \frac{a(a+1)b(b+1)}{2c(c+1)} a_0$$

$$a_{n+1} = \frac{(a+n)(b+n)}{(n+1)(c+n)} a_n. \tag{3}$$

With these coefficients and by letting $a_0 = 1$, the solution (2) becomes

$$y = 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{2c(c+1)} x^2 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)\cdots(c+n-1)} x^n. \tag{4}$$

This is known as the *hypergeometric series*, and is denoted by the symbol $F(a, b, c, x)$. It is called by this name because it generalizes the familiar geometric series as follows:

- When $a = 1$ and $c = b$: $F(1, b, b, x) = 1 + x + x^2 + \cdots = \dfrac{1}{1-x}$.

Gauss's Hypergeometric
ooooo●ooooooooooooooooooooooo

Legendre's
oooooooooooooooooooooooo

Bessel's
oooooooooooooooooooooooo

What is the Hypergeometric Series?

- The **Gauss hypergeometric series** is

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!},$$

  where the coefficients α_n are given by $\alpha_n = \dfrac{(a)_n (b)_n}{(c)_n}, \quad n \geq 0.$

- Here $(q)_n$ is the **Pochhammer symbol** (rising factorial):

$$(q)_n = q(q+1)(q+2) \cdots (q+n-1), \qquad (q)_0 := 1.$$

- **Familiar Functions as Special Cases of** $F(a, b; c; x)$:

Function	Hypergeometric Form	Comment
e^x	$F(1, 1; 1; x)$	Standard series expansion
$(1-x)^{-a}$	$F(a, 1; 1; x)$	Binomial theorem
$\ln(1+x)$	$x\, F(1, 1; 2; -x)$	Integral representation
$\arcsin x$	$x\, F(1/2, 1/2; 3/2; x^2)$	Trigonometric inverse
$\arctan x$	$x\, F(1/2, 1; 3/2; -x^2)$	Another inverse trig example
$\frac{1}{1-x}$	$F(1, 1; 1; x)$	Geometric series

Gauss's Hypergeometric
○○○○○○●○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

Solution corresponding to the second exponent $m = 1 - c$

- If $1 - c$ is neither zero nor a negative integer, i.e., c is not a positive integer, then Theorem A also tells us that there is a second independent solution of GHE (1) near $x = 0$ with exponent $m = 1$.

- This solution can be found directly by substituting

$$y = x^{1-c}\big(a_0 + a_1 x + a_2 x^2 + \cdots\big)$$

  into GHE (1) and calculating the coefficients.

- The other way of finding the solution is to change the dependent variable in (1) from y to z by writing $y = x^{1-c}z$. When the necessary computations are performed, equation (1) becomes

$$x(1-x)z'' + \Big[(2-c) - \big([a-c+1] + [b-c+1] + 1\big)x\Big]z' - (a-c+1)(b-c+1)z = 0 \tag{5}$$

  which is the hypergeometric equation with the constants a, b and c are replaced by $a - c + 1$, $b - c + 1$ and $2 - c$.

Gauss's Hypergeometric
○○○○○○○●○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○○○○

- We already know that (5) (from the previous Subsection) has the power series solution

$$z = F(a - c + 1, b - c + 1, 2 - c, x)$$

near the origin, so our desired second solution is

$$y = x^{1-c} z = x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x).$$

- Accordingly, when c is not an integer, we have two independent Frobenius series solutions and hence

$$y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x) \qquad (6)$$

is the general solution of the hypergeometric equation (1) near the sigular point $x = 0$.

Gauss's Hypergeometric
○○○○○○○●○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○

- We already know that (5) (from the previous Subsection) has the power series solution
$$z = F(a - c + 1, b - c + 1, 2 - c, x)$$

  near the origin, so our desired second solution is
$$y = x^{1-c} z = x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x).$$

- Accordingly, when c is not an integer, we have two independent Frobenius series solutions and hence
$$y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x) \qquad (6)$$

  is the general solution of the hypergeometric equation (1) near the sigular point $x = 0$.

- Note that the above solution is only valid near the origin. We now solve GHE (1) near another singular point $x = 1$.

Gauss's Hypergeometric
○○○○○○○○○●○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○

General solution of GHE at the regular point $x = 1$

- The simplest procedure is to obtain this solution from the one already found, by introducing a new independent variable $t = 1 - x$. ($x = 1 - t$, $dy/dx = -dy/dt$ and $d^2y/dx^2 = d^2y/dt^2$).

- This makes $x = 1$ correspond to $t = 0$ and transforms (1) into

$$t(1 - t)y'' + \big[(a + b - c + 1) - (a + b + 1)t\big]y' - ab\,y = 0$$

  where the primes denotes the derivatives with respect to t.

- Since the above is a hypergeometric equation, its general solution near $t = 0$ can be written down at once from (6) by replacing x by t and c by $a + b - c + 1$ and then we replace t by $1 - x$ to get the general solution of GHE (1) near $x = 1$:

$$y = c_1 F(a, b, a + b - c + 1, 1 - x)$$
$$+ c_2(1 - x)^{c-a-b} F(c - b, c - a, c - a - b + 1, 1 - x). \qquad (7)$$

  (In this case it is necessary to assume that $c - a - b$ is not an integer).

Some comments, remarks and applications of hypergeometric functions F

- Formulas (6) and (7) show that the adaptability of the constants in equation GHE (1) makes it possible to express the general solution of this equation near each of its singular points in terms of the single function F.
- Moreover, the general solution of a wide class of differential equations can be written in terms of hypergeometric functions F.
- Any differential equation in which the coefficients of y'', y' and y are polynomials of degree 2, 1 and 0 respectively and also the first of these polynomials has distinct real roots can be brought into hypergeometric form by a linear change of the independent variable, and hence can be solved near its singular point in terms of the hypergeometric function.
- The above remarks can be explained as follows. We consider the class of the equation of this type:

$$(x - A)(x - B)y'' + (C + Dx)y' + Ey = 0 \qquad (8)$$

where $A \neq B$.

- If we change the independent variable from x to t linearly by means of $(x - A)(x - B) = kt(t - 1)$ for some constant k i.e.

$$t = \frac{x - A}{B - A}$$

  then $x = A$ corresponds to $t = 0$ and $x = B$ to $t = 1$.

- After changing the independent variable from x to t the equation (8) assumes the form

$$t(1 - t)\, y'' + (F + Gt)\, y' + Hy = 0,$$

  where F, G and H are certain combinations of the constants in (8) and the primes indicate derivatives with respect to t.

- This a hypergeometric equation with constants a, b and c defined by $F = c$, $G = -(a + b + 1)$ and $H = -ab$ and can therefore be solved near $t = 0$ and $t = 1$ in terms of the hypergeometric function.

- This means that (8) can be solved in terms of the same function near $x = A$ and $x = B$.

Gauss's Hypergeometric
○○○○○○○○○○○○●○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

Problems

Find the general solution of each of the following differential equations near the indicated singular point in terms of hypergeometric function.

1. $x(1-x)y'' + \left(\dfrac{3}{2} - 2x\right)y' + 2y = 0, \quad x = 0;$

2. $(2x^2 + 2x)y'' + (1 + 5x)y' + y = 0, \quad x = 0;$

3. $(x^2 - 1)y'' + (5x + 4)y' + 4y = 0, \quad x = -1;$

4. $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0, \quad x = 3.$

Hint: Convert the equation into hypergeometric equation with a suitable change of independent variable as explained in the previous slides.
Before we start to solve these problems, we recall hypergeometric equation and its general solutions near its regular singular points.

Gauss's Hypergeometric
○○○○○○○○○○○○○●○○○○○○○○○○○○
Legendre's
○○○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○○○○○○○○○○○○○○○○○○○○○○

Hypergeometric equation and solutions

Hypergeometric equation:

$$x(1-x)y'' + [c - (a+b+1)x]y' - ab\,y = 0$$

General solution at $x = 0$:

$$y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, c-2, x)$$

General solution at $x = 1$:

$$\begin{aligned}
y = {} & c_1 F(a,b,a+b-c+1, 1-x) \\
& + c_2 (1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-x)
\end{aligned}$$

Hypergeometric function:

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{n!\,c(c+1)\cdots(c+n-1)}\, x^n$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○●○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○

Solution of Problem 1

$$x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0, \quad x = 0.$$

If we compare this equation with the hypergeometric equation we have the coefficients

$$c = \frac{3}{2}; \quad a+b+1 = 2; \quad ab = -2.$$

Or $c = \frac{3}{2}$; $a+b = 1$; $ab = -2$. And then $a-b$ can be found by using the formula

$$a - b = \sqrt{(a+b)^2 - 4ab} = \sqrt{1 - 4(-2)} = 3.$$

By solving these equations we will get $a = 2$, $b = -1$, $c = \frac{3}{2}$.

Therefore the General solution at $x = 0$ is given by

$$
\begin{aligned}
y &= c_1 F(a, b, c, x) + c_2 x^{1-c} F(a-c+1, b-c+1, c-2, x) \\
&= c_1 F\left(2, -1, \frac{3}{2}, x\right) + c_2 x^{-1/2} F\left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, x\right) \\
&= c_1\left(1 - \frac{4}{3}x\right) + c_2 x^{-1/2} F\left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, x\right).
\end{aligned}
$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○●○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○

Solution of Problem 4

$$(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0, \quad x = 3.$$

We use the change of independent variable from x to t by $t = \dfrac{x - A}{B - A}$, where A and B are roots of the polynomial $x^2 - x - 6$ which is a coefficient of y'' in given equation.

We see that $A = 3$ and $B = -2$ so that $t = \dfrac{3 - x}{5}$ or $x = -5t + 3$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{1}{5}\frac{dy}{dx}; \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{1}{5}\frac{dy}{dx}\right) = \frac{1}{25}\frac{d^2y}{dt^2}.$$

This transformation reduces the equation into

$$(-5t)(-5t + 5)\left(\frac{1}{25}\right)y'' + (5 - 15t + 9)\left(-\frac{1}{5}\right)y' + y = 0$$

$$t(1 - t)y'' + \left(\frac{14}{5} - 3t\right)y' - y = 0.$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○●○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

If we compare this equation with the hypergeometric equation we have the coefficients

$$c = \frac{14}{5}; \quad a + b + 1 = 3; \quad ab = 1.$$

$a + b = 2$; $ab = 1$ and hence $a - b = \sqrt{4 - 4} = 0$. Therefore

$$a = 1, \quad b = 1, \quad c = \frac{14}{5}.$$

The General solution at $t = 0$ or $x = 3$ is given by

$$
\begin{aligned}
y &= c_1 F(a, b, c, t) + c_2 t^{1-c} F(a - c + 1, b - c + 1, c - 2, t) \\
&= c_1 F\left(1, 1, \frac{14}{5}, t\right) + c_2 t^{-9/5} F\left(-\frac{4}{5}, -\frac{4}{5}, -\frac{4}{5}, t\right) \quad \left(\text{since } t = \frac{3 - x}{5}\right) \\
&= c_1 F\left(1, 1, \frac{14}{5}, \frac{3 - x}{5}\right) + c_2 \left(\frac{3 - x}{5}\right)^{-9/5} F\left(-\frac{4}{5}, -\frac{4}{5}, -\frac{4}{5}, \frac{3 - x}{5}\right).
\end{aligned}
$$

Problem 5

Some differential equations are of the hypergeometric type even though they may not appear to be so. Find the general solution of

$$(1 - e^x)y'' + \frac{1}{2}y' + e^x y = 0$$

near $x = 0$ by changing the independent variable to $t = e^x$.

Sol. The transformation $t = e^x$ implies $dt/dx = e^x = t$ and

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = t\frac{dy}{dt}; \qquad \frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

We substitute all these in given equation to get

$$t^2(1-t)\frac{d^2y}{dt^2} + t\left(\frac{3}{2} - t\right)\frac{dy}{dt} + ty = 0.$$

Or

$$t(1-t)\frac{d^2y}{dt^2} + \left(\frac{3}{2} - t\right)\frac{dy}{dt} + y = 0.$$

After comparing with standard hypergeometric equation we get the constants $c = \frac{3}{2}$, $a + b + 1 = 1$ and $ab = -1$. If solve these equations. then we will get $a = 1$, $b = -1$, $c = \frac{3}{2}$.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○●○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

The General solution at $t = 1$ or $x = 0$ is given by

$$y = c_1 F(a, b, a + b - c + 1, 1 - t) + c_2(1 - t)^{c-a-b} F(c - b, c - a, c - a - b + 1, 1 - t)$$

$$= c_1 F\left(1, -1, -\frac{1}{2}, 1 - t\right) + c_2(1 - t)^{3/2} F\left(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}, 1 - t\right)$$

$$= c_1 F\left(1, -1, -\frac{1}{2}, 1 - e^x\right) + c_2(1 - e^x)^{3/2} F\left(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}, 1 - e^x\right).$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○●○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○

Problem 6

Consider the Chebyshev's equation

$$(1 - x^2)y'' - xy' + p^2 y = 0,$$

where p is a nonnegative constant. Transform it into a hypergeometric equation by changing the independent variable from x to $t = \frac{1}{2}(1 - x)$, and show that its general solution near $x = 1$ is

$$y = c_1 F\left(p, -p, \frac{1}{2}, \frac{1-x}{2}\right) + c_2 \left(\frac{1-x}{2}\right)^{1/2} F\left(p + \frac{1}{2}, -p + \frac{1}{2}, \frac{3}{2}, \frac{1-x}{2}\right).$$

Problem 7

Consider the differential equation

$$x(1 - x)y'' + [p - (p + 2)x]y' - py = 0$$

where p is a constant.

- (a) If p is not an integer, find the general solution near $x = 0$ in terms of hypergeometric functions.
- (b) Write the general solution found in (a) in terms of elementary functions.
- (c) When $p = 1$, the differential equation becomes

$$x(x - 1)y'' + (1 - 3x)y' - y = 0$$

and the solution in (b) is no longer the general solution. Find the general solution in this case by the method of the use of known solution to find another.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○●○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○

Problem 8. Consider the Legendre's equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (9)$$

where n is understood to be nonnegative integer.

We have found analytic solutions near ordinary point $x = 0$. However, the solutions most useful in the applications are those bounded near $x = 1$.

(a) Find the general solution near $x = 1$ by changing the independent variable from x to $\frac{1}{2}(1 - x)$

(b) Find the solutions from (a) that are bounded near $x = 1$.

Sol. The change of independent variable x to $\frac{1}{2}(1 - x)$ makes $x = 1$ correspond to $t = 0$ and transforms the equation (9) into

$$t(1 - t)y'' + (1 - 2t)y' + n(n+1)y = 0 \qquad (10)$$

where the primes denotes derivatives with respect to t.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○●○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○

This is hypergeometric equation with $a = -n$, $b = n+1$, and $c = 1$, so it has the following polynomial solution near $t = 0$:

$$y_1 = F(-n, n+1, 1, t). \tag{11}$$

Since the exponents of the equation (10) at the origin $t = 0$ are both zero ($m_1 = 0$ and $m_2 = 1 - c = 0$), we seek a second solution via the formula $y_2(t) = y_1(t)v(t)$ with $v(t)$ is given by

$$v(t) = \int \frac{1}{y_1^2} e^{-\int P(t)\, dt} dt$$

where $P(t) = \dfrac{1 - 2t}{t(1 - t)}$. Note that

$$P(t) = \frac{1 - 2t}{t(1 - t)} = \frac{1}{t} - \frac{1}{1 - t}$$

therefore

$$e^{-\int P(t)\, dt} = e^{-\ln t - \ln(1-t)} = \frac{1}{t(1 - t)}.$$

Gauss's Hypergeometric
ooooooooooooooooooooooooooo●oo

Legendre's
ooooooooooooooooooooooooo

Bessel's
ooooooooooooooooooooooooo

So we have that

$$v(t) = \int \frac{1}{y_1^2} \frac{1}{t(1-t)} \, dt = \int \frac{1}{t} \left[\frac{1}{y_1^2} (1-t) \right] dt$$

Since y_1 is a polynomial with constant term 1, the bracket expression on right is an analytic function of the form $1 + a_1 t + a_2 t^2 + \cdots$. This yields $v(t) = \log t + a_1 t + \cdots$, so

$$y_2 = y_1 (\log t + a_1 t + \cdots).$$

and the general solution is given by

$$y = c_1 y_1 + c_2 y_2. \tag{12}$$

If we replace t in (12) by $\frac{1}{2}(1-x)$, we will get the general solution of Legendre's equation.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○●○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○○

Because of the presence of the term $\log t$ in y_2, it is clear that (12) is bounded if and only if $c_2 = 0$.

If we replace t in (11) by $\frac{1}{2}(1 - x)$, it follows that the solutions of Legendre's equation (9) are bounded near $x = 1$ are precisely constant multiple of the polynomial

$$F\left[-n,\, n + 1,\, 1,\, \tfrac{1}{2}(1 - x)\right].$$

Here the polynomial

$$F\left[-n,\, n + 1,\, 1,\, \tfrac{1}{2}(1 - x)\right]$$

is called n-th Legendre polynomial.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○●
Legendre's
○○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○○○○○○○○○○○○○○○○○○○

Problem 9. Verify the following identities

(a) $(1+x)^p = F(-p, b, b, -x)$

(b) $e^x = \lim\limits_{b \to \infty} F\left(a, b, a, \dfrac{x}{b}\right)$

(c) $\log(1+x) = xF(1, 1, 2, -x)$

(d) $\cos x = \lim\limits_{a \to \infty} F\left(a, a, \dfrac{1}{2}, -\dfrac{x^2}{4a^2}\right)$

Legendre's equation

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○●○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre Polynomials

- The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0,$$

  where n is a constant.

- How will you try to find its solution at $x = 0$?

- **Series Solution at an Ordinary Point:** Here $x = 0$ is an ordinary point and the general solution is

$$y(x) = a_0 \left[1 - \frac{n(n+1)}{2!}\, x + \frac{n(n-2)(n+1)(n+3)}{3!}\, x^4 - \cdots \right]$$
$$+ a_1 \left[x - \frac{(n-1)(n+2)}{3!}\, x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}\, x^5 - \cdots \right].$$

  Note:
  - If n is not an integer, then each series is convergent in $-1 < x < 1$.
  - If n is a non-negative integer, one of the series terminates and thus is a polynomial.

Solution near $x = 1$

- However, the solutions, most useful in the applications, are those bounded near $x = 1$.
- So our interest is to solve the Legendre equation near $x = 1$.
- For this we proceed by the approach of Hypergeometric equation.

The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0,$$

where n is a constant.

Solution near $x = 1$

- However, the solutions, most useful in the applications, are those bounded near $x = 1$.
- So our interest is to solve the Legendre equation near $x = 1$.
- For this we proceed by the approach of Hypergeometric equation.

The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n + 1)\, y = 0,$$

where n is a constant.

- What are its regular singular points?

Gauss's Hypergeometric
ooooooooooooooooooooooooooo

Legendre's
ooo●oooooooooooooooooooooo

Bessel's
oooooooooooooooooooooo

Solution near $x = 1$

The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n + 1)\, y = 0,$$

where n is a constant.

- Looking at coefficients of y'', y', y we notice that we can transform this into a hypergeometric equation in standard form.

Solution near $x = 1$

The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0,$$

where n is a constant.

- Looking at coefficients of y'', y', y we notice that we can transform this into a hypergeometric equation in standard form.
- What transformation will be helpful to find the solution near to $x = 1$?
- Can we find the transformed Differential equation?

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○
Legendre's
○○○○●○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○○○

Solution near $x = 1$

The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0,$$

where n is a constant.

- We change the independent variable from x to t by

$$t = \frac{x - 1}{(-1) - 1} = \frac{1}{2}(1 - x).$$

- This makes $x = 1$ correspond to $t = 0$ and transforms the equation into

$$t(1 - t)y'' + (1 - 2t)y' + n(n+1)y = 0,$$

where the primes represent derivatives with respect to t.

What are its solutions?

Solution near $x = 1$

The Transformed Equation is

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0,$$

where n is a constant.

- This is hypergeometric equation with $a = -n, b = n+1, c = 1$.
- When $1 - c$ is a real number but not an integer, then the general solution of the above equation can be found as

$$y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x) \qquad (6)$$

- But in our case we have $1 - c = 0$. Infact both exponents are 0 ($m_1 = 0, m_2 = 0$.)
- So we need an alternative approach.
- Note, the first solution is $y_1 = F(-n, n+1, 1, t)$.

The Transformed Equation is

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0,$$

where n is a constant. Note that we have $P(x) = \dfrac{(1-2t)}{t(1-t)}$.

- We seek a second solution by the method of Section 16. This second solution is $y_2 = vy_1$, where

$$v' = \frac{1}{y_1^2} e^{-\int p\, dt} = \frac{1}{y_1^2} e^{\int \frac{(2t-1)}{t(1-t)}\, dt} = \frac{1}{y_1^2 t(1-t)} = \frac{1}{t}\left[\frac{1}{y_1^2(1-t)}\right].$$

- On solving, we get

$$y_2 = y_1\big(\log t + a_1 t + \cdots\big)$$

and the general solution of (2) near the origin is

$$y = c_1\, y_1 + c_2\, y_2.$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○●○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

Bounded Solutions Near $x = 1$

- Because of the presence of the term $\log t$ in y_2, it is clear that the general solution

$$y = c_1 y_1 + c_2 y_2$$

  is bounded near $t = 0$ if and only if $c_2 = 0$.

- Replace t by

$$t = \tfrac{1}{2}(1 - x),$$

- The solutions of Legendre's equation that are bounded near $x = 1$ are precisely constant multiples of the polynomial

$$F(-n, \, n + 1, \, 1, \, \tfrac{1}{2}(1 - x)).$$

- This brings us to the fundamental definition. The nth Legendre polynomial is denoted by $P_n(x)$ and defined by

$$P_n(x) = F\Big(-n, \, n + 1, \, 1, \, \tfrac{1}{2}(1 - x)\Big)$$

Definition of $P_n(x)$ via F

Definition (Legendre polynomial). The nth Legendre polynomial $P_n(x)$ is defined by

$$P_n(x) = F\left(-n,\ n+1,\ 1,\ \frac{1-x}{2}\right).$$

$$P_n(x) = F\left(-n,\ n+1,\ 1,\ \frac{1-x}{2}\right)$$

$$= 1 + \frac{(-n)(n+1)}{(1!)^2}\left(\frac{1-x}{2}\right) + \frac{(-n)(-n+1)(n+1)(n+2)}{(2!)^2}\left(\frac{1-x}{2}\right)^2 + \cdots$$

$$+ \frac{(-n)(-n+1)(-n+(n-1))(n+1)(n+2)\cdots(2n)}{(n!)^2}\left(\frac{1-x}{2}\right)^n$$

$$P_n(x) = 1 + \frac{n(n+1)}{(1!)^2\,2}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2\,2^2}(x-1)^2 + \cdots + \frac{(2n)!}{(n!)^2\,2^n}(x-1)^n.$$

Remark: The above formula is a very inconvenient tool to use in studying $P_n(x)$, so we look for something simpler.

Rodrigues' formula

The nth Legendre polynomial can equivalently be defined by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \big[(x^2 - 1)^n\big], \qquad n = 0, 1, 2, \dots$$

Remark. The above provides a relatively easy method for computing successive Legendre polynomials.

- **First few polynomials (by Rodrigues):**

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1), \qquad P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x), \ \dots$$

Properties of Legendre polynomials

- The sequence of Legendre polynomials $P_0(x), P_1(x), \ldots, P_n(x), \ldots$
- **Orthogonality:** On $[-1, 1]$, the sequence $\{P_n\}_{n \geq 0}$ is orthogonal:

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0, & m \neq n, \\ \dfrac{2}{2n+1}, & m = n. \end{cases}$$

$$OR$$

The sequence of Legendre polynomials forms a sequence of orthogonal functions on the interval $-1 \leq x \leq 1$.

Proof: Orthogonality of Legendre polynomials

- Let

$$I = \int_{-1}^{1} f(x) P_n(x)\, dx \qquad (1)$$

  where $f(x)$ be any function with at least n continuous derivatives on $[-1, 1]$.

- By Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

- Therefore, (1) becomes

$$I = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○●○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

Proof: Orthogonality of Legendre polynomials

- Let

$$I = \int_{-1}^{1} f(x) P_n(x)\, dx \qquad (1)$$

  where $f(x)$ be any function with at least n continuous derivatives on $[-1, 1]$.

- By Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

- Therefore, (1) becomes

$$I = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

- Integrating by parts, we get

$$I = \frac{(-1)^1}{2^n n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n\, dx$$

Proof: Orthogonality of Legendre polynomials

- Integrating repeatedly, we get

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} f^{(n)}(x)(x^2 - 1)^n \, dx \qquad (2)$$

- Let $m < n$ and $f(x) = P_m(x)$ in (1), then $f^{(n)}(x) = 0$ and therefore $I = 0$.
- Similarly, we can show that

$$\int_{-1}^{1} P_m(x)P_n(x) \, dx = 0, \quad \text{for } m > n.$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○●○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○○○

Proof: Orthogonality of Legendre polynomials

- If $m = n$ let $f(x) = P_n(x)$. We already have
$$P_n^{(n)}(x) = \frac{1}{2^n n!} \frac{d^{2n}}{dx^{2n}} (x^2 - 1)^n$$

- Thus
$$P_n^{(n)}(x) == \frac{1}{2^n n!} \frac{d^{2n}}{dx^{2n}} \left(x^{2n} + \text{terms with lower exponent than } 2n \right) = \frac{(2n)!}{2^n n!},$$

- Therefore, using (2), $I = \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (1 - x^2)^n \, dx$

- Since $(1 - x^2)^n$ is an even function, $I = \frac{2(2n)!}{2^{2n}(n!)^2} \int_{0}^{1} (1 - x^2)^n \, dx$

- Change the variable $x = \sin \theta$, then recall the formula
$$\int_{0}^{\pi/2} \cos^{2n+1} \theta \, d\theta = \frac{2^{2n}(n!)^2}{(2n)!(2n + 1)}.$$

- Hence, we conclude that in this case
$$I = \frac{2}{2n + 1}, \quad \text{hence proved.}$$

Properties of Legendre polynomials

- **Legendre Series**: Many problems of the potential theory depend on the possibility of expanding a given function in a series of Legendre polynomials.
- *If the given function is a polynomial, it is easy to see that it can be expressed as the sum of the Legendre polynomials.*
- For example:

$$1 = P_0(x), \qquad x = P_1(x),$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1) \;\Rightarrow\; x^2 = \tfrac{1}{3} + \tfrac{2}{3}P_2(x) = \tfrac{1}{3}P_0(x) + \tfrac{2}{3}P_2(x),$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \;\Rightarrow\; x^3 = \tfrac{3}{5}x + \tfrac{2}{5}P_3(x) = \tfrac{3}{5}P_1(x) + \tfrac{2}{5}P_3(x).$$

Properties of Legendre polynomials

- Third degree polynomial $p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ can be written as

$$p(x) = b_0 P_0(x) + b_1 P_1(x) + b_2 \left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)\right] + b_3 \left[\frac{3}{5} P_1(x) + \frac{2}{5} P_3(x)\right]$$
$$= \left(b_0 + \frac{b_2}{3}\right) P_0(x) + \left(b_1 + \frac{3b_3}{5}\right) P_1(x) + \frac{2b_2}{3} P_2(x) + \frac{2b_3}{5} P_3(x)$$
$$= \sum_{n=0}^{3} a_n P_n(x).$$

- Therefore, any polynomial $p(x)$ of degree k can be written as

$$p(x) = \sum_{n=0}^{k} a_n P_n(x).$$

Properties of Legendre polynomials

- **What about an arbitrary function?**
- An arbitrary function $f(x)$ can be expressed as **Legendre series**

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

  - Need to calculate coefficients a_n for the above expression.
  - Multiply $P_m(x)$ on both sides and integrate from -1 to 1, we get

  $$\int_{-1}^{1} f(x) P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\, dx$$

  - In view of our orthogonality result, this collapses to

  $$\int_{-1}^{1} f(x) P_m(x)\, dx = \frac{2 a_m}{2m+1}$$

- And therefore, we have

$$a_n = \left(n + \tfrac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx.$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○●○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○

Generating Function of the Legendre Polynomials

$$g(x,t) := \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

is called the **generating function** of the Legendre polynomials. Therefore, $P_n(x)$ is the coefficient of t^n in the expansion of

$$(1 - 2xt + t^2)^{-1/2}$$

in ascending powers of t.

- We can prove the following

$$\text{(i)} \quad P_n(1) = 1 \quad \text{and} \quad P_n(-1) = (-1)^n.$$

$$\text{(ii)} \quad P_{2n+1}(0) = 0 \quad \text{and} \quad P_{2n}(0) = (-1)^n \, \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n \, n!}.$$

- Recursion formula to generate Legendre polynomials:

$$(n + 1)P_{n+1} = (2n + 1)x P_n - n P_{n-1}, \quad n = 1, 2, \ldots$$

- Assume that $P_0(x) = 1$ and $P_1(x) = x$ are known, and one can use the recursion formula to calculate $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○●○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○

Generating Function of the Legendre Polynomials

Evaluate $P_n(-1)$. This is a simpler problem. In this case we have

$$g(-1, t) = \frac{1}{\sqrt{1 + 2t + t^2}} = \frac{1}{1 + t} = 1 - t + t^2 - t^3 + \cdots$$

Therefore,

$$P_n(-1) = (-1)^n.$$

Properties of Legendre Polynomials: Least Squares Approximation

- Let $f(x)$ be a function defined on the interval $[-1, 1]$.
- Now the problem is to approximate the function $f(x)$ by a polynomial $p(x)$ of degree $\leq n$ as closely as possible in the sense of least squares, i.e.

> Given a function $f(x)$, continuous on $[-1, 1]$, find a polynomial $p(x)$ of degree at most n such that the integral of the square of the error is minimized. That is,
>
> $$I = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$$
>
> is minimized.

- It turns out that the minimizing polynomial is precisely the sum of the first $n + 1$ terms of the Legendre series:

$$p(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots + a_n P_n(x)$$

where

$$a_n = \left(n + \tfrac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x) \, dx.$$

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○●○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

- We have $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- The coefficients of the Legendre series are given by

$$a_n = \left(n + \tfrac{1}{2}\right) \int_{-1}^{1} e^x P_n(x)\, dx, \qquad n = 0, 1, 2, \ldots$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○●○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

- We have $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- The coefficients of the Legendre series are given by

$$a_n = \left(n + \tfrac{1}{2}\right) \int_{-1}^{1} e^x P_n(x)\, dx, \qquad n = 0, 1, 2, \ldots$$

- **Linear approximation:** Using $P_0(x) = 1$, $P_1(x) = x$,

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

- We have $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- The coefficients of the Legendre series are given by

$$a_n = \left(n + \tfrac{1}{2}\right) \int_{-1}^{1} e^x P_n(x)\, dx, \qquad n = 0, 1, 2, \ldots$$

- **Linear approximation:** Using $P_0(x) = 1$, $P_1(x) = x$,

$$a_0 = \frac{1}{2} \int_{-1}^{1} e^x \, dx = \frac{1}{2}\left(e - e^{-1}\right), \qquad a_1 = \frac{3}{2} \int_{-1}^{1} x e^x \, dx = 3e^{-1}.$$

Hence $p_1(x) = a_0 P_0(x) + a_1 P_1(x) = \frac{1}{2}(e - e^{-1}) + 3e^{-1}x$.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○●○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

- We have $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- The coefficients of the Legendre series are given by

$$a_n = \left(n + \tfrac{1}{2}\right) \int_{-1}^{1} e^x P_n(x)\, dx, \qquad n = 0, 1, 2, \ldots$$

- **Linear approximation:** Using $P_0(x) = 1$, $P_1(x) = x$,

$$a_0 = \frac{1}{2} \int_{-1}^{1} e^x \, dx = \tfrac{1}{2}\left(e - e^{-1}\right), \qquad a_1 = \frac{3}{2} \int_{-1}^{1} x e^x \, dx = 3e^{-1}.$$

Hence $p_1(x) = a_0 P_0(x) + a_1 P_1(x) = \frac{1}{2}(e - e^{-1}) + 3e^{-1}x$.

- **Quadratic approximation:** With $P_2(x) = \frac{1}{2}(3x^2 - 1)$,

$$a_2 = \frac{5}{2} \int_{-1}^{1} e^x \, \tfrac{1}{2}(3x^2 - 1) \, dx = \frac{5}{2}(e - 7e^{-1}).$$

Thus $p_2(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$,

$$p_2(x) = \frac{-3e + 33e^{-1}}{4} + 3e^{-1}x + \frac{15}{4}(e - 7e^{-1})x^2.$$

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

$y = e^{x}$

$l = 0.5 \cdot \left(e - e^{-1}\right) + 3e^{-1}x$

$q = \frac{\left(-3 \cdot e + 33 \cdot e^{-1}\right)}{4} + 3 \cdot e^{-1}x + \frac{15}{4}\left(e - 7e^{-1}\right)x^{2}$

Bessel's equation (and functions)

Bessel Functions: An Introduction

The Core Idea

Bessel functions are the standard solutions to physical problems involving **circular or cylindrical symmetry**.

- They are like the "sines and cosines" for cylindrical shapes.
- They appear when solving differential equations (like the wave or heat equation) in cylindrical coordinates.
- We will explore some applications where they are essential.

Application 1: Vibrating Drumheads
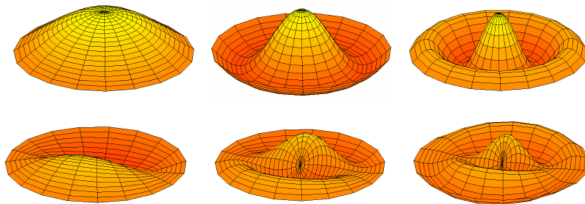
Drumhead Vibration Modes

- The motion of a circular drum after being struck is described by the wave equation.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○
Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○●○○○○○○○○○○○○○○○○○○○○○○

Application 1: Vibrating Drumheads

Drumhead Vibration Modes

- The motion of a circular drum after being struck is described by the wave equation.
- In polar coordinates, this equation simplifies to the Bessel equation.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○●○○○○○○○○○○○○○○○○○○○○○

Application 1: Vibrating Drumheads

Drumhead Vibration Modes

- The motion of a circular drum after being struck is described by the wave equation.
- In polar coordinates, this equation simplifies to the Bessel equation.
- Bessel functions describe the "modes" of vibration - the beautiful, complex patterns the drum surface makes.

Application 2: Waves in Pipes and Cables
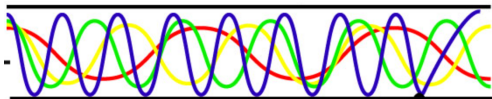
Bessel functions are essential for describing how waves travel through cylindrical guides:

- **Sound waves** in a circular pipe.

Application 2: Waves in Pipes and Cables
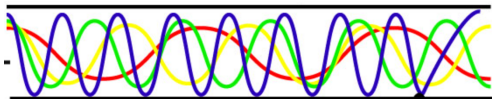
Bessel functions are essential for describing how waves travel through cylindrical guides:

- **Sound waves** in a circular pipe.
- **Electromagnetic waves** in a coaxial cable or waveguide.

Application 2: Waves in Pipes and Cables
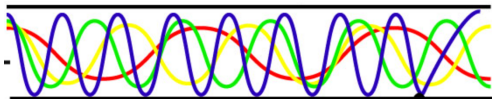
Bessel functions are essential for describing how waves travel through cylindrical guides:

- **Sound waves** in a circular pipe.
- **Electromagnetic waves** in a coaxial cable or waveguide.
- **Light** traveling through an **optical fiber**. The functions describe the different "modes," or patterns of light.

Bessel's equation and function

- **Bessel's Equation of order** p:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

Bessel's equation and function

- **Bessel's Equation of order** p:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

- To analyze the point $x = 0$, we rewrite it in the **standard form** $y'' + P(x)y' + Q(x)y = 0$ by dividing by x^2:

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

- From this, we identify the coefficients:
  - $P(x) = \dfrac{1}{x}$ $Q(x) = \dfrac{x^2 - p^2}{x^2}$

- A point x_0 is a **regular singular point** if either $P(x)$ or $Q(x)$ is not analytic at x_0.

- **Conclusion:** Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$, the origin is a regular **singular point** of Bessel's equation.

Bessel's equation and function

- The **indicial equation** is $m^2 - p^2 = 0$, and the exponents are $m_1 = p$ and $m_2 = -p$.
- It follows from **Theorem 30-A** that Bessels' equation has a solution of the form

$$y(x) = x^p \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+p},$$

where $a_0 \neq 0$ and the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x.

Bessel's equation (Solution)

Bessel's equation : $x^2 y'' + xy' + (x^2 - p^2)y = 0.$

- We get $y(x) = \sum_{n=0}^{\infty} a_n x^{n+p}$, $y'(x) = \sum_{n=0}^{\infty} a_n (n+p) x^{n+p-1}$,

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+p)(n+p-1) x^{n+p-2}.$$

Bessel's equation (Solution)

Bessel's equation : $x^2 y'' + xy' + (x^2 - p^2)y = 0$.

- We get $y(x) = \sum_{n=0}^{\infty} a_n x^{n+p}$, $y'(x) = \sum_{n=0}^{\infty} a_n(n+p)x^{n+p-1}$,

$$y''(x) = \sum_{n=0}^{\infty} a_n(n+p)(n+p-1)x^{n+p-2}.$$

- The terms of the equation become

$$-p^2 y = \sum_{n=0}^{\infty} -p^2 a_n x^{n+p}, \qquad x^2 y = \sum_{n=0}^{\infty} a_{n-2} x^{n+p},$$

$$xy' = \sum_{n=0}^{\infty} a_n(n+p)x^{n+p}, \quad x^2 y'' = \sum_{n=0}^{\infty} a_n(n+p)(n+p-1)x^{n+p}.$$

- Hence

$$a_0(p(p-1) + p - p^2) + a_1((p+1)^2 - p^2) + \sum_{n=2}^{\infty} [a_n n(n+2p) + a_{n-2}]x^n = 0$$

Bessel's equation (Solution)

Bessel's equation : $x^2 y'' + xy' + (x^2 - p^2)y = 0$.

- Inserting into the equation and equating the coefficients of x^{n+p} to zero, we get the following recursion formula

$$a_n = -\frac{a_{n-2}}{n(2p+n)}$$

Bessel's equation (Solution)

Bessel's equation : $x^2 y'' + xy' + (x^2 - p^2)y = 0$.

- Inserting into the equation and equating the coefficients of x^{n+p} to zero, we get the following recursion formula

$$a_n = -\frac{a_{n-2}}{n(2p + n)}$$

- Since $a_0 \neq 0$ and $a_1(2p + 1) = 0$ tells us that $a_1 = 0$, and
- Repeated application of this recursion formula yields the fact that $a_n = 0$ for every odd subscript n.

Bessel's equation (Solution)

Bessel's equation : $x^2 y'' + xy' + (x^2 - p^2)y = 0.$

- The nonzero coefficients of our solution are therefore

$$a_2 = -\frac{a_0}{2(2p+2)} = \frac{a_0}{1 \cdot 2^2(p+1)},$$

$$a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_0}{2 \cdot 4(2p+2)(2p+4)} = \frac{(-1)^2 a_0}{2!2^4(p+1)(p+2)},$$

$$a_6 = -\frac{a_4}{6(2p+6)}$$

$$= -\frac{a_0}{2 \cdot 4 \cdot 6(2p+2)(2p+4)(2p+6)} = \frac{(-1)^3 a_0}{3!2^6(p+1)(p+2)(p+3)}, \cdots$$

Bessel's equation (Solution)

$$\text{Bessel's equation} : x^2 y'' + x y' + (x^2 - p^2)y = 0.$$

- The nonzero coefficients of our solution are therefore

$$a_2 = -\frac{a_0}{2(2p+2)} = \frac{a_0}{1 \cdot 2^2(p+1)},$$

$$a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_0}{2 \cdot 4(2p+2)(2p+4)} = \frac{(-1)^2 a_0}{2! 2^4(p+1)(p+2)},$$

$$a_6 = -\frac{a_4}{6(2p+6)}$$

$$= -\frac{a_0}{2 \cdot 4 \cdot 6(2p+2)(2p+4)(2p+6)} = \frac{(-1)^3 a_0}{3! 2^6(p+1)(p+2)(p+3)}, \cdots$$

And the solution is

$$y = a_0 x^p \left[1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4 2!(p+1)(p+2)} - \frac{x^6}{2^6 3!(p+1)(p+2)(p+3)} + \ldots \right]$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○
Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○○○○○○○○○○○●○○○○○○○○○○○○○

The Bessel Function of the First Kind

The solution is

$$y = a_0 x^p \left[1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4 2!(p+1)(p+2)} - \frac{x^6}{2^6 3!(p+1)(p+2)(p+3)} + \cdots \right]$$

- Or we have

$$y = a_0 x^p \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)(p+2)\cdots(p+n)}.$$

- If we replace the arbitrary constant a_0 by $1/(2^p p!)$, we obtain a particular solution of Bessel's equation which is denoted by $J_p(x)$ and is known as **Bessel function of the first kind of order** p.

Definition

$$J_p(x) = \frac{x^p}{2^p p!} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)(p+2)\cdots(p+n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2} \right)^{2n+p}$$

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○
Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○○○○○○○○○○○○●○○○○○○○○○○○○

Important Bessel Functions: Order 0 and 1

The solution is

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- The most useful Bessel functions are those of order 0 and 1 which are

Important Bessel Functions: Order 0 and 1

The solution is

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- The most useful Bessel functions are those of order 0 and 1 which are

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Important Bessel Functions: Order 0 and 1

The solution is

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- The most useful Bessel functions are those of order 0 and 1 which are

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

- and

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} = \frac{x}{2} - \frac{1}{1!2!} \left(\frac{x}{2}\right)^3 + \frac{1}{2!3!} \left(\frac{x}{2}\right)^5 - \cdots$$

Graphs of J_0, J_1, J_2, J_3

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$
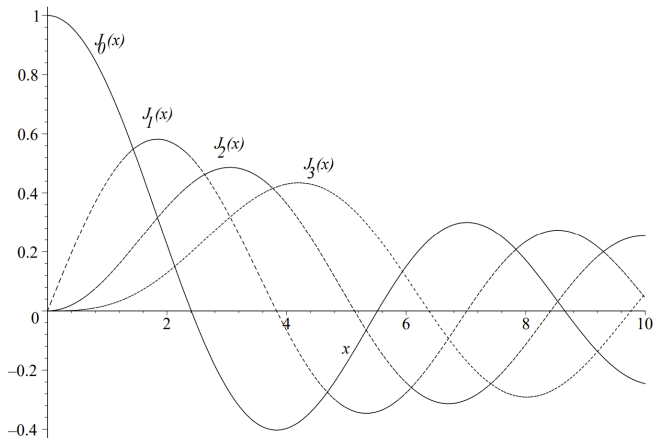
Properties and Challenges of Bessel Functions

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- These graphs display several interesting properties of the functions $J_0(x)$ and $J_1(x)$.
- Each has a damped oscillating behavior producing an infinite number of **positive zeros**; and these zeros occur alternatively, in a manner suggesting the functions $\cos x$ and $\sin x$. (**Positive zeros** - positive real numbers for which the function $J_p(x)$ vanishes)

Properties and Challenges of Bessel Functions

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- These graphs display several interesting properties of the functions $J_0(x)$ and $J_1(x)$.
- Each has a damped oscillating behavior producing an infinite number of **positive zeros**; and these zeros occur alternatively, in a manner suggesting the functions $\cos x$ and $\sin x$. (**Positive zeros** - positive real numbers for which the function $J_p(x)$ vanishes)
- Also
$$\frac{d}{dx} J_0(x) = -J_1(x) \quad \text{and} \quad \frac{d}{dx} x J_1(x) = x J_0(x).$$
- In the denominator of $J_p(x)$ there is a term $(p+n)!$, but it is meaningless if p is not a positive integer.
- Now our next turn is to **overcome this difficulty**.

Gauss's Hypergeometric ooooooooooooooooooooooooooo
Legendre's oooooooooooooooooooooo
Bessel's oooooooooooooooooo●oooooo

The Gamma Function

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- The purpose of gamma function is to give a reasonable and useful meaning to $p!$ [and more generally to $(p+n)!$ for $n = 0, 1, 2, \ldots$].
- The **gamma function** is defined by

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t} dt, \quad p > 0 \qquad (A)$$

- The factor $e^{-t} \to 0$ so rapidly as $t \to \infty$, that is, the integral converges at the upper limit regardless of the value of p.
- However at the lower limit we have $e^{-t} \to 1$ and the factor $t^{p-1} \to \infty$ whenever $p < 1$ as $t \to 0$.
- Therefore, the restriction that p must be positive is necessary in order to guarantee convergence at the lower limit.

Properties of the Gamma Function

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt, \quad p > 0 \qquad (A)$$

- $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$
- We see that

$$\Gamma(p+1) = p\Gamma(p)$$

- For integration by parts yields

$$\Gamma(p+1) = \lim_{b \to \infty} \int_0^b t^p e^{-t} dt$$

$$= \lim_{b \to \infty} \left(-t^p e^{-t} \Big|_0^b + p \int_0^b t^{p-1} e^{-t} dt \right)$$

$$= p \left(\lim_{b \to \infty} \int_0^b t^{p-1} e^{-t} dt \right) = p\Gamma(p).$$

- Since $\frac{b^p}{e^b} \to 0$ as $b \to \infty$.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○●○○○○

Gamma Function and Factorials

- For $p \in \mathbb{N}$, $\Gamma(p + 1) = p\Gamma(p)$ yields

Gamma Function and Factorials

- For $p \in \mathbb{N}$, $\Gamma(p+1) = p\Gamma(p)$ yields

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2 \cdot 1 = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2 \cdot 1 = 3!$$

- In general $\Gamma(n+1) = n!$ for any integer greater than or equal to zero.

- **One important fact:** $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

- Define $p!$ by

$$p! = \Gamma(p+1)$$

  for all values of p except negative integers.

- Also $\dfrac{1}{p!} = \dfrac{1}{\Gamma(p+1)}$ is defined for all values of p and has the value 0 whenever p is a negative integer.

General Solution of Bessel's Equation

- Recall, the **indicial equation** is $m^2 - p^2 = 0$, and the exponents are $m_1 = p$ and $m_2 = -p$.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○●○○○

General Solution of Bessel's Equation

- Recall, the **indicial equation** is $m^2 - p^2 = 0$, and the exponents are $m_1 = p$ and $m_2 = -p$.
- Found one solution for the exponent $m_1 = p$, namely $J_p(x)$.

General Solution of Bessel's Equation

- Recall, the **indicial equation** is $m^2 - p^2 = 0$, and the exponents are $m_1 = p$ and $m_2 = -p$.

- Found one solution for the exponent $m_1 = p$, namely $J_p(x)$.

- To get the general solution, we must construct a second linearly independent solution (known as ***Bessel function of second kind***).

- For the second Linearly Independent solution, the procedure is to try the other exponent $m_2 = -p$.

- But in doing this we expect to encounter *difficulties* whenever the difference $m_1 - m_2 = 2p$ is zero or a positive integer (i.e. when the nonnegative integer p is an integer or half of an odd integer).

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○
Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○○
Bessel's
○○○○○○○○○○○○○○○○○○○○○○●○○

Case: p is not an Integer

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- Assume p is **not an integer**.
- Replace p by $-p$ in our earlier treatment.
- We get **second solution** as

$$J_{-p}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n-p}}{n!(-p+n)!}.$$

- First term of the series is $\frac{1}{(-p)!} \left(\frac{x}{2}\right)^{-p}$. So $J_{-p}(x)$ is unbounded near $x = 0$.
- Since $J_p(x)$ is bounded near $x = 0$, these two solutions are independent and

$$y = c_1 J_p(x) + c_2 J_{-p}(x), \quad p \text{ not an integer}$$

is the *general solution of Bessel's equation*.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○○○●○

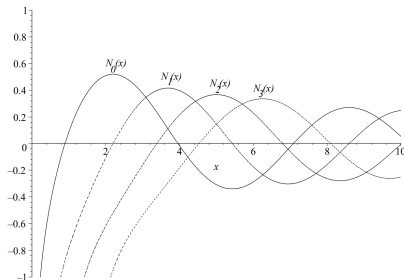## The General Solution for All Cases

- The **standard <u>Bessel function of second kind</u>** is defined by

$$Y_p(x) = \frac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi} \qquad \text{(also denoted by } N_p(x))$$

- Then the general solution of the Bessel's equation is

$$y = c_1 J_p(x) + c_2 Y_p(x)$$

in all cases, whether p is an integer or not.

Gauss's Hypergeometric
○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

Legendre's
○○○○○○○○○○○○○○○○○○○○○○○○

Bessel's
○○○○○○○○○○○○○○○○○○○○○○○●

Thank you for your attention