Recall: Orthogonality & Fourier Coefficients on $[-\pi, \pi]$

Orthogonality of $1,\ \sin(nx),\ \cos(nx)$

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0\ (n \neq 0), \qquad \int_{-\pi}^{\pi} \sin(nx)\,dx = 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \qquad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0 \quad \text{(all } m, n\text{)}$$

Fourier Series: $$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)}$$

Euler Formulas (coefficients)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Fourier Series: Even and Odd Functions; Cosine and Sine Series

A Symmetry-Based View of Fourier Series

Section 35 notes by Deepak Bhoriya

November 14, 2025

Outline

Motivation and Symmetry

Why Symmetry Matters in Fourier Series

- Our Fourier work could be based on any interval of length 2π.
- The interval $[-\pi, \pi]$ is *symmetric* around 0.
- Symmetry allows us to exploit even and odd properties of $f(x)$.
- This dramatically simplifies:
  - the structure of the Fourier series (sine vs. cosine only),
  - the computations of the coefficients,
  - conceptual understanding of convergence and parity.

The whole point: symmetry turns many integrals into 0 or into $2\times$ a smaller one.

Even and Odd Functions

Definitions: Even and Odd Functions

Definition (Even function)

A function f defined on a symmetric interval (e.g. $[-\pi, \pi]$) is *even* if

$$f(-x) = f(x) \quad \text{for all } x.$$

Definition (Odd function)

A function f is *odd* if

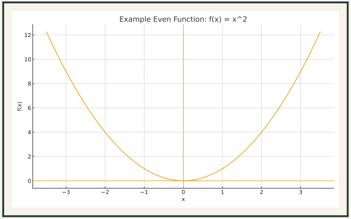
$$f(-x) = -f(x) \quad \text{for all } x.$$

- Examples of even functions: x^2, $\cos x$.
- Examples of odd functions: x^3, $\sin x$.

Geometrically: even $\Rightarrow$ mirror symmetry across y-axis; odd $\Rightarrow$ 180° rotational symmetry.

Geometric Picture: Even Functions

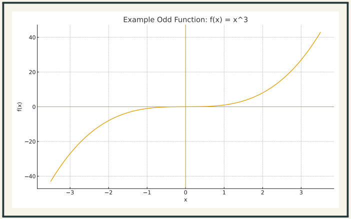
- Graph is symmetric about the y-axis.
- Area from $-a$ to 0 equals area from 0 to a.
- Evenness implies $f(-x) = f(x)$ for all x.
- Examples: Figure shows $f(x) = x^2$.



Think: fold the graph along y-axis; an even function overlaps with itself.

Geometric Picture: Odd Functions

- Graph has skew symmetry: rotate by 180° around the origin.
- Always satisfies $f(0) = 0$ (put $x = 0$ in $f(-x) = -f(x)$).
- Positive area on $(0, a)$ cancels negative area on $(-a, 0)$.
- Examples: Figure shows $f(x) = x^3$.



For odd functions, every point $(x, f(x))$ has a partner $(-x, -f(x))$.

Integral Properties of Even and Odd Functions

Let f be integrable on $[-a, a]$.

Key facts

1. If f is even, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

2. If f is odd, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

- These facts can be seen from *signed areas* under the curve.
- But we might also prove them analytically (as in the text's Problem 3).

This is exactly the behavior of symmetric positive/negative contributions.

Analytic Proof: Even Case

Assume f is even: $f(-x) = f(x)$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

In the first term, substitute $x = -t$ (so $dx = -dt$):

$$\int_{-a}^{0} f(x)\,dx = \int_{t=a}^{t=0} f(-t)(-dt) = \int_{0}^{a} f(-t)\,dt.$$

Evenness: $f(-t) = f(t)$, so

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(t)\,dt.$$

Therefore

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

This matches equation (3) in the textbook.

Analytic Proof: Odd Case

Assume f is odd: $f(-x) = -f(x)$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-t)(-dt) = \int_{0}^{a} f(-t)\,dt.$$

Using oddness $f(-t) = -f(t)$,

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} -f(t)\,dt = -\int_{0}^{a} f(t)\,dt.$$

Hence

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx = 0.$$

This matches equation (4) in the textbook.

Parity of Products

Parity Rules for Products

Product symmetry

- (even)·(even) = even
- (even)·(odd) = odd
- (odd)·(odd) = even

- These match the sign rules $(+1)(+1) = +1$, $(+1)(-1) = -1$, $(-1)(-1) = +1$.
- Example: $x^3 \cos(nx)$ is odd because x^3 is odd, $\cos(nx)$ is even.
- Therefore
$$\int_{-\pi}^{\pi} x^3 \cos(nx)\, dx = 0,$$
immediately, by the odd-integral property.

This is a powerful shortcut: no integration by parts needed.

Proof Example: (even)·(odd) = odd

Let f be even and g be odd. Consider $F(x) = f(x)g(x)$.

$$\begin{aligned} F(-x) &= f(-x)g(-x) \\ &= f(x) \cdot [-g(x)] \quad (f \text{ even, } g \text{ odd}) \\ &= -f(x)g(x) \\ &= -F(x). \end{aligned}$$

Therefore F is odd. *The other two rules are proved similarly.*

Cosine and Sine Series via Symmetry

Fourier Series on $[-\pi, \pi]$

The general Fourier series of an integrable function f on $[-\pi, \pi]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

with

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

Now we will see how parity kills either the sine or cosine part.

Theorem: Even Functions $\Rightarrow$ Cosine Series Only

Theorem (first part)

Let f be integrable on $[-\pi, \pi]$ and even. Then its Fourier series has only cosine terms and the coefficients can be computed as

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) \, dx, \qquad b_n = 0.$$

We now prove carefully that $b_n = 0$ when f is even.

Proof that $b_n = 0$ for Even f

If f is even, then $f(x)\sin(nx)$ is odd (even·odd)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx. = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \int_{0}^{\pi} f(x)\sin(nx)\,dx\right]$$

In the first integral, substitute $x = -t$:

$$\begin{aligned}\int_{-\pi}^{0} f(x)\sin(nx)\,dx &= \int_{\pi}^{0} f(-t)\sin(-nt)(-dt)\\ &= \int_{0}^{\pi} f(-t)\big(-\sin(nt)\big)\,dt.\end{aligned}$$

Evenness: $f(-t) = f(t)$. Thus

$$\int_{-\pi}^{0} f(x)\sin(nx)\,dx = -\int_{0}^{\pi} f(t)\sin(nt)\,dt.$$

Adding the two halves:

$$b_n = \frac{1}{\pi}\left[-\int_{0}^{\pi} f(t)\sin(nt)\,dt + \int_{0}^{\pi} f(x)\sin(nx)\,dx\right] = 0.$$

So for even f the sine part vanishes completely.

Formula for a_n in the Even Case

Still assuming f is even:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx.$$

Here $f(x)\cos(nx)$ is even $\Rightarrow$ use the even integral property:

$$\begin{aligned} a_n &= \frac{1}{\pi}\cdot 2\int_0^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx. \end{aligned}$$

This matches the formula stated in the theorem.

Theorem: Odd Functions $\Rightarrow$ Sine Series Only

Theorem (second part)

Let f be integrable on $[-\pi, \pi]$ and odd. Then its Fourier series has only sine terms and the coefficients can be computed as

$$a_n = 0, \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx.$$

Now we prove that $a_n = 0$ for odd f.

Proof that $a_n = 0$ for Odd f

If f is odd, then $f(x)\cos(nx)$ is odd (odd·even).

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx.$$

Because the integrand is odd, by the odd-integral property,

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0,$$

so $a_n = 0$. *This gives a clean explanation of why all cosine terms disappear.*

Formula for b_n in the Odd Case

For odd f,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx.$$

Now $f(x) \sin(nx)$ is even (odd·odd). Therefore,

$$\begin{aligned} b_n &= \frac{1}{\pi} \cdot 2 \int_0^{\pi} f(x) \sin(nx) \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx, \end{aligned}$$

matching the theorem.

Example 1(a): $f(x) = x$ on $[-\pi, \pi]$

Example 1(a): $f(x) = x$ is Odd

Consider $f(x) = x$ on $[-\pi, \pi]$.

- ◇ $f(-x) = -x = -f(x)$, so f is odd.
- ◇ Its Fourier series is therefore a pure sine series.
- ◇ We do not need to compute any cosine coefficients: $a_n = 0$.

We now derive the sine series.

Computing the Sine Coefficients for $f(x) = x$

Since f is odd,

$$x \sim \sum_{n=1}^{\infty} b_n \sin(nx).$$

We have

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx.$$

Use integration by parts: let $u = x$, $dv = \sin(nx)\, dx$.Then $du = dx$ and $v = -\frac{1}{n}\cos(nx)$, so

$$\int_0^{\pi} x \sin(nx)\, dx = \left[-\frac{x}{n}\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos(nx)\, dx.$$

Finishing b_n for $f(x) = x$

Continue:

$$\int_0^\pi x\sin(nx)\,dx = -\frac{\pi}{n}\cos(n\pi) + 0 + \frac{1}{n}\left[\frac{\sin(nx)}{n}\right]_0^\pi$$
$$= -\frac{\pi}{n}(-1)^n + 0.$$

So

$$\int_0^\pi x\sin(nx)\,dx = -\frac{\pi}{n}(-1)^n.$$

Hence

$$b_n = \frac{2}{\pi}\cdot\left(-\frac{\pi}{n}(-1)^n\right) = -\frac{2(-1)^n}{n}.$$

Therefore

$$x = 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

with alternating signs; this matches the series form in the textbook.

Example 1(b): $f(x) = |x|$ on $[-\pi, \pi]$

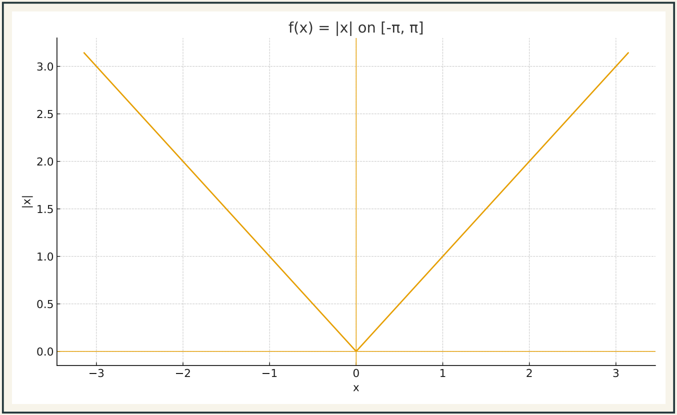
Example 1(b): $f(x) = |x|$ is Even

Consider $f(x) = |x|$ on $[-\pi, \pi]$.

- ◇ $f(-x) = |-x| = |x|$, so f is even.
- ◇ Hence its Fourier series is a pure cosine series.
- ◇ We do not need to compute any sine coefficients: $b_n = 0$.

This is Example 1(b) in the textbook.

Graph of $f(x) = |x|$ on $[-\pi, \pi]$



Symmetric V-shape: clear evenness.

Computing a_0 for $f(x) = |x|$

By evenness,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx.$$

Compute:

$$\int_0^{\pi} x\, dx = \left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}.$$

Thus

$$a_0 = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi.$$

So the constant term in the cosine series is $\dfrac{a_0}{2} = \dfrac{\pi}{2}$.

Computing a_n for $n \geq 1$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx)\, dx.$$

Integration by parts: let $u = x$, $dv = \cos(nx)\, dx$.Then $du = dx$, $v = \frac{1}{n} \sin(nx)$. So

$$\int_0^\pi x \cos(nx)\, dx = \left. \frac{x}{n} \sin(nx) \right|_0^\pi - \frac{1}{n} \int_0^\pi \sin(nx)\, dx.$$

The boundary term vanishes (since $\sin(n\pi) = 0$), so

$$\int_0^\pi x \cos(nx)\, dx = -\frac{1}{n} \int_0^\pi \sin(nx)\, dx.$$

Finishing a_n for $f(x) = |x|$

Continue:

$$\int_0^\pi \sin(nx)\,dx = \left[-\frac{\cos(nx)}{n}\right]_0^\pi = -\frac{\cos(n\pi) - \cos(0)}{n}$$

$$= -\frac{(-1)^n - 1}{n}.$$

Thus

$$\int_0^\pi x\cos(nx)\,dx = -\frac{1}{n}\left(-\frac{(-1)^n - 1}{n}\right) = \frac{(-1)^n - 1}{n^2}.$$

Hence

$$a_n = \frac{2}{\pi} \cdot \frac{(-1)^n - 1}{n^2}.$$

Notice:

$$(-1)^n - 1 = \begin{cases} 0, & n \text{ even}, \\ -2, & n \text{ odd}. \end{cases}$$

Simplifying a_n and Final Cosine Series

Therefore,

$$a_n = \begin{cases} 0, & n \text{ even}, \\ -\dfrac{4}{\pi n^2}, & n \text{ odd}. \end{cases}$$

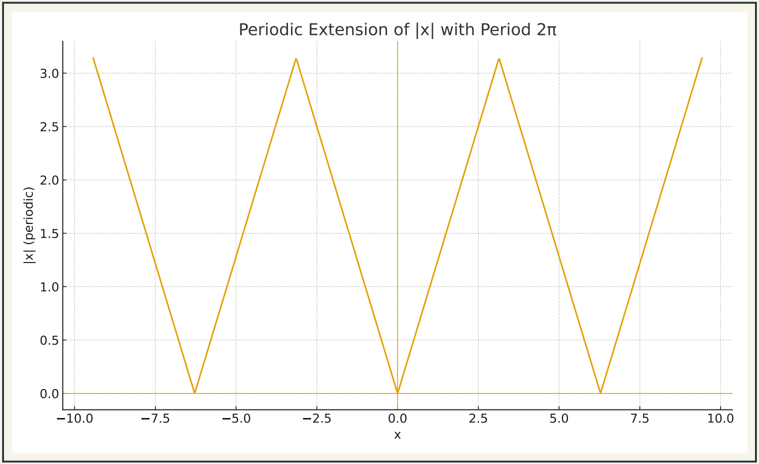
Writing $n = 2k + 1$ for odd indices,

$$a_{2k+1} = -\frac{4}{\pi(2k+1)^2}.$$

Thus the cosine series for $|x|$ is

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos\big((2k+1)x\big)}{(2k+1)^2}, \quad -\pi \leq x \leq \pi.$$

Periodic Extension of $|x|$



The series converges to this periodic extension, with period 2π.

Sine and Cosine Series on $[0, \pi]$

Two Different Series for the Same Function on $[0, \pi]$

On $[0, \pi]$ we have $|x| = x$ since $x \geq 0$.

- The series from Example 1(a) gives a sine expansion of x.
- The series from Example 1(b) gives a cosine expansion of x.

Both are valid on $0 < x < \pi$. *This motivates the terminology "Fourier sine series" and "Fourier cosine series".*

Sine and Cosine Series: General Construction on $[0, \pi]$

Let f be defined on $[0, \pi]$ and satisfy Dirichlet conditions.

- ◇ To obtain the sine series for f on $[0, \pi]$:
  1. Redefine $f(0)$ and $f(\pi)$ (if needed) so that $f(0) = f(\pi) = 0$.
  2. Extend f to $[-\pi, \pi]$ as an *odd* function:

$$f(x) = -f(-x), \quad -\pi \leq x < 0.$$

  3. Compute the Fourier series on $[-\pi, \pi]$. Only sine terms appear.
- ◇ To obtain the cosine series for f on $[0, \pi]$:
  1. Extend f to $[-\pi, \pi]$ as an *even* function:

$$f(x) = f(-x), \quad -\pi \leq x < 0.$$

  2. Compute the Fourier series; only cosine terms appear.

Example 2: $f(x) = \cos x$ on $[0, \pi]$

Sine Series for $\cos x$ on $[0, \pi]$

We want the **sine series** of the function

$$f(x) = \cos x \qquad \text{on } [0, \pi].$$

For a sine series, the coefficients are

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\, dx.$$

Thus we must compute

$$b_n = \frac{2}{\pi} \int_0^\pi \cos x \, \sin(nx)\, dx.$$

This integral requires a trigonometric identity to simplify the product.

Step 1: Use a Trigonometric Identity

Use the product-to-sum identity:

$$\cos x \, \sin(nx) = \tfrac{1}{2}\big[\sin((n+1)x) - \sin((n-1)x)\big].$$

Thus,

$$b_n = \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi} [\sin((n+1)x) \; - \; \sin((n-1)x)] \, dx.$$

Now compute the two integrals separately.

Step 2: Evaluate the General Integral

For any integer $k \neq 0$,

$$\int_0^\pi \sin(kx)\,dx = \left[-\frac{\cos(kx)}{k}\right]_0^\pi.$$

This gives

$$\int_0^\pi \sin(kx)\,dx = \frac{1-\cos(k\pi)}{k} = \frac{1-(-1)^k}{k}.$$

We now apply this to $k = n+1$ and $k = n-1$.

Step 3: Substitute $k = n + 1$ and $k = n - 1$

Using the previous result:

$$\int_0^\pi \sin((n+1)x)\,dx = \frac{1-(-1)^{n+1}}{n+1},$$

$$\int_0^\pi \sin((n-1)x)\,dx = \frac{1-(-1)^{n-1}}{n-1}.$$

Thus

$$b_n = \frac{1}{\pi}\left[\frac{1-(-1)^{n+1}}{n+1} - \frac{1-(-1)^{n-1}}{n-1}\right].$$

Next we simplify the parity expressions.

Step 4: Simplify the Signs

Recall:

$$(-1)^{n+1} = -(-1)^n, \qquad (-1)^{n-1} = -(-1)^n.$$

Therefore:

$$1 - (-1)^{n+1} = 1 + (-1)^n, \qquad 1 - (-1)^{n-1} = 1 + (-1)^n.$$

Factor out the common term:

$$b_n = \frac{1 + (-1)^n}{\pi} \left(\frac{1}{n+1} - \frac{1}{n-1} \right).$$

Now simplify the bracket.

Step 5: Final Simplification

Compute

$$\frac{1}{n+1} - \frac{1}{n-1} = \frac{n-1-(n+1)}{(n+1)(n-1)} = \frac{-2}{n^2-1}.$$

Thus

$$b_n = \frac{1+(-1)^n}{\pi}\left(\frac{-2}{n^2-1}\right) = \frac{2n}{\pi(n^2-1)}\left[(-1)^n+1\right].$$

Interpretation: - If n is odd: $(-1)^n + 1 = 0$, so $b_n = 0$. - If n is even: coefficient is nonzero.

This exactly reproduces the book's conclusion, but with full explicit proof.

Cosine Series for $\cos x$ on $[0, \pi]$

To compute a **cosine series**, we even-extend the function to $[-\pi, \pi]$.

But $\cos x$ is already an even function:

$$\cos(-x) = \cos x.$$

Therefore: - All sine terms automatically vanish: $b_n = 0$. - The cosine coefficients reproduce the function exactly.

Thus

$$\cos x = \cos x$$

is already its own Fourier cosine series.

Decomposition into Even and Odd Parts

Any Function = Even Part + Odd Part

Show (Problem 2) that any f on a symmetric interval can be written as

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x),$$

where

$$f_{\text{even}}(x) = \frac{1}{2}\big[f(x) + f(-x)\big], \quad f_{\text{odd}}(x) = \frac{1}{2}\big[f(x) - f(-x)\big].$$

- f_{even} is even, f_{odd} is odd.
- This is a decomposition of f into symmetric and antisymmetric parts.
- Fourier series naturally split into cosine (even) and sine (odd) contributions.

Conceptually: the cosine series approximates the even part, the sine series the odd part.

Summary and Takeaways

Big Picture Summary

- $\diamond$ Symmetry on $[-\pi, \pi]$ is a powerful tool in Fourier analysis.
- $\diamond$ Even functions $\Rightarrow$ cosine series only ($b_n = 0$).
- $\diamond$ Odd functions $\Rightarrow$ sine series only ($a_n = 0$).
- $\diamond$ Product parity rules let you zero out many integrals instantly.
- $\diamond$ Examples $f(x) = x$ and $f(x) = |x|$ illustrate pure sine/pure cosine behavior.
- $\diamond$ On $[0, \pi]$, sine and cosine series correspond to odd and even extensions.
- $\diamond$ Any function can be decomposed into even and odd parts, matching cosine and sine components of its Fourier series.

These notes follow and expand Section 35 of Differential Equations with Applications and Historical Notes.

Fourier Series on Arbitrary Intervals

From $[-\pi, \pi)$ to $[-L, L)$

Section 36 notes by Deepak Bhoriya

November 14, 2025

Outline

From $[-L, L)$ to $[-\pi, \pi)$

Final Form of the Series

Example: A Shifted Square Wave

Takeaways

Why Fourier Series on $[-L, L)$?

- In theory, Fourier series are introduced on the canonical interval $[-\pi, \pi)$.
- In applications, functions naturally live on problem-specific intervals:
  - vibrating strings of length $2L$,
  - heat conduction on a bar $[0, L]$,
  - signal windows of duration $2L$, etc.
- We want to keep all the orthogonality magic of sines and cosines *without* being stuck to $[-\pi, \pi)$.
- The solution: a **change of scale** on the horizontal axis.

Think of zooming/stretching the x-axis so that $[-L, L)$ becomes $[-\pi, \pi)$.

Connections to Differential Equations

Where this really matters

- ◇ Solving the heat equation on $[-L, L]$ via separation of variables.
- ◇ Solving the wave equation on a finite string of length $2L$.
- ◇ Solving Poisson/Laplace equations with boundary data on $[-L, L]$.

- ◇ The eigenfunctions of these boundary-value problems involve $\sin(\frac{n\pi x}{L})$ and $\cos(\frac{n\pi x}{L})$.
- ◇ So we *need* Fourier expansions adapted to $[-L, L)$.

The scaling parameter L carries the geometry of the physical problem.

From $[-L, L)$ to $[-\pi, \pi)$

The Scaling Idea

Suppose f is defined on $[-L, L)$, with $L > 0$.

- ◇ We want to build a Fourier series for f using familiar formulas.
- ◇ Introduce a new variable t so that

$$-L \leq x < L \quad \Longleftrightarrow \quad -\pi \leq t < \pi.$$

- ◇ The natural choice is

$$t = \frac{\pi x}{L} \quad \Longleftrightarrow \quad x = \frac{Lt}{\pi}.$$

Linear rescaling: x scaled by $\frac{\pi}{L}$ to fit into $[-\pi, \pi)$.

Defining the Rescaled Function

Step 1: Transfer f to the t-world

$$g(t) := f\left(\frac{Lt}{\pi}\right), \qquad -\pi \le t < \pi.$$

- ◇ If f satisfies the Dirichlet conditions on $[-L, L)$, then g satisfies the Dirichlet conditions on $[-\pi, \pi)$.
- ◇ Therefore we may expand g in an ordinary Fourier series:

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right).$$

Conceptually: we solve the problem in the t-variable, then return to x.

Fourier Coefficients for $g(t)$

The coefficients for g are the usual ones:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt.$$

Substitute $g(t) = f\left(\frac{Lt}{\pi}\right)$.

Change of variable $x = \dfrac{Lt}{\pi}$

- Then $t = \dfrac{\pi x}{L}$ and $dt = \dfrac{\pi}{L}\,dx$.
- The integral limits $t = -\pi$ and $t = \pi$ correspond to $x = -L$ and $x = L$.

Now we rewrite everything back in terms of the original variable x.

Coefficients in Terms of $f(x)$

Using $x = \dfrac{Lt}{\pi}$, we get:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) \cos(nt)\, dt \\ &= \frac{1}{\pi} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \frac{\pi}{L}\, dx \\ &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx. \end{aligned}$$

Similarly,

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

The orthogonality basis has simply changed frequency from n to $n\pi/L$.

Final Form of the Series

Fourier Series on $[-L, L)$

Putting everything together, we have:

Fourier expansion of f on $[-L, L)$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

This is the standard formula you will see in PDE and Fourier analysis courses.

Orthogonality on $[-L, L]$

The functions

$$\cos\left(\frac{n\pi x}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)$$

inherit orthogonality from the $[-\pi, \pi)$ basis.

Key orthogonality relations

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \neq 0, \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}$$

Exactly as on $[-\pi, \pi]$, but with L playing the role of π.

Example: A Shifted Square Wave

Example Setup: A Simple Step Function

Consider $L = 2$ and define

$$f(x) = \begin{cases} 0, & -2 \leq x < 0, \\ 1, & 0 \leq x < 2. \end{cases}$$

- ◇ This describes a half-on, half-off signal on $[-2, 2)$.
- ◇ We extend f periodically with period 4.
- ◇ Goal: Find its Fourier series on $[-2, 2)$ using our general formulas.

This is the standard "square wave" shifted to the interval $[-2, 2)$.

Rescaling to $g(t)$

For $L = 2$ we use

$$t = \frac{\pi x}{2}, \qquad x = \frac{2t}{\pi}.$$

Then

$$g(t) = f\left(\frac{2t}{\pi}\right) = \begin{cases} 0, & -\pi \le t < 0, \\ 1, & 0 \le t < \pi. \end{cases}$$

- g is a classic square wave on $[-\pi, \pi)$.
- We already know how to expand such functions in Fourier series.

All the hard work is shifted to a familiar template.

Fourier Coefficients for $g(t)$

Compute a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\,dt = \frac{1}{\pi}\int_{0}^{\pi} 1\,dt = 1.$$

For $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{0}^{\pi} \cos(nt)\,dt = 0.$$

For b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{0}^{\pi} \sin(nt)\,dt \\ &= \frac{1}{\pi}\left[-\frac{\cos(nt)}{n}\right]_0^{\pi} = \frac{1}{\pi n}(1-\cos(n\pi)) \\ &= \frac{2}{\pi n}(1-(-1)^n). \end{aligned}$$

Thus $b_n = 0$ for even n and $b_n = \dfrac{4}{\pi n}$ for odd n.

Series for $g(t)$ and Then $f(x)$

Since only odd n contribute, write $n = 2k + 1$:

$$g(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\big((2k+1)t\big).$$

Recall $t = \frac{\pi x}{2}$, so

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left(\frac{(2k+1)\pi x}{2}\right), \qquad -2 < x < 2.$$

This series converges to the periodic extension (period 4) of f.

Takeaways

Key Takeaways

- The interval $[-\pi, \pi)$ is a *convenient model*, not a restriction.
- A simple linear rescaling $t = \dfrac{\pi x}{L}$ moves us between $[-L, L)$ and $[-\pi, \pi)$.
- Orthogonality of sines and cosines survives under this rescaling.
- The resulting Fourier series on $[-L, L)$ is tailor-made for PDE problems on finite intervals.
- Examples like the step function illustrate how the general formulas work in practice.

Once you are comfortable with the scaling idea, any interval becomes "standard".

Thank you!

Questions or discussion?