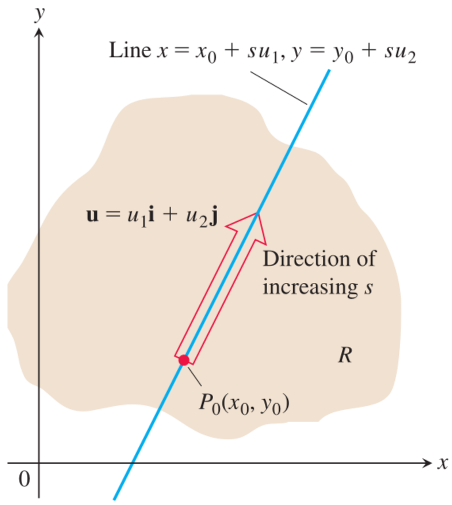
- How to find the derivative in a certain direction? ((1,0) or (0,1))



DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \tag{1}$$

provided the limit exists.

DEFINITION The **gradient vector** (or **gradient**) of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Taylor's Formula for $f(x, y)$ at the Origin

$$\begin{aligned} f(x,y) = {} & f(0,0) + xf_x + yf_y + \frac{1}{2!}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right) \\ & + \frac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y\frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right) \\ & + \frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,cy)} \end{aligned}$$

Those numbers (**5, 10, etc.**) are **binomial coefficients**, coming from the expansion of $(x+y)^n$.

For example:

- At **4th order** (degree 4): coefficients are from
  $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
  That's why we get **1, 4, 6, 4, 1**.
- At **5th order** (degree 5): coefficients are from
  $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.
  That's why we get **1, 5, 10, 10, 5, 1**.

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line

$$x = x_0 + su_1, \quad y = y_0 + su_2, \tag{2}$$

through $P_0(x_0, y_0)$, parametrized with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then, by the Chain Rule we find

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} &= \left.\frac{\partial f}{\partial x}\right|_{P_0} \frac{dx}{ds} + \left.\frac{\partial f}{\partial y}\right|_{P_0} \frac{dy}{ds} && \text{Chain Rule for differentiable } f \\
&= \left.\frac{\partial f}{\partial x}\right|_{P_0} u_1 + \left.\frac{\partial f}{\partial y}\right|_{P_0} u_2 && \text{From Eqs. (2), } dx/ds = u_1 \text{ and } dy/ds = u_2 \\
&= \underbrace{\left[\left.\frac{\partial f}{\partial x}\right|_{P_0} \mathbf{i} + \left.\frac{\partial f}{\partial y}\right|_{P_0} \mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}. && (3)
\end{aligned}$$

Equation (3) says that the derivative of a differentiable function f in the direction of $\mathbf{u}$ at P_0 is the dot product of $\mathbf{u}$ with a special vector, which we now define.

THEOREM 9—The Directional Derivative Is a Dot Product

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \nabla f|_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 with the vector $\mathbf{u}$. In brief, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution Recall that the direction of a vector $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = \left(e^y - y\sin(xy)\right)\Big|_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = \left(xe^y - x\sin(xy)\right)\Big|_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.29). The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$\begin{aligned}
D_{\mathbf{u}}f|_{(2,0)} &= \nabla f|_{(2,0)} \cdot \mathbf{u} && \text{Eq. (4) with the } D_{\mathbf{u}}f|_{P_0} \text{ notation} \\
&= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.
\end{aligned}$$

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\end{aligned}$$

Evaluating the dot product in the brief version of Equation (4) gives

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f, and reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Q3. Find the maximum rate of increase and decrease for the given function

$$f(x, y, z) = xy^2 - yz^2 + zx^2$$

at $P(1, -1, -1)$. Is there a direction $\mathbf{u}$ in which the rate of change of this function at P is -4? Justify.

Solution. We have

$$\nabla f(x, y, z) = \left(y^2 + 2zx\right)\mathbf{i} + \left(2xy - z^2\right)\mathbf{j} + \left(-2yz + x^2\right)\mathbf{k}.$$

At the point $(1, -1, -1)$,

$$\nabla f(1, -1, -1) = (-1)\mathbf{i} - 3\mathbf{j} - 1\mathbf{k}.$$

The magnitude of the gradient is

$$|\nabla f(1, -1, -1)| = \sqrt{(-1)^2 + (-3)^2 + (-1)^2} = \sqrt{1 + 9 + 1} = \sqrt{11}.$$

• The maximum rate of increase of f at $(1, -1, -1)$ is $|\nabla f| = \sqrt{11} = 3.32$,
• and the maximum rate of decrease is $-|\nabla f| = -\sqrt{11} \approx -3.32$.
Since $-4 < -\sqrt{11}$, it is not possible to have a directional derivative equal to -4.

$$\boxed{\text{Therefore, there is no direction in which the rate of change of } f \text{ is } -4.}$$