Tutorial sheet 3

Smoothness, Angle in a Space Curve, etc

(dbhoriya.github.io/teaching_F101/tutorial_3.pdf) or SCAN:

Key definitions:

- **Smooth Curve:** A curve $\mathbf{r}(t) = (f(t), g(t), h(t))$ is *smooth* on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ throughout.
- **Velocity (Tangent Vector):** $\mathbf{v}(t) = \mathbf{r}'(t)$, represents the tangent direction and speed.
- **Acceleration:** $\mathbf{a}(t) = \mathbf{r}''(t)$, derivative of velocity.
- **Dot Product:** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta.$$

- **Angle Between Vectors:**

$$\theta = \arccos \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Question 1: Consider the space curve:

$$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + (t^2 + \cos t)\mathbf{j} + (t^3 - \pi t^2)\mathbf{k}, \quad t \neq 0,$$

with

$$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.$$

(i) Find all points t at which $\mathbf{r}(t)$ is **smooth**.

(ii) Find the **angle** between the **tangent** and **acceleration** vectors at $t = \pi$.

Question-1

Answer: Given:

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + (t^2 + \cos t)\mathbf{j} + (t^3 - \pi t^2)\mathbf{k}, \quad t \neq 0.$

- Velocity: $\mathbf{r}'(t) = (\cos t,\ 2t - \sin t,\ 3t^2 - 2\pi t)$.
- For smoothness:
  - $\mathbf{r}(t)$ must be continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.
  - At $t = 0$, $\lim_{t\to 0} \mathbf{r}(t) = (1, 1, 0) \neq \mathbf{r}(0) = (0, 0, 0) \Rightarrow$ **not smooth**.
  - For $t \neq 0$, $\mathbf{r}'(t) = \mathbf{0}$ would require:

$$\cos t = 0,\ 2t - \sin t = 0,\ 3t^2 - 2\pi t = 0.$$

    No common $t \neq 0$ satisfies all $\Rightarrow \mathbf{r}'(t) \neq \mathbf{0}$.
- **Conclusion:** Curve is smooth for all $t \neq 0$.

Question-1

- Velocity (tangent) at $t = \pi$:

$$\mathbf{r}'(\pi) = (-1, 2\pi, \pi^2).$$

- Acceleration at $t = \pi$:

$$\mathbf{r}''(t) = (-\sin t, 2 - \cos t, 6t - 2\pi) \Rightarrow \mathbf{r}''(\pi) = (0, 3, 4\pi).$$

- Dot product:

$$\mathbf{r}'(\pi) \cdot \mathbf{r}''(\pi) = 6\pi + 4\pi^3.$$

- Magnitudes:

$$|\mathbf{r}'(\pi)| = \sqrt{1 + 4\pi^2 + \pi^4}, \quad |\mathbf{r}''(\pi)| = \sqrt{9 + 16\pi^2}.$$

- Angle:

$$\theta = \arccos \frac{6\pi + 4\pi^3}{\sqrt{1 + 4\pi^2 + \pi^4}\sqrt{9 + 16\pi^2}}.$$

- Numerically, $\theta \approx 20^\circ$.

Question 2: Solve the initial value problem:

$$\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

with initial conditions:

$$\mathbf{r}(0) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \qquad \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{0},$$

where $a, b, c \in \mathbb{R}$ are constants.

Question-2: Solution Approach

- Start with the second-order ODE:

$$\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

- Integrate once to find velocity:

$$\frac{d\mathbf{r}}{dt} = -t(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}_1$$

- Apply initial condition $\frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$.

- Integrate velocity to find position:

$$\mathbf{r}(t) = -\frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}_2$$

- Apply initial condition
  $\mathbf{r}(0) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \Rightarrow \mathbf{C}_2 = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- Final solution:

$$\mathbf{r}(t) = \left(a - \frac{t^2}{2}\right)\mathbf{i} + \left(b - \frac{t^2}{2}\right)\mathbf{j} + \left(c - \frac{t^2}{2}\right)\mathbf{k}.$$

Motivation: Finding the Direction

Step 1: Intuition

- Imagine standing at a starting point A. To reach point B, you simply **look towards** B **and move in that line**.
- Direction is nothing mysterious – it is just the **arrow from** A **to** B.
- Everyday example: walking from home to school or flying a drone to a target $\rightarrow$ you follow the straight arrow joining the two points.

Step 2: Mathematics

- In 3D: moving from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ $\rightarrow$

$$\overrightarrow{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

How Speed Fixes the Velocity Vector

General Idea: Velocity = Direction + Speed.

- Let the **direction vector** be $\mathbf{d} = (d_x, d_y, d_z)$.
- Its magnitude is

$$|\mathbf{d}| = \sqrt{d_x^2 + d_y^2 + d_z^2}.$$

- The **unit vector** in this direction is

$$\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|}.$$

- If the given speed is v_0, then the velocity vector must be

$$\mathbf{v}(0) = v_0 \cdot \hat{\mathbf{d}}.$$

Reason: Direction fixes the orientation, speed fixes the length. Together, they uniquely determine the vector.

Question-3

Question 3: A particle travelling in a straight line is located at the point $(1, -1, 2)$ and has speed 2 at time $t = 0$. The particle moves towards the point $(3, 0, 3)$ with a constant acceleration of $2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find its position vector $\mathbf{r}(t)$ at time t.

Answer: Given initial point $\mathbf{r}(0) = (1, -1, 2)$, speed $|\mathbf{v}(0)| = 2$ at $t = 0$, target $B = (3, 0, 3)$, constant acceleration $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Question-3

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Answer: Given initial point $\mathbf{r}(0) = (1, -1, 2)$, speed $|\mathbf{v}(0)| = 2$ at $t = 0$, target $B = (3, 0, 3)$, constant acceleration $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

- **Find $\mathbf{v}(t)$ by integrating acceleration:**

$$\mathbf{v}(t) = \int \mathbf{a}\, dt = (2t + C_1,\ t + C_2,\ t + C_3).$$

- **Direction (arrow from A to B):**

$$\mathbf{d} = \overrightarrow{AB} = (3 - 1,\ 0 - (-1),\ 3 - 2) = (2, 1, 1).$$

- **Unit vector in direction $\mathbf{d}$:**

$$|\mathbf{d}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}, \qquad \hat{\mathbf{d}} = \frac{\mathbf{d}}{\sqrt{6}} = \frac{1}{\sqrt{6}}(2, 1, 1).$$

Question-3

- **Initial velocity vector:** the speed 2 along $\hat{\mathbf{d}}$ gives

$$\mathbf{v}(0) = 2\,\hat{\mathbf{d}} = \frac{2}{\sqrt{6}}(2,1,1) = \Big(\frac{4}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\Big).$$

- **Velocity at time** t (constant acceleration):

$$\mathbf{v}(t) = \Big(2t + \frac{4}{\sqrt{6}},\ t + \frac{2}{\sqrt{6}},\ t + \frac{2}{\sqrt{6}}\Big).$$

- **Position at time** t (integrate once; use $\mathbf{r}(0)$ as constant):

$$\mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\mathbf{i} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t\right)\mathbf{j} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t\right)\mathbf{k} + \mathbf{D}.$$

- Given that $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, we get $\mathbf{D} = \mathbf{r}(0)$.

$$\boxed{\mathbf{r}(t) = \Big(t^2 + \frac{4}{\sqrt{6}}t + 1\Big)\mathbf{i} + \Big(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t - 1\Big)\mathbf{j} + \Big(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t + 2\Big)\mathbf{k}}$$

Question-4

Question 4: Find the arc lengths of the following curves:

$$\text{(i)} \quad \mathbf{r}(t) = \cos(3t)\,\mathbf{i} + \sin(3t)\,\mathbf{j} + 6t\,\mathbf{k}, \qquad 0 \le t \le \frac{2\pi}{3},$$

$$\text{(ii)} \quad \mathbf{r}(t) = \cos\left(\frac{t}{5}\right)\mathbf{i} + \sin\left(\frac{t}{5}\right)\mathbf{j} + \frac{2t}{5}\,\mathbf{k}, \qquad 0 \le t \le 10\pi,$$

$$\text{(iii)} \quad \mathbf{r}(t) = \cos\left(\frac{2t}{3}\right)\mathbf{i} - \sin\left(\frac{2t}{3}\right)\mathbf{j} - \frac{4t}{3}\,\mathbf{k}, \quad -3\pi \le t \le 0.$$

Is the answer the same for each curve? If so, why?

- Recall: Arc Length in $\mathbb{R}^3$. For a C^1 curve $\mathbf{r}(t)$, $t \in [a, b]$,

$$L = \int_a^b \|\mathbf{r}'(t)\|\,dt.$$

  - Compute the speed $\|\mathbf{r}'(t)\|$.
  - If the speed is constant, then $L = (\text{speed}) \times (b - a)$.

(i) $\mathbf{r}(t) = (\cos 3t,\ \sin 3t,\ 6t)$

- $\mathbf{r}'(t) = (-3\sin 3t,\ 3\cos 3t,\ 6)$.
- Speed: $\|\mathbf{r}'(t)\| = \sqrt{9(\sin^2 3t + \cos^2 3t) + 36} = 3\sqrt{5}$ (constant).
- Interval length: $\Delta t = \frac{2\pi}{3}$.

$$\boxed{L_1 = 3\sqrt{5} \cdot \frac{2\pi}{3} = 2\pi\sqrt{5}}$$

(ii) $\mathbf{r}(t) = \left(\cos\frac{t}{5},\ \sin\frac{t}{5},\ \frac{2t}{5} \right)$

- $\mathbf{r}'(t) = \left(-\frac{1}{5}\sin\frac{t}{5},\ \frac{1}{5}\cos\frac{t}{5},\ \frac{2}{5} \right)$.
- Speed: $\|\mathbf{r}'(t)\| = \sqrt{\frac{1}{25}(\sin^2 + \cos^2) + \frac{4}{25}} = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5}$ (constant).
- Interval length: $\Delta t = 10\pi$.

$$\boxed{L_2 = \frac{\sqrt{5}}{5} \cdot 10\pi = 2\pi\sqrt{5}}$$

(iii) $\mathbf{r}(t) = \left(\cos\frac{2t}{3},\ -\sin\frac{2t}{3},\ -\frac{4t}{3}\right)$

- $\mathbf{r}'(t) = \left(-\frac{2}{3}\sin\frac{2t}{3},\ -\frac{2}{3}\cos\frac{2t}{3},\ -\frac{4}{3}\right)$.
- Speed: $\|\mathbf{r}'(t)\| = \sqrt{\frac{4}{9}(\sin^2+\cos^2)+\frac{16}{9}} = \frac{2}{3}\sqrt{5}$ (constant).
- Interval length: $\Delta t = 3\pi$.

$$\boxed{L_3 = \frac{2}{3}\sqrt{5}\cdot 3\pi = 2\pi\sqrt{5}}$$

Why are all three lengths equal?

- Write θ for the planar angle: (i) $\theta = 3t$, (ii) $\theta = \frac{t}{5}$, (iii) $\theta = \frac{2t}{3}$.
- In each case the z–coordinate satisfies $z = 2\theta$ (up to orientation), so each curve traces **the same geometric helix**:

$$x = \cos\theta, \quad y = \sin\theta, \quad z = 2\theta, \qquad 0 \le \theta \le 2\pi.$$

- The arc length over one full turn of this helix is

$$L = \int_0^{2\pi} \sqrt{1^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{1 + 2^2}\, d\theta = 2\pi\sqrt{5}.$$

- The three parametrizations only change the *speed*/direction (reparameterizations), not the traced path.

Curvature and Normal Vectors of a Curve

- **Arc Length:** $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ on $[a, b]$.

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |v(t)| dt.$$

- **Unit tangent vector:** Unit tangent vectors are unit vectors (vectors with length of 1) that are tangent to the curve at certain points.

$$\mathbf{T} = \frac{d\mathbf{r}}{dt}, \qquad \hat{\mathbf{T}} = \frac{\mathbf{T}}{|\mathbf{T}|} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}.$$

- **Normal Vector of a Curve:** A unit normal vector of a curve, by its definition, is perpendicular to the curve at given point.

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt}$$

- **Unit normal vector:** $\hat{\mathbf{N}} = \dfrac{d\hat{\mathbf{T}}/dt}{\left|d\hat{\mathbf{T}}/dt\right|}.$

- **Curvature:** Curvature is a measure of how much the curve deviates from a straight line. (Will be done in Lecture class)

$$\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}.$$

- **Equation of circle of radius R:** $x^2 + y^2 = R^2$

$$\mathbf{r}(t) = R\cos t\,\hat{i} + R\sin t\,\hat{j}$$

Derivatives:

$$\mathbf{r}'(t) = -R\sin t\,\hat{i} + R\cos t\,\hat{j}, \quad \|\mathbf{r}'(t)\| = R$$

$$\mathbf{r}''(t) = -R\cos t\,\hat{i} - R\sin t\,\hat{j}$$

Curvature:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{R^2}{R^3} = \frac{1}{R}$$

$$\boxed{\text{Curvature of circle} = \tfrac{1}{R} \text{ (constant)}}$$

Curvature of a Plane Curve

- **Curvature of a Plane Curve**: If a curve resides only in the $xy-$plane and is defined by the function $y = f(x)$.

$$\mathbf{r}(t) = t\hat{i} + f(t)\hat{j}$$

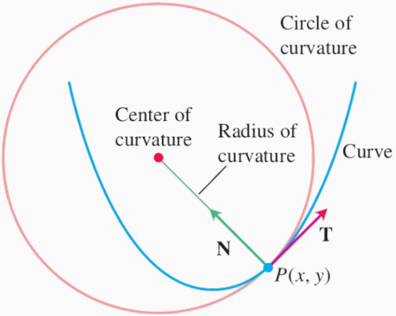
Derivatives: $\mathbf{r}'(t) = \hat{i} + f'(t)\hat{j}, \qquad \mathbf{r}''(t) = f''(t)\hat{j}$
Cross product:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = f''(t)\hat{k}, \quad \|\mathbf{r}' \times \mathbf{r}''\| = |f''(t)|$$

Curvature formula:

$$\kappa(t) = \frac{|f''(t)|}{\left(1 + (f'(t))^2\right)^{3/2}}$$

Circle of Curvature/osculating circle



The **circle of curvature** or **osculating circle** at a point P on a plane curve, where $\kappa \neq 0$ is the circle in the plane of the curve that

- is tangent to the curve at P (has the same tangent line the curve has)
- has the same curvature the curve has at P
- has center that lies toward the concave or inner side of the curve

Circle of Curvature/osculating circle

- **Radius of curvature**: The radius of curvature of the curve at P is the radius of the circle of curvature

$$\text{Radius of curvature} = \rho(t) = R(t) = \frac{1}{\kappa}.$$

- **Center of the osculating circle**: The center of the osculating circle at the point t_0 is given by

$$\mathbf{C} = \mathbf{r}(t_0) + R(t_0)\mathbf{N}(t_0)$$

Question-5: Find the curvature of the curve $y = e^x$ at any point. Also, determine the point where the curvature is maximum.

Step 1: Formula $\kappa(x) = \frac{|y''|}{(1+(y')^2)^{3/2}}$

Step 2: Derivatives for $y = e^x : y' = e^x, \quad y'' = e^x$

Step 3: Substitution $\kappa(x) = \frac{e^x}{(1+e^{2x})^{3/2}}$

Step 4: Maximum curvature Differentiate:

$$\kappa'(x) = \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{5/2}} = 0 \qquad \Longrightarrow \qquad x = -\frac{1}{2}\ln 2$$

Step 5: Point of maximum curvature: $y = e^{-\frac{1}{2}\ln 2} = \frac{1}{\sqrt{2}}$. So the point is $\left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$

Step 6: As $x \to \infty$, $\kappa(x) \to 0$

Question 6

Question 6: For the curve $r(t) = t\mathbf{i} + t^2\mathbf{j}$, find the coordinates of the center of the osculating circle at the point t_0.

Step 1: Derivatives.

$$r'(t) = \mathbf{i} + 2t\mathbf{j}, \quad r''(t) = 2\mathbf{j}.$$

Step 2: Curvature.

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{2}{(1+4t^2)^{3/2}}.$$

Step 3: Radius of curvature.

$$R(t) = \frac{1}{\kappa(t)} = \frac{(1+4t^2)^{3/2}}{2}.$$

Step 4: Unit tangent vector.

$$\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{1+4t^2}}(\mathbf{i} + 2t\mathbf{j}).$$

Step 5: Derivative of $\mathbf{T}(t)$.

$$\frac{d\mathbf{T}(t)}{dt} = \frac{-4t\mathbf{i} + 2\mathbf{j}}{(1+4t^2)^{3/2}}, \quad \left|\frac{d\mathbf{T}(t)}{dt}\right| = \frac{2}{1+4t^2}.$$

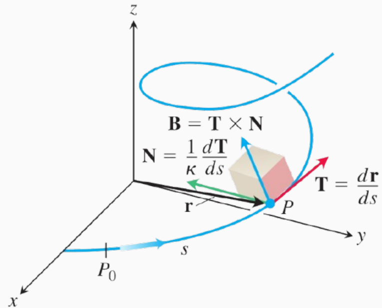
Step 6: Unit normal vector.

$$\mathbf{N}(t) = \frac{\frac{d\mathbf{T}(t)}{dt}}{\left|\frac{d\mathbf{T}(t)}{dt}\right|} = \frac{1}{\sqrt{1+4t^2}}(-2t\mathbf{i} + \mathbf{j}).$$

Step 7: Osculating circle center.

$$\mathbf{C} = r(t_0) + R(t_0)\mathbf{N}(t_0) = -4t_0^3\mathbf{i} + \left(3t_0^2 + \tfrac{1}{2}\right)\mathbf{j}.$$

Binormal vector and torsion



Binormal vector: The binormal vector of a curve in space is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, which is a unit vector that is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

Torsion: Tells how much a curve twists out of the plane:

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \quad (\text{ if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0})$$

Question 7

Find T, N, B, κ and τ for

$$r(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} - 4\mathbf{k}, \quad t > 0$$

Further, find the equations of osculating, normal and rectifying planes at $t_0 > 0$.

Solution:

$$r'(t) = (t\sin t)\mathbf{i} + (t\cos t)\mathbf{j}$$

Thus,

$$\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{(t\sin t)\mathbf{i} + (t\cos t)\mathbf{j}}{t} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}, \quad t \neq 0$$

Normal Vector:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\cos t\,\mathbf{i} - \sin t\,\mathbf{j}}{\sqrt{\cos^2 t + \sin^2 t}} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

Question 7 (contd.)

Binormal Vector:

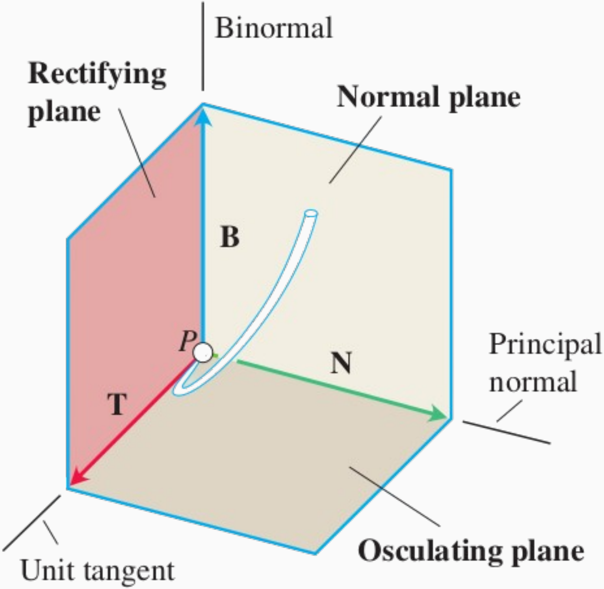
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\mathbf{B}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ \cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k}(-\sin^2 t - \cos^2 t) = -\mathbf{k}$$

Curvature:

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|r'(t)|} = \frac{1}{t}$$

Torsion: Since third (or z) component of $\mathbf{r}(t)$ is constant, we conclude that $\tau(t) = 0$.


Binormal
Rectifying plane
Normal plane
B
P
N
T
Principal normal
Unit tangent
Osculating plane

Question 7 (contd.): Planes Associated with a Space Curve

Given: A space curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ with unit tangent $\mathbf{T}(t)$, unit normal $\mathbf{N}(t)$, and binormal $\mathbf{B}(t)$. At a point $\mathbf{r}(t_0) = (x_0, y_0, z_0)$:

- **Equation of a plane:** For normal vector $\mathbf{n} = (a, b, c)$,

  $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0 \iff \mathbf{n}\cdot(\mathbf{r}(t)-\mathbf{r}(t_0)) = 0$

- **Osculating plane:** Normal $\mathbf{B}(t_0)$

$$\mathbf{B}(t_0) \cdot (\mathbf{r} - \mathbf{r}(t_0)) = 0$$

- **Normal plane:** Normal $\mathbf{T}(t_0)$

$$\mathbf{T}(t_0) \cdot (\mathbf{r} - \mathbf{r}(t_0)) = 0$$

- **Rectifying plane:** Normal $\mathbf{N}(t_0)$

$$\mathbf{N}(t_0) \cdot (\mathbf{r} - \mathbf{r}(t_0)) = 0$$