Tutorial sheet 3

Smoothness, Angle in a Space Curve, etc

(dbhoriya.github.io/teaching_F101/tutorial_3.pdf) or SCAN:

Key definitions:

- **Smooth Curve:** A curve $\mathbf{r}(t) = (f(t), g(t), h(t))$ is *smooth* on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ throughout.
- **Velocity (Tangent Vector):** $\mathbf{v}(t) = \mathbf{r}'(t)$, represents the tangent direction and speed.
- **Acceleration:** $\mathbf{a}(t) = \mathbf{r}''(t)$, derivative of velocity.
- **Dot Product:** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta.$$

- **Angle Between Vectors:**

$$\theta = \arccos \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Question-1

Question 1: Consider the space curve:

$$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + (t^2 + \cos t)\mathbf{j} + (t^3 - \pi t^2)\mathbf{k}, \quad t \neq 0,$$

with

$$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.$$

(i) Find all points t at which $\mathbf{r}(t)$ is **smooth**.

(ii) Find the **angle** between the **tangent** and **acceleration** vectors at $t = \pi$.

Question-1

Answer: Given:

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + (t^2 + \cos t)\mathbf{j} + (t^3 - \pi t^2)\mathbf{k}, \quad t \neq 0.$

- Velocity: $\mathbf{r}'(t) = (\cos t,\ 2t - \sin t,\ 3t^2 - 2\pi t)$.
- For smoothness:
  - $\mathbf{r}(t)$ must be continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.
  - At $t = 0$, $\lim_{t\to 0} \mathbf{r}(t) = (1, 1, 0) \neq \mathbf{r}(0) = (0, 0, 0) \Rightarrow$ **not smooth**.
  - For $t \neq 0$, $\mathbf{r}'(t) = \mathbf{0}$ would require:

    $$\cos t = 0,\ 2t - \sin t = 0,\ 3t^2 - 2\pi t = 0.$$

    No common $t \neq 0$ satisfies all $\Rightarrow \mathbf{r}'(t) \neq \mathbf{0}$.
- **Conclusion:** Curve is smooth for all $t \neq 0$.

Question-1

- Velocity (tangent) at $t = \pi$:

$$\mathbf{r}'(\pi) = (-1, 2\pi, \pi^2).$$

- Acceleration at $t = \pi$:

$$\mathbf{r}''(t) = (-\sin t, 2 - \cos t, 6t - 2\pi) \Rightarrow \mathbf{r}''(\pi) = (0, 3, 4\pi).$$

- Dot product:

$$\mathbf{r}'(\pi) \cdot \mathbf{r}''(\pi) = 6\pi + 4\pi^3.$$

- Magnitudes:

$$|\mathbf{r}'(\pi)| = \sqrt{1 + 4\pi^2 + \pi^4}, \quad |\mathbf{r}''(\pi)| = \sqrt{9 + 16\pi^2}.$$

- Angle:

$$\theta = \arccos \frac{6\pi + 4\pi^3}{\sqrt{1 + 4\pi^2 + \pi^4}\sqrt{9 + 16\pi^2}}.$$

- Numerically, $\theta \approx 20^\circ$.

Question 2: Solve the initial value problem:

$$\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

with initial conditions:

$$\mathbf{r}(0) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \qquad \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{0},$$

where $a, b, c \in \mathbb{R}$ are constants.

- Start with the second-order ODE:

$$\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

- Integrate once to find velocity:

$$\frac{d\mathbf{r}}{dt} = -t(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}_1$$

- Apply initial condition $\frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$.

- Integrate velocity to find position:

$$\mathbf{r}(t) = -\frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}_2$$

- Apply initial condition
  $\mathbf{r}(0) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \Rightarrow \mathbf{C}_2 = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- Final solution:

$$\mathbf{r}(t) = \left(a - \frac{t^2}{2}\right)\mathbf{i} + \left(b - \frac{t^2}{2}\right)\mathbf{j} + \left(c - \frac{t^2}{2}\right)\mathbf{k}.$$

Question-4

Question 4: Find the arc lengths of the following curves:

$$\text{(i)} \quad \mathbf{r}(t) = \cos(3t)\,\mathbf{i} + \sin(3t)\,\mathbf{j} + 6t\,\mathbf{k}, \qquad 0 \le t \le \frac{2\pi}{3},$$

$$\text{(ii)} \quad \mathbf{r}(t) = \cos\left(\frac{t}{5}\right)\mathbf{i} + \sin\left(\frac{t}{5}\right)\mathbf{j} + \frac{2t}{5}\,\mathbf{k}, \qquad 0 \le t \le 10\pi,$$

$$\text{(iii)} \quad \mathbf{r}(t) = \cos\left(\frac{2t}{3}\right)\mathbf{i} - \sin\left(\frac{2t}{3}\right)\mathbf{j} - \frac{4t}{3}\,\mathbf{k}, \quad -3\pi \le t \le 0.$$

Is the answer the same for each curve? If so, why?

- Recall: Arc Length in $\mathbb{R}^3$. For a C^1 curve $\mathbf{r}(t)$, $t \in [a, b]$,

$$L = \int_a^b \|\mathbf{r}'(t)\|\,dt.$$

  - Compute the speed $\|\mathbf{r}'(t)\|$.
  - If the speed is constant, then $L = (\text{speed}) \times (b - a)$.

(i) $\mathbf{r}(t) = (\cos 3t,\ \sin 3t,\ 6t)$

- $\mathbf{r}'(t) = (-3\sin 3t,\ 3\cos 3t,\ 6)$.
- Speed: $\|\mathbf{r}'(t)\| = \sqrt{9(\sin^2 3t + \cos^2 3t) + 36} = 3\sqrt{5}$ (constant).
- Interval length: $\Delta t = \frac{2\pi}{3}$.

$$\boxed{L_1 = 3\sqrt{5} \cdot \frac{2\pi}{3} = 2\pi\sqrt{5}}$$

(ii) $\mathbf{r}(t) = \left(\cos\frac{t}{5},\ \sin\frac{t}{5},\ \frac{2t}{5}\right)$

- $\mathbf{r}'(t) = \left(-\frac{1}{5}\sin\frac{t}{5},\ \frac{1}{5}\cos\frac{t}{5},\ \frac{2}{5}\right)$.
- Speed: $\|\mathbf{r}'(t)\| = \sqrt{\frac{1}{25}(\sin^2 + \cos^2) + \frac{4}{25}} = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5}$ (constant).
- Interval length: $\Delta t = 10\pi$.

$$\boxed{L_2 = \frac{\sqrt{5}}{5} \cdot 10\pi = 2\pi\sqrt{5}}$$

(iii) $\mathbf{r}(t) = \left(\cos\frac{2t}{3},\ -\sin\frac{2t}{3},\ -\frac{4t}{3}\right)$

- $\mathbf{r}'(t) = \left(-\frac{2}{3}\sin\frac{2t}{3},\ -\frac{2}{3}\cos\frac{2t}{3},\ -\frac{4}{3}\right)$.
- Speed: $\|\mathbf{r}'(t)\| = \sqrt{\frac{4}{9}(\sin^2 + \cos^2) + \frac{16}{9}} = \frac{2}{3}\sqrt{5}$ (constant).
- Interval length: $\Delta t = 3\pi$.

$$\boxed{L_3 = \frac{2}{3}\sqrt{5} \cdot 3\pi = 2\pi\sqrt{5}}$$

Why are all three lengths equal?

- Write θ for the planar angle: (i) $\theta = 3t$, (ii) $\theta = \frac{t}{5}$, (iii) $\theta = \frac{2t}{3}$.
- In each case the z–coordinate satisfies $z = 2\theta$ (up to orientation), so each curve traces **the same geometric helix**:

$$x = \cos\theta, \quad y = \sin\theta, \quad z = 2\theta, \qquad 0 \le \theta \le 2\pi.$$

- The arc length over one full turn of this helix is

$$L = \int_0^{2\pi} \sqrt{1^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{1 + 2^2}\, d\theta = 2\pi\sqrt{5}.$$

- The three parametrizations only change the *speed*/direction (reparameterizations), not the traced path.