Sine & Cosine at Key Angles

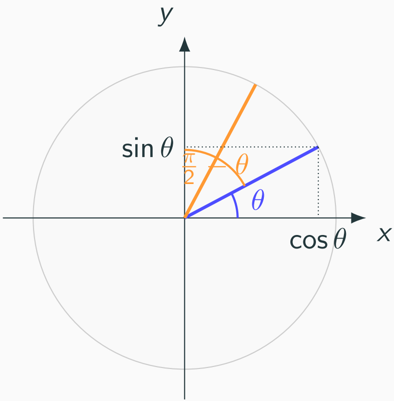
with Cofunction Identities

(dbhoriya.github.io/teaching_F101/tutorial_2.pdf) or SCAN:

Some Trigonometry identities:

Exact Values at Common Angles

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0



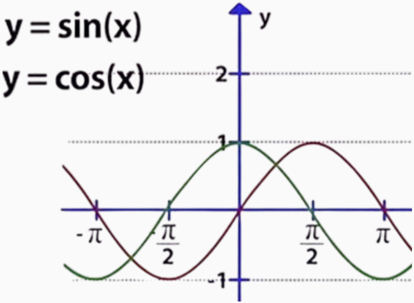
Some identities:

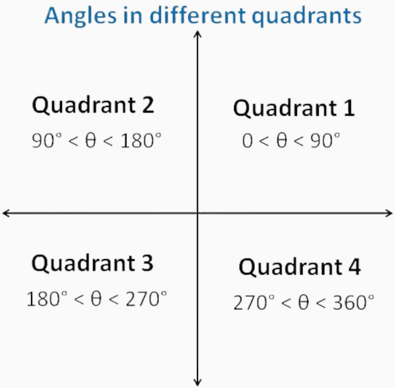
$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta.$$

Example: $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}, \quad \cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}.$

Sin-Cos in various quadrants





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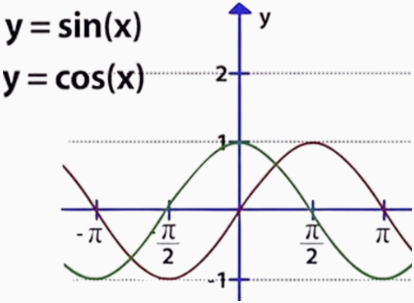
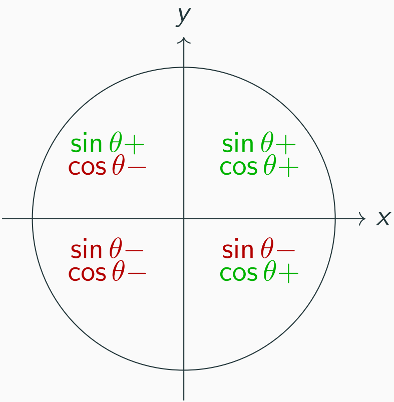




Polar co-ordinates

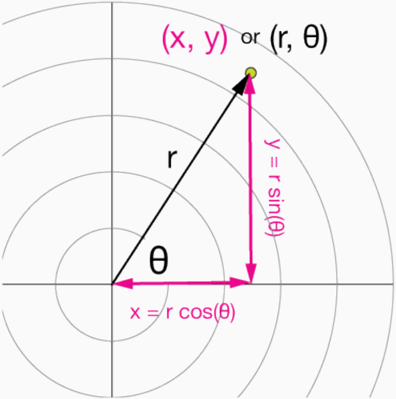
Definition and motivation

Polar Coordinates: Definition & Formulas

- A point (x, y) is represented as (r, θ):
  - $r =$ distance from origin
  - $\theta =$ angle from positive x-axis
- Conversion to Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

- Conversion from Cartesian coordinates:



$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x} = \tan^{-1}\left(\frac{y}{x}\right)$$

Why Do We Need Polar Coordinates? Real-Life Motivation

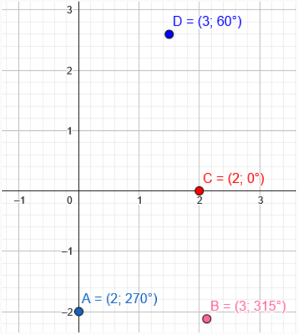
- Some situations are easier to describe with **distance + angle** than (x, y).
- **Examples:**
  - GPS: "5 km at 30° NE"
  - Radar/Sonar: Track objects by angle & range
  - Lighthouses: Light beam at a specific distance & direction
  - CCTV: Camera coverage via rotation & reach

Engineering & Math Usefulness

- **Engineering:**
  - AC circuits: voltages/currents as polar vectors
  - Mechanical systems: gears, turbines, rotating parts
  - Robotics: arm positions via joint angles
  - Civil structures: domes, arches, tunnels
- **Mathematics:**
  - Circle equation: $x^2 + y^2 = a^2 \Rightarrow r = a$ (simpler in polar)
  - Ideal for problems with circular or rotational symmetry

Polar Coordinates: Tutorial Sheet-2

Question 1: Plot the following points, given in polar coordinates, and find all polar coordinates for each point.



A: $(-2, \pi/2)$, **C:** $(-2, -5\pi)$

B: $(3, -\pi/4)$, **D:** $(3, \pi/3)$

Hint: Remember that a point (r, θ) is equivalent to $(-r, \theta + \pi)$ and that angles can be expressed modulo 2π.

Answer:

A: $(-2, \pi/2 + 2n\pi), (2, -\pi/2 + 2n\pi)$ where $n \in \mathbb{Z}$

B: $(3, -\pi/4 + 2n\pi), (-3, 3\pi/4 + 2n\pi)$ where $n \in \mathbb{Z}$.

C: $(-2, -\pi + 2n\pi), (2, 2n\pi)$ where $n \in \mathbb{Z}$.

D: $(3, \pi/3 + 2n\pi), (-3, -2\pi/3 + 2n\pi)$ where$n \in \mathbb{Z}$.

Question 2: Find the polar coordinates of the following points.

(a) $(\sqrt{3}, -1)$, with $0 \leq \theta < 2\pi$ and $r \leq 0$.

(b) $(\sqrt{3}, -1)$, with $-\pi \leq \theta < \pi$ and $r \geq 0$.

Answer: **(a)** $(-2, \pi - \pi/6)$ **(b)** $(2, -\pi/6)$

Question 3: Convert the following into Cartesian equations:

(a) $r^2 + 2r^2 \cos\theta \sin\theta = 1$ **(b)** $r \sin\theta = \ln r + \ln\cos\theta$.

Polar Coordinates: Tutorial Sheet-2

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Answer:

(a) $x^2 + y^2 + 2rcos\theta r \sin\theta = 1 \iff x^2 + y^2 + 2xy = 1$

(b) $r \sin\theta = \ln r + \ln \cos\theta \Longrightarrow r \sin\theta = \ln r \cos\theta$.

This gives $y = \ln x$

Question 5: Sketch the region defined by the inequalities
$-1 \leq r \leq 1$ and $-\pi/4 \leq \theta \leq \pi/4$

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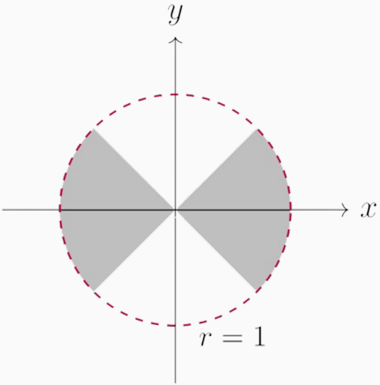
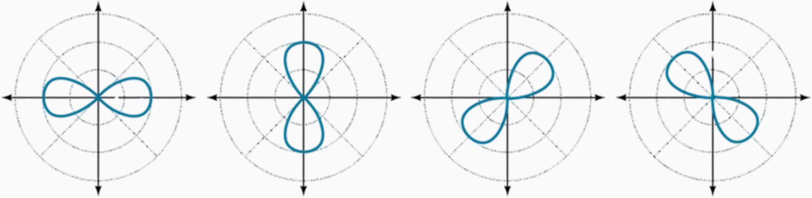


Polar Coordinates: lemniscate (a polar curve resembling the infinity symbol ∞)



$r^2 = a^2 cos(2\theta)$
(a)

$r^2 = -a^2 cos(2\theta)$
(b)

$r^2 = a^2 sin(2\theta)$
(c)

$r^2 = -a^2 sin(2\theta)$
(d)

Question 4: Draw the graph of the curve $r^2 = -\sin 2\theta$.

Answer:

- Equation $r^2 = -\sin(2\theta) \implies \sin(2\theta) \leq 0.$
- Therefore, $\pi \leq 2\theta \leq 2\pi$. That is $\pi/2 \leq \theta \leq \pi.$
- Also, the graph is symmetric about the origin because if (r, θ) lies on the graph, then $(-r, \theta)$ also lies on the graph.
- At origin, $r = 0$. Therefore $\sin(2\theta) = 0$. We get $2\theta = \pi, 2\pi$. Therefore $\theta = \pi/2, \pi$
- Slopes at original $(r = 0)$ is given by $\tan(\theta)$ (see next slides for the proof)

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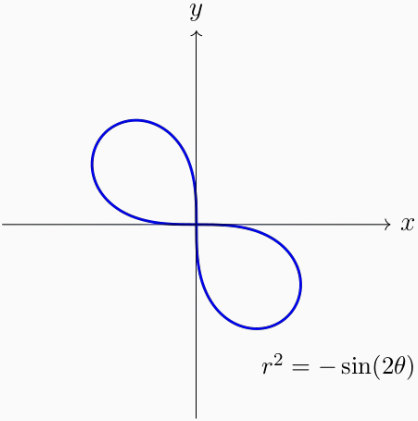
- Slopes at original $(r = 0)$ is given by $\tan(\theta)$ (see next slides for the proof)
- At $\theta = \pi/2$, it has a vertical tangent since slope is infinite, and at $\theta = \pi$, it has a horizontal slope $= 0$.

Answer:



Tangents at the Pole for $r^2 = -\sin(2\theta)$

Curve: $r^2 = -\sin(2\theta)$.

Write coordinates explicitly as functions of θ:

$$x(\theta) = r(\theta)\cos\theta, \qquad y(\theta) = r(\theta)\sin\theta.$$

Differentiate (product rule):

$$\frac{dx}{d\theta} = r'(\theta)\cos\theta - r(\theta)\sin\theta, \qquad \frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta.$$

Hence the slope:

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}.$$

Tangent at the Pole ($r(\theta_0) = 0$)

Suppose the curve passes through the origin at angle θ_0 (i.e. $r(\theta_0) = 0$). Then

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0} = \frac{r'(\theta_0)\sin\theta_0}{r'(\theta_0)\cos\theta_0} = \tan\theta_0,$$

provided $r'(\theta_0) \neq 0$ (the same limit also holds if $|r'(\theta_0)| = \infty$; the r-terms vanish and the common factor $r'(\theta_0)$ cancels).

Geometric view: near the pole the position vector is

$$(x, y) = r(\theta)(\cos\theta, \sin\theta),$$

so the approach direction is the unit vector $(\cos\theta_0, \sin\theta_0)$; hence the tangent line at the pole forms angle θ_0 with the x-axis.

Apply to $r^2 = -\sin(2\theta)$ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$

Zeros at the pole occur when $\theta = \frac{\pi}{2}, \qquad \theta = \pi.$

Case $\theta = \frac{\pi}{2}$:

$$\frac{dy}{dx} = \tan\left(\frac{\pi}{2}\right) = \infty \quad \Rightarrow \quad \text{vertical tangent } x = 0.$$

Case $\theta = \pi$:

$$\frac{dy}{dx} = \tan(\pi) = 0 \quad \Rightarrow \quad \text{horizontal tangent } y = 0.$$

Recall: Area and length of the curve in polar coordinates:

- The formula for area in polar coordinates:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \; d\theta$$

- The length of a curve given in polar coordinates is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Question 6: Find the area inside the lemniscate $r^2 = 6cos(2\theta)$ and outside the circle $r = \sqrt{3}$.

Answer: At $\theta = 0$, $r^2 = 6cos(2\theta)$ gives $r = \pm 3$.

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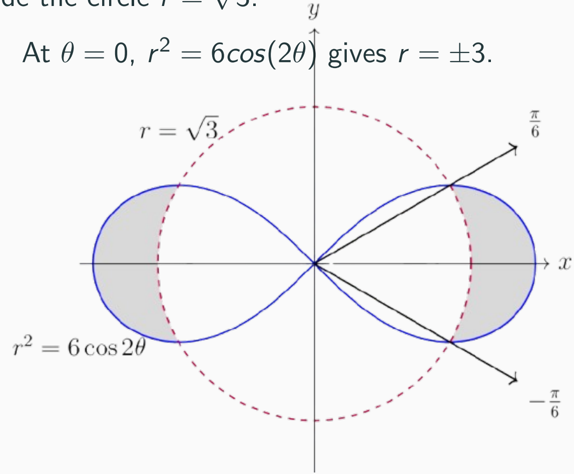

Intersection at: $6cos(2\theta) = 3 \implies \theta = \pm\pi/6$.

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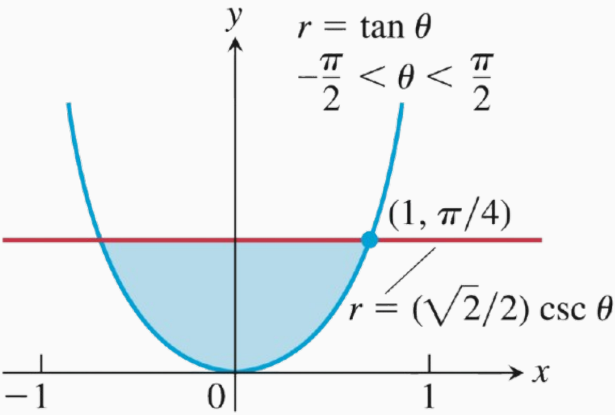
- Intersection at: $6cos(2\theta) = 3 \implies \theta = \pm\pi/6$.

By symmetry, the required area is

$$
\begin{aligned}
\mathrm{A} &= 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{6}} \left(6\cos 2\theta - (\sqrt{3})^2\right) d\theta \\
&= 2\left[3\sin 2\theta - 3\theta\right]_0^{\frac{\pi}{6}} \\
&= 3\sqrt{3} - \pi.
\end{aligned}
$$

Question 7: Find the area of the shaded region in the following figure:



Answer: $\mathrm{Area} = 2\left[\int_0^{\pi/4} \frac{1}{2}\tan^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\left(\frac{\sqrt{2}}{2}\csc\theta\right)^2 d\theta\right]$

Question 7: Find the area of the shaded region in the following figure:

Answer:

$$\begin{aligned}
\text{Area} &= 2\left[\int_0^{\pi/4} \frac{1}{2}\tan^2\theta d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\left(\frac{\sqrt{2}}{2}\csc\theta\right)^2 d\theta\right] \\
&= \int_0^{\pi/4} \tan^2\theta d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\csc^2\theta d\theta \\
&= [-\theta + \tan\theta]_0^{\pi/4} + \frac{1}{2}[-\cot\theta]_{\pi/4}^{\pi/2} \\
&= \frac{-\pi}{4} + 1 + \frac{1}{2}[0+1] \\
&= \frac{3}{2} - \frac{\pi}{4} \text{ sq. units}
\end{aligned}$$

Question 8: Find the length of the curve given by $r = \sqrt{1 - \sin 2\theta}$ where θ lies in the interval $[0, \pi/2]$.

Answer: The length of a curve given in polar coordinates is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

- Since $r = \sqrt{1 - \sin 2\theta}$,

$$\frac{dr}{d\theta} = \frac{1}{2\sqrt{1 - \sin 2\theta}} \cdot (-2\cos 2\theta) = \frac{-\cos 2\theta}{\sqrt{1 - \sin 2\theta}}.$$

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Answer:

- Since $r = \sqrt{1 - \sin 2\theta}$,

$$\frac{dr}{d\theta} = \frac{1}{2\sqrt{1 - \sin 2\theta}} \cdot (-2\cos 2\theta) = \frac{-\cos 2\theta}{\sqrt{1 - \sin 2\theta}}.$$

Therefore,

$$\begin{aligned} \mathrm{L} &= \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin 2\theta + \frac{\cos^2 2\theta}{1 - \sin 2\theta}}\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1 - \sin 2\theta)^2 + \cos^2 2\theta}{1 - \sin 2\theta}}\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{2(1 - \sin 2\theta)}{1 - \sin 2\theta}}\, d\theta = \frac{\pi}{2}\sqrt{2}. \end{aligned}$$