Birla Institute of Technology & Science, Pilani (Raj.)

First Semester, 2025-26, MATH F101 (Multivariable Calculus)

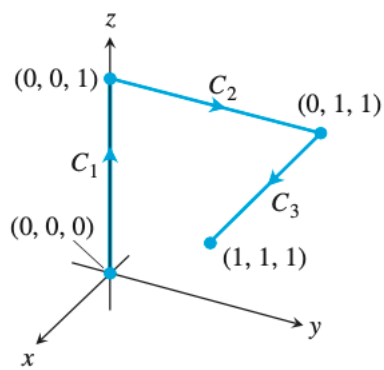
Solutions: Tutorial Sheet 7

Q.1 Evaluate the line integral of the scalar field

$$f(x,y,z) = x + \sqrt{y} - z^2$$

along the piecewise smooth curve C from $(0,0,0)$ to $(1,1,1)$, where C consists of the following segments:

$$\begin{aligned} C_1 &: \ \mathbf{r}(t) = t\mathbf{k}, \quad 0 \le t \le 1 \\ C_2 &: \ \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1 \\ C_3 &: \ \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1 \end{aligned}$$



Solution: We will evaluate the integral using the formula:

$$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))|\boldsymbol{v}(t)|dt$$

where, $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k},\ a \le t \le b$ gives the smooth parametrization of the curve C.

$$\begin{aligned} \text{Along } C_1 : \frac{d\mathbf{r}}{dt} &= \mathbf{k},\ i.e.\ \mathbf{v}(t) = \mathbf{k} \implies |\mathbf{v}(t)| = 1; \\ x + \sqrt{y} - z^2 &= 0 + 0 - t^2 = -t^2; \\ \int_{C_1} f(x,y,z)ds &= \int_0^1 (-t^2)(1)dt = -\frac{1}{3}. \end{aligned}$$

$$\begin{aligned} \text{Along } C_2 : \frac{d\mathbf{r}}{dt} &= \mathbf{j},\ i.e.\ \mathbf{v}(t) = \mathbf{j} \implies |\mathbf{v}(t)| = 1; \\ x + \sqrt{y} - z^2 &= 0 + \sqrt{t} - 1 = \sqrt{t} - 1; \\ \int_{C_2} f(x,y,z)ds &= \int_0^1 (\sqrt{t} - 1)(1)dt = -\frac{1}{3}. \end{aligned}$$

$$\begin{aligned} \text{Along } C_3 : \frac{d\mathbf{r}}{dt} &= \mathbf{i},\ i.e.\ \mathbf{v}(t) = \mathbf{i} \implies |\mathbf{v}(t)| = 1; \\ x + \sqrt{y} - z^2 &= t + \sqrt{1} - 1 = t; \\ \int_{C_3} f(x,y,z)ds &= \int_0^1 (t)(1)dt = \frac{1}{2}. \end{aligned}$$

$$\implies \int_C f(x,y,z)ds = \int_{C_1} fds + \int_{C_2} fds + \int_{C_3} fds = -\frac{1}{3} - \frac{1}{3} + \frac{1}{2} = -\frac{1}{6}$$

Q.2 Evaluate $\int_C (4x^2 + y^2)\, dy$, where C is the ellipse $4x^2 + y^2 = 4$, oriented counterclockwise.

Solution: Here $x = \cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. Hence, $dy = 2\cos t\, dt$, implies that $\int_C (4x^2 + y^2)dy = \int_0^{2\pi} 4(2\cos t)dt = 0$.

Q.3 Find the work done by the force field

$$\mathbf{F}(x, y) = x^2 e^y\, \mathbf{i} + y e^x\, \mathbf{j}$$

in moving a particle along the curve C given by $y = x^3$ from the point $(0, 0)$ to $(1, 1)$. That is, evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution: Consider $x = t, y = t^3$, $0 \le t \le 1$. Implies that $dx = dt$ and $dy = 3t^2 dt$. So, work done is determined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x^2 e^y dx + y e^x dy = \int_0^1 (t^2 e^{t^3} + (t^3 e^t)3t^2)dt = \frac{1079}{3} - \frac{395\, e}{3}.$$

Q.4 Let $\mathbf{F}(x, y, z) = y\big(\cos(xy) + z\big)\, \mathbf{i} + \big(zx + z\cos(yz) + x\cos(xy)\big)\, \mathbf{j} + y\big(x + \cos(yz)\big)\, \mathbf{k}$.

(a) Show that $\mathbf{F}$ is conservative in $\mathbb{R}^3$, and hence find a potential function f such that $f(1, 1, 1) = 0$.

(b) Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the piecewise linear path oriented from $(0, 0, 0)$ to $(1, 1, 0)$ and then from $(1, 1, 0)$ to $(1, 1, 1)$. (Hint: use the potential from part (a) and the Fundamental Theorem for Line Integrals.)

Solution: A vector field $\mathbf{F}(x, y, z) = M\, \mathbf{i} + N\, \mathbf{j} + P\, \mathbf{k}$, defined on a simply connected domain, is conservative if and only if its components M, N, and P have continuous first-order partial derivatives and satisfy the following conditions:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.$$

For the given $\mathbf{F}$:

$$M = y\cos xy + yz, \quad N = zx + z\cos yz + x\cos xy, \quad P = xy + y\cos yz,$$

$$M_y = \cos xy - xy\sin xy + z,\ N_x = z + \cos xy - xy\sin xy,$$

$$N_z = x + \cos yz - yz\sin yz,\ P_y = x + \cos yz - yz\sin yz,\ M_z = y, P_x = y.$$

Hence, F is conservative and there exists f such that $\nabla f = F$.

$$\frac{\partial f}{\partial x} = y\cos xy + yz, \tag{1}$$

$$\frac{\partial f}{\partial y} = xz + z\cos yz + x\cos xy,$$

$$\frac{\partial f}{\partial z} = xy + y\cos yz.$$

On integrating (1), we get

$$f(x, y, z) = \sin xy + xyz + g(y, z).$$

$$\frac{\partial f}{\partial y} = x\cos xy + xz + g_y(y, z) = xz + z\cos yz + x\cos xy.$$

So, $g_y(y, z) = z\cos yz \Rightarrow g(y, z) = \sin yz + h(z)$. Hence

$$f(x, y, z) = \sin xy + xyz + \sin yz + h(z).$$

$$\frac{\partial f}{\partial z} = xy + y\cos yz + h'(z) = xy + y\cos yz.$$

So, $h'(z) = 0 \Rightarrow h(z) = C$. Hence,

$$f(x, y, z) = \sin xy + xyz + \sin yz + C.$$

$f(1, 1, 1) = 0 \implies 2\sin(1) + 1 + C = 0 \implies C = -(1 + 2\sin(1))$. Hence,

$$f(x, y, z) = \sin xy + xyz + \sin yz - (1 + 2\sin(1)).$$

(b) Since F is conservative, the line integral $\int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r}$ is path independent. Thus

$$\begin{aligned}\int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r} &= f(1, 1, 1) - f(0, 0, 0)\\ &= 1 + 2\sin(1) - 1 - 2\sin(1) + 1 + 2\sin(1)\\ &= 1 + 2\sin(1).\end{aligned}$$

Q.5 Find all $a, b \in \mathbb{R}$ such that the vector field

$$\mathbf{F}(x, y, z) = \ln(1 + y^2 + z^2)\,\mathbf{i} + \frac{(b - a^2)\,xy}{1 + y^2 + z^2}\,\mathbf{j} + \frac{a\,xz}{1 + y^2 + z^2}\,\mathbf{k}$$

is conservative in $\mathbb{R}^3$, and hence find a potential function for $\mathbf{F}$.

Solution: A vector field $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, defined on a simply connected domain, is conservative if and only if its components M, N, and P have continuous first-order partial derivatives and satisfy the following conditions:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.$$

For the given $\mathbf{F}$

$$M = \ln(1 + y^2 + z^2), \quad N = \frac{(b - a^2)\,xy}{1 + y^2 + z^2}, \quad P = \frac{a\,xz}{1 + y^2 + z^2}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \frac{2y}{1 + y^2 + z^2} = \frac{(b - a^2)\,y}{1 + y^2 + z^2} \Rightarrow (b - a^2 - 2)y = 0.$$

Thus, we have

$$b - a^2 = 2. \tag{2}$$

$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow \frac{2z}{1 + y^2 + z^2} = \frac{a\,z}{1 + y^2 + z^2} \Rightarrow a = 2$$

$$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \Rightarrow -\frac{2z\,(b - a^2)\,xy}{(1 + y^2 + z^2)^2} = -\frac{2y\,a\,xz}{(1 + y^2 + z^2)^2}. \Rightarrow (b - a^2) = a$$

Since $a = 2$, we get $b = 6$. This is consistent with (2).

Thus, the only values of a and b for which the vector field is conservative are:

$$a = 2, \quad b = 6.$$

Now, we need to find a potential function $\phi(x, y, z)$ such that $\mathbf{F} = \nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k}$, i.e.

$$\phi_x(x, y, z) = \ln(1 + y^2 + z^2), \quad \phi_y(x, y, z) = \frac{2\,xy}{1 + y^2 + z^2}, \quad \phi_z(x, y, z) = \frac{2\,xz}{1 + y^2 + z^2}.$$

On integrating ϕ_x, we get

$$\phi(x, y, z) = \int \ln(1 + y^2 + z^2)\,dx = x\ln(1 + y^2 + z^2) + C(y, z).$$

Differentiating ϕ with respect to y yields

$$\phi_y(x,y,z) = \frac{2xy}{1+y^2+z^2} + C_y(y,z),$$

which implies that $C_y(y,z) = 0$ or $C(y,z) = C_1(z)$. Similarly, on differentiating ϕ with respect to z and substituting it equal to P, yields $C_1(z) = \text{Constant}$. Thus

$$\phi(x,y,z) = x\ln(1+y^2+z^2) + \text{Constant}.$$

Q.6 Find the outward flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x-y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Solution: If C is a smooth, simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the flux of $\mathbf{F}$ across C is given by:

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C (M\,dy - N\,dx)\,.$$

Using $x = a\cos t$ and $y = a\sin t$,

$$\mathbf{F}_1 = (2a\cos t)\,\mathbf{i} - (3a\sin t)\,\mathbf{j},$$
$$\mathbf{F}_2 = (2a\cos t)\,\mathbf{i} + (a\cos t - a\sin t)\,\mathbf{j}.$$

The flux of $\mathbf{F}_1$ is:

$$\text{Flux}(\mathbf{F}_1) = \int_0^{2\pi} \left(2a^2\cos^2 t - 3a^2\sin^2 t\right)\,dt = -\pi a^2.$$

The flux of $\mathbf{F}_2$ is:

$$\text{Flux}(\mathbf{F}_2) = \int_0^{2\pi} \left(2a^2\cos^2 t + a^2\cos t\sin t - a^2\sin^2 t\right)\,dt = \pi a^2.$$

Q.7 Show that the differential form in the given integral is exact, and then evaluate the integral

$$\int_{(1,1,2)}^{(3,5,0)} \sin z\,dx + z^2e^{yz^2}\,dy + (x\cos z + 2yze^{yz^2})dz.$$

Solution. Comparing it with expression $Mdx + Ndy + Pdz$, here we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0,$$

$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} = \cos z,$$

and

$$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} = e^{yz^2}(2z + 2yz^3).$$

So, the differential form is exact. Hence,

$$f_x = \sin z, f_y = z^2e^{yz^2}, f_z = x\cos z + 2yze^{yz^2}.$$

Integrating the first term with respect to x, treating y and z as constants, gives

$$f(x,y,z) = x\sin z + g(y,z).$$

This leads to

$$f_y = \frac{\partial g}{\partial y} = z^2e^{yz^2}.$$

So, $g(y,z) = e^{yz^2} + h(z)$. Now, differentiating $f = x\sin z + e^{yz^2} + h(z)$ with respect to z and comparing with $f_z = x\cos z + 2yze^{yz^2}$ provide

$$x\cos z + 2yze^{yz^2} + h'(z) = x\cos z + 2yze^{yz^2},$$

and so $h'(z) = 0 \implies h(z) = C \implies f = x\sin z + e^{yz^2} + C$. Hence,

$$\int_{(1,1,2)}^{(3,5,0)} \sin z\, dx + z^2 e^{yz^2}\, dy + (x\cos z + 2yze^{yz^2})dz = f(3,5,0) - f(1,1,2) = 1 - \sin 2 - e^4.$$

Q.8 Find the flow of the velocity field $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$ along each of the following paths from $(1,0)$ to $(-1,0)$ in the xy-plane:

(a) The upper half of the circle $x^2+y^2=1$

(b) The line segment from $(1,0)$ to $(-1,0)$

(c) The line segment from $(1,0)$ to $(0,-1)$ followed by the line segment from $(0,-1)$ to $(-1,0)$

Solution. (a). $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le \pi \implies \dfrac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$$\mathbf{F} = (\cos t + \sin t)\mathbf{i} - (cos^2 t + sin^2 t)\mathbf{j} \implies \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\sin t\cos t - sin^2 t - \cos t$$

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{\pi} (-\sin t\cos t - sin^2 t - \cos t)dt = -\frac{\pi}{2}$$

(b). $\mathbf{r} = (1-2t)\mathbf{i}, \quad 0 \le t \le 1 \implies \dfrac{d\mathbf{r}}{dt} = -2\mathbf{i}$

$$\mathbf{F} = (1-2t)\mathbf{i} - (1-2t)^2\mathbf{j} \implies \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 4t-2$$

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^1 (4t-2)dt = 0$$

(c). $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j}, \quad 0 \le t \le 1 \implies \dfrac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$

$$\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j} \implies \mathbf{F}\cdot\frac{d\mathbf{r}_1}{dt} = 2t^2$$

$$\int_{C_1} \mathbf{F}\cdot\mathbf{T}ds = \int_0^1 2t^2 dt = \frac{2}{3}$$

$$\mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j}, \quad 0 \le t \le 1 \implies \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j}$$

$$\mathbf{F} = -\mathbf{i} - (1-2t+2t^2)\mathbf{j} \implies \mathbf{F}\cdot\frac{d\mathbf{r}_2}{dt} = 2t - 2t^2$$

$$\int_{C_2} \mathbf{F}\cdot\mathbf{T}ds = \int_0^1 2t - 2t^2 dt = \frac{1}{3}$$

Total flow $= \frac{2}{3} + \frac{1}{3} = 1$.

Q.9 Use Green's theorem to evaluate the integral

$$I = \oint_C \left(\frac{x}{x^2+1} - y\right) dx + \left(3x - 4\tan\frac{y}{2}\right) dy,$$

where C is the portion of $y = x^2$ from $(-1,1)$ to $(1,1)$, followed by the portion of $y = 2 - x^2$ from $(1,1)$ to $(-1,1)$.

Solution: By Green's theorem,

$$I = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA,$$

where $M(x,y) = \frac{x}{x^2+1} - y$ and $N(x,y) = 3x - 4\tan\frac{y}{2}$.
We first compute the partial derivatives:

$$\frac{\partial N}{\partial x} = 3, \quad \frac{\partial M}{\partial y} = -1.$$

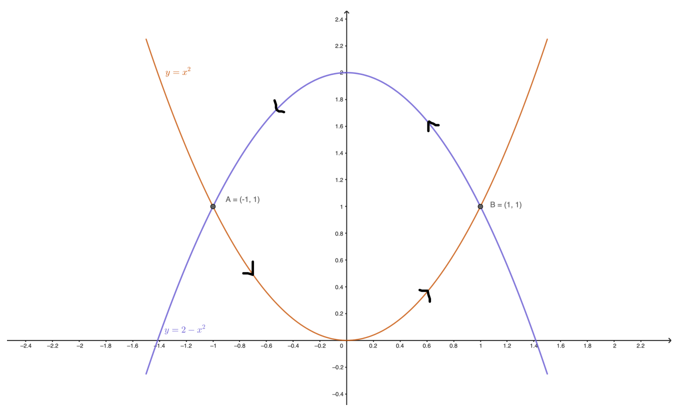
Thus,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3 - (-1) = 4.$$

The integral becomes

$$I = 4 \iint_R dA = 4\,\mathrm{Area}(R).$$

The region R is enclosed by the curves $y = x^2$ from $(-1,1)$ to $(1,1)$ and $y = 2 - x^2$ from $(1,1)$ to $(-1,1)$. The area is:



$$\mathrm{Area}(R) = \int_{-1}^{1}\int_{x^2}^{2-x^2} dy dx = \int_{-1}^{1} \left((2-x^2) - x^2\right)\, dx = \int_{-1}^{1} \left(2 - 2x^2\right)\, dx = \frac{8}{3}.$$

Hence

$$I = 4 \times \frac{8}{3} = \frac{32}{3}.$$

Q.10 Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

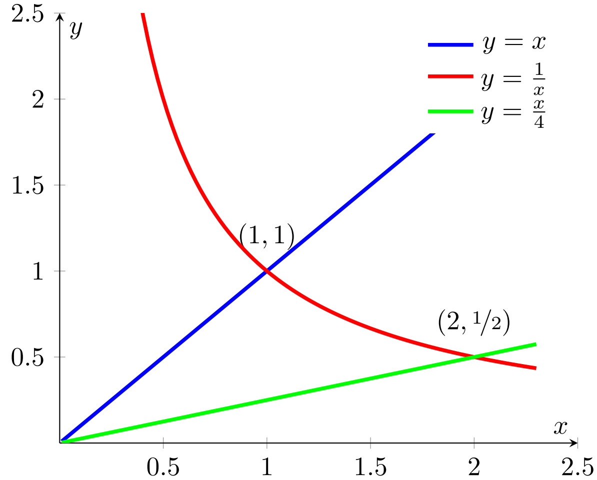
$$y = x, \quad y = \frac{1}{x}, \quad y = \frac{x}{4}.$$

Solution. Using Green's theorem, the area can be expressed as

$$\mathrm{Area} = \frac{1}{2}\oint_C \left(x\, dy - y\, dx\right),$$

where C is a positively oriented, closed curve enclosing the region, consisting of the following three segments:

(a) From $x = 0$ to $x = 2$ along $y = \frac{x}{4}$.
(b) From $y = \frac{1}{2}$ to $y = 1$ along $y = \frac{1}{x}$.
(c) From $x = 1$ to $x = 0$ along $y = x$.



Along $y = \frac{x}{4}$, from $x = 0$ to $x = 2$, we have $dy = \frac{1}{4}\,dx$. The integral becomes

$$\int_0^2 \left(\frac{x}{4}\,dx - x \cdot \frac{1}{4}\,dx \right) = 0.$$

Next, along $y = \frac{1}{x}$, from $y = \frac{1}{2}$ to $y = 1$, we have, $dx = -\frac{1}{y^2}\,dy$. The integral is

$$\int_{1/2}^1 \left(\frac{1}{y}\,dy - y \cdot \left(-\frac{1}{y^2} \right)\,dy \right) = \int_{1/2}^1 \frac{2}{y}\,dy = 2\ln(2).$$

Finally, along $y = x$, from $x = 1$ to $x = 0$, we have $dy = dx$. The integral becomes

$$\int_1^0 (-x\,dx + x\,dx) = 0.$$

The total area is

$$\text{Area} = \frac{1}{2}(0 + 2\ln(2) + 0) = \ln(2).$$

Path Independence

If A and B are two points in an open region D in space, the line integral of $\mathbf{F}$ along C from A to B for a field $\mathbf{F}$ defined on D usually depends on the path C taken, as we saw in Section 16.1. For some special fields, however, the integral's value is the same for all paths from A to B.

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for $\mathbf{F}$**.

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

THEOREM 1—Fundamental Theorem of Line Integrals
Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

EXAMPLE 2 Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla f, \quad \text{where} \quad f(x, y, z) = xyz,$$

in moving an object along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

With $f(x, y, z) = xyz$, we have

$$\begin{aligned} \cdot\, d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} \\ &= f(B) - f(A) \\ &= xyz|_{(1,6,-4)} - xyz|_{(-1,3,9)} \\ &= (1)(6)(-4) - (-1)(3)(9) \end{aligned}$$

Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$