

# Tutorial Sheet 8: Multivariable Calculus

## Surface Integrals & Flux

Mathematics-I  
BITS-Pilani

Verify everything!

Dr. Deepak Bhoriya

# The Concept: Shadow vs. Surface

## The Big Question

Why isn't the surface area just the area of the limits?

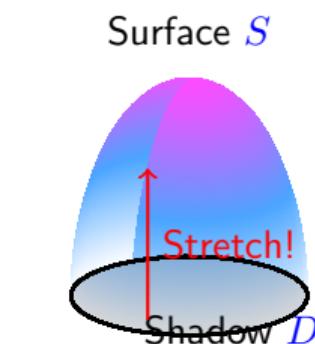
1. **The Shadow ( $\iint 1 \, dA$ )** This calculates the flat "footprint" on the floor.

$$\text{Area} = \pi R^2$$

2. **The Surface ( $\iint \text{Stretch } dA$ )** The real surface is curved (tilted). We must multiply the shadow area by a "**Stretch Factor**" (or Tilt Factor) to get the true area.

$$\text{Area} = 2\pi R^2$$

*The Double Integral sums up these tiny stretched patches.*



# Method 1: The Projection Formula

When to use?

When the surface is given as an equation  $f(x, y, z) = c$  (e.g.,  $x^2 + y^2 + z^2 = R^2$ ).

The Formula

$$\text{Stretch Factor} = \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|}$$

$$S = \iint_D \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

**Example (Hemisphere):**  $f = x^2 + y^2 + z^2 - R^2 = 0$ .

$$\nabla f = \langle 2x, 2y, 2z \rangle \implies |\nabla f| = 2R$$

$$|\nabla f \cdot \hat{k}| = 2z$$

$$d\sigma = \frac{2R}{2z} dA = \frac{R}{z} dA$$

## Method 2: The Parametric Formula

### When to use?

When the surface is given as a vector function  $\mathbf{r}(u, v)$ . (Useful for cylinders, spheres, cones).

**The Logic:** We create a tiny “grid” on the curved surface using tangent vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$ .

The area of one tiny curved patch is the area of the parallelogram formed by these vectors.

$$\text{Patch Area} = |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

### The Formula

$$S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

*This avoids the “shadow” entirely and works directly on the skin.*

# Example: Hemisphere via Parametrization

## 1. Define the Vector Function (Spherical Coords)

$$\mathbf{r}(\phi, \theta) = \langle R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi \rangle$$

## 2. Find Tangent Vectors

$$\mathbf{r}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = \langle R \cos \phi \cos \theta, R \cos \phi \sin \theta, -R \sin \phi \rangle$$

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -R \sin \phi \sin \theta, R \sin \phi \cos \theta, 0 \rangle$$

## 3. The Cross Product (The Area Element)

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} = \dots \text{Standard Result} \dots = (R^2 \sin \phi) \hat{e}_r$$

$$d\sigma = |\mathbf{r}_\phi \times \mathbf{r}_\theta| d\phi d\theta = R^2 \sin \phi d\phi d\theta$$

# Comparison: Hemisphere Area ( $R$ )

## Method 1: Projection

$$S = \iint_D \frac{R}{z} dA$$

Convert to Polar ( $z = \sqrt{R^2 - r^2}$ ):

$$S = \int_0^{2\pi} \int_0^R \frac{R}{\sqrt{R^2 - r^2}} r dr d\theta$$

$$\dots = 2\pi R^2$$

## Method 2: Parametric

$$S = \iint_D R^2 \sin \phi d\phi d\theta$$

Limits:  $0 \leq \phi \leq \pi/2$ ,  $0 \leq \theta \leq 2\pi$ .

$$S = R^2 \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \phi d\phi$$

$$= R^2(2\pi)(1) = 2\pi R^2$$

**Result:** Both methods give the same answer ( $2\pi R^2$ ), which is exactly twice the shadow area ( $\pi R^2$ ).

# Recall: Surface Area & Projection Tricks

## General Formula

To find the area of surface  $S$  ( $f(x, y, z) = c$ ), project it onto a domain  $D$  with normal  $\hat{p}$ :

$$A_S = \iint_D \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

## The Trick: Choosing the Projection Plane ( $\hat{p}$ )

### Standard (XY)

Use  $\hat{p} = \hat{k}$

Best when  $z$  is easily isolated:

$$z = f(x, y)$$

### Trick (XZ)

Use  $\hat{p} = \hat{j}$

Best when  $y$  is constrained or isolated:

$$y = f(x, z) \quad (\text{like } y \geq 0)$$

### Trick (YZ)

Use  $\hat{p} = \hat{i}$

Best when  $x$  is constrained or isolated:

$$x = f(y, z)$$

# Question 1

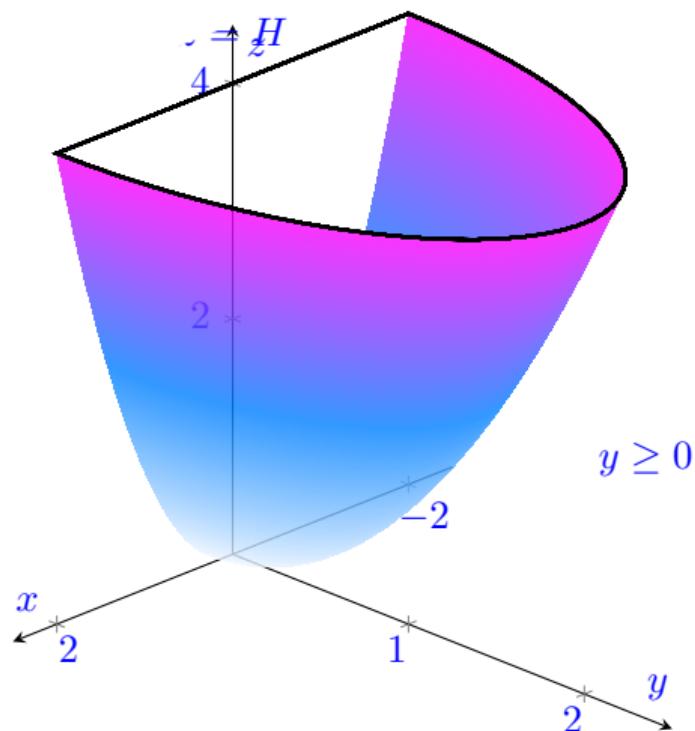
## Problem Statement

Find the surface area of the region cut from the paraboloid

$$x^2 + y^2 - z = 0$$

by the plane  $z = 0$  and  $z = H$ , that lies on the side  $y \geq 0$ .

# Visualization (Geometry)



## Solution: Setup and Projection

**1. Surface and Gradient** Let  $f(x, y, z) = x^2 + y^2 - z = 0$ .

$$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$|\nabla f| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4z + 1}$$

**2. Apply the “Trick” (Choice of  $\hat{p}$ )** Since the problem specifies  $y \geq 0$  and we have boundaries in  $z$ , we project onto the **XZ-plane**.

$$\hat{p} = \hat{j}$$

$$|\nabla f \cdot \hat{p}| = |2y| = 2y$$

Substituting  $y = \sqrt{z - x^2}$  from the surface equation:

$$|\nabla f \cdot \hat{p}| = 2\sqrt{z - x^2}$$

# Solution: The Integral Setup

3. Determine Domain  $D$  In the XZ-plane,  $y$  is real when  $z - x^2 \geq 0 \implies x^2 \leq z$ .

$$D = \{(x, z) : 0 \leq z \leq H, -\sqrt{z} \leq x \leq \sqrt{z}\}$$

4. Formulate the Integral

$$S = \iint_D \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA = \int_0^H \int_{-\sqrt{z}}^{\sqrt{z}} \frac{\sqrt{4z+1}}{2\sqrt{z-x^2}} dx dz$$

Factor out terms constant w.r.t  $x$ :

$$S = \int_0^H \frac{\sqrt{4z+1}}{2} \left[ \int_{-\sqrt{z}}^{\sqrt{z}} \frac{dx}{\sqrt{z-x^2}} \right] dz$$

# Solution: Evaluation

## 5. Evaluate Inner Integral

$$\int_{-\sqrt{z}}^{\sqrt{z}} \frac{dx}{\sqrt{z-x^2}} = \left[ \sin^{-1} \left( \frac{x}{\sqrt{z}} \right) \right]_{-\sqrt{z}}^{\sqrt{z}} = \pi$$

## 6. Final Integration

$$S = \frac{\pi}{2} \int_0^H \sqrt{4z+1} dz$$

Use substitution  $u = 4z + 1$ :

$$S = \frac{\pi}{8} \left[ \frac{2}{3} u^{3/2} \right]_1^{4H+1}$$

$$S = \frac{\pi}{12} \left( (4H+1)^{3/2} - 1 \right)$$

# Recall: Scalar Fields & Vertical Planes

## Scalar Surface Integral

To integrate a function  $G(x, y, z)$  over surface  $S$ :

$$\iint_S G \, d\sigma = \iint_D G(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} \, dA$$

**The Strategy: “Vertical” Surfaces** If the surface is a cylinder or plane like  $x + y = 1$ :

- The normal  $\nabla f$  is horizontal ( $\hat{i} + \hat{j}$ ).
- $\nabla f \cdot \hat{k} = 0$ .
- **Result:** You cannot project onto the XY-plane (division by zero).

## The Fix

Switch your projection!

**Option A (XZ-Plane):** Use  $\hat{p} = \hat{j}$ .  
Eliminates  $y$ .

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \hat{j}|} dx dz$$

**Option B (YZ-Plane):** Use  $\hat{p} = \hat{i}$ .  
Eliminates  $x$ .

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \hat{i}|} dy dz$$

## Question 2

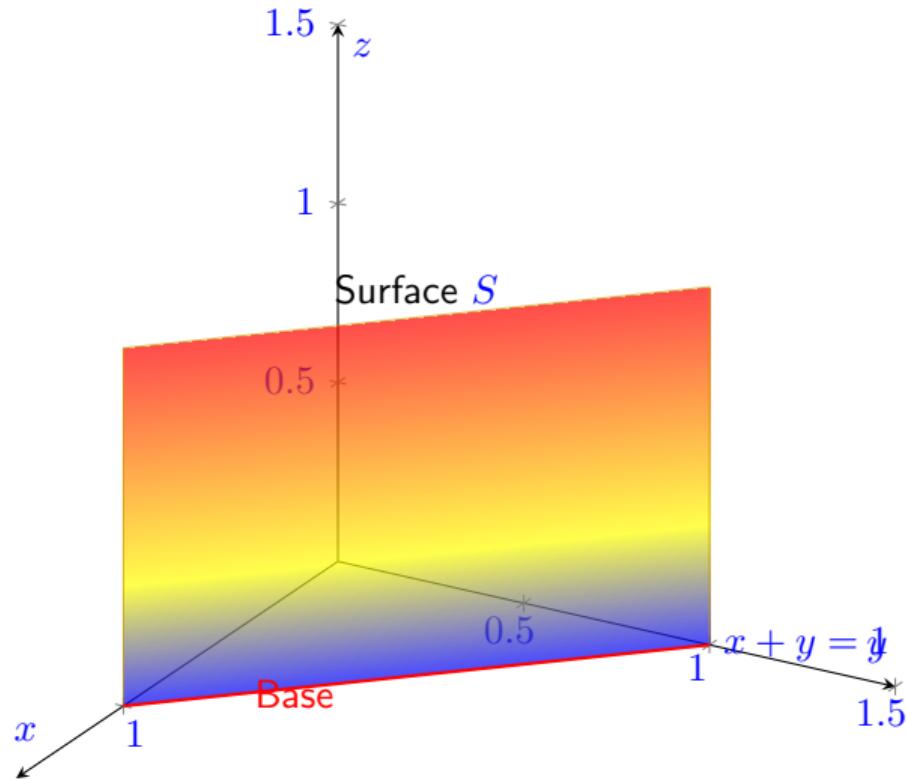
### Problem Statement

Integrate  $G(x, y, z) = x - y - z$  over the portion of the plane

$$x + y = 1$$

in the first octant between  $z = 0$  and  $z = 1$ .

# Visualization (Geometry)



# Solution: Setup and Geometry

**1. Surface and Gradient** Define the surface  $f(x, y, z) = x + y - 1 = 0$ .

$$\nabla f = \hat{i} + \hat{j}$$

$$|\nabla f| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

**2. Choose Projection (The Trick)** Since the plane is vertical, we cannot use  $\hat{k}$ . We choose the **XZ-plane** (normal  $\hat{p} = \hat{j}$ ).

$$|\nabla f \cdot \hat{p}| = |(\hat{i} + \hat{j}) \cdot \hat{j}| = 1$$

Calculate the surface element  $d\sigma$ :

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA = \frac{\sqrt{2}}{1} dx dz = \sqrt{2} dx dz$$

# Solution: The Integral

**3. Setup the Integral** Substitute  $y = 1 - x$  into  $G(x, y, z)$ :

$$G = x - y - z = x - (1 - x) - z = 2x - 1 - z$$

The domain  $D$  in the XZ-plane is the rectangle:  $0 \leq x \leq 1$ ,  $0 \leq z \leq 1$ .

**4. Evaluate**

$$\iint_S G \, d\sigma = \int_0^1 \int_0^1 (2x - 1 - z)\sqrt{2} \, dz \, dx$$

$$= \sqrt{2} \int_0^1 \left[ (2x - 1)z - \frac{z^2}{2} \right]_0^1 \, dx$$

$$= \sqrt{2} \int_0^1 \left( 2x - 1 - \frac{1}{2} \right) \, dx = \sqrt{2} \int_0^1 \left( 2x - \frac{3}{2} \right) \, dx$$

$$= \sqrt{2} \left[ x^2 - \frac{3}{2}x \right]_0^1 = \sqrt{2}(1 - 1.5) = -\frac{\sqrt{2}}{2}$$

# Recall: Flux Formulas (2D vs 3D)

## 2D Flux (Across a Curve)

For  $\mathbf{F} = M\hat{i} + N\hat{j}$  crossing a curve  $C$ :

$$\text{Flux} = \oint_C \mathbf{F} \cdot \hat{n} \, ds$$

**Coordinate Form:**

$$\text{Flux} = \oint_C (M \, dy - N \, dx)$$

## 3D Flux (Across a Surface)

For  $\mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k}$  crossing surface  $S$ :

$$\text{Flux} = \iint_S \mathbf{F} \cdot \hat{n} \, d\sigma$$

**Calculation Formula:**

$$\iint_D \mathbf{F} \cdot \left( \frac{\nabla g}{|\nabla g \cdot \hat{p}|} \right) \, dA$$

Just as  $Mdy - Ndx$  is the standard form for line flux,  
the  $\mathbf{F} \cdot \nabla g$  method is the standard for surface flux.

# Recall: Flux Derivations (2D)

## 2D Flux: The Short Proof

Let curve vector be  $d\mathbf{r} = \langle dx, dy \rangle$ .

- **Normal Vector:** Rotate  $d\mathbf{r}$  by  $90^\circ$  clockwise to get outward normal direction:

$$\hat{n} ds = \langle dy, -dx \rangle$$

- **Flux Dot Product:**

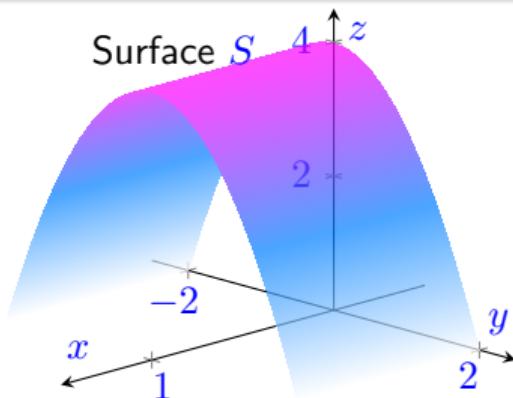
$$\mathbf{F} \cdot \hat{n} ds = \langle M, N \rangle \cdot \langle dy, -dx \rangle$$

$$= M dy - N dx$$

# Question 3

## Problem Statement

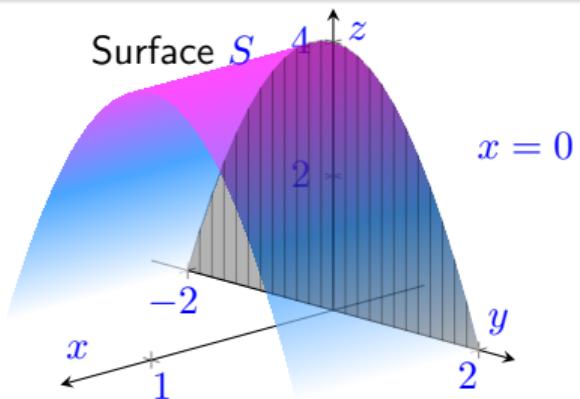
Find the outward flux of  $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$  through the surface cut from the cylinder  $z = 4 - y^2$  by planes  $x = 0$ ,  $x = 1$ , and  $z = 0$ .



# Question 3

## Problem Statement

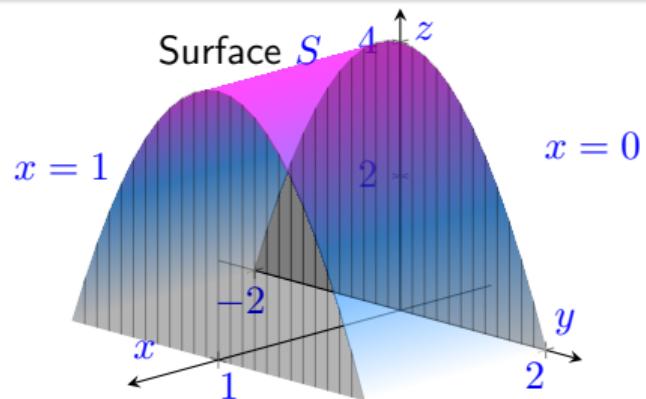
Find the outward flux of  $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$  through the surface cut from the cylinder  $z = 4 - y^2$  by planes  $x = 0$ ,  $x = 1$ , and  $z = 0$ .



# Question 3

## Problem Statement

Find the outward flux of  $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$  through the surface cut from the cylinder  $z = 4 - y^2$  by planes  $x = 0$ ,  $x = 1$ , and  $z = 0$ .



# Solution: Setup & Calculation

**1. Surface & Gradient** We have  $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$

$$g = z + y^2 - 4 = 0 \implies \nabla g = 2y\hat{j} + \hat{k}$$

Project on XY-plane ( $\hat{p} = \hat{k}$ ). Outward normal points up ( $+\hat{k}$ ):

$$\hat{n}d\sigma = \frac{\nabla g}{|\nabla g \cdot \hat{k}|} dA = (2y\hat{j} + \hat{k}) dx dy$$

**2. Integrand ( $\mathbf{F} \cdot \hat{n}d\sigma$ )**

$$\mathbf{F} \cdot \nabla g = (z^2\hat{i} + x\hat{j} - 3z\hat{k}) \cdot (2y\hat{j} + \hat{k}) = 2xy - 3z$$

Substitute surface  $z = 4 - y^2$ :

$$\text{Integrand} = 2xy - 3(4 - y^2) = 2xy - 12 + 3y^2$$

**3. Final Integral** (Limits:  $0 \leq x \leq 1, -2 \leq y \leq 2$ )

$$\text{Flux} = \int_0^1 \int_{-2}^2 (2xy - 12 + 3y^2) dy dx = -32$$

3D Flux (Across a Surface)

For  $\mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k}$  crossing surface  $S$ :

$$\text{Flux} = \iint_S \mathbf{F} \cdot \hat{n} d\sigma$$

**Calculation Formula:**

$$\iint_D \mathbf{F} \cdot \left( \frac{\nabla g}{|\nabla g \cdot \hat{p}|} \right) dA$$

# Recall: Surface Area via Parametrization

## Parametric Surface Area Formula

If a surface  $S$  is defined by a vector function  $\mathbf{r}(u, v)$  over a domain  $D$ :

$$\text{Area} = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

## Steps to Solve:

- ➊ **Parametrize:** Choose variables (e.g.,  $r, \theta$ ) to define  $\mathbf{r}(r, \theta)$ .
- ➋ **Tangents:** Calculate partial vectors  $\mathbf{r}_r$  and  $\mathbf{r}_\theta$ .
- ➌ **Normal:** Find the cross product  $\mathbf{r}_r \times \mathbf{r}_\theta$ .
- ➍ **Integrate:** Compute magnitude and integrate.

## Why use this?

The “Projection Method” ( $\frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}$ ) works for graphs  $z = f(x, y)$ . This “Parametric Method” is more general and often easier for surfaces with **cylindrical symmetry** (like cones and paraboloids).

# Question 4

## Problem Statement

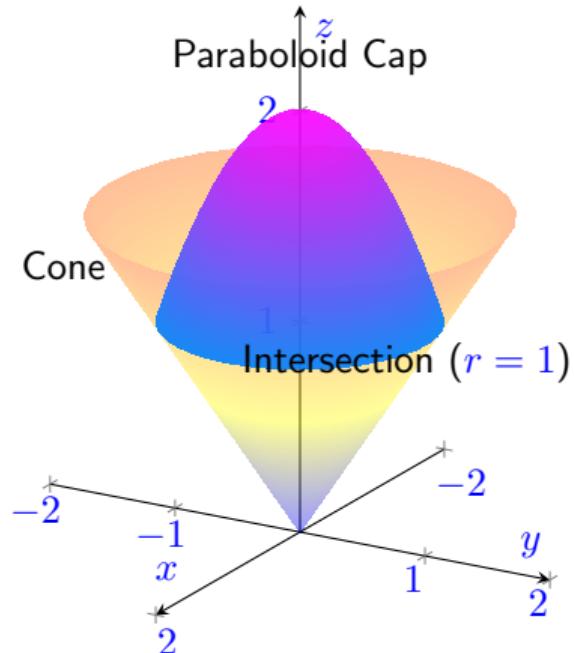
Use a parametrization to express the area of the cap cut from the paraboloid

$$z = 2 - x^2 - y^2$$

by the cone

$$z = \sqrt{x^2 + y^2}$$

Evaluate the integral to find the surface area.



# Solution: Intersection & Parametrization

**1. Find Intersection (Region  $D$ )** The cap is cut from the paraboloid by the cone.

$$\text{Paraboloid: } z = 2 - r^2, \quad \text{Cone: } z = r$$

Intersection occurs when:

$$2 - r^2 = r \implies r^2 + r - 2 = 0 \implies (r + 2)(r - 1) = 0$$

Since  $r > 0$ , we have  $r = 1$ . The cap exists for  $0 \leq r \leq 1$ .

**2. Define Vector Function  $\mathbf{r}(r, \theta)$**  Use polar coordinates for  $x$  and  $y$ .

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = 2 - r^2$$

$$\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 2 - r^2 \rangle$$

Limits:  $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$ .

# Solution: Tangent Vectors

## 3. Calculate Partial Derivatives

$$\mathbf{r}_r = \frac{\partial \mathbf{r}}{\partial r} = \langle \cos \theta, \sin \theta, -2r \rangle$$

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

## 4. Compute Cross Product ( $\mathbf{r}_r \times \mathbf{r}_\theta$ )

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix}$$

$$= \hat{i}(0 - (-2r^2 \cos \theta)) - \hat{j}(0 - (2r^2 \sin \theta)) + \hat{k}(r \cos^2 \theta + r \sin^2 \theta)$$

$$= \langle 2r^2 \cos \theta, 2r^2 \sin \theta, r \rangle$$

# Solution: Integration

## 5. Magnitude and Integral

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(2r^2 \cos \theta)^2 + (2r^2 \sin \theta)^2 + r^2}$$

$$= \sqrt{4r^4(\cos^2 \theta + \sin^2 \theta) + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}$$

## 6. Evaluate

$$\text{Area} = \int_0^{2\pi} \int_0^1 r \sqrt{4r^2 + 1} dr d\theta$$

$$= 2\pi \int_0^1 r(4r^2 + 1)^{1/2} dr$$

Let  $u = 4r^2 + 1 \implies du = 8r dr$ . Limits:  $1 \rightarrow 5$ .

$$= 2\pi \cdot \frac{1}{8} \int_1^5 u^{1/2} du = \frac{\pi}{4} \left[ \frac{2}{3} u^{3/2} \right]_1^5$$

$$= \frac{\pi}{6} (5^{3/2} - 1)$$

# Recall: Stokes' Theorem (Flux of Curl)

## Stokes' Theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n} d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

**Direct Calculation (RHS of Theorem):** If asked for “Flux of Curl”, calculate the surface integral directly:

$$\text{Flux} = \iint_D (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) d\phi d\theta$$

**Application:** If calculating the line integral is hard, switch to the surface integral (or vice versa)!

## Spherical Normal

For sphere radius  $R$ :

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = (R^2 \sin \phi) \hat{e}_r$$

Vector points radially outward.

## Question 5

### Problem Statement

Use the surface integral to calculate the flux of the curl of the field

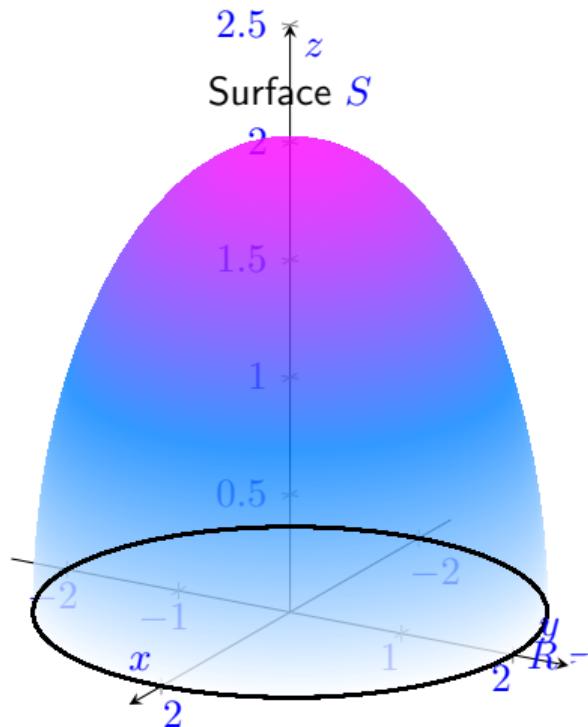
$$\mathbf{F} = y^2 \hat{i} + z^2 \hat{j} + x \hat{k}$$

across  $S$ :

$$\mathbf{r}(\phi, \theta) = (2 \sin \phi \cos \theta) \hat{i} + (2 \sin \phi \sin \theta) \hat{j} + (2 \cos \phi) \hat{k}$$

where  $0 \leq \theta \leq 2\pi$  and  $0 \leq \phi \leq \pi/2$  (Upper Hemisphere).

# Visualization (Geometry)



# Solution: Curl and Normal Vector

## 1. Calculate Curl $\mathbf{F}$

$$\mathbf{F} = \langle y^2, z^2, x \rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & z^2 & x \end{vmatrix}$$

$$= \hat{i}(0 - 2z) - \hat{j}(1 - 0) + \hat{k}(0 - 2y) = \langle -2z, -1, -2y \rangle$$

## 2. Calculate Normal Vector ( $\mathbf{N}$ )

For sphere  $R = 2$ , the normal vector  $\mathbf{r}_\phi \times \mathbf{r}_\theta$  is the position vector scaled by  $R \sin \phi$ :

$$\mathbf{N} = (2 \sin \phi) \mathbf{r} = \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin \phi \cos \phi \rangle$$

Alternatively, computed via cross product determinant.

# Solution: The Integral

**3. Setup Dot Product** Substitute  $x, y, z$  into Curl:

$$\nabla \times \mathbf{F} = \langle -4 \cos \phi, -1, -4 \sin \phi \sin \theta \rangle$$

Compute  $(\nabla \times \mathbf{F}) \cdot \mathbf{N}$ :

$$= (-4 \cos \phi)(4 \sin^2 \phi \cos \theta) + (-1)(4 \sin^2 \phi \sin \theta) + (-4 \sin \phi \sin \theta)(4 \sin \phi \cos \phi)$$

$$= -16 \sin^2 \phi \cos \phi \cos \theta - 4 \sin^2 \phi \sin \theta - 16 \sin^2 \phi \cos \phi \sin \theta$$

**4. Integrate**

$$I = \int_0^{\pi/2} \int_0^{2\pi} [-16 \sin^2 \phi \cos \phi (\cos \theta + \sin \theta) - 4 \sin^2 \phi \sin \theta] d\theta d\phi$$

# Solution: Evaluation (Symmetry)

**5. Evaluate Inner Integral ( $d\theta$ )** Notice that  $\int_0^{2\pi} \cos \theta d\theta = 0$  and  $\int_0^{2\pi} \sin \theta d\theta = 0$ . Every term in our integrand contains either  $\sin \theta$  or  $\cos \theta$  (or both linearly)!

$$\int_0^{2\pi} (\dots \cos \theta + \dots \sin \theta) d\theta = 0$$

Therefore:

$$\text{Flux} = \int_0^{\pi/2} (0) d\phi = 0$$

## Conclusion

The total flux of the curl through this hemisphere is **zero**.

# Recall: The Two Sides of Stokes' Theorem

## The Theorem

$$\underbrace{\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n} d\sigma}_{\text{Surface Integral}} = \underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Line Integral}}$$

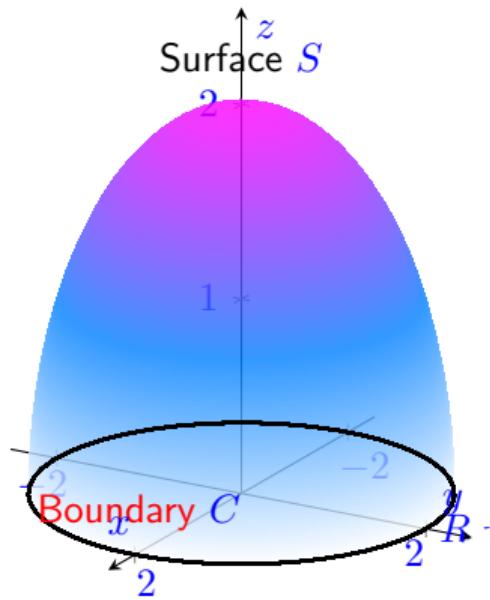
### Method A: Surface Integral

- Used when asked for “Flux of Curl”.
- Requires: Curl  $\nabla \times \mathbf{F}$ , Normal  $\mathbf{N}$ , Double Integral.
- **This Question requires Method A.**

### Method B: Line Integral

- Used when asked to “Verify” or “Evaluate using Stokes”.
- Requires: Boundary Curve  $C$ , Parametrization  $\mathbf{r}(t)$ , Single Integral.
- Usually faster!

# Visualization (Geometry)



# Alternative: What if we used the Line Integral?

## Hypothetical Scenario

If the question allowed: "Use Stokes' Theorem...", we could calculate the circulation around the boundary curve  $C$ .

**1. Identify Boundary  $C$**  The edge of the hemisphere is the circle  $x^2 + y^2 = 4$  on the plane  $z = 0$ .

**2. Evaluate Field on  $C$**  Since  $z = 0$ , the field  $\mathbf{F} = y^2\hat{i} + z^2\hat{j} + x\hat{k}$  simplifies:

$$\mathbf{F}_{\text{on } C} = y^2\hat{i} + 0\hat{j} + x\hat{k}$$

Also, along  $C$ ,  $dz = 0$ .

**3. Calculate Line Integral**

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (y^2 dx + 0 dy + x \underbrace{dz}_0) = \oint_C y^2 dx$$

Using parametrization  $x = 2 \cos t, y = 2 \sin t$ :

$$\int_0^{2\pi} (2 \sin t)^2 (-2 \sin t) dt = -8 \int_0^{2\pi} \sin^3 t dt = 0$$

## Question 6

### Problem Statement

Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field:

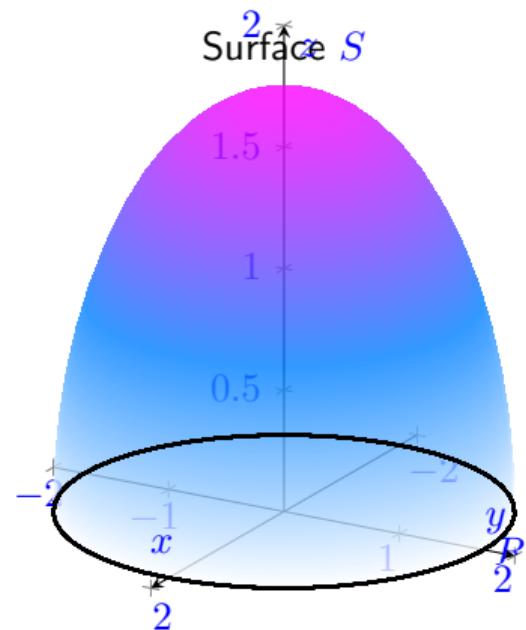
$$\mathbf{F} = 3y\hat{i} + (5 - 2x)\hat{j} + (z^2 - 2)\hat{k}$$

across the surface  $S$  (Hemisphere  $R = \sqrt{3}$ ):

$$\mathbf{r}(\phi, \theta) = (\sqrt{3} \sin \phi \cos \theta)\hat{i} + (\sqrt{3} \sin \phi \sin \theta)\hat{j} + (\sqrt{3} \cos \phi)\hat{k}$$

where  $0 \leq \phi \leq \pi/2$  and  $0 \leq \theta \leq 2\pi$ .

# Visualization (Geometry)



# Solution: Curl Calculation

## 1. Calculate the Curl ( $\nabla \times \mathbf{F}$ )

$$\mathbf{F} = \langle \underbrace{3y}_M, \underbrace{5 - 2x}_N, \underbrace{z^2 - 2}_P \rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3y & 5 - 2x & z^2 - 2 \end{vmatrix}$$

- $\hat{i}$ -comp:  $\partial_y(z^2 - 2) - \partial_z(5 - 2x) = 0 - 0 = 0$
- $\hat{j}$ -comp:  $\partial_z(3y) - \partial_x(z^2 - 2) = 0 - 0 = 0$
- $\hat{k}$ -comp:  $\partial_x(5 - 2x) - \partial_y(3y) = -2 - 3 = -5$

$$\nabla \times \mathbf{F} = -5\hat{k}$$

# Solution: The Integral

**2. Normal Vector  $\mathbf{N}$**  For sphere  $R = \sqrt{3}$ ,  $\mathbf{N} = (R \sin \phi) \mathbf{r}$ .

$$\mathbf{N}_z = (R \sin \phi)(z) = (R \sin \phi)(R \cos \phi) = 3 \sin \phi \cos \phi$$

**3. Setup Dot Product**

$$(\nabla \times \mathbf{F}) \cdot \mathbf{N} = (-5\hat{k}) \cdot \mathbf{N} = -5(\mathbf{N}_z) = -15 \sin \phi \cos \phi$$

**4. Evaluate**

$$\begin{aligned}\text{Flux} &= \int_0^{2\pi} \int_0^{\pi/2} -15 \sin \phi \cos \phi \, d\phi \, d\theta \\ &= 2\pi(-15) \int_0^{\pi/2} \sin \phi \cos \phi \, d\phi \quad (\text{Let } u = \sin \phi) \\ &= -30\pi \left[ \frac{\sin^2 \phi}{2} \right]_0^{\pi/2} = -30\pi \left( \frac{1}{2} \right) = -15\pi\end{aligned}$$

# Recall: Divergence Theorem (Gauss's Theorem)

## The Formula

The outward flux of  $\mathbf{F}$  across a closed surface  $S$  is equal to the triple integral of the divergence over the volume  $V$ :

$$\iint_S \mathbf{F} \cdot \hat{n} d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) dV$$

**Integration Tip (Elliptical Coords):** For

regions bounded by  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ :

- $x = ar \cos \theta$
- $y = br \sin \theta$
- $dV = abr dz dr d\theta$  (Don't forget the Jacobian!)

## When to use this?

When the surface is **closed** (e.g., bounded by coordinate planes, cylinders, and caps).

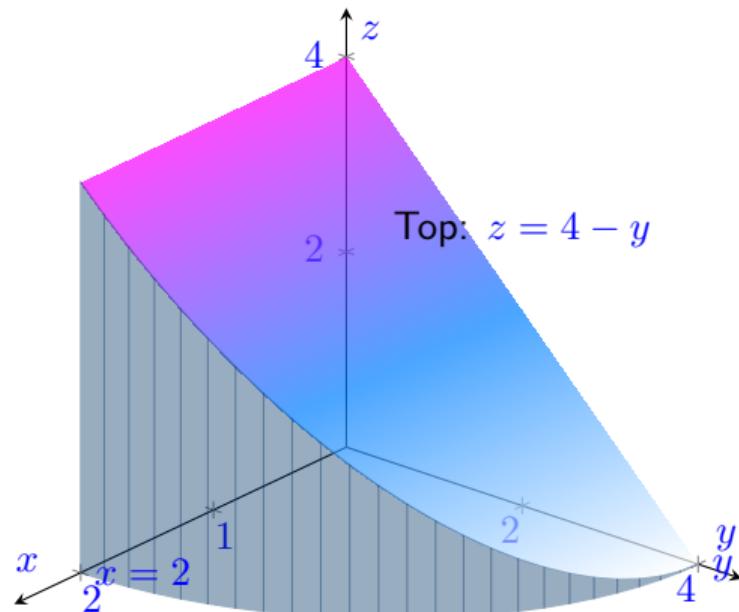
Calculating 5 separate surface integrals (4 faces + curved part) is hard. One volume integral is easy!

## Question 7

### Problem Statement

Use Divergence theorem to find the flux of  $\mathbf{F} = 2xz\hat{i} - xy\hat{j} - z^2\hat{k}$  across the boundary of the region  $D$ : the wedge cut from the first octant by the plane  $y + z = 4$ , and the elliptical cylinder  $4x^2 + y^2 = 16$ .

# Visualization (The Wedge)



# Solution: Divergence & Coordinates

## 1. Calculate Divergence

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(-z^2) \\ &= 2z - x - 2z = -x\end{aligned}$$

2. Setup Coordinate System Boundary:  $4x^2 + y^2 = 16 \implies \frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$ . Use Elliptical Coordinates:

$$x = 2r \cos \theta, \quad y = 4r \sin \theta, \quad z = z$$

$$\text{Jacobian } J = (2)(4)r = 8r$$

## 3. Determine Limits (First Octant)

- $r: 0 \rightarrow 1$  (Inside the cylinder)
- $\theta: 0 \rightarrow \pi/2$  (First Octant)
- $z: 0 \rightarrow 4 - y \implies 0 \rightarrow 4 - 4r \sin \theta$

# Solution: The Integral

## 4. Setup Triple Integral

$$\text{Flux} = \iiint_V (-x) dV = \int_0^{\pi/2} \int_0^1 \int_0^{4-4r\sin\theta} \underbrace{(-2r\cos\theta)}_{\text{Div}} \underbrace{(8r)}_{\text{Jac}} dz dr d\theta$$

## 5. Evaluate ( $z$ integral first)

$$= -16 \int_0^{\pi/2} \int_0^1 r^2 \cos\theta [z]_0^{4-4r\sin\theta} dr d\theta$$

$$= -16 \int_0^{\pi/2} \int_0^1 r^2 \cos\theta (4 - 4r\sin\theta) dr d\theta$$

$$= -64 \int_0^{\pi/2} \cos\theta \left[ \int_0^1 (r^2 - r^3 \sin\theta) dr \right] d\theta$$

# Solution: Final Evaluation

## 6. Evaluate ( $r$ and $\theta$ integrals)

$$[\dots]_r = \left[ \frac{r^3}{3} - \frac{r^4}{4} \sin \theta \right]_0^1 = \frac{1}{3} - \frac{1}{4} \sin \theta$$

Now integrate w.r.t  $\theta$ :

$$\begin{aligned}\text{Flux} &= -64 \int_0^{\pi/2} \cos \theta \left( \frac{1}{3} - \frac{1}{4} \sin \theta \right) d\theta \\ &= -64 \left[ \frac{1}{3} \sin \theta - \frac{1}{4} \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} \\ &= -64 \left( \frac{1}{3}(1) - \frac{1}{8}(1) \right) = -64 \left( \frac{8-3}{24} \right) = -64 \left( \frac{5}{24} \right) = -\frac{40}{3}\end{aligned}$$

**Final Answer:** The flux is  $-40/3$ .

# Recall: Green's Theorem (2D)

## The Theorem

For a positively oriented, simple closed curve  $C$  enclosing a region  $R$ :

$$\oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

**Why is it useful here?** Direct line integration is hard when terms like  $\tan(y/2)$  or  $\frac{x}{x^2+1}$  are present.

However, if these terms are in the "wrong" component ( $M$  or  $N$ ), their partial derivatives often vanish!

## Calculation Strategy

1. Identify  $M$  (attached to  $dx$ ) and  $N$  (attached to  $dy$ ). 2. Compute  $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ . 3. Set up double integral limits for  $R$ .

# Question 8

## Problem Statement

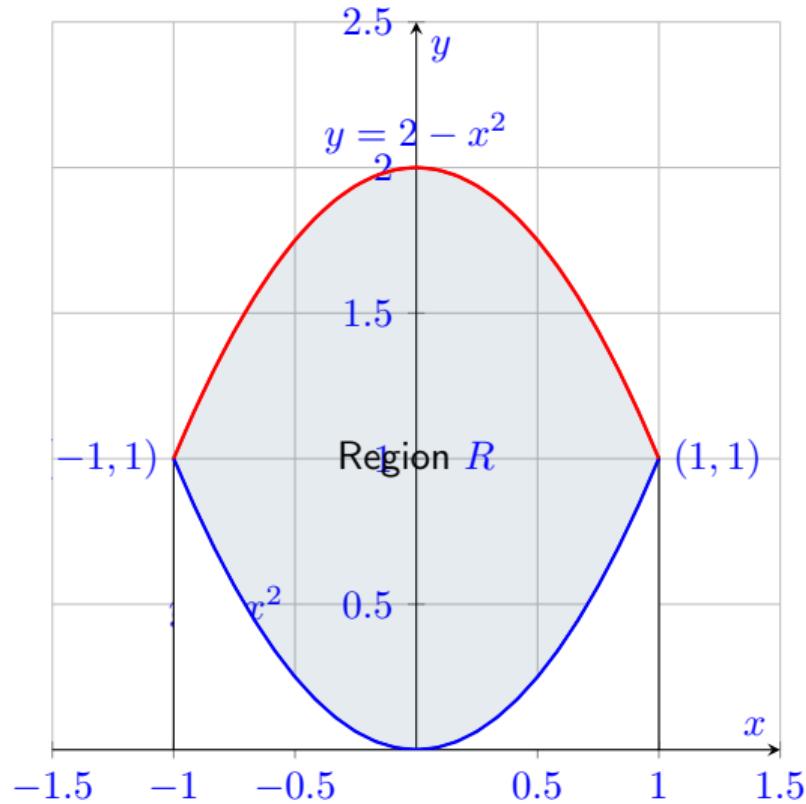
Evaluate the line integral using Green's Theorem:

$$\oint_C \left( \frac{x}{x^2 + 1} - y \right) dx + \left( 3x - 4 \tan \frac{y}{2} \right) dy$$

where  $C$  is the boundary enclosed by:

- Lower curve:  $y = x^2$  from  $(-1, 1)$  to  $(1, 1)$
- Upper curve:  $y = 2 - x^2$  from  $(1, 1)$  to  $(-1, 1)$

# Visualization Q8 (The Region R)



# Solution Q8

## 1. Partial Derivatives

$$M = \frac{x}{x^2 + 1} - y \implies \frac{\partial M}{\partial y} = -1 \quad N = 3x - 4 \tan(y/2) \implies \frac{\partial N}{\partial x} = 3$$

$$\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = 3 - (-1) = 4$$

2. Setup Double Integral Limits for  $R$ :  $x$  goes from  $-1$  to  $1$ . For each  $x$ ,  $y$  goes from  $x^2$  to  $2 - x^2$ .

$$I = \iint_R 4 \, dA = 4 \int_{-1}^1 \int_{x^2}^{2-x^2} dy \, dx$$

## 3. Evaluate

$$= 4 \int_{-1}^1 [y]_{x^2}^{2-x^2} dx = 4 \int_{-1}^1 (2 - x^2 - x^2) dx$$

$$= 4 \int_{-1}^1 (2 - 2x^2) dx = 8 \int_0^1 (2 - 2x^2) dx \quad (\text{Even function}) = 8 \left[ 2x - \frac{2x^3}{3} \right]_0^1 = \frac{32}{3}$$

# Recall: The Divergence Theorem

## Statement of the Theorem

Let  $D$  be a solid region with a closed boundary surface  $S$  oriented outward.

$$\iint_S \mathbf{F} \cdot \hat{n} d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) dV$$

**Physical Intuition:** Total outward flux through the boundary = Sum of all "sources" and "sinks" inside the volume.

## Strategy for Cylinders

If  $D$  is defined by  $x^2 + y^2 \leq R^2$ :

- ① Compute  $\nabla \cdot \mathbf{F}$ .
- ② Switch to **Cylindrical Coords.**  
That is, for  $x^2 + y^2 \leq 1$   
bounded by  $z = \pm 1$ , use:  
 $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, -1 \leq z \leq 1$
- ③ **Crucial:** Use  $dV = r dz dr d\theta$ .

## Question 9

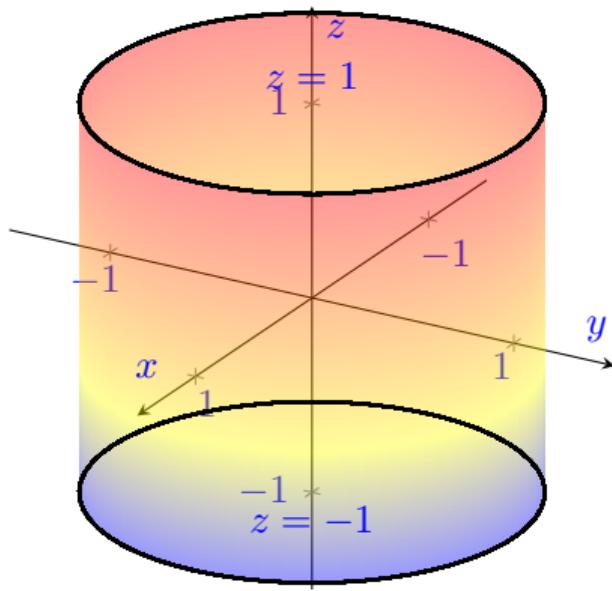
### Problem Statement

Using the Divergence Theorem, find the outward flux of the field

$$\mathbf{F} = x^2y^2\hat{i} + 2x^3y\hat{j} + y^3\hat{k}$$

across the boundary of the region enclosed by the cylinder  $x^2 + y^2 = 1$  and the planes  $z = 1$  and  $z = -1$ .

# Visualization Q9 (Cylinder)



## Solution Q9

1. Calculate Divergence:  $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(2x^3y) + \frac{\partial}{\partial z}(y^3)$   
 $= 2xy^2 + 2x^3 + 0 = 2x(y^2 + x^2)$

2. Convert to Cylindrical Coordinates:  $x = r \cos \theta, \quad x^2 + y^2 = r^2$   
 $\nabla \cdot \mathbf{F} = 2(r \cos \theta)(r^2) = 2r^3 \cos \theta$

$dV = r \, dz \, dr \, d\theta$

3. Evaluate Integral: Flux  $= \int_0^{2\pi} \int_0^1 \int_{-1}^1 (2r^3 \cos \theta)r \, dz \, dr \, d\theta$   
 $= \left( \int_{-1}^1 dz \right) \left( \int_0^1 2r^4 dr \right) \left( \int_0^{2\pi} \cos \theta d\theta \right)$   
 $= [z]_{-1}^1 \cdot \left[ \frac{2r^5}{5} \right]_0^1 \cdot [\sin \theta]_0^{2\pi}$   
 $= (2) \cdot (2/5) \cdot (0) = 0$