Laplace Transform Tricks for Improper Integrals

Prepared for class

Recall: Laplace identities (parameter $p > 0$)

For a function $f(x)$ with Laplace transform

$$\mathcal{L}\{f\}(p) = F(p) = \int_0^\infty e^{-px} f(x)\,dx,$$

the following hold (when integrals converge):

Linearity: $\mathcal{L}\{af + bg\} = aF + bG.$

Frequency shift: $\mathcal{L}\{e^{cx} f(x)\}(p) = F(p - c).$

Multiplication by x^n: $\mathcal{L}\{(-1)^n x^n f(x)\}(p) = F^{(n)}(p).$

Division by x: $\mathcal{L}\left\{\frac{f(x)}{x}\right\}(p) = \int_p^\infty F(s)\,ds.$

Key corollary (set $p = 0$): $\boxed{\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(p)\,dp.}$

5(a) $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx,\ a, b > 0$

5(b) $\displaystyle\int_0^\infty \frac{e^{-ax}\sin(bx)}{x}\,dx,\ a, b > 0$

Recall: $\boxed{\displaystyle\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(p)\,dp.}$

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Solution 5(a). Let $f(x) = e^{-ax} - e^{-bx}$. Then $F(p) = \dfrac{1}{p+a} - \dfrac{1}{p+b}$. So

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$$5(a)\ \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx,\ a,b>0 \qquad 5(b)\ \int_0^\infty \frac{e^{-ax}\sin(bx)}{x}\,dx,\ a,b>0$$

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Solution 5(b). Let $f(x) = e^{-ax}\sin(bx)$. Then

$$\int_0^\infty \frac{e^{-ax}\sin(bx)}{x}\,dx = \int_0^\infty F(p)\,dp = \int_0^\infty \frac{b}{(p+a)^2+b^2}\,dp = \arctan\frac{b}{a}.$$

6(a) $\int_0^\infty J_0(x)\,dx = 1$

Recall: $\boxed{\mathcal{L}\{J_0(x)\}(p) = \dfrac{1}{\sqrt{p^2+1}}}$

Solution. Now $\mathcal{L}\{J_0(x)\}(p) = \displaystyle\int_0^\infty e^{-px} J_0(x)\,dx = \frac{1}{\sqrt{p^2+1}}.$

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Let $p \to 0$:

$$\int_0^\infty J_0(x)\,dx = \frac{1}{\sqrt{0^2+1}} = 1.$$

6(b) $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x\cos t)\,dt$

Solution. Define $g(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos t)\,dt$. Then, interchanging integrals (absolute convergence),

$$\mathcal{L}\{g\}(p) = \frac{1}{\pi}\int_0^\pi\int_0^\infty e^{-px}\cos(x\cos t)\,dx\,dt = \frac{1}{\pi}\int_0^\pi \frac{p}{p^2+\cos^2 t}\,dt = \frac{1}{\sqrt{p^2+1}}.$$

Since $\mathcal{L}\{g\} = \mathcal{L}\{J_0\}$ and both are bounded at 0, uniqueness gives $g(x) = J_0(x)$.

$$\boxed{J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos t)\,dt.}$$

Easy proof of $\frac{1}{\pi}\int_0^\pi \frac{p}{p^2+\cos^2 t}\,dt = \frac{1}{\sqrt{p^2+1}}$

Let

$$I(p) := \frac{1}{\pi}\int_0^\pi \frac{p}{p^2+\cos^2 t}\,dt \qquad (p > 0).$$

By symmetry $\cos^2 t = \cos^2(\pi - t)$,

$$I(p) = \frac{2}{\pi}\int_0^{\pi/2} \frac{p}{p^2+\cos^2 t}\,dt.$$

Set $u = \tan t$ so that $dt = \dfrac{du}{1+u^2}$ and $\cos^2 t = \dfrac{1}{1+u^2}$:

$$I(p) = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2(1+u^2)+1}\,du = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2u^2+(p^2+1)}\,du.$$

Let $v = \dfrac{p}{\sqrt{p^2+1}}u$. Then $du = \dfrac{\sqrt{p^2+1}}{p}\,dv$ and

$$I(p) = \frac{2}{\pi}\cdot\frac{1}{\sqrt{p^2+1}}\int_0^\infty \frac{dv}{1+v^2} = \frac{2}{\pi}\cdot\frac{1}{\sqrt{p^2+1}}\cdot\frac{\pi}{2} = \boxed{\frac{1}{\sqrt{p^2+1}}}.$$

7(a) $\displaystyle\int_0^\infty \frac{\sin(xt)}{t}\,dt = \frac{\pi}{2}$ for $x > 0$

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Recall: $\boxed{\displaystyle\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(p)\,dp.}$

Solution. Let $f(t) = \sin(xt)$ (viewed as a function of t). Its Laplace transform in t is

$$F(p) = \frac{x}{p^2 + x^2}.$$

Hence, by the corollary,

$$\int_0^\infty \frac{\sin(xt)}{t}\,dt = \int_0^\infty F(p)\,dp = \int_0^\infty \frac{x}{p^2 + x^2}\,dp = \left[\arctan\left(\frac{p}{x}\right)\right]_0^\infty = \frac{\pi}{2}.$$

7(b) $\displaystyle\int_0^\infty \frac{\cos(xt)}{1+t^2}\,dt = \frac{\pi}{2}e^{-x}$ for $x > 0$

Solution. Let $f(x) = \displaystyle\int_0^\infty \frac{\cos(xt)}{1+t^2}\,dt.$

7(b) $\displaystyle\int_0^\infty \frac{\cos(xt)}{1+t^2}\,dt = \frac{\pi}{2}e^{-x}$ for $x > 0$

Solution. Let $f(x) = \displaystyle\int_0^\infty \frac{\cos(xt)}{1+t^2}\,dt$. Then

$$\mathcal{L}\{f\}(p) = \int_0^\infty \frac{p}{(1+t^2)(p^2+t^2)}\,dt = \frac{\pi}{2}\cdot\frac{1}{p+1}.$$

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$$\mathcal{L}\{f\}(p) = \int_0^\infty \frac{p}{(1+t^2)(p^2+t^2)}\,dt = \frac{\pi}{2}\cdot\frac{1}{p+1}.$$

Thus $f(x) = \mathcal{L}^{-1}\left\{\frac{\pi}{2}\cdot\frac{1}{p+1}\right\} = \frac{\pi}{2}e^{-x}$.

$$\boxed{\int_0^\infty \frac{\cos(xt)}{1+t^2}\,dt = \frac{\pi}{2}e^{-x}.}$$

7(b) Deriving $\mathcal{L}_x\{F\}(p) = \dfrac{\pi}{2(p+1)}$

Define $F(x) = \displaystyle\int_0^\infty \frac{\cos(xt)}{1+t^2}\,dt$ for $x \geq 0$.

$$\begin{aligned}
\mathcal{L}_x\{F\}(p) &= \int_0^\infty \int_0^\infty \frac{e^{-px}\cos(xt)}{1+t^2}\,dx\,dt \\
&= \int_0^\infty \frac{1}{1+t^2}\left(\int_0^\infty e^{-px}\cos(xt)\,dx\right)dt \\
&= \int_0^\infty \frac{1}{1+t^2}\cdot\frac{p}{p^2+t^2}\,dt = \int_0^\infty \frac{p\,dt}{(t^2+p^2)(t^2+1)}.
\end{aligned}$$

Partial fractions:

$$\frac{p}{(t^2+p^2)(t^2+1)} = \frac{p}{1-p^2}\left(\frac{1}{t^2+p^2} - \frac{1}{t^2+1}\right).$$

Hence, using $\displaystyle\int_0^\infty \frac{dt}{t^2+a^2} = \frac{\pi}{2a}$, $\boxed{\mathcal{L}_x\{F\}(p) = \dfrac{p}{1-p^2}\left(\dfrac{\pi}{2p} - \dfrac{\pi}{2}\right) = \dfrac{\pi}{2(p+1)}, \quad p > 0.}$

8(a) If $f(x)$ is periodic with period a, so that $f(x+a) = f(x)$, show that

$$F(p) = \frac{1}{1-e^{-ap}} \int_0^a e^{-px} f(x)\, dx.$$

Proof. Start with the definition of the Laplace transform: $F(p) = \int_0^\infty f(x) e^{-px}\, dx$. Divide the integral into equal intervals of length a:

$$F(p) = \sum_{n=0}^{\infty} \int_{na}^{(n+1)a} f(x) e^{-px}\, dx.$$

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Now substitute $u = x - na$, so that $x = u + na$ and $du = dx$:

$$F(p) = \sum_{n=0}^{\infty} \int_0^a f(u+na)e^{-p(u+na)}\,du = \sum_{n=0}^{\infty} e^{-nap} \int_0^a f(u+na)e^{-pu}\,du.$$

Since $f(x)$ is periodic with period a, we have $f(u+na) = f(u)$.

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Since $f(x)$ is periodic with period a, we have $f(u+na) = f(u)$. Therefore,

$$F(p) = \sum_{n=0}^{\infty} e^{-nap} \int_0^a f(u)e^{-pu}\, du.$$

8(a) Proof for periodic $f(x)$ with period a (Part 2)

Taking the common factor $\displaystyle\int_0^a f(u)e^{-pu}\,du$,

$$F(p) = \left(\int_0^a f(u)e^{-pu}\,du\right)\sum_{n=0}^{\infty} e^{-nap}.$$

The series $\displaystyle\sum_{n=0}^{\infty} e^{-nap}$ is geometric with ratio e^{-ap}. For $\mathrm{Re}(p) > 0$, this converges to

$$\sum_{n=0}^{\infty} e^{-nap} = \frac{1}{1-e^{-ap}}.$$

Substituting this result gives

$$F(p) = \frac{1}{1-e^{-ap}}\int_0^a f(u)e^{-pu}\,du = \boxed{\frac{1}{1-e^{-ap}}\int_0^a e^{-px}f(x)\,dx.}$$

Hence, proved.

Question 8(b): Find $F(p)$ if $f(x) = 1$ in the intervals from 0 to 1, 2 to 3, 4 to 5, etc., and $f(x) = 0$ in the remaining intervals.

Answer: Here the period is $a = 2$ and on one period,

$$f(x) = \begin{cases} 1, & x \in [0, 1), \\ 0, & x \in [1, 2). \end{cases} \qquad \int_0^2 e^{-px} f(x)\, dx = \int_0^1 e^{-px}\, dx = \frac{1 - e^{-p}}{p}.$$

Using $F(p) = \dfrac{1}{1 - e^{-2p}} \displaystyle\int_0^2 e^{-px} f(x)\, dx$, we get

$$\boxed{F(p) = \frac{1}{p(1 + e^{-p})}}.$$