Tutorial Sheet 7

Multivariable Calculus / Line Integrals & Green's Theorem

Might be completely! Verify! Ask Doubts!

November 20, 2025

Overview I

Overview II

7. Q7: Exact Differential Forms

8. Q8: Flow along paths

9. Q9: Green's Theorem

10. Q10: Area via Green's Theorem

Question 1: Problem Statement

Evaluate the line integral of the scalar field

$$f(x, y, z) = x + \sqrt{y} - z^2$$

along the piecewise smooth curve C from $(0, 0, 0)$ to $(1, 1, 1)$, where C consists of the following segments:

- $C_1 : \vec{r}(t) = t\hat{k}, \quad 0 \leq t \leq 1$
- $C_2 : \vec{r}(t) = t\hat{j} + \hat{k}, \quad 0 \leq t \leq 1$
- $C_3 : \vec{r}(t) = t\hat{i} + \hat{j} + \hat{k}, \quad 0 \leq t \leq 1$

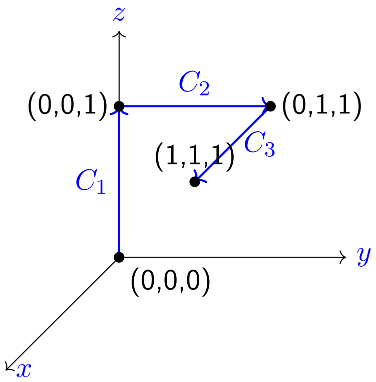
Recall: Scalar Line Integral

Formula

$$\int_C f(x, y, z)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

Strategy: We will calculate the integral for each segment (C_1, C_2, C_3) individually and sum them up to get the total integral over C.

Question 1: Visualization



Question 1: Solution (Segments 1 & 2)

Segment C_1: $\vec{r}(t) = t\hat{k}$ $(0 \leq t \leq 1)$.

- $\vec{v}(t) = \vec{r}'(t) = \hat{k} \implies |\vec{v}(t)| = 1$.
- On C_1: $x = 0, y = 0, z = t$.
- $f(x, y, z) = 0 + \sqrt{0} - t^2 = -t^2$.

$$\int_{C_1} f\, ds = \int_0^1 (-t^2)(1)\, dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3}$$

Question 1: Solution (Segments 1 & 2)

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- On C_1: $x = 0, y = 0, z = t$.
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$$\int_{C_1} f\, ds = \int_0^1 (-t^2)(1)\, dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3}$$

Segment C_2: $\vec{r}(t) = t\hat{j} + \hat{k}$ $(0 \le t \le 1)$.

- $\vec{v}(t) = \hat{j} \implies |\vec{v}(t)| = 1$.
- On C_2: $x = 0, y = t, z = 1$.
- $f(x, y, z) = 0 + \sqrt{t} - 1^2 = \sqrt{t} - 1$.

$$\int_{C_2} f\, ds = \int_0^1 (\sqrt{t} - 1)\, dt = \left[\frac{2}{3}t^{3/2} - t\right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3}$$

Question 1: Solution (Segment 3 & Total)

Segment C_3: $\vec{r}(t) = t\hat{i} + \hat{j} + \hat{k}$ $(0 \leq t \leq 1)$.

- $\vec{v}(t) = \hat{i} \implies |\vec{v}(t)| = 1$.
- On C_3: $x = t, y = 1, z = 1$.
- $f(x, y, z) = t + \sqrt{1 - 1^2} = t$.

$$\int_{C_3} f\, ds = \int_0^1 (t)\, dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

Segment C_3: $\vec{r}(t) = t\hat{i} + \hat{j} + \hat{k}$ $(0 \leq t \leq 1)$.

- $\vec{v}(t) = \hat{i} \implies |\vec{v}(t)| = 1$.
- On C_3: $x = t, y = 1, z = 1$.
- $f(x, y, z) = t + \sqrt{1 - 1^2} = t$.

$$\int_{C_3} f\,ds = \int_0^1 (t)\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

Total Integral:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds$$

$$= \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) + \frac{1}{2} = -\frac{2}{3} + \frac{1}{2} = \frac{-4+3}{6} = -\frac{1}{6}$$

Question 2: Problem Statement

Evaluate

$$\int_C (4x^2 + y^2)\, dy$$

where C is the ellipse $4x^2 + y^2 = 4$ oriented counterclockwise.

Question 2: Solution

Step 1: Parametrization Rewrite the ellipse equation as $x^2 + \frac{y^2}{4} = 1$. We can set:

$$x = \cos t, \quad y = 2\sin t, \quad 0 \leq t \leq 2\pi$$

Then, the differential dy is:

$$dy = \frac{d}{dt}(2\sin t)\,dt = 2\cos t\,dt$$

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Step 2: Substitution Substitute x and y into the integrand $4x^2 + y^2$:

$$4x^2 + y^2 = 4(\cos^2 t) + (2\sin t)^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$$

Step 3: Integration

$$\int_C (4x^2 + y^2)\, dy = \int_0^{2\pi} (4)(2\cos t\, dt) = 8\int_0^{2\pi} \cos t\, dt$$

$$= 8[\sin t]_0^{2\pi} = 8(0 - 0) = 0$$

Question 3: Problem Statement

Find the work done by the force field

$$\vec{F}(x,y) = x^2 e^y \hat{i} + y e^x \hat{j}$$

in moving a particle along the curve C given by $y = x^3$ from the point $(0,0)$ to $(1,1)$. That is, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

Question 3: Solution (Setup)

Parametrization: Since $y = x^3$, let $x = t$. Then $y = t^3$ for $0 \leq t \leq 1$.

- $dx = dt$
- $dy = 3t^2\,dt$

Substitute into Integral:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (x^2 e^y\,dx + ye^x\,dy)$$

$$= \int_0^1 \left[(t^2 e^{t^3}) dt + (t^3 e^t)(3t^2 dt) \right]$$

$$= \int_0^1 t^2 e^{t^3}\,dt + \int_0^1 3t^5 e^t\,dt$$

Question 3: Solution (Calculation)

First Term: $\int_0^1 t^2 e^{t^3} dt$ Let $u = t^3 \implies du = 3t^2 dt \implies t^2 dt = du/3$.

$$\int_0^1 t^2 e^{t^3} dt = \frac{1}{3} \int_0^1 e^u du = \frac{1}{3} [e^u]_0^1 = \frac{e-1}{3}$$

Second Term: $3 \int_0^1 t^5 e^t dt$ Using tabular integration or reduction ($I_n = e - nI_{n-1}$), we evaluate $\int_0^1 t^5 e^t dt$:

$$= [e^t (t^5 - 5t^4 + 20t^3 - 60t^2 + 120t - 120)]_0^1$$

$$= e(1 - 5 + 20 - 60 + 120 - 120) - (1)(0 - 0 + 0 - 0 + 0 - 120)$$

$$= e(-44) - (-120) = 120 - 44e$$

Multiplying by 3: $3(120 - 44e) = 360 - 132e$.

Total Work:

$$W = \frac{e-1}{3} + 360 - 132e = \frac{e - 1 + 1080 - 396e}{3} = \frac{1079 - 395e}{3}$$

Recall: Conservative Vector Fields

A field $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative if $\nabla \times \vec{F} = \vec{0}$.

Conditions

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

If conservative, $\vec{F} = \nabla f$. Fundamental Theorem of Line Integrals: $\int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$.

Question 4: Problem Statement

Let $\vec{F}(x, y, z) = y(\cos(xy) + z)\hat{i} + (zx + z\cos(yz) + x\cos(xy))\hat{j} + y(x + \cos(yz))\hat{k}$.

ⓐ Show that $\vec{F}$ is conservative in $\mathbb{R}^3$, and hence find a potential function f such that $f(1, 1, 1) = 0$.

ⓑ Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the piecewise linear path from $(0, 0, 0)$ to $(1, 1, 0)$ and then to $(1, 1, 1)$.

Question 4(a): Checking Conservative Property

Identify components:

$$\begin{aligned} M &= y\cos(xy) + yz \\ N &= zx + z\cos(yz) + x\cos(xy) \\ P &= xy + y\cos(yz) \end{aligned}$$

Check Partials:

- $M_y = \cos(xy) - xy\sin(xy) + z$
- $N_x = z + \cos(xy) - xy\sin(xy)$ ✓Match
- $M_z = y, \quad P_x = y$ ✓Match
- $N_z = x + \cos(yz) - yz\sin(yz)$
- $P_y = x + \cos(yz) - yz\sin(yz)$ ✓Match

Thus, $\vec{F}$ is conservative.

Question 4(a): Finding Potential f

We integrate $\nabla f = \vec{F}$ component by component.

1. Integrate $f_z = P = xy + y\cos(yz)$ w.r.t z:

$$f(x,y,z) = xyz + \sin(yz) + g(x,y)$$

2. Differentiate w.r.t x and match with M:

$$f_x = yz + g_x(x,y) = y\cos(xy) + yz$$

$$\implies g_x = y\cos(xy) \implies g(x,y) = \sin(xy) + h(y)$$

3. Current f: $f = xyz + \sin(yz) + \sin(xy) + h(y)$.
4. Differentiate w.r.t y and match with N:

$$f_y = xz + z\cos(yz) + x\cos(xy) + h'(y) = N$$

All terms match N, so $h'(y) = 0 \implies h(y) = C$.

$$f(x,y,z) = \sin(xy) + xyz + \sin(yz) + C$$

Using $f(1,1,1) = 0$: $\sin(1) + 1 + \sin(1) + C = 0 \implies C = -(1 + 2\sin(1))$

Question 4(b): Evaluating Integral

Since $\vec{F}$ is conservative, the line integral is path independent.

$$\int_{(0,0,0)}^{(1,1,1)} \vec{F} \cdot d\vec{r} = f(1,1,1) - f(0,0,0)$$

We are given $f(1,1,1) = 0$.
Now evaluate $f(0,0,0)$:

$$f(0,0,0) = \sin(0) + 0 + \sin(0) + C = C$$

$$\text{Integral} = 0 - C = -(-(1 + 2\sin(1)))$$

Answer

$$1 + 2\sin(1)$$

Question 5: Problem Statement

Find all $a, b \in \mathbb{R}$ such that the vector field

$$\vec{F}(x,y,z) = \ln\left(1 + y^2 + z^2\right)\hat{i} + \frac{(b - a^2)xy}{1 + y^2 + z^2}\hat{j} + \frac{axz}{1 + y^2 + z^2}\hat{k}$$

is conservative in $\mathbb{R}^3$, and hence find a potential function for $\vec{F}$.

Question 5: Solution (Finding Constants)

For $\vec{F}$ to be conservative, $\nabla \times \vec{F} = \vec{0}$.

Condition 1: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$M_y = \frac{2y}{1+y^2+z^2}, \quad N_x = \frac{(b-a^2)y}{1+y^2+z^2}$$

$$\implies 2 = b - a^2$$

Condition 2: $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$

$$M_z = \frac{2z}{1+y^2+z^2}, \quad P_x = \frac{az}{1+y^2+z^2}$$

$$\implies 2 = a$$

Substitute $a = 2$ into first equation: $b - 4 = 2 \implies b = 6$.

Constants

$a = 2, \quad b = 6$

Question 5: Solution (Potential Function)

Using $a = 2, b = 6$, we integrate to find $\phi(x, y, z)$.

$$\phi_x = \ln\left(1 + y^2 + z^2\right)$$

$$\phi = \int \ln\left(1 + y^2 + z^2\right) dx = x \ln\left(1 + y^2 + z^2\right) + g(y, z)$$

Differentiating w.r.t y:

$$\phi_y = \frac{2xy}{1 + y^2 + z^2} + g_y$$

Matches $N = \frac{2xy}{1+y^2+z^2}$, so $g_y = 0$. Similarly $g_z = 0$, so $g(y, z) = K$.

Potential Function

$$\phi(x, y, z) = x \ln\left(1 + y^2 + z^2\right) + K$$

Question 6: Problem Statement

Find the outward flux of the fields

$$\vec{F}_1 = 2x\hat{i} - 3y\hat{j} \quad \text{and} \quad \vec{F}_2 = 2x\hat{i} + (x - y)\hat{j}$$

across the circle $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}, \quad 0 \le t \le 2\pi$.

Question 6: Solution (Flux of $\vec{F}_1$)

Formula: Flux $= \oint_C (M\,dy - N\,dx)$. Here $x = a\cos t, y = a\sin t$.
For $\vec{F}_1$: $M = 2x, N = -3y$.

$$\text{Flux} = \int_0^{2\pi} [(2a\cos t)(a\cos t dt) - (-3a\sin t)(-a\sin t dt)]$$

$$= \int_0^{2\pi} (2a^2\cos^2 t - 3a^2\sin^2 t)dt$$

$$= a^2 \int_0^{2\pi} (2\cos^2 t - 3\sin^2 t)dt$$

Using $\int_0^{2\pi} \cos^2 t dt = \pi$ and $\int_0^{2\pi} \sin^2 t dt = \pi$:

$$= a^2(2\pi - 3\pi) = -\pi a^2$$

Question 6: Solution (Flux of $\vec{F_2}$)

For $\vec{F}_2$: $M = 2x, N = x - y$.

$$\text{Flux} = \int_0^{2\pi} [(2a\cos t)(a\cos t) - (a\cos t - a\sin t)(-a\sin t)]dt$$

$$= a^2 \int_0^{2\pi} (2\cos^2 t + \sin t\cos t - \sin^2 t)dt$$

Integrate term by term:

- $\int 2\cos^2 t dt = 2\pi$
- $\int \sin t \cos t dt = 0$ (periodicity)
- $\int -\sin^2 t dt = -\pi$

$$\text{Flux} = a^2(2\pi + 0 - \pi) = \pi a^2$$

Question 7: Problem Statement

Show that the differential form in the integral is exact, and then evaluate:

$$\int_{(1,1,2)}^{(3,5,0)} \sin z \, dx + z^2 e^{yz^2} \, dy + \left(x \cos z + 2yze^{yz^2}\right) dz$$

Step 1: Exactness Let $\vec{F} = \langle \sin z, \quad z^2 e^{yz^2}, \quad x\cos z + 2yze^{yz^2} \rangle$. Check curl components:

- $M_z = \cos z, \quad P_x = \cos z$ ✓
- $N_z = 2ze^{yz^2} + z^2(2yz)e^{yz^2}, \quad P_y = 2ze^{yz^2} + 2yz(z^2)e^{yz^2}$ ✓
- $M_y = 0, \quad N_x = 0$ ✓

Step 2: Potential Function $f_x = \sin z \implies f = x\sin z + g(y,z)$.
$f_y = g_y = z^2 e^{yz^2} \implies g = e^{yz^2} + h(z)$. $f_z = x\cos z + 2yze^{yz^2} + h'(z)$. Comparing with P, we see $h'(z) = 0 \implies h(z) = C$.

$$f(x,y,z) = x\sin z + e^{yz^2}$$

Step 3: Evaluation

$$f(3,5,0) - f(1,1,2) = (3\sin(0) + e^0) - (1\sin(2) + e^4)$$

$$= 1 - (\sin 2 + e^4)$$

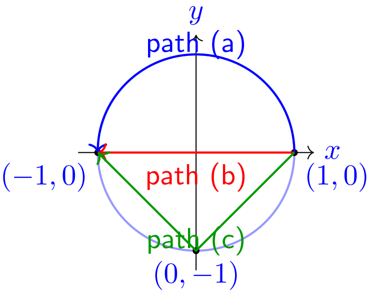
Question 8: Problem Statement

Find the flow of the velocity field $\vec{F} = (x+y)\hat{i} - (x^2+y^2)\hat{j}$ along each of the following paths from $(1,0)$ to $(-1,0)$ in the xy-plane:

(a) The upper half of the circle $x^2+y^2=1$.

(b) The line segment from $(1,0)$ to $(-1,0)$.

(c) The line segment from $(1,0)$ to $(0,-1)$ followed by the line segment from $(0,-1)$ to $(-1,0)$.



Question 8: Solution (a)

Path (a): Upper Semicircle $\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j}, \quad 0 \leq t \leq \pi$.

$$\vec{F} = (\cos t + \sin t)\hat{i} - (1)\hat{j}$$

$$d\vec{r} = (-\sin t\hat{i} + \cos t\hat{j})dt$$

Dot Product:

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= [(\cos t + \sin t)(-\sin t) - (1)(\cos t)]dt \\ &= [-\sin t \cos t - \sin^2 t - \cos t]dt\end{aligned}$$

Integrate from 0 to π:

$$\begin{aligned}&\int_0^\pi (-\frac{1}{2}\sin 2t - \sin^2 t - \cos t)dt \\ &= 0 - \int_0^\pi \sin^2 t dt - 0 = -\frac{\pi}{2}\end{aligned}$$

Question 8: Solution (b)

Path (b): Line Segment Along the x-axis from 1 to -1: $y = 0, dy = 0$. $r(t) = t\hat{i}$.

$$\vec{F} = x\hat{i} - x^2\hat{j}$$

$$d\vec{r} = dx\hat{i}$$

$$\int \vec{F} \cdot d\vec{r} = \int_1^{-1} x\, dx$$

$$= \left[\frac{x^2}{2}\right]_1^{-1} = \frac{(-1)^2}{2} - \frac{1^2}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

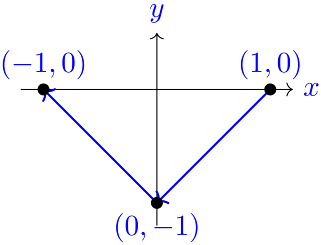
Question 8: Solution (c)

Path (c): Two Segments

- C_1: $(1,0) \to (0,-1)$. $\vec{r}_1 = (1-t)\hat{i} - t\hat{j}$.
- C_2: $(0,-1) \to (-1,0)$. $\vec{r}_2 = -t\hat{i} + (t-1)\hat{j}$.

Integrate C_1**:** $\vec{F} \cdot d\vec{r}_1 = \cdots = 2t^2 dt$.

$$\int_0^1 2t^2 dt = 2/3$$



Integrate C_2**:** $\vec{F} \cdot d\vec{r}_2 = \cdots = (2t - 2t^2)dt$.

$$\int_0^1 (2t - 2t^2)dt = 1 - 2/3 = 1/3$$

Total Flow: $2/3 + 1/3 = 1$.

Recall: Green's Theorem

Theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Question 9: Problem Statement

Use Green's theorem to evaluate the integral

$$I = \oint_C \left(\frac{x}{x^2+1} - y \right) dx + \left(3x - 4\tan\frac{y}{2} \right) dy$$

where C is the portion of $y = x^2$ from $(-1, 1)$ to $(1, 1)$, followed by the portion of $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$.

Question 9: Solution

Identify $M = \frac{x}{x^2+1} - y$ and $N = 3x - 4\tan\frac{y}{2}$.
Calculate partials:

$$\frac{\partial N}{\partial x} = 3, \quad \frac{\partial M}{\partial y} = -1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3 - (-1) = 4$$

Using Green's Theorem:

$$I = \iint_R 4\,dA = 4 \times \text{Area}(R)$$

Question 9: Area Calculation

Region R: Bounded by $y = x^2$ (bottom) and $y = 2 - x^2$ (top) between $x = -1$ and $x = 1$.

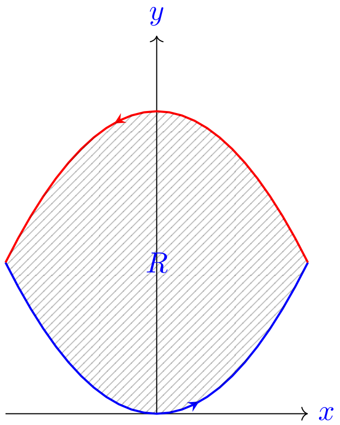
$$\text{Area} = \int_{-1}^{1} [(2 - x^2) - x^2] dx$$

$$= \int_{-1}^{1} (2 - 2x^2) dx$$

$$= 2 \int_{0}^{1} (2 - 2x^2) dx$$

$$= 4[x - \frac{x^3}{3}]_0^1 = 4(1 - 1/3) = \frac{8}{3}$$

$$\text{Total Integral} = 4 \times \frac{8}{3} = \frac{32}{3}$$



Recall: Area Formula via Green's Theorem

Green's Theorem: $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$

To find the **Area (A)** of region R, we want the double integral to become $\iint_R 1\,dA$. We need to choose M and N such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$.

Possible choices:

- Let $M = -y/2, N = -x/2 \implies \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 \implies A = \oint_C x\,dy$.
- Let $M = -y, N = 0 \implies \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 - (-1) = 1 \implies A = -\oint_C y\,dx$.

Symmetric Form (Average of the two):

$$A = \frac{1}{2} \left(\oint_C x\,dy + \oint_C (-y)\,dx \right) = \frac{1}{2} \oint_C (x\,dy - y\,dx)$$

Question 10: Problem Statement

Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

$$y = x, \quad y = \frac{1}{x}, \quad y = \frac{x}{4}$$

Question 10: Solution (Setup)

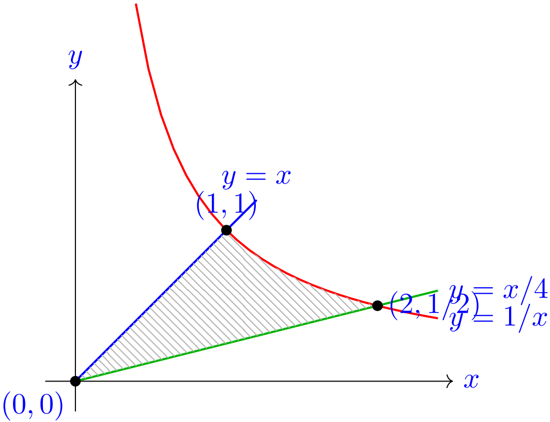
Area Formula via Green's Theorem: $A = \frac{1}{2} \oint_C (xdy - ydx)$.

Intersection Points:

1. $y = x$ and $y = 1/x \implies x^2 = 1 \implies (1, 1)$.
2. $y = x/4$ and $y = 1/x \implies x^2 = 4 \implies (2, 1/2)$.
3. $y = x$ and $y = x/4 \implies (0, 0)$.

The boundary C consists of three segments.

Question 10: Visualizing the Region



Question 10: Solution (Integration)

1. Along $y = x/4$ **from** 0 **to** 2**:** $dy = \frac{1}{4}dx$. Integrand: $x(\frac{1}{4}dx) - (\frac{x}{4})dx = 0$.
2. Along $y = 1/x$ **from** 2 **to** 1**:** $x = 1/y \implies dx = -dy/y^2$. Integrand: $(\frac{1}{y})dy - y(-\frac{dy}{y^2}) = \frac{2}{y}dy$.

$$\int_{1/2}^{1} \frac{2}{y}dy = [2\ln y]_{1/2}^{1} = 2(0 - \ln(1/2)) = 2\ln 2$$

3. Along $y = x$ **from** 1 **to** 0**:** $dy = dx$. Integrand $xdx - xdx = 0$.
Total Area:

$$A = \frac{1}{2}(0 + 2\ln 2 + 0) = \ln 2$$

Answer

$$\ln 2$$

Question 10: Alternative Method (Area $= \oint x\,dy$)

We can also use the formula $A = \oint_C x\,dy$.

1. Along $y = x/4$ **from** 0 **to** 2 **(**C_1**):** $y = t/4 \implies dy = dt/4$. $x = t$. Limits $0 \to 2$.

$$\int_0^2 t\left(\frac{1}{4}dt\right) = \frac{1}{4}\left[\frac{t^2}{2}\right]_0^2 = \frac{1}{4}(2) = \frac{1}{2}$$

2. Along $y = 1/x$ **(**C_2**):** From $(2, 1/2)$ to $(1, 1)$. $x = 1/y \implies \oint(1/y)dy$. Limits $y : 1/2 \to 1$.

$$\int_{1/2}^1 \frac{1}{y}dy = [\ln y]_{1/2}^1 = 0 - (-\ln 2) = \ln 2$$

3. Along $y = x$ **(**C_3**):** From $(1, 1)$ to $(0, 0)$. $y = t \implies dy = dt$. $x = t$. Limits $1 \to 0$.

$$\int_1^0 t\,dt = \left[\frac{t^2}{2}\right]_1^0 = 0 - \frac{1}{2} = -\frac{1}{2}$$

Total Area: $\frac{1}{2} + \ln 2 - \frac{1}{2} = \ln 2$.

Thank You