# Sine & Cosine at Key Angles

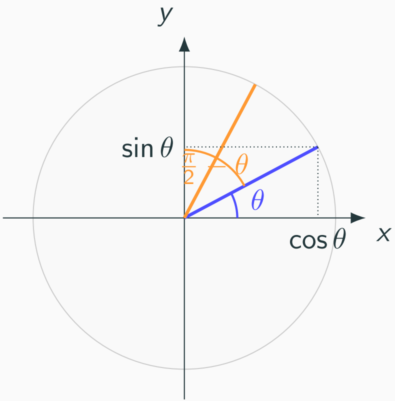
with Cofunction Identities

(dbhoriya.github.io/teaching_F101/tutorial_2.pdf) or SCAN:

## Some Trigonometry identities:

### Exact Values at Common Angles

| $\theta$ | $0^\circ$ | $30^\circ$ | $45^\circ$ | $60^\circ$ | $90^\circ$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |



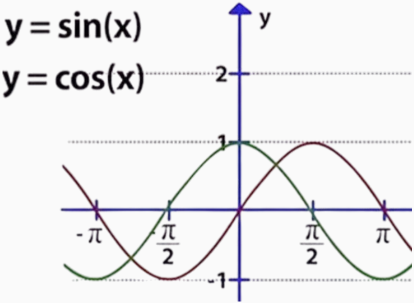

### Some identities:

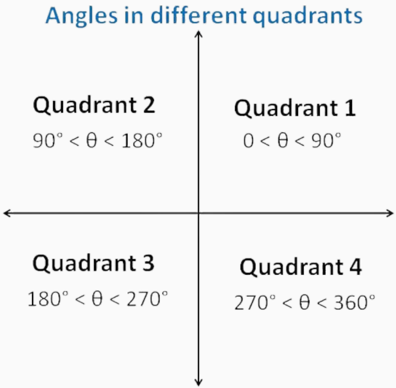
$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta.$$

Example: $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}, \quad \cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}.$

# Sin-Cos in various quadrants



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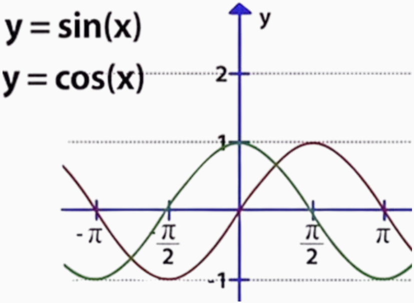


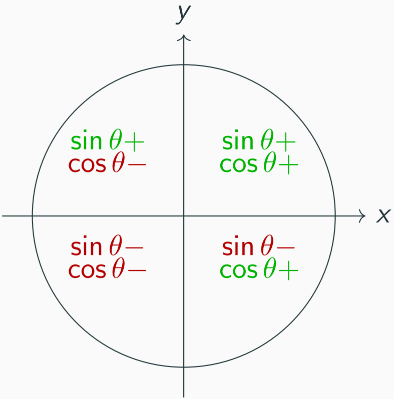

# Polar co-ordinates

Definition and motivation

# Polar Coordinates: Definition & Formulas

- A point $(x, y)$ is represented as $(r, \theta)$:
  - $r =$ distance from origin
  - $\theta =$ angle from positive x-axis
- Conversion to Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

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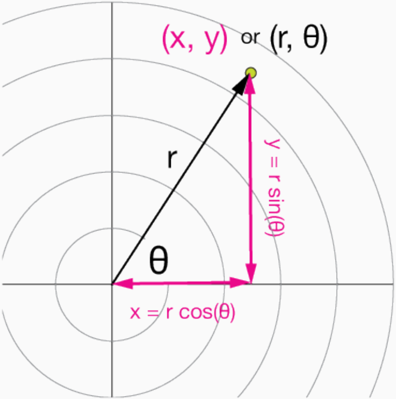



$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x} = \tan^{-1}\left(\frac{y}{x}\right)$$

# Why Do We Need Polar Coordinates? Real-Life Motivation

- Some situations are easier to describe with **distance + angle** than $(x, y)$.
- **Examples:**
  - GPS: “5 km at 30° NE”
  - Radar/Sonar: Track objects by angle & range
  - Lighthouses: Light beam at a specific distance & direction
  - CCTV: Camera coverage via rotation & reach

## Engineering & Math Usefulness

- **Engineering:**
  - AC circuits: voltages/currents as polar vectors
  - Mechanical systems: gears, turbines, rotating parts
  - Robotics: arm positions via joint angles
  - Civil structures: domes, arches, tunnels
- **Mathematics:**
  - Circle equation: $x^2 + y^2 = a^2 \Rightarrow r = a$ (simpler in polar)
  - Ideal for problems with circular or rotational symmetry

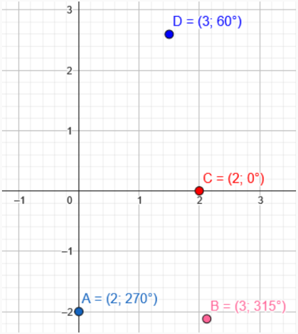
# Polar Coordinates: Tutorial Sheet-2

**Question 1:** Plot the following points, given in polar coordinates, and find all polar coordinates for each point.

**A:** $(-2, \pi/2)$, **C:** $(-2, -5\pi)$

**B:** $(3, -\pi/4)$, **D:** $(3, \pi/3)$

*Hint: Remember that a point $(r, \theta)$ is equivalent to $(-r, \theta+\pi)$ and that angles can be expressed modulo $2\pi$.*




**Answer:**

**A:** $(-2, \pi/2 + 2n\pi), (2, -\pi/2 + 2n\pi)$ where $n \in \mathbb{Z}$

**B:** $(3, -\pi/4 + 2n\pi), (-3, 3\pi/4 + 2n\pi)$ where $n \in \mathbb{Z}$.

**C:** $(-2, -\pi + 2n\pi), (2, 2n\pi)$ where $n \in \mathbb{Z}$.

**D:** $(3, \pi/3 + 2n\pi), (-3, -2\pi/3 + 2n\pi)$ where$n \in \mathbb{Z}$.

# Polar Coordinates: Tutorial Sheet-2

**Question 2:** Find the polar coordinates of the following points.

**(a)** $(\sqrt{3}, -1)$, with $0 \leq \theta < 2\pi$ and $r \leq 0$.

**(b)** $(\sqrt{3}, -1)$, with $-\pi \leq \theta < \pi$ and $r \geq 0$.

**Answer:** **(a)** $(-2, \pi - \pi/6)$ **(b)** $(2, -\pi/6)$

**Question 3:** Convert the following into Cartesian equations:

**(a)** $r^2 + 2r^2 \cos\theta \sin\theta = 1$ **(b)** $r \sin\theta = \ln r + \ln \cos\theta$.

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**Answer:**

**(a)** $x^2 + y^2 + 2rcos\theta r\sin\theta = 1 \iff x^2 + y^2 + 2xy = 1$

**(b)** $r\sin\theta = \ln r + \ln\cos\theta \implies r\sin\theta = \ln r\cos\theta$.

This gives $y = \ln x$

**Question 5:** Sketch the region defined by the inequalities
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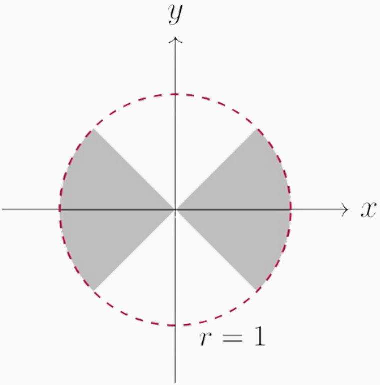
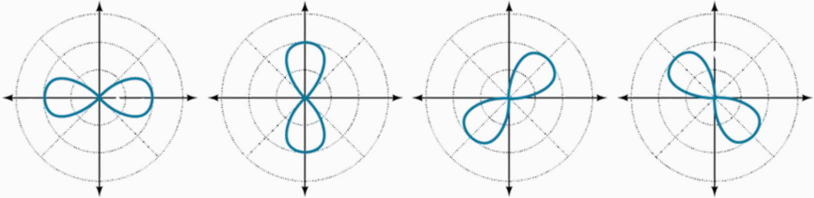


## Polar Coordinates: lemniscate (a polar curve resembling the infinity symbol $\infty$)



$r^2 = a^2 cos(2\theta)$ (a) $r^2 = -a^2 cos(2\theta)$ (b) $r^2 = a^2 sin(2\theta)$ (c) $r^2 = -a^2 sin(2\theta)$ (d)

**Question 4:** Draw the graph of the curve $r^2 = -\sin 2\theta$.

**Answer:**

- Equation $r^2 = -\sin(2\theta) \implies \sin(2\theta) \le 0.$
- Therefore, $\pi \le 2\theta \le 2\pi$. That is $\pi/2 \le \theta \le \pi$.
- Also, the graph is symmetric about the origin because if $(r, \theta)$ lies on the graph, then $(-r, \theta)$ also lies on the graph.
- At origin, $r = 0$. Therefore $\sin(2\theta) = 0$. We get $2\theta = \pi, 2\pi$. Therefore $\theta = \pi/2, \pi$
- Slopes:

**Answer:**

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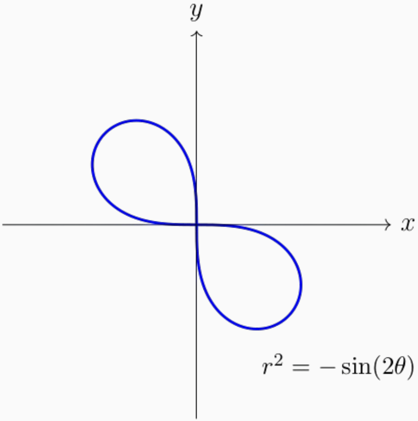
- Also, the graph is symmetric about the origin because if $(r, \theta)$ lies on the graph, then $(-r, \theta)$ also lies on the graph.
- At origin, $r = 0$. Therefore $\sin(2\theta) = 0$. We get $2\theta = \pi, 2\pi$. Therefore $\theta = \pi/2, \pi$
- Slopes: $dr/d\theta = -\frac{\cos 2\theta}{\sqrt{-\sin 2\theta}}$
- At $\theta = \pi/2$, it has a vertical tangent since slope is infinite, and at $\theta = \pi$, it has a horizontal slope $= 0$.

**Answer:**



**Recall:** Area and length of the curve in polar coordinates:

- The formula for area in polar coordinates:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

- The length of a curve given in polar coordinates is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

**Question 6:** Find the area inside the lemniscate $r^2 = 6cos(2\theta)$ and outside the circle $r = \sqrt{3}$.

**Answer:** At $\theta = 0$, $r^2 = 6cos(2\theta)$ gives $r = \pm 3$.

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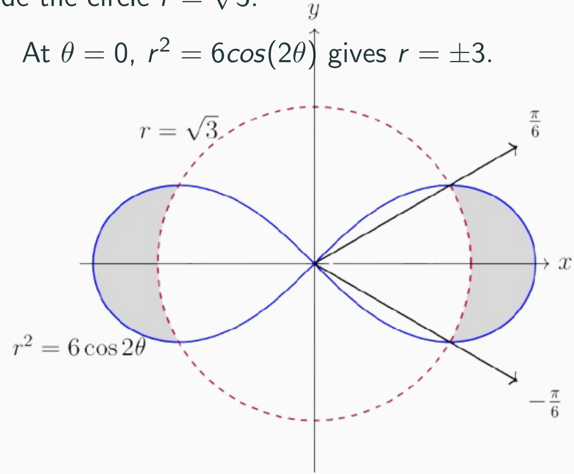


Intersection at: $6cos(2\theta) = 3 \implies \theta = \pm\pi/6$.

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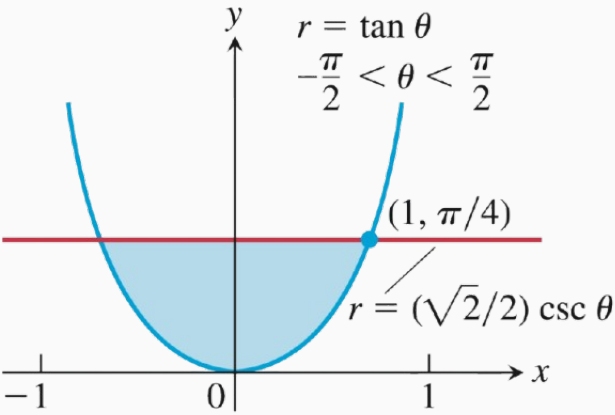
- Intersection at: $6cos(2\theta) = 3 \implies \theta = \pm\pi/6$.

By symmetry, the required area is

$$\begin{aligned} \mathrm{A} &= 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{6}} \left(6\cos 2\theta - (\sqrt{3})^2\right) d\theta \\ &= 2\left[3\sin 2\theta - 3\theta\right]_0^{\frac{\pi}{6}} \\ &= 3\sqrt{3} - \pi. \end{aligned}$$

**Question 7:** Find the area of the shaded region in the following figure:




**Answer:** $\text{Area} = 2\left[\int_0^{\pi/4} \frac{1}{2}\tan^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\left(\frac{\sqrt{2}}{2}\csc\theta\right)^2 d\theta\right]$

**Question 7:** Find the area of the shaded region in the following figure:

**Answer:**

$$\begin{aligned}
\text{Area} &= 2\left[\int_0^{\pi/4} \frac{1}{2}\tan^2\theta d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\left(\frac{\sqrt{2}}{2}\csc\theta\right)^2 d\theta\right] \\
&= \int_0^{\pi/4} \tan^2\theta d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\csc^2\theta d\theta \\
&= [-\theta + \tan\theta]_0^{\pi/4} + \frac{1}{2}\left[-\cot\theta\right]_{\pi/4}^{\pi/2} \\
&= \frac{-\pi}{4} + 1 + \frac{1}{2}[0+1] \\
&= \frac{3}{2} - \frac{\pi}{4} \text{ sq. units}
\end{aligned}$$

**Question 8:** Find the length of the curve given by $r = \sqrt{1 - \sin 2\theta}$ where $\theta$ lies in the interval $[0, \pi/2]$.

**Answer:** The length of a curve given in polar coordinates is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

- Since $r = \sqrt{1 - \sin 2\theta}$,

$$\frac{dr}{d\theta} = \frac{1}{2\sqrt{1 - \sin 2\theta}} \cdot (-2\cos 2\theta) = \frac{-\cos 2\theta}{\sqrt{1 - \sin 2\theta}}.$$

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**Answer:**

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$$\frac{dr}{d\theta} = \frac{1}{2\sqrt{1 - \sin 2\theta}} \cdot (-2\cos 2\theta) = \frac{-\cos 2\theta}{\sqrt{1 - \sin 2\theta}}.$$

Therefore,

$$\begin{aligned} \mathrm{L} &= \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin 2\theta + \frac{\cos^2 2\theta}{1 - \sin 2\theta}}\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1 - \sin 2\theta)^2 + \cos^2 2\theta}{1 - \sin 2\theta}}\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{2(1 - \sin 2\theta)}{1 - \sin 2\theta}}\, d\theta = \frac{\pi}{2}\sqrt{2}. \end{aligned}$$