ODE notes

- Gauss's Hypergeometric Equation
- Special functions:
  - Legendre Polynomials
  - Bessel functions
  - Gamma functions

Solving a Differential Equation using Hypergeometric Functions

Gauss's Hypergeometric Equation (GHE)

The differential equation

$$x(1-x)\,y'' + \left[c-(a+b+1)x\right]y' - ab\,y = 0 \tag{1}$$

where a, b, and c are constants, is called Gauss's Hypergeometric equation (GHE).

Learnings:?

- We will find the general solution of GHE at its regular points.
- As an application, in particular, we find the general solutions of the equation

$$(x-A)(x-B)\,y'' + (C+Dx)\,y' + Ey = 0$$

where $A \neq B$ by using the general solution of GHE.

Singularities

If we simplify GHE in the form of a standard second-order linear equation, we obtain

$$P(x) = \frac{c - [a+b+1]x}{x(1-x)} \quad \text{and} \quad Q(x) = \frac{-abx}{x(1-x)}.$$

What are the singular points: $x = 0$ and $x = 1$ are only singular points.
Are these regular points? Yes, $x = 0$ and $x = 1$ are regular points, since

$$\begin{aligned} xP(x) &= \frac{c - [a+b+1]x}{(1-x)} = [c - (a+b+1)x]\,(1 + x + x^2 + \cdots) \\ &= c + [c - (a+b+1)]x + \cdots \end{aligned}$$

and

$$x^2 Q(x) = \frac{-abx}{1-x} = -abx(1 + x + x^2 + \cdots) = -abx - abx^2 - \cdots .$$

Similarly, we can show that $x = 1$ is a regular point. Aim to find the general solutions of GHE at $x = 0$ and $x = 1$.

General solution at the regular point $x = 0$

- From the above expansions of $xP(x)$ and $x^2Q(x)$, $p_0 = c$ and $q_0 = 0$ so the indicial equation is

$$m(m-1) + mp_0 + q_0 = 0 \quad \Rightarrow \quad m[m-(1-c)] = 0$$

  and the exponents are $m = 0$ and $m = 1 - c$.

- **Solution corresponding to the exponent m = 0:** The exponent $m = 0$ corresponds to a Frobenius series solution by **Theorem A** if $1 - c < 0$ or the difference $(1-c) - 0 = 1 - c$ is not a positive integer.

  Note that the second condition implies the first one, so if $1 - c$ is not a positive integer, equivalently c is neither zero nor a negative integer, then $m = 0$ corresponds to a Frobenius series solution of the form

$$y = x^0 \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (2)$$

  where a_0 is nonzero.

On substituting the above into (1) and equating the coefficients of x^n to zero, we obtain the following relations:

$$a_1 = \frac{ab}{c}\,a_0; \quad a_2 = \frac{(a+1)(b+1)}{2(c+1)}\,a_1 = \frac{a(a+1)b(b+1)}{2c(c+1)}\,a_0$$

$$a_{n+1} = \frac{(a+n)(b+n)}{(n+1)(c+n)}\,a_n. \tag{3}$$

With these coefficients and by letting $a_0 = 1$, the solution (2) becomes

$$\begin{aligned} y &= 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{2c(c+1)}x^2 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\; b(b+1)\cdots(b+n-1)}{n!\, c(c+1)\cdots(c+n-1)}\, x^n. \end{aligned} \tag{4}$$

This is known as the *hypergeometric series*, and is denoted by the symbol $F(a, b, c, x)$. It is called by this name because it generalizes the familiar geometric series as follows:

- When $a = 1$ and $c = b$: $F(1, b, b, x) = 1 + x + x^2 + \cdots = \dfrac{1}{1-x}$.

What is the Hypergeometric Series?

- The **Gauss hypergeometric series** is

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!},$$

where the coefficients α_n are given by $\alpha_n = \dfrac{(a)_n (b)_n}{(c)_n}, \quad n \geq 0.$

- Here $(q)_n$ is the **Pochhammer symbol** (rising factorial):

$$(q)_n = q(q+1)(q+2)\cdots(q+n-1), \qquad (q)_0 := 1.$$

- **Familiar Functions as Special Cases of $F(a,b;c;x)$:**

Function	Hypergeometric Form	Comment
e^x	$F(1,1;1;x)$	Standard series expansion
$(1-x)^{-a}$	$F(a,1;1;x)$	Binomial theorem
$\ln(1+x)$	$x\,F(1,1;2;-x)$	Integral representation
$\arcsin x$	$x\,F(1/2,1/2;3/2;x^2)$	Trigonometric inverse
$\arctan x$	$x\,F(1/2,1;3/2;-x^2)$	Another inverse trig example
$\frac{1}{1-x}$	$F(1,1;1;x)$	Geometric series

Solution corresponding to the second exponent $m = 1 - c$

- If $1 - c$ is neither zero nor a negative integer, i.e., c is not a positive integer, then Theorem A also tells us that there is a second independent solution of GHE (1) near $x = 0$ with exponent $m = 1$.
- This solution can be found directly by substituting

$$y = x^{1-c}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

into GHE (1) and calculating the coefficients.

- The other way of finding the solution is to change the dependent variable in (1) from y to z by writing $y = x^{1-c} z$. When the necessary computations are performed, equation (1) becomes

$$x(1-x)z'' + \Big[(2-c) - \big([a-c+1] + [b-c+1] + 1\big)x\Big]z' - (a-c+1)(b-c+1)z = 0 \tag{5}$$

which is the hypergeometric equation with the constants a, b and c are replaced by $a - c + 1$, $b - c + 1$ and $2 - c$.

- We already know that (5) (from the previous Subsection) has the power series solution

$$z = F(a - c + 1, b - c + 1, 2 - c, x)$$

near the origin, so our desired second solution is

$$y = x^{1-c} z = x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x).$$

- Accordingly, when c is not an integer, we have two independent Frobenius series solutions and hence

$$y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x) \qquad (6)$$

is the general solution of the hypergeometric equation (1) near the sigular point $x = 0$.

- Note that the above solution is only valid near the origin. We now solve GHE (1) near another singular point $x = 1$.

General solution of GHE at the regular point $x = 1$

- The simplest procedure is to obtain this solution from the one already found, by introducing a new independent variable $t = 1 - x$. ($x = 1 - t$, $dy/dx = -dy/dt$ and $d^2y/dx^2 = d^2y/dt^2$).
- This makes $x = 1$ correspond to $t = 0$ and transforms (1) into

$$t(1-t)y'' + \big[(a+b-c+1) - (a+b+1)t\big]y' - ab\,y = 0$$

where the primes denotes the derivatives with respect to t.

- Since the above is a hypergeometric equation, its general solution near $t = 0$ can be written down at once from (6) by replacing x by t and c by $a+b-c+1$ and then we replace t by $1-x$ to get the general solution of GHE (1) near $x = 1$:

$$\begin{aligned} y &= c_1 F(a, b, a+b-c+1, 1-x) \\ &\quad + c_2 (1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-x). \end{aligned} \tag{7}$$

(In this case it is necessary to assume that $c - a - b$ is not an integer).

Some comments, remarks and applications of hypergeometric functions F

- Formulas (6) and (7) show that the adaptability of the constants in equation GHE (1) makes it possible to express the general solution of this equation near each of its singular points in terms of the single function F.
- Moreover, the general solution of a wide class of differential equations can be written in terms of hypergeometric functions F.
- Any differential equation in which the coefficients of y'', y' and y are polynomials of degree 2, 1 and 0 respectively and also the first of these polynomials has distinct real roots can be brought into hypergeometric form by a linear change of the independent variable, and hence can be solved near its singular point in terms of the hypergeometric function.
- The above remarks can be explained as follows. We consider the class of the equation of this type:

$$(x - A)(x - B)y'' + (C + Dx)y' + Ey = 0 \tag{8}$$

where $A \neq B$.

- If we change the independent variable from x to t linearly by means of $(x-A)(x-B)=kt(t-1)$ for some constant k i.e.

$$t=\frac{x-A}{B-A}$$

then $x=A$ corresponds to $t=0$ and $x=B$ to $t=1$.

- After changing the independent variable from x to t the equation (8) assumes the form

$$t(1-t)\,y''+(F+Gt)\,y'+Hy=0,$$

where F, G and H are certain combinations of the constants in (8) and the primes indicate derivatives with respect to t.

- This a hypergeometric equation with constants a, b and c defined by $F=c$, $G=-(a+b+1)$ and $H=-ab$ and can therefore be solved near $t=0$ and $t=1$ in terms of the hypergeometric function.
- This means that (8) can be solved in terms of the same function near $x=A$ and $x=B$.

Problems

Find the general solution of each of the following differential equations near the indicated singular point in terms of hypergeometric function.

1. $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0, \quad x = 0;$
2. $(2x^2 + 2x)y'' + (1 + 5x)y' + y = 0, \quad x = 0;$
3. $(x^2 - 1)y'' + (5x + 4)y' + 4y = 0, \quad x = -1;$
4. $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0, \quad x = 3.$

Hint: Convert the equation into hypergeometric equation with a suitable change of independent variable as explained in the previous slides.
Before we start to solve these problems, we recall hypergeometric equation and its general solutions near its regular singular points.

Hypergeometric equation and solutions

Hypergeometric equation:

$$x(1-x)y'' + [c-(a+b+1)x]y' - ab\,y = 0$$

General solution at $x = 0$:

$$y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, c-2, x)$$

General solution at $x = 1$:

$$\begin{aligned} y = {} & c_1 F(a,b,a+b-c+1,1-x) \\ & + c_2(1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-x) \end{aligned}$$

Hypergeometric function:

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{n!\,c(c+1)\cdots(c+n-1)}\,x^n$$

Solution of Problem 1

$$x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0, \quad x = 0.$$

If we compare this equation with the hypergeometric equation we have the coefficients

$$c = \frac{3}{2}; \quad a + b + 1 = 2; \quad ab = -2.$$

Or $c = \frac{3}{2}$; $a + b = 1$; $ab = -2$. And then $a - b$ can be found by using the formula

$$a - b = \sqrt{(a+b)^2 - 4ab} = \sqrt{1 - 4(-2)} = 3.$$

By solving these equations we will get $a = 2$, $b = -1$, $c = \frac{3}{2}$.
Therefore the General solution at $x = 0$ is given by

$$\begin{aligned} y &= c_1 F(a, b, c, x) + c_2 x^{1-c} F(a - c + 1, b - c + 1, c - 2, x) \\ &= c_1 F\left(2, -1, \frac{3}{2}, x\right) + c_2 x^{-1/2} F\left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, x\right) \\ &= c_1 \left(1 - \frac{4}{3}x\right) + c_2 x^{-1/2} F\left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, x\right). \end{aligned}$$

Solution of Problem 4

$$(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0, \quad x = 3.$$

We use the change of independent variable from x to t by $t = \dfrac{x - A}{B - A}$, where A and B are roots of the polynomial $x^2 - x - 6$ which is a coefficient of y'' in given equation.

We see that $A = 3$ and $B = -2$ so that $t = \dfrac{3 - x}{5}$ or $x = -5t + 3$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{1}{5}\frac{dy}{dx}; \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{1}{5}\frac{dy}{dx}\right) = \frac{1}{25}\frac{d^2y}{dt^2}.$$

This transformation reduces the equation into

$$(-5t)(-5t + 5)\left(\frac{1}{25}\right)y'' + (5 - 15t + 9)\left(-\frac{1}{5}\right)y' + y = 0$$

$$t(1 - t)y'' + \left(\frac{14}{5} - 3t\right)y' - y = 0.$$

If we compare this equation with the hypergeometric equation we have the coefficients

$$c = \frac{14}{5}; \quad a + b + 1 = 3; \quad ab = 1.$$

$a + b = 2;\ ab = 1$ and hence $a - b = \sqrt{4-4} = 0$. Therefore

$$a = 1, \quad b = 1, \quad c = \frac{14}{5}.$$

The General solution at $t = 0$ or $x = 3$ is given by

$$\begin{aligned} y &= c_1 F(a,b,c,t) + c_2 t^{1-c} F(a-c+1, b-c+1, c-2, t) \\ &= c_1 F\left(1, 1, \frac{14}{5}, t\right) + c_2 t^{-9/5} F\left(-\frac{4}{5}, -\frac{4}{5}, -\frac{4}{5}, t\right) \quad \left(\text{since } t = \frac{3-x}{5}\right) \\ &= c_1 F\left(1, 1, \frac{14}{5}, \frac{3-x}{5}\right) + c_2 \left(\frac{3-x}{5}\right)^{-9/5} F\left(-\frac{4}{5}, -\frac{4}{5}, -\frac{4}{5}, \frac{3-x}{5}\right). \end{aligned}$$

Problem 5

Some differential equations are of the hypergeometric type even though they may not appear to be so. Find the general solution of

$$(1-e^x)y'' + \frac{1}{2}y' + e^x y = 0$$

near $x = 0$ by changing the independent variable to $t = e^x$.
Sol. The transformation $t = e^x$ implies $dt/dx = e^x = t$ and

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = t\frac{dy}{dt}; \qquad \frac{d^2y}{dt^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

We substitute all these in given equation to get

$$t^2(1-t)\frac{d^2y}{dt^2} + t\left(\frac{3}{2} - t\right)\frac{dy}{dt} + ty = 0.$$

Or

$$t(1-t)\frac{d^2y}{dt^2} + \left(\frac{3}{2} - t\right)\frac{dy}{dt} + y = 0.$$

After comparing with standard hypergeometric equation we get the constants $c = \frac{3}{2}$, $a + b + 1 = 1$ and $ab = -1$. If solve these equations. then we will get $a = 1$, $b = -1$, $c = \frac{3}{2}$.

The General solution at $t = 1$ or $x = 0$ is given by

$$
\begin{aligned}
y &= c_1 F(a, b, a+b-c+1, 1-t) + c_2 (1-t)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-t) \\
&= c_1 F\left(1, -1, -\frac{1}{2}, 1-t\right) + c_2 (1-t)^{3/2} F\left(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}, 1-t\right) \\
&= c_1 F\left(1, -1, -\frac{1}{2}, 1-e^x\right) + c_2 (1-e^x)^{3/2} F\left(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}, 1-e^x\right).
\end{aligned}
$$

Problem 6

Consider the Chebyshev's equation

$$(1-x^2)y'' - xy' + p^2 y = 0,$$

where p is a nonnegative constant. Transform it into a hypergeometric equation by changing the independent variable from x to $t = \frac{1}{2}(1-x)$, and show that its general solution near $x = 1$ is

$$y = c_1 F\left(p, -p, \frac{1}{2}, \frac{1-x}{2}\right) + c_2 \left(\frac{1-x}{2}\right)^{1/2} F\left(p + \frac{1}{2}, -p + \frac{1}{2}, \frac{3}{2}, \frac{1-x}{2}\right).$$

Problem 7

Consider the differential equation

$$x(1-x)y'' + [p-(p+2)x]y' - py = 0$$

where p is a constant.

(a) If p is not an integer, find the general solution near $x = 0$ in terms of hypergeometric functions.

(b) Write the general solution found in (a) in terms of elementary functions.

(c) When $p = 1$, the differential equation becomes

$$x(x-1)y'' + (1-3x)y' - y = 0$$

and the solution in (b) is no longer the general solution. Find the general solution in this case by the method of the use of known solution to find another.

Problem 8. Consider the Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{9}$$

where n is understood to be nonnegative integer.
We have found analytic solutions near ordinary point $x = 0$. However, the solutions most useful in the applications are those bounded near $x = 1$.

(a) Find the general solution near $x = 1$ by changing the independent variable from x to $\frac{1}{2}(1-x)$

(b) Find the solutions from (a) that are bounded near $x = 1$.

Sol. The change of independent variable x to $\frac{1}{2}(1-x)$ makes $x = 1$ correspond to $t = 0$ and transforms the equation (9) into

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0 \tag{10}$$

where the primes denotes derivatives with respect to t.

This is hypergeometric equation with $a = -n$, $b = n + 1$, and $c = 1$, so it has the following polynomial solution near $t = 0$:

$$y_1 = F(-n, n+1, 1, t). \tag{11}$$

Since the exponents of the equation (10) at the origin $t = 0$ are both zero ($m_1 = 0$ and $m_2 = 1 - c = 0$), we seek a second solution via the formula $y_2(t) = y_1(t)v(t)$ with $v(t)$ is given by

$$v(t) = \int \frac{1}{y_1^2} e^{-\int P(t)\,dt} dt$$

where $P(t) = \dfrac{1-2t}{t(1-t)}$. Note that

$$P(t) = \frac{1-2t}{t(1-t)} = \frac{1}{t} - \frac{1}{1-t}$$

therefore

$$e^{-\int P(t)\,dt} = e^{-\ln t - \ln(1-t)} = \frac{1}{t(1-t)}.$$

So we have that

$$v(t) = \int \frac{1}{y_1^2} \frac{1}{t(1-t)}\, dt = \int \frac{1}{t} \left[\frac{1}{y_1^2}(1-t) \right] dt$$

Since y_1 is a polynomial with constant term 1, the bracket expression on right is an analytic function of the form $1 + a_1 t + a_2 t^2 + \cdots$. This yields $v(t) = \log t + a_1 t + \cdots$, so

$$y_2 = y_1(\log t + a_1 t + \cdots).$$

and the general solution is given by

$$y = c_1 y_1 + c_2 y_2. \tag{12}$$

If we replace t in (12) by $\frac{1}{2}(1-x)$, we will get the general solution of Legendre's equation.

Because of the presence of the term $\log t$ in y_2, it is clear that (12) is bounded if and only if $c_2 = 0$.
If we replace t in (11) by $\frac{1}{2}(1-x)$, it follows that the solutions of Legendre's equation (9) are bounded near $x = 1$ are precisely constant multiple of the polynomial

$$F\left[-n,\, n+1,\, 1,\, \tfrac{1}{2}(1-x)\right].$$

Here the polynomial

$$F\left[-n,\, n+1,\, 1,\, \tfrac{1}{2}(1-x)\right]$$

is called n-th Legendre polynomial.

Problem 9. Verify the following identities

(a) $(1+x)^p = F(-p, b, b, -x)$

(b) $e^x = \lim_{b\to\infty} F\left(a, b, a, \frac{x}{b}\right)$

(c) $\log(1+x) = xF(1, 1, 2, -x)$

(d) $\cos x = \lim_{a\to\infty} F\left(a, a, \tfrac{1}{2}, -\frac{x^2}{4a^2}\right)$

Legendre's equation

Legendre Polynomials

- The Legendre Equation is

$$(1-x^2)\,y'' - 2x\,y' + n(n+1)\,y = 0,$$

where n is a constant.

- How will you try to find its solution at $x = 0$?
- **Series Solution at an Ordinary Point:** Here $x = 0$ is an ordinary point and the general solution is

$$y(x) = a_0\left[1 - \frac{n(n+1)}{2!}\,x + \frac{n(n-2)(n+1)(n+3)}{3!}\,x^4 - \cdots\right] + a_1\left[x - \frac{(n-1)(n+2)}{3!}\,x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}\,x^5 - \cdots\right].$$

Note:

- If n is not an integer, then each series is convergent in $-1 < x < 1$.
- If n is a non-negative integer, one of the series terminates and thus is a polynomial.

Solution near $x = 1$

- However, the solutions, most useful in the applications, are those bounded near $x = 1$.
- So our interest is to solve the Legendre equation near $x = 1$.
- For this we proceed by the approach of Hypergeometric equation.

The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0,$$

where n is a constant.

- What are its regular singular points?

Solution near $x = 1$

The Legendre Equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0,$$

where n is a constant.

- Looking at coefficients of y'', y', y we notice that we can transform this into a hypergeometric equation in standard form.
- What transformation will be helpful to find the solution near to $x = 1$?
- Can we find the transformed Differential equation?

Solution near $x = 1$

The Legendre Equation is

$$(1-x^2)\,y'' - 2x\,y' + n(n+1)\,y = 0,$$

where n is a constant.

- We change the independent variable from x to t by

$$t = \frac{x-1}{(-1)-1} = \frac{1}{2}(1-x).$$

- This makes $x = 1$ correspond to $t = 0$ and transforms the equation into

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0,$$

where the primes represent derivatives with respect to t.

What are its solutions?

Solution near $x = 1$

The Transformed Equation is

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0,$$

where n is a constant.

- This is hypergeometric equation with $a = -n, b = n+1, c = 1$.
- When $1-c$ is a real number but not an integer, then the general solution of the above equation can be found as

$$y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x) \qquad (6)$$

- But in our case we have $1-c=0$. Infact both exponents are 0 $(m_1 = 0, m_2 = 0.)$
- So we need an alternative approach.
- Note, the first solution is $y_1 = F(-n, n+1, 1, t)$.

The Transformed Equation is

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0,$$

where n is a constant. Note that we have $P(x) = \dfrac{(1-2t)}{t(1-t)}$.

- We seek a second solution by the method of Section 16. This second solution is $y_2 = vy_1$, where

$$v' = \frac{1}{y_1^2} e^{-\int p\,dt} = \frac{1}{y_1^2} e^{\int \frac{(2t-1)}{t(1-t)}\,dt} = \frac{1}{y_1^2 t(1-t)} = \frac{1}{t}\left[\frac{1}{y_1^2(1-t)}\right].$$

- On solving, we get

$$y_2 = y_1\left(\log t + a_1 t + \cdots\right)$$

and the general solution of (2) near the origin is

$$y = c_1\, y_1 + c_2\, y_2.$$

Bounded Solutions Near $x = 1$

- Because of the presence of the term $\log t$ in y_2, it is clear that the general solution

$$y = c_1 y_1 + c_2 y_2$$

  is bounded near $t = 0$ if and only if $c_2 = 0$.

- Replace t by

$$t = \tfrac{1}{2}(1 - x),$$

- The solutions of Legendre's equation that are bounded near $x = 1$ are precisely constant multiples of the polynomial

$$F(-n,\, n+1,\, 1,\, \tfrac{1}{2}(1 - x)).$$

- This brings us to the fundamental definition. The n^{th} Legendre polynomial is denoted by $P_n(x)$ and defined by

$$P_n(x) = F\Big(-n,\, n+1,\, 1,\, \tfrac{1}{2}(1 - x)\Big)$$

Definition of $P_n(x)$ via F

Definition (Legendre polynomial). The n^{th} Legendre polynomial $P_n(x)$ is defined by

$$P_n(x) = F\left(-n,\, n+1,\, 1,\, \frac{1-x}{2}\right).$$

$$\begin{aligned} P_n(x) &= F\left(-n,\, n+1,\, 1,\, \frac{1-x}{2}\right) \\ &= 1 + \frac{(-n)(n+1)}{(1!)^2}\left(\frac{1-x}{2}\right) + \frac{(-n)(-n+1)(n+1)(n+2)}{(2!)^2}\left(\frac{1-x}{2}\right)^2 + \cdots \\ &\quad + \frac{(-n)(-n+1)(-n+(n-1))(n+1)(n+2)\cdots(2n)}{(n!)^2}\left(\frac{1-x}{2}\right)^n \end{aligned}$$

$$P_n(x) = 1 + \frac{n(n+1)}{(1!)^2\,2}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2\,2^2}(x-1)^2 + \cdots + \frac{(2n)!}{(n!)^2\,2^n}(x-1)^n.$$

Remark: The above formula is a very inconvenient tool to use in studying $P_n(x)$, so we look for something simpler.

Rodrigues' formula

The n^{th} Legendre polynomial can equivalently be defined by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right], \qquad n = 0, 1, 2, \ldots$$

Remark. The above provides a relatively easy method for computing successive Legendre polynomials.

- **First few polynomials (by Rodrigues):**

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \tfrac{1}{2}(3x^2-1), \qquad P_3(x) = \tfrac{1}{2}(5x^3-3x),$$

$$P_4(x) = \tfrac{1}{8}(35x^4-30x^2+3), \quad P_5(x) = \tfrac{1}{8}(63x^5-70x^3+15x), \ \ldots$$

Properties of Legendre polynomials

- The sequence of Legendre polynomials $P_0(x), P_1(x), \dots, P_n(x), \dots$
- **Orthogonality:** On $[-1, 1]$, the sequence $\{P_n\}_{n\geq 0}$ is orthogonal:

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0, & m \neq n, \\ \dfrac{2}{2n+1}, & m = n. \end{cases}$$

$$OR$$

The sequence of Legendre polynomials forms a sequence of orthogonal functions on the interval $-1 \leq x \leq 1$.

Proof: Orthogonality of Legendre polynomials

- Let

$$I = \int_{-1}^{1} f(x) P_n(x)\, dx \tag{1}$$

where $f(x)$ be any function with at least n continuous derivatives on $[-1, 1]$.

- By Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

- Therefore, (1) becomes

$$I = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

- Integrating by parts, we get

$$I = \frac{(-1)^1}{2^n n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n\, dx$$

Proof: Orthogonality of Legendre polynomials

- Integrating repeatedly, we get

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} f^{(n)}(x)(x^2-1)^n \, dx \tag{2}$$

- Let $m < n$ and $f(x) = P_m(x)$ in (1), then $f^{(n)}(x) = 0$ and therefore $I = 0$.
- Similarly, we can show that

$$\int_{-1}^{1} P_m(x)P_n(x) \, dx = 0, \quad \text{for } m > n.$$

Proof: Orthogonality of Legendre polynomials

- If $m = n$ let $f(x) = P_n(x)$. We already have
  $P_n^{(n)}(x) = \dfrac{1}{2^n n!}\dfrac{d^{2n}}{dx^{2n}}(x^2-1)^n$
- Thus

$$P_n^{(n)}(x) == \frac{1}{2^n n!}\frac{d^{2n}}{dx^{2n}}\left(x^{2n} + \text{terms with lower exponent than } 2n\right) = \frac{(2n)!}{2^n n!},$$

- Therefore, using (2), $I = \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx$
- Since $(1-x^2)^n$ is an even function, $I = \frac{2(2n)!}{2^{2n}(n!)^2}\int_0^1(1-x^2)^n\,dx$
- Change the variable $x = \sin\theta$, then recall the formula

$$\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta = \frac{2^{2n}(n!)^2}{(2n)!(2n+1)}.$$

- Hence, we conclude that in this case

$$I = \frac{2}{2n+1}, \quad \text{hence proved.}$$

Properties of Legendre polynomials

- **Legendre Series:** Many problems of the potential theory depend on the possibility of expanding a given function in a series of Legendre polynomials.
- *If the given function is a polynomial, it is easy to see that it can be expressed as the sum of the Legendre polynomials.*
- For example:

$$1 = P_0(x), \qquad x = P_1(x),$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \;\Rightarrow\; x^2 = \frac{1}{3} + \frac{2}{3}P_2(x) = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x),$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \;\Rightarrow\; x^3 = \frac{3}{5}x + \frac{2}{5}P_3(x) = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Properties of Legendre polynomials

- Third degree polynomial $p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ can be written as

$$\begin{aligned} p(x) =& b_0 P_0(x) + b_1 P_1(x) + b_2 \left[\tfrac{1}{3} P_0(x) + \tfrac{2}{3} P_2(x)\right] + b_3 \left[\tfrac{3}{5} P_1(x) + \tfrac{2}{5} P_3(x)\right] \\ =& \left(b_0 + \tfrac{b_2}{3}\right) P_0(x) + \left(b_1 + \tfrac{3b_3}{5}\right) P_1(x) + \tfrac{2b_2}{3} P_2(x) + \tfrac{2b_3}{5} P_3(x) \\ =& \sum_{n=0}^{3} a_n P_n(x). \end{aligned}$$

- Therefore, any polynomial $p(x)$ of degree k can be written as

$$p(x) = \sum_{n=0}^{k} a_n P_n(x).$$

Properties of Legendre polynomials

- ***What about an arbitrary function?***
- An arbitrary function $f(x)$ can be expressed as ***Legendre series***

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

  - Need to calculate coefficients a_n for the above expression.
  - Multiply $P_m(x)$ on both sides and integrate from -1 to 1, we get

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x)P_n(x)\,dx$$

  - In view of our orthogonality result, this collapses to

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \frac{2a_m}{2m+1}$$

- And therefore, we have

$$a_n = \left(n + \tfrac{1}{2}\right) \int_{-1}^{1} f(x)P_n(x)\,dx.$$

Generating Function of the Legendre Polynomials

$$g(x,t) := \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

is called the **generating function** of the Legendre polynomials. Therefore, $P_n(x)$ is the coefficient of t^n in the expansion of

$$(1-2xt+t^2)^{-1/2}$$

in ascending powers of t.

- We can prove the following

$$\text{(i)} \quad P_n(1) = 1 \quad \text{and} \quad P_n(-1) = (-1)^n.$$

$$\text{(ii)} \quad P_{2n+1}(0) = 0 \quad \text{and} \quad P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n\, n!}.$$

- Recursion formula to generate Legendre polynomials:

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}, \quad n = 1, 2, \ldots$$

- Assume that $P_0(x) = 1$ and $P_1(x) = x$ are known, and one can use the recursion formula to calculate $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

Generating Function of the Legendre Polynomials

Evaluate $P_n(-1)$. This is a simpler problem. In this case we have

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

Therefore,

$$P_n(-1) = (-1)^n.$$

Properties of Legendre Polynomials: Least Squares Approximation

- Let $f(x)$ be a function defined on the interval $[-1, 1]$.
- Now the problem is to approximate the function $f(x)$ by a polynomial $p(x)$ of degree $\leq n$ as closely as possible in the sense of least squares, i.e.

> Given a function $f(x)$, continuous on $[-1, 1]$, find a polynomial $p(x)$ of degree at most n such that the integral of the square of the error is minimized. That is,
>
> $$I = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$$
>
> is minimized.

- It turns out that the minimizing polynomial is precisely the sum of the first $n + 1$ terms of the Legendre series:

$$p(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots + a_n P_n(x)$$

where

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{1} f(x) P_n(x) \, dx.$$

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

- We have $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- The coefficients of the Legendre series are given by

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{1} e^x P_n(x)\, dx, \qquad n = 0, 1, 2, \ldots$$

- **Linear approximation:** Using $P_0(x) = 1$, $P_1(x) = x$,

$$a_0 = \frac{1}{2} \int_{-1}^{1} e^x\, dx = \tfrac{1}{2}\,(e - e^{-1}), \qquad a_1 = \frac{3}{2} \int_{-1}^{1} x e^x\, dx = 3e^{-1}.$$

  Hence $p_1(x) = a_0 P_0(x) + a_1 P_1(x) = \frac{1}{2}(e - e^{-1}) + 3e^{-1}x$.
- **Quadratic approximation:** With $P_2(x) = \frac{1}{2}(3x^2 - 1)$,

$$a_2 = \frac{5}{2} \int_{-1}^{1} e^x\, \tfrac{1}{2}(3x^2 - 1)\, dx = \frac{5}{2}(e - 7e^{-1}).$$

  Thus $p_2(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$,

$$p_2(x) = \frac{-3e + 33e^{-1}}{4} + 3e^{-1}x + \frac{15}{4}(e - 7e^{-1})x^2.$$

Properties of Legendre Polynomials: Least Squares Approximation

Example: Find linear and quadratic least-squares approximations to $f(x) = e^x$ using Legendre polynomials.

```
y = e^{x}

l = 0.5\cdot\left(e-e^{-1}\right)+3e^{-1}x

q = \frac{\left(-3\cdot e+33\cdot e^{-1}\right)}{4}+3\cdot e^{-1}x+\frac{15}{4}\left(e-7e^{-1}\right)x^{2}
```

Bessel's equation (and functions)

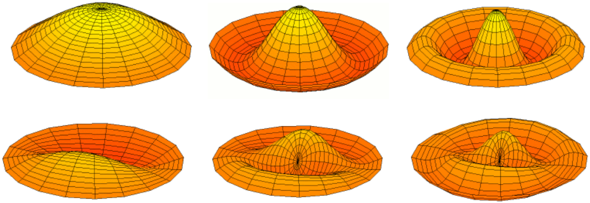
Bessel Functions: An Introduction

The Core Idea

Bessel functions are the standard solutions to physical problems involving **circular or cylindrical symmetry**.

- They are like the "sines and cosines" for cylindrical shapes.
- They appear when solving differential equations (like the wave or heat equation) in cylindrical coordinates.
- We will explore some applications where they are essential.

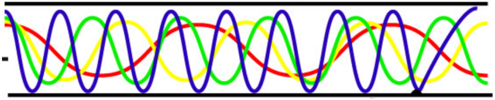
Application 1: Vibrating Drumheads



Drumhead Vibration Modes

- The motion of a circular drum after being struck is described by the wave equation.
- In polar coordinates, this equation simplifies to the Bessel equation.
- Bessel functions describe the "modes" of vibration - the beautiful, complex patterns the drum surface makes.

Application 2: Waves in Pipes and Cables



Bessel functions are essential for describing how waves travel through cylindrical guides:

- **Sound waves** in a circular pipe.
- **Electromagnetic waves** in a coaxial cable or waveguide.
- **Light** traveling through an **optical fiber**. The functions describe the different "modes," or patterns of light.

Bessel's equation and function

- **Bessel's Equation of order p:**

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

- To analyze the point $x = 0$, we rewrite it in the **standard form** $y'' + P(x)y' + Q(x)y = 0$ by dividing by x^2:

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

- From this, we identify the coefficients:
  - $P(x) = \frac{1}{x} \qquad Q(x) = \frac{x^2 - p^2}{x^2}$
- A point x_0 is a **singular point** if either $P(x)$ or $Q(x)$ is not analytic at x_0.
- **Conclusion:** Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$, the origin is a **singular point** of Bessel's equation.
- It's a regular singular point because $xP(x)$ and $x^2Q(x)$ are analytic.

Bessel's equation and function

- The **indicial equation** is $m^2 - p^2 = 0$, and the exponents are $m_1 = p$ and $m_2 = -p$.
- It follows from **Theorem 30-A** that Bessels' equation has a solution of the form

$$y(x) = x^p \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+p},$$

where $a_0 \neq 0$ and the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x.

Bessel's equation (Solution)

Bessel's equation : $x^2y'' + xy' + (x^2 - p^2)y = 0.$

- We get $y(x) = \sum_{n=0}^{\infty} a_n x^{n+p}$, $y'(x) = \sum_{n=0}^{\infty} a_n(n+p)x^{n+p-1}$,

$$y''(x) = \sum_{n=0}^{\infty} a_n(n+p)(n+p-1)x^{n+p-2}.$$

- The terms of the equation become

$$-p^2y = \sum_{n=0}^{\infty} -p^2 a_n x^{n+p}, \qquad x^2y = \sum_{n=0}^{\infty} a_{n-2}x^{n+p},$$

$$xy' = \sum_{n=0}^{\infty} a_n(n+p)x^{n+p}, \quad x^2y'' = \sum_{n=0}^{\infty} a_n(n+p)(n+p-1)x^{n+p}.$$

- Hence

$$a_0(p(p-1) + p - p^2) + a_1((p+1)^2 - p^2) + \sum_{n=2}^{\infty}[a_n n(n+2p) + a_{n-2}]x^n = 0$$

Bessel's equation (Solution)

Bessel's equation : $x^2y'' + xy' + (x^2 - p^2)y = 0.$

- Inserting into the equation and equating the coefficients of x^{n+p} to zero, we get the following recursion formula

$$a_n = -\frac{a_{n-2}}{n(2p+n)}$$

- Since $a_0 \neq 0$ and $a_1(2p+1) = 0$ tells us that $a_1 = 0$, and
- Repeated application of this recursion formula yields the fact that $a_n = 0$ for every odd subscript n.

Bessel's equation (Solution)

Bessel's equation : $x^2y'' + xy' + (x^2 - p^2)y = 0.$

- The nonzero coefficients of our solution are therefore

$$a_2 = -\frac{a_0}{2(2p+2)} = \frac{a_0}{1\cdot 2^2(p+1)},$$

$$a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_0}{2\cdot 4(2p+2)(2p+4)} = \frac{(-1)^2 a_0}{2!2^4(p+1)(p+2)},$$

$$a_6 = -\frac{a_4}{6(2p+6)}$$

$$= -\frac{a_0}{2\cdot 4\cdot 6(2p+2)(2p+4)(2p+6)} = \frac{(-1)^3 a_0}{3!2^6(p+1)(p+2)(p+3)},\dots$$

And the solution is

$$y = a_0x^p\left[1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4 2!(p+1)(p+2)} - \frac{x^6}{2^6 3!(p+1)(p+2)(p+3)} + \dots\right]$$

The Bessel Function of the First Kind

The solution is

$$y = a_0 x^p \left[1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4 2!(p+1)(p+2)} - \frac{x^6}{2^6 3!(p+1)(p+2)(p+3)} + \dots\right]$$

- Or we have

$$y = a_0 x^p \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)(p+2)\cdots(p+n)}.$$

- If we replace the arbitrary constant a_0 by $1/(2^p p!)$, we obtain a particular solution of Bessel's equation which is denoted by $J_p(x)$ and is known as **Bessel function of the first kind of order** p.

Definition

$$J_p(x) = \frac{x^p}{2^p p!} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)(p+2)\cdots(p+n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

Important Bessel Functions: Order 0 and 1

The solution is

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- The most useful Bessel functions are those of order 0 and 1 which are

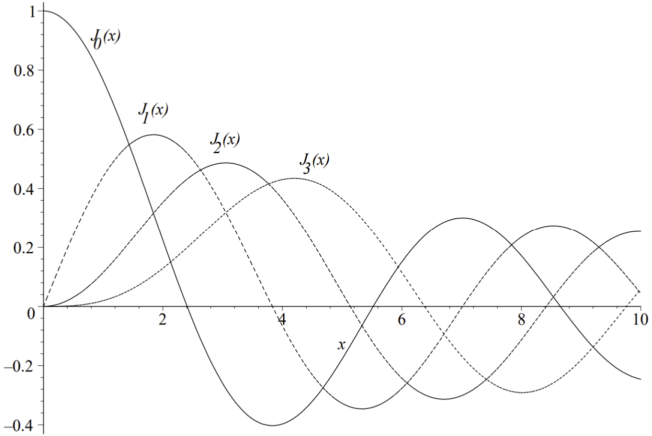
$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

- and

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} = \frac{x}{2} - \frac{1}{1!2!} \left(\frac{x}{2}\right)^3 + \frac{1}{2!3!} \left(\frac{x}{2}\right)^5 - \cdots$$

Graphs of J_0, J_1, J_2, J_3

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$



Properties and Challenges of Bessel Functions

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- These graphs display several interesting properties of the functions $J_0(x)$ and $J_1(x)$.
- Each has a damped oscillating behavior producing an infinite number of **positive zeros**; and these zeros occur alternatively, in a manner suggesting the functions $\cos x$ and $\sin x$. (**Positive zeros** - positive real numbers for which the function $J_p(x)$ vanishes)
- Also
$$\frac{d}{dx} J_0(x) = -J_1(x) \quad \text{and} \quad \frac{d}{dx} x J_1(x) = x J_0(x).$$
- In the denominator of $J_p(x)$ there is a term $(p+n)!$, but it is meaningless if p is not a positive integer.
- Now our next turn is to ***overcome this difficulty***.

The Gamma Function

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- The purpose of gamma function is to give a reasonable and useful meaning to $p!$ [and more generally to $(p+n)!$ for $n = 0, 1, 2, \ldots$].
- The **gamma function** is defined by

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t} dt, \quad p > 0 \qquad (A)$$

- The factor $e^{-t} \to 0$ so rapidly as $t \to \infty$, that is, the integral converges at the upper limit regardless of the value of p.
- However at the lower limit we have $e^{-t} \to 1$ and the factor $t^{p-1} \to \infty$ whenever $p < 1$ as $t \to 0$.
- Therefore, the restriction that p must be positive is necessary in order to guarantee convergence at the lower limit.

Properties of the Gamma Function

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt, \quad p > 0 \qquad (A)$$

- $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$
- We see that

$$\Gamma(p+1) = p\Gamma(p)$$

- For integration by parts yields

$$\begin{aligned}\Gamma(p+1) &= \lim_{b\to\infty}\int_0^b t^p e^{-t}dt \\ &= \lim_{b\to\infty}\left(-t^p e^{-t}\Big|_0^b + p\int_0^b t^{p-1}e^{-t}dt\right) \\ &= p\left(\lim_{b\to\infty}\int_0^b t^{p-1}e^{-t}dt\right) = p\Gamma(p).\end{aligned}$$

- Since $\frac{b^p}{e^b} \to 0$ as $b \to \infty$.

Gamma Function and Factorials

- For $p \in \mathbb{N}$, $\Gamma(p+1) = p\Gamma(p)$ yields

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2 \cdot 1 = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2 \cdot 1 = 3!$$

- In general $\Gamma(n+1) = n!$ for any integer greater than or equal to zero.
- Define $p!$ by

$$p! = \Gamma(p+1)$$

  for all values of p except negative integers.
- Also $\frac{1}{p!} = \frac{1}{\Gamma(p+1)}$ is defined for all values of p and has the value 0 whenever p is a negative integer.
- <u>**One important fact:**</u> $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

General Solution of Bessel's Equation

- Recall, the **indicial equation** is $m^2 - p^2 = 0$, and the exponents are $m_1 = p$ and $m_2 = -p$.
- Found one solution for the exponent $m_1 = p$, namely $J_p(x)$.
- To get the general solution, we must construct a second linearly independent solution (known as ***Bessel function of second kind***).
- For the second Linearly Independent solution, the procedure is to try the other exponent $m_2 = -p$.
- But in doing this we expect to encounter *difficulties* whenever the difference $m_1 - m_2 = 2p$ is zero or a positive integer (i.e. when the nonnegative integer p is an integer or half of an odd integer).

Case: p is not an Integer

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- Assume p is **not an integer**.
- Replace p by $-p$ in our earlier treatment.
- We get **second solution** as

$$J_{-p}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n-p}}{n!(-p+n)!}.$$

- First term of the series is $\frac{1}{(-p)!}\left(\frac{x}{2}\right)^{-p}$. So $J_{-p}(x)$ is unbounded near $x = 0$.
- Since $J_p(x)$ is bounded near $x = 0$, these two solutions are independent and

$$y = c_1 J_p(x) + c_2 J_{-p}(x), \quad p \text{ not an integer}$$

is the *general solution of Bessel's equation*.

The General Solution for All Cases

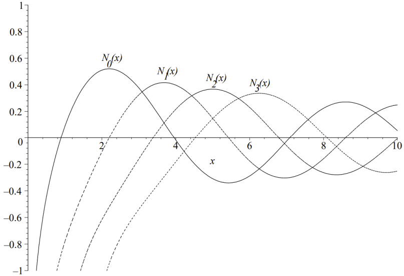
- The **standard Bessel function of second kind** is defined by

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} \qquad \text{(also denoted by } N_p(x)\text{)}$$

- Then the general solution of the Bessel's equation is

$$y = c_1 J_p(x) + c_2 Y_p(x)$$

in all cases, whether p is an integer or not.



Properties of Bessel Functions

- The Bessel function $J_p(x)$ is defined for any real number p by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p} \qquad \left(= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p} \right)$$

- Some identities:
  1. $\dfrac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$
  2. $\dfrac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$
  3. $J_p'(x) - \dfrac{p}{x} J_p(x) = -J_{p+1}(x)$
  4. $J_p'(x) + \dfrac{p}{x} J_p(x) = J_{p-1}(x)$
  5. $\dfrac{2p}{x} J_p(x) = J_{p+1}(x) + J_{p-1}(x)$
  6. $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$

Proof of Identity (i): $\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p} \qquad \left(= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p} \right)$$

We start by multiplying the series for $J_p(x)$ by x^p:

$$\begin{aligned}
\frac{d}{dx}\left[x^p J_p(x)\right] &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{2^{2n+p} n!(p+n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p) x^{2n+2p-1}}{2^{2n+p} n!(p+n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+p) x^{2n+2p-1}}{2^{2n+p} n!(p+n)(p+n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p-1}}{2^{2n+p-1} n!(p+n-1)!} \\
&= x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!((p-1)+n)!} \left(\frac{x}{2}\right)^{2n+p-1} = x^p J_{p-1}(x).
\end{aligned}$$

Proof of Identity (ii): $\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x)$

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!}\left(\frac{x}{2}\right)^{2n+p} \quad \left(= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p}\right)$$

Similarly, we multiply the series for $J_p(x)$ by x^{-p}:

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+p}n!(p+n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n+p}n(n-1)!(p+n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2x^{2n-1}}{2^{2n+p}(n-1)!(p+n)!}, \quad \text{Let } k = n-1 \text{ to get}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2x^{2(k+1)-1}}{2^{2(k+1)+p}k!(p+k+1)!}$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+p+1}k!((p+1)+k)!}$$

$$= -x^{-p}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!((p+1)+k)!}\left(\frac{x}{2}\right)^{2k+p+1} = -x^{-p}J_{p+1}(x).$$

Proof of Identity (iii): $J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$

We already proved:

1. $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$
2. $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

We will then algebraically combine the resulting equations.

- We start with identity (ii) and apply the product rule to the left-hand side:

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -px^{-p-1} J_p(x) + x^{-p} J_p'(x)$$

- Setting this equal to the right-hand side of identity (ii):

$$-px^{-p-1} J_p(x) + x^{-p} J_p'(x) = -x^{-p} J_{p+1}(x)$$

- Dividing the entire equation by x^{-p} gives us the final identity:

$$-\frac{p}{x} J_p(x) + J_p'(x) = -J_{p+1}(x)$$

Proof of Identity (iv): $J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$

We already proved:

1. $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$
2. $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

We will then algebraically combine the resulting equations.

- We start with identity (i) and apply the product rule to the left-hand side:

$$\frac{d}{dx}\left[x^p J_p(x)\right] = p x^{p-1} J_p(x) + x^p J_p'(x)$$

- Now, we set this equal to the right-hand side of identity (i):

$$p x^{p-1} J_p(x) + x^p J_p'(x) = x^p J_{p-1}(x)$$

- Dividing the entire equation by x^p yields the desired result:

$$\frac{p}{x} J_p(x) + J_p'(x) = J_{p-1}(x)$$

Proof of Identity (v): $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

- We take the two identities we just proved:

$$J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x) \qquad (iv)$$

$$J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x) \qquad (iii)$$

- Now, we subtract equation (iii) from equation (iv):

$$\left(J_p'(x) + \frac{p}{x} J_p(x)\right) - \left(J_p'(x) - \frac{p}{x} J_p(x)\right) = J_{p-1}(x) - (-J_{p+1}(x))$$

- The $J_p'(x)$ terms cancel, and simplifying gives the result:

$$2\frac{p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$$

Proof of Identity (vi): $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$

- Again, we start with the same two identities:

$$J_p'(x) + \frac{p}{x}J_p(x) = J_{p-1}(x) \qquad (iv)$$

$$J_p'(x) - \frac{p}{x}J_p(x) = -J_{p+1}(x) \qquad (iii)$$

- This time, we add equation (iv) and equation (iii):

$$\left(J_p'(x) + \frac{p}{x}J_p(x)\right) + \left(J_p'(x) - \frac{p}{x}J_p(x)\right) = J_{p-1}(x) + (-J_{p+1}(x))$$

- The $\frac{p}{x}J_p(x)$ terms cancel, leading to the final identity:

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$$

Differentiation and Integration Formulas

- The differentiation formulas

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x), \quad \frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

  can be written in the form

$$\int x^p J_{p-1}(x)dx = x^p J_p(x) + c$$

  and

$$\int x^{-p} J_{p+1}(x)dx = -x^{-p} J_p(x) + c$$

- Then they serve for the integration of many simple expressions containing Bessel functions. For example, when $p = 1$, we get

$$\int x J_0(x)dx = x J_1(x) + c.$$

Bessel Functions of Half-Integer Order

- Function $J_{m+1/2}(x)$:

  (a) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and (b) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

  (c) $J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

  (d) $J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{\cos x}{x} - \sin x \right)$

- It can be continued indefinitely, and therefore every Bessel function $J_{m+1/2}(x)$ (where m is an integer) is elementary.

Proof of $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

We start with the series definition of the Bessel function, using the Gamma function $\Gamma(z)$:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

Let $p = 1/2$. We need the values $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(z+1) = z\Gamma(z)$.

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n+1/2}$$

Expanding the series:

$$\begin{aligned} J_{1/2}(x) &= \frac{1}{0!\Gamma(3/2)} \left(\frac{x}{2}\right)^{1/2} - \frac{1}{1!\Gamma(5/2)} \left(\frac{x}{2}\right)^{5/2} + \cdots \\ &= \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{1}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^2 + \cdots\right] \\ &= \sqrt{\frac{x}{2}} \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{3 \cdot 2} + \frac{x^4}{5 \cdot 4 \cdot 3 \cdot 2} - \cdots\right] \\ &= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = \sqrt{\frac{2}{\pi x}} \sin x \end{aligned}$$

Proof of $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Let $p = -1/2$. The series definition becomes:

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1/2)} \left(\frac{x}{2}\right)^{2n-1/2}$$

Expanding the series:

$$\begin{aligned} J_{-1/2}(x) &= \frac{1}{0!\Gamma(1/2)} \left(\frac{x}{2}\right)^{-1/2} - \frac{1}{1!\Gamma(3/2)} \left(\frac{x}{2}\right)^{3/2} + \cdots \\ &= \left(\frac{x}{2}\right)^{-1/2} \left[\frac{1}{\sqrt{\pi}} - \frac{1}{\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^2 + \cdots\right] \\ &= \sqrt{\frac{2}{x}} \frac{1}{\sqrt{\pi}} \left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right] \\ &= \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] \\ &= \sqrt{\frac{2}{\pi x}} \cos x \end{aligned}$$

Proofs using Recurrence Relations

We use the recurrence relation $J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$.

Proof of Identity (c)

Let $p = 1/2$. Then $p + 1 = 3/2$ and $p - 1 = -1/2$.

$$\begin{aligned} J_{3/2}(x) &= \frac{2(1/2)}{x} J_{1/2}(x) - J_{-1/2}(x) \\ &= \frac{1}{x}\left(\sqrt{\frac{2}{\pi x}}\sin x\right) - \sqrt{\frac{2}{\pi x}}\cos x = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \end{aligned}$$

Proof of Identity (d)

Rearranging the relation: $J_{p-1}(x) = \frac{2p}{x} J_p(x) - J_{p+1}(x)$.
Let $p = -1/2$. Then $p - 1 = -3/2$ and $p + 1 = 1/2$.

$$\begin{aligned} J_{-3/2}(x) &= \frac{2(-1/2)}{x} J_{-1/2}(x) - J_{1/2}(x) \\ &= -\frac{1}{x}\left(\sqrt{\frac{2}{\pi x}}\cos x\right) - \sqrt{\frac{2}{\pi x}}\sin x = \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right) \end{aligned}$$

RECALL: Important Bessel Functions: Order 0 and 1

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$

- The most useful Bessel functions are those of order 0 and 1 which are

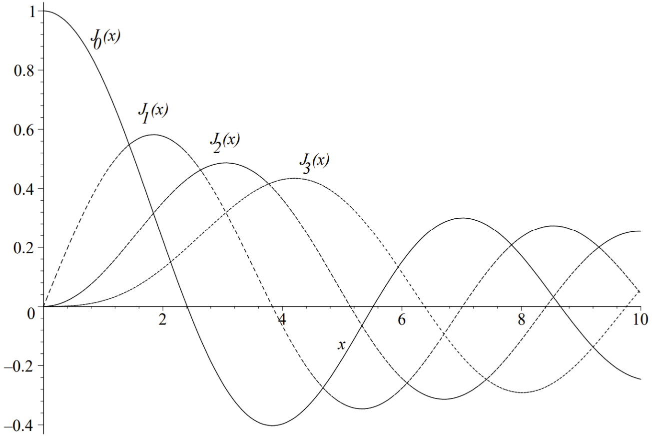
$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

- and

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} = \frac{x}{2} - \frac{1}{1!2!} \left(\frac{x}{2}\right)^3 + \frac{1}{2!3!} \left(\frac{x}{2}\right)^5 - \cdots$$

RECALL: Graphs of J_0, J_1, J_2, J_3

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}$$



Zeros and Bessel Series

- For every value of p, the function $J_p(x)$ has an infinite number of positive zeros (and are of high degree of accuracy).
- For $J_0(x)$ the first five positive zeros are

  2.4048, 5.5201, 8.6537, 11.7915, and 14.9309.

  Successive differences are: 3.1153, 3.1336, 3.1378, 3.1394 $\to \pi$
- For $J_1(x)$ the first five positive zeros are

  3.8317, 7.0156, 10.1735, 13.3237, and 16.4706.

  Successive differences are: 3.1839, 3.1579, 3.1502, 3.1469 $\to \pi$
- **The purpose of concern of these zeros of $J_p(x)$ is:** to expand a given function in terms of Bessel functions.

The Fourier-Bessel Series Expansion

- Simple and most useful expansion of this kind is the series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x) = a_1 J_p(\lambda_1 x) + a_2 J_p(\lambda_2 x) + a_3 J_p(\lambda_3 x) + \cdots$$

  where $f(x)$ is defined on the interval $0 \leq x \leq 1$ and λ_n are the positive zeros of some fixed Bessel function $J_p(x)$ with $p \geq 0$.
- As in Legendre series, here also we need to determine the coefficients of the expansion, which depend on certain integral properties of the function

$$J_p(\lambda_n x).$$

The Fourier-Bessel Series Expansion

- A function $f(x)$ defined on the interval $[0, 1]$ can be expanded in a series of Bessel functions:
$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x) = a_1 J_p(\lambda_1 x) + a_2 J_p(\lambda_2 x) + \cdots$$
where λ_n are the positive zeros of a fixed Bessel function $J_p(x)$ with $p \geq 0$.
- The key to determining the coefficients a_n is the **orthogonality property** of Bessel functions.

Orthogonality Relation

The functions $J_p(\lambda_n x)$ are orthogonal on the interval $[0, 1]$. This is expressed by the integral:
$$\int_0^1 x J_p(\lambda_m x) J_p(\lambda_n x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{1}{2}[J_{p+1}(\lambda_n)]^2 & \text{if } m = n. \end{cases}$$

- By multiplying the series for $f(x)$ by $x J_p(\lambda_m x)$ and integrating from 0 to 1, the orthogonality causes all terms in the sum to become zero except for the term where $n = m$. This allows us to solve for the coefficients:
$$a_n = \frac{2}{[J_{p+1}(\lambda_n)]^2} \int_0^1 x f(x) J_p(\lambda_n x) dx$$

Orthogonality of Bessel Functions

- Here we need the fact

$$\int_0^1 xJ_p(\lambda_m x)J_p(\lambda_n x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{1}{2}J_{p+1}(\lambda_n)^2 & \text{if } m = n. \end{cases}$$

- The functions $J_p(\lambda_n x)$ are said to be orthogonal with respect to the **weight function x** on the interval $0 \leq x \leq 1$.
- If the expansion is assumed to be possible then multiplying through $xJ_p(\lambda_m x)$, and integrating term by term from 0 to 1, and using the above fact, we get

$$\int_0^1 xf(x)J_p(\lambda_m x)dx = \frac{a_m}{2}J_{p+1}(\lambda_m)^2$$

The Fourier-Bessel Series Expansion and Bessel Expansion Theorem

The series

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x) = a_1 J_p(\lambda_1 x) + a_2 J_p(\lambda_2 x) + a_3 J_p(\lambda_3 x) + \cdots$$

with the coefficients given by the previous page formula is known as **Bessel series** (or sometimes the **Fourier-Bessel series**) of the function $f(x)$.

The following theorem (without proof) tells under what conditions the series actually converges:

- **Theorem A (Bessel expansion theorem):**
  Assume that $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities on the interval $0 \leq x \leq 1$. If $0 < x < 1$, then the Bessel series converges to $f(x)$ when x is a point of continuity of this function and when x is a point of discontinuity, it converges to $\frac{1}{2}[f(x-) + f(x+)]$.
- At $x = 1$, series converges to 0 regardless of the nature of function.
- At $x = 0$, the series also converges to 0 if $p > 0$ and to $f(0+)$ if $p = 0$.

Exercise 1: Problem Statement

Goal

As an illustration, we will compute the Bessel series of the function $f(x) = 1$ for the interval $0 \leq x \leq 1$.

- The expansion will be in terms of the functions $J_0(\lambda_n x)$, where it is understood that the λ_n are the positive zeros of the Bessel function $J_0(x)$.
- This means we are using the case where the order $p = 0$.
- The series has the general form:

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n x)$$

- Our task is to find the specific values for the coefficients a_n.

Exercise 1: Step 1: Evaluating the Integral

- The general formula for the coefficients is:

$$a_n = \frac{2}{[J_{p+1}(\lambda_n)]^2} \int_0^1 x f(x) J_p(\lambda_n x) dx$$

- Substituting $p = 0$ and $f(x) = 1$, the formula becomes:

$$a_n = \frac{2}{[J_1(\lambda_n)]^2} \int_0^1 x J_0(\lambda_n x) dx$$

- First, we must solve the integral. We use the recurrence relation:

$$\frac{d}{dt}[tJ_1(t)] = tJ_0(t) \implies \int tJ_0(t)dt = tJ_1(t)$$

- To evaluate our specific integral, let $t = \lambda_n x$, so $dt = \lambda_n dx$.

$$\int xJ_0(\lambda_n x)dx = \int \frac{t}{\lambda_n} J_0(t) \frac{dt}{\lambda_n} = \frac{1}{\lambda_n^2} \int tJ_0(t)dt$$

$$= \frac{1}{\lambda_n^2}[tJ_1(t)] = \frac{1}{\lambda_n^2}[\lambda_n x J_1(\lambda_n x)] = \frac{x}{\lambda_n} J_1(\lambda_n x)$$

Exercise 1: Step 2: Finalizing the Coefficient a_n

- Now we can evaluate the definite integral from the previous step:

$$\int_0^1 xJ_0(\lambda_n x)dx = \left[\frac{x}{\lambda_n}J_1(\lambda_n x)\right]_0^1$$
$$= \left(\frac{1}{\lambda_n}J_1(\lambda_n \cdot 1)\right) - \left(\frac{0}{\lambda_n}J_1(\lambda_n \cdot 0)\right)$$
$$= \frac{J_1(\lambda_n)}{\lambda_n}$$

- Now, we substitute this result back into our formula for a_n:

$$a_n = \frac{2}{[J_1(\lambda_n)]^2}\left(\frac{J_1(\lambda_n)}{\lambda_n}\right)$$

- After canceling terms, we get the final expression for the coefficients:

$$a_n = \frac{2}{\lambda_n J_1(\lambda_n)}$$

Exercise 1: Step 3: The Desired Bessel Series

- We have the function we want to represent:

$$f(x) = 1$$

- We have the general form of the series:

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n x)$$

- And we have calculated the coefficients:

$$a_n = \frac{2}{\lambda_n J_1(\lambda_n)}$$

- It follows that by substituting the coefficients into the series, we obtain the desired result:

$$1 = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_1(\lambda_n)} J_0(\lambda_n x) \quad (0 \leq x < 1)$$

Note on Convergence

According to the Bessel Expansion Theorem, the series converges to $f(x) = 1$ for all x in the interval $[0, 1)$. At the endpoint $x = 1$, the series converges to 0.

Exercise 2: Series for $f(x) = x^p$: Step 1 - Setup

Goal

We want to find the Bessel series expansion for the function $f(x) = x^p$ on the interval $0 \le x \le 1$. The series has the form:

$$x^p = \sum_{k=1}^{\infty} a_k J_p(\lambda_k x)$$

where λ_k are the positive zeros of $J_p(x)$.

- The general formula for the coefficients a_k is:

$$a_k = \frac{2}{[J_{p+1}(\lambda_k)]^2} \int_0^1 x f(x) J_p(\lambda_k x) dx$$

- Substituting our function $f(x) = x^p$, the formula becomes:

$$a_k = \frac{2}{[J_{p+1}(\lambda_k)]^2} \int_0^1 x \cdot x^p \cdot J_p(\lambda_k x) dx$$

- Our main task is to evaluate the following integral:

$$I = \int_0^1 x^{p+1} J_p(\lambda_k x) dx$$

Exercise 2: Series for $f(x) = x^p$: Step 2 - Calculation

- To solve the integral $I = \int_0^1 x^{p+1} J_p(\lambda_k x)dx$, we use the integration identity:

$$\int t^{p+1} J_p(t)dt = t^{p+1} J_{p+1}(t)$$

- We use the substitution $t = \lambda_k x$, which means $x = t/\lambda_k$ and $dx = dt/\lambda_k$.

$$I = \int_0^{\lambda_k} \left(\frac{t}{\lambda_k}\right)^{p+1} J_p(t)\frac{dt}{\lambda_k} = \frac{1}{\lambda_k^{p+2}} \int_0^{\lambda_k} t^{p+1} J_p(t)dt$$
$$= \frac{1}{\lambda_k^{p+2}} \left[t^{p+1} J_{p+1}(t)\right]_0^{\lambda_k} = \frac{1}{\lambda_k^{p+2}} [\lambda_k^{p+1} J_{p+1}(\lambda_k) - 0] = \frac{J_{p+1}(\lambda_k)}{\lambda_k}$$

- Substitute this result back into the formula for a_k:

$$a_k = \frac{2}{[J_{p+1}(\lambda_k)]^2} \cdot \frac{J_{p+1}(\lambda_k)}{\lambda_k} = \frac{2}{\lambda_k J_{p+1}(\lambda_k)}$$

Final Result

The Bessel series for x^p is given by:

$$x^p = 2\sum_{k=1}^{\infty} \frac{J_p(\lambda_k x)}{\lambda_k J_{p+1}(\lambda_k)}$$

Exercise 3:

Problem

Prove that $\frac{d}{dx}\left[xJ_p(x)J_{p+1}(x)\right] = x\left[J_p^2(x) - J_{p+1}^2(x)\right]$.

- **Strategy:** Instead of a brute-force product rule on the original expression, we can use a more elegant method by defining two auxiliary functions based on fundamental Bessel identities.
- Let's define $U(x)$ and $V(x)$ as follows:
    - $U(x) = x^{p+1}J_{p+1}(x)$, for which we know the derivative is $U'(x) = x^{p+1}J_p(x)$.
    - $V(x) = x^{-p}J_p(x)$, for which we know the derivative is $V'(x) = -x^{-p}J_{p+1}(x)$.
- Notice that the product of these two functions is exactly the term we need to differentiate:

$$U(x)V(x) = (x^{p+1}J_{p+1}(x))(x^{-p}J_p(x)) = xJ_p(x)J_{p+1}(x)$$

Exercise 3:

- Now, we apply the standard product rule to $U(x)V(x)$:

$$\frac{d}{dx}[U(x)V(x)] = U'(x)V(x) + U(x)V'(x)$$

- Substitute the known functions and their derivatives from the previous slide:

$$\frac{d}{dx}[xJ_pJ_{p+1}] = (x^{p+1}J_p(x))(x^{-p}J_p(x)) + (x^{p+1}J_{p+1}(x))(-x^{-p}J_{p+1}(x))$$

$$= x^{p+1-p}J_p^2(x) - x^{p+1-p}J_{p+1}^2(x)$$

$$= xJ_p^2(x) - xJ_{p+1}^2(x)$$

Conclusion

Factoring out x gives the final result:

$$\frac{d}{dx}\left[xJ_p(x)J_{p+1}(x)\right] = x\left[J_p^2(x) - J_{p+1}^2(x)\right] \quad \blacksquare$$

Exercise 4: Expressing $J_2(x)$ and $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$

- We use the recurrence relation:

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$$

Expressing $J_2(x)$

Set $p = 1$:

$$J_2(x) = \frac{2(1)}{x} J_1(x) - J_{1-1}(x) = \frac{2}{x} J_1(x) - J_0(x)$$

Expressing $J_3(x)$

Set $p = 2$:

$$J_3(x) = \frac{2(2)}{x} J_2(x) - J_{2-1}(x) = \frac{4}{x} J_2(x) - J_1(x)$$

Substitute our expression for $J_2(x)$:

$$= \frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

Exercise 5: A Zero of J_{p+1} Between Zeros of J_p

Goal

Prove that between any two consecutive positive zeros of $J_p(x)$, there is at least one zero of $J_{p+1}(x)$.

- Let α and β be two consecutive positive zeros of $J_p(x)$, with $0 < \alpha < \beta$. This means $J_p(\alpha) = 0$ and $J_p(\beta) = 0$.
- Consider the auxiliary function $f(x) = x^{-p} J_p(x)$. At the zeros, we have:

$$f(\alpha) = \alpha^{-p} J_p(\alpha) = 0 \qquad f(\beta) = \beta^{-p} J_p(\beta) = 0$$

- Since $f(\alpha) = f(\beta)$, by **Rolle's Theorem**, there must exist a point γ in the interval (α, β) such that $f'(\gamma) = 0$.
- We use the known recurrence relation for the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$$

- At the point γ, we have $f'(\gamma) = -\gamma^{-p} J_{p+1}(\gamma) = 0$. Since $\gamma > 0$, it must be that $J_{p+1}(\gamma) = 0$.

Conclusion of Part 1

We have shown that at least one zero of $J_{p+1}(x)$ exists between any two consecutive zeros of $J_p(x)$. A similar argument proves the reverse.

Exercise 5: Proving Uniqueness

Goal

Prove that there is *exactly* one zero of $J_{p+1}(x)$ between consecutive zeros of $J_p(x)$. We use proof by contradiction.

- **Assume for contradiction** that there are two (or more) zeros of $J_{p+1}(x)$ between two consecutive zeros of $J_p(x)$.
- Let α and β be consecutive positive zeros of $J_p(x)$.
- Let γ_1 and γ_2 be two distinct zeros of $J_{p+1}(x)$ such that $\alpha < \gamma_1 < \gamma_2 < \beta$.
- From a similar argument to Part 1 (using the function $g(x) = x^{p+1} J_{p+1}(x)$), we know that between any two consecutive zeros of $J_{p+1}(x)$, there must be at least one zero of $J_p(x)$.
- Therefore, between γ_1 & γ_2, there must exist a zero of $J_p(x)$; call it α'.
- This means we have found a zero α' such that $\alpha < \alpha' < \beta$.
- This is a **contradiction**! We started by defining α and β as *consecutive* zeros, meaning no other zero of $J_p(x)$ can exist between them.

Final Conclusion

The assumption that there can be more than one zero must be false. Therefore, between each pair of consecutive positive zeros of $J_p(x)$ there is **exactly one** zero of $J_{p+1}(x)$. The same logic proves the converse, so their zeros interlace.

Thank you for your attention