Partial Differential Equations & Boundary Value Problems

Solution to Problem 1: Eigenvalues & Eigenfunctions

Based on G.F. Simmons, Section 40

November 24, 2025

Recall: The Theory of the Vibrating String

The Boundary Value Problem

We seek a nontrivial solution to the differential equation:

$$y'' + \lambda y = 0$$

subject to boundary conditions where y vanishes at the endpoints.

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Key Results

- The parameter λ must be **positive** ($\lambda > 0$).
- The general solution is:

$$y(x) = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x)$$

- Applying $y(0) = 0$ eliminates the cosine term ($c_2 = 0$), leaving:

$$y(x) = c_1 \sin(\sqrt{\lambda} x)$$

Problem 1(a)

Conditions: $y(0) = 0, \quad y(\pi/2) = 0$

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Step 1: Apply the first condition Since $y(0) = 0$, we know $y(x) = c_1 \sin(\sqrt{\lambda}x)$.

Step 2: Apply the second condition

$$y(\pi/2) = c_1 \sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0$$

For a nontrivial solution $(c_1 \neq 0)$, the argument must be an integer multiple of π:

$$\sqrt{\lambda}\frac{\pi}{2} = n\pi \quad (n = 1, 2, \dots)$$

Solution

$$\sqrt{\lambda} = 2n \implies \lambda_n = 4n^2$$
$$y_n(x) = \sin(2nx)$$

Problem 1(b)

Conditions: $y(0) = 0, \quad y(2\pi) = 0$

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Derivation: Using $y(x) = c_1 \sin(\sqrt{\lambda} x)$, apply the second condition:

$$\sin(\sqrt{\lambda} \cdot 2\pi) = 0$$

Solving for λ:

$$\sqrt{\lambda}(2\pi) = n\pi \implies \sqrt{\lambda} = \frac{n}{2}$$

Solution

$$\lambda_n = \frac{n^2}{4}$$
$$y_n(x) = \sin\left(\frac{nx}{2}\right)$$

Problem 1(c)

Conditions: $y(0) = 0, \quad y(1) = 0$

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Conditions: $y(0) = 0, \quad y(1) = 0$

Derivation:

$$\sin(\sqrt{\lambda} \cdot 1) = 0$$

$$\sqrt{\lambda} = n\pi$$

Solution

$$\lambda_n = n^2\pi^2$$

$$y_n(x) = \sin(n\pi x)$$

Problem 1(d): General Interval Length L

Conditions: $y(0) = 0, \quad y(L) = 0$ where $L > 0$

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Conditions: $y(0) = 0, \quad y(L) = 0$ where $L > 0$

This is the standard case derived in the text.

$$\sin(\sqrt{\lambda}L) = 0 \implies \sqrt{\lambda}L = n\pi$$

Solution

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$
$$y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Problem 1(e): Symmetric Interval

Conditions: $y(-L) = 0, \quad y(L) = 0$ where $L > 0$

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Strategy: The interval has length $2L$. We can shift the coordinate system to match case 1(d). Let $t = x + L$. Then $x = -L \implies t = 0$ and $x = L \implies t = 2L$.

Using the result from 1(d) with length $2L$:

$$\lambda_n = \frac{n^2\pi^2}{(2L)^2} = \frac{n^2\pi^2}{4L^2}$$

The eigenfunction in terms of t is $\sin(\frac{n\pi t}{2L})$. Substituting $t = x + L$:

Solution

$$\lambda_n = \frac{n^2\pi^2}{4L^2}, \quad y_n(x) = \sin\left(\frac{n\pi(x+L)}{2L}\right)$$

Problem 1(f): General Shifted Interval

Conditions: $y(a) = 0, \quad y(b) = 0$ where $a < b$

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Conditions: $y(a) = 0, \quad y(b) = 0$ where $a < b$

Strategy: The interval length is $L_{eff} = b - a$. We shift the coordinate system by defining $t = x - a$.

- At $x = a$, $t = 0$.
- At $x = b$, $t = b - a$.

Applying the general formula for length $(b - a)$:

Solution

$$\lambda_n = \frac{n^2\pi^2}{(b-a)^2}$$

$$y_n(x) = \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$