1. Guass's Hypergeometric Equation (GHE): The famous differential equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0 \qquad (0.1)$$

where $a,\ b$ and c are constants, is called *Guass's Hypergoemetric equation*.

What will we learn in this section?

- In this section we will find the general solution of GHE at its regular points.
- We realise that many equations are of the form GHE after a suitable change of independen variable.
- In particular, we find the general solutions of the equation

$$(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0$$

where $A \neq B$ by using the general solution of GHE.

If we write GHE in the form of standard second order linear equation, we get

$$P(x) = \frac{c - [a+b+1]x}{x(1-x)} \quad \text{and} \quad Q(x) = \frac{-abx}{x(1-x)}.$$

What are the sigular points: $x = 0$ and $x = 1$ are only singular points.

Are these regular points? Yes, $x = 0$ and $x = 1$ are regular points, since

$$\begin{aligned} xP(x) &= \frac{c - [a+b+1]x}{(1-x)} = [c - (a+b+1)x](1 + x + x^2 + \cdots) \\ &= c + [c - (a+b+1)]x + \cdots \qquad \text{and} \\ x^2 Q(x) &= \frac{-abx}{1-x} = -abx(1 + x + x^2 + \cdots) \\ &= -abx - abx^2 - \cdots . \end{aligned}$$

Similarly we can show that $x = 1$ is a regular point. We will now find the general solutions of GHE at $x = 0$ and $x = 1$.

2. General solution at the regular point $x = 0$: From the above expansions of $xP(x)$ and $x^2Q(x)$, $p_0 = c$ and $q_0 = 0$ so the indicial equation is

$$m(m-1) + mc = 0 \quad \text{and} \quad m[m-(1-c)] = 0$$

and the *exponents* are $m = 0$ and $m = 1 - c$.

2.1. Solution correspoding to the exponent $m = 0$: The exponent $m = 0$ corresponds a Frobenius series solution by Theorem A if $1 - c < 0$ or the difference $(1-c) - 0 = 1 - c$ is not a positive integer.

Note that the second condition implies the first one, so if $1 - c$ is not a positive integer equivalently c is neither zero nor negative integer, then $m = 0$ corresponds a Frobenius series solution of the form

$$y = x^0 \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{0.2}$$

where a_0 is nonzero.

On substituting the above into (0.1) and equating the coefficients of x^n to zero, we obtaion the following relations:

$$a_1 = \frac{ab}{c}\, a_0; \quad a_2 = \frac{(a+1)(b+1)}{2(c+1)}\, a_1 = \frac{a(a+1)b(b+1)}{2c(c+1)}\, a_0$$

$$a_{n+1} = \frac{(a+n)(b+n)}{(n+1)(c+n)}\, a_n. \tag{0.3}$$

With these coefficients and by letting $a_0 = 1$, the solution (0.2) becomes

$$\begin{aligned} y &= 1 + \frac{ab}{c}\, x + \frac{a(a+1)b(b+1)}{2c(c+1)}\, x^2 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}\, x^n \end{aligned} \tag{0.4}$$

This is known as the *hypergeometric series*, and is denoted by the symbol $F(a, b, c, x)$. It is called by this name because it generalise the familiar geometric series as follows:

when $a = 1$ and $c = b$, we obtain

$$F(1, b, b, x) = 1 + x + x^2 + \cdots = \frac{1}{1 - x}$$

If either a or b is either zero or negative integer, the series (0.4) breaks off and is a polynomial; otherwise the ratio tests shows that it converges for $|x| < 1$, since (0.3) gives

$$\left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \left| \frac{(a+n)(b+n)}{(n+1)(c+n)} \right| |x| \to |x| \quad \text{as} \quad n \to \infty.$$

When c is neither zero nor negative integer, $F(a, b, c, x)$ is an analytic function called *hypergeometric function* on the interval $|x| < 1$.

It is the simplest paritcular solution of the hypergeometrc equation and it has a great many properties, of which the most obvious is that it is unaltered when a and b are interchanged: $F(a, b, c, x) = F(b, a, c, x)$.

2.2. Solution corresponding to the second exponent $m = 1 - c$: If $1 - c$ is neither zero nor negative integer i.e., c is not a positive integer, then Theorem A also tells us that there is second independent solution of GHE (0.1) near $x = 0$ with exponent $m = 1 - c$.

This solution can be found directly, by substituting

$$y = x^{1-c}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

into GHE (0.1) and calculating the coefficients.

The other way of finding the solution is to change the dependent variable in (0.1) from y to z by writting $y = x^{1-c} z$.

When the necessary computations are performed, equation (0.1) becomes

$$\begin{aligned} x(1-x)z'' + \big[(2-c) - ([a-c+1] + [b-c+1] + 1)x\big] z' \\ - (a-c+1)(b-c+1)z = 0 \quad (0.5) \end{aligned}$$

which is the hypergeometric equation with the costants a, b and c are replace by $a - c + 1$, $b - c + 1$ and $2 - c$.

We already know that (0.5) (from the previous Subsection) has the power series solution

$$z = F(a - c + 1, b - c + 1, 2 - c, x)$$

near the origin, so our desired second solution is

$$y = x^{1-c} z = x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x).$$

Accordingly, when c is not an integer, we have two independent Frobenius series solutions and hence

$$y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a - c + 1, b - c + 1, 2 - c, x) \qquad (0.6)$$

is the general solution of the hypergeometric equation (0.1) near the sigular point $x = 0$.

Note that the above solution is only valid near the origin. We now solve GHE (0.1) near another singular point $x = 1$.

3. General solution of GHE at the regular point $x = 1$: The simplest procedure is to obtain this solution from the one already found, by introducing a new independent variable $t = 1 - x$.
($x = 1 - t$, $dy/dx = -dy/dt$ and $d^2y/dx^2 = d^2y/dt^2$).

This makes $x = 1$ correspond to $t = 0$ and transforms (0.1) into

$$t(1-t)y'' + \big[(a+b-c+1) - (a+b+1)t\big]y' - aby = 0$$

where the primes denotes the derivatives with respect to t.

Since the above is a hypergeometric equation, its general solution near $t = 0$ can be written down at once from (0.6) by replacing x by t and c by $a + b - c + 1$ and then we replace t by $1 - x$ to get the general solution of GHE (0.1) near $x = 1$:

$$\begin{aligned} y = {} & c_1 F(a, b, a+b-c+1, 1-x) \\ & + c_2(1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-x) \end{aligned} \quad (0.7)$$

In this case it is necessary to assume that $c - a - b$ is not an integer.

Some comments and remarks on hypergeometric functions F:
Formulas (0.6) and (0.7) show that the adaptability of the constants in equation GHE (0.1) makes it possible to express the general solution of this equation near each of its singular points in terms of the single function F.

More than this is true. The general solution of a wide class of differential equations can be written in terms of hypergeometric functions F.

Any differential equation in which the coefficients of y'', y' and y are polynomials of degree 2, 1 and 0 repsectively and also the first of these polynomials has distinct real roots can be brought into hypergeometric form by a linear change of the independent variable, and hence can be solved near its singular point in terms of the hypergeometric function.

We brief the above remarks in a concrete way, we consider the class of the equation of this type:

$$(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0 \tag{0.8}$$

where $A \neq B$.

If we change the independent variable from x to t linearly by means of $(x-A)(x-B) = kt(t-1)$ for some constant k i.e.

$$t = \frac{x-A}{B-A}$$

then $x = A$ corresponds to $t = 0$ and $x = B$ to $t = 1$.

After changing the independendet variable from x to t the equation (0.8) assumes the form

$$t(1-t)y'' + (F + Dt)y' + Hy = 0,$$

where F, G and H are certain combinations of the constants in (0.8) and the primes indicate derivatives with respect to t.

This a hypergeometric equation with constants a, b and c defined by $F = c$, $G = -(a+b+1)$ and $H = -ab$ and can therefore be solved near $t = 0$ and $t = 1$ in terms of the hypergeometric function.

But this means that (0.8) can be solved in terms of the same function near $x = A$ and $x = B$.

Problems:

Find the general solution of each of the following differential equations near the indicated singular point in terms of hypergeometric function.

1. $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0, \qquad x = 0;$
2. $(2x^2 + 2x)y'' + (1 + 5x)y' + y = 0, \qquad x = 0;$
3. $(x^2 - 1)y'' + (5x + 4)y' + 4y = 0, \qquad x = -1;$
4. $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0, \qquad x = 3.$

Hint: Convert the equation into hypergeometric equation with a suitable change of indipendent variable as explained in the previous slides.

Before we start to solve these problems, we recall hypergeometric equation and its general solutions near its regual singular points.

Hypergeometric equation:

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

General solution at $x = 0$:

$$y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, c-2, x)$$

General solution at $x = 1$:

$$\begin{aligned} y = {} & c_1 F(a,b,a+b-c+1,1-x) \\ & + c_2 (1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-x) \end{aligned}$$

Hypergeometric function:

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)} x^n$$

Solution of Problem 1: $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0, \qquad x = 0;$

If we compare this equation with the hypergeometric equation we have the coefficients

$$c = \frac{3}{2}; \quad a + b + 1 = 2; \quad ab = -2.$$

Or $c = \dfrac{3}{2}; \quad a + b = 1; \quad ab = -2$. And then $a - b$ can be found by using the formula $a - b = \sqrt{(a+b)^2 - 4ab} = \sqrt{1 - 4(-2)} = 3.$

By solving these equations we will get $a = 2, \;\; b = -1, \;\; c = \frac{3}{2}$.

Therefore the General solution at $x = 0$ is given by

$$\begin{aligned} y =& c_1 F(a, b, c, x) + c_2 x^{1-c} F(a - c + 1, b - c + 1, c - 2, x) \\ =& c_1 F\left(2, -1, \frac{3}{2}, x\right) + c_2 x^{-\frac{1}{2}} F\left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, x\right) \\ =& c_1 \left(1 - \frac{4}{3}x\right) + c_2 x^{-\frac{1}{2}} F\left(\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, x\right). \end{aligned}$$

Solution of Problem 4: $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0, \qquad x = 3.$

We use the change of independent variable from x to t by $t = \frac{(x-A)}{(B-A)}$, where A and B are roots of the polynomial $x^2 - x - 6$ which is a coefficient of y'' in given equation.

We see that $A = 3$ and $B = -2$ so that $t = \dfrac{3-x}{5}$ or $x = -5t + 3$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-1}{5}\frac{dy}{dx}; \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{-1}{5}\frac{dy}{dx}\right) = \frac{1}{25}\frac{d^2y}{dt^2}.$$

This transformtion reduces the equation into

$$(-5t)(-5t+5)\left(\frac{1}{25}\right)\frac{d^2y}{dt^2} + (5 - 15t + 9)\left(\frac{-1}{5}\right)\frac{dy}{dx} + y = 0$$

$$t(1-t)y'' + \left(\frac{14}{5} - 3t\right)y' - y = 0.$$

If we compare this equation with the hypergeometric equation we have the coefficients

$$c = \frac{14}{5}; \quad a + b + 1 = 3; \quad ab = 1.$$

$a + b = 2;\ ab = 1$ and hence $a - b = \sqrt{4 - 4} = 0$. Therefore

$$a = 1, \quad b = 1, \quad c = \frac{14}{5}.$$

The General solution at $t = 0$ or $x = 3$ is given by

$$\begin{aligned} y =& c_1 F(a, b, c, t) + c_2 t^{1-c} F(a - c + 1, b - c + 1, c - 2, t) \\ =& c_1 F\left(1, 1, \frac{14}{5}, t\right) + c_2 t^{-\frac{9}{5}} F\left(\frac{-4}{5}, \frac{-4}{5}, \frac{-4}{5}, t\right) \quad \left(\text{since } \ t = \frac{3 - x}{5}\right) \\ =& c_1 F\left(1, 1, \frac{14}{5}, \frac{3 - x}{5}\right) + c_2 \left(\frac{3 - x}{5}\right)^{-\frac{9}{5}} F\left(\frac{-4}{5}, \frac{-4}{5}, \frac{-4}{5}, \frac{3 - x}{5}\right) \end{aligned}$$

Problme 5. Some differential equations are of the hypergeometric type even though they may not appear to be so. Find the general solution of

$$(1-e^x)y'' + \frac{1}{2}y' + e^x y = 0$$

near $x = 0$ by changing the independent varaible to $t = e^x$.

Sol. The transformation $t = e^x$ implies $dt/dx = e^x = t$ and

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = t\,\frac{dy}{dt}; \quad \frac{d^2y}{dt^2} = t^2\,\frac{d^2y}{dt^2} + t\,\frac{dy}{dt}$$

We subsitute all these in given equation to get

$$t^2(1-t)\frac{d^2y}{dt^2} + t\left(\frac{3}{2} - t\right)\frac{dy}{dt} + ty = 0.$$

Or

$$t(1-t)\frac{d^2y}{dt^2} + \left(\frac{3}{2} - t\right)\frac{dy}{dt} + y = 0.$$

After comparing with standard hypergeometric equation we get the constants $c = \frac{3}{2}$, $a + b + 1 = 1$ and $ab = -1$. If solve these equations. then we will get $a = 1$, $b = -1$, $c = \frac{3}{2}$.

The General solution at $t=1$ or $x=0$ is given by

$$\begin{aligned} y =& c_1 F(a,b,a+b-c+1,1-t) \\ & \qquad + c_2(1-t)^{c-a-b}F(c-b,c-a,c-a-b+1,1-t) \\ =& c_1 F\left(1,-1,-\frac{1}{2},1-t\right) + c_2(1-t)^{\frac{3}{2}}F\left(\frac{5}{2},\frac{1}{2},\frac{5}{2},1-t\right) \\ =& c_1 F\left(1,-1,-\frac{1}{2},1-e^x\right) + c_2(1-e^x)^{\frac{3}{2}}F\left(\frac{5}{2},\frac{1}{2},\frac{5}{2},1-e^x\right). \end{aligned}$$

Problem 6. Consider the Chebyshev's equation

$$(1-x^2)y'' - xy' + p^2 y = 0,$$

where p is a nonegative constant. Tranform it into a hypergeometric equation by changing the independent variable from x to $t=\frac{1}{2}(1-x)$, and show that its general solution near $x=1$ is

$$y = c_1 F\left(p,-p,\frac{1}{2},\frac{1-x}{2}\right) + c_2\left(\frac{1-x}{2}\right)^{1/2} F\left(p+\frac{1}{2},-p+\frac{1}{2},\frac{3}{2},\frac{1-x}{2}\right).$$

Problem 7. Consider the differential equation

$$x(1-x)y'' + [p-(p+2)x]y' - py = 0$$

where p is a constant.

(a) If p is not an integer, find the general solution near $x=0$ in terms of hypergeometric functions.

(b) Write the general solution found in (a) in terms of elementary functions.

(c) When $p=1$, the differential equation becomes

$$x(x-1)y'' + (1-3x)y' - y = 0$$

and the solution in (b) is no longer the general solution. Find the general solution in this case by the method of the use of known solution to find another.

Problem 8. Consider the *Legender's equation*:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{0.9}$$

where n is understood to be nonnegative integer.

We have found analytic solutions near ordinary poitn $x = 0$. However, the solutions most useful in the applications are those bounded near $x = 1$.

(a) Find the general solution near $x = 1$ by changing the independent variable from x to $\frac{1}{2}(1-x)$

(b) Find the solutions from (a) that are bounded near $x = 1$.

Sol. The change of independent variable x to $\frac{1}{2}(1-x)$ makes $x = 1$ correspond to $t = 0$ and transforms the equation (0.9) into

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0 \tag{0.10}$$

where the primes denotes derivatives with repsect to t.

This is hypergeometric equation with $a = -n$, $b = n+1$, and $c = 1$, so it has the following polynomial solution near $t = 0$:

$$y_1 = F(-n, n+1, 1, t). \tag{0.11}$$

Since the exponents of the equation (0.10) at the origin $t = 0$ are both zero ($m_1 = 0$ and $m_2 = 1 - c = 0$), we seek a second solution via the formula $y_2(t) = y_1(t)v(t)$ with $v(t)$ is given by

$$v(t) = \int \frac{1}{y_1^2} e^{-\int P(t)dt} dt$$

where $P(t) = (1-2t)/t(1-t)$. Note that

$$P(t) = \frac{1-2t}{t(1-t)} = \frac{1}{t} - \frac{1}{1-t}$$

therefore

$$e^{-\int P(t)dt} = e^{-\ln t - \ln(1-t)} = \frac{1}{t(1-t)}.$$

So we have that

$$v(t) = \int \frac{1}{y_1^2} \frac{1}{t(1-t)} dt = \int \frac{1}{t} \left[\frac{1}{y_1^2(1-t)} \right] dt$$

Since y_1 is a polynomial with constant term 1, the bracket expression on right is an analtyic function of the form $1 + a_1 t + a_2 t^2 + \cdots$. This yields $v(t) = \log t + a_1 t + \cdots$, so

$$y_2 = y_1(\log t + a_1 t + \cdots).$$

and the general solution is given by

$$y = c_1 y_1 + c_2 y_2. \tag{0.12}$$

If we replace t in (0.12) by $\frac{1}{2}(1-x)$, we will get the general solution of Legendre's equation.

Because of the presence of the term $\log t$ in y_2, it is clear that (0.12) is bounded if and only if $c_2 = 0$.

If we replace t in (0.11) by $\frac{1}{2}(1-x)$, it follows that the solutions of Legendre's equation (0.9) are bounded near $x = 1$ are precisely constant multiple of the polynomial $F\left[-n, n+1, 1, \frac{1}{2}(1-x)\right]$.

Here the polynomial $F\left[-n, n+1, 1, \frac{1}{2}(1-x)\right]$ is called *n-th Legendre polynomial*.

Problem 9. Verify the following identities:

(a) $(1+x)^p = F(-p, b, b, -x)$

(b) $e^x = \lim_{b\to\infty} F\left(a, b, a, \frac{x}{b}\right)$

(c) $\log(1+x) = xF(1, 1, 2, -x)$

(d) $\cos x = \lim_{a\to\infty} F\left(a, a, \frac{1}{2}, \frac{-x^2}{4a^2}\right)$.

Thank you for your attention