Sine & Cosine at Key Angles

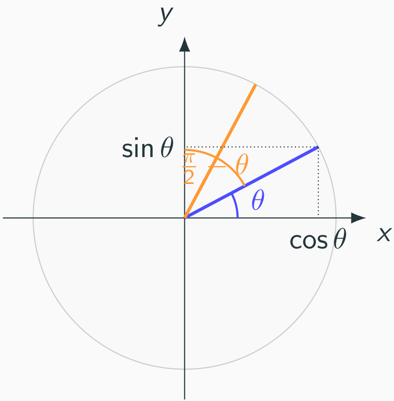
with Cofunction Identities

(This slide is also available in the "teaching section" of the website dbhoriya.github.io)

Some Trigonometry identities:

Exact Values at Common Angles

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0



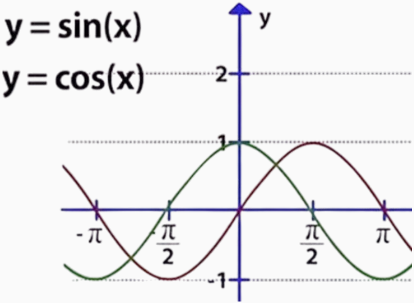
Some identities:

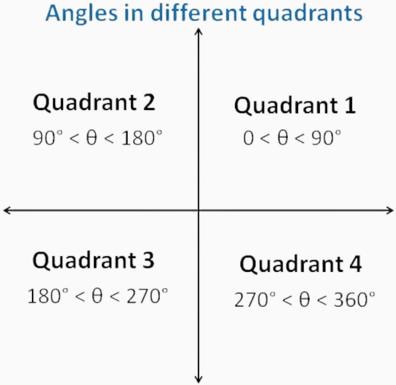
$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta.$$

Example: $\sin 30° = \cos 60° = \frac{1}{2}, \quad \cos 30° = \sin 60° = \frac{\sqrt{3}}{2}.$

Sin-Cos in various quadrants





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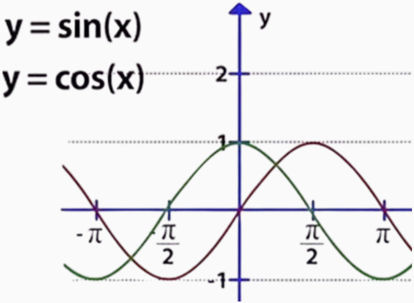
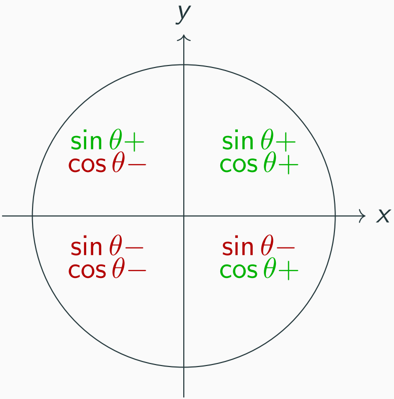




Polar co-ordinates

Definition and motivation

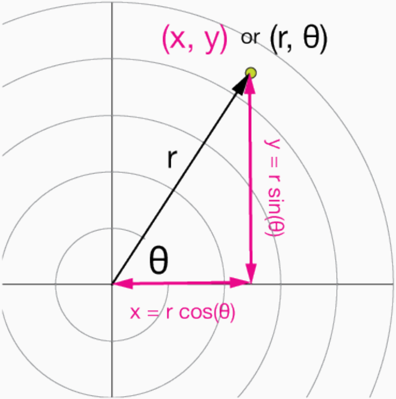
Polar Coordinates: Definition & Formulas

- A point (x, y) is represented as (r, θ):
  - r = distance from origin
  - θ = angle from positive x-axis
- Conversion to Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

- Conversion from Cartesian coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x} = \tan^{-1}\left(\frac{y}{x}\right)$$



Why Do We Need Polar Coordinates? Real-Life Motivation

- Some situations are easier to describe with **distance + angle** than (x, y).
- **Examples:**
  - GPS: "5 km at 30° NE"
  - Radar/Sonar: Track objects by angle & range
  - Lighthouses: Light beam at a specific distance & direction
  - CCTV: Camera coverage via rotation & reach

Engineering & Math Usefulness

- **Engineering:**
  - AC circuits: voltages/currents as polar vectors
  - Mechanical systems: gears, turbines, rotating parts
  - Robotics: arm positions via joint angles
  - Civil structures: domes, arches, tunnels
- **Mathematics:**
  - Circle equation: $x^2 + y^2 = a^2 \Rightarrow r = a$ (simpler in polar)
  - Ideal for problems with circular or rotational symmetry

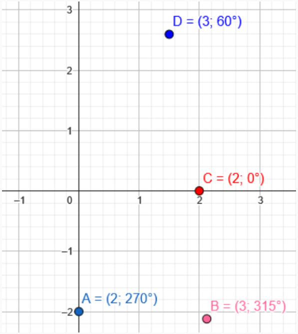
Polar Coordinates: Tutorial Sheet-2

Plot the following points, given in polar coordinates, and find all polar coordinates for each point.

A: $(-2, \pi/2)$, **C:** $(-2, -5\pi)$

B: $(3, -\pi/4)$, **D:** $(3, \pi/3)$

Hint: Remember that a point (r, θ) is equivalent to $(-r, \theta+\pi)$ and that angles can be expressed modulo 2π.



Answer:

A: $(-2, \pi/2 + 2n\pi), (2, -\pi/2 + 2n\pi)$ where $n \in \mathbb{Z}$

B: $(3, -\pi/4 + 2n\pi), (-3, 3\pi/4 + 2n\pi)$ where $n \in \mathbb{Z}$.

C: $(-2, -\pi + 2n\pi), (2, 2n\pi)$ where $n \in \mathbb{Z}$.

D: $(3, \pi/3 + 2n\pi), (-3, -2\pi/3 + 2n\pi)$ where$n \in \mathbb{Z}$.

Question 2: Find the polar coordinates of the following points.

(a) $(\sqrt{3}, -1)$, with $0 \leq \theta < 2\pi$ and $r \leq 0$.

(b) $(\sqrt{3}, -1)$, with $-\pi \leq \theta < \pi$ and $r \geq 0$.

Answer: **(a)** $(-2, \pi - \pi/6)$ **(b)** $(2, -\pi/6)$

Question 3: Convert the following into Cartesian equations:

(a) $r^2 + 2r^2 \cos\theta \sin\theta = 1$ **(b)** $r \sin\theta = \ln r + \ln\cos\theta$.

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Answer:

(a) $x^2 + y^2 + 2rcos\theta r \sin\theta = 1 \iff x^2 + y^2 + 2xy = 1$

(b) $r \sin\theta = \ln r + \ln\cos\theta \Longrightarrow r \sin\theta = \ln r \cos\theta$.

This gives $y = \ln x$

Question 5: Sketch the region defined by the inequalities
$-1 \leq r \leq 1$ and $-\pi/4 \leq \theta \leq \pi/4$

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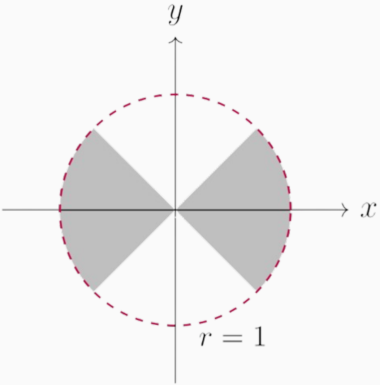
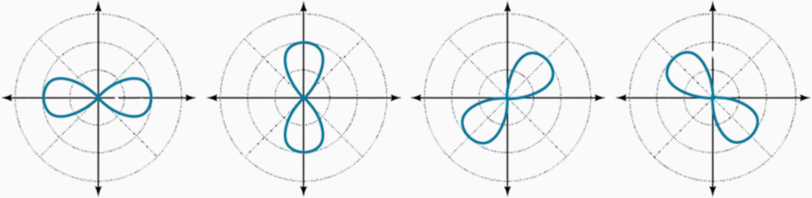
Answer:



Polar Coordinates: lemniscate (a polar curve resembling the infinity symbol ∞)



$r^2 = a^2 cos(2\theta)$
(a)

$r^2 = -a^2 cos(2\theta)$
(b)

$r^2 = a^2 sin(2\theta)$
(c)

$r^2 = -a^2 sin(2\theta)$
(d)

Question 4: Draw the graph of the curve $r^2 = -\sin 2\theta$.