Tutorial sheet 3

Smoothness, Angle in a Space Curve, etc

(dbhoriya.github.io/teaching_F101/tutorial_3.pdf) or SCAN:

Key definitions:

- **Smooth Curve:** A curve $\mathbf{r}(t) = (f(t), g(t), h(t))$ is *smooth* on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ throughout.
- **Velocity (Tangent Vector):** $\mathbf{v}(t) = \mathbf{r}'(t)$, represents the tangent direction and speed.
- **Acceleration:** $\mathbf{a}(t) = \mathbf{r}''(t)$, derivative of velocity.
- **Dot Product:** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta.$$

- **Angle Between Vectors:**

$$\theta = \arccos \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Question 1: Consider the space curve:

$$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + (t^2 + \cos t)\mathbf{j} + (t^3 - \pi t^2)\mathbf{k}, \quad t \neq 0,$$

with

$$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.$$

(i) Find all points t at which $\mathbf{r}(t)$ is **smooth**.

(ii) Find the **angle** between the **tangent** and **acceleration** vectors at $t = \pi$.

Tutorial Sheet-3: Q1 Smoothness of the Curve

Answer: Given:

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + (t^2 + \cos t)\mathbf{j} + (t^3 - \pi t^2)\mathbf{k}, \quad t \neq 0.$

- Velocity: $\mathbf{r}'(t) = (\cos t,\ 2t - \sin t,\ 3t^2 - 2\pi t)$.
- For smoothness:
  - $\mathbf{r}(t)$ must be continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.
  - At $t = 0$, $\lim_{t\to 0} \mathbf{r}(t) = (1, 1, 0) \neq \mathbf{r}(0) = (0, 0, 0) \Rightarrow$ **not smooth**.
  - For $t \neq 0$, $\mathbf{r}'(t) = \mathbf{0}$ would require:

    $$\cos t = 0,\ 2t - \sin t = 0,\ 3t^2 - 2\pi t = 0.$$

    No common $t \neq 0$ satisfies all $\Rightarrow \mathbf{r}'(t) \neq \mathbf{0}$.
- **Conclusion:** Curve is smooth for all $t \neq 0$.

Tutorial Sheet-3: Q1 Smoothness of the Curve

- Velocity (tangent) at $t = \pi$:

$$\mathbf{r}'(\pi) = (-1, 2\pi, \pi^2).$$

- Acceleration at $t = \pi$:

$$\mathbf{r}''(t) = (-\sin t, 2 - \cos t, 6t - 2\pi) \Rightarrow \mathbf{r}''(\pi) = (0, 3, 4\pi).$$

- Dot product:

$$\mathbf{r}'(\pi) \cdot \mathbf{r}''(\pi) = 6\pi + 4\pi^3.$$

- Magnitudes:

$$|\mathbf{r}'(\pi)| = \sqrt{1 + 4\pi^2 + \pi^4}, \quad |\mathbf{r}''(\pi)| = \sqrt{9 + 16\pi^2}.$$

- Angle:

$$\theta = \arccos \frac{6\pi + 4\pi^3}{\sqrt{1 + 4\pi^2 + \pi^4}\sqrt{9 + 16\pi^2}}.$$

- Numerically, $\theta \approx 20^\circ$.