

Tutorial Sheet 8: Multivariable Calculus

Surface Integrals & Flux

Mathematics-I
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Verify everything!

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The Concept: Shadow vs. Surface

The Big Question

Why isn't the surface area just the area of the limits?

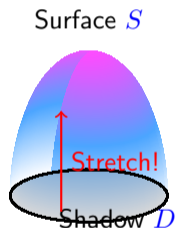
1. The Shadow ($\iint 1 dA$) This calculates the flat “footprint” on the floor.

$$\text{Area} = \pi R^2$$

2. The Surface ($\iint \text{Stretch} dA$) The real surface is curved (tilted). We must multiply the shadow area by a “**Stretch Factor**” (or Tilt Factor) to get the true area.

$$\text{Area} = 2\pi R^2$$

The Double Integral sums up these tiny stretched patches.



Method 1: The Projection Formula

When to use?

When the surface is given as an equation $f(x, y, z) = c$ (e.g., $x^2 + y^2 + z^2 = R^2$).

$$\text{Stretch Factor} = \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|}$$

The Formula

$$S = \iint_D \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

Example (Hemisphere): $f = x^2 + y^2 + z^2 - R^2 = 0$.

$$\nabla f = \langle 2x, 2y, 2z \rangle \implies |\nabla f| = 2R$$

$$|\nabla f \cdot \hat{k}| = 2z$$

$$d\sigma = \frac{2R}{2z} dA = \frac{R}{z} dA$$

Method 2: The Parametric Formula

When to use?

When the surface is given as a vector function $\mathbf{r}(u, v)$. (Useful for cylinders, spheres, cones).

The Logic: We create a tiny “grid” on the curved surface using tangent vectors \mathbf{r}_u and \mathbf{r}_v . The area of one tiny curved patch is the area of the parallelogram formed by these vectors.

$$\text{Patch Area} = |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

The Formula

$$S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

This avoids the “shadow” entirely and works directly on the skin.

Example: Hemisphere via Parametrization

1. Define the Vector Function (Spherical Coords)

$$\mathbf{r}(\phi, \theta) = \langle R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi \rangle$$

2. Find Tangent Vectors

$$\mathbf{r}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = \langle R \cos \phi \cos \theta, R \cos \phi \sin \theta, -R \sin \phi \rangle$$

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -R \sin \phi \sin \theta, R \sin \phi \cos \theta, 0 \rangle$$

3. The Cross Product (The Area Element)

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} = \dots \text{Standard Result} \dots = (R^2 \sin \phi) \hat{e}_r$$

$$d\sigma = |\mathbf{r}_\phi \times \mathbf{r}_\theta| d\phi d\theta = R^2 \sin \phi d\phi d\theta$$

Comparison: Hemisphere Area (R)

Method 1: Projection

$$S = \iint_D \frac{R}{z} dA$$

Convert to Polar ($z = \sqrt{R^2 - r^2}$):

$$S = \int_0^{2\pi} \int_0^R \frac{R}{\sqrt{R^2 - r^2}} r dr d\theta$$

$$\dots = 2\pi R^2$$

Method 2: Parametric

$$S = \iint_D R^2 \sin \phi d\phi d\theta$$

Limits: $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$.

$$S = R^2 \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \phi d\phi$$

$$= R^2 (2\pi)(1) = 2\pi R^2$$

Result: Both methods give the same answer ($2\pi R^2$), which is exactly twice the shadow area (πR^2).

Recall: Surface Area & Projection Tricks

General Formula

To find the area of surface S ($f(x, y, z) = c$), project it onto a domain D with normal \hat{p} :

$$A_S = \iint_D \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

The Trick: Choosing the Projection Plane (\hat{p})

Standard (XY)

Use $\hat{p} = \hat{k}$

Best when z is easily isolated:

$$z = f(x, y)$$

Trick (XZ)

Use $\hat{p} = \hat{j}$

Best when y is constrained or isolated:

$$y = f(x, z) \quad (\text{like } y \geq 0)$$

Trick (YZ)

Use $\hat{p} = \hat{i}$

Best when x is constrained or isolated:

$$x = f(y, z)$$

Question 1

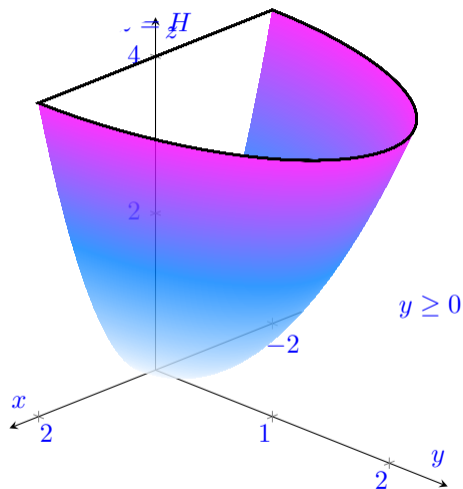
Problem Statement

Find the surface area of the region cut from the paraboloid

$$x^2 + y^2 - z = 0$$

by the plane $z = 0$ and $z = H$, that lies on the side $y \geq 0$.

Visualization (Geometry)



Solution: Setup and Projection

1. Surface and Gradient Let $f(x, y, z) = x^2 + y^2 - z = 0$.

$$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$|\nabla f| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4z + 1}$$

2. Apply the “Trick” (Choice of \hat{p}) Since the problem specifies $y \geq 0$ and we have boundaries in z , we project onto the **XZ-plane**.

$$\hat{p} = \hat{j}$$

$$|\nabla f \cdot \hat{p}| = |2y| = 2y$$

Substituting $y = \sqrt{z - x^2}$ from the surface equation:

$$|\nabla f \cdot \hat{p}| = 2\sqrt{z - x^2}$$

Solution: The Integral Setup

3. **Determine Domain D** In the XZ-plane, y is real when $z - x^2 \geq 0 \implies x^2 \leq z$.

$$D = \{(x, z) : 0 \leq z \leq H, -\sqrt{z} \leq x \leq \sqrt{z}\}$$

4. **Formulate the Integral**

$$S = \iint_D \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA = \int_0^H \int_{-\sqrt{z}}^{\sqrt{z}} \frac{\sqrt{4z+1}}{2\sqrt{z-x^2}} dx dz$$

Factor out terms constant w.r.t x :

$$S = \int_0^H \frac{\sqrt{4z+1}}{2} \left[\int_{-\sqrt{z}}^{\sqrt{z}} \frac{dx}{\sqrt{z-x^2}} \right] dz$$

Solution: Evaluation

5. Evaluate Inner Integral

$$\int_{-\sqrt{z}}^{\sqrt{z}} \frac{dx}{\sqrt{z-x^2}} = \left[\sin^{-1} \left(\frac{x}{\sqrt{z}} \right) \right]_{-\sqrt{z}}^{\sqrt{z}} = \pi$$

6. Final Integration

$$S = \frac{\pi}{2} \int_0^H \sqrt{4z+1} dz$$

Use substitution $u = 4z + 1$:

$$S = \frac{\pi}{8} \left[\frac{2}{3} u^{3/2} \right]_1^{4H+1}$$

$$S = \frac{\pi}{12} \left((4H+1)^{3/2} - 1 \right)$$

Recall: Scalar Fields & Vertical Planes

Scalar Surface Integral

To integrate a function $G(x, y, z)$ over surface S :

$$\iint_S G d\sigma = \iint_D G(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

The Strategy: “Vertical” Surfaces If the surface is a cylinder or plane like $x + y = 1$:

- The normal ∇f is horizontal ($\hat{i} + \hat{j}$).
- $\nabla f \cdot \hat{k} = 0$.
- **Result:** You cannot project onto the XY-plane (division by zero).

The Fix

Switch your projection!

Option A (XZ-Plane): Use $\hat{p} = \hat{j}$.
Eliminates y .

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \hat{j}|} dx dz$$

Option B (YZ-Plane): Use $\hat{p} = \hat{i}$.
Eliminates x .

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \hat{i}|} dy dz$$

Question 2

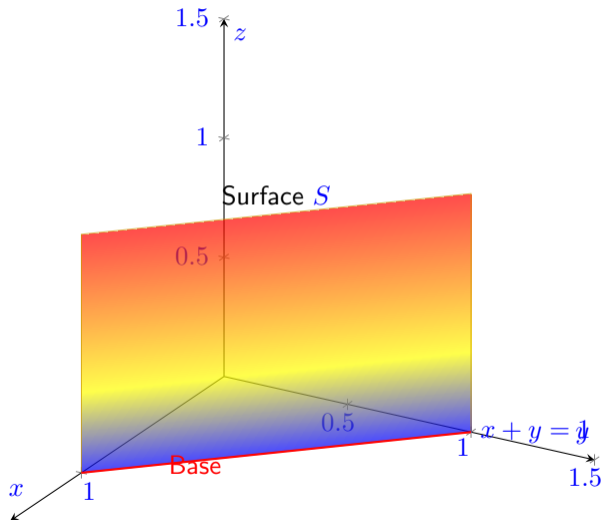
Problem Statement

Integrate $G(x, y, z) = x - y - z$ over the portion of the plane

$$x + y = 1$$

in the first octant between $z = 0$ and $z = 1$.

Visualization (Geometry)



Solution: Setup and Geometry

1. Surface and Gradient Define the surface $f(x, y, z) = x + y - 1 = 0$.

$$\nabla f = \hat{i} + \hat{j}$$

$$|\nabla f| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

2. Choose Projection (The Trick) Since the plane is vertical, we cannot use \hat{k} . We choose the **XZ-plane** (normal $\hat{p} = \hat{j}$).

$$|\nabla f \cdot \hat{p}| = |(\hat{i} + \hat{j}) \cdot \hat{j}| = 1$$

Calculate the surface element $d\sigma$:

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA = \frac{\sqrt{2}}{1} dx dz = \sqrt{2} dx dz$$

Solution: The Integral

3. **Setup the Integral** Substitute $y = 1 - x$ into $G(x, y, z)$:

$$G = x - y - z = x - (1 - x) - z = 2x - 1 - z$$

The domain D in the XZ-plane is the rectangle: $0 \leq x \leq 1$, $0 \leq z \leq 1$.

4. **Evaluate**

$$\begin{aligned}\iint_S G \, d\sigma &= \int_0^1 \int_0^1 (2x - 1 - z) \sqrt{2} \, dz \, dx \\ &= \sqrt{2} \int_0^1 \left[(2x - 1)z - \frac{z^2}{2} \right]_0^1 dx \\ &= \sqrt{2} \int_0^1 \left(2x - 1 - \frac{1}{2} \right) dx = \sqrt{2} \int_0^1 \left(2x - \frac{3}{2} \right) dx \\ &= \sqrt{2} \left[x^2 - \frac{3}{2}x \right]_0^1 = \sqrt{2} (1 - 1.5) = -\frac{\sqrt{2}}{2}\end{aligned}$$

Recall: Flux Formulas (2D vs 3D)

2D Flux (Across a Curve)

For $\mathbf{F} = M\hat{i} + N\hat{j}$ crossing a curve C :

$$\text{Flux} = \oint_C \mathbf{F} \cdot \hat{n} \, ds$$

Coordinate Form:

$$\text{Flux} = \oint_C (M \, dy - N \, dx)$$

3D Flux (Across a Surface)

For $\mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k}$ crossing surface S :

$$\text{Flux} = \iint_S \mathbf{F} \cdot \hat{n} \, d\sigma$$

Calculation Formula:

$$\iint_D \mathbf{F} \cdot \left(\frac{\nabla g}{|\nabla g \cdot \hat{p}|} \right) dA$$

*Just as $Mdy - Ndx$ is the standard form for line flux,
the $\mathbf{F} \cdot \nabla g$ method is the standard for surface flux.*

Recall: Flux Derivations (2D)

2D Flux: The Short Proof

Let curve vector be $d\mathbf{r} = \langle dx, dy \rangle$.

- **Normal Vector:** Rotate $d\mathbf{r}$ by 90° clockwise to get outward normal direction:

$$\hat{n} ds = \langle dy, -dx \rangle$$

- **Flux Dot Product:**

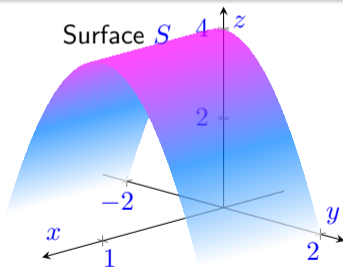
$$\mathbf{F} \cdot \hat{n} ds = \langle M, N \rangle \cdot \langle dy, -dx \rangle$$

$$= M dy - N dx$$

Question 3

Problem Statement

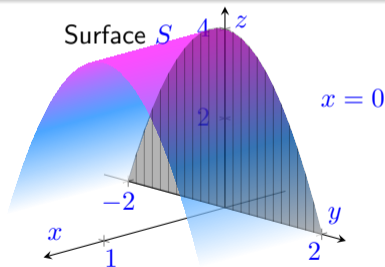
Find the outward flux of $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$ through the surface cut from the cylinder $z = 4 - y^2$ by planes $x = 0$, $x = 1$, and $z = 0$.



Question 3

Problem Statement

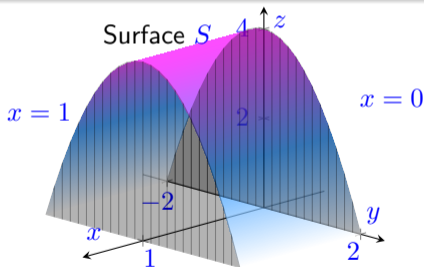
Find the outward flux of $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$ through the surface cut from the cylinder $z = 4 - y^2$ by planes $x = 0$, $x = 1$, and $z = 0$.



Question 3

Problem Statement

Find the outward flux of $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$ through the surface cut from the cylinder $z = 4 - y^2$ by planes $x = 0$, $x = 1$, and $z = 0$.



Solution: Setup & Calculation

1. Surface & Gradient We have $\mathbf{F} = z^2\hat{i} + x\hat{j} - 3z\hat{k}$

$$g = z + y^2 - 4 = 0 \implies \nabla g = 2y\hat{j} + \hat{k}$$

Project on XY-plane ($\hat{p} = \hat{k}$). Outward normal points up ($+\hat{k}$):

$$\hat{n}d\sigma = \frac{\nabla g}{|\nabla g \cdot \hat{k}|} dA = (2y\hat{j} + \hat{k}) dxdy$$

2. Integrand ($\mathbf{F} \cdot \hat{n}d\sigma$)

$$\mathbf{F} \cdot \nabla g = (z^2\hat{i} + x\hat{j} - 3z\hat{k}) \cdot (2y\hat{j} + \hat{k}) = 2xy - 3z$$

Substitute surface $z = 4 - y^2$:

$$\text{Integrand} = 2xy - 3(4 - y^2) = 2xy - 12 + 3y^2$$

3. Final Integral (Limits: $0 \leq x \leq 1, -2 \leq y \leq 2$)

$$\text{Flux} = \int_0^1 \int_{-2}^2 (2xy - 12 + 3y^2) dy dx = -32$$

3D Flux (Across a Surface)

For $\mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k}$ crossing surface S :

$$\text{Flux} = \iint_S \mathbf{F} \cdot \hat{n} d\sigma$$

Calculation Formula:

$$\iint_D \mathbf{F} \cdot \left(\frac{\nabla g}{|\nabla g \cdot \hat{p}|} \right) dA$$

Recall: Surface Area via Parametrization

Parametric Surface Area Formula

If a surface S is defined by a vector function $\mathbf{r}(u, v)$ over a domain D :

$$\text{Area} = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

Steps to Solve:

- 1 **Parametrize:** Choose variables (e.g., r, θ) to define $\mathbf{r}(r, \theta)$.
- 2 **Tangents:** Calculate partial vectors \mathbf{r}_r and \mathbf{r}_θ .
- 3 **Normal:** Find the cross product $\mathbf{r}_r \times \mathbf{r}_\theta$.
- 4 **Integrate:** Compute magnitude and integrate.

Why use this?

The “Projection Method” ($\frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}$) works for graphs $z = f(x, y)$. This “Parametric Method” is more general and often easier for surfaces with **cylindrical symmetry** (like cones and paraboloids).

Question 4

Problem Statement

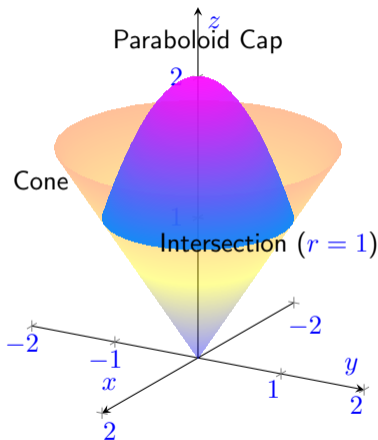
Use a parametrization to express the area of the cap cut from the paraboloid

$$z = 2 - x^2 - y^2$$

by the cone

$$z = \sqrt{x^2 + y^2}$$

Evaluate the integral to find the surface area.



Solution: Intersection & Parametrization

1. Find Intersection (Region D) The cap is cut from the paraboloid by the cone.

$$\text{Paraboloid: } z = 2 - r^2, \quad \text{Cone: } z = r$$

Intersection occurs when:

$$2 - r^2 = r \implies r^2 + r - 2 = 0 \implies (r + 2)(r - 1) = 0$$

Since $r > 0$, we have $r = 1$. The cap exists for $0 \leq r \leq 1$.

2. Define Vector Function $\mathbf{r}(r, \theta)$ Use polar coordinates for x and y .

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = 2 - r^2$$

$$\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 2 - r^2 \rangle$$

Limits: $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$.

Solution: Tangent Vectors

3. Calculate Partial Derivatives

$$\mathbf{r}_r = \frac{\partial \mathbf{r}}{\partial r} = \langle \cos \theta, \sin \theta, -2r \rangle$$

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

4. Compute Cross Product ($\mathbf{r}_r \times \mathbf{r}_\theta$)

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix}$$

$$= \hat{i}(0 - (-2r^2 \cos \theta)) - \hat{j}(0 - (2r^2 \sin \theta)) + \hat{k}(r \cos^2 \theta + r \sin^2 \theta)$$

$$= \langle 2r^2 \cos \theta, 2r^2 \sin \theta, r \rangle$$

Solution: Integration

5. Magnitude and Integral

$$\begin{aligned} |\mathbf{r}_r \times \mathbf{r}_\theta| &= \sqrt{(2r^2 \cos \theta)^2 + (2r^2 \sin \theta)^2 + r^2} \\ &= \sqrt{4r^4(\cos^2 \theta + \sin^2 \theta) + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1} \end{aligned}$$

6. Evaluate

$$\begin{aligned} \text{Area} &= \int_0^{2\pi} \int_0^1 r\sqrt{4r^2 + 1} \, dr \, d\theta \\ &= 2\pi \int_0^1 r(4r^2 + 1)^{1/2} \, dr \end{aligned}$$

Let $u = 4r^2 + 1 \implies du = 8r \, dr$. Limits: $1 \rightarrow 5$.

$$\begin{aligned} &= 2\pi \cdot \frac{1}{8} \int_1^5 u^{1/2} \, du = \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right]_1^5 \\ &= \frac{\pi}{6} (5^{3/2} - 1) \end{aligned}$$

Recall: Stokes' Theorem (Flux of Curl)

Stokes' Theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Direct Calculation (RHS of Theorem): If asked for “Flux of Curl”, calculate the surface integral directly:

$$\text{Flux} = \iint_D (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, d\phi \, d\theta$$

Application: If calculating the line integral is hard, switch to the surface integral (or vice versa)!

Spherical Normal

For sphere radius R :

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = (R^2 \sin \phi) \hat{e}_r$$

Vector points radially outward.

Question 5

Problem Statement

Use the surface integral to calculate the flux of the curl of the field

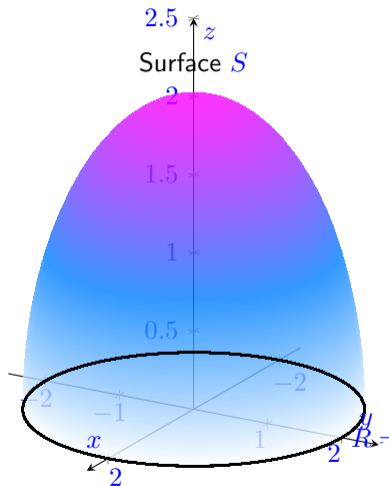
$$\mathbf{F} = y^2 \hat{i} + z^2 \hat{j} + x \hat{k}$$

across S :

$$\mathbf{r}(\phi, \theta) = (2 \sin \phi \cos \theta) \hat{i} + (2 \sin \phi \sin \theta) \hat{j} + (2 \cos \phi) \hat{k}$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$ (Upper Hemisphere).

Visualization (Geometry)



Solution: Curl and Normal Vector

1. Calculate Curl \mathbf{F}

$$\mathbf{F} = \langle y^2, z^2, x \rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & z^2 & x \end{vmatrix}$$

$$= \hat{i}(0 - 2z) - \hat{j}(1 - 0) + \hat{k}(0 - 2y) = \langle -2z, -1, -2y \rangle$$

2. Calculate Normal Vector (\mathbf{N}) For sphere $R = 2$, the normal vector $\mathbf{r}_\phi \times \mathbf{r}_\theta$ is the position vector scaled by $R \sin \phi$:

$$\mathbf{N} = (2 \sin \phi) \mathbf{r} = \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin \phi \cos \phi \rangle$$

Alternatively, computed via cross product determinant.

Solution: The Integral

3. Setup Dot Product Substitute x, y, z into Curl:

$$\nabla \times \mathbf{F} = \langle -4 \cos \phi, -1, -4 \sin \phi \sin \theta \rangle$$

Compute $(\nabla \times \mathbf{F}) \cdot \mathbf{N}$:

$$= (-4 \cos \phi)(4 \sin^2 \phi \cos \theta) + (-1)(4 \sin^2 \phi \sin \theta) + (-4 \sin \phi \sin \theta)(4 \sin \phi \cos \phi)$$

$$= -16 \sin^2 \phi \cos \phi \cos \theta - 4 \sin^2 \phi \sin \theta - 16 \sin^2 \phi \cos \phi \sin \theta$$

4. Integrate

$$I = \int_0^{\pi/2} \int_0^{2\pi} [-16 \sin^2 \phi \cos \phi (\cos \theta + \sin \theta) - 4 \sin^2 \phi \sin \theta] d\theta d\phi$$

Solution: Evaluation (Symmetry)

5. Evaluate Inner Integral ($d\theta$) Notice that $\int_0^{2\pi} \cos \theta \, d\theta = 0$ and $\int_0^{2\pi} \sin \theta \, d\theta = 0$. Every term in our integrand contains either $\sin \theta$ or $\cos \theta$ (or both linearly)!

$$\int_0^{2\pi} (\dots \cos \theta + \dots \sin \theta) \, d\theta = 0$$

Therefore:

$$\text{Flux} = \int_0^{\pi/2} (0) \, d\phi = 0$$

Conclusion

The total flux of the curl through this hemisphere is **zero**.

Recall: The Two Sides of Stokes' Theorem

The Theorem

$$\underbrace{\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n} \, d\sigma}_{\text{Surface Integral}} = \underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Line Integral}}$$

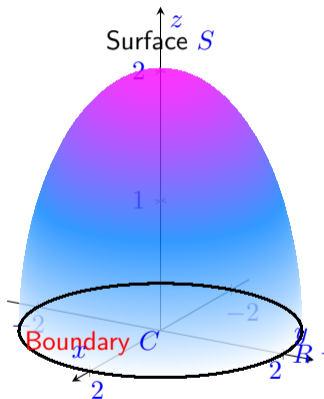
Method A: Surface Integral

- Used when asked for “Flux of Curl”.
- Requires: Curl $\nabla \times \mathbf{F}$, Normal \mathbf{N} , Double Integral.
- **This Question requires Method A.**

Method B: Line Integral

- Used when asked to “Verify” or “Evaluate using Stokes”.
- Requires: Boundary Curve C , Parametrization $\mathbf{r}(t)$, Single Integral.
- Usually faster!

Visualization (Geometry)



Alternative: What if we used the Line Integral?

Hypothetical Scenario

If the question allowed: “Use Stokes’ Theorem...”, we could calculate the circulation around the boundary curve C .

1. Identify Boundary C The edge of the hemisphere is the circle $x^2 + y^2 = 4$ on the plane $z = 0$.

2. Evaluate Field on C Since $z = 0$, the field $\mathbf{F} = y^2\hat{i} + z^2\hat{j} + x\hat{k}$ simplifies:

$$\mathbf{F}_{on\ C} = y^2\hat{i} + 0\hat{j} + x\hat{k}$$

Also, along C , $dz = 0$.

3. Calculate Line Integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (y^2 dx + 0 dy + x \underbrace{dz}_0) = \oint_C y^2 dx$$

Using parametrization $x = 2 \cos t, y = 2 \sin t$:

$$\int_0^{2\pi} (2 \sin t)^2 (-2 \sin t) dt = -8 \int_0^{2\pi} \sin^3 t dt = 0$$

Question 6

Problem Statement

Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field:

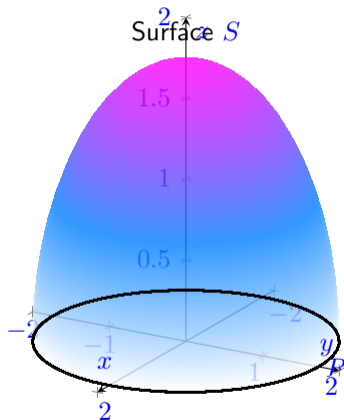
$$\mathbf{F} = 3y\hat{i} + (5 - 2x)\hat{j} + (z^2 - 2)\hat{k}$$

across the surface S (Hemisphere $R = \sqrt{3}$):

$$\mathbf{r}(\phi, \theta) = (\sqrt{3} \sin \phi \cos \theta)\hat{i} + (\sqrt{3} \sin \phi \sin \theta)\hat{j} + (\sqrt{3} \cos \phi)\hat{k}$$

where $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$.

Visualization (Geometry)



Solution: Curl Calculation

1. Calculate the Curl ($\nabla \times \mathbf{F}$)

$$\mathbf{F} = \langle \underbrace{3y}_M, \underbrace{5-2x}_N, \underbrace{z^2-2}_P \rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3y & 5-2x & z^2-2 \end{vmatrix}$$

- \hat{i} -comp: $\partial_y(z^2-2) - \partial_z(5-2x) = 0 - 0 = 0$
- \hat{j} -comp: $\partial_z(3y) - \partial_x(z^2-2) = 0 - 0 = 0$
- \hat{k} -comp: $\partial_x(5-2x) - \partial_y(3y) = -2 - 3 = -5$

$$\nabla \times \mathbf{F} = -5\hat{k}$$

Solution: The Integral

2. **Normal Vector \mathbf{N}** For sphere $R = \sqrt{3}$, $\mathbf{N} = (R \sin \phi)\mathbf{r}$.

$$\mathbf{N}_z = (R \sin \phi)(z) = (R \sin \phi)(R \cos \phi) = 3 \sin \phi \cos \phi$$

3. **Setup Dot Product**

$$(\nabla \times \mathbf{F}) \cdot \mathbf{N} = (-5\hat{k}) \cdot \mathbf{N} = -5(\mathbf{N}_z) = -15 \sin \phi \cos \phi$$

4. **Evaluate**

$$\begin{aligned}\text{Flux} &= \int_0^{2\pi} \int_0^{\pi/2} -15 \sin \phi \cos \phi \, d\phi \, d\theta \\ &= 2\pi(-15) \int_0^{\pi/2} \sin \phi \cos \phi \, d\phi \quad (\text{Let } u = \sin \phi) \\ &= -30\pi \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} = -30\pi \left(\frac{1}{2} \right) = -15\pi\end{aligned}$$

Recall: Divergence Theorem (Gauss's Theorem)

The Formula

The outward flux of \mathbf{F} across a closed surface S is equal to the triple integral of the divergence over the volume V :

$$\iint_S \mathbf{F} \cdot \hat{n} \, d\sigma = \iiint_V (\nabla \cdot \mathbf{F}) \, dV$$

Integration Tip (Elliptical Coords): For regions bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

- $x = ar \cos \theta$
- $y = br \sin \theta$
- $dV = abr \, dz \, dr \, d\theta$ (Don't forget the Jacobian!)

When to use this?

When the surface is **closed** (e.g., bounded by coordinate planes, cylinders, and caps).

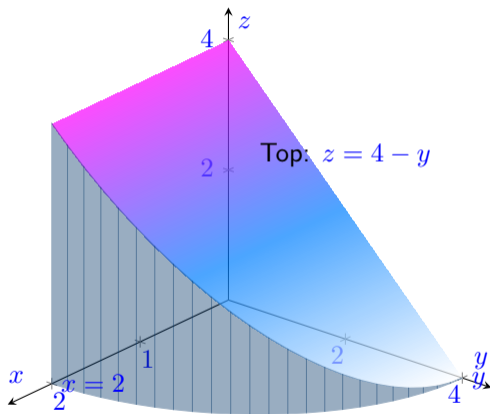
Calculating 5 separate surface integrals (4 faces + curved part) is hard. One volume integral is easy!

Question 7

Problem Statement

Use Divergence theorem to find the flux of $\mathbf{F} = 2xz\hat{i} - xy\hat{j} - z^2\hat{k}$ across the boundary of the region D : the wedge cut from the first octant by the plane $y + z = 4$, and the elliptical cylinder $4x^2 + y^2 = 16$.

Visualization (The Wedge)



Solution: Divergence & Coordinates

1. Calculate Divergence

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(-z^2) \\ &= 2z - x - 2z = -x\end{aligned}$$

2. Setup Coordinate System Boundary: $4x^2 + y^2 = 16 \implies \frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$. Use Elliptical Coordinates:

$$x = 2r \cos \theta, \quad y = 4r \sin \theta, \quad z = z$$

$$\text{Jacobian } J = (2)(4)r = 8r$$

3. Determine Limits (First Octant)

- $r: 0 \rightarrow 1$ (Inside the cylinder)
- $\theta: 0 \rightarrow \pi/2$ (First Octant)
- $z: 0 \rightarrow 4 - y \implies 0 \rightarrow 4 - 4r \sin \theta$

Solution: The Integral

4. Setup Triple Integral

$$\text{Flux} = \iiint_V (-x) dV = \int_0^{\pi/2} \int_0^1 \int_0^{4-4r \sin \theta} \underbrace{(-2r \cos \theta)}_{\text{Div}} \underbrace{(8r)}_{\text{Jac}} dz dr d\theta$$

5. Evaluate (z integral first)

$$\begin{aligned} &= -16 \int_0^{\pi/2} \int_0^1 r^2 \cos \theta [z]_0^{4-4r \sin \theta} dr d\theta \\ &= -16 \int_0^{\pi/2} \int_0^1 r^2 \cos \theta (4 - 4r \sin \theta) dr d\theta \\ &= -64 \int_0^{\pi/2} \cos \theta \left[\int_0^1 (r^2 - r^3 \sin \theta) dr \right] d\theta \end{aligned}$$

Solution: Final Evaluation

6. Evaluate (r and θ integrals)

$$[\dots]_r = \left[\frac{r^3}{3} - \frac{r^4}{4} \sin \theta \right]_0^1 = \frac{1}{3} - \frac{1}{4} \sin \theta$$

Now integrate w.r.t θ :

$$\begin{aligned} \text{Flux} &= -64 \int_0^{\pi/2} \cos \theta \left(\frac{1}{3} - \frac{1}{4} \sin \theta \right) d\theta \\ &= -64 \left[\frac{1}{3} \sin \theta - \frac{1}{4} \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} \\ &= -64 \left(\frac{1}{3}(1) - \frac{1}{8}(1) \right) = -64 \left(\frac{8-3}{24} \right) = -64 \left(\frac{5}{24} \right) = -\frac{40}{3} \end{aligned}$$

Final Answer: The flux is $-\frac{40}{3}$.

Recall: Green's Theorem (2D)

The Theorem

For a positively oriented, simple closed curve C enclosing a region R :

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Why is it useful here? Direct line integration is hard when terms like $\tan(y/2)$ or $\frac{x}{x^2+1}$ are present.

However, if these terms are in the "wrong" component (M or N), their partial derivatives often vanish!

Calculation Strategy

1. Identify M (attached to dx) and N (attached to dy).
2. Compute $\partial_x N - \partial_y M$.
3. Set up double integral limits for R .

Question 8

Problem Statement

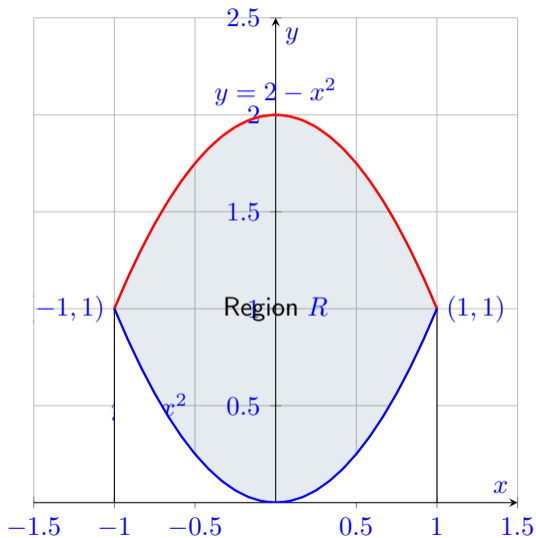
Evaluate the line integral using Green's Theorem:

$$\oint_C \left(\frac{x}{x^2 + 1} - y \right) dx + \left(3x - 4 \tan \frac{y}{2} \right) dy$$

where C is the boundary enclosed by:

- Lower curve: $y = x^2$ from $(-1, 1)$ to $(1, 1)$
- Upper curve: $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$

Visualization Q8 (The Region R)



Solution Q8

1. Partial Derivatives

$$M = \frac{x}{x^2 + 1} - y \implies \frac{\partial M}{\partial y} = -1 \quad N = 3x - 4 \tan(y/2) \implies \frac{\partial N}{\partial x} = 3$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = 3 - (-1) = 4$$

2. Setup Double Integral Limits for R : x goes from -1 to 1 . For each x , y goes from x^2 to $2 - x^2$.

$$I = \iint_R 4 \, dA = 4 \int_{-1}^1 \int_{x^2}^{2-x^2} dy \, dx$$

3. Evaluate

$$= 4 \int_{-1}^1 [y]_{x^2}^{2-x^2} dx = 4 \int_{-1}^1 (2 - x^2 - x^2) dx$$

$$= 4 \int_{-1}^1 (2 - 2x^2) dx = 8 \int_0^1 (2 - 2x^2) dx \quad (\text{Even function}) = 8 \left[2x - \frac{2x^3}{3} \right]_0^1 = \frac{32}{3}$$

Recall: The Divergence Theorem

Statement of the Theorem

Let D be a solid region with a closed boundary surface S oriented outward.

$$\iint_S \mathbf{F} \cdot \hat{n} \, d\sigma = \iiint_D (\nabla \cdot \mathbf{F}) \, dV$$

Physical Intuition: Total outward flux through the boundary = Sum of all "sources" and "sinks" inside the volume.

Strategy for Cylinders

If D is defined by $x^2 + y^2 \leq R^2$:

- 1 Compute $\nabla \cdot \mathbf{F}$.
- 2 Switch to **Cylindrical Coords**.
That is, for $x^2 + y^2 \leq 1$
bounded by $z = \pm 1$, use:
 $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, $-1 \leq z \leq 1$
- 3 **Crucial:** Use $dV = r \, dz \, dr \, d\theta$.

Question 9

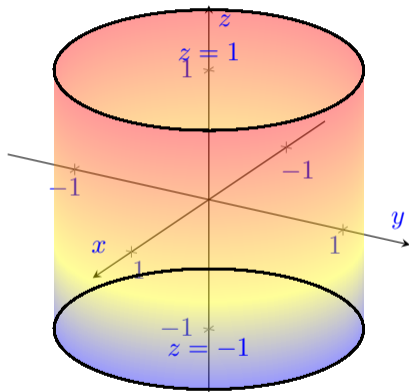
Problem Statement

Using the Divergence Theorem, find the outward flux of the field

$$\mathbf{F} = x^2y^2\hat{i} + 2x^3y\hat{j} + y^3\hat{k}$$

across the boundary of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 1$ and $z = -1$.

Visualization Q9 (Cylinder)



Solution Q9

1. **Calculate Divergence:** $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(2x^3y) + \frac{\partial}{\partial z}(y^3)$

$$= 2xy^2 + 2x^3 + 0 = 2x(y^2 + x^2)$$

2. **Convert to Cylindrical Coordinates:** $x = r \cos \theta, \quad x^2 + y^2 = r^2$

$$\nabla \cdot \mathbf{F} = 2(r \cos \theta)(r^2) = 2r^3 \cos \theta$$

$$dV = r \, dz \, dr \, d\theta$$

3. **Evaluate Integral:** $\text{Flux} = \int_0^{2\pi} \int_0^1 \int_{-1}^1 (2r^3 \cos \theta) r \, dz \, dr \, d\theta$

$$= \left(\int_{-1}^1 dz \right) \left(\int_0^1 2r^4 \, dr \right) \left(\int_0^{2\pi} \cos \theta \, d\theta \right)$$

$$= [z]_{-1}^1 \cdot \left[\frac{2r^5}{5} \right]_0^1 \cdot [\sin \theta]_0^{2\pi}$$

$$= (2) \cdot (2/5) \cdot (0) = 0$$