

Tutorial Sheet 7

Multivariable Calculus / Line Integrals & Green's Theorem

Might be completely! Verify! Ask Doubts!

November 21, 2025

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Overview II

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Question 1: Problem Statement

Evaluate the line integral of the scalar field

$$f(x, y, z) = x + \sqrt{y} - z^2$$

along the piecewise smooth curve C from $(0, 0, 0)$ to $(1, 1, 1)$, where C consists of the following segments:

- $C_1 : \vec{r}(t) = t\hat{k}, \quad 0 \leq t \leq 1$
- $C_2 : \vec{r}(t) = t\hat{j} + \hat{k}, \quad 0 \leq t \leq 1$
- $C_3 : \vec{r}(t) = t\hat{i} + \hat{j} + \hat{k}, \quad 0 \leq t \leq 1$

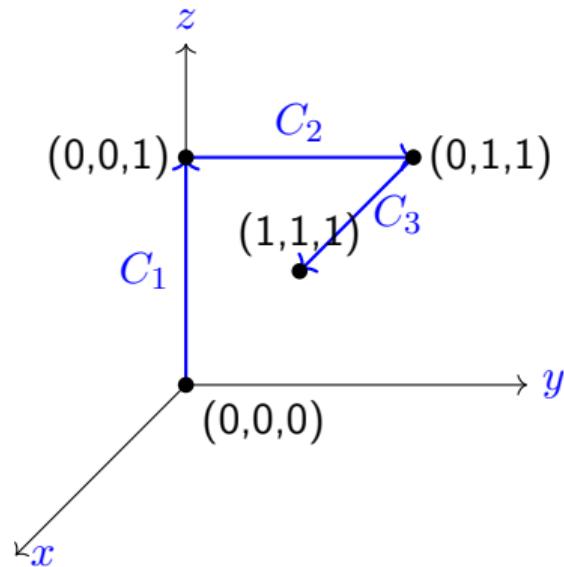
Recall: Scalar Line Integral

Formula

$$\int_C f(x, y, z) \, ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| \, dt$$

Strategy: We will calculate the integral for each segment (C_1, C_2, C_3) individually and sum them up to get the total integral over C .

Question 1: Visualization



Question 1: Solution (Segments 1 & 2)

Segment C_1 : $\vec{r}(t) = t\hat{k}$ ($0 \leq t \leq 1$).

- $\vec{v}(t) = \vec{r}'(t) = \hat{k} \implies |\vec{v}(t)| = 1$.
- On C_1 : $x = 0, y = 0, z = t$.
- $f(x, y, z) = 0 + \sqrt{0} - t^2 = -t^2$.

$$\int_{C_1} f \, ds = \int_0^1 (-t^2)(1) \, dt = \left[-\frac{t^3}{3} \right]_0^1 = -\frac{1}{3}$$

Question 1: Solution (Segments 1 & 2)

Segment C_1 : $\vec{r}(t) = t\hat{k}$ ($0 \leq t \leq 1$).

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$$\int_{C_1} f \, ds = \int_0^1 (-t^2)(1) \, dt = \left[-\frac{t^3}{3} \right]_0^1 = -\frac{1}{3}$$

Segment C_2 : $\vec{r}(t) = t\hat{j} + \hat{k}$ ($0 \leq t \leq 1$).

- $\vec{v}(t) = \hat{j} \implies |\vec{v}(t)| = 1$.
- On C_2 : $x = 0, y = t, z = 1$.
- $f(x, y, z) = 0 + \sqrt{t} - 1^2 = \sqrt{t} - 1$.

$$\int_{C_2} f \, ds = \int_0^1 (\sqrt{t} - 1) \, dt = \left[\frac{2}{3}t^{3/2} - t \right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3}$$

Question 1: Solution (Segment 3 & Total)

Segment C_3 : $\vec{r}(t) = t\hat{i} + \hat{j} + \hat{k}$ ($0 \leq t \leq 1$).

- $\vec{v}(t) = \hat{i} \implies |\vec{v}(t)| = 1$.
- On C_3 : $x = t, y = 1, z = 1$.
- $f(x, y, z) = t + \sqrt{1 - 1^2} = t$.

$$\int_{C_3} f \, ds = \int_0^1 (t) \, dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2}$$

Question 1: Solution (Segment 3 & Total)

Segment C_3 : $\vec{r}(t) = t\hat{i} + \hat{j} + \hat{k}$ ($0 \leq t \leq 1$).

- $\vec{v}(t) = \hat{i} \implies |\vec{v}(t)| = 1$.
- On C_3 : $x = t, y = 1, z = 1$.
- $f(x, y, z) = t + \sqrt{1 - 1^2} = t$.

$$\int_{C_3} f \, ds = \int_0^1 (t) \, dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2}$$

Total Integral:

$$\begin{aligned} \int_C f \, ds &= \int_{C_1} f \, ds + \int_{C_2} f \, ds + \int_{C_3} f \, ds \\ &= \left(-\frac{1}{3} \right) + \left(-\frac{1}{3} \right) + \frac{1}{2} = -\frac{2}{3} + \frac{1}{2} = \frac{-4 + 3}{6} = -\frac{1}{6} \end{aligned}$$

Question 2: Problem Statement

Evaluate

$$\int_C (4x^2 + y^2) dy$$

where C is the ellipse $4x^2 + y^2 = 4$ oriented counterclockwise.

Question 2: Solution

Step 1: Parametrization Rewrite the ellipse equation as $x^2 + \frac{y^2}{4} = 1$. We can set:

$$x = \cos t, \quad y = 2 \sin t, \quad 0 \leq t \leq 2\pi$$

Then, the differential dy is:

$$dy = \frac{d}{dt}(2 \sin t) dt = 2 \cos t dt$$

Question 2: Solution

Step 1: Parametrization Rewrite the ellipse equation as $x^2 + \frac{y^2}{4} = 1$. We can set:

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Then, the differential dy is:

$$dy = \frac{d}{dt}(2 \sin t) dt = 2 \cos t dt$$

Step 2: Substitution Substitute x and y into the integrand $4x^2 + y^2$:

$$4x^2 + y^2 = 4(\cos^2 t) + (2 \sin t)^2 = 4 \cos^2 t + 4 \sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$$

Step 3: Integration

$$\begin{aligned} \int_C (4x^2 + y^2) dy &= \int_0^{2\pi} (4)(2 \cos t dt) = 8 \int_0^{2\pi} \cos t dt \\ &= 8[\sin t]_0^{2\pi} = 8(0 - 0) = 0 \end{aligned}$$

Question 3: Problem Statement

Find the work done by the force field

$$\vec{F}(x, y) = x^2 e^y \hat{i} + y e^x \hat{j}$$

in moving a particle along the curve C given by $y = x^3$ from the point $(0, 0)$ to $(1, 1)$. That is, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

Question 3: Solution (Setup)

Parametrization: Since $y = x^3$, let $x = t$. Then $y = t^3$ for $0 \leq t \leq 1$.

- $dx = dt$
- $dy = 3t^2 dt$

Substitute into Integral:

$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{r} = \int_C (x^2 e^y dx + y e^x dy) \\ &= \int_0^1 \left[(t^2 e^{t^3})dt + (t^3 e^t)(3t^2 dt) \right] \\ &= \int_0^1 t^2 e^{t^3} dt + \int_0^1 3t^5 e^t dt \end{aligned}$$

Question 3: Solution (Calculation)

First Term: $\int_0^1 t^2 e^{t^3} dt$ Let $u = t^3 \implies du = 3t^2 dt \implies t^2 dt = du/3.$

$$\int_0^1 t^2 e^{t^3} dt = \frac{1}{3} \int_0^1 e^u du = \frac{1}{3} [e^u]_0^1 = \frac{e - 1}{3}$$

Second Term: $3 \int_0^1 t^5 e^t dt$ Using tabular integration or reduction ($I_n = e - nI_{n-1}$), we evaluate $\int_0^1 t^5 e^t dt:$

$$\begin{aligned} &= [e^t(t^5 - 5t^4 + 20t^3 - 60t^2 + 120t - 120)]_0^1 \\ &= e(1 - 5 + 20 - 60 + 120 - 120) - (1)(0 - 0 + 0 - 0 + 0 - 120) \\ &= e(-44) - (-120) = 120 - 44e \end{aligned}$$

Multiplying by 3: $3(120 - 44e) = 360 - 132e.$

Total Work:

$$W = \frac{e - 1}{3} + 360 - 132e = \frac{e - 1 + 1080 - 396e}{3} = \frac{1079 - 395e}{3}$$

Recall: Conservative Vector Fields

A field $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative if $\nabla \times \vec{F} = \vec{0}$.

Conditions

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

If conservative, $\vec{F} = \nabla f$. Fundamental Theorem of Line Integrals: $\int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$.

Question 4: Problem Statement

Let $\vec{F}(x, y, z) = y(\cos(xy) + z)\hat{i} + (zx + z \cos(yz) + x \cos(xy))\hat{j} + y(x + \cos(yz))\hat{k}$.

- (a) Show that \vec{F} is conservative in \mathbb{R}^3 , and hence find a potential function f such that $f(1, 1, 1) = 0$.
- (b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the piecewise linear path from $(0, 0, 0)$ to $(1, 1, 0)$ and then to $(1, 1, 1)$.

Question 4(a): Checking Conservative Property

Identify components:

$$M = y \cos(xy) + yz$$

$$N = zx + z \cos(yz) + x \cos(xy)$$

$$P = xy + y \cos(yz)$$

Check Partials:

- $M_y = \cos(xy) - xy \sin(xy) + z$ ✓ Match
- $N_x = z + \cos(xy) - xy \sin(xy)$ ✓ Match
- $M_z = y, P_x = y$ ✓ Match
- $N_z = x + \cos(yz) - yz \sin(yz)$
- $P_y = x + \cos(yz) - yz \sin(yz)$ ✓ Match

Thus, \vec{F} is conservative.

Question 4(a): Finding Potential f

We integrate $\nabla f = \vec{F}$ component by component.

1. Integrate $f_z = P = xy + y \cos(yz)$ w.r.t z :

$$f(x, y, z) = xyz + \sin(yz) + g(x, y)$$

2. Differentiate w.r.t x and match with M :

$$f_x = yz + g_x(x, y) = y \cos(xy) + yz$$

$$\implies g_x = y \cos(xy) \implies g(x, y) = \sin(xy) + h(y)$$

3. Current f : $f = xyz + \sin(yz) + \sin(xy) + h(y)$.

4. Differentiate w.r.t y and match with N :

$$f_y = xz + z \cos(yz) + x \cos(xy) + h'(y) = N$$

All terms match N , so $h'(y) = 0 \implies h(y) = C$.

$$f(x, y, z) = \sin(xy) + xyz + \sin(yz) + C$$

Using $f(1, 1, 1) = 0$: $\sin(1) + 1 + \sin(1) + C = 0 \implies C = -(1 + 2 \sin(1))$

Question 4(b): Evaluating Integral

Since \vec{F} is conservative, the line integral is path independent.

$$\int_{(0,0,0)}^{(1,1,1)} \vec{F} \cdot d\vec{r} = f(1, 1, 1) - f(0, 0, 0)$$

We are given $f(1, 1, 1) = 0$.

Now evaluate $f(0, 0, 0)$:

$$f(0, 0, 0) = \sin(0) + 0 + \sin(0) + C = C$$

$$\text{Integral} = 0 - C = -(-(1 + 2 \sin(1)))$$

Answer

$$1 + 2 \sin(1)$$

Question 5: Problem Statement

Find all $a, b \in \mathbb{R}$ such that the vector field

$$\vec{F}(x, y, z) = \ln(1 + y^2 + z^2) \hat{i} + \frac{(b - a^2)xy}{1 + y^2 + z^2} \hat{j} + \frac{axz}{1 + y^2 + z^2} \hat{k}$$

is conservative in \mathbb{R}^3 , and hence find a potential function for \vec{F} .

Question 5: Solution (Finding Constants)

For \vec{F} to be conservative, $\nabla \times \vec{F} = \vec{0}$.

Condition 1: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$M_y = \frac{2y}{1+y^2+z^2}, \quad N_x = \frac{(b-a^2)y}{1+y^2+z^2}$$
$$\implies 2 = b - a^2$$

Condition 2: $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$

$$M_z = \frac{2z}{1+y^2+z^2}, \quad P_x = \frac{az}{1+y^2+z^2}$$
$$\implies 2 = a$$

Substitute $a = 2$ into first equation: $b - 4 = 2 \implies b = 6$.

Constants

$$a = 2, \quad b = 6$$

Question 5: Solution (Potential Function)

Using $a = 2, b = 6$, we integrate to find $\phi(x, y, z)$.

$$\phi_x = \ln(1 + y^2 + z^2)$$

$$\phi = \int \ln(1 + y^2 + z^2) dx = x \ln(1 + y^2 + z^2) + g(y, z)$$

Differentiating w.r.t y :

$$\phi_y = \frac{2xy}{1 + y^2 + z^2} + g_y$$

Matches $N = \frac{2xy}{1+y^2+z^2}$, so $g_y = 0$. Similarly $g_z = 0$, so $g(y, z) = K$.

Potential Function

$$\phi(x, y, z) = x \ln(1 + y^2 + z^2) + K$$

Question 6: Problem Statement

Find the outward flux of the fields

$$\vec{F}_1 = 2x\hat{i} - 3y\hat{j} \quad \text{and} \quad \vec{F}_2 = 2x\hat{i} + (x - y)\hat{j}$$

across the circle $\vec{r}(t) = (a \cos t)\hat{i} + (a \sin t)\hat{j}$, $0 \leq t \leq 2\pi$.

Question 6: Solution (Flux of \vec{F}_1)

Formula: Flux = $\oint_C (M dy - N dx)$. Here $x = a \cos t, y = a \sin t$.

For \vec{F}_1 : $M = 2x, N = -3y$.

$$\text{Flux} = \int_0^{2\pi} [(2a \cos t)(a \cos t dt) - (-3a \sin t)(-a \sin t dt)]$$

$$= \int_0^{2\pi} (2a^2 \cos^2 t - 3a^2 \sin^2 t) dt$$

$$= a^2 \int_0^{2\pi} (2 \cos^2 t - 3 \sin^2 t) dt$$

Using $\int_0^{2\pi} \cos^2 t dt = \pi$ and $\int_0^{2\pi} \sin^2 t dt = \pi$:

$$= a^2(2\pi - 3\pi) = -\pi a^2$$

Question 6: Solution (Flux of \vec{F}_2)

For \vec{F}_2 : $M = 2x, N = x - y$.

$$\begin{aligned}\text{Flux} &= \int_0^{2\pi} [(2a \cos t)(a \cos t) - (a \cos t - a \sin t)(-a \sin t)] dt \\ &= a^2 \int_0^{2\pi} (2 \cos^2 t + \sin t \cos t - \sin^2 t) dt\end{aligned}$$

Integrate term by term:

- $\int 2 \cos^2 t dt = 2\pi$
- $\int \sin t \cos t dt = 0$ (periodicity)
- $\int -\sin^2 t dt = -\pi$

$$\text{Flux} = a^2(2\pi + 0 - \pi) = \pi a^2$$

Question 7: Problem Statement

Show that the differential form in the integral is exact, and then evaluate:

$$\int_{(1,1,2)}^{(3,5,0)} \sin z \, dx + z^2 e^{yz^2} \, dy + (x \cos z + 2yze^{yz^2}) \, dz$$

Question 7: Solution

Step 1: Exactness Let $\vec{F} = \langle \sin z, z^2 e^{yz^2}, x \cos z + 2yze^{yz^2} \rangle$. Check curl components:

- $M_z = \cos z, P_x = \cos z \checkmark$
- $N_z = 2ze^{yz^2} + z^2(2yz)e^{yz^2}, P_y = 2ze^{yz^2} + 2yz(z^2)e^{yz^2} \checkmark$
- $M_y = 0, N_x = 0 \checkmark$

Step 2: Potential Function $f_x = \sin z \implies f = x \sin z + g(y, z)$.

$f_y = g_y = z^2 e^{yz^2} \implies g = e^{yz^2} + h(z)$. $f_z = x \cos z + 2yze^{yz^2} + h'(z)$. Comparing with P , we see $h'(z) = 0 \implies h(z) = C$.

$$f(x, y, z) = x \sin z + e^{yz^2}$$

Step 3: Evaluation

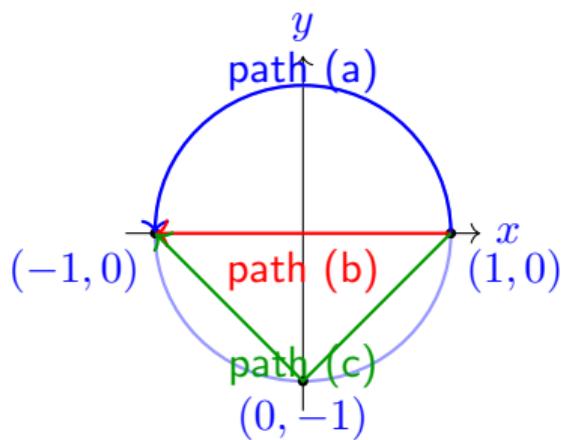
$$f(3, 5, 0) - f(1, 1, 2) = (3 \sin(0) + e^0) - (1 \sin(2) + e^4)$$

$$= 1 - (\sin 2 + e^4)$$

Question 8: Problem Statement

Find the flow of the velocity field $\vec{F} = (x + y)\hat{i} - (x^2 + y^2)\hat{j}$ along each of the following paths from $(1, 0)$ to $(-1, 0)$ in the xy -plane:

- Ⓐ The upper half of the circle $x^2 + y^2 = 1$.
- Ⓑ The line segment from $(1, 0)$ to $(-1, 0)$.
- Ⓒ The line segment from $(1, 0)$ to $(0, -1)$ followed by the line segment from $(0, -1)$ to $(-1, 0)$.



Question 8: Solution (a)

Path (a): Upper Semicircle $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j}, \quad 0 \leq t \leq \pi.$

$$\vec{F} = (\cos t + \sin t) \hat{i} - (1) \hat{j}$$

$$d\vec{r} = (-\sin t \hat{i} + \cos t \hat{j}) dt$$

Dot Product:

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= [(\cos t + \sin t)(-\sin t) - (1)(\cos t)] dt \\ &= [-\sin t \cos t - \sin^2 t - \cos t] dt\end{aligned}$$

Integrate from 0 to π :

$$\begin{aligned}\int_0^\pi &\left(-\frac{1}{2} \sin 2t - \sin^2 t - \cos t\right) dt \\ &= 0 - \int_0^\pi \sin^2 t dt - 0 = -\frac{\pi}{2}\end{aligned}$$

Question 8: Solution (b)

Path (b): Line Segment Along the x-axis from 1 to -1: $y = 0, dy = 0, r(t) = t\hat{i}$.

$$\vec{F} = x\hat{i} - x^2\hat{j}$$

$$d\vec{r} = dx\hat{i}$$

$$\begin{aligned}\int \vec{F} \cdot d\vec{r} &= \int_1^{-1} x \, dx \\ &= \left[\frac{x^2}{2} \right]_1^{-1} = \frac{(-1)^2}{2} - \frac{1^2}{2} = \frac{1}{2} - \frac{1}{2} = 0\end{aligned}$$

Question 8: Solution (c)

Path (c): Two Segments

- $C_1: (1, 0) \rightarrow (0, -1)$. $\vec{r}_1 = (1-t)\hat{i} - t\hat{j}$.
- $C_2: (0, -1) \rightarrow (-1, 0)$. $\vec{r}_2 = -t\hat{i} + (t-1)\hat{j}$.

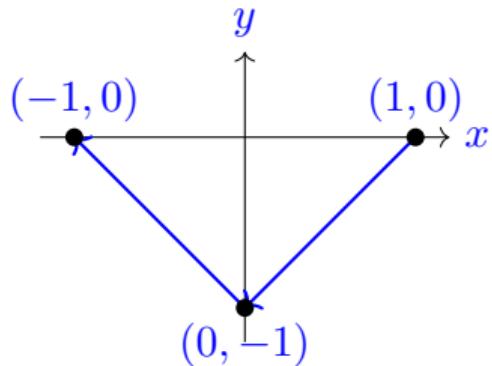
Integrate C_1 : $\vec{F} \cdot d\vec{r}_1 = \dots = 2t^2 dt$.

$$\int_0^1 2t^2 dt = 2/3$$

Integrate C_2 : $\vec{F} \cdot d\vec{r}_2 = \dots = (2t - 2t^2)dt$.

$$\int_0^1 (2t - 2t^2)dt = 1 - 2/3 = 1/3$$

Total Flow: $2/3 + 1/3 = 1$.



Recall: Green's Theorem

Theorem

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Question 9: Problem Statement

Use Green's theorem to evaluate the integral

$$I = \oint_C \left(\frac{x}{x^2 + 1} - y \right) dx + \left(3x - 4 \tan \frac{y}{2} \right) dy$$

where C is the portion of $y = x^2$ from $(-1, 1)$ to $(1, 1)$, followed by the portion of $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$.

Question 9: Solution

Identify $M = \frac{x}{x^2+1} - y$ and $N = 3x - 4 \tan \frac{y}{2}$.

Calculate partials:

$$\frac{\partial N}{\partial x} = 3, \quad \frac{\partial M}{\partial y} = -1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3 - (-1) = 4$$

Using Green's Theorem:

$$I = \iint_R 4 \, dA = 4 \times \text{Area}(R)$$

Question 9: Area Calculation

Region R: Bounded by $y = x^2$ (bottom) and $y = 2 - x^2$ (top) between $x = -1$ and $x = 1$.

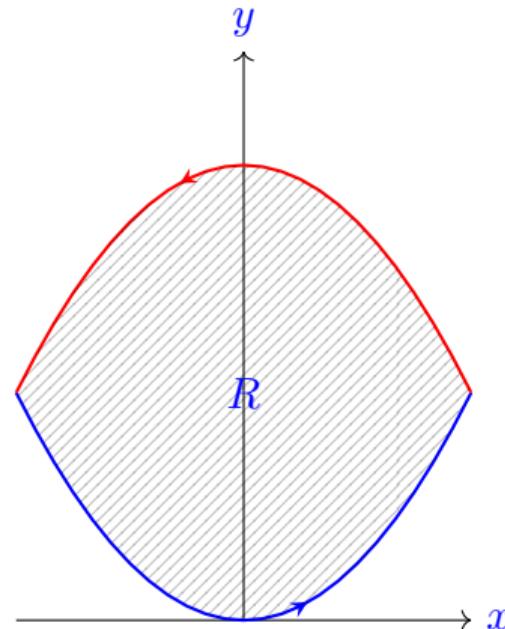
$$\text{Area} = \int_{-1}^1 [(2 - x^2) - x^2] dx$$

$$= \int_{-1}^1 (2 - 2x^2) dx$$

$$= 2 \int_0^1 (2 - 2x^2) dx$$

$$= 4[x - \frac{x^3}{3}]_0^1 = 4(1 - 1/3) = \frac{8}{3}$$

$$\text{Total Integral} = 4 \times \frac{8}{3} = \frac{32}{3}$$



Recall: Area Formula via Green's Theorem

Green's Theorem: $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$

To find the **Area (A)** of region R , we want the double integral to become $\iint_R 1 dA$. We need to choose M and N such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$.

Possible choices:

- Let $M = 0, N = x \implies \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 \implies A = \oint_C x dy.$
- Let $M = -y, N = 0 \implies \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 - (-1) = 1 \implies A = -\oint_C y dx.$

Symmetric Form (Average of the two):

$$A = \frac{1}{2} \left(\oint_C x dy + \oint_C (-y) dx \right) = \frac{1}{2} \oint_C (x dy - y dx)$$

Question 10: Problem Statement

Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

$$y = x, \quad y = \frac{1}{x}, \quad y = \frac{x}{4}$$

Question 10: Solution (Setup)

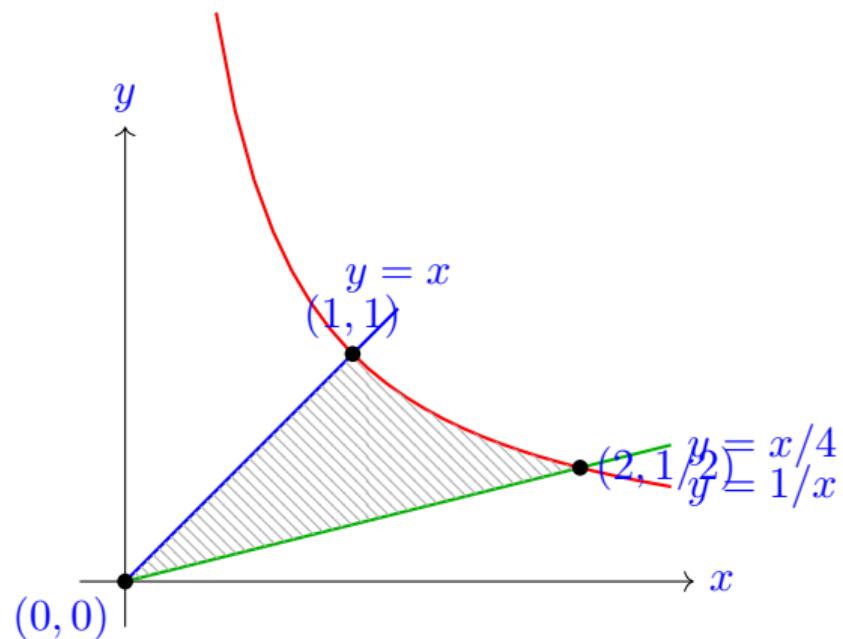
Area Formula via Green's Theorem: $A = \frac{1}{2} \oint_C (xdy - ydx)$.

Intersection Points:

1. $y = x$ and $y = 1/x \Rightarrow x^2 = 1 \Rightarrow (1, 1)$.
2. $y = x/4$ and $y = 1/x \Rightarrow x^2 = 4 \Rightarrow (2, 1/2)$.
3. $y = x$ and $y = x/4 \Rightarrow (0, 0)$.

The boundary C consists of three segments.

Question 10: Visualizing the Region



Question 10: Solution (Integration)

1. **Along $y = x/4$ from 0 to 2:** $dy = \frac{1}{4}dx$. Integrand: $x(\frac{1}{4}dx) - (\frac{x}{4})dx = 0$.
2. **Along $y = 1/x$ from 2 to 1:** $x = 1/y \implies dx = -dy/y^2$. Integrand:
 $(\frac{1}{y})dy - y(-\frac{dy}{y^2}) = \frac{2}{y}dy$.

$$\int_{1/2}^1 \frac{2}{y} dy = [2 \ln y]_{1/2}^1 = 2(0 - \ln(1/2)) = 2 \ln 2$$

3. **Along $y = x$ from 1 to 0:** $dy = dx$. Integrand $xdx - xdx = 0$.

Total Area:

$$A = \frac{1}{2}(0 + 2 \ln 2 + 0) = \ln 2$$

Answer

$\ln 2$

Question 10: Alternative Method (Area = $\oint x \, dy$)

We can also use the formula $A = \oint_C x \, dy$.

1. **Along $y = x/4$ from 0 to 2 (C_1):** $y = t/4 \implies dy = dt/4$. $x = t$. Limits $0 \rightarrow 2$.

$$\int_0^2 t \left(\frac{1}{4} dt \right) = \frac{1}{4} \left[\frac{t^2}{2} \right]_0^2 = \frac{1}{4}(2) = \frac{1}{2}$$

2. **Along $y = 1/x$ (C_2):** From $(2, 1/2)$ to $(1, 1)$. $x = 1/y \implies \oint (1/y) dy$. Limits $y : 1/2 \rightarrow 1$.

$$\int_{1/2}^1 \frac{1}{y} dy = [\ln y]_{1/2}^1 = 0 - (-\ln 2) = \ln 2$$

3. **Along $y = x$ (C_3):** From $(1, 1)$ to $(0, 0)$. $y = t \implies dy = dt$. $x = t$. Limits $1 \rightarrow 0$.

$$\int_1^0 t \, dt = \left[\frac{t^2}{2} \right]_1^0 = 0 - \frac{1}{2} = -\frac{1}{2}$$

Total Area: $\frac{1}{2} + \ln 2 - \frac{1}{2} = \ln 2$.

Thank You