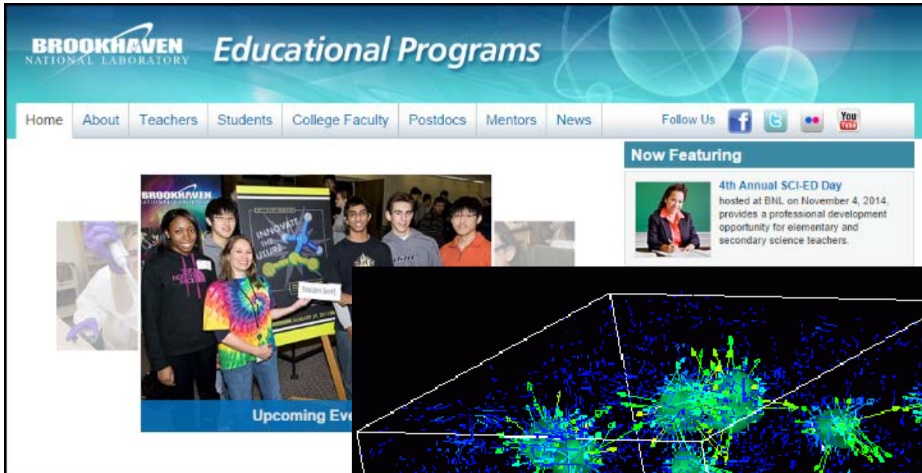
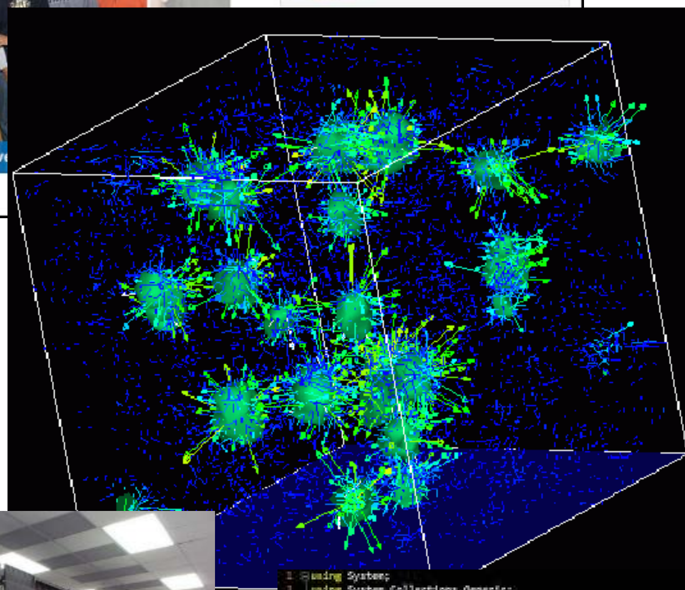
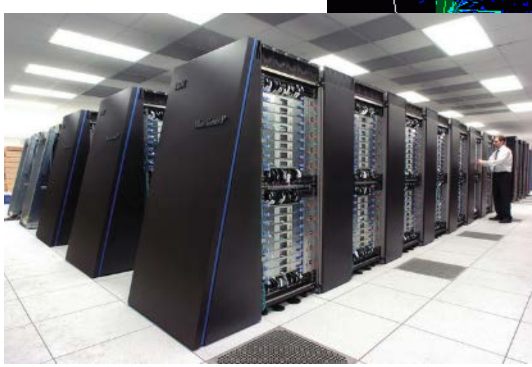






Survey of Scientific Computing

(SciComp 301)

Dave Biersach
Brookhaven National Laboratory
dbiersach@bnl.gov

Session 09

Functional Equations, Equilibrium Simulation

Session Goals

- More practice with the **for()** and **while()** loop constructs
- Understand the C++ **class** idiom
- Review "is-a" and "has-a" class relationships
- Calculate center of mass of a **Jenga** pile
- Model the construction of a *cantilever*
- Develop **functional equations** to confirm simulation results

A Shortcut

Carl Friedrich Gauss
(1777 – 1855)

Sum the integers from 1 to 100

1
2
3
4
5
6
7
8
9
10

1	9
2	8
3	7
4	6
5	
10	

4 matched rows that each sum to 10

1 row that is = 10 / 2 = 5

1 row that is = n = 10

$$n\left(\frac{n}{2}-1\right)+\frac{n}{2}+n \quad = \frac{n*(n+1)}{2}$$

= 55

Gaussian Summation

- Create a program to sum the first **1,000** natural numbers
- Write a **for()** loop so it runs while $\mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{n}$, and increments k **by one** after each iteration
- Implement the **functional equation** for Gaussian summation:

$$\sum_{k=1}^{n} k = \frac{n * (n+1)}{2}$$

Edit Lab 1 – Gaussian Summation

- Create a program to sum the first **1,000** natural numbers
- Write a **for()** loop so it runs while $1 \le k \le n$, and increments k by one after each iteration
- Implement the functional equation for Gaussian summation

main.cpp

```cpp
#include "stdafx.h"

using namespace std;

int main()
{
    double n = 1000;

    double sumByLooping = 0;

    for (int k{}; k < 0; k)
        sumByLooping += k;

    cout.imbue(std::locale(""));

    cout << "Manual sum of first " << n
         << " natural numbers = "
         << sumByLooping << endl;

    double sumByGauss = 0;

    cout << "Gaussian sum of first " << n
         << " natural numbers = "
         << sumByGauss << endl;

    return 0;
}
```

Run Lab 1 – Gaussian Summation

- Create a program to sum the first **1,000** natural numbers
- Write a **for()** loop so it runs while $1 \leq k \leq n$, and increments k by one after each iteration
- Implement the functional equation for Gaussian summation

main.cpp

```
#include "stdafx.h"

using namespace std;

int main()
{
    double n = 1000;

    double sumByLooping = 0;

    for (int k{1}; k <= n; ++k)
        sumByLooping += k;

    cout.imbue(std::locale(""));

    cout << "Manual sum of first " << n
         << " natural numbers = "
         << sumByLooping << endl;

    double sumByGauss = (n * (n + 1)) / 2;

    cout << "Gaussian sum of first " << n
                      numbers = "
                        << endl;
```

gauss-sum

File Edit View Terminal Tabs Help

```
Manual sum of first 1,000 natural numbers = 500,500
Gaussian sum of first 1,000 natural numbers = 500,500

Process returned 0 (0x0)   execution time : 0.017 s
Press ENTER to continue.
```

Another Shortcut

Sum of first **n** *natural* numbers:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

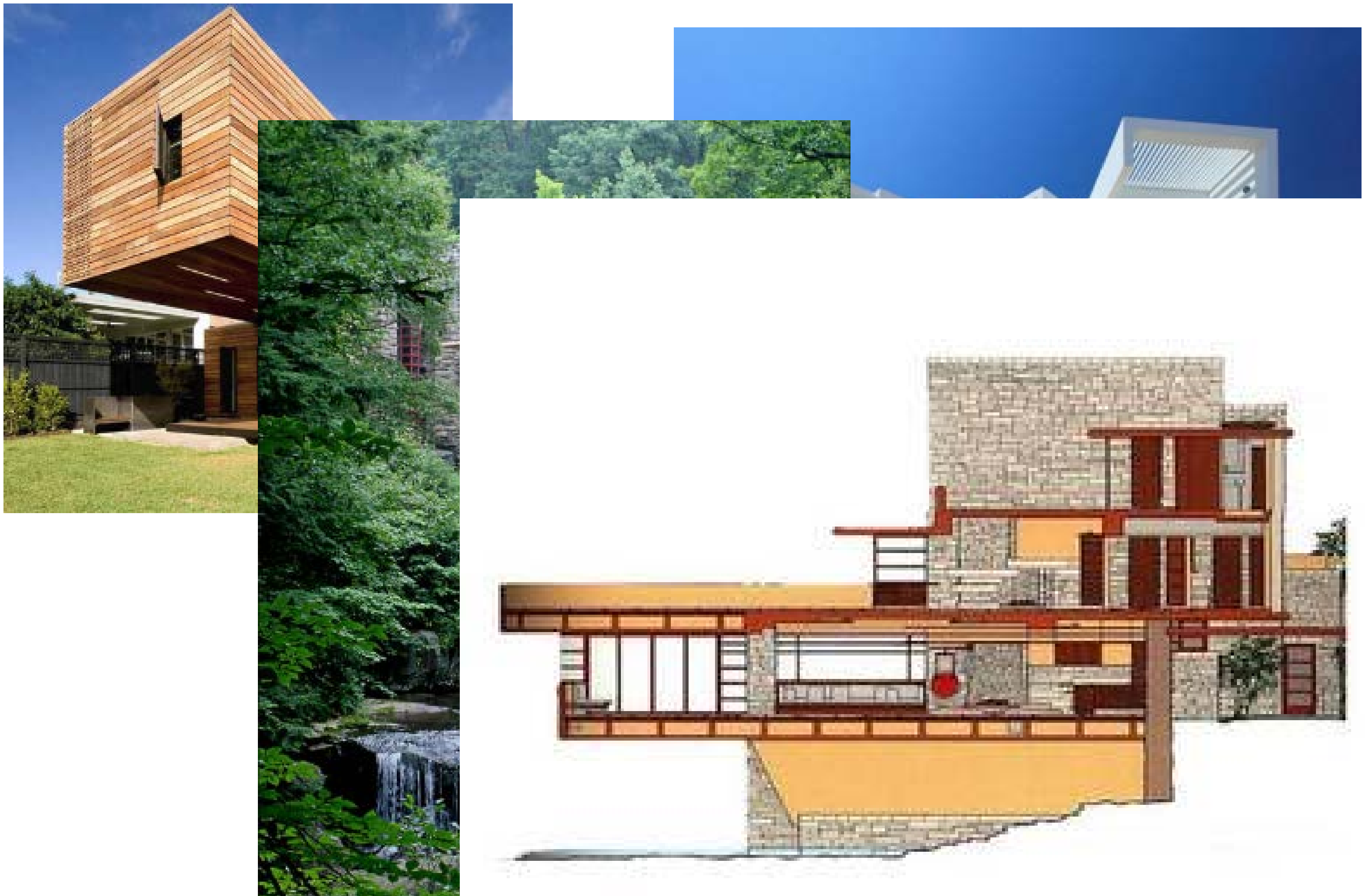
Sum of <u>squares</u> of first **n** *natural* numbers:

n	n^2	Sum
1	1	1
2	4	5
3	9	14
4	16	30
5	25	55
6	36	91
7	49	140
8	64	204
9	81	285
10	100	385

$$P_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \boxed{\frac{2n^3 + 3n^2 + n}{6}.}$$

These are functional equations - we can now calculate the sums *immediately* without having to loop over every element!

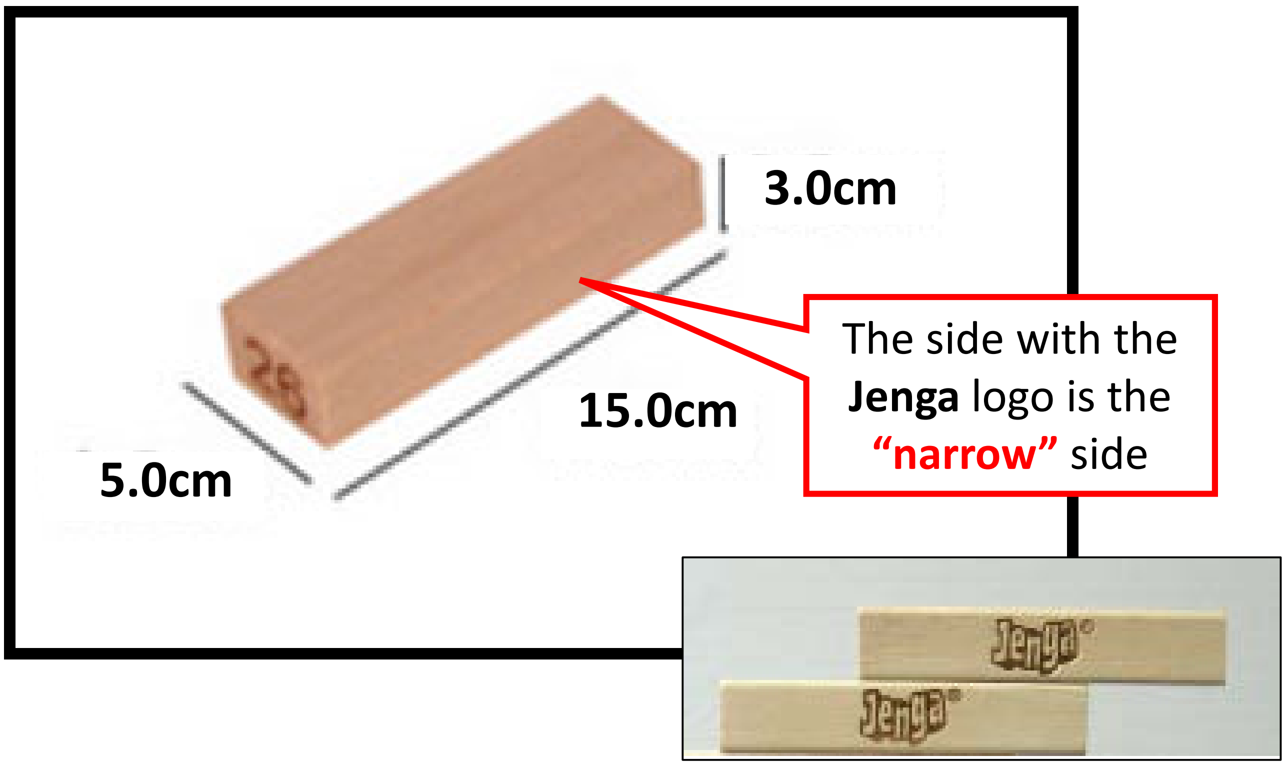
Cantilever Building Design



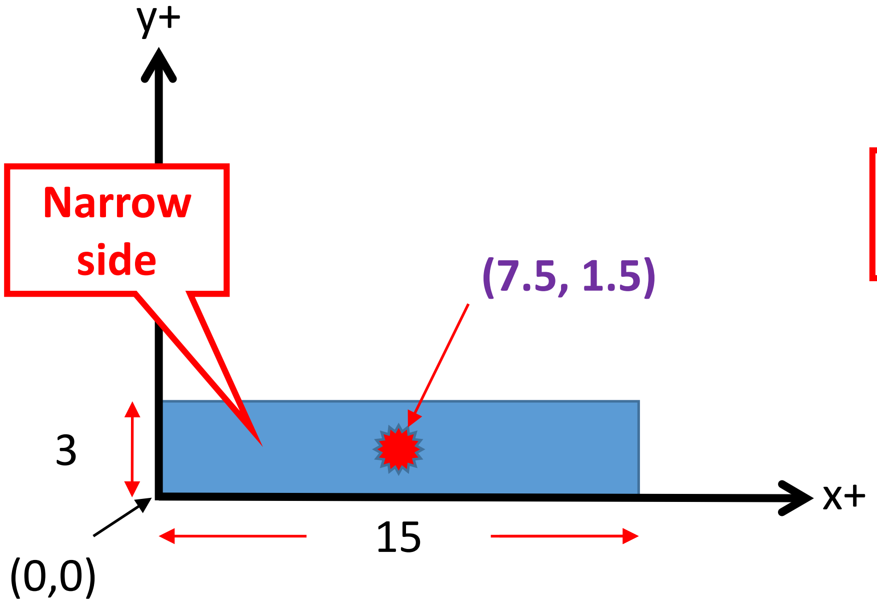
Jenga Block Dimensions

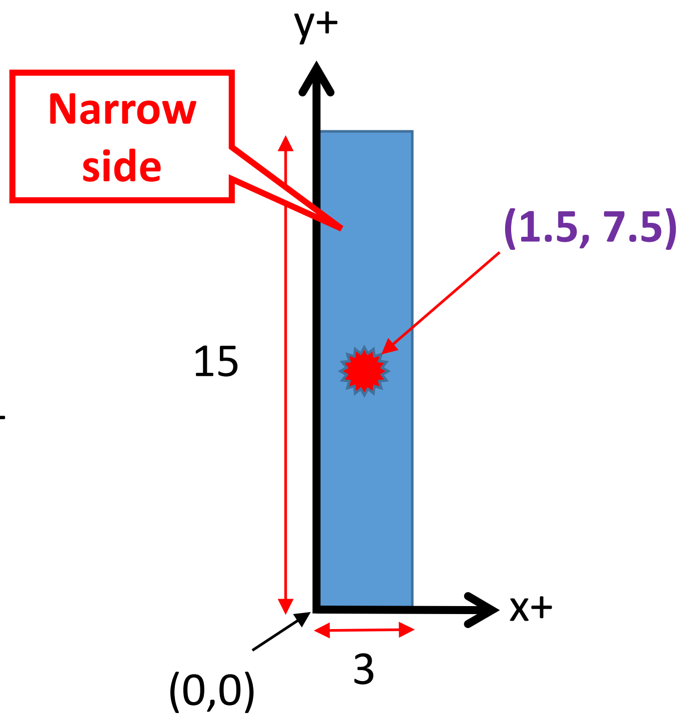
Leslie Scott created Jenga in 1983

Jenga Block Dimensions



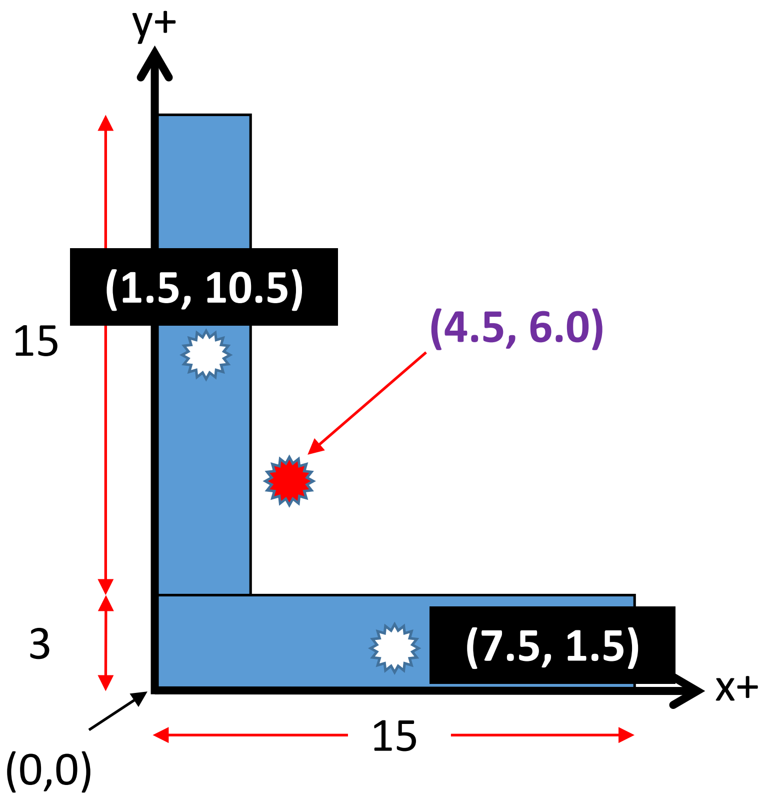
Center of Mass (COM)





Center of Mass = Average of All Corner Coordinates (in **each** **X** & **Y** dimensions)

1^{st} Ensemble *(first two blocks)*

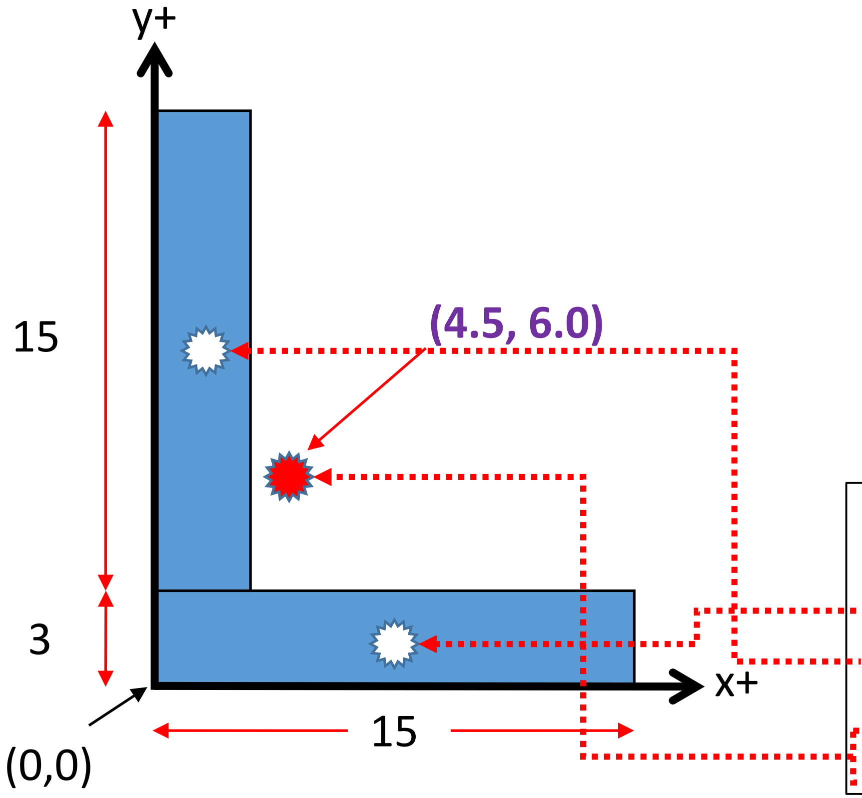


Center of Mass = Average of the centers of all blocks (in **each** **X** & **Y** dimension)

$$C_x = \frac{1.5 + 7.5}{2} = 4.5$$

$$C_y = \frac{10.5 + 1.5}{2} = 6.0$$

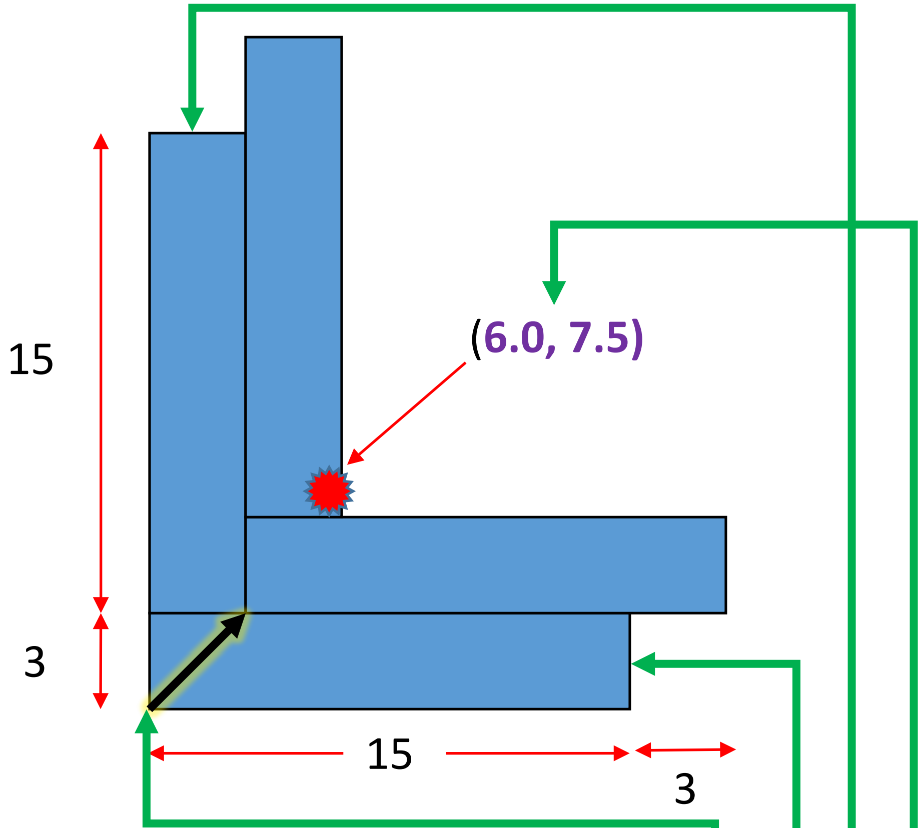
1st Ensemble *(first two blocks)*



Center of Mass = Average of the centers of all blocks (in each **X** & **Y** dimension)

```
BlockList* blocks = new BlockList();

blocks->AddBlock(7.5, 1.5);
blocks->AddBlock(1.5, 10.5);

blocks->CalcCenter();
```

2^{nd} Ensemble *(next two blocks)*

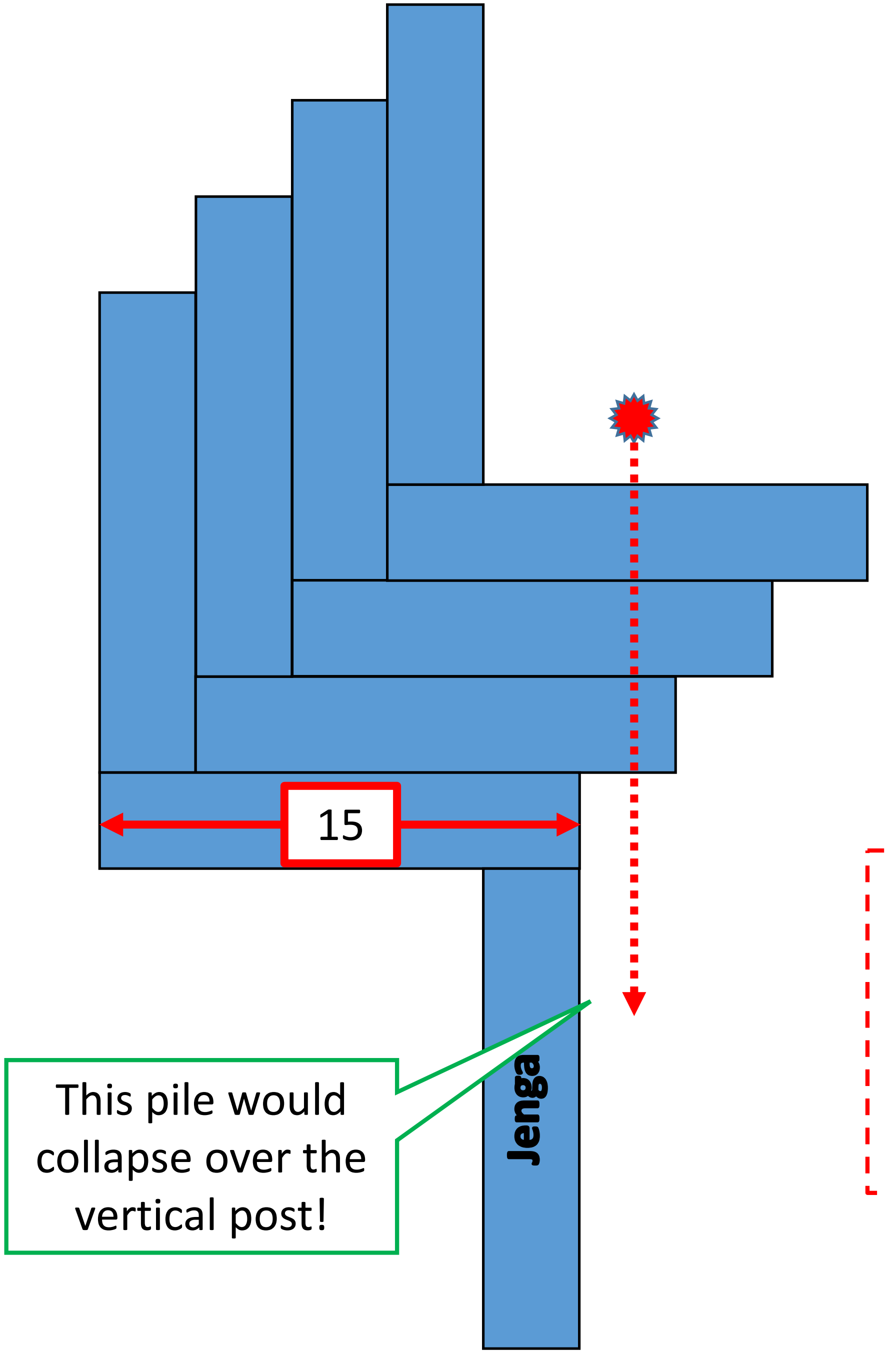



```
blocks->MoveBlocks(3, 3);
blocks->AddBlock(7.5, 1.5);
blocks->AddBlock(1.5, 10.5);
blocks->CalcCenter();
```

Before adding next ensemble, we first **move** each *existing* block by $+\Delta x, +\Delta y$

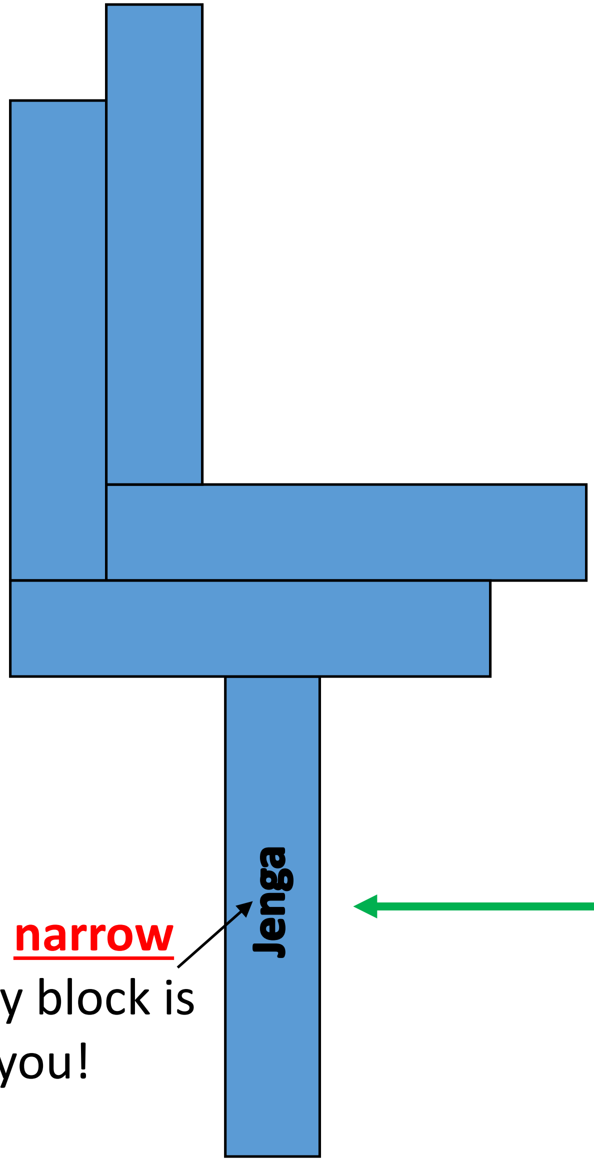
The effect is to move all *existing* blocks **upwards and to the right**, *before* the next iteration.

$$\Delta x = +3, \Delta y = +3$$



How many **2-block** ensembles can be balanced on a single vertical post?

The center of mass (of the entire pile) in the **x-dimension** must remain **< 15.0** or else the bottom *horizontal* block **cannot** be supported by a vertical post!



How many **2-block** ensembles can be balanced on a vertical post?

Ensure the **narrow** side of every block is facing you!

Try this now with your blocks

C++ Class Hierarchy

```
class Point2D
{
public:
    Point2D();
    Point2D(double x, double y);
    ~Point2D();
    double x, y;
};
```

- **A block "has-a" center *(of mass)***
- **A center "is-a" Point2D**

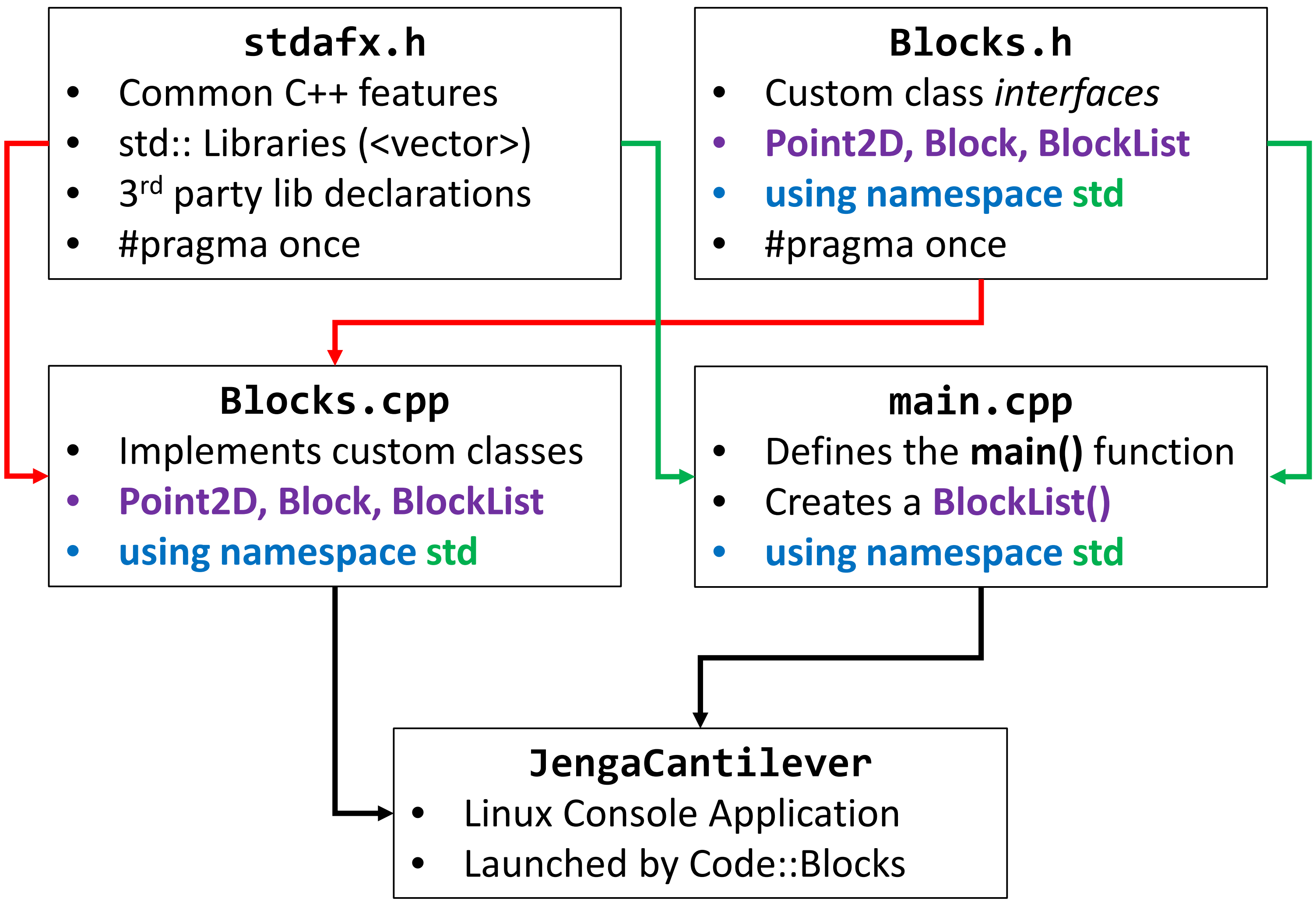
```
class Block
{
public:
    Block();
    Block(const Block &rhs);
    Block(double x, double y);
    ~Block();
    void Move(double deltaX, double deltaY);
    Point2D* center;
};
```

C++ Class Hierarchy

```
class BlockList
{
public:
    BlockList();
    ~BlockList();
    int Count();
    void CalcCenter();
    void AddBlock(double x, double y);
    void MoveBlocks(double deltaX, double deltaY);
    Point2D* center;
private:
    vector<Block>* blocks;
};
```

- **BlockList "has-a" vector of Blocks**
- **BlockList "has-a" center *(of mass)***

Header and Source File Dependency



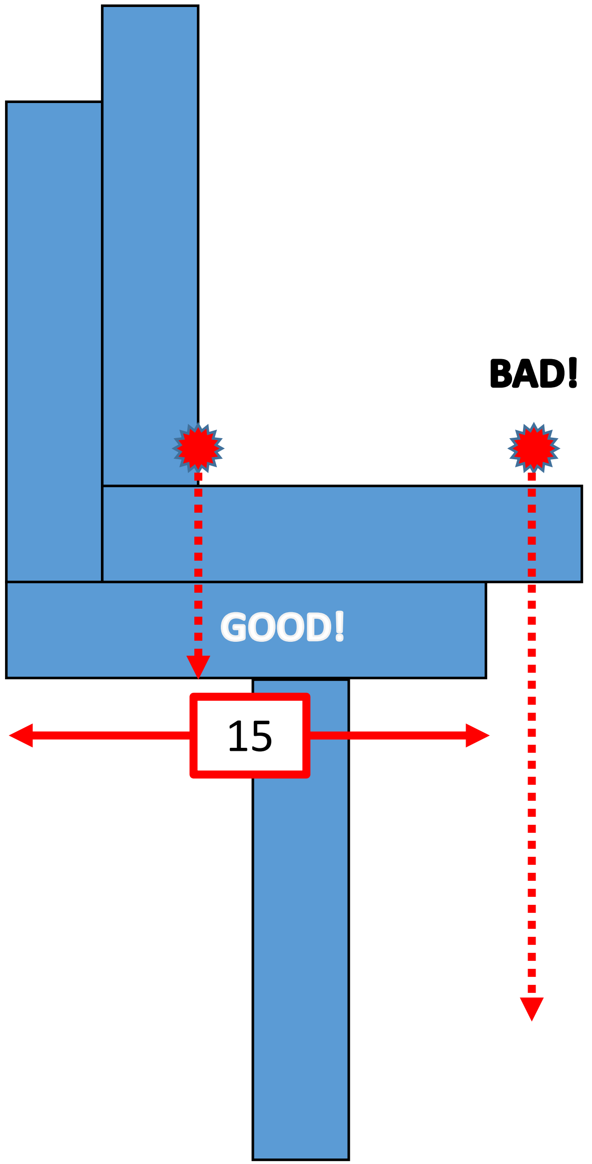
The Center of Mass is a Mean

```
void BlockList::CalcCenter()
{
    center->x = 0;
    center->y = 0;
    for (const auto &block : *blocks)
    {
        center->x += block.center->x;
        center->y += block.center->y;
    }
    center->x /= blocks->size();
    center->y /= blocks->size();

    cout << "Blocks: ";
    cout << setw(3) << blocks->size();
    cout << "\tCenter X:";
    cout << setw(7) << setprecision(2) << fixed;
    cout << center->x;
    cout << "\tCenter Y:";
    cout << setw(7) << setprecision(2) << fixed;
    cout << center->y;
    cout << endl;
}
```

The center of mass of a pile is just the **mean** of the centers of every block

Open Lab 2 – Jenga-14

```
#include "stdafx.h"
#include "blocks.h"

using namespace std;

int main()
{
    BlockList* blocks = new BlockList();

    blocks->AddBlock(7.5, 1.5);
    blocks->AddBlock(1.5, 10.5);

    blocks->CalcCenter();

    while (blocks->center->x < 15)
    {
        blocks->MoveBlocks(3, 3);
        blocks->AddBlock(7.5, 1.5);
        blocks->AddBlock(1.5, 10.5);
        blocks->CalcCenter();
    }

    return 0;
}
```



Run Lab 2
14 Block Jenga Cantilever

lab1-jenga14

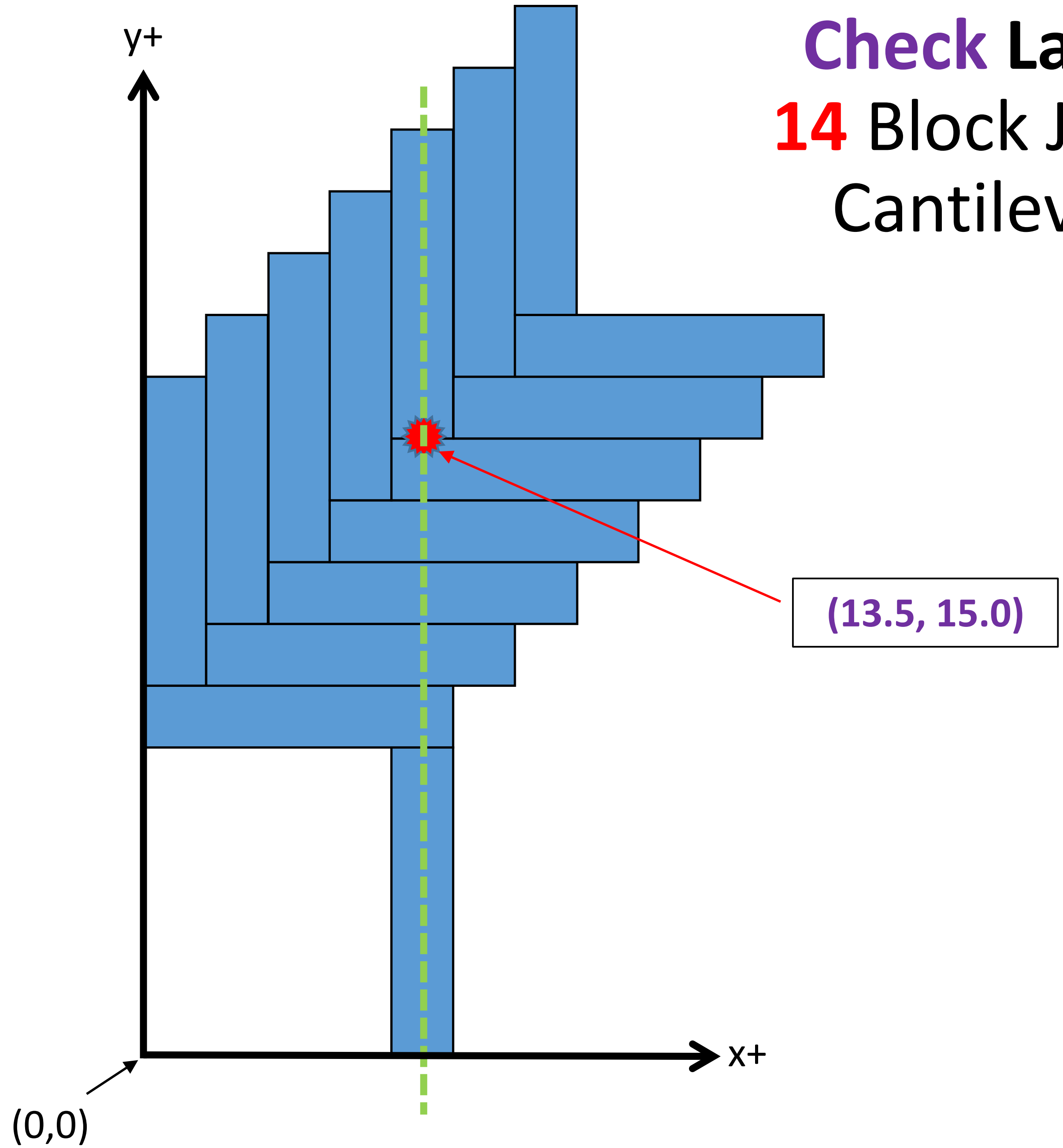
File Edit View Terminal Tabs Help

```
Blocks:   2      Center X:   4.50        Center Y:   6.00
Blocks:   4      Center X:   6.00        Center Y:   7.50
Blocks:   6      Center X:   7.50        Center Y:   9.00
Blocks:   8      Center X:   9.00        Center Y:  10.50
Blocks:  10      Center X:  10.50        Center Y:  12.00
Blocks:  12      Center X:  12.00        Center Y:  13.50
Blocks:  14      Center X:  13.50        Center Y:  15.00
Blocks:  16      Center X:  15.00        Center Y:  16.50

Process returned 0 (0x0)   execution time : 0.011 s
Press ENTER to continue.
```

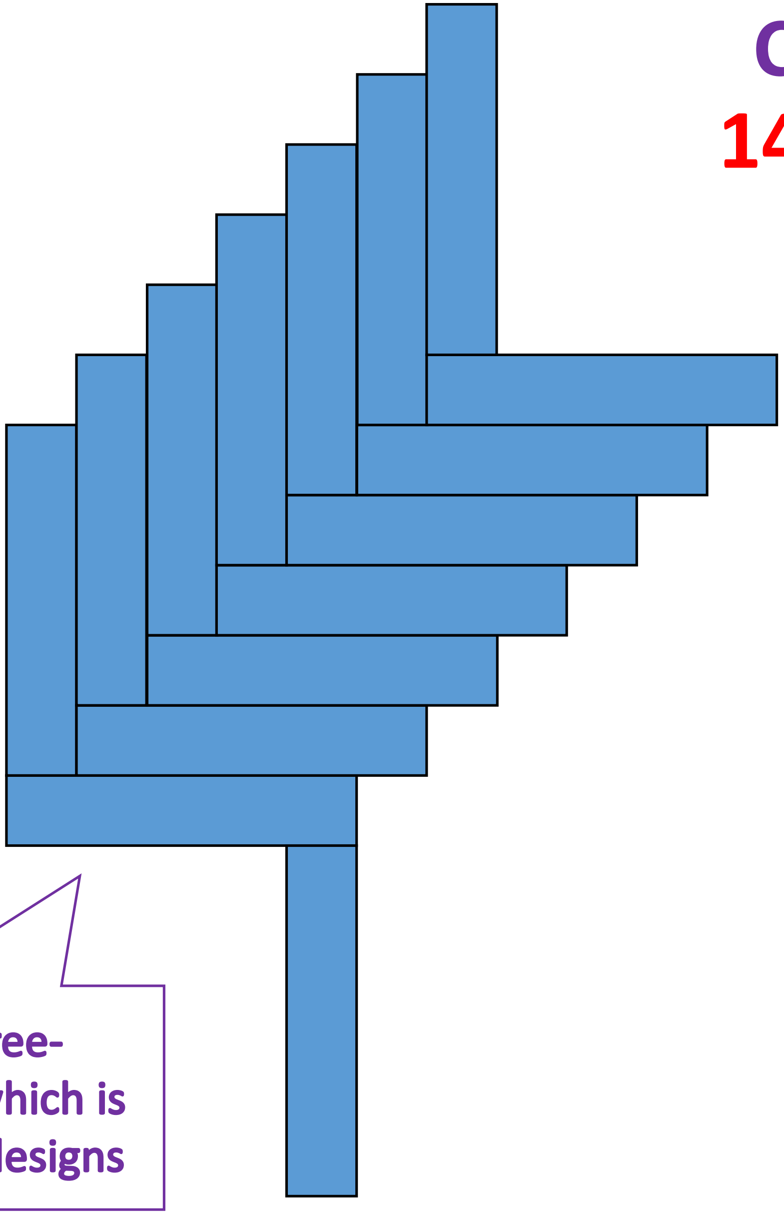
A 16 block pile in this configuration is not possible since $Center_x$ must be <u>less than</u> 15.0 to be balancing somewhere over the vertical post

Check Lab 1

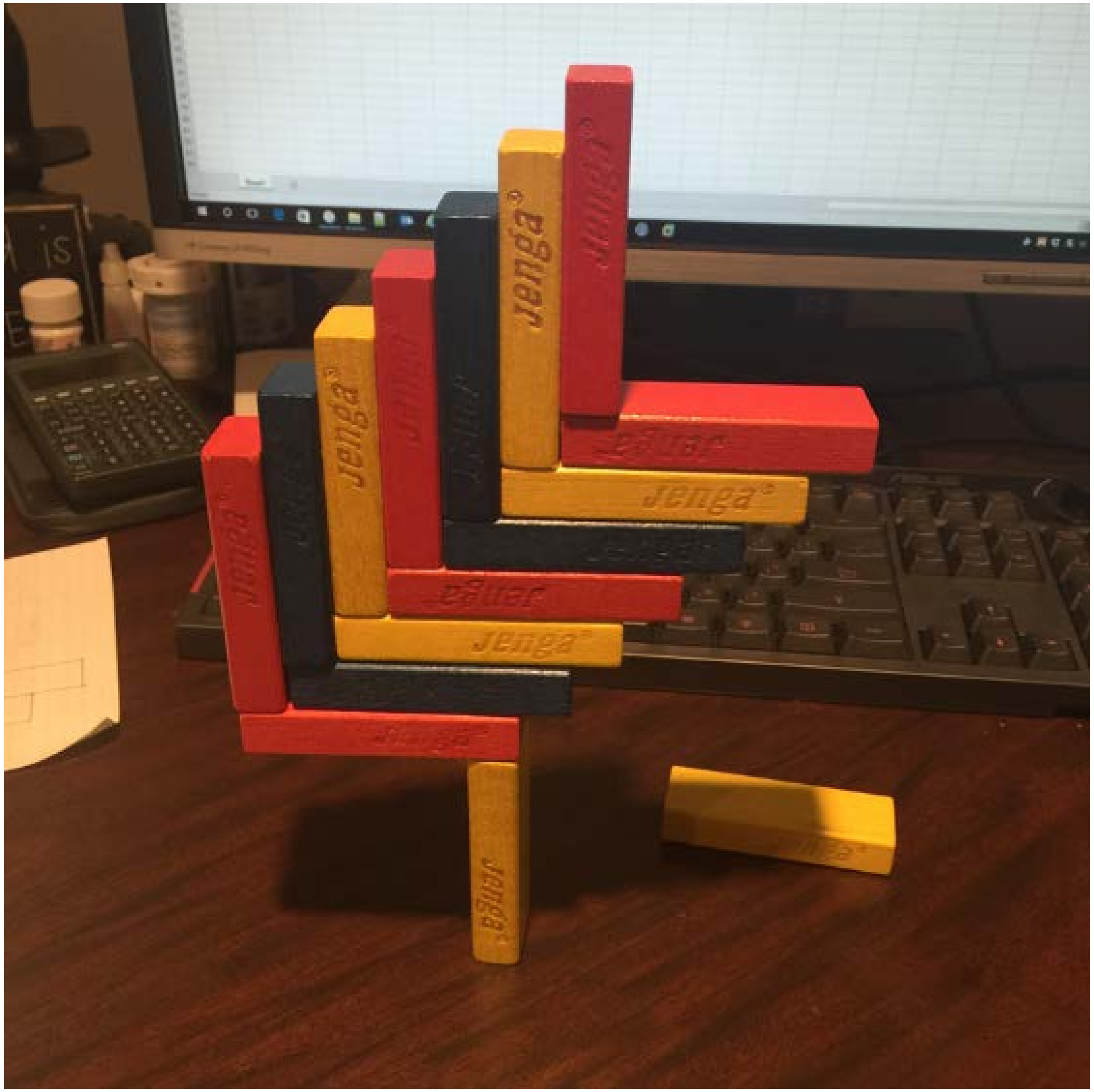
14 Block Jenga Cantilever



Check **Lab 1**
14 Block Jenga Cantilever

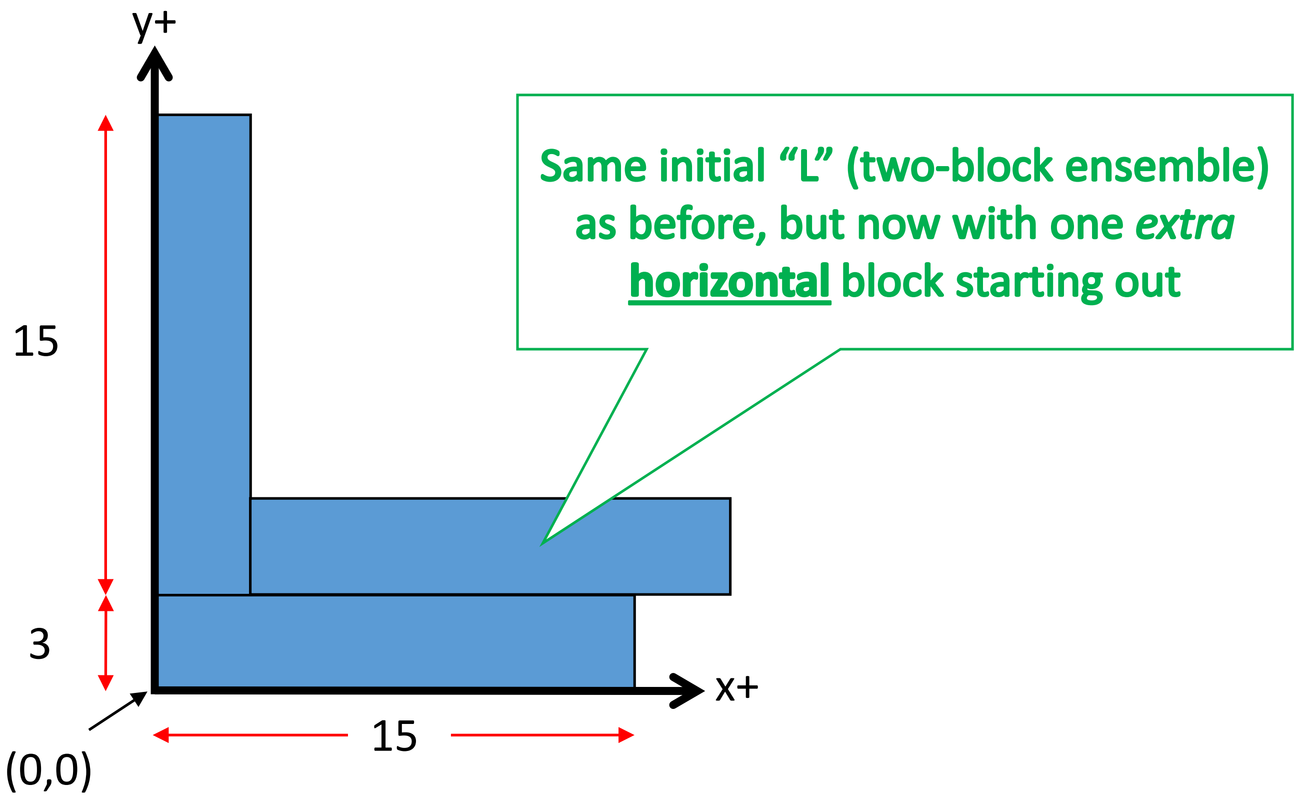


Notice the large free-standing overhang, which is typical of <u>cantilever</u> designs

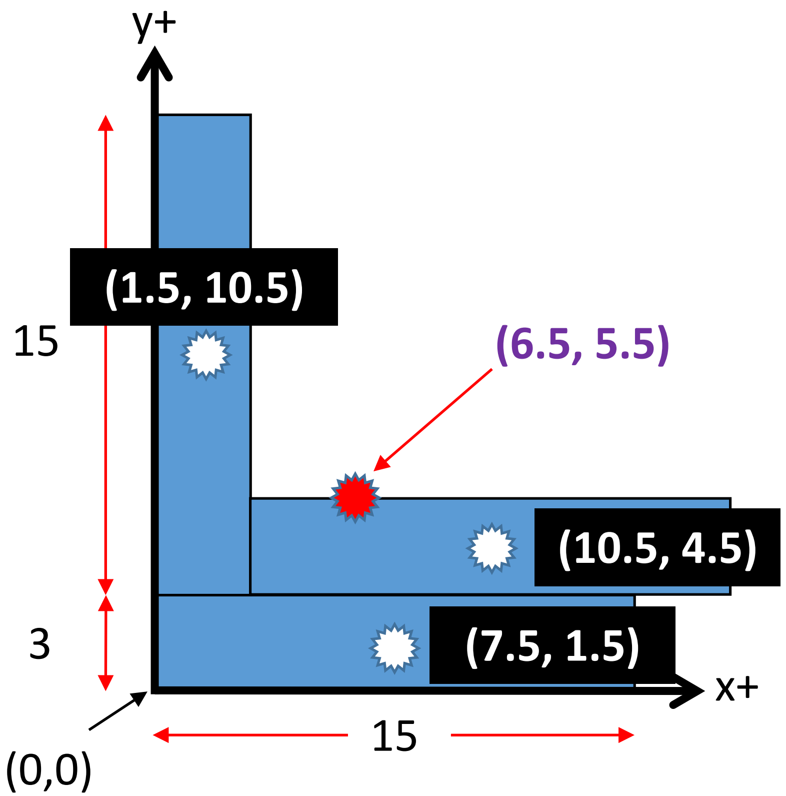


Can we ***simulate*** the construction of a 15 block Jenga cantilever?

15 Block Pile: 1st Ensemble *(3 blocks)*



15 Block Pile: 1st Ensemble *(3 blocks)*



Center of Mass = Average of the centers of all **3** blocks (in **each** **X** & **Y** dimension)

$$C_x = \frac{1.5 + 7.5 + 10.5}{3} = 6.5$$

$$C_y = \frac{10.5 + 1.5 + 4.5}{3} = 5.5$$

Open
Lab 3 - Jenga-15

main.cpp

```cpp
#include "stdafx.h"
#include "blocks.h"

using namespace std;

int main()
{
    BlockList* blocks = new BlockList();

    blocks->AddBlock(7.5, 1.5);
    blocks->AddBlock(1.5, 10.5);
    blocks->AddBlock();

    blocks->CalcCenter();

    while (blocks->center->x < 15)
    {
        blocks->MoveBlocks(3, 3);
        blocks->AddBlock(7.5, 1.5);
        blocks->AddBlock(1.5, 10.5);
        blocks->CalcCenter();
    }

    return 0;
}

```

Add these two ***Center of Mass*** coordinates as we will now begin with a **3-block** ensemble

Edit

Lab 3 - Jenga-15

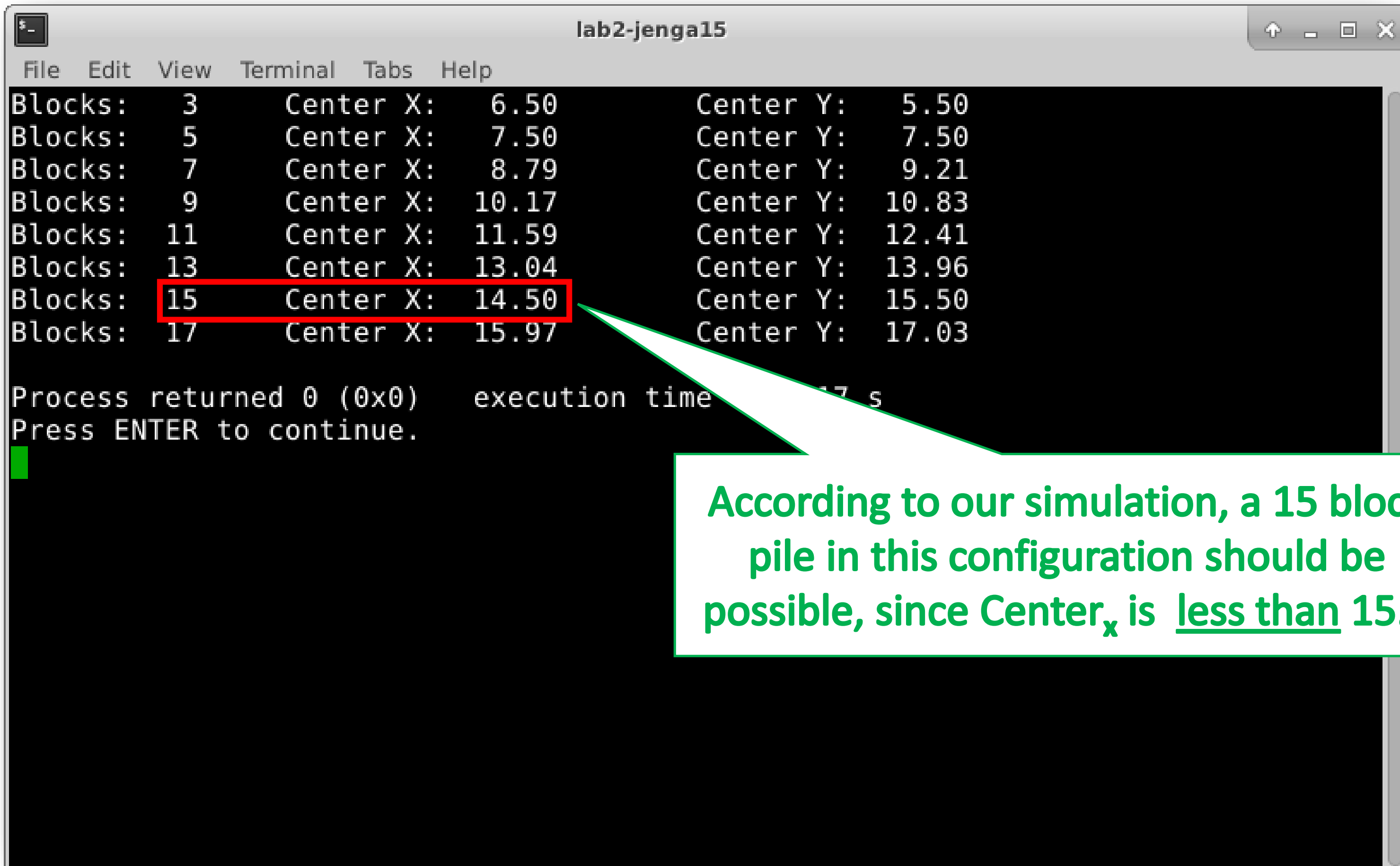
main.cpp

```cpp
#include "stdafx.h"
#include "blocks.h"

using namespace std;

int main()
{
    BlockList* blocks = new BlockList();

    blocks->AddBlock(7.5, 1.5);
    blocks->AddBlock(1.5, 10.5);
    blocks->AddBlock(10.5, 4.5);

    blocks->CalcCenter();

    while (blocks->center->x < 15)
    {
        blocks->MoveBlocks(3, 3);
        blocks->AddBlock(7.5, 1.5);
        blocks->AddBlock(1.5, 10.5);
        blocks->CalcCenter();
    }

    return 0;
}
```

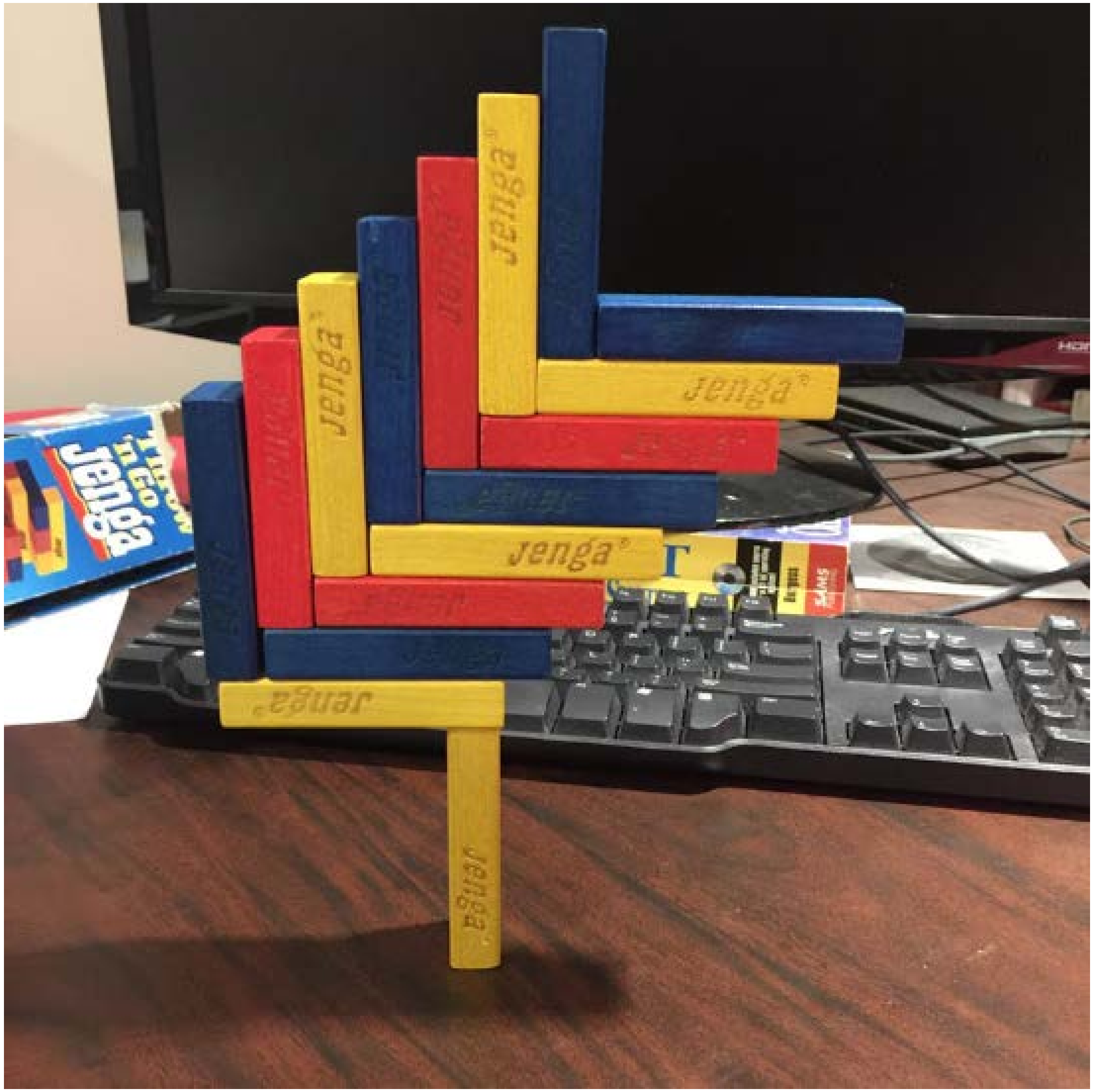
Add these two *Center of Mass* coordinates as we will now begin with a **3-block** ensemble

Run

Lab 3 - Jenga-15



According to our simulation, a 15 block pile in this configuration should be possible, since Center$_x$ is <u>less than</u> 15.0



Can you build a **15** Block Jenga Cantilever ??

Try this now with your blocks

Functional Equation

$$Pile\ Center\ of\ Mass_X = \frac{\sum_{n=1}^{blocks} Center\ Coordinate_x\ of\ Block_n}{Number\ of\ Blocks}$$

$$C_x = \frac{1}{4}(B_{1x} + B_{2x} + B_{3x} + B_{4x})$$

Assume we have **4** blocks in the pile

$$= \frac{1}{4}\big((B_{1x} + \Delta x) + (B_{2x} + \Delta x) + (B_{3x} + \Delta x) + (B_{4x} + \Delta x)\big)$$

Moving all blocks by Δx just moves C_x by Δx

$$= \frac{1}{4}\big(4\Delta x + (B_{1x} + B_{2x} + B_{3x} + B_{4x})\big)$$

$$C'_x = \Delta x + \frac{1}{4}(B_{1x} + B_{2x} + B_{3x} + B_{4x})$$

Functional Equation – 14 Block Cantilever

$$\sum X_{centers} \text{ (of a 2 block ensemble) } = 7.5 + 1.5 = \mathbf{9}$$

$$\sum \Delta X_{centers} \text{ (after moving 2 block ensemble) } = 3 + 3 = \mathbf{6}$$

2 Blocks	9
4 Blocks	$(9 + 6) + 9$
6 Blocks	$(9 + 6 + 6) + (9 + 6) + 9$
8 Blocks	$(9 + 6 + 6 + 6) + (9 + 6 + 6) + (9 + 6) + 9$
10 Blocks	$(9 + 6 + 6 + 6 + 6) + (9 + 6 + 6 + 6) + (9 + 6 + 6) + (9 + 6) + 9$

Functional Equation – 14 Block Cantilever

# of 9's	**# of 6's**	$\sum Center_x\ of$ **Ensembles**
1	**0**	9
2	**1**	$(9+6)+9$
3	**3**	$(9+6+6)+(9+6)+9$
4	**6**	$(9+6+6+6)+(9+6+6)+(9+6)+9$
5	**10**	$(9+6+6+6+6)+(9+6+6+6)+(9+6+6)+(9+6)+9$

Functional Equation – 14 Block Cantilever

# of 9's	# of 6's	n
1	0	1st Ensemble
2	1	2nd Ensemble
3	3	3rd Ensemble
4	6	4th Ensemble
5	10	5th Ensemble

$$9n + 6\left(\frac{n^2 - n}{2}\right)$$

$$9n + 3(n^2 - n)$$

$$3(3n + n^2 - n) = 3n(n + 2)$$

$$\boxed{Center\ of\ Mass_x = \frac{3n(n+2)}{2n}}$$

There are two blocks per ensemble

Functional Equations

Jenga 14 Block Cantilever

$$Center\ of\ Mass_X = \frac{3n(n+2)}{2n}$$

$$n \equiv Number\ of\ Ensembles$$

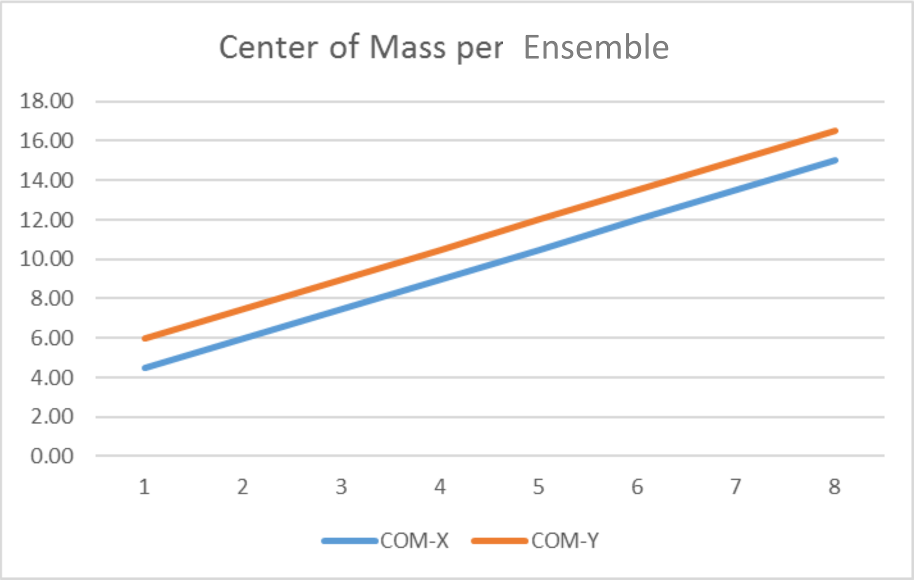
Jenga 15 Block Cantilever

$$Center\ of\ Mass_X = \frac{19.5 + 3(n-1)(n+4)}{2n+1}$$

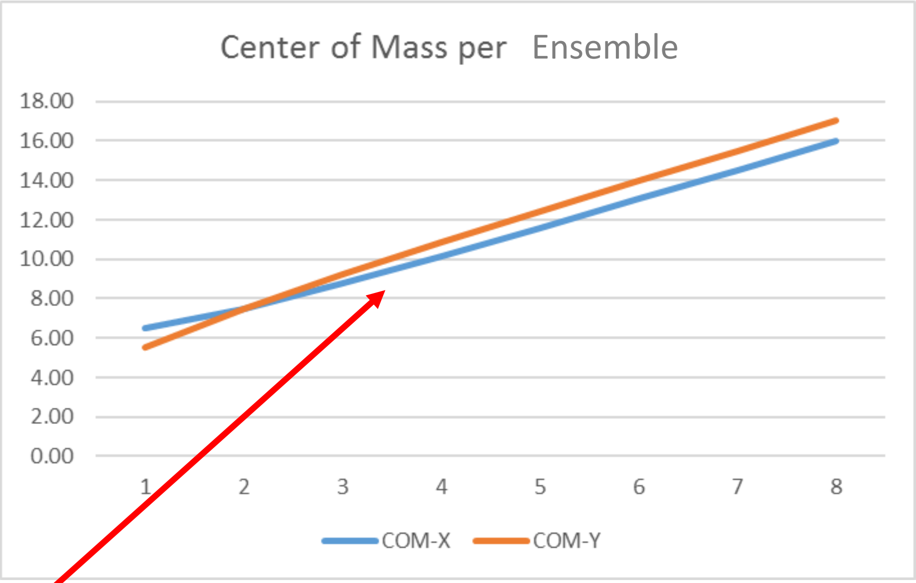
$$n \equiv Number\ of\ Ensembles$$

Functional Equations

Jenga 14 Block Cantilever

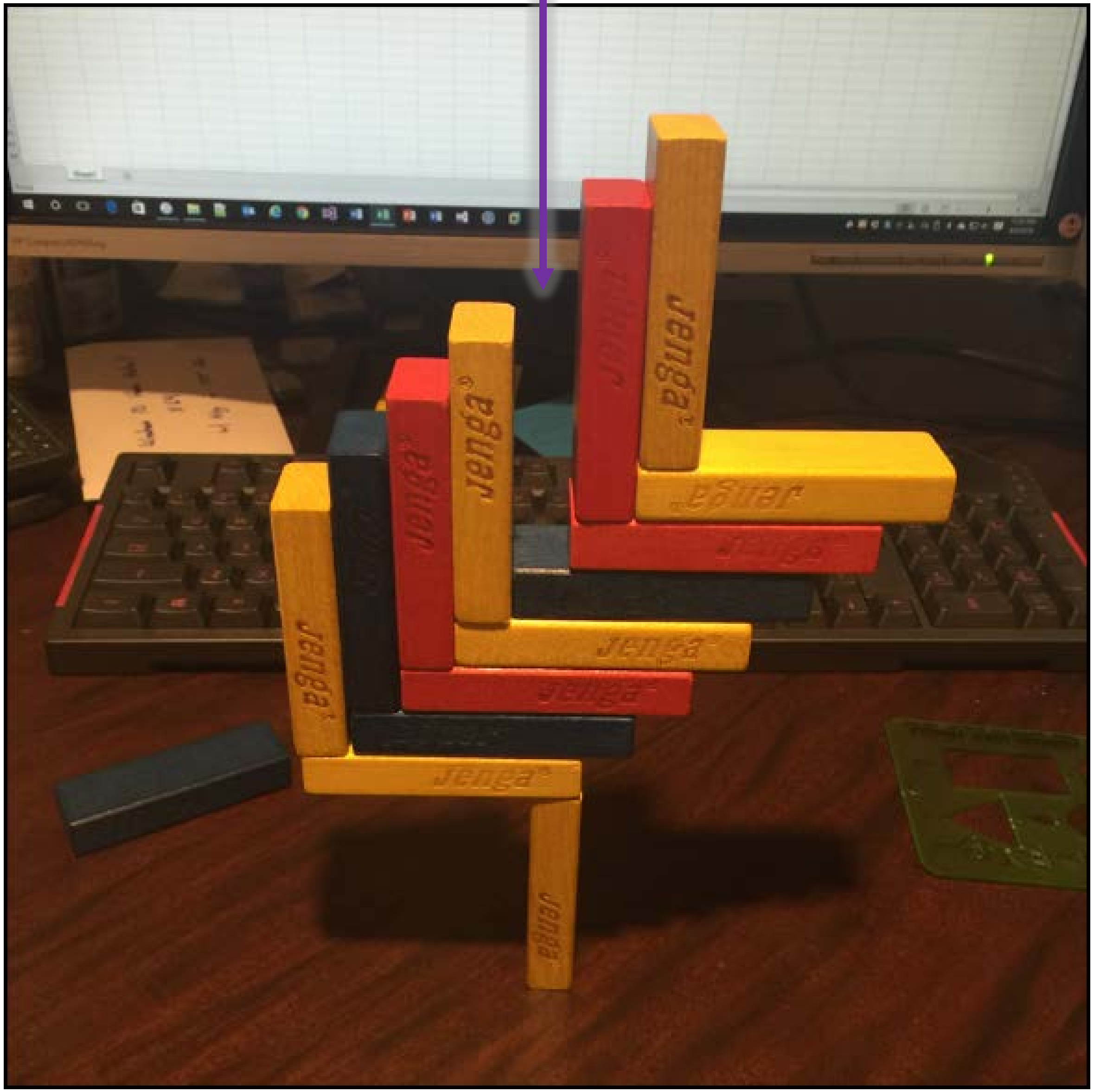


Jenga 15 Block Cantilever

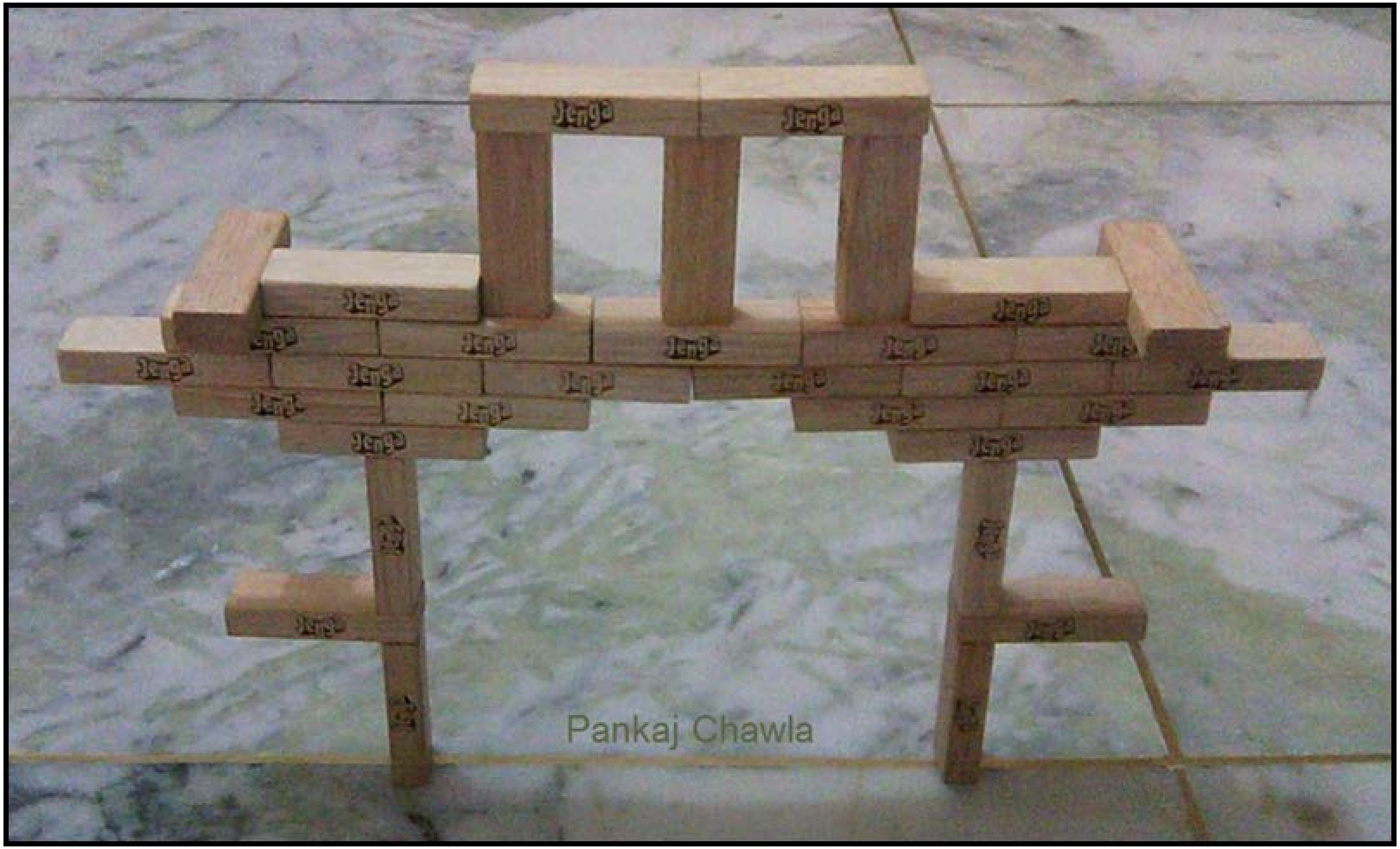


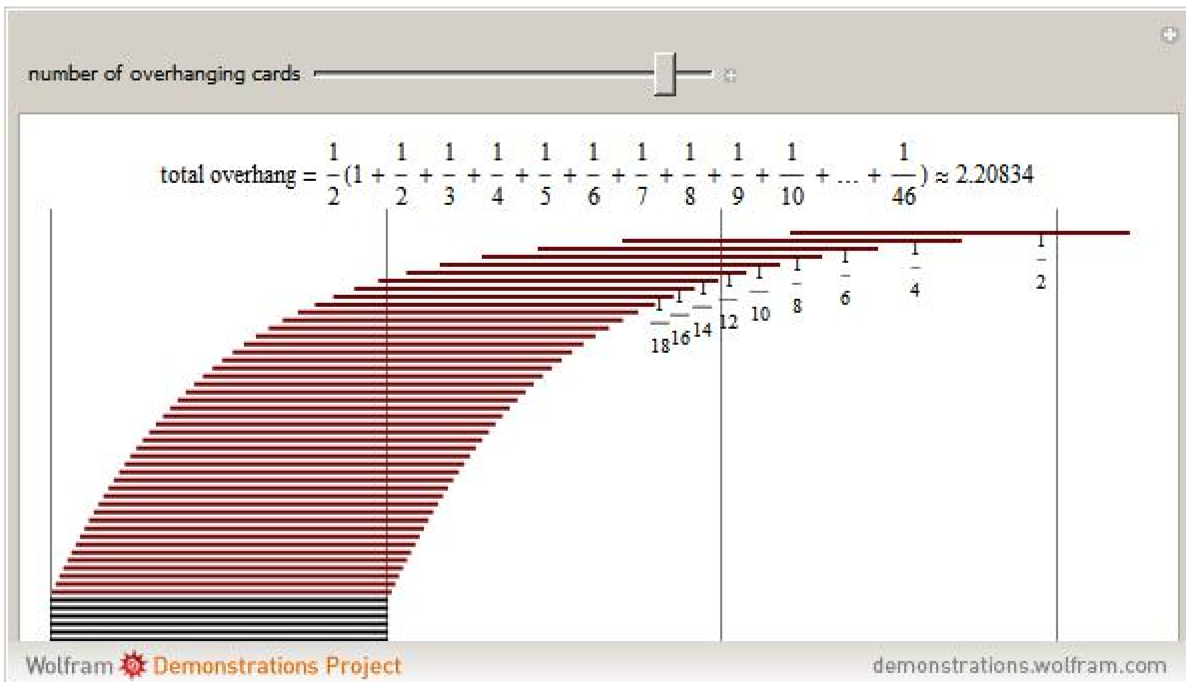
The center-of-mass in the X & Y dimensions are closer in a 15 block cantilever so it can rotate (tip over) more easily than a 14 block cantilever

Cantilever Building Design



13 Block Jenga Cantilever (missing middle)


Pankaj Chawla

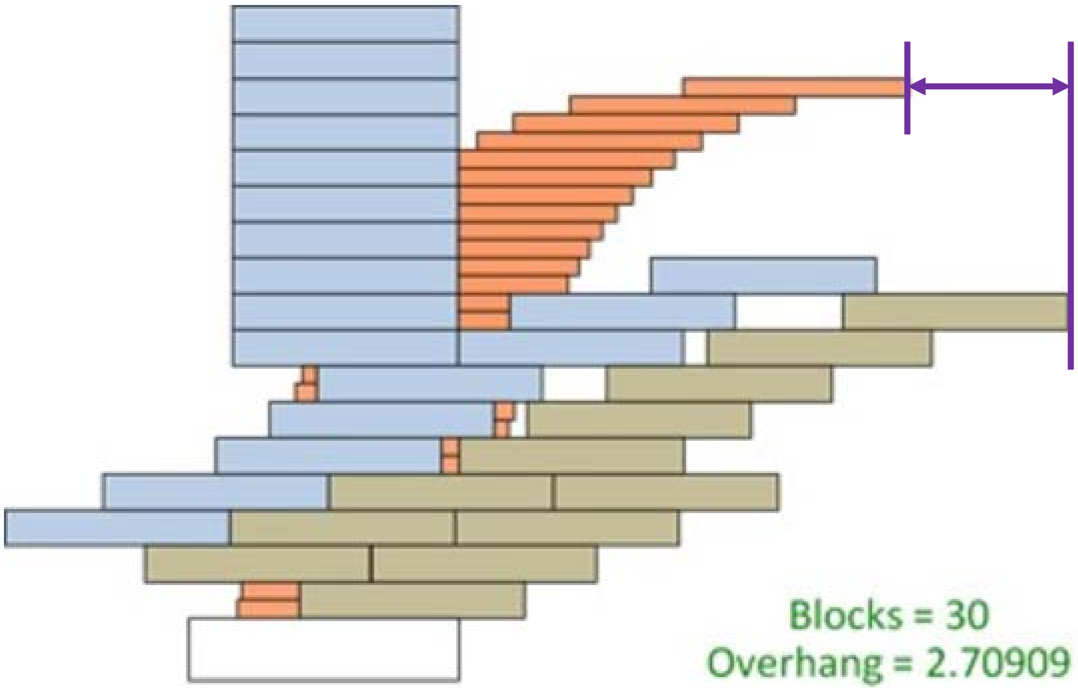

number of overhanging cards
total overhang = $\frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{46}) \approx 2.20834$
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{6}$
$\frac{1}{8}$
$\frac{1}{10}$
$\frac{1}{12}$
$\frac{1}{14}$
$\frac{1}{16}$
$\frac{1}{18}$
Wolfram Demonstrations Project
demonstrations.wolfram.com

Maximum Overhang, Optimum Reward

March 3, 2011 | Posted by Microsoft Research Blog

By Janie Chang, Writer, Microsoft Research

Yuval Peres, princi... s advocates both healthy skepticism ... prised when a paper he co-autho... merica (MAA) for taking a completel...



A classic stack of 30 blocks (in orange) compared to the same number of blocks configured as a jagged wall (in gray) held stable by counterweights (in blue).

Now you know…

- Looping with **for()** and **while()** statements
- Using a **class** to group data elements and functions
- Identity = "is-a" while ownership (contains) = "has-a"
- A **vector<>** template is a flexible container of similar types
- SciComp often involves ***simulating*** the real world before spending $$$
- Developing a **functional equation** can often produce the answer faster than looping to calculate it
- A complex construction of objects balances on its **virtual center of mass**
- Cantilever designs can reduce load on external walls to increase building **survivability**