Survey of Scientific Computing

(SciComp 301)

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Session 10

Matrix Algebra, Number Theory

Session Goals

- Learn how to declare and define a **2D** matrix in C++
- Implement a function to perform **matrix multiplication**
- Develop **recursive** code to calculate the **determinant** of a 2D matrix of any size
- Implement **Cramer's rule** to solve a system of linear solutions
- Verify **Goldbach's Conjecture** within a given range by developing a vector of primes

Matrices

- A “matrix” in C++ is represented as a **vector of vectors**
  - A matrix can hold any number of elements (in each dimension) but **all** elements must be of the same data type, such as **int** or **double**
  - Matrix elements are accessed by their index numbers, which starts at **zero** and correspond to each dimension
- Matrix nomenclature
  - In mathematics, a matrix size is written as **(Rows x Columns)**
  - In ***older*** C++ code a matrix (aka an **array**) is written using **commas** instead of multiplication crosses, and using **square brackets** instead of parenthesis: **[Row][Column]**
  - An example ***older*** C++ array is **[5][7]** which has **5 rows** and **7 columns**

Matrix Multiplication

- When rendering computer graphics, we often need to find the **product** of two matrices, e.g.

$$\begin{pmatrix} 4 & 5 & 8 \\ 1 & 9 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 6 & 1 \\ 5 & 9 \end{pmatrix} = ?$$

- There is a standard algorithm to do this multiplication
- It involves multiplying each element in the **rows** of first matrix by each element in the **columns** of the second matrix

Matrix Multiplication

- In order to multiply two matrices together, the number of *columns* in matrix **A** **must equal** the number of *rows* in matrix **B** (Cols **A** = Rows **B**)
- The resulting matrix will have as many rows as there were rows in matrix **A**, and as many columns as there were columns in matrix **B** (Rows **A** x Cols **B**)

Matrix Multiplication

- Example
  - Matrix **A** has dimension (2 x 3) = 2 rows, 3 columns
  - Matrix **B** has dimension (3 x 2) = 3 rows, 2 columns

The <u>inner</u> values must match!

$$\begin{pmatrix} 4 & 5 & 8 \\ 1 & 9 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 6 & 1 \\ 5 & 9 \end{pmatrix} = ?$$

$$(2 \times 3) \bullet (3 \times 2)$$

The <u>outer</u> values will be the dimension of the final matrix product!

Matrix Multiplication

- *Matrix* multiplication is **not** commutative!
  - If we multiple **A** x **B** we will get a different matrix than if we multiply **B** x **A**

    (3 x 2) • **(2 x 3)** = result is a (3 x 3) matrix

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- Welcome to the world of **non-commutative algebra**
  - This is very strange – it catches even great physicists by surprise!
  - This asymmetry is the foundation of the matrix formulation of **quantum mechanics**

Matrix Multiplication

- The algorithm is simple but tedious for **A** x **B** = **C**
- **Sum** the **product** of every element in each row of matrix **A** and the corresponding element in each column of matrix **B**
- That *sum* becomes just ***one element*** in new matrix **C**
- Continue this process for all rows in matrix **A**
  - Every row in **A** gets multiplied by every column in **B**
  - The resulting matrix will have dimensions (Rows **A** x Cols **B**)

Matrix Multiplication

Matrix A
(2 rows x 3 cols)

	Col 1	Col 2	Col 3
Row 1	**4**	**5**	**8**
Row 2	**1**	**9**	**7**

X

Matrix B
(3 rows x 2 cols)

	Col 1	Col 2
Row 1	**2**	**4**
Row 2	**6**	**1**
Row 3	**5**	**9**

Product Cell (**1,1**) = A Row 1 x B Col 1 = (4 x 2 + 5 x 6 + 8 x 5) = **78**
Product Cell (**1,2**) = A Row 1 x B Col 2 = (4 x 4 + 5 x 1 + 8 x 9) = **93**
Product Cell (**2,1**) = A Row 2 x B Col 1 = (1 x 2 + 9 x 6 + 7 x 5) = **91**
Product Cell (**2,2**) = A Row 2 x B Col 2 = (1 x 4 + 9 x 1 + 7 x 9) = **76**

Matrix C
(2 rows x 2 cols)

78	**93**
91	**76**

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Open Lab 1 – Matrix Multiply

- Review the key functions **main**(), **DisplayMatrix**(), and **MultiplyMatrices**()
- The code will display the product matrix in row, col format

```
int main()
{
    matrix A{{4,5,8},{1,9,7}};
    matrix B{{2,4},{6,1},{5,9}};
    matrix C = MultiplyMatrices(A, B);

    cout << "Matrix A = " << endl;
    DisplayMatrix(A);

    cout << "Matrix B = " << endl;
    DisplayMatrix(B);

    cout << "Matrix C = " << endl;
    DisplayMatrix(C);

    return 0;
}
```

We can define matrices using the consistent C++ value initialization syntax

Passing Objects To/From Functions in C++

- In C++ inbound and outbound (**return**) function parameters are passed **by value**, even for object types
  - However, for maximum speed, it is always best to pass large objects by reference between functions than to pass them by *value*
  - This is because passing by value forces the computer to make a complete copy of the object between caller ↔ callee
- However, passing **by reference** may allow the called function to unexpectedly modify the object
  - Therefore if your called function does not make any changes to the object, then prefix the reference type with the keyword **const**
  - Specifying **const** will get the compiler to verify the promise to the caller that the called function **will not modify** the passed object

Lab 1 – Matrix Multiply

```
// Receiving matrix A by a constant reference assures
// the caller that no changes will be made to the matrix,
// while gaining the performance benefit of not causing
// the copy constructor to be called on the passed in matrix
void DisplayMatrix(const matrix& A)
{
    const size_t rowsA = A.size();
    const size_t colsA = A.at(0).size();

    for (size_t row{}; row < rowsA; row++)
    {
        for (size_t col{}; col < colsA; col++)
            cout << setw(5) << A.at(row).at(col);
        cout << endl;
    }
    cout << endl;
}
```

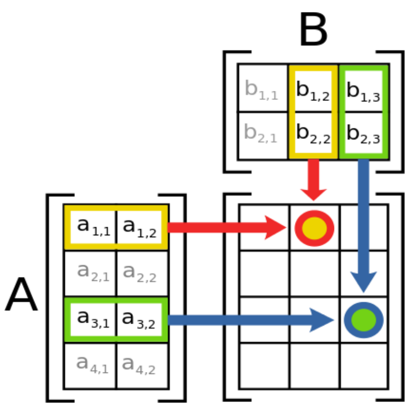
Matrix A is received as a constant reference (const matrix&)

Declaring function parameters as **const &** is a promise your function will **not** modify the variable passed to it

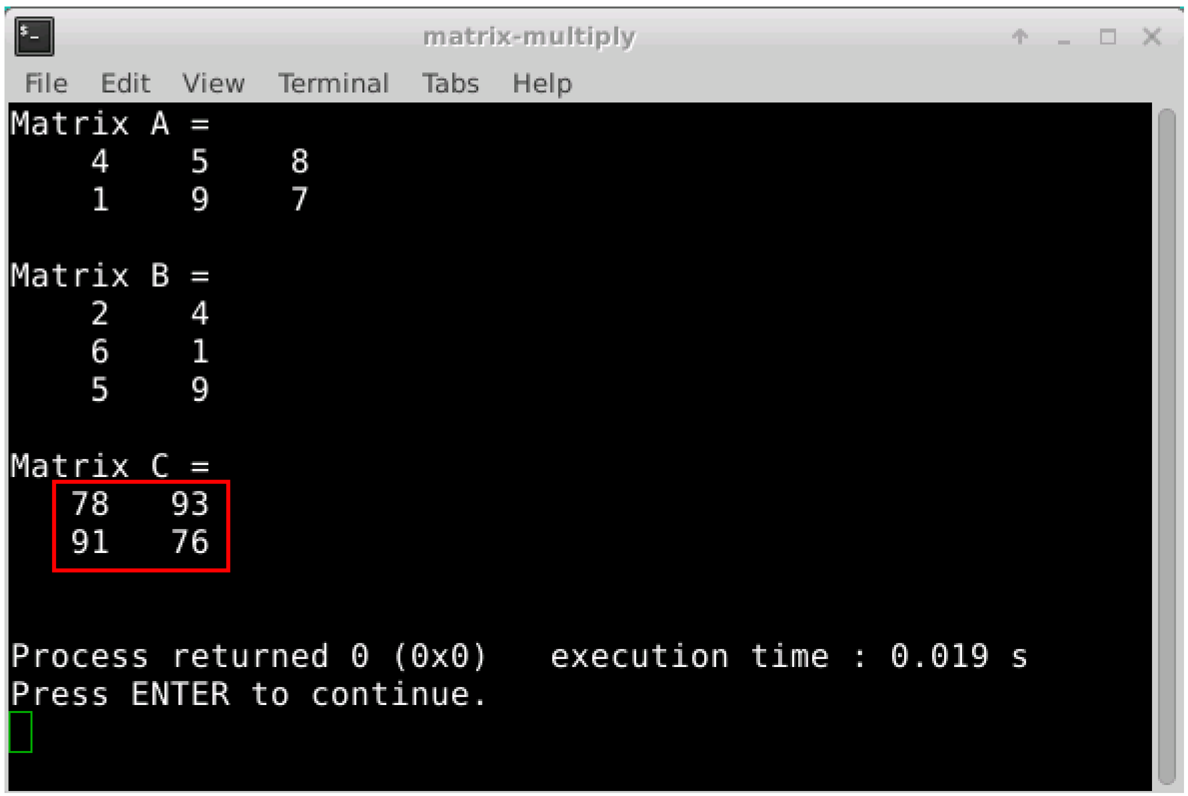
Lab 1 – Matrix Multiply

- C++ uses **copy <u>e</u>lision** (rhymes with decision) to avoid duplicating an entire object when returning to the caller
- C++ implements **RVO** (return value optimization) where *locally* scoped variables can be returned **directly** to the caller without having to be copied

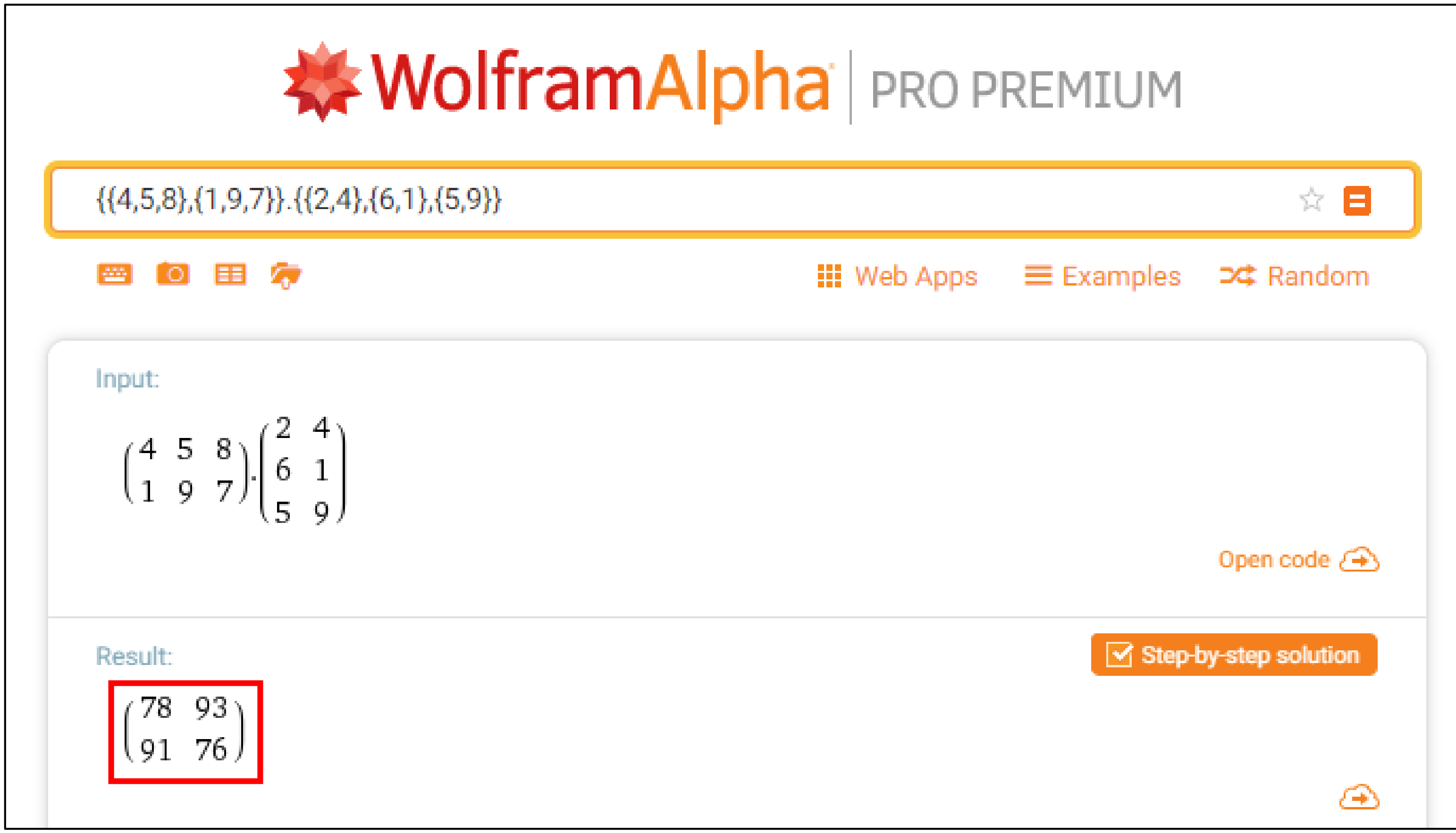
```
matrix MultiplyMatrices(const matrix& A, const matrix& B)
{
    const size_t rowsA = A.size();
    const size_t colsA = A.at(0).size();
    const size_t colsB = B.at(0).size();

    matrix c(rowsA, vector<int>(colsB, 0));

    for (size_t i{}; i < rowsA; ++i)
        for (size_t j{}; j < colsB; ++j)
            for (size_t k{}; k < colsA; ++k)
                c.at(i).at(j) += A.at(i).at(k) * B.at(k).at(j);

    return c;
}
```



Run Lab 1 – Matrix Multiply



Check Lab 1 - Matrix Multiply



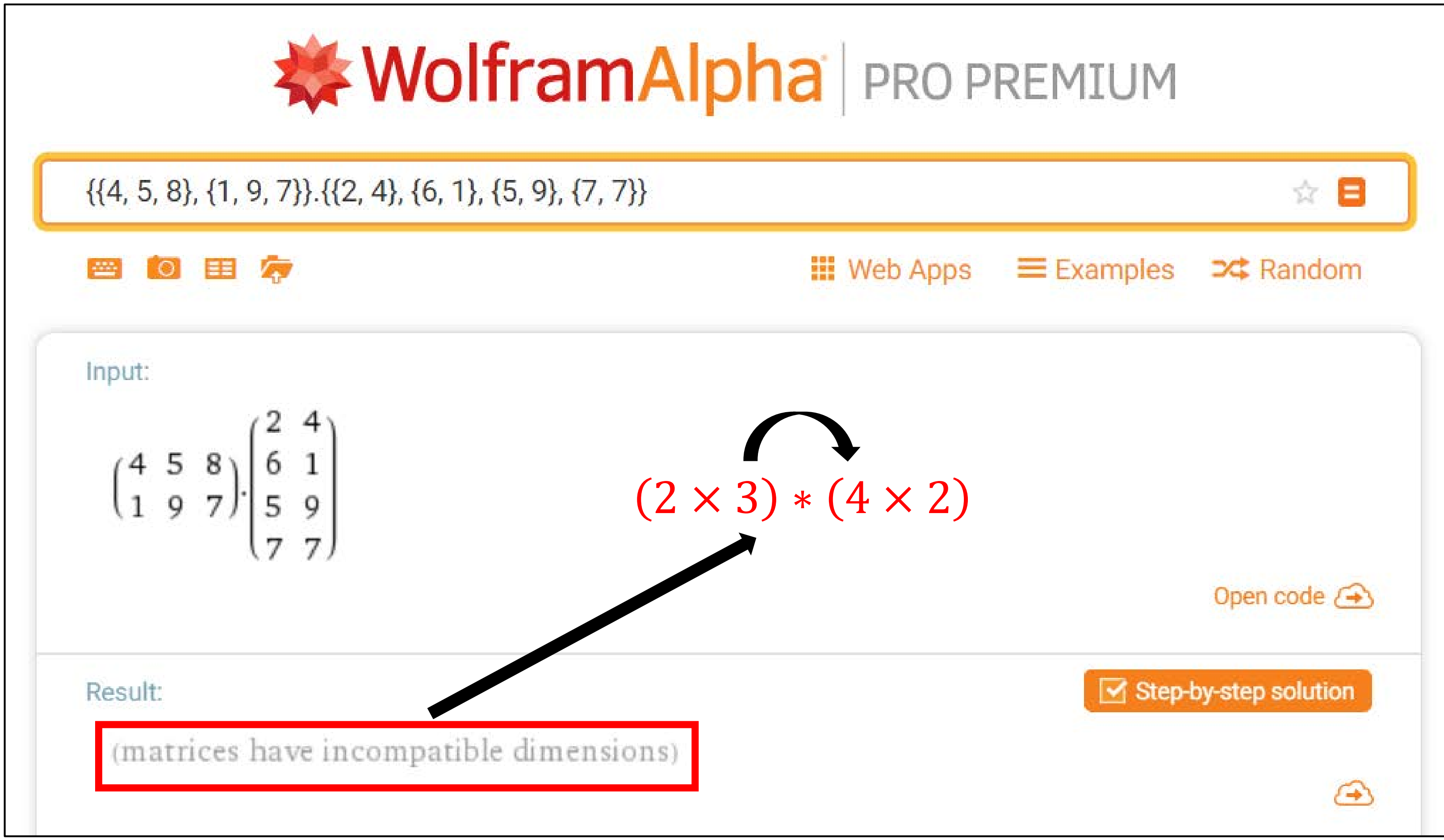
Edit Lab 1 – Matrix Multiply

- Now change the code to perform $B \times A$ – what is the output?
- Then change **matrix B** so it is declared with an extra row of the values {7,7}

```
    matrix A{{4,5,8},{1,9,7}};
    matrix B{{2,4},{6,1},{5,9},{7,7}};
    matrix C = MultiplyMatrices(A, B);
```

- Rerun Lab 1 – what is the output? **Is this valid?**
- How could we strengthen the code to prevent anomalies?

Check Lab 1 – Matrix Multiply



Determinant of a Matrix

- In linear algebra, the **determinant** is a value that can be computed from the elements of a **square matrix**
- The determinant can be used:
  - To solve a system of linear equations when those equations are represented by a matrix
  - To find the Jacobian of the matrix of all first-order partial derivatives of a vector-valued function
  - To calculate the characteristic polynomial of a matrix which is essential for eigenvalue problems
  - To express the signed *n*-dimensional volumes of *n*-dimensional parallelepipeds in analytic geometry

Determinant of a Matrix

Calculating the Determinant of a **2 x 2** Matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \det \boldsymbol{A} = |\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \boxed{a_{11} \cdot a_{22}} \boxed{-} a_{12} \cdot a_{21}$$

$$det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \cdot a_{22} - \boxed{a_{12} \cdot a_{21}}$$

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$$det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$det \begin{bmatrix} 8 & 3 \\ 4 & 2 \end{bmatrix} = 8 \cdot 2 \boxed{-} 3 \cdot 4 = 16 - 12 = 4$$

Determinant of a Matrix

Calculating the Determinant of a **3 x 3** Matrix

$$\det \boldsymbol{A} = |\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|\boldsymbol{A}| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

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Determinant of a Matrix

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Determinant of a Matrix

Calculating the Determinant of a **4 x 4** Matrix

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the definition of determinant is inherently **recursive**:
The determinant of a $\boldsymbol{n}\ x\ \boldsymbol{n}$ matrix is calculated from the determinants of $\boldsymbol{n}$ **reduced matrices** of size $(\boldsymbol{n}-\mathbf{1}) \times (\boldsymbol{n}-\mathbf{1})$,
and so on and so on, getting down to simple 2 x 2 matrices

Open Lab 2 – Matrix Determinant

- Your scientist wants you to write a program to calculate the determinant of a **10 x 10** matrix
- Each element in the matrix should be a **uniform** random **integer** between **-10** and **10** inclusively
- Implement the matrix as a C++ **vector of vectors**

These type aliases declare the same structure

```
typedef vector<double> matrix1D;
typedef vector<matrix1D> matrix2D;
```

```
typedef vector<vector<double>> matrix;
```

View Lab 2 – Matrix Determinant

```
int main()
{
    matrix2D A = CreateRandomMatrix(10,10);
    DisplayMatrix(A);

    double det{};
    CalcDeterminant(A, det);

    cout.imbue(std::locale(""));
    cout << fixed << setprecision(4);
    cout << "det = " << det << endl;

    return 0;
}
```

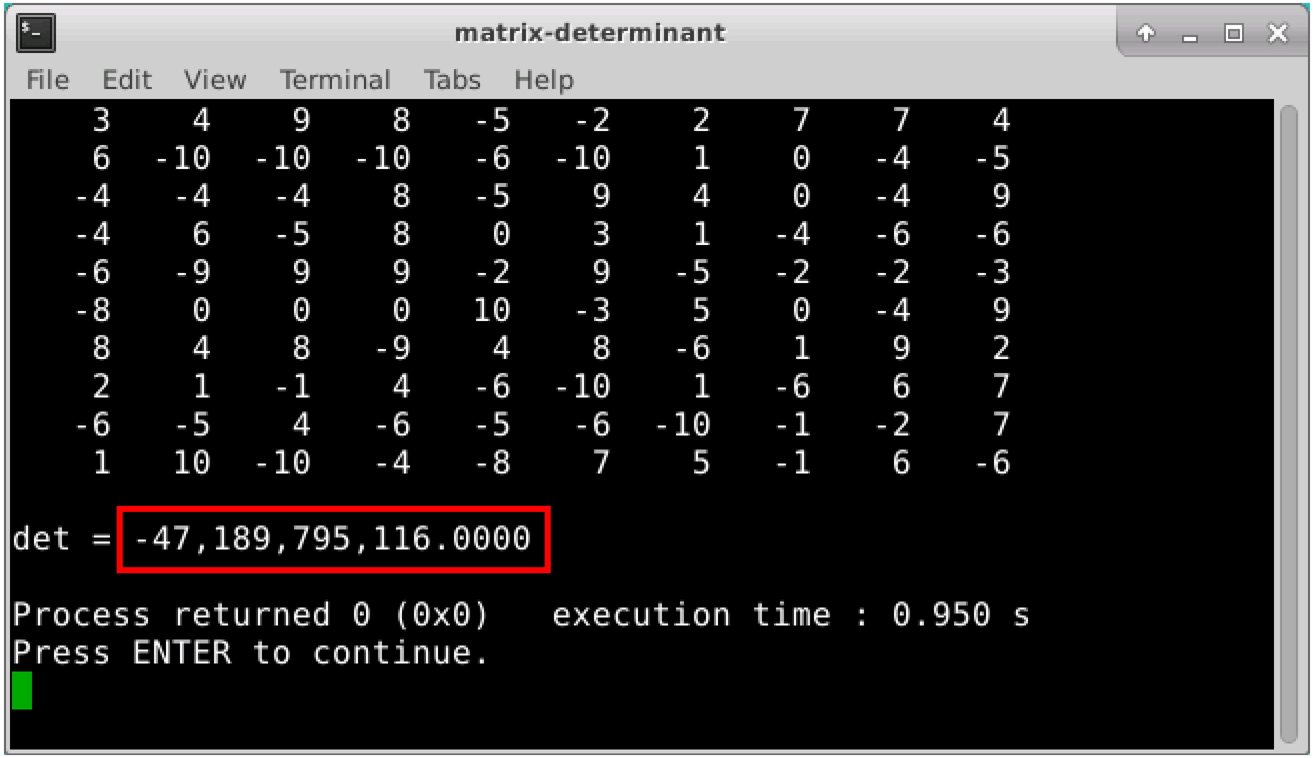
```
matrix2D CreateRandomMatrix(size_t rows, size_t cols)
{
    seed_seq seed{ 2016 };
    default_random_engine generator{ seed };
    uniform_int_distribution<> distribution(-10, 10);

    matrix2D A;
    A.resize(rows, matrix1D(cols));

    for (size_t row{}; row < rows; row++)
        for (size_t col{}; col < cols; col++)
            A.at(row).at(col) = distribution(generator);

    return A;
}
```

View Lab 2 – Matrix Determinant

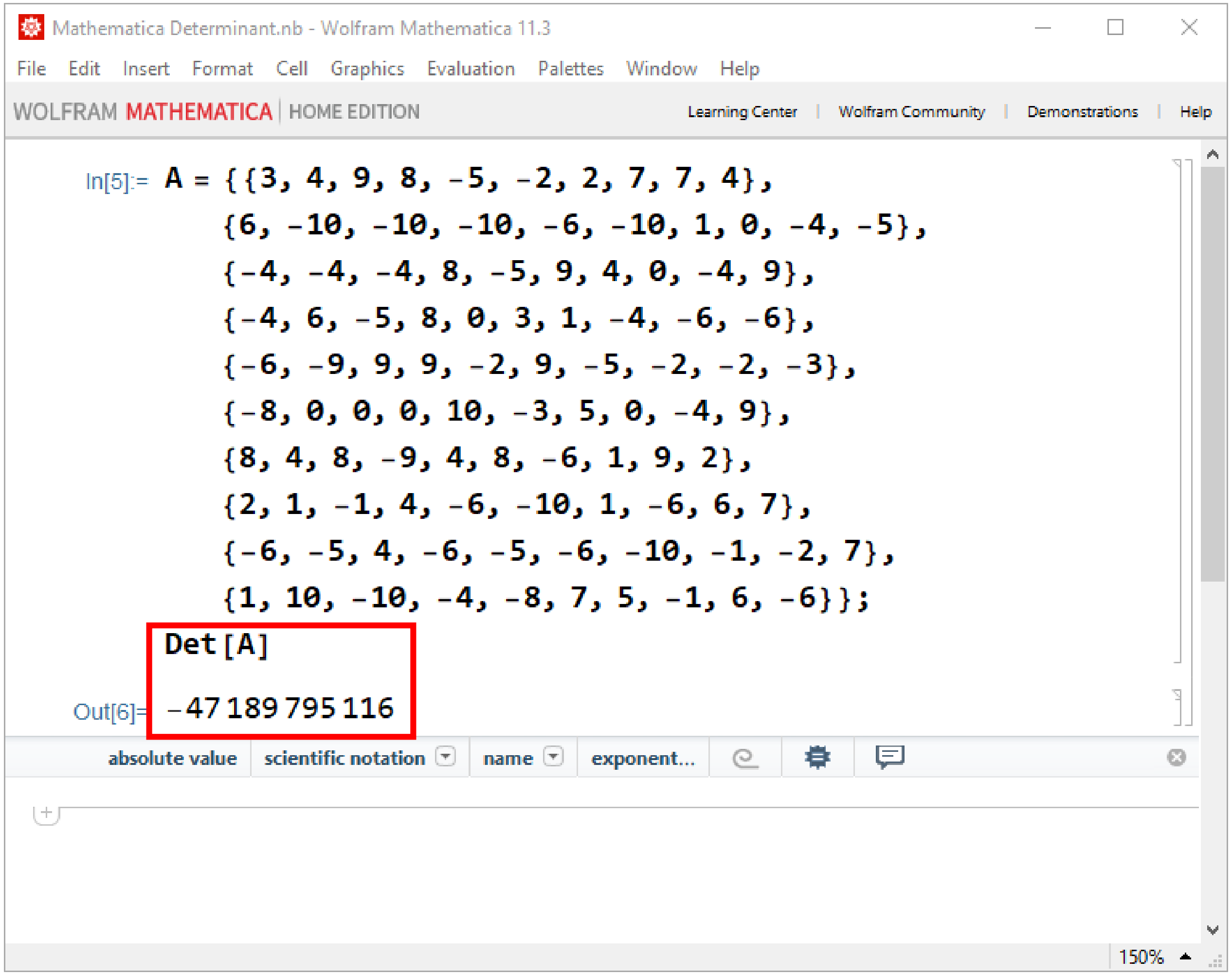
```cpp
void CalcDeterminant(const matrix2D& A, double& det, double f = 1)
{
    size_t rowsA = A.size();
    size_t colsA = A[0].size();

    if (rowsA == 2 && colsA == 2)
        det += f * (A[0][0] * A[1][1] - A[0][1] * A[1][0]);
    else
    {
        for (size_t rowA{}; rowA < rowsA; rowA++)
        {
            matrix2D B = CreateReducedMatrix(A, rowA, 0);

            double f2 = A[rowA][0];
            if (rowA % 2 == 1) f2 *= -1;
            CalcDeterminant(B, det, f * f2);
        }
    }
}
```

```cpp
matrix2D CreateReducedMatrix(const matrix2D& A, size_t skipRow, size_t skipCol)
{
    size_t rowsA = A.size();
    size_t colsA = A.at(0).size();

    matrix2D B(rowsA - 1, matrix1D(colsA - 1, 0));

    size_t rowB{};
    for (size_t rowA{}; rowA < rowsA; rowA++)
    {
        if (rowA == skipRow)
            continue;
        size_t colB{};
        for (size_t colA{}; colA < colsA; colA++)
        {
            if (colA == skipCol)
                continue;
            B[rowB][colB] = A[rowA][colA];
            colB++;
        }
        rowB++;
    }

    return B;
}
```

Run Lab 2 – Matrix Determinant



Check Lab 2 – Matrix Determinant



Goal: Solve a System of Linear Equations

Two Equations and Two Unknowns

$$ax + by = c$$
$$dx + ey = f$$

$$ax = c - by$$

$$x = \frac{c - by}{a}$$

$$\frac{d(c - by)}{a} + ey = f$$

$$cd - bdy + aey = af$$

$$y(ae - bd) = af - cd$$

$$y = \boxed{\frac{af - cd}{ae - bd}}$$

Goal: Solve a System of Linear Equations

Two Equations and Two Unknowns

$$ax + by = c$$
$$dx + ey = f$$

$$by = c - ax$$

$$y = \frac{c - ax}{b}$$

$$dx + \frac{e(c - ax)}{b} = f$$

$$bdx + ce - aex = bf$$

$$x(bd - ae) = bf - ce$$

$$x = \frac{bf - ce}{bd - ae}\left(\frac{-1}{-1}\right) = \boxed{\frac{ce - bf}{ae - bd}}$$

Goal: Solve a System of Linear Equations

Two Equations and Two Unknowns

$$\begin{aligned} \boxed{a}x + \boxed{b}y &= c \\ \boxed{d}x + \boxed{e}y &= f \end{aligned}$$

$$x = \frac{ce - bf}{ae - bd}$$

$$y = \frac{af - cd}{ae - bd}$$

$$\det \boldsymbol{C} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

Goal: Solve a System of Linear Equations

Two Equations and Two Unknowns

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$$dx + \boxed{e}y = \boxed{f}$$

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Goal: Solve a System of Linear Equations

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$$\det \boldsymbol{B} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd$$

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Goal: Solve a System of Linear Equations

Two Equations and Two Unknowns

$$ax + by = c$$
$$dx + ey = f$$

$$x = \frac{ce - bf}{ae - bd}$$

$$y = \frac{af - cd}{ae - bd}$$

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$$\det \boldsymbol{B} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd$$

$$\det \boldsymbol{C} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

$$\boxed{x = \frac{|\boldsymbol{A}|}{|\boldsymbol{C}|}}$$

$$\boxed{y = \frac{|\boldsymbol{B}|}{|\boldsymbol{C}|}}$$

Goal: Solve a System of Linear Equations

Three Equations and Three Unknowns

$$
\begin{aligned}
4x + 5y - 2z &= -14 \\
7x - \;\; y + 2z &= 42 \\
3x + \;\; y + 4z &= 28
\end{aligned}
$$

Open Lab 3 – Cramer's Rule

- Develop a console mode C++ application that uses Cramer's Rule to solve a system of three linear equations with three unknowns
  - The code encodes the system of equations as a 2D coefficient matrix and value vector (a 1D array)
  - The code already handles **linearly dependent** systems
- The code will display the equations using standard algebraic formatting rules (aka "pretty print")
  - No coefficients of 1, such as **1**x + 5y + 32z = 49
  - No plus signed followed by a negative sign, such as 2x **+ -**5y + 9 = -13
  - Don't display an unknown if that term's coefficient is **0**

Create a Coefficient Matrix & Value Vector

Coefficients

Values

$$4x + 5y - 2z = -14$$
$$7x - \; y + 2z = 42$$
$$3x + \; y + 4z = 28$$

NOTE: This is the **old school** way of specifying 2D arrays

```
int main()
{
    double coeffMatrix[3][3]
    { { 4,5,-2 },
    { 7,-1,2 },
    { 3,1,4 } };

    double valueVector[3]{ -14,42,28 };
```

Cramer's Rule

Given the system:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

with

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

then the solution of this system is:

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

Cramer’s Rule

Calculating the Determinant of a 3 x 3 Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

$$= (aei + bfg + cdh) - (afh + bdi + ceg)$$

```
double Determinant(double matrix[3][3])
{
    double a = matrix[0][0];
    double b = matrix[0][1];
    double c = matrix[0][2];
    double d = matrix[1][0];
    double e = matrix[1][1];
    double f = matrix[1][2];
    double g = matrix[2][0];
    double h = matrix[2][1];
    double i = matrix[2][2];

    double det = a * (e * i - f * h)
        - (b * (d * i - f * g))
        + c * (d * h - e * g);

    return det;
}
```

Coefficients

Values

$$\begin{aligned} 4x + 5y - 2z &= -14 \\ 7x - y + 2z &= 42 \\ 3x + y + 4z &= 28 \end{aligned}$$

Cramer's Rule

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Overlay the values onto the **coeffMatrix**

```
void OverlayValues(double coeffMatrix[3][3], double valueVector[3],
    int col, double newMatrix[3][3])
{
    // Copy existing coeffMatrix to newMatrix
    for (int i{}; i < 3; ++i)
        for (int j{}; j < 3; ++j)
            newMatrix[i][j] = coeffMatrix[i][j];

    // Overlay the valueVector on the specified column
    for (int i{}; i < 3; ++i)
        newMatrix[i][col] = valueVector[i];
}
```

Cramer’s Rule

Create new matrices mX, mY, mZ

```
double mX[3][3];
OverlayValues(coeffMatrix, valueVector, 0, mX);
double detX = Determinant(mX);

double mY[3][3];
OverlayValues(coeffMatrix, valueVector, 1, mY);
double detY = Determinant(mY);

double mZ[3][3];
OverlayValues(coeffMatrix, valueVector, 2, mZ);
double detZ = Determinant(mZ);

cout << "DetCoeff = " << detCoeff << endl;
cout << endl;

cout << "DetX = " << detX << endl;
cout << "DetY = " << detY << endl;
cout << "DetZ = " << detZ << endl;
cout << endl;

cout << "X = " << detX / detCoeff << endl;
cout << "Y = " << detY / detCoeff << endl;
cout << "Z = " << detZ / detCoeff << endl;
```

Run Lab 3 - Cramer's Rule

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

What does it mean if D = 0?

cramers-rule

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```
4x + 5y -2z = -14
7x -y + 2z = 42
3x + y + 4z = 28

DetCoeff = -154

DetX = -616
DetY = 616
DetZ = -770

X = 4
Y = -4
Z = 5

Process returned 0 (0x0)   execution time : 0.015 s
Press ENTER to continue.
```

Edit Lab 3 – Cramer's Rule

- Update the **main**() function to solve these two systems of linear equations:

$$
\begin{aligned}
-6r + 5s + 2t &= -11 \\
-2r + s + 4t &= -9 \\
4r - 5s + 5t &= -4
\end{aligned}
$$

$$
\begin{aligned}
-3a - b - 3c &= -8 \\
-5a + 3b + 6c &= -4 \\
-6a - 4b + c &= -20
\end{aligned}
$$

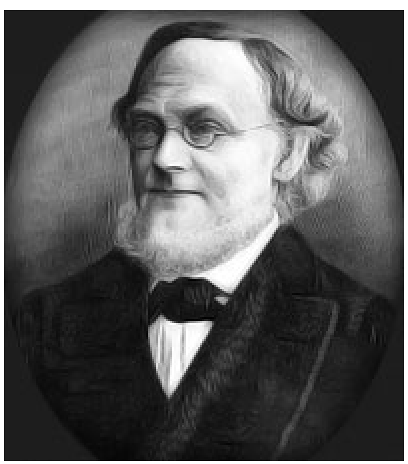
Goldbach's Conjecture

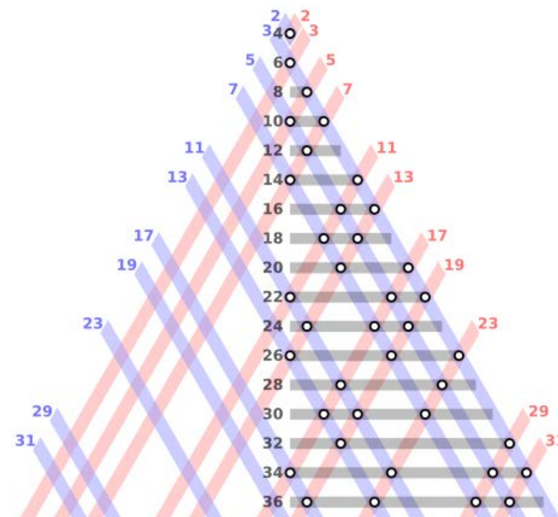
Goldbach's conjecture

From Wikipedia, the free encyclopedia

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics.

On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII)[6] in which he proposed the following conjecture:





Every even integer greater than 4 can be written as the sum of two odd primes!

Open Lab 4 – Goldbach's Conjecture

- Goldbach's Conjecture (1742): **All even integers greater than 4 are the sum of just two odd primes**
- Develop a console mode C++ application that verifies Goldbach's Conjecture for all even integers $\mathbf{6 \le x \le 458}$
  - You must first create a vector of **odd primes** up to 229
  - Then you must check if all even integers in the above specified range are the sum of two odd primes
  - Display on screen **any violations** of the conjecture
- Despite its simplicity, this remains a **conjecture** as it has never been proven nor disproven

Lab 4
Goldbach's Conjecture

```
vector<int> GeneratePrimes(size_t numPrimes)
{
    vector<int> primes;

    primes.push_back(2);

    int n = 3;
    while (primes.size() < numPrimes)
    {
        int r = 0;
        for (int p : primes)
        {
            r = n % p;
            if (r == 0) break;
        }
        if (r > 0)
            primes.push_back(n);
        n = n + 2;
    }

    return primes;
}
```

primes	**0**	**1**	**2**	**3**	4
.at()	2				

n	3	
p	2	
r	1	

primes	**0**	**1**	**2**	**3**	4
.at()	2	3			

n	5	
p	2	3
r	1	2

primes	**0**	**1**	**2**	**3**	4
.at()	2	3	5		

primes	**0**	**1**	**2**	**3**	4
.at()	2	3	5	7	

n	9	
p	2	3
r	1	0

n	11

primes	**0**	**1**	**2**	**3**	4
.at()	2	3	5	7	11

Lab 4
Goldbach's Conjecture

```cpp
int main()
{
    vector<int> primes = GeneratePrimes(50);

    // Remove the first (only even) prime which is 2
    primes.erase(primes.begin());

    cout << "For primes " << primes.front()
         << " to " << primes.back()
         << ": " << endl;

    vector<bool> goldbachNumbers(primes.back() * 2 + 1);

    // Pair up each prime with itself, and then to each successive prime.
    // Sum these two numbers, and use this sum as an index into a boolean
    // vector that records if a sum has occurred at least once
    for (size_t i{0}; i < primes.size(); i++)
        for (size_t j{i}; j < primes.size(); j++)
            goldbachNumbers.at(primes.at(i) + primes.at(j)) = true;

    // Verify all evens #s > 4 are the sum of two odd primes
    for (size_t k{6}; k < goldbachNumbers.size(); k += 2)
        if (goldbachNumbers.at(k) == false)
            cout << "Goldbach violation at " << k << endl;

    return 0;
}
```

Every even integer greater than 4 can be written as the sum of two odd primes!

primes .at()	0	1	2	3	4	5	6	7
	2	3	5	7	11	13	17	19

primes .at()	0	1	2	3	4	5	6	7
	3	5	7	11	13	17	19	23

goldbachNumbers .at()	0	1	2	3	4	5	6	7
	false	false	false	false	false	false	false	false

i 0
j 0
primes.at(i) 3
primes.at(j) 3

goldbachNumbers .at()	6	7	8	9	10	11	12	13
	TRUE	false	false	false	false	false	false	false

i 0
j 1
primes.at(i) 3
primes.at(j) 5

goldbachNumbers .at()	6	7	8	9	10	11	12	13
	TRUE	false	TRUE	false	false	false	false	false

i 2
j 2
primes.at(i) 5
primes.at(j) 5

goldbachNumbers .at()	6	7	8	9	10	11	12	13
	TRUE	false	TRUE	false	TRUE	false	false	false

Run Lab 4 – Goldbach's Conjecture

goldbach-conjecture

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```
For primes 3 to 229:
Goldbach violation at 412
Goldbach violation at 430
Goldbach violation at 432
Goldbach violation at 436
Goldbach violation at 442
Goldbach violation at 444
Goldbach violation at 448

Process returned 0 (0x0)   execution time : 0.015 s
Press ENTER to continue.
```

How can this be? Where did we go wrong?

List of first 100 prime numbers

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97
101	103	107	109	113
127	131	137	139	149
151	157	163	167	173
179	181	191	193	197
199	211	223	227	$\boxed{229}$
233	239	241	251	257
263	269	271	277	281
283	293	307	311	313
317	331	337	347	349
353	359	367	373	379
383	389	397	401	409
419	$\boxed{421}$	431	433	439
443	449	457	461	463
467	479	487	491	499
503	509	521	523	541

WolframAlpha

what two primes sum to 432?

Input interpretation:

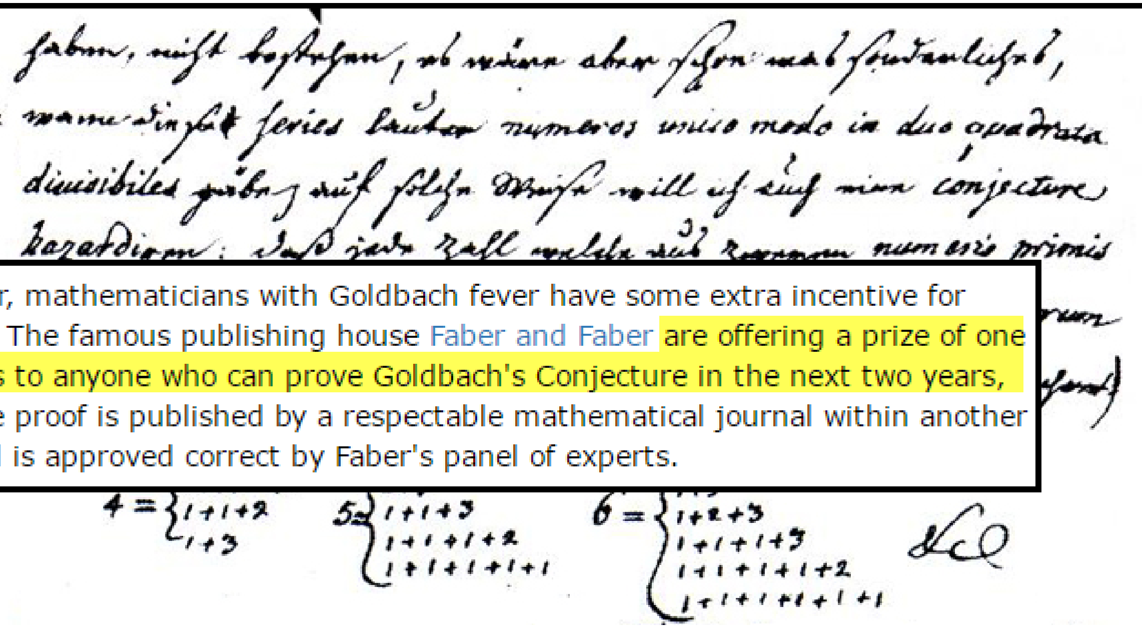
432	as the sum of exactly	2	primes

Result:

$421 + 11 = 432$

Verify for all even integers $6 \leq x \leq 458$

Goldbach's Conjecture



As of this year, mathematicians with Goldbach fever have some extra incentive for their labours. The famous publishing house Faber and Faber are offering a prize of one million dollars to anyone who can prove Goldbach's Conjecture in the next two years, as long as the proof is published by a respectable mathematical journal within another two years and is approved correct by Faber's panel of experts.

PRIME NUMBERS: THE 271 YEAR OLD PUZZLE RESOLVED

STORY BY ARTEM KAZNATCHEEV

Published: May 13, 2013

The odd Goldbach conjecture, a two-hundred and seventy-one year open problem of mathematics, has been resolved. Earlier today, H.A. Helfgott proved that any odd number greater than 5 can be written as the sum of 3 primes.

Now you know…

- To multiply two matrices, the **# of columns** in the first matrix must match the **# of rows** in the second matrix
  - The **product matrix** will have the same # of rows as the first matrix, and the same # of columns as the second matrix
  - Matrix multiplication is the sum of the element by element products of the rows in the first matrix and the columns in the second matrix
- **Cramer's Rule** is a step-by-step algorithmic way to solve systems of linear equations ***without*** the tedious algebra of back substitution – it uses **determinants**
  - This method is easy to run in parallel as each CPU core can calculate a separate unknown at the same time
- **Goldbach is waiting for you!** ☺