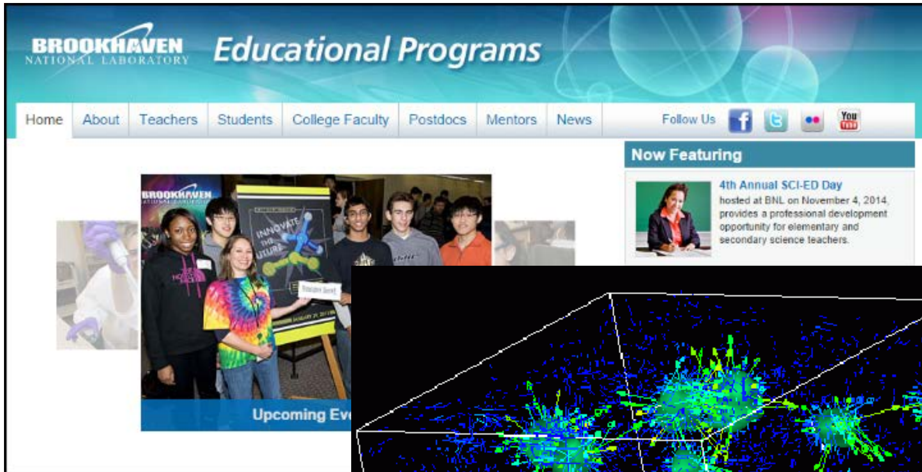

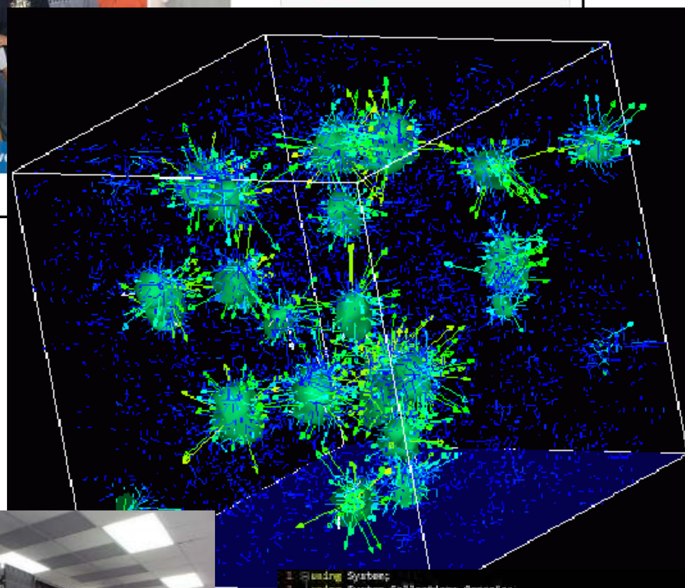

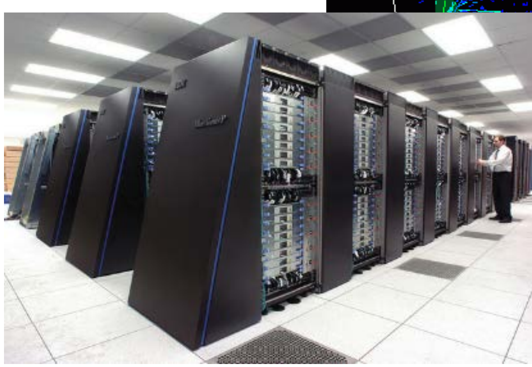

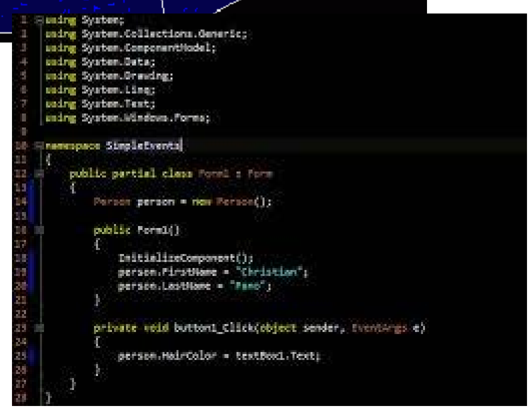

Survey of Scientific Computing

(SciComp 301)

Dave Biersach
Brookhaven National Laboratory
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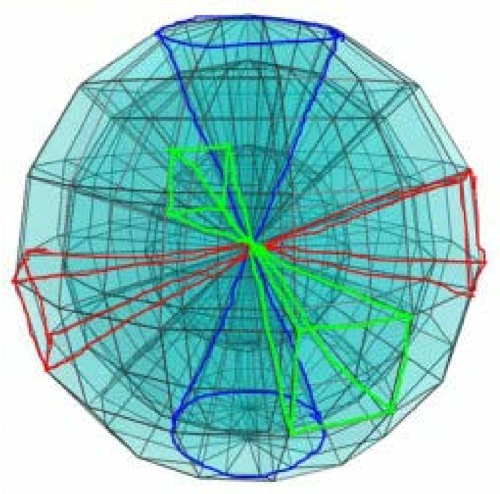
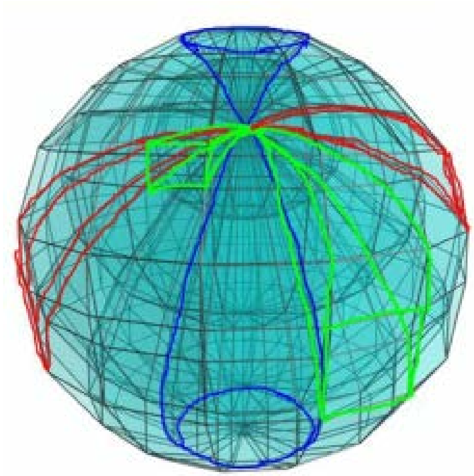
Session 20

Monte Carlo Method

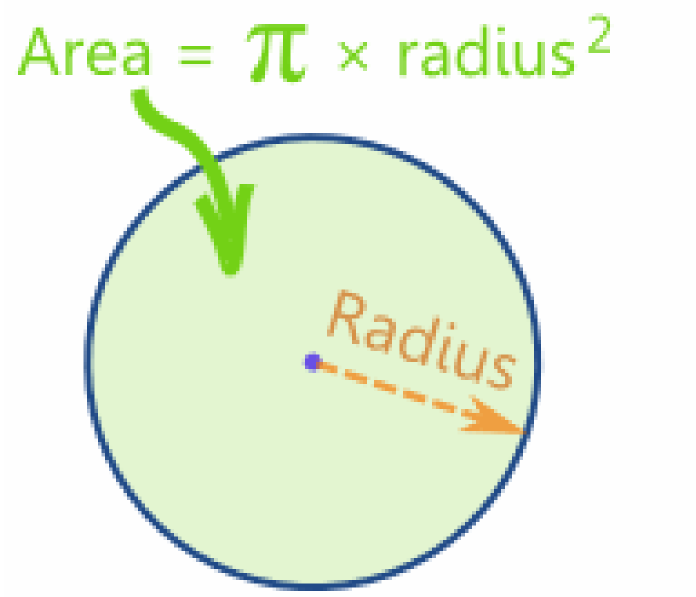
An *Interesting* Question

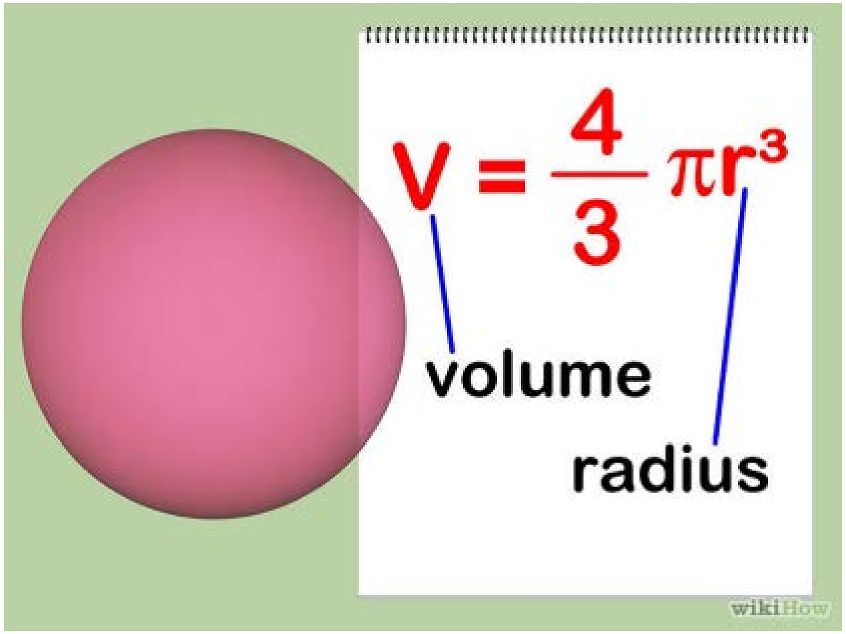
- What is the ***volume*** of a **four-dimensional** **unit** hypersphere?
  - What does a 4D sphere "look" like?
  - What is a "unit" sphere?
  - Where do I even start?
- Break down complex questions into simpler steps:
  - How can we calculate the area of a 2D circle?
  - How can we calculate the volume of a 3D sphere?
  - How do we move from 3D to 4D?



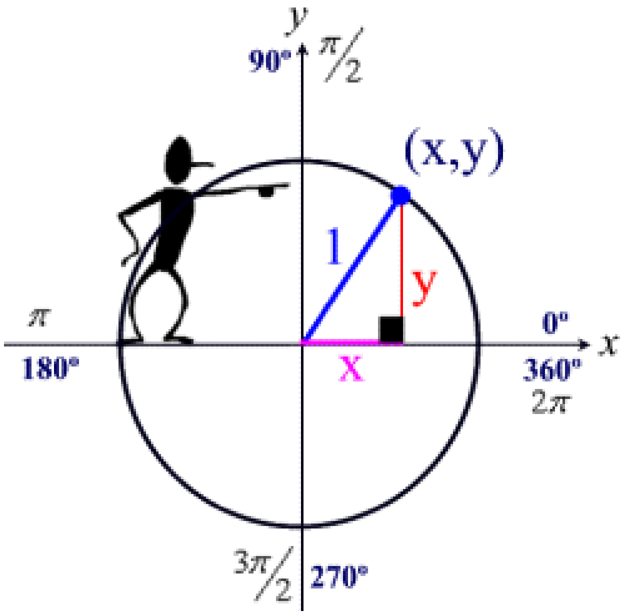


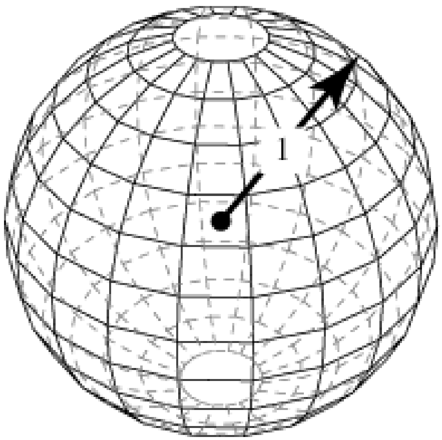
Area and Volume



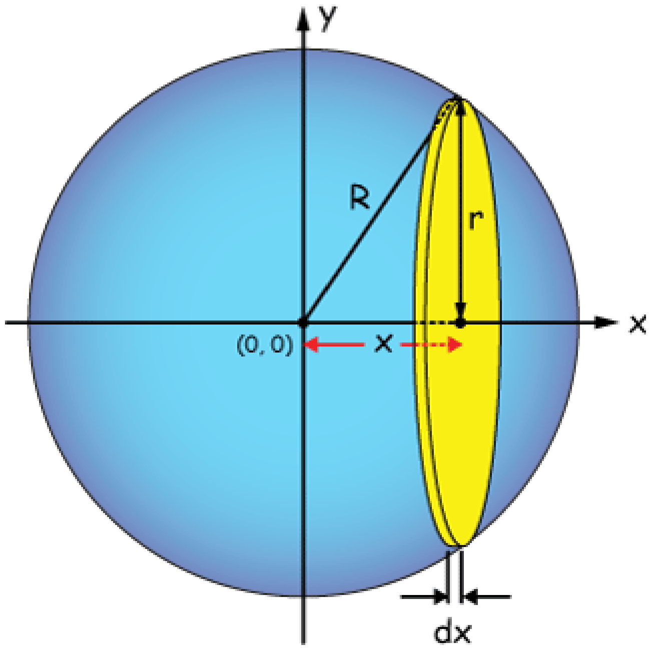


A **Unit** Circle and **Unit** Sphere

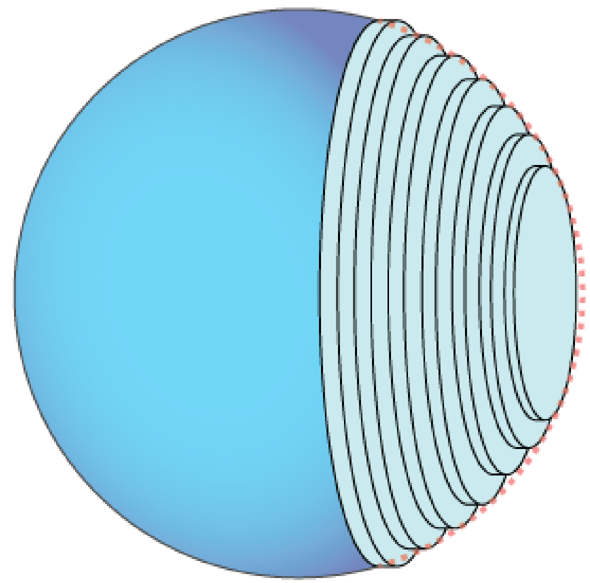




2-D Area → 3-D Volume

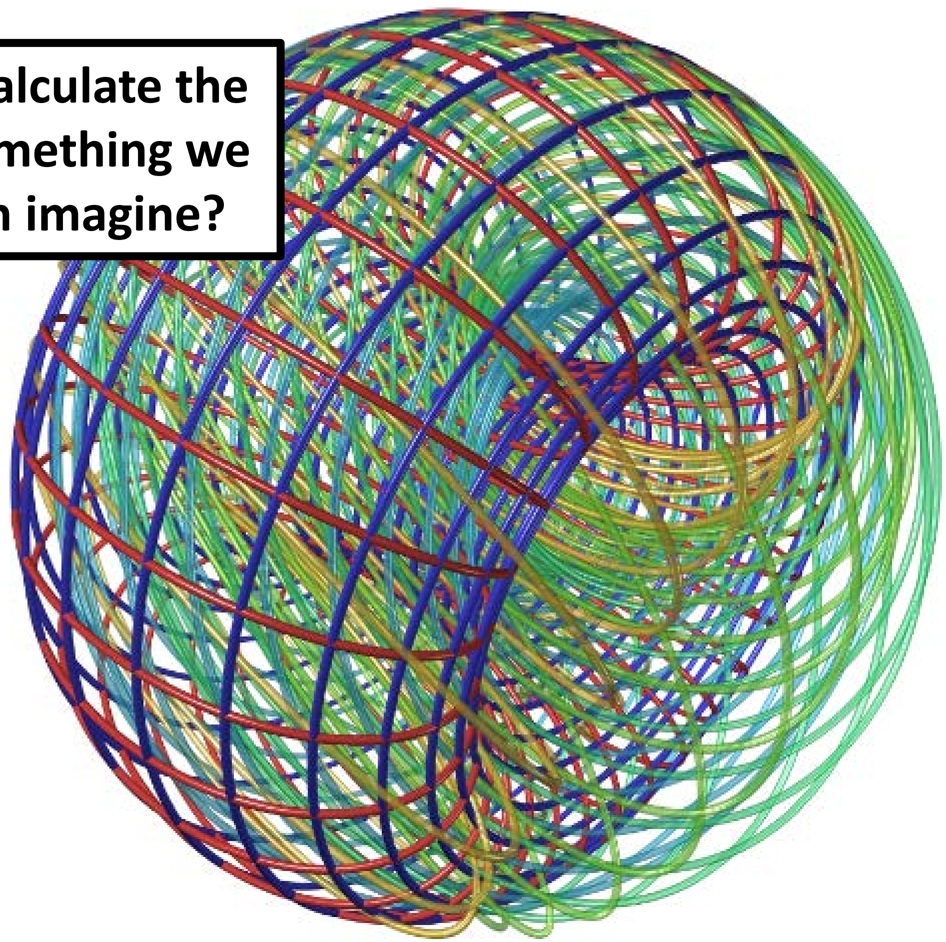


Volume of the disk:
$\pi r^2 \cdot dx = \pi(R^2 - x^2) \cdot dx$

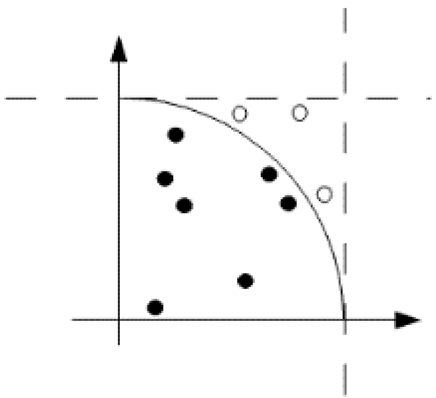


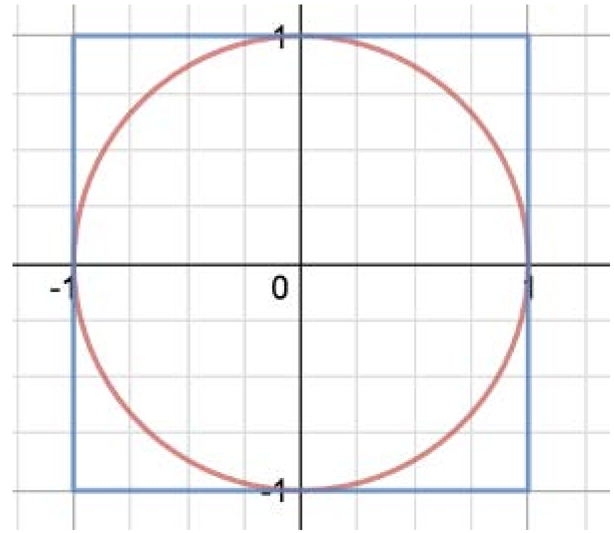
A 4-D Hypersphere

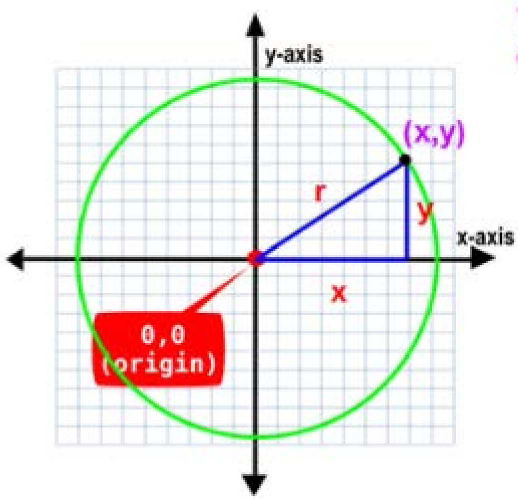
How do we calculate the volume of something we can not even imagine?



Area as a "Ratio" of Inside vs. Total Dots

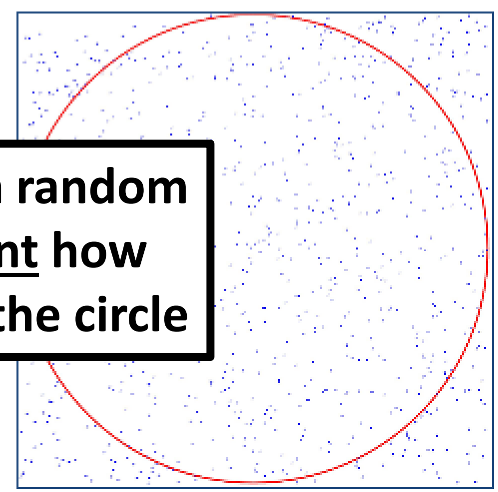






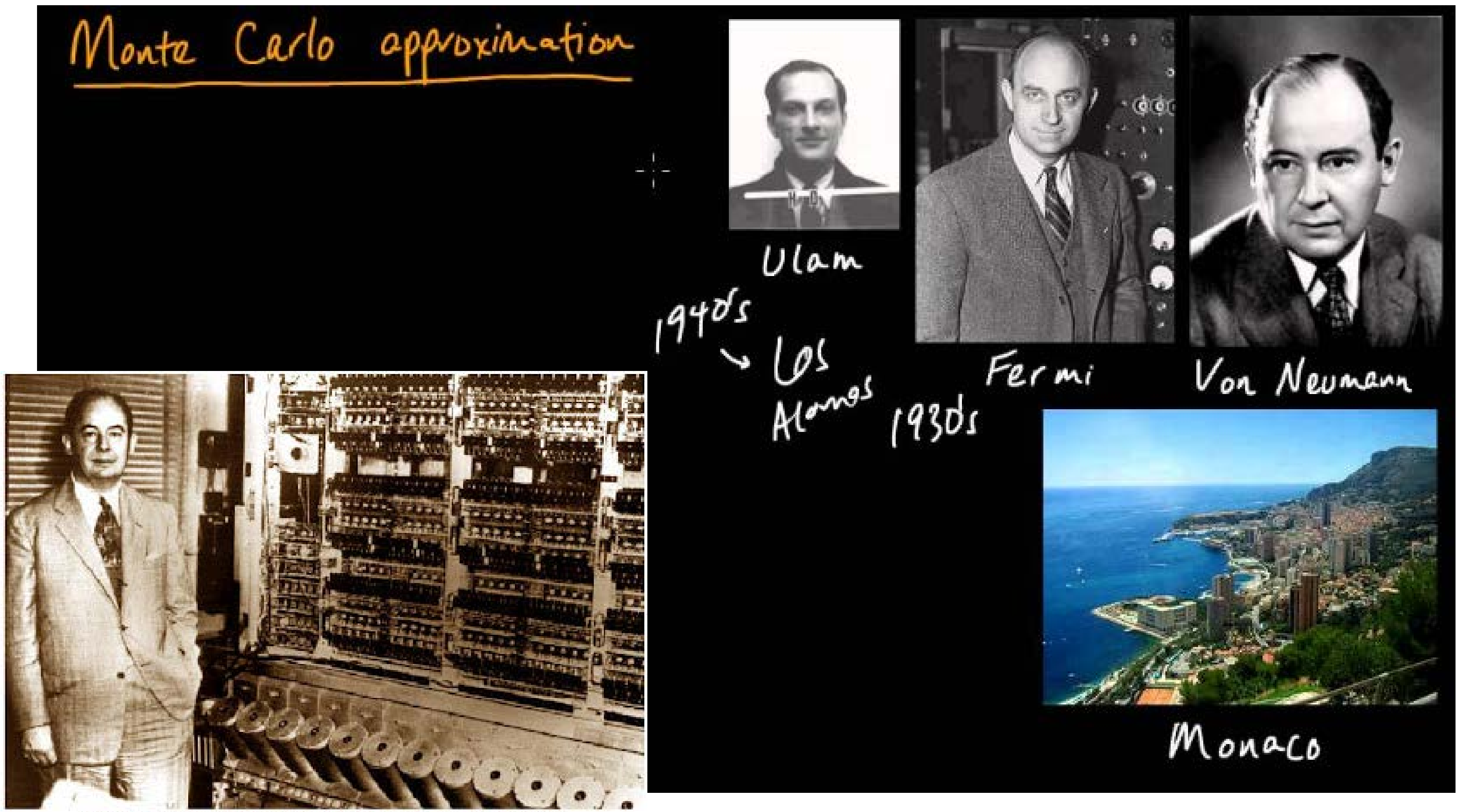
The equation of a circle centered at the origin

$$x^2 + y^2 = r^2$$



We pick a million random points and <u>count</u> how many are inside the circle

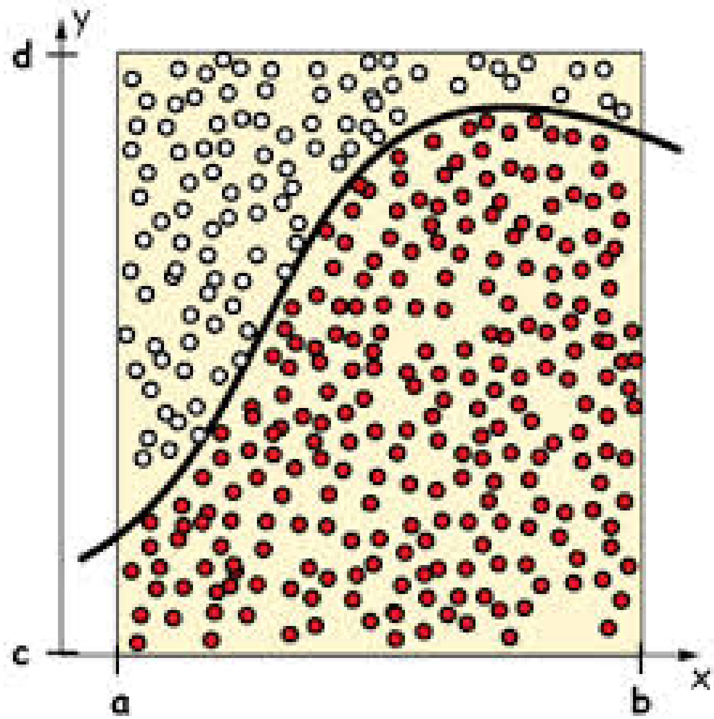
The Monte Carlo Method



Johnny von Neumann [1903-1957] alongside the Maniac computer at the Institute for Advanced Studies, Princeton.

The Monte Carlo Method

- With Monte Carlo, we **randomly sample points** across the entire space and count how many are below the curve
- The **ratio** between points below the curve to ***total points*** is an estimate of the **area** (integral)
- Monte Carlo is a **non-deterministic** approach because it uses a pseudo-random number generator



2-D Area as a "Ratio" of Dots

```
void draw(SimpleScreen& ss)
{
    ss.DrawAxes();
    ss.DrawCircle(0, 0, 1, "green", 2);

    seed_seq seed{ 2017 };
    default_random_engine generator{ seed };
    uniform_real_distribution<double> distribution{ 0, 1 };

    const int iterations = 100000;
    int count{};

    ss.LockDisplay();

    for (int i{};i < iterations;++i) {
        double x = distribution(generator) * 2.0 - 1.0;
        double y = distribution(generator) * 2.0 - 1.0;
        if (x*x + y*y <= 1.0) {
            ss.DrawPoint(x, y, "red");
            count++;
        }
        else
            ss.DrawPoint(x, y, "blue");
    }

    ss.UnlockDisplay();

    double area = (double)count / iterations * 4.0;
    double err = (M_PI - area) / M_PI * 100;

    cout << "2D Circle Area PRNG" << endl
        << "Iterations = " << iterations << endl
        << "Est. Area  = " << area << endl
        << "Act. Area  = " << M_PI << endl
        << "Abs. % Err = " << abs(err) << endl;

}
```

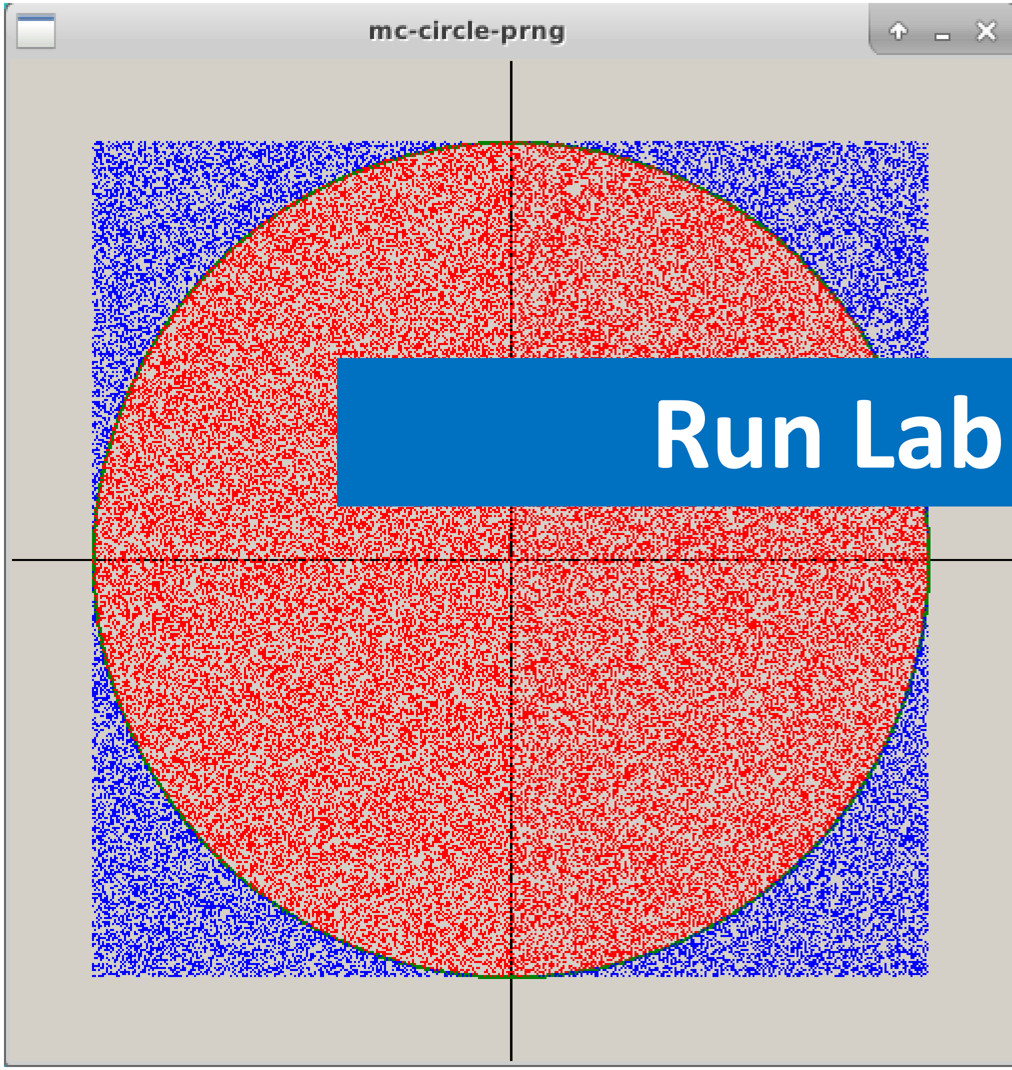
Open Lab 1

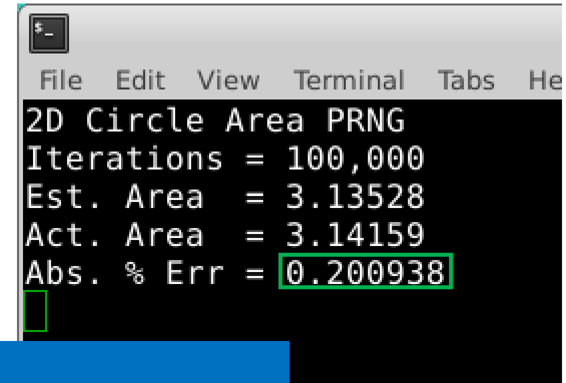
$$\frac{dots_{inside}}{dots_{total}} = \frac{area_{circle}}{area_{square}}$$

$$area_{square} = 2 \times 2 = 4$$

$$area_{circle} = \frac{count}{iterations} \times 4$$

2-D Area as a “Ratio” of Dots

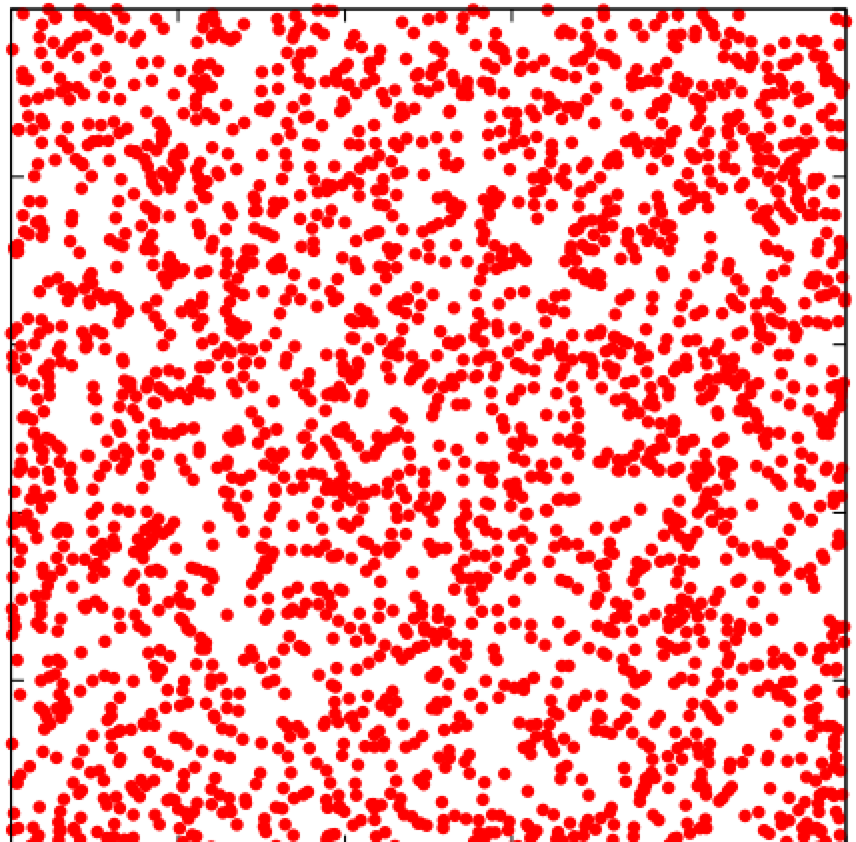




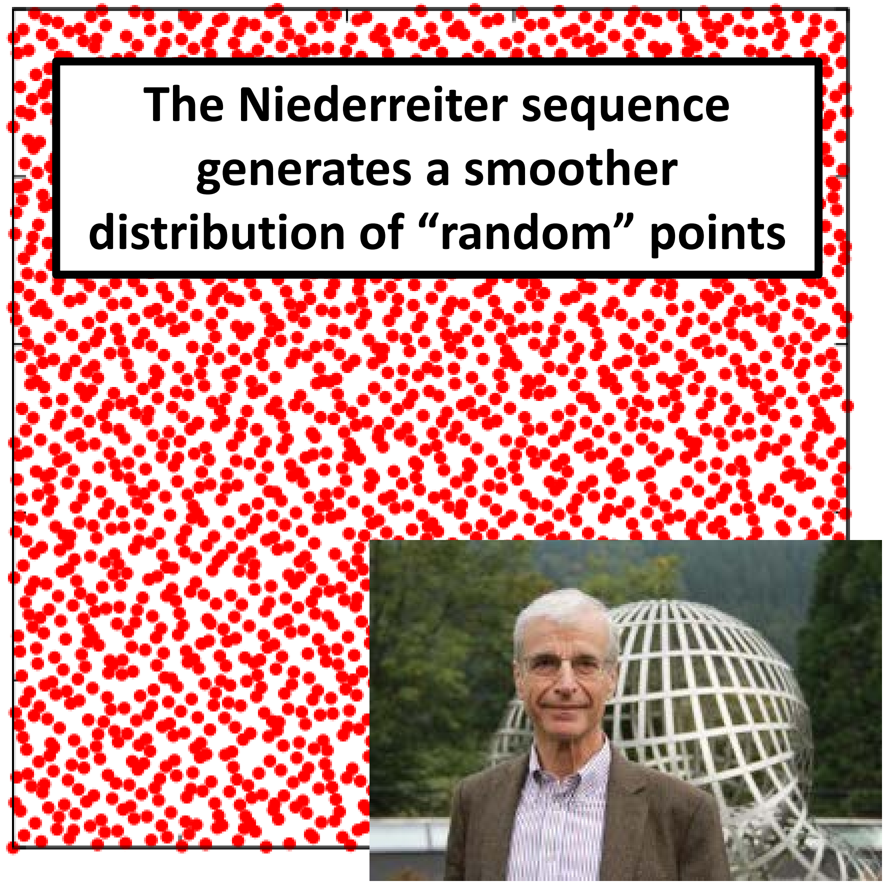
Run Lab 1

Comparing “Random” Number Generators

Standard PRNG

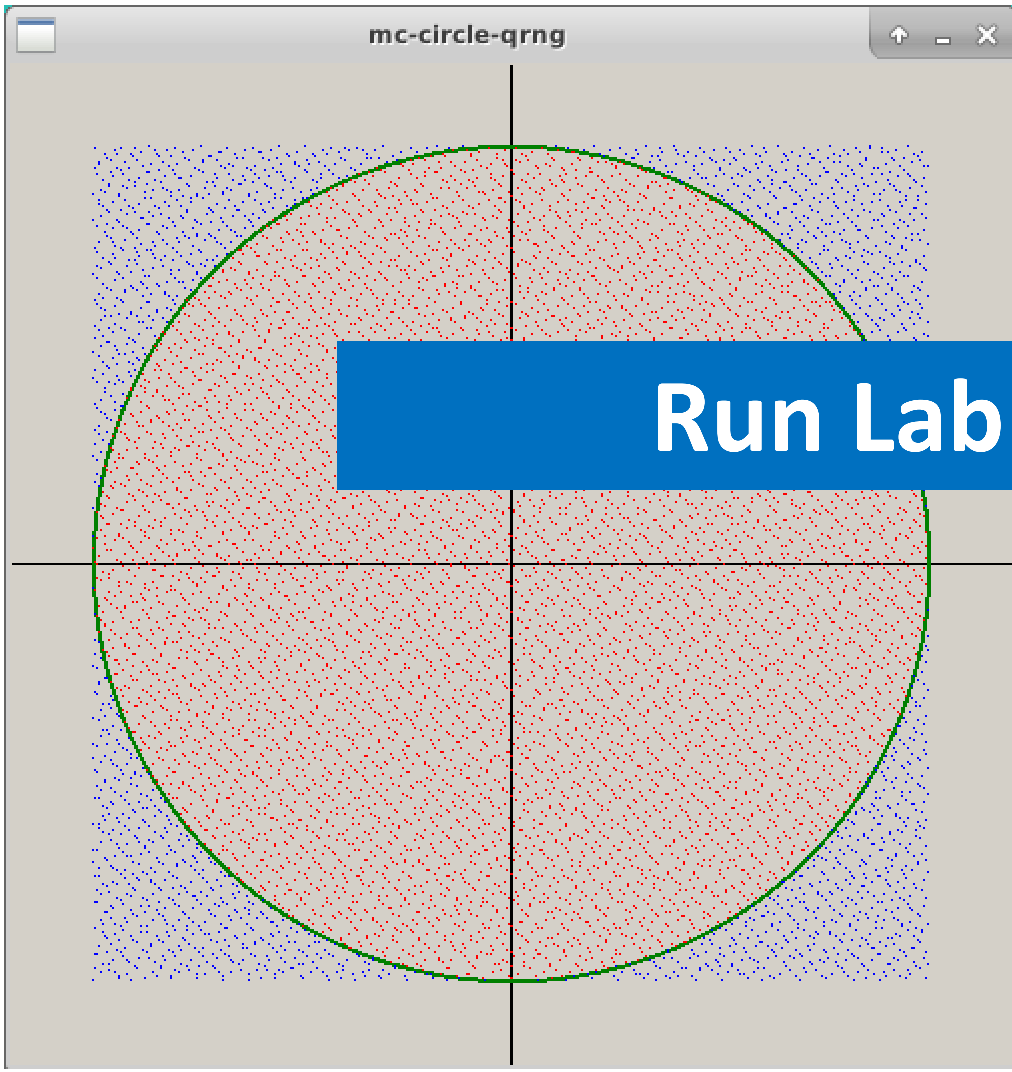


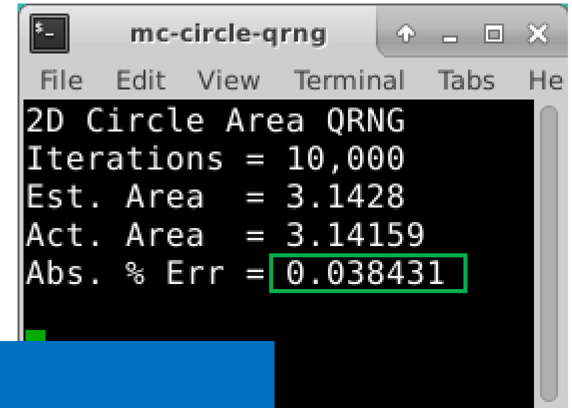
Niederreiter QRNG



The Niederreiter sequence generates a smoother distribution of “random” points

Improved 2-D Monte Carlo Estimator

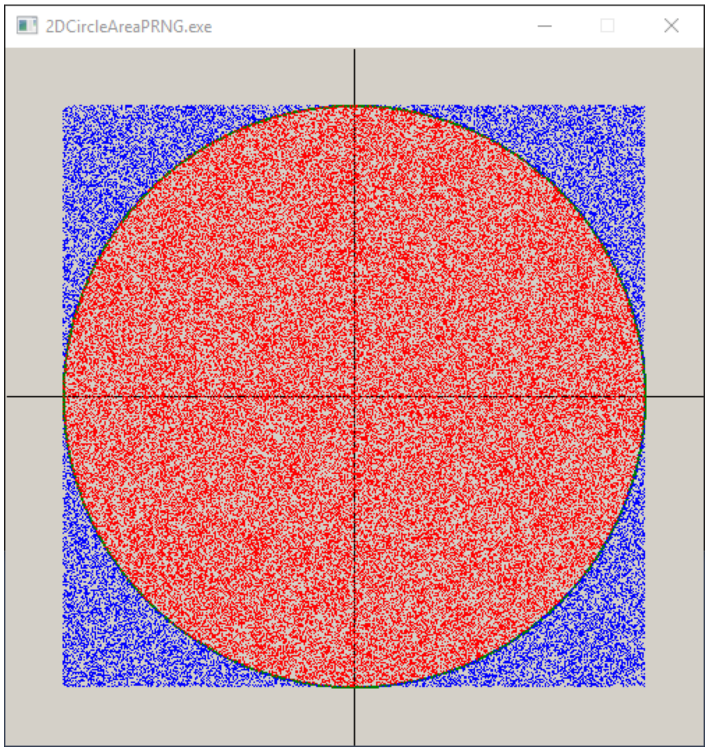




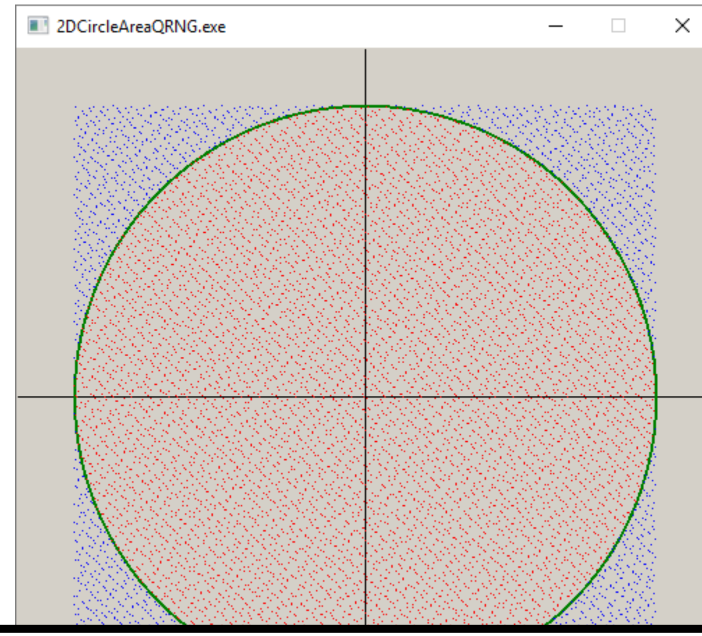
Run Lab 2

Improved 2-D Monte Carlo Estimator

PRNG % Error = .2009



QRNG % Error = .0384



The Niederreiter sequence provides a 6X increase in accuracy of the estimate using 10X fewer points!

3-D Unit Sphere Volume Estimator

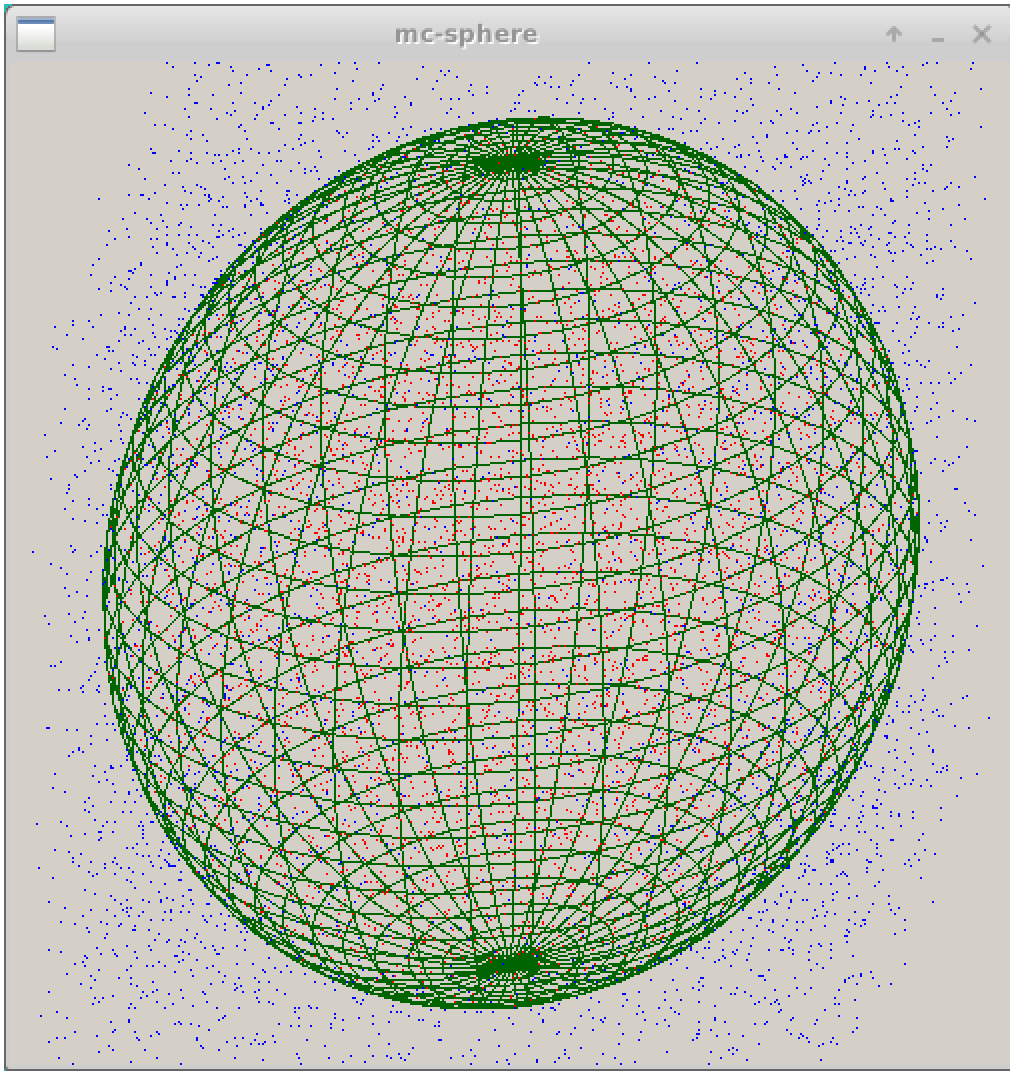
```
Niederreiter2 qrng;
double r[3];
int seed{};

const int iterations = 10000;
int count{};

ss.LockDisplay();

for (int i{};i < iterations;++i) {
    qrng.Next(3, &seed, r);

    double x = r[0] * -2.0 - 1.0;
    double y = r[1] * -2.0 - 1.0;
    double z = r[2] * -2.0 - 1.0;
    if (x*x + y*y + z*z <= 1.0) {
        ss.DrawPoint3D(x, y, z, "red");
        count++;
    }
    else
        ss.DrawPoint3D(x, y, z, "blue");
}

ss.UnlockDisplay();

double estVol = (double)count / iterations * 8;
double actVol = 4.0 / 3.0 * M_PI;
double err = (actVol - estVol) / actVol * 100;

cout << "3D Sphere Volume QRNG" << endl
    << "Iterations = " << iterations << endl
    << "Est. Volume  = " << estVol << endl
    << "Act. Volume  = " << actVol << endl
    << "Abs. % Error = " << abs(err) << endl << endl;
```

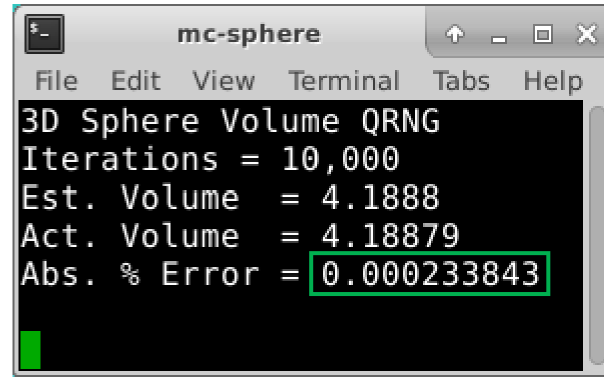
$$\frac{dots_{inside}}{dots_{total}} = \frac{volume_{sphere}}{volume_{cube}}$$

$$volume_{cube} = 2 \times 2 \times 2 = 8$$

$$volume_{sphere} = \frac{count}{iterations} \times 8$$

3-D Unit Sphere Volume Estimator





3

$$= \frac{4\pi}{3}$$

The Halton Sequence

```
double Halton(int n, int p)
{
    double h = 0;
    double f = 1;
    for (int i = n; i > 0; i /= p) {
        f = f / p;
        h += f * (i % p);
    }
    return h;
}
```

Mathematical Proceedings of the Cambridge Philosophical Society

Mathematical Proceedings of the Cambridge Philosophical
/ Issue 02 | April 1965, pp 497-498

DOI: http://dx.doi.org/10.1017/S0305004100004059 (About
online: 24 October 2008

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Research Article

On the relative merits of correlated and importance sampling for Monte Carlo integration

John H. Halton[a1]

[a1] Brookhaven National Laboratory, Upton, New York

Accommodating the 4^{th} Dimension

```
int main()
{
    int iterations = int(1e7);

    double count = 0;

    for (int i = 0; i < iterations; i++) {
        double x = Halton(i, primes[0]);

        if (distance <= 1.0)
            count++;
    }

    double volume = count / iterations * 8;

    cout << fixed << setprecision(4)
        << volume << endl;

    return 0;
}
```

We need to update this code to include the 4^{th}

on!

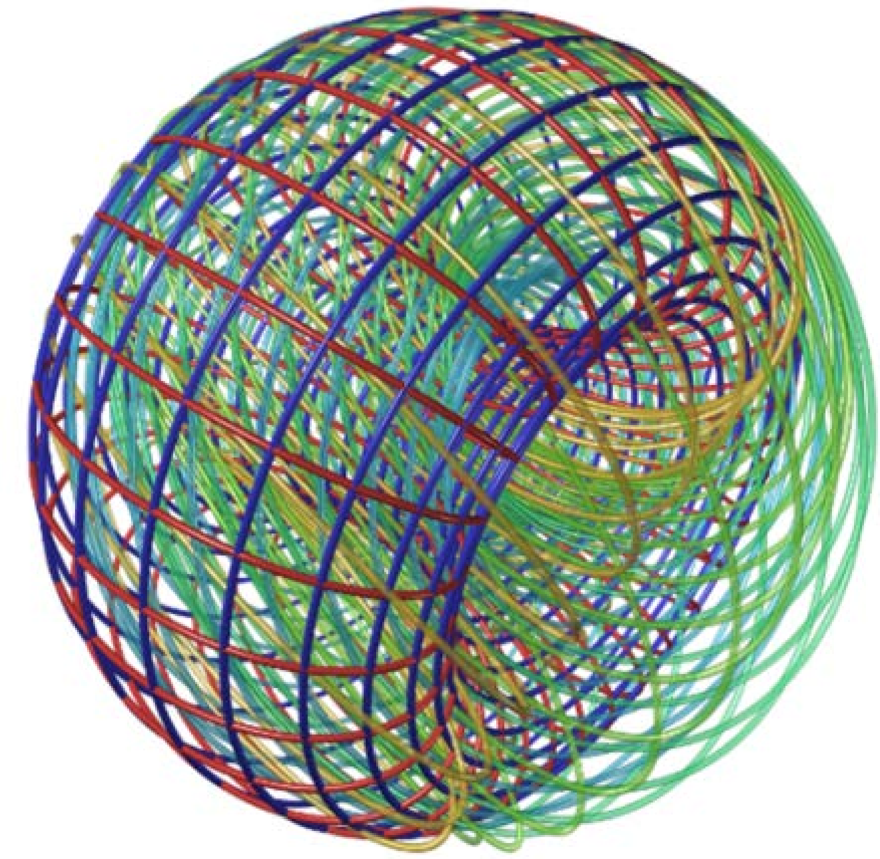
Open and edit Lab 4

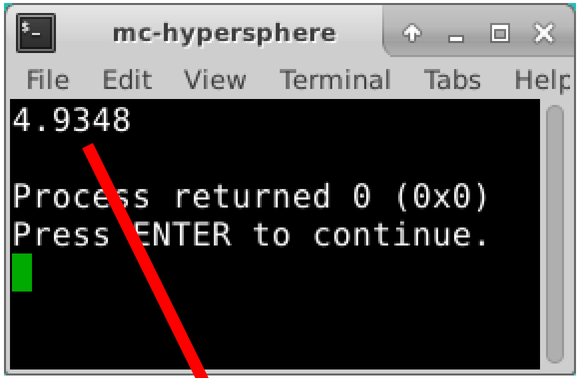
Accommodating the 4th Dimension

```
int main()
{
    int iterations = int(1e7);

    double count = 0;

    for (int i = 0; i < iterations; i++) {
        double x = Halton(i, primes[0]);
        double y = Halton(i, primes[1]);
        double z = Halton(i, primes[2]);
        double w = Halton(i, primes[3]);

        double distance = x * x + y * y + z * z + w * w;

        if (distance <= 1.0)
            count++;
    }

    double volume = count / iterations * 16;

    cout << fixed << setprecision(4)
        << volume << endl;

    return 0;
}
```

Add all the code in **red** then run the application

What is the content of a 4-D unit hypersphere?





$$= \frac{\pi^2}{2}$$

What lurks beyond the 4th dimension?

```
int main()
{
    int iterations = int(1e7);
    for (int dimension{ 2 }; dimension < 13; ++dimension) {
        double count{};
        for (int i{}; i < iterations; ++i) {
            double distance = 0;
            for (int d{}; d < dimension; ++d) {
                double v = Halton(i, primes[d]);
                distance = distance + v * v;
                if (distance > 1.0)
                    break;
            }
            if (distance <= 1.0)
                count++;
        }
        double volume = count / iterations * pow(2, dimension);
        cout << fixed << right << setw(2) << dimension << ", "
            << setprecision(4) << volume
            << endl;
    }

    return 0;
}
```

This code will estimate the volume up to the 12th dimension!

$$volume_{sphere} = \frac{count}{iterations} \times 2^{dimensions}$$

What lurks beyond the 4th dimension?

```
int main()
{
    int iterations = int(1e7);
    for (int dimension{ 2 }; dimension < 13; ++dimension) {
        double count{};
        for (int i{}; i < iterations; ++i) {
            double distance = 0;
            for (int d{}; d < dimension; ++d) {
                double v = Halton(i, primes[d]);
                distance = distance + v * v;
                if (distance > 1.0)
                    break;
            }
            if (distance <= 1.0)
                count++;
        }
        double volume =
        cout << fixed <<
            << setprecis
            << endl;
    }

    return 0;
}
```

This code will estimate the volume up to the 12th dimension!

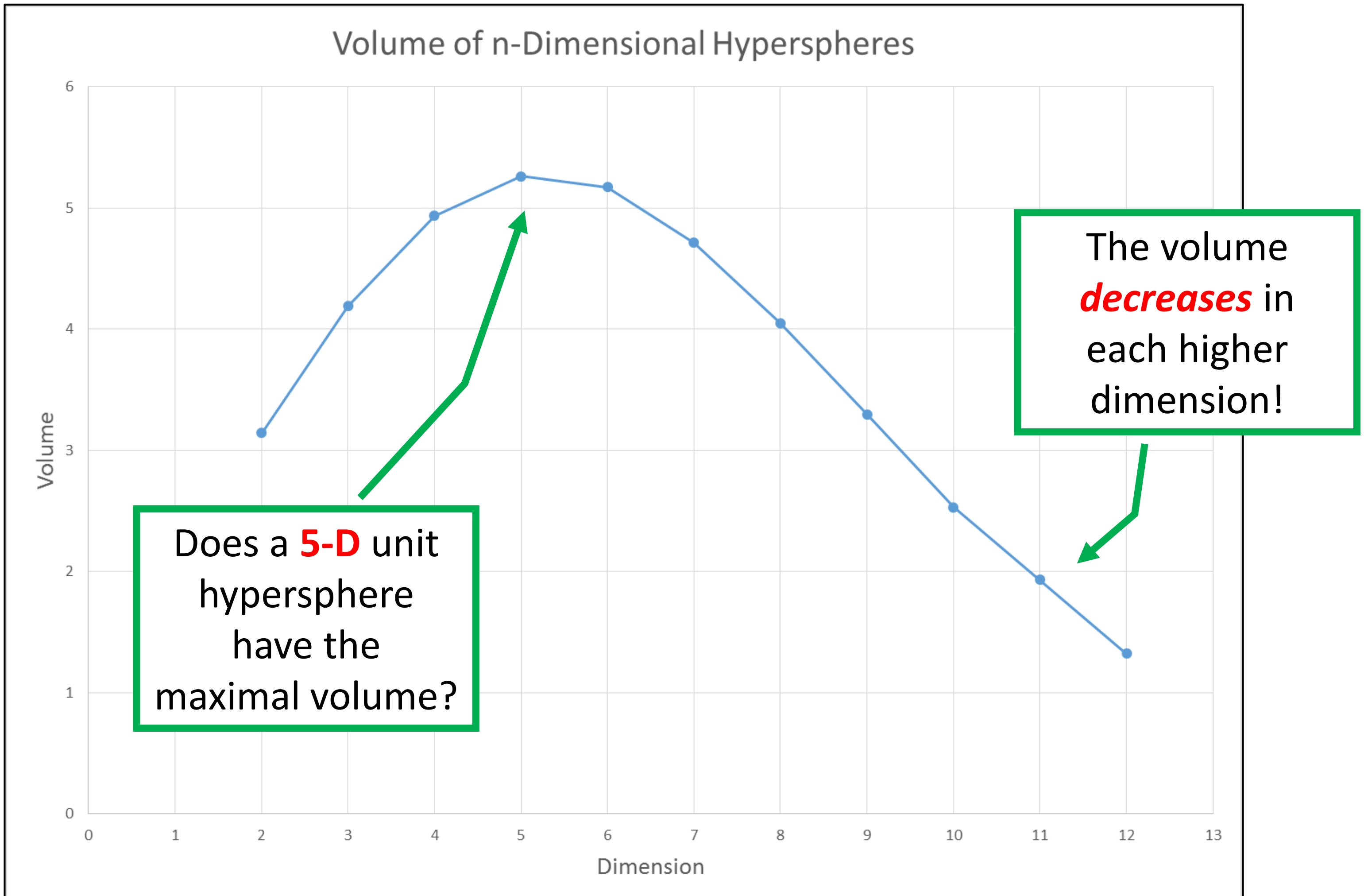
Run Lab 5

mc-highdimensional

File Edit View Terminal Tabs Help

```
 2, 3.1416
 3, 4.1888
 4, 4.9348
 5, 5.2630
 6, 5.1692
 7, 4.7208
 8, 4.0605
 9, 3.2935
10, 2.5463
11, 1.8883
12, 1.3443

Process returned 0 (0x0)
Press ENTER to continue.
```

What lurks beyond the 4^{th} dimension?



The Power Of Monte Carlo Integration

$$
\begin{aligned}
\mathbf{F}^{(n)} = {} & \frac{\mu}{8\pi} \int_{r_1}^{r_2} \int_{s_1}^{s_2} \int_{y_1}^{y_2} \left(\frac{2}{R_a^3} + \frac{3a^2}{R_a^5} \right) \{ (\mathbf{R} \times \mathbf{b}) \, (\mathbf{t} \cdot \mathbf{n}) + \mathbf{t} \, [(\mathbf{R} \times \mathbf{b}) \cdot \mathbf{n}] \} \\
& \times \left(\frac{r - r_1}{r_2 - r_1} \frac{s - s_1}{s_2 - s_1} \right) \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}y \\
& - \frac{\mu}{4\pi \, (1 - \nu)} \int_{r_1}^{r_2} \int_{s_1}^{s_2} \int_{y_1}^{y_2} \left(\frac{1}{R_a^3} + \frac{3a^2}{R_a^5} \right) [(\mathbf{R} \times \mathbf{b}) \cdot \mathbf{t}] \, \mathbf{n} \left(\frac{r - r_1}{r_2 - r_1} \frac{s - s_1}{s_2 - s_1} \right) \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}y \\
& + \frac{\mu}{4\pi \, (1 - \nu)} \int_{r_1}^{r_2} \int_{s_1}^{s_2} \int_{y_1}^{y_2} \frac{1}{R_a^3} \{ (\mathbf{b} \times \mathbf{t}) \, (\mathbf{R} \cdot \mathbf{n}) + \mathbf{R} \, [(\mathbf{b} \times \mathbf{t}) \cdot \mathbf{n}] \} \left(\frac{r - r_1}{r_2 - r_1} \frac{s - s_1}{s_2 - s_1} \right) \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}y \\
& - \frac{\mu}{4\pi \, (1 - \nu)} \int_{r_1}^{r_2} \int_{s_1}^{s_2} \int_{y_1}^{y_2} \frac{3}{R_a^5} [(\mathbf{R} \times \mathbf{b}) \cdot \mathbf{t}] \, (\mathbf{R} \cdot \mathbf{n}) \, \mathbf{R} \left(\frac{r - r_1}{r_2 - r_1} \frac{s - s_1}{s_2 - s_1} \right) \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}y.
\end{aligned}
$$

A Functional Equation for the Factorial

- Consider the classic factorial function:

$$n! = n * (n-1) * (n-2) * (n-3) * \cdots * 1$$

$$5! = 5 \ * 4 \ * 3 \ * 2 \ * 1 = 120$$

- We wish to find a **functional equation** that provides a ***shortcut*** to compute the factorial without having to iterate through the product of every term
- A closed form (analytic) **Riemann Integral** is the functional equation of an infinite series of diminishing rectangles under a curve within a given interval
- Can we express the **factorial function** as an ***integral***?

Euler’s **Gamma Function**

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx \quad \{s \in \mathbb{C}, \boldsymbol{Re}(s) > 0\}$$

$$\Gamma(n) = (n-1)! \quad \{n \in \mathbb{Z}^{+}\}$$

$$\Gamma(6) = (6-1)! = 5! = 120$$

Euler's **Gamma Function**

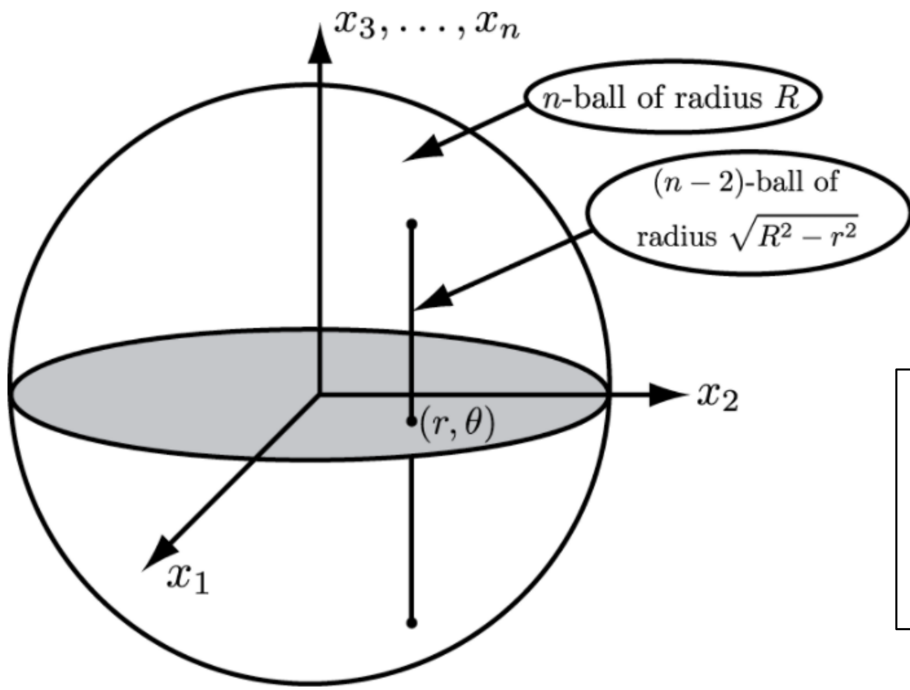
- Euler was a mathematical **pathfinder** – he liked to bend the rules and push the boundaries of existing functions
  - He asked "what is the factorial of a **fraction**?"
  - He also asked "what is the factorial of a **negative** number?"
- Using his Gamma function, Euler proved these two gems:

$$\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2} = 0.8862269 \ldots$$

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi} = 1.7724538 \ldots$$

Analytic Solution

Observe that the intersection of the (x_1, x_2)-plane with the $\boldsymbol{n}$-ball is a disk of radius $\boldsymbol{R}$ centered at the origin.



The perpendicular cross section of the n-ball at the point (r, θ) is an $(n-2)$-ball of radius $\sqrt{R^2 - r^2}$

A Proportionality Relation

If a solid in n-dimensional space is scaled by a factor of $\boldsymbol{k}$, then its volume increases by a factor of $\boldsymbol{k^n}$

$$V_n(R) \propto R^n$$

$$V_n(R) = V_n(1)R^n$$

The scaling factor is the ***volume*** *of the* ***unit*** *sphere for that dimension*

The volume of the n-ball is the integral of volumes of $(n-1)$ balls:

$$V_n(R) = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2 - x^2}\right) dx$$

due to the inductive assumption

$$V_n(R) = R^{n-1}\int_{-R}^{R} V_{n-1}\left(\sqrt{1-\left(\frac{x}{R}\right)^2}\right) dx$$

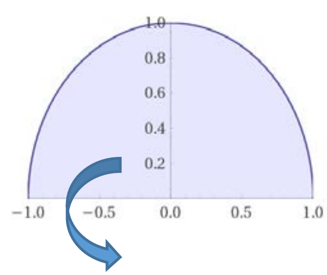
$$t = \frac{x}{R}$$

$$V_1(1) = 2$$

$$V_n(R) = R^n \int_{-1}^{1} V_{n-1}\left(\sqrt{1-t^2}\right) dt$$

$$V_2(1) = ?$$

$$R = 1 \qquad V_1(1) = 2 \qquad \int_{-1}^{1}\sqrt{1-t^2}\, dt = \frac{\pi}{2}$$

$V_2(1) =$ 

$$\times\, 2 = \frac{2}{1} \times \frac{\pi}{2} = R^2\pi$$

$$= (1)^2\pi$$

$$= \pi$$

A Proportionality Relation

If a solid in n-dimensional space is scaled by a factor of $\boldsymbol{k}$, then its volume increases by a factor of $\boldsymbol{k^n}$

$$V_n(R) \propto R^n$$

*The scaling factor is the **volume** of the **unit** sphere for that dimension*

The volume of the n-ball is the integral of volumes of $(n-1)$ balls:

$$V_n(R) = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2 - x^2}\right) dx$$

$$V_n(R) = R^{n-1} \int_{-R}^{R} V_{n-1}\left(\sqrt{1 - \left(\frac{x}{R}\right)^2}\right) dx$$

$$t = \frac{x}{R}$$

$$V_n(R) = R^n \int_{-1}^{1} V_{n-1}\left(\sqrt{1 - t^2}\right) dt$$

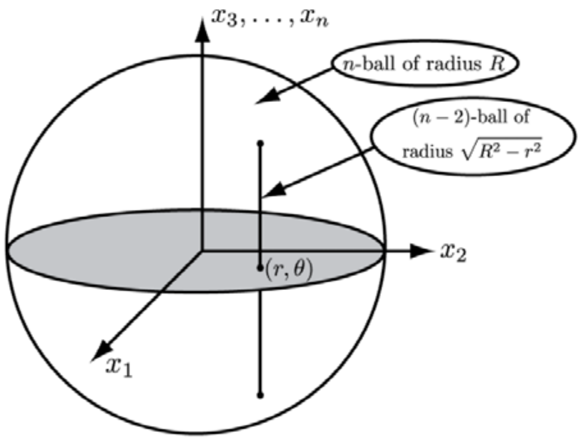
$$\boxed{V_n(R) = V_n(1)R^n}$$

$$V_2(R) = V_2(1)R^2 = \pi R^2$$

$$V_3(R) = V_3(1)R^3 = \frac{4\pi}{3} R^3$$

A Recurrence Relationship

We can compute $V_n(R)$ by integrating $V_{\mathbf{n-2}}\left(\sqrt{R^2-r^2}\right)$ over the disk using polar coordinates:



$$V_n(R) = \int_0^R \int_0^{2\pi} V_{n-2}\left(\sqrt{R^2-r^2}\right) r\, d\theta\, dr$$

$$= \int_0^R \int_0^{2\pi} V_{n-2}(1)\left(\sqrt{R^2-r^2}\right)^{n-2} r\, d\theta\, dr$$

$$= V_{n-2}(1) \int_0^R r(R^2-r^2)^{\frac{n-2}{2}}\, \theta\Big|_0^{2\pi}\, dr$$

$$= 2\pi V_{n-2}(1) \int_0^R r(R^2-r^2)^{\frac{n-2}{2}}\, dr$$

$$= -\frac{2\pi}{n} V_{n-2}(1)\,(R^2-r^2)^{\frac{n}{2}}\Big|_0^R$$

$$= 2\pi V_{n-2}(1)\,\frac{R^n}{n}$$

$$\boxed{V_{\mathbf{n}}(R) = \frac{2\pi R^2}{n} V_{\mathbf{n-2}}(R)}$$

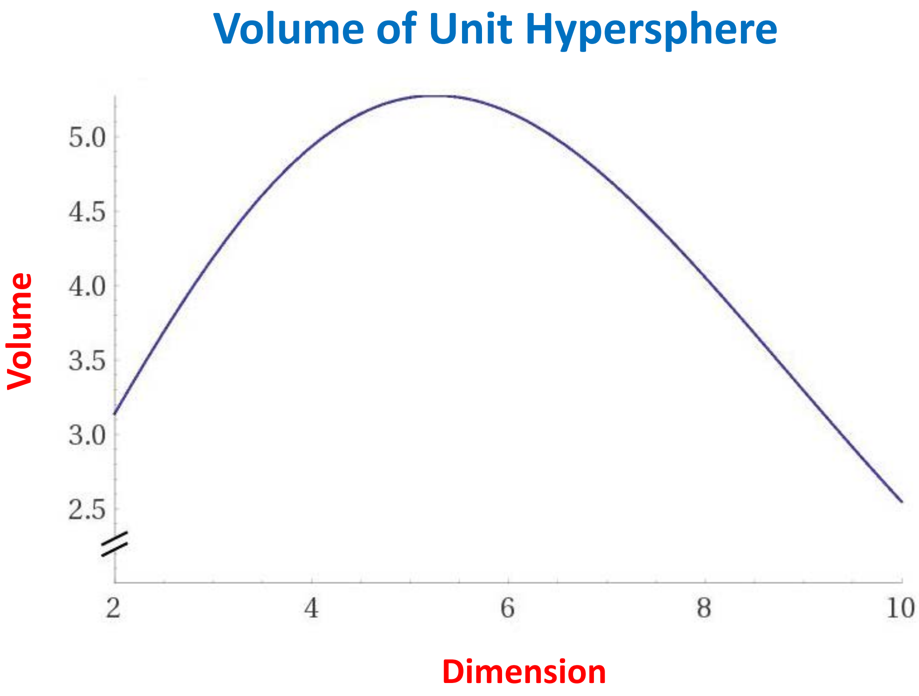
Volume via the Gamma Function

$$n_{even} \rightarrow V_n(R) = \frac{2^{\frac{n}{2}} \pi^{\frac{n}{2}} R^n}{2 \cdot 4 \cdot 6 \cdots n}$$

$$n_{odd} \rightarrow V_n(R) = \frac{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} R^n}{1 \cdot 3 \cdot 5 \cdots n}$$

$$\boxed{V_n(R) = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma\left(\frac{n}{2} + 1\right)}}$$

This is the key formula!



Volume via the Gamma Function

$$V_n(R) = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$\Gamma(n) = (n-1)!$$
$$n! = \Gamma(n+1)$$

$$V_2(R) = \frac{\pi R^2}{\Gamma\left(\frac{2}{2}+1\right)} = \frac{\pi R^2}{\Gamma(2)} = \frac{\pi R^2}{(2-1)!} = \boxed{\pi R^2}$$

$$V_3(R) = \frac{\pi^{\frac{3}{2}} R^3}{\Gamma\left(\frac{3}{2}+1\right)} = \frac{\pi R^3}{\Gamma\left(\frac{5}{3}\right)} = \frac{\pi^{\frac{3}{2}} R^3}{\left(\frac{3\sqrt{\pi}}{4}\right)} = \pi^{\frac{3}{2}} R^3 \left(\frac{4}{3\sqrt{\pi}}\right) = \boxed{\frac{4}{3}\pi R^3}$$

$$V_4(R) = \frac{\pi^{\frac{4}{2}} R^4}{\Gamma\left(\frac{4}{2}+1\right)} = \frac{\pi^2 R^2}{\Gamma(3)} = \frac{\pi^2 R^2}{(3-1)!} = \boxed{\frac{\pi^2 R^2}{2}}$$

Challenges of Numerical Analysis

$$V_{12}(1) = \frac{\pi^{\frac{12}{2}} 1^{12}}{\Gamma\left(\frac{12}{2}+1\right)} = \boxed{1.335262}$$

Did we use enough dots?

mc-highdimensional

File Edit View Terminal Tabs Help

```
 2, 3.1416
 3, 4.1888
 4, 4.9348
 5, 5.2630
 6, 5.1692
 7, 4.7208
 8, 4.0605
 9, 3.2935
10, 2.5463
11, 1.8883
12, 1.3443

Process returned 0 (0x0)
Press ENTER to continue.
```

From Lab 5

Curse of dimensionality

The curse of dimensionality refers to various phenomena that arise when analyzing and organizing data in high-dimensional spaces (often with hundreds or thousands of dimensions) that do not occur in low-dimensional settings such as the three-dimensional physical space of everyday experience. The expression was coined by Richard E. Bellman when considering problems in dynamic optimization.[1][2]

There are multiple phenomena referred to by this name in domains such as numerical analysis, sampling, combinatorics, machine learning, data mining, and databases. The common theme of these problems is that when the dimensionality increases, the volume of the space increases so fast that the available data become sparse. This sparsity is problematic for any method that requires statistical significance. In order to obtain a statistically sound and reliable result, the amount of data needed to support the result often grows exponentially with the dimensionality.

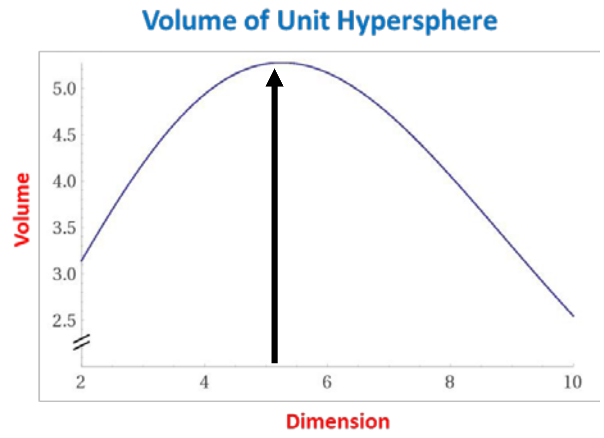
Volume via the Gamma Function

$$V_n(R) = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

As the Gamma function can extends its domain to include $\boldsymbol{n} \in \mathbb{R}$, we can use this analytic solution to compute the volume of hyperspheres having **fractional** (non-integer) dimensions!

$$V_{\mathbf{7.89}}(5.12) = \frac{\pi^{\frac{7.89}{2}} 5.12^{7.89}}{\Gamma\left(\frac{7.89}{2}+1\right)} = \mathbf{1,633,106.2809}$$

It appears $V_n(1)$ has a maximum somewhere *between* **4** and **6** dimensions. Does a fractional dimension contain the largest unit hypersphere?



Open Lab 6 – nball-volume

nball-volume.cpp

```
// nball-volume.cpp

#include "stdafx.h"

using namespace std;

// Euler's Gamma kernel
inline double f(double x, double n)
{
    return pow(x,n-1) * exp(-x);
}

// Find Gamma using Simpson's integration
double gamma(double n)
{
    double a{ 0 };
    double b{ 1e3 };
    int intervals = 1e5;

    double dx{ (b - a) / intervals };
    double sum{ f(a,n) + f(b,n) };
    a += dx;
    for (int i{ 1 }; i < intervals; ++i, a += dx)
        sum += f(a, n)*(2 * (i % 2 + 1));
    return (dx / 3)*sum;
}

// Find volume of unit ball
// See https://en.wikipedia.org/wiki/N-sphere
double v(double x)
{
    double halfx = x / 2.0;
    return pow(M_PI, halfx) / gamma(halfx + 1);
}
```

- The code currently uses Simpson's Rule to calculate Euler's Gamma function:

$$\Gamma(s) = \int_0^\infty \boxed{x^{s-1}e^{-x}\,dx}$$

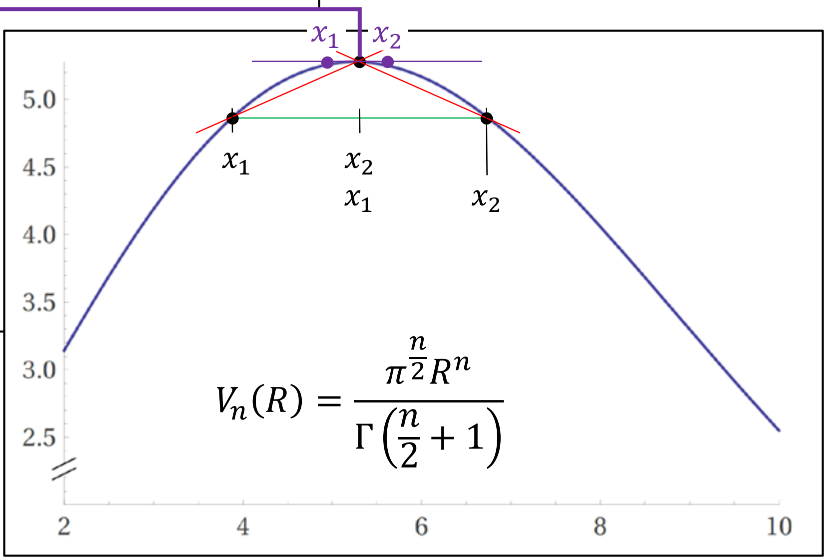
- From there, it uses Gamma to find the volume of **unit** hyperspheres with **integer** dimensions

$$\boxed{V_n(1) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}}$$

View Lab 6 – nball-volume

```
// This returns the x and y values for the maximum value of a function
// when both the analytic & numerical first derivative are unavailable.
// Note: This only works for concave down (negative second derivative) functions
tuple<double, double> find_max(double x1, double x2, double epsilon)
{
    double max = numeric_limits<double>::min();
    while (true)
    {
        double dx = (x2 - x1) / 97;
        double y1 = v(x1);
        double y2 = v(x1 + dx);
        while (y1 < y2)
        {
            x1 = x1 + dx;
            y1 = v(x1);
            y2 = v(x1 + dx);
        }
        if (abs(max - y1) < epsilon)
            break;
        x1 = x1 - dx;
        x2 = x1 + dx;
        max = v(x1);
    }
    return tuple<double, double>(x1, v((x1 + x2) / 2));
}
```

A **tuple<>** is a compound type that can hold **two** data items $\langle x_{max} | y_{max} \rangle$



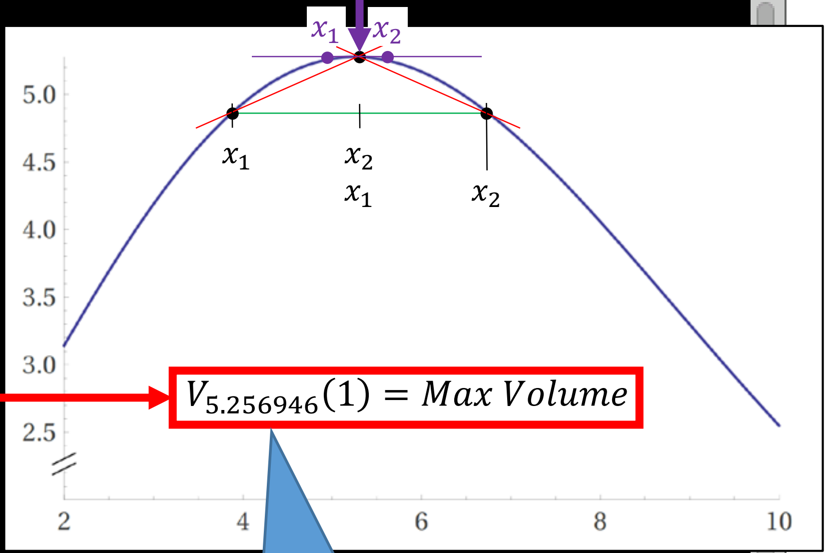
This ***hill climbing*** algorithm steps across the curve in **decreasing** brackets until the maximum point is located

Run Lab 6 – nball-volume

nball-volume

File Edit View Terminal Tabs Help

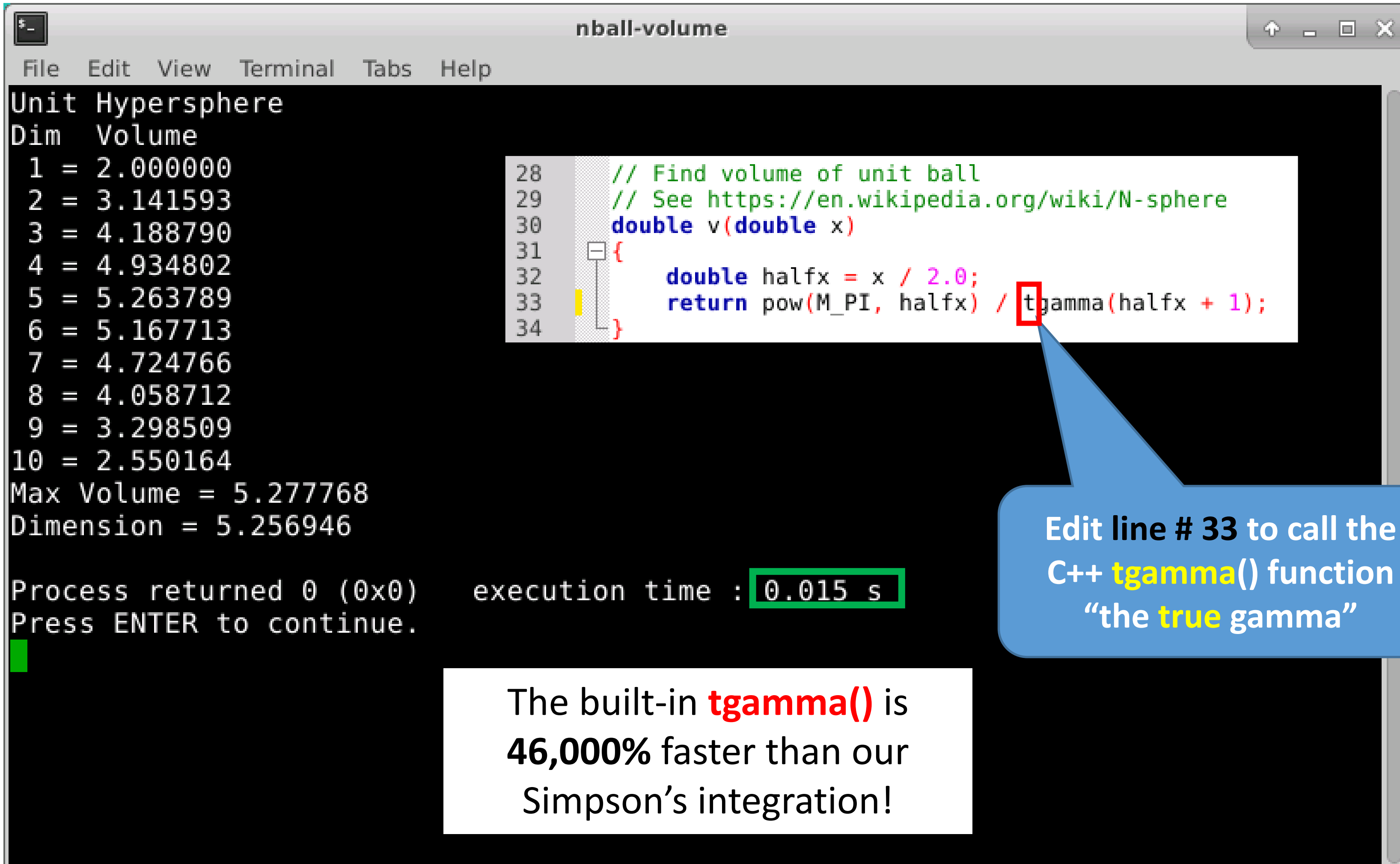
```
Unit Hypersphere
Dim  Volume
 1 = 2.000184
 2 = 3.141593
 3 = 4.188790
 4 = 4.934802
 5 = 5.263789
 6 = 5.167713
 7 = 4.724766
 8 = 4.058712
 9 = 3.298509
10 = 2.550164
Max Volume = 5.277768
Dimension = 5.256946

Process returned 0 (0x0)   execution time : 6.991 s
Press ENTER to continue.
```



$V_{5.256946}(1) = Max\ Volume$

A unit hypersphere with a *fractional* dimension has the maximum volume!

Edit Lab 6 – nball-volume



Edit line # 33 to call the C++ tgamma() function "the true gamma"

The built-in **tgamma()** is **46,000%** faster than our Simpson's integration!

Now you know…

https://dlmf.nist.gov/5

Gamma gets its own chapter!

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U.S. Department of Commerce

Chapter 5 Gamma Function

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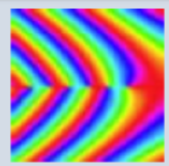

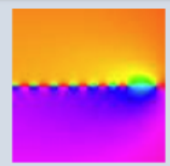

In C++ use **tgamma()**

Now you know…

- Monte Carlo integration uses **random sampling**
  - The method calculates the ratio of the points below the curve to the total number of points – **the final ratio is the “area”**
  - It requires millions/billions of samples to provide a few decimal points in accuracy
  - It may be the *only way* to take the integral of a very complex function
- We can calculate volume of any solid via the ratio of points that fall within a solid versus the total number of points
  - The volume of a 4-D unit hypersphere $= \frac{\pi^2}{2}$
  - A *fractional* **5-dimensional** unit sphere has maximum volume
  - In ever increasing dimensions the volume of **all** hyperspheres approaches zero!