Survey of Scientific Computing

(SciComp 301)

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Session 11

Complex Algebra

Section Goals

- Write code to perform **complex algebra**
- Factor primes over the **Gaussian Integers**
- Use a **Taylor Series** to approximate e^x $\{x \in \mathbb{C}\}$
- Calculate and display **Euler's Identity**
- Derive **Euler's Formula** in Complex Analysis
- Develop a functional equation for an infinite series
- Numerically calculate **Euler's Gamma** Function
- Explore the famous **Riemann Hypothesis**
- Debate what it means for two functions to be considered ***equivalent***

Complex Numbers

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$\sqrt{-100} = \sqrt{100}\sqrt{-1} = 10\sqrt{-1}$$

$$\sqrt{-5} = \sqrt{5}\ \sqrt{-1}$$

$$\sqrt{-290} = \sqrt{290}\ \sqrt{-1} \quad etc.$$

$$\sqrt{-5} = i\sqrt{5}$$

$$\sqrt{-81} = 9i$$

$i^0 = \mathbf{1}$ (anything raised to 0 = 1)

$i^1 = \boldsymbol{i}$ (anything raised to 1 = itself)

$i^2 = -\mathbf{1}$ (definition of i^2)

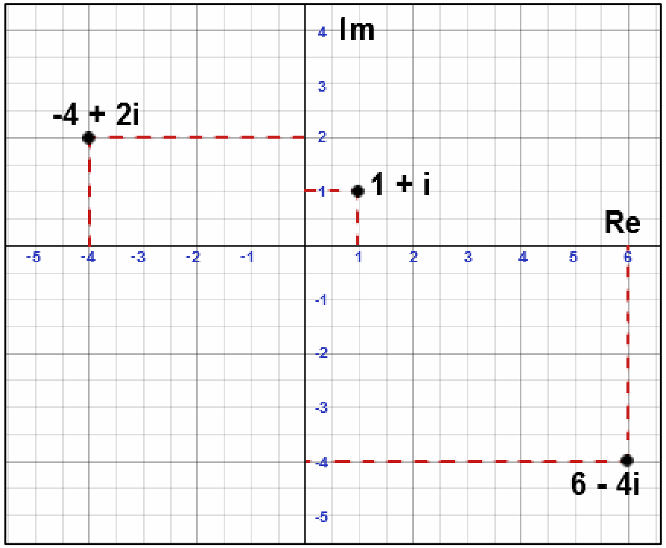
$i^3 = i^2 \times i = -1 \times i = -\boldsymbol{i}$

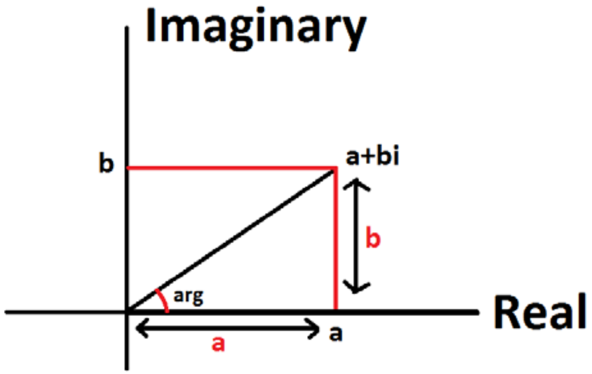
$i^4 = i^2 \times i^2 = -1 \times -1 = \mathbf{1}$

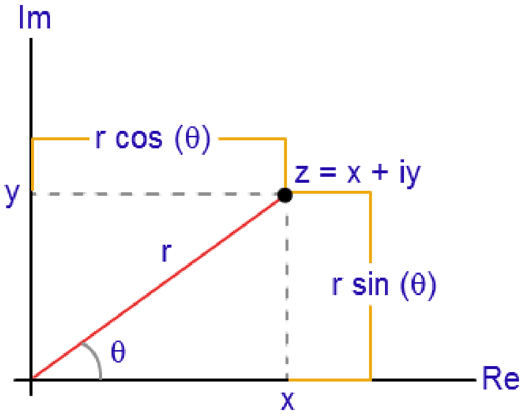
Complex Numbers

$$i = \sqrt{-1}$$

$$i^2 = -1$$







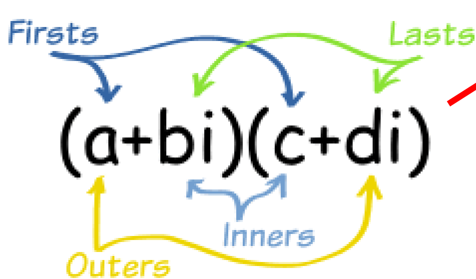
Complex Algebra

Sum: $$(4+3i)+(5-4i) = (4+5)+(3-4)i$$
$$= 9 - i$$

Difference: $$(4+3i)-(5-4i) = (4-5)+\big(3-(-4)\big)i$$
$$= -1 + 7i$$

Product: $$(4+3i)(5-4i) = 20 - 16i + 15i - 12i^2$$
$$= 20 - i + 12$$
$$= 32 - i$$

$i^2 = -1$



Open Lab 1 – Complex Algebra

- Write code that leverages C++ **built-in** support for complex numbers
- Given two complex numbers $\boldsymbol{z_1}, \boldsymbol{z_2} \in \mathbb{C}$, calculate
  - Addition $(z_1 + z_2)$
  - Subtraction $(z_1 - z_2)$
  - Multiplication $(z_1 \times z_2)$
  - Division $\left(\frac{z_1}{z_2}\right)$
- Raise a complex number to an **integer** power: $(z_1)^{\boldsymbol{n}}$
  - The built-in C++ **pow()** function can directly raise a complex number to a complex power, but in Lab 1 we will only raise a complex number to an **integer** power

Run Lab 1
Complex Algebra

```cpp
int main()
{
    complex<double> z1(-5.9, -7.5);
    complex<double> z2(sqrt(2), M_PI);

    cout << "z1 = " << z1 << endl
         << "z2 = " << z2 << endl << endl;

    cout << "z1 + z2 = "
         << z1 + z2 << endl;

    cout << "z1 - z2 = "
         << z1 - z2 << endl;

    cout << "z1 * z2 = "
         << z1 * z2 << endl;

    cout << "z1 / z2 = "
         << z1 / z2 << endl;

    cout << endl << "z1^2 = "
         << pow(z1, 2) << endl << endl;

    return 0;
}
```

$$z_1 = -5.9 - 7.5\boldsymbol{i} = (-5.9, -7.5)$$

$$z_2 = \sqrt{2} + \pi\boldsymbol{i} = (\sqrt{2}, \pi)$$

File Edit View Terminal Tabs Help

```
z1 = (-5.9,-7.5)
z2 = (1.41421,3.14159)

z1 + z2 = (-4.48579,-4.35841)
z1 - z2 = (-7.31421,-10.6416)
z1 * z2 = (15.2181,-29.142)
z1 / z2 = (-2.68803,0.667992)

z1^2 = (-21.44,88.5)
```

$$z_1{}^2 = -21.44 + 88.5\boldsymbol{i}$$

Representation Theory

- Any integer N $\{N \in \mathbb{Z}^{+}\}$ must be one of these four forms

$$N = 4n$$
$$N = 4n + 1$$
$$N = 4n + 2$$
$$N = 4n + 3$$

- If $N \in \{primes\}$, then N can only be one of these two forms

$$\cancel{N = 4n}$$
$$N = \mathbf{4n + 1}$$
$$\cancel{N = 4n + 2}$$
$$N = \mathbf{4n + 3}$$

$$N\ prime \Rightarrow \boldsymbol{N}\ \%\ \mathbf{4} = \{\mathbf{1}, \mathbf{3}\}$$

Unique Factorization Domains

- When restricting the **factorization domain** to just **positive integers**, certain numbers are **primes**

$$\{2, 3, 5, 7, 11, 13, 17, 23, 29, \dots\}$$

- Now consider positive **Gaussian integers**, which are complex numbers having both ***integer*** **real** and ***integer*** **imaginary** components

$$\{2 - 7i, 13 + 5i, 1 - 2i, 12 + 202i, \dots\}$$

- If we expand the factorization domain to include Gaussian integers, then what was *previously* a prime may *now* be a **composite** number

$$5 = (2 + i)(2 - i)$$

Unique Factorization Domains

- A $prime\ p$ over the integers $\mathbb{Z}$ is **composite** over the Gaussian integers $\mathbb{Z}[i]$ when $\boldsymbol{p}$ **is the sum of two squares**

$$p = a^2 + b^2 = (a + bi)(a - bi)$$

- To find all primes **p** which are **composite** over $\mathbb{Z}[i]$
  - $\forall_p$, try all $a, where\ 1 \leq a \leq \sqrt{p}$
  - Set $b = \sqrt{p - a^2}$
  - If $(\lfloor b \rfloor == b) \therefore p = (a + bi)(a - bi)$

$\boxed{\lfloor x \rfloor \equiv \text{floor of x}}$

- Let's write code to check the first odd **25** primes $(p < 100)$
- What do these "**weak primes**" have in common?

Open Lab 2 – Complex Factorization

```
int main()
{
    GeneratePrimes(25);

    FindSumOfSquares();

    return 0;
}
```

```
void GeneratePrimes(int count)
{
    primes.push_back(2);
    int n = 3;
    while (primes.size() < count)
    {
        if (n % 2 == 1)
        {
            bool isPrime = true;
            for (size_t p{}; p < primes.size(); p++)
                if (n % primes.at(p) == 0)
                {
                    isPrime = false;
                    break;
                }
            if (isPrime)
                primes.push_back(n);
        }
        n += 2;
    }
    primes.erase(primes.begin());
}
```

This removes the first prime, which is 2, as we only want **odd** primes

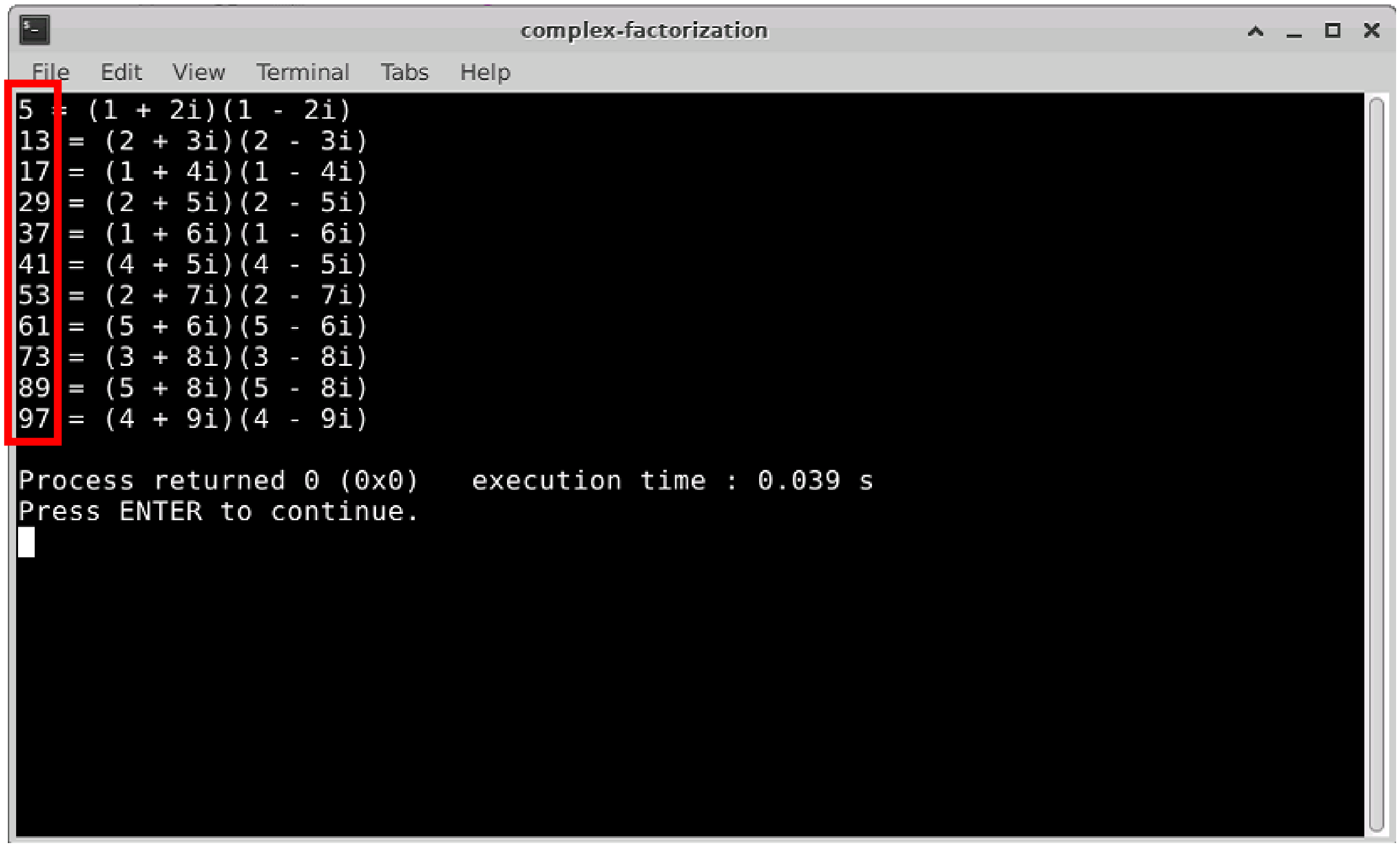
View Lab 2 – Complex Factorization

```cpp
int main()
{
    GeneratePrimes(25);

    FindSumOfSquares();

    return 0;
}
```

```cpp
void FindSumOfSquares()
{
    for (int p : primes)
    {
        for (int a = 1; a <= sqrt(p); a++)
        {
            double b = sqrt(p - a*a);
            if (floor(b) == b)
            {
                cout << p << " = "
                     << "(" << a << " + " << b << "i)"
                     << "(" << a << " - " << b << "i)"
                     << endl;
                break;
            }
        }
    }
}
```

- Try all $a, where\ 1 \leq a \leq \sqrt{p}$
- Set $b = \sqrt{p - a^2}$
- If $(\lfloor b \rfloor == b) \Rightarrow p = (a + bi)(a - bi)$

Run Lab 2 – Complex Factorization



What do these 11 primes have in common?

Hint: see slide #8

Research Questions

1. If we know $(a+bi)$ & $(a-bi)$ are factors of p, what **two other factors** do we know *automatically*? Why?

$$5 = (2+i)(2-i) \therefore \text{what others?}$$

$\boldsymbol{a = 2}$
$\boldsymbol{b = 1}$

$$(a+bi)(a-bi) = (b+ai)(b-ai)$$
$$a^2 + b^2 = b^2 + a^2$$

$$5 = (1+2i)(1-2i)$$

$$5 = (-i)\boxed{(2+i)(1+2i)}$$
$$5 = (-i)(2+4i+i-2)$$
$$(-i)(5i) = 5$$

When listing factors of Gaussian Integers:

- We don't include $(\pm i)$ just like we don't include (± 1) in the list of normal integer factors – it is *implied*
- We only include the complex factors having **positive** components

Research Questions

1. If we know $(a+bi)$ & $(a-bi)$ are factors of p, what **two other factors** do we know *automatically*? Why?

$$5 = (2+i)(2-i) \therefore \text{what others?}$$

2. There are **24** odd integer primes < 100, but **11** are composite (weak primes) when factored over the domain of Gaussian integers - **what do these 11 primes have in common**?

3. Are all **Pythagorean primes** "strong" primes over $\mathbb{Z}[i]$?

4. Who was **Pierre de Fermat** - and what was his theorem on the **sums of two squares**?

Why is e so special?

Take an item of size n and divide it into m parts
$\therefore$ the size of each part $p = \frac{n}{m}$
Q: What value of m maximizes p^m ?
A: When $m = e$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$\int_1^e \frac{1}{t} dt = 1$$

e is the base of the natural logarithm

= 2.71828182845904 5...

$$\ln e = 1$$

Why is *e* so special?

$c \equiv$ a constant

$$ce^{x} = c + cx + \frac{cx^{2}}{2!} + \frac{cx^{3}}{3!} + \frac{cx^{4}}{4!} + \frac{cx^{5}}{5!} + \frac{cx^{6}}{6!} + \frac{cx^{7}}{7!} + \cdots$$

$$\frac{d}{dx}(ce^{x}) = \frac{d}{dx}(c) + \frac{d}{dx}(cx) + \frac{d}{dx}\left(\frac{cx^{2}}{2!}\right) + \frac{d}{dx}\left(\frac{cx^{3}}{3!}\right) + \cdots$$

$$\frac{d}{dx}(ce^{x}) = 0 + c + cx + \frac{cx^{2}}{2!} + \frac{x^{3}}{3!} + \frac{x^{4}}{4!} + \frac{x^{5}}{5!} + \frac{x^{6}}{6!} + \frac{x^{7}}{7!} + \cdots$$

$$\boxed{\frac{d}{dx}(ce^{x}) = ce^{x}}$$

ce^{x} is the only function which is the derivative of *<u>itself</u>* !

Euler's Identity

- Calculate an approximation of $\boldsymbol{e^z}$ where $\mathbf{z} \in \mathbb{C}$, using its Taylor Series expansion to **20 terms**

$$\boldsymbol{e^z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \cdots$$

- Use the above ***power series*** to display the value of $\boldsymbol{e^{\pi i}}$

$$(e^z \; where \; z = \mathbf{0} + \boldsymbol{\pi i})$$

- Because the *denominators* grow at a **factorial** rate, you must store them using data type **`uintmax_t`** which has a range of 0 to 18,446,744,073,709,551,615

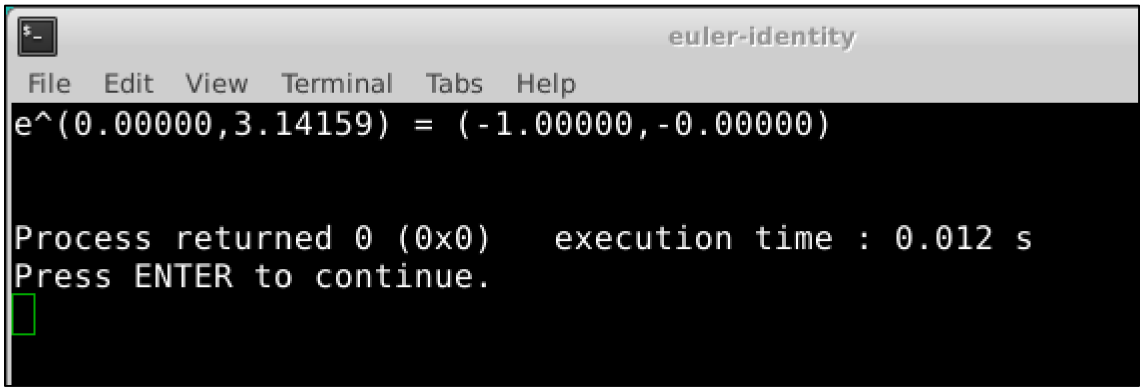
Open Lab 3 - Euler's Identity

$$z = \boldsymbol{\pi i}$$

$$\boldsymbol{e^z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \cdots$$

```
int main()
{
    complex<double> z(0, M_PI);

    complex<double> ez(1, 0);

    uintmax_t fact = 1;

    for (int p = 1; p < 21; p = p + 1)
    {
        ez = ez + pow(z, p) / complex<double>(fact, 0);
        fact = fact * (p + 1);
    }

    cout << fixed << setprecision(5)
         << "e^" << z << " = " << ez
         << endl << endl;

    return 0;
}
```

Run Lab 3 - Euler's Identity

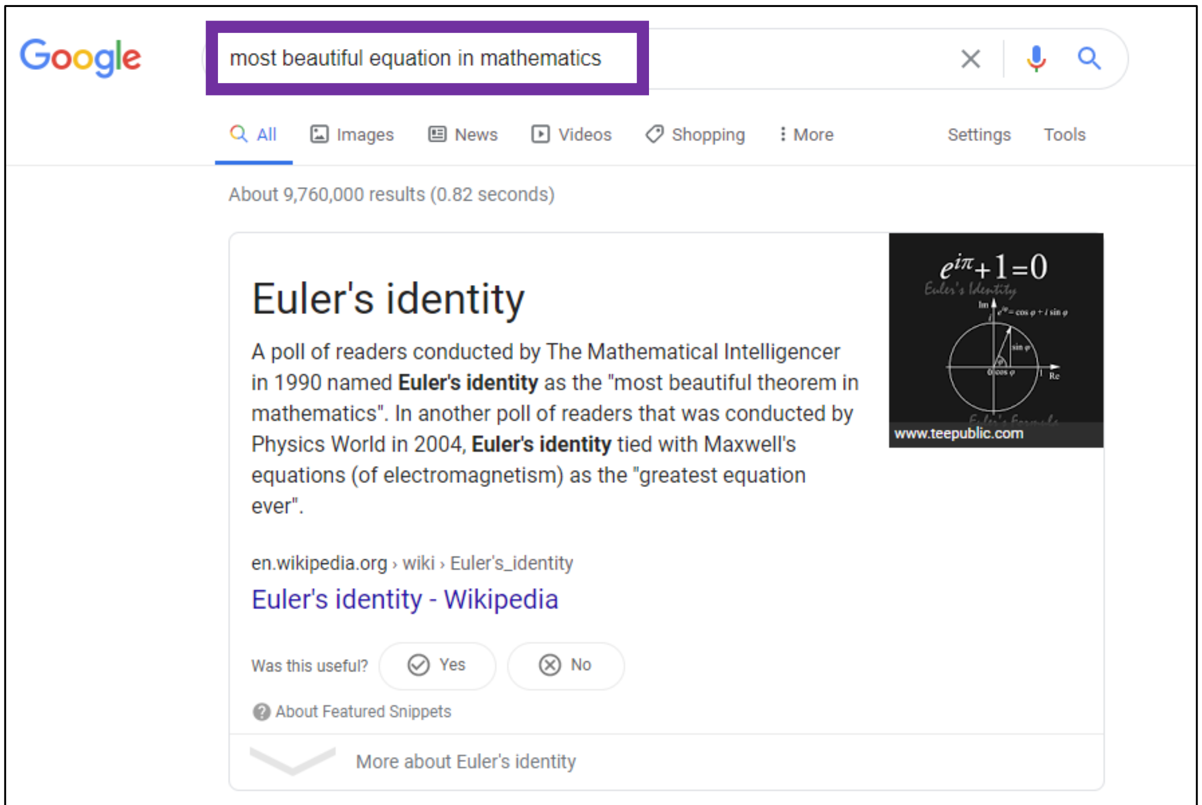


$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

Euler’s Identity

$$e^{i\pi} + 1 = 0$$



$$\boxed{i^i = ?}$$

$$\left(a^b\right)^c = a^{bc}$$

$$(2^3)^4 = 2^{3\times 4} = 2^{12}$$

$$e^{\pi i} = -1$$

$$-1 = e^{\pi i}$$

$$(-1)^{\frac{1}{2}} = \left(e^{\pi i}\right)^{\frac{1}{2}}$$

$$\sqrt{-1} = e^{\frac{\pi i}{2}}$$

$$i = e^{\frac{\pi i}{2}}$$

$$i^i = \left(e^{\frac{\pi i}{2}}\right)^i$$

$$i^i = e^{\frac{\pi i^2}{2}}$$

$$i^i = e^{\frac{-\pi}{2}}$$

$$\boldsymbol{i^i \cong 0.20787} \in \mathbb{R}$$

Run Lab 4 - Euler's Formula

```
void epow(double x)
{
    complex<double> z(0, x);

    complex<double> ez(1, 0);

    uintmax_t fact = 1;

    for (int p = 1; p < 21; p = p + 1)
    {
        ez = ez + pow(z, p) / complex<double>(fact, 0);
        fact = fact * (p + 1);
    }

    cout << fixed << setprecision(5)
         << "e^" << z << " = " << ez
         << endl << endl;
}

int main()
{
    epow(0);                    // Theta = PI * 0
    epow(M_PI / 2.);            // Theta = PI * 1/2
    epow(M_PI);                 // Theta = PI * 1
    epow(3. * M_PI / 2.);       // Theta = PI * 3/2

    return 0;
}
```

Run Lab 4 code to evaluate $\boldsymbol{e^{\theta i}}$ at these values for θ:

$z(0,\theta)$	$e^{i\theta}$: $Real$	$e^{i\theta}$: Img
0		
$\frac{\pi}{2}$		
π	-1	0
$\frac{3\pi}{2}$		

What **trigonometric** functions can produce these *specific* **real** & **imaginary** components at each $\boldsymbol{\theta}$?

Check Lab 4 - Euler's Formula

Input	Output	
$z(0, \boldsymbol{\theta})$	$\boldsymbol{e^{i\theta}}$: $\boldsymbol{Real}$	$\boldsymbol{e^{i\theta}}$: $\boldsymbol{Img}$
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1

What **trigonometric** functions can produce these *specific* **real** & **imaginary** components at each $\boldsymbol{\theta}$?

$$\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\boxed{\sqrt{i}+\sqrt{-i}=?}$$

$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$e^{\frac{\pi}{2}i}=i \qquad e^{\frac{-\pi}{2}i}=-i$$

$$\sqrt{e^{\frac{\pi}{2}i}}+\sqrt{e^{\frac{-\pi}{2}i}}$$

$$\left(e^{\frac{\pi}{2}i}\right)^{\frac{1}{2}}+\left(e^{\frac{-\pi}{2}i}\right)^{\frac{1}{2}}$$

$$e^{\frac{\pi}{4}i}+e^{\frac{-\pi}{4}i}$$

$$\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)+\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right)$$

$$\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}+\frac{-i}{\sqrt{2}}\right)$$

$$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

$$\boldsymbol{\sqrt{i}+\sqrt{-i}=\sqrt{2}}$$

Functional Equations

- Consider the following series:

$$M_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n = \sum_{k=0}^{n} x^k$$

- For example: $M_2(x) = 1 + x + x^2$
- As $n \to \infty$ such that $M(x) = \sum_{k=0}^{\infty} x^k$, what is the domain interval where $M(x)$ converges?

$$M(1) = 1 + 1 + 1 + 1 + \cdots = \infty \text{ (diverges)}$$

$$M(0) = 1 + 0 + 0 + 0 + \cdots = 1 \text{ (converges)}$$

- What about $\mathbf{M(-1)}$?

Functional Equations

$$M(-1) = 1 + (-1) + (-1)^2 + (-1)^3 + (-1)^4 + \cdots = ?$$

$$M_{\mathbf{5}}(-1) = (1 - 1) + (1 - 1) + (1 - 1) = \mathbf{0}$$

$$M_{\mathbf{6}}(-1) = (1 - 1) + (1 - 1) + (1 - 1) + \mathbf{1} = \mathbf{1}$$

$$\mu = \frac{(M_5 + M_6)}{2} = \frac{(0 + 1)}{2} = \frac{1}{2} \therefore M(-1) = 0.5\ ?$$

Note: This approach of adding partial terms of a series is called **Cesàro** summation

Functional Equations

$$M_2(x)(1-x) = (1)(1-x) + (x)(1-x) + (x^2)(1-x)$$

$$M_2(x)(1-x) = 1 - x + x - x^2 + x^2 - x^3$$

$$M_2(x)(1-x) = 1 + (-x + x) + (-x^2 + x^2) - x^3$$

$$M_2(x)(1-x) = 1 - x^3$$

$$\lim_{n\to\infty} x^n = 0 \Leftrightarrow |x| < 1$$

$$M_2(x) = \frac{1-x^3}{1-x} \therefore \boldsymbol{M(x)} = \frac{1-x^{\infty}}{1-x} = \frac{\mathbf{1-0}}{\mathbf{1}-\boldsymbol{x}} \{x \in \mathbb{R}\ (-1,1)\}$$

Functional Equations

$$M(x) = \frac{1 - x^{\infty}}{1 - x} = \frac{1}{1 - x} \Leftrightarrow -1 < x < 1$$

$M(-1) = \frac{1}{1-(-1)} = \frac{1}{2}$ **?!** (*debatable* as -1^{∞} is *undefined*)

$$M(x) = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \; \{x \in \mathbb{R} \; (-1,1)\}$$

- $M(x) = \frac{1}{1-x}$ is the **functional equation** for the infinite series **only** over the exclusive domain **(-1, 1)**
- We no longer need to add an infinite number of terms to get the sum within that domain – we can use this limit as a shortcut!

A Functional Equation for the Factorial

- Consider the classic factorial function:

$$n! = n * (n-1) * (n-2) * (n-3) * \cdots * 1$$

$$5! = 5 \ * 4 \ * 3 \ * 2 \ * 1 = 120$$

- We wish to find a **functional equation** that provides a shortcut to compute the factorial without having to iterate through the product of every term
- A closed form (analytic) **Riemann Integral** is the functional equation of an infinite series of diminishing rectangles under a curve within a given interval
- Can we express the **factorial function** as an ***integral***?

Euler's **Gamma Function**

$$\Gamma(n) = (n-1)! \quad \{n \in \mathbb{Z}^{+}\}$$

$$\Gamma(6) = (6-1)! = 5! = 120$$

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx \quad \{s \in \mathbb{C}, \boldsymbol{Re}(s) > 0\}$$

$$\Gamma(6) = \int_0^{\infty} x^{5} e^{-x}\, dx = ?$$

Euler's Gamma Function

$$\Gamma(6) = \int_0^\infty x^5 e^{-x}\, dx$$

- Recall integration by parts (from *differential* product rule)

$$\int u\, dv = uv - \int v\, du$$

$$u = x^5, dv = e^{-x}\, dx$$

$$du = 5x^4 dx, v = -e^{-x}$$

$$(x^5)(-e^{-x})\Big|_0^\infty - \int_0^\infty (-e^{-x})(5x^4 dx)$$

Euler's Gamma Function

$$\boxed{(x^5)(-e^{-x})\Big|_0^\infty}\boxed{-\int_0^\infty(-e^{-x})(5x^4dx)}$$

$$\boxed{\lim_{b\to\infty}(b^5)\left(-e^{-b}\right)-(0^5)(-e^{-0})}=\lim_{b\to\infty}\frac{b^5}{-e^b}\boxed{=0}$$

$$\boxed{-\int_0^\infty(-e^{-x})(5x^4dx)}=5\int_0^\infty x^4e^{-x}\,dx$$

$$\Gamma(6)=5\int_0^\infty x^4e^{-x}\,dx=?$$

Euler's Gamma Function

$$\Gamma(6) = 5\int_0^\infty x^4 e^{-x}\,dx$$

$$\Gamma(6) = 20\int_0^\infty x^3 e^{-x}\,dx$$

$$\Gamma(6) = 60\int_0^\infty x^2 e^{-x}\,dx$$

$$\Gamma(6) = 120\int_0^\infty x^1 e^{-x}\,dx$$

$$\Gamma(6) = 120\int_0^\infty x^0 e^{-x}\,dx = (120)(1)$$

$$\Gamma(6) = 120 = \mathbf{5}!$$

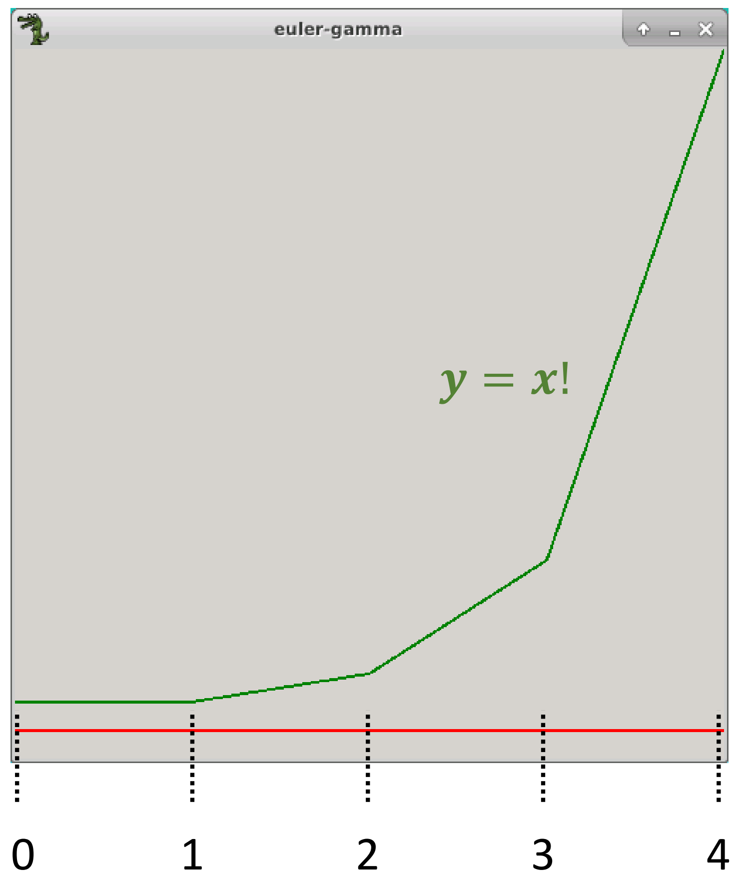
Euler's Gamma Function

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx \quad \{s \in \mathbb{C}, Re(s) > 0\}$$

- Let's graph this integral over the domain of **real numbers**
- However ***first*** we'll only consider $s \in \mathbb{Z}^{+}$
- Realize the real Gamma function is just an integral that we can numerically compute using **Simpson's Rule**
- First we will populate a **PointSet** with integer Cartesian coordinates $\mathbf{0} \leq \boldsymbol{x} \leq \mathbf{4}$ and $\boldsymbol{y} = \boldsymbol{x}!$
- Then we will use **SimpleScreen** to draw the "polynomial" that plots the factorial function using an integer domain

Run Lab 5 - Euler's Gamma Function

- Verify the growth of the integer factorial function



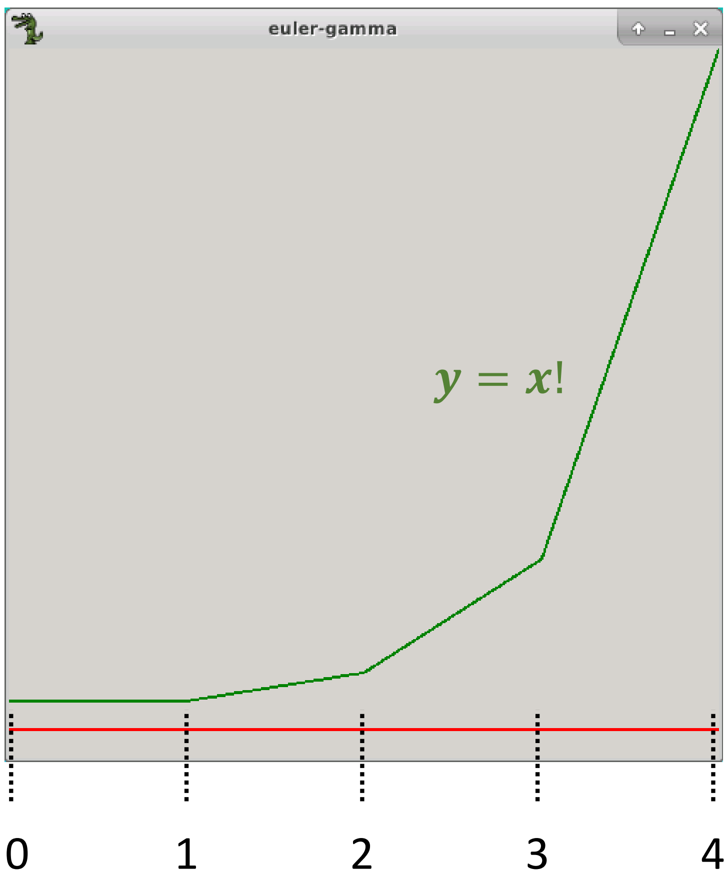
Edit Lab 5 - Euler's Gamma Function

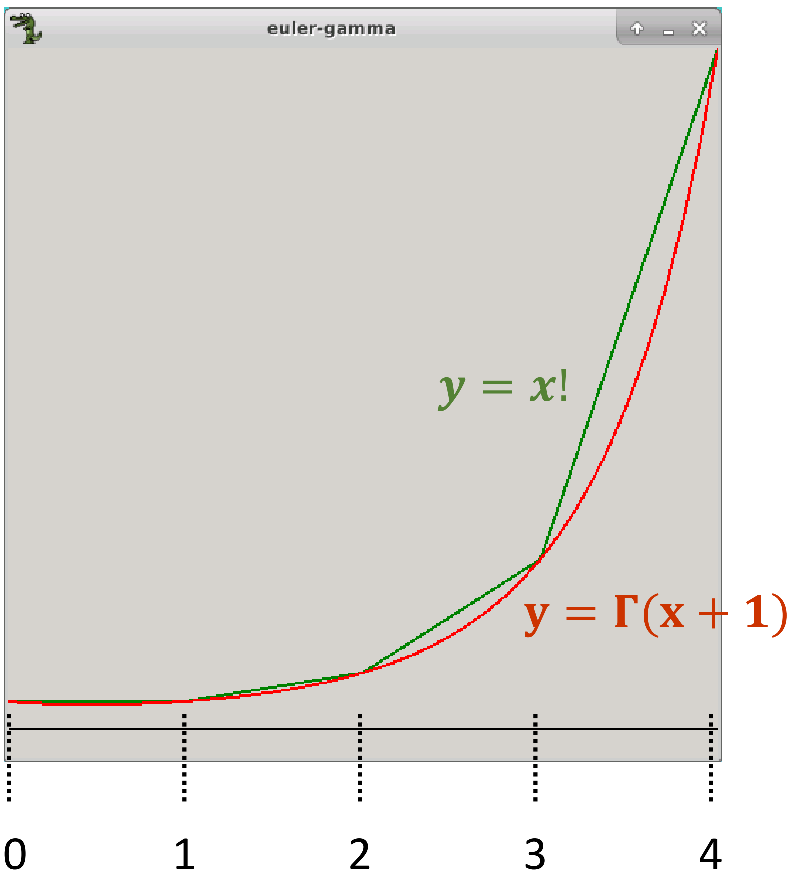
- Verify the growth of the integer factorial function
- Then change line **#12** to **return 1 *** instead of **0 ***

main.cpp

```cpp
1  #include "stdafx.h"
2  #include "simplescreen.h"
3
4  using namespace std;
5
6  PointSet psFactorial;
7  PointSet psGamma;
8
9  inline double f(double x, double n)
10 {
11     // Euler's Gamma Function
12     return 1 * pow(x,n-1) * exp(-x);
13 }
14
```

Run Lab 5 - Euler's Gamma Function



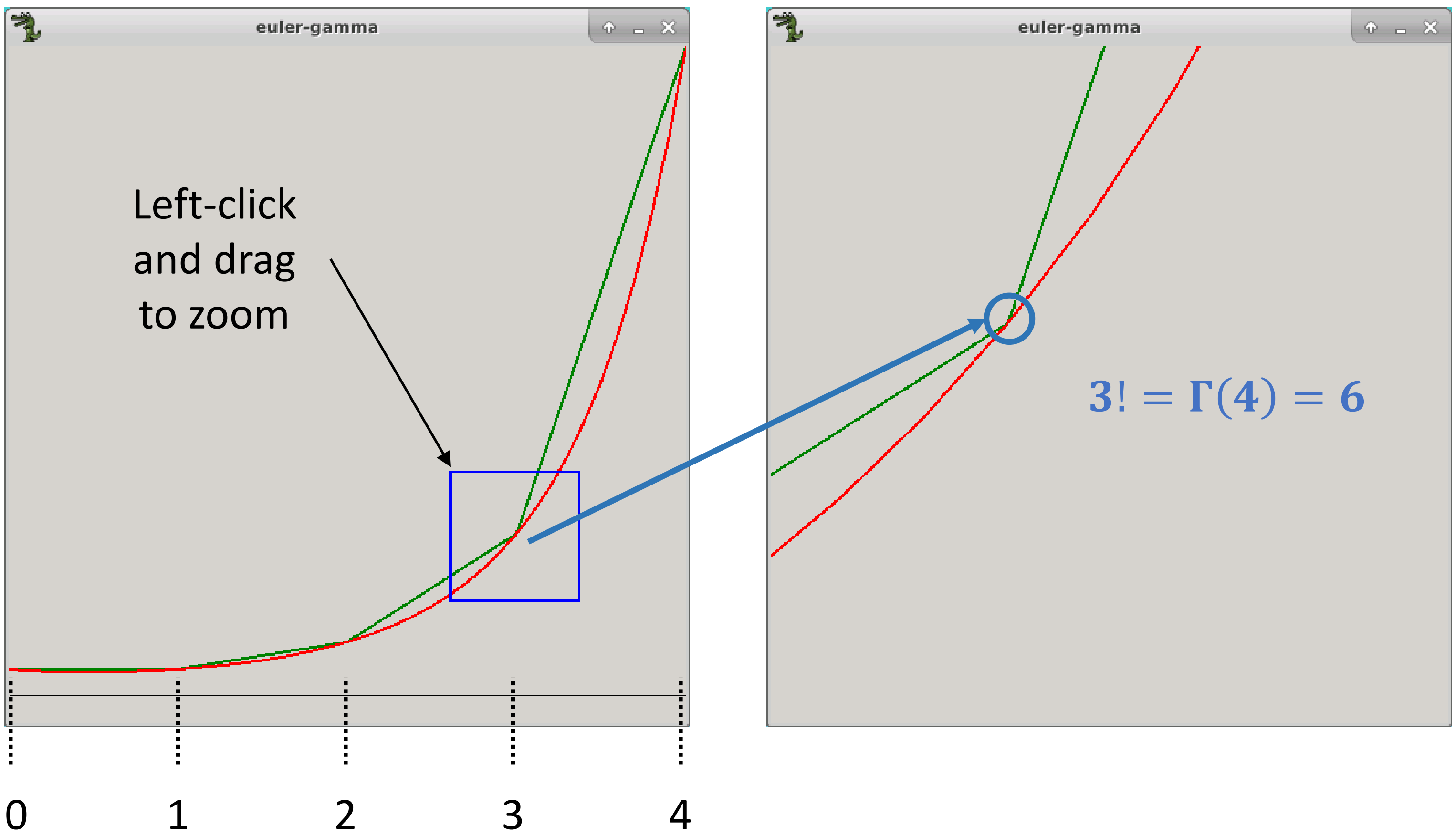


After editing line **#12**

Run Lab 5 - Euler's Gamma Function

- Verify the growth of the integer factorial function
- Then change line **#12** to **return 1** $*$ instead of **0** $*$
- Zoom in on the point at $x = 3$ to confirm $\Gamma(4) = 3!$

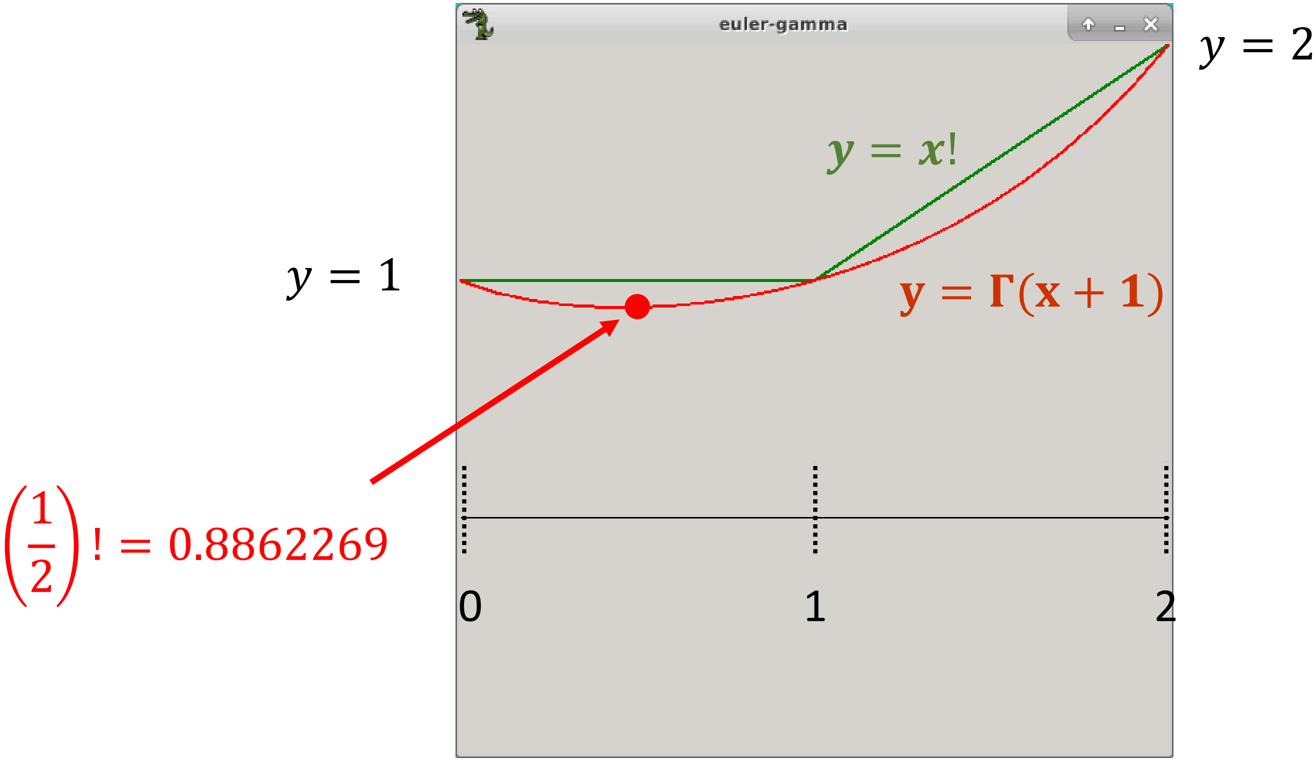
Check Lab 5 - Euler's Gamma Function



Run Lab 5 - Euler's Gamma Function

- Verify the growth of the integer factorial function
- Then change line **#12** to **return 1** $*$ instead of **0** $*$
- Zoom in on the point at $x = 3$ to confirm $\Gamma(4) = 3!$
- At the lattice points (where $x \in \mathbb{Z}^+$) the Gamma integral "equals" the integer factorial function – **but are they truly the *same* equation?**
- Change line **#42** so **max_n** = 2 and then run the lab again
- Look at the curve between $x = 0$ and $x = 1$
- Consider the range at $x = 1/2$ as it dips below $y = 1$
- But $n!$ is not defined for non-integers – so what is $(1/2)!$?

Check Lab 5 - Euler's Gamma Function



Via the Gamma Function we can now calculate the factorial of ***fractions***!

Euler's Gamma Function

- Euler was a mathematical **pathfinder** – he liked to bend the rules and push the boundaries of existing functions
- He asked "what is the factorial of a **fraction**?"
- He also asked "what is the factorial of a **negative** number?"
- You can see graphically in lab 5 that $\left(\frac{1}{2}\right)! < 1$
- Euler proved these two gems:

$$\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2} = 0.8862269\ldots$$

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi} = 1.7724538\ldots$$

The Riemann Zeta Function

- Recall the Harmonic Series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

- Nicole Oresme (*O-rays-mah*) proved this diverges to ∞ in **1360**

- Recall the Basel Problem:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\mathbf{2}}} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots$$

- Euler proved this converged to $\boldsymbol{\pi/6}$ in **1735**

The Riemann Zeta Function

- Bernhard Riemann considered in **1859** what happens to the series if we extend the domain beyond natural numbers to the **complex** domain – he used the Greek letter **zeta**:

$$\xi(\boldsymbol{s}) = \sum_{n=1}^{\infty} \frac{1}{n^{\boldsymbol{s}}} = 1 + \frac{1}{2^{\boldsymbol{s}}} + \frac{1}{3^{\boldsymbol{s}}} + \frac{1}{4^{\boldsymbol{s}}} + \cdots \quad \boldsymbol{s} \in \mathbb{C}$$

- He was actually trying to come up with a **functional equation** (a shortcut) that would analytically determine the <u>exact</u> number of primes less than a given number, but *without* having to count each individual prime
- This "**prime counting function**" is often expressed as $\pi(x)$
- For example $\pi(1{,}000{,}000) = 78{,}498$

The Dirichlet Eta Function

- Riemann immediately faced a problem because the standard **Zeta** function converges only for complex numbers having a **real part > 1**
- Fortunately the series can be slightly modified to help it converge more easily. This is called the **eta** function:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} \ldots$$

- The sign alternates between successive *terms* – all terms with an even $\mathbf{n}$ are now subtracted
- This simple change extends its domain so $\eta(s)$ converges for all complex numbers having a **real part > 0**

The Dirichlet Eta Function

- The Eta function has some interesting values which you can write code to numerically compute:
  - $\eta(2) = \frac{\pi^2}{12}$ which is one-half of Euler's Basel sum
  - $\eta(1) = \ln 2$ which is called the **alternating** harmonic series
  - $\eta(0) = \frac{1}{2}$ which is the Abel sum of Grandi's series
  - $\eta(0) = 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}\,??$ *(see slide #29)*
- Fortunately $\eta(s)$ helps us extend the domain of $\boldsymbol{\xi(s)}$ as it converges for complex numbers having a **real part** $\mathbf{> 0}$
- But how?

Zeta in terms of Eta

$$\xi(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} \dots$$

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} \dots$$

$$\xi(s) - \eta(s) = \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \frac{2}{8^s} + \frac{2}{10^s} \dots$$

$$\xi(s) - \eta(s) = \left(\frac{2}{2^s}\right)\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \dots\right)$$

$$\xi(s) - \eta(s) = (2^{1-s})\xi(s)$$

$$\xi(s) - (2^{1-s})\xi(s) = \eta(s)$$

$$\xi(s)\left(1 - (2^{1-s})\right) = \eta(s)$$

$$\boxed{\xi(s) = \frac{\eta(s)}{(1 - 2^{1-s})}}$$

We can now calculate Zeta using Eta for all complex numbers in the <u>right</u> plane

(except at s = 1 which is the divergent harmonic series)

The Riemann Zeta Function

- The *extended* **Zeta function** has some interesting values that appear in many branches of math & physics:
  - $\xi(0+0i) = \frac{1}{2}$ (Grandi's series)
  - $\xi\left(\frac{3}{2}+0i\right) \approx 2.612375$ (appears when calculating the critical temperature for a **Bose-Einstein condensate**)
  - $\xi(2+0i) = \frac{\pi^2}{6}$ (Euler's Basel sum)
  - $\xi(4+0i) \approx 1.082323$ (appears when integrating Planck's law to derive the **Stefan-Boltzmann law** for black body radiation)
  - $\xi(-1+0i) = -\frac{1}{12}$ which "suggests" something Ramanujan independently discovered: $\mathbf{1+2+3+4+\cdots=-\frac{1}{12}}$ (this series appears in **string theory**)

The Riemann Hypothesis

- To find his prime counting function, Riemann needed to determine what complex numbers make the Zeta function **converge to zero**:

$$\xi(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} \ldots = \mathbf{0}$$

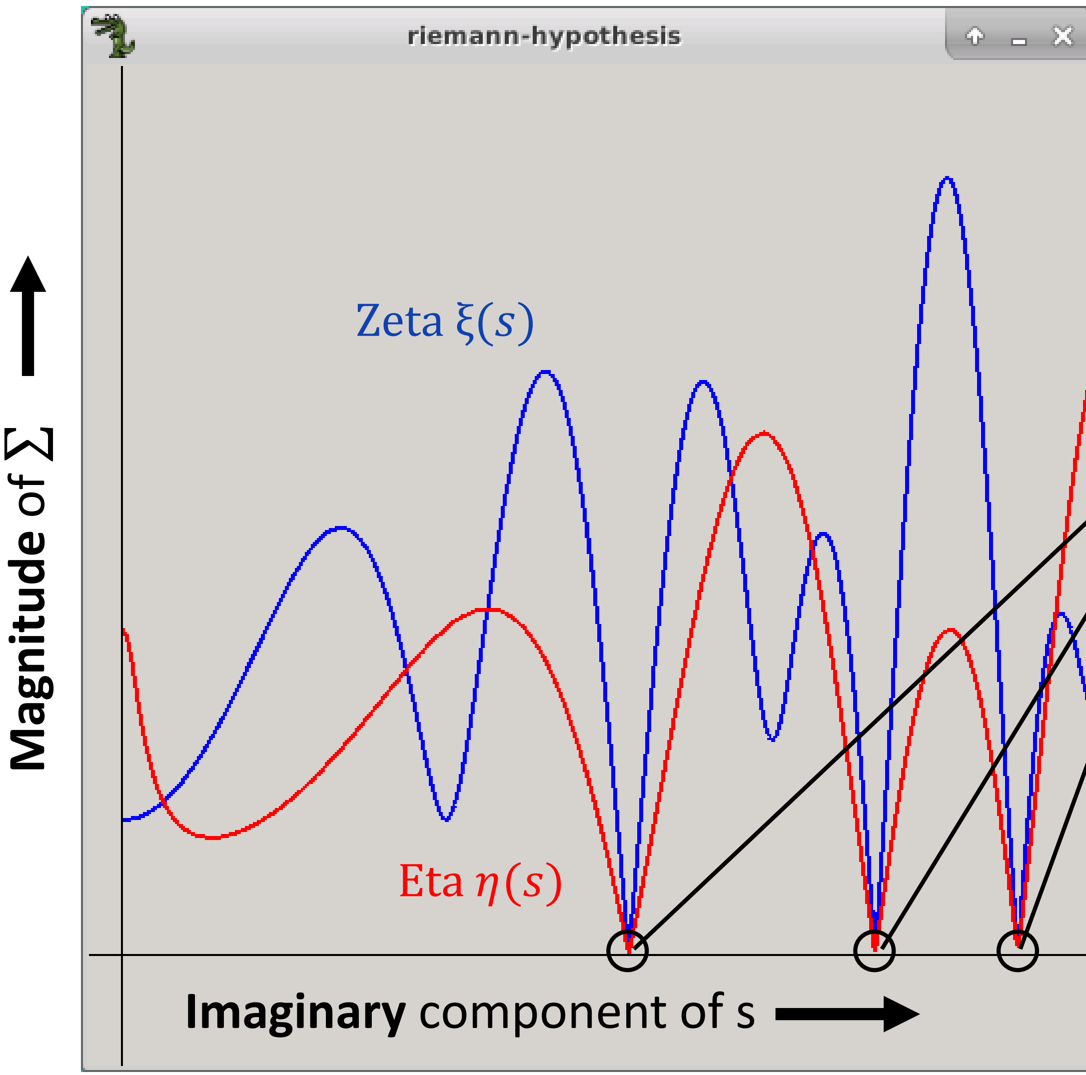
- He then discovered something unexpected - all the **zeta zeroes** seemed to have a **real part = ½**
  - He could not offer a proof and this idea has become the famous **Riemann Hypothesis**
  - No one has been able to prove or disprove that Zeta zeroes can only exist on that single vertical line in the complex plane (Re=1/2)
  - It is the **most important unsolved problem in Mathematics** because it is intricately linked to the distribution of prime numbers

Run Lab 6 – Riemann Hypothesis

- Run Lab 6 to compare the Zeta and Eta functions
- The Zeta(s) function is in **red**
- The Eta(s) function is in **blue**
- The x-axis (domain) is the *imaginary* component of the complex number s where 0 < s < 27
- The y-axis (range) is the **magnitude** (absolute value) of the respective series
- Riemann found the first three zeta zeroes are located near

$$\xi_1\left(\frac{1}{2}+\mathbf{14}.134725i\right)\ \xi_2\left(\frac{1}{2}+\mathbf{21}.022040i\right)\ \xi_3\left(\frac{1}{2}+\mathbf{25}.010858i\right)$$

Check Lab 6 – Riemann Hypothesis



These are the first **three** zeroes of Riemann's Zeta

Do all zeroes have real part $\frac{1}{2}$?

Check Lab 6 – Riemann Hypothesis

- Recall Riemann was only interested in the zeta **zeroes**
- Why is it the case that wherever $\eta(s) = 0 \rightarrow \xi(s) = 0$?
- In Riemann's narrow pursuit are **Eta** and **Zeta** therefore equivalent (the "same") functions?
  - If you only look (care about) at the points where two functions happen to be equal to each other, will you consider them as equal functions?
- Think back to the Gamma function vs. Integer Factorial…
  - Does it matter how the two functions behave where you are "not" looking?
  - Who defines what makes two functions equivalent?

Now you know…

- Only the set of complex numbers $\mathbb{C}$ is closed under both division and radicals

$$\frac{1}{2} \notin \mathbb{Z}, \sqrt{2} \notin \mathbb{Q}, \sqrt{-1} \notin \mathbb{R}$$

- The numerators in the Taylor series for e^x can be complex numbers, but fortunately **each numerator has only a positive integer <u>exponent</u>** which we can easily expand
- It is difficult to evaluate the power series expansion for e^x for many terms in software because the factorial in the denominator grows at a hyper-exponential rate!
- Euler's Identity shows a deep relationship between **the five most important constants** in all of Mathematics!

Now you know…

- Napier's logarithm down converts **multiplication** into easier addition:

$$\log AN = \log A + \log B$$

- De Moivre's Formula down converts **exponentiation** into easier multiplication:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\, sin\, n\theta$$

- Euler's Formula is considered the most useful equation in all of mathematics!

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Now you know…

- Functional equations essentially summarize the behavior of an infinite series
  - They provide a shortcut to determine the converged limit without having to loop through every element
  - They often allow you to **extend the domain** of the series to evaluate points that at first seem impossible
  - What is means for two algebraically different functions **to be the same** is a tricky question – especially when you are only interested in certain points along the domain!
- It is interesting to break the rules and **insert unexpected values** into existing formulas to see what happens – be a mathematical **renegade**!