Expectation Values & Operators

$$\langle \ \rangle = \int_{-\infty}^{\infty} \psi^* [\ \ ] \psi \, dx$$

↑ operator goes here to get expectation value of desired quantity.

For example, average position operator is x.

In general, operator is written $\hat{Q}$

$$\langle Q \rangle = \int_{-\infty}^{\infty} \psi^* \hat{Q} \psi \, dx = \langle \psi | \hat{Q} \psi \rangle$$

While $\hat{Q}$ can be anything not all operators correspond to real Physics. Any observable must be real. $\langle Q \rangle = \langle Q \rangle^*$. This means that Q must be hermetian.

$$\langle Q \rangle^* = \langle \psi | \hat{Q} \psi \rangle^* = \langle \hat{Q} \psi | \psi \rangle$$

$$\langle Q \rangle = \langle \psi | \hat{Q} \psi \rangle$$

$$\Rightarrow \langle \psi | \hat{Q} \psi \rangle = \langle \hat{Q} \psi | \psi \rangle$$

if Q represents an observable. Such an operator $\hat{Q}$ is called Hermitian.

$$\hat{Q}^{\dagger} = \hat{Q} \qquad ^{\dagger} = \text{complex transpose.} = T^*$$

Momentum Operator, and others

$$\hat{p} = \left(\frac{\hbar}{i}\right) \frac{\partial}{\partial x} \qquad \text{Momentum}$$

Note! Squaring an operator means applying it twice!

$$\langle p \rangle = \int \psi^* \frac{\hbar}{i} \frac{\partial}{\partial x} \psi \, dx$$

Why?. $\langle p \rangle = m \dfrac{d \langle x \rangle}{dt}$

$$\frac{d \langle x \rangle}{dt} = \frac{d}{dt}\left[\int_{-\infty}^{\infty} \psi^* x \psi \, dx\right]$$

$$= \int \left(\frac{\partial \psi^*}{\partial t}\right) x \psi \, dx + \int \psi^* x \frac{\partial \psi}{\partial t} \, dx$$

There is no $\dfrac{\partial x}{\partial t}$ because x is the operator. It doesn't change for this operator (Though in general it might).

Use Schrö'd: $\dfrac{\partial \psi}{\partial t} = \dfrac{i\hbar}{2m} \dfrac{\partial^2 \psi}{\partial x^2} - \dfrac{i}{\hbar} V \psi \Rightarrow^* \dfrac{\partial \psi^*}{\partial t} = \dfrac{-i\hbar}{2m} \dfrac{\partial^2 \psi^*}{\partial x^2} + \dfrac{i}{\hbar} V \psi^*$

... Poof! $\dfrac{d \langle x \rangle}{dt} = \dfrac{-i\hbar}{m} \displaystyle\int \psi^* \frac{\partial \psi}{\partial x} dx = \frac{1}{m} \frac{\hbar}{i} \int \psi^* \frac{\partial}{\partial x} \psi \, dx$