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Notes on General Relativity

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1 Preliminaries

Wald Ch. 2, Carroll Ch. 2-3

An n-dimensional C^∞ real **manifold** M is defined as a set with a collection of subsets $\{O_a\}$ that satisfy:

1. Each $p \in M$ lies in at least one O_a; the $\{O_a\}$ cover M.
2. For each α, there is a bijective map $\psi_a : O_a \to U_a$, where U_a is open in $\mathbb{R}^n$.
3. If $O_a \cap O_\beta \neq \emptyset$, then $\psi_\beta \circ \psi_\alpha^{-1}$ is C^∞.

Some examples include $\mathbb{R}^3$, $\mathbb{R}^d$, S^2, and T^2. Anti-examples include cones and discrete manifolds.

In $\mathbb{R}^n$, a vector $v = (v^1, \cdots, v^n)$ defines the directional derivatives and vice versa. For a manifold, **Tangent vectors** are maps $v : \mathcal{F} \to \mathbb{R}$ characterized by linearity and Leibnitz rule; for $\mathcal{F}$, a field of $C^\infty : M \to \mathbb{R}$,

1. $v(af + bg) = av(f) + bv(g)$, $\forall f, g \in \mathcal{F}$; $a, b \in \mathbb{R}$.
2. $v(fg) = f(p)v(g) + g(p)v(f)$.

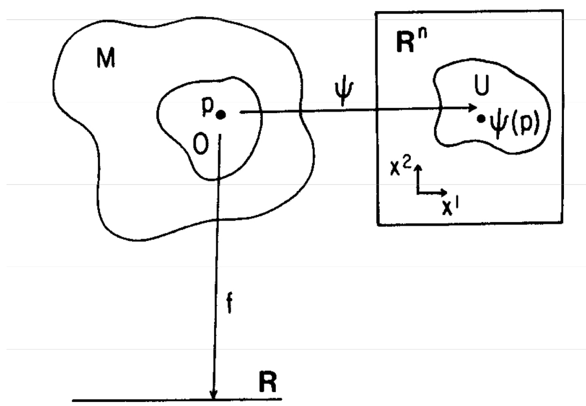

Figure 1. A diagram illustrating the definition of the directional derivatives X_μ

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Now, we construct the basis of $V_p = \{v\}$. We define $X_\mu : \mathcal{F} \to \mathbb{R}$ by

$$X_\mu(f) = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\bigg|_{\psi(p)}.$$

Using Haadmard's lemma (Wald thm. 2.2.1)

$$v = \sum_{\mu=1}^{n} v^\mu X_\mu, \quad v^\mu = v(x^\mu \circ \psi).$$

Typically, we denote X_μ as $\frac{\partial}{\partial x^\mu}$.

Using the chain rule, we can show the vector transformation law:

$$X_\mu = \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu}\bigg|_{\psi(p)} X'_\nu, \quad v'^\nu = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^\nu}{\partial x^\mu}.$$

Let $C : \mathbb{R} \to M$ be a C^∞ map - think worldlines. At point $p \in M$, C is related to a tangent vector T as $T(f) = \frac{\mathrm{d} f \circ C}{\mathrm{d}t}$:

$$T(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ C) = \sum_\mu \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}) \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \sum_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}t} X_\mu(f) \implies T^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}.$$

Let V be a finite dimensional vector space. Consider the vector space V^* of linear maps $f : V \to \mathbb{R}$. For $v_1, \cdots, v_n$ basis of V, define $v^{1*}, \cdots, v^{n*} \in V^*$ by

$$v^{\mu*}(v_\nu) = \delta^\mu_\nu.$$

Now, we define a tensor T of type (k, l) over V as

$$T : V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}.$$

Obviously, the vector space $\mathbb{T}(k, l)$ is n^{k+l}.

Let $\{v_\mu\}$ be a basis of V and $\{v^{\nu*}\}$ its dual basis; from multilinearity, a basis of $\mathbb{T}(k, l)$ can be $\{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1*} \otimes \cdots \otimes v^{\nu_l*}\}$:

$$T = \sum_{\mu_1, \cdots, \nu_l = 1}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_1} v_{\mu_1} \otimes \cdots \otimes v^{\nu_l*}.$$

We now define two operators on tensors. The first operation is the **contraction**; defined with the ith and jth slots is the map $C : \mathbb{T}(k, l) \to \mathbb{T}(k-1, l-1)$, defined

$$CT = \sum_{\sigma=1}^{n} T(\cdots, v^{\sigma*}, \cdots; \cdots, v_\sigma, \cdots),$$

where $\{v_\sigma\}$ is a basis of V, $\{v^{\sigma*}\}$ is its dual basis.

The second operation is the **outer product**. Given a tensor $T(k, l)$ and another tensor $T'(k', l')$, we construct tensor $T \otimes T'(k+k', l+l')$. It is simply defined

$$T \otimes T' = T(v^{1*}, \cdots, v^{k*}; w_1, \cdots, w_l) T'(v^{k+1*}, \cdots, v^{k+k'*}; w_{l+1}, \cdots, w_{l+l'}).$$

In the case that V_p is the tangent space to manifold M at point p, V_p^* is the cotangent space. If the coordinate basis of V_p is $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right\}$, the associated dual basis is $\mathrm{d}x^1, \cdots, \mathrm{d}x^n$. It is defined that $\mathrm{d}x^\mu \left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$.

It follows that the dual vector ω, in the dual basis $\{\mathrm{d}x^\mu\}$, transforms

$$\omega'_{\mu'} = \sum_{\mu=1}^{n} \omega_\mu \frac{\partial x^\mu}{\partial x'^\mu}.$$

Likewise,

$$T'^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = \sum_{\mu_1, \cdots, \nu_l = 1}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x'^{\nu'_l}}.$$

As an example, the metric is a rank $(0,2)$ tensor.

In a metric space, the metric is defined; one can think of it as the "infinitesimal square distance" or as something that defines the inner product in the tangent space (which, in turn, is the natural isomorphism between the tangent space and the cotangent space)

$$g = \sum_{\mu,\nu} g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \quad \mathrm{d}s^2 = \sum_{\mu,\nu} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu.$$

We can always find a coordinate system such that $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and $\partial_\sigma g_{\mu\nu}(p) = 0$, making the metric locally inertial.

A tensor is said to be symmetric if it's unchanged under exchange of its indices; $S_{\mu\nu\rho} = S_{\nu\mu\rho}$. It's antisymmetric (about given indices) if it changes sign when those indices are exchanged: $A_{\mu\nu\rho} = -A_{\rho\nu\mu}$. One can symmetrize/antisymmetrize tensors by taking the sums:

It is useful to define the **levi-civita symbol** as a tensor. It is defined (note that here, $\tilde{\epsilon}$ is the symbol):

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}, \quad \epsilon^{\mu_1\mu_2\cdots\mu_n} = \frac{1}{\sqrt{|g|}}\tilde{\epsilon}^{\mu_1\mu_2\cdots\mu_n}.$$

For p and q forms, the **wedge product**, $\wedge : \omega_p, \omega_q \to \omega_{p+q}$ is defined

$$(A \wedge B)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}]}.$$

The **exterior derivative** $d : \omega_p \to \omega_{p+1}$ is defined

$$(dA)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}.$$

The **Hodge symbol** $* : \omega_p \to \omega_{n-p}$, where n is the dimensionality of the maniofld, is defined

$$(*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}} A_{\nu_1\cdots\nu_p} \implies **A = (-1)^{s+p(n-p)}A,$$

where s is the number of minus signs in the eigenvalues of the metric.

We define the **covariant derivative** as a covariant map $\nabla : T^\mu_\nu \to T^\mu_{\nu+1}$, having properties:

1. Linearity: $\nabla(T+S) = \nabla T + \nabla S$;
2. Leibniz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$;
3. Commutes with contractions: $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$;
4. Reduces to partial on scalars: $\nabla_\mu \phi = \partial_\mu \phi$.

A covariant derivative satisfying such properties takes the form

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \quad \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda.$$

In general relativity, we impose that the metric satisfies two more conditions:

5. Torsion free - connection symmetric in lower indices: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$;
6. Metric compatibility: $\nabla_\rho g_{\mu\nu} = 0$.

In this case, the connection coefficients are called the **Chrisftoffel symbols**:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}).$$

We define the parallel transport of the tensor T along the path $x^\mu(\lambda)$ to be the requirement that the covariant derivative along the path vanishes:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}V^\rho = 0.$$

We define a geodesic to be a curve that parallel-transports its own tangent vector:

$$\frac{D}{d\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = 0 \implies \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0.$$

Note that we have an affine parameter; the geodesic equation is invariant under a linear transformation $\lambda \to a\lambda + b$. With this freedom, we choose

1. for massive particles, $p^\mu = mu^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$,
2. for massive particles, $p^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$, $p^2 = 0$.

Let k^μ be a vector at point p. It defines a geodesic curve x^μ. We define **Riemann normal coordinates** as $x^\mu(\lambda) = \lambda k^\mu$, which is locally inertial.

We define an isometry[1] as a map $f : M \to M$ with $d(f(p_1), f(p_2)) = d(p_1, p_2)$. Infinitesimally, we can parametrize f by a vector field ζ^μ: for $p \to f(p)$, we have $x^\mu \to x^\mu + \zeta^\mu(x)$ and hence $\delta x^\mu = \zeta^\mu(x)$. Isometry requires that $\delta(ds^2) = 0$, giving us

$$\delta(\mathrm{d}s^2) = \delta(g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu) = \partial_\alpha g_{\mu\nu}\delta x^\alpha \mathrm{d}x^\mu\mathrm{d}x^\nu + 2g_{\mu\nu}\,\mathrm{d}(\delta x^\mu)\,\mathrm{d}x^\nu = (\delta_\zeta g_{\mu\nu})\mathrm{d}x^\mu\mathrm{d}x^\nu,$$

where we have defined

$$\mathcal{L}_\zeta g_{\mu\nu} = \delta_\zeta g_{\mu\nu} = \zeta^\alpha\partial_\alpha g_{\mu\nu} + \partial_\mu\zeta^\alpha g_{\alpha\nu} + \partial_\nu\zeta^\alpha g_{\mu\alpha},$$

which is known as the **Lie derivative**. We can then rewrite the isometry statement in two ways:

$$\delta(\mathrm{d}s^2) = 0 \implies \mathcal{L}_\zeta g_{\mu\nu} = 0, \quad \nabla_{(\mu}\zeta_{\nu)} = 0.$$

The second equation is called the **Killing equation**. The coordinates in which locally, $\zeta = \partial_x$ hae obvious symmetries - in these coordinates, the isometries, finite or infinitesimal, are trivial translations.

Each Killing vector corresponds to a conserved quantity, $p_\zeta = p^\alpha\zeta_\alpha$ along geodesics;

$$\frac{\mathrm{d}p_\zeta}{\mathrm{d}\tau} = p^\mu\partial_\mu(p^\alpha\zeta_\alpha) = p^\mu\nabla_\mu(p^\alpha\zeta_\alpha) = (p^\mu\nabla_\mu p^\alpha)\zeta_\alpha + p^\mu p^\alpha\nabla_\mu\zeta_\alpha.$$

The first term is zero from the geodesic equation and the second from the Killing equation. As an example, $E = -\zeta^\alpha p_\mu$ as a conserved quantity[2].

[1] In case of $\mathbb{R}^n$, there are n translations and $\frac{1}{2}n(n+1)$ rotations, giving us $\frac{n}{2}(n+1)$ killing vectors. Manifolds with $\frac{n}{2}(n+1)$ isometries are called **maximally symmetric**. Other maximally symmetric manifolds include: $\mathbb{R}^n$, S^n, $\mathbb{H}^n$ spatially, and $\mathbb{R}^{n-1,1}$, $\mathrm{d}S_n$, and $\mathrm{Ad}S_n$.

[2] Since the observed energy is $-u^\alpha p_\alpha$ where u is the velocitiy of the observer; hence, $E_{\mathrm{obs}} = E$ only if ζ is timelike.

2 Intro

Add section for SR? - Carroll ch.1, Hartle ch.4

The **postulates of SR** state

1. Spacetime is the geometry with the (Minkowski) metric $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$; speed of light is the same in all frames. This was realized from electromagnetism.

2. The laws of physics are tensor equations; physics is the same in all inertial frames.

The symmetries in Minkowski spacetime are called Poincare transformations, defined $x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha$, with 4 transformations in a^α, and 3 rotations and 3 boosts in Lorentz transformations, defined $\eta = \Lambda^\intercal \eta \Lambda$. These form the Lorentz group, $O(3,1)$, consisting of combinations of proper/restricted Lorentz transformation, time reversal T, parity reversal P, and PT.

Using the definition, we can find the generators of the Lorentz group: for an infinitesimal transformation $\Lambda^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta$, $\Lambda^\intercal \eta \Lambda = \eta \implies \omega^\intercal = -\omega$; hence, since ω is antisymmetric, there are 6 independent generators;

$$\Lambda = \mathbb{1} + \sum_i \psi_i K_i + \sum_i \theta_i J_i,$$

with K_i boosts and J_i rotations. Each generator can also be represented as a vector field, ζ^α, defined

$$x' = (\mathbb{1} + \omega)x \implies x'^\alpha = x^\alpha - \zeta^\alpha.$$

For example, the vector field ζ^α for $K_1 = x\partial_t + t\partial_x$ points towards the light cone, corresponding to a boost, and the vector field for $J_3 = -y\partial_x + x\partial_y$ points around in a circle. These give another representation of the Lorentz group, defined with the Lie bracket:

$$[X, Y]^\mu = X^\alpha \partial_\alpha Y^\mu - Y^\alpha \partial_\alpha X^\mu.$$

Furthermore, the Lorentz group is a Lie group, with

$$[J_i, J_j] = -\epsilon_{ijk} J_k, \quad [J_i, K_j] = -\epsilon_{ijk} K_k, \quad [K_i, K_j] = \epsilon_{ijk} J_k.$$

The **equivalence principle** states one of the following:

1. The inertial mass is equivalent to the gravitational mass - verified down to 10^{-14};

2. Physics in accelerating frame is indistinguishable in a gravitational field, local in time and space;

3. Free fall is indistinguishable from inertial motion - spacetime is locally flat; $g'_{\mu\nu}(x_0) = \eta_{mu\nu}$, $\partial'_\alpha g'_{\mu\nu}(x_0) = 0$.

The **postulates of GR** states the following:

1. Spacetime is curved;

2. Matter produces curvature;

3. Objects move on locally straight lines.

We characterize the motion of a point particle with $x^\mu(\lambda) : \mathbb{R} \to M$. If we reparameterize x^μ by the proper time, the tangent vector is known as the **four-velocity**:

$$U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad \eta_{\mu\nu} U^\mu U^\nu = -1.$$

We can further define the **momentum** and force as, analogous to classical mechanics, $p^\mu = mU^\mu$, $f^\mu = m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}p^\mu$.

Another useful tensor is the **stress-energy tensor**, $T^{\mu\nu}$. Its components are defined as the flux of p^μ in the x^ν direction. More specifically,

1. T^{00} is the rest energy density;
2. $T^{0i} = T^{i0}$ are the k-momentum density;
3. T^{ij} are the shearing/pressure terms.

The **perfect fluid** - no conductivity, no viscosity - and its conservation law has the form

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + p\eta^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad \partial_\mu T^{\mu\nu} = 0.$$