0.1 Random Walks

Consider a random walk with steps $\mathbf{l}_N$ of size L in a random direction. The displacement after N such steps is $\mathbf{s}_N = \sum_N l_N$.

The mean square displacement of a random walk containing N steps is

$$\left\langle \mathbf{s}_N^2 \right\rangle = \left\langle (\mathbf{s}_N + \mathbf{l})^2 \right\rangle = \left\langle \mathbf{s}_{N-1}^2 \right\rangle + \cancel{2 \left\langle \mathbf{s}_N \cdot \mathbf{l} \right\rangle} + \left\langle \mathbf{l}^2 \right\rangle = NL^2 \implies \sqrt{\left\langle \mathbf{s}_N^2 \right\rangle} = \sqrt{N} L.$$

Note that $\langle \mathbf{s}_N \cdot l \rangle = \sum \langle \mathbf{l}_N \rangle \cdot \langle \mathbf{l} \rangle = 0$, since $\mathbf{l}$ are independent.

0.2 Diffusion Equation

The continuum limit of an ensemble of random walks can be described by the diffusion equation.

Consider a random walk with step size described by a probability distribution $\chi(l)$. Its mean is 0 with standard deviation a^2:

$$\int \chi(z) \mathrm{d}z = 1, \quad \int z\chi(z) \mathrm{d}z = 0, \quad \int z^2 \chi(z) \mathrm{d}z = a^2.$$

Thus, the density at x and $t + \Delta t$ can be written

$$\begin{aligned}
\rho(x, t + \Delta t) &= \int \rho(x', t)\chi(x - x')\mathrm{d}x' \\
&= \int \rho(x - z, t)\chi(z)\mathrm{d}z \\
&\approx \int \left[\rho(x, t) - z\frac{\partial \rho}{\partial x} + \frac{1}{2} z^2 \frac{\partial^2 \rho}{\partial x^2} \right] \chi(z)\mathrm{d}z \\
&= \rho(x, t) + \frac{a^2}{2} \frac{\partial^2 \rho}{\partial x^2}.
\end{aligned}$$

For a sufficiently small Δt,

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial t^2}, \qquad D = \frac{a^2}{2\Delta t}. \tag{0.1}$$

We now define the current to be

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}, \tag{0.2}$$

which allows us to write the diffusion equation as

$$J_{\text{diffusion}} = -D \frac{\partial \rho}{\partial x}. \tag{0.3}$$

To solve the diffusion equation, we can consider the plane wave solutions $\rho(x, t) = \tilde{\rho}_k(t)e^{ikx}$. From the diffusion equation,

$$\frac{\mathrm{d}\tilde{\rho}_k}{\mathrm{d}t} = -Dk^2 \tilde{\rho}_k \implies \tilde{\rho}_k(t) = \tilde{\rho}_k(0)e^{-D^2 kt}.$$

Reversing this, we get

$$\rho(x, t) = \frac{1}{2\pi} \int \tilde{\rho}_k(0)e^{ikx} e^{-Dk^2 t} \mathrm{d}k, \quad \tilde{\rho}_k(0) = \int \rho(x, 0)e^{-ikx} \mathrm{d}x.$$

The Green's function for the diffusion equation is

$$G(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}. \tag{0.4}$$

1 Temperature

We define the configuration space $\mathbb{Q} = (x_1, y_1, z_1, \cdots, x_N, y_N, z_N)$ and the momenta space $\mathbb{P} = (p_1, \cdots, p_{3N})$ of the particles.

We define the function $\Omega(E)$ to be the phase-space volume of the states in a shell $(E, E + \delta E)$ where $\delta E \to 0$:

$$\Omega(E) = \frac{1}{\delta E} \int_{E<\mathcal{H}(\mathbb{P},\mathbb{Q})<E+\delta E} \mathrm{d}\mathbb{P}\mathrm{d}\mathbb{Q}. \tag{1.1}$$