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Notes on Electrodynamics

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1 Preliminaries

Goldbarch Ch.2

Complete orthonormal functions A set of functions $\{U_n\} \in L^2(S)$ (the inner product is $\langle f, g\rangle = \int_S w f^* g \mathrm{d}x$, where w is the weighing function) is **orthonormal** if

$$\langle U_n, U_m\rangle = \delta_{nm}.$$

The set $\{U_n\} \in L^2(S)$ is **complete** if minimum square error $E_n = \|f - (c_1U_1 + \cdots c_nU_n)\|_S$ approaches zero as $n \to \infty$. Therefore, we have a convergent expansion and completeness relation

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi), \quad a_n = \int_S U_n^*(\xi) f(\xi) \mathrm{d}\xi \implies \sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi' - \xi),$$

For example, $U_k = (2\pi)^{-1}\exp(ikx)$ is complete and orthonormal in $L^2(\mathbb{R})$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} \mathrm{d}k, \quad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \mathrm{d}x.$$

Orthogonality and completeness relations are

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} \mathrm{d}x = \delta(k - k'), \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \mathrm{d}k = \delta(x - x').$$

Green's functions Green's function, for Poisson's equation, is defined

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \implies G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}'), \qquad \nabla'^2 F(\mathbf{x}, \mathbf{x}) = 0.$$

From Green's theorem,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') \mathrm{d}^3x' + \frac{1}{4\pi} \oint_S \left[G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] \mathrm{d}a'.$$

From the free parameter $F(\mathbf{x}, \mathbf{x}')$, we can let either $G(\mathbf{x}, \mathbf{x}') = 0$ or $\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} = 0$.

Uniqueness theorem Let $U = \Phi_2 - \Phi_1$ where Φ_1 and Φ_2 both satisfy the same boundary conditions. From Green's first theorem,

$$\int_V (U\nabla^2 U + \nabla U \cdot \nabla U) \mathrm{d}^3x = \oint_S U \frac{\partial U}{\partial n} \mathrm{d}a.$$

For Dirichlet ($U = 0$ on S) or Neumann ($\frac{\partial U}{\partial n} = 0$ on S)conditions,

$$\int_V |\nabla U|^2 \mathrm{d}^3x = 0,$$

meaning U is a constant.

2 Elecrostatics

Jackson Chapter 1 (except 1.12-1.13)

2.1 Axioms and Definitions

Coulomb's law Coulomb's law states: the force between two small charged bodies separated in air a distance large compared to their dimensions

1. varies directly as the magnitude of each charge,
2. varies inversely as the square of the distance between them,
3. is directed along the line joining the bodies, and
4. is attractive if the bodies are oppositely charged and repulsive if the bodies have the same type of charge.

This can be written as

$$\mathbf{E} = kq_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}.$$

where $\mathbf{F} = q\mathbf{E}$ by definition.

Electrical forces follow the principle of superposition; for a discrete/continuous distribution of charges,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|^3} = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \mathrm{d}^3 x'.$$

Gauss' law From Coulomb's law,

$$\mathbf{E} \cdot \mathbf{n} \, \mathrm{d}a = \frac{q}{4\pi\epsilon_0} \frac{\cos\theta}{r^2} \mathrm{d}a = \frac{q}{4\pi\epsilon_0} \mathrm{d}\Omega,$$

for some surface S. Therefore,

$$\oint_S \mathbf{E} \cdot \, \mathrm{d}a = \frac{q}{\epsilon_0} \implies \nabla \cdot \mathbf{E} = \rho/\epsilon_0,$$

from the divergence theorem.

Potential We can write the Coulomb's law as

$$\mathbf{E} = -\nabla \left(\frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3 x' \right) = -\nabla\Phi,$$

where Φ is some scalar quantity. We define this to be the electric potential.

Boundary conditions For a charged sheet, the discontinuity in the electrical field is

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n} = \sigma/\epsilon_0.$$

For a charged dipole layer Jackson page 32, we have

$$\Phi_2 - \Phi_1 = D/\epsilon_0.$$

Poisson equations Poisson's equation states that

$$\nabla^2\Phi = -\rho/\epsilon_0.$$

It is easy to see that $\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'$ satisfies Poisson's equation using $\boldsymbol{\nabla}\cdot(1/r) = 4\pi\delta(\mathbf{x})$.

Energy Consider a charge brought from infinity. The work done is

$$W = -\int \mathbf{F}\cdot\mathrm{d}\mathbf{l} = q\int \nabla\Phi\cdot\mathrm{d}\mathbf{l} = q\Phi(\mathbf{x}),$$

where we took the potential at infinity to zero. It can be written that, for a charge distribution,

$$\begin{aligned} W &= \frac{1}{8\pi\epsilon_0}\sum_i\sum_j \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \\ &= \frac{1}{8\pi\epsilon_0}\iint \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x\,\mathrm{d}^3x' \\ &= \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\,\mathrm{d}^3x \\ &= \frac{-\epsilon_0}{2}\int \Phi\nabla^2\Phi\,\mathrm{d}^3x \\ &= \frac{\epsilon_2}{2}\int_{\mathbb{R}^3} |\nabla\Phi|^2\,\mathrm{d}^3x, \end{aligned}$$

where we have used Green's first theorem for the last part. From the last result, the energy density is $w = \frac{\epsilon_0}{2}|\mathbf{E}|^2$. However, be warned that this also contains self energy contributions - the interaction potential energy, however, is meaningful in this case (Jackson p. 42.)

Green's theorems The first Green's theorem can be obtained by applying the div. theorem on $\mathbf{A} = \phi\nabla\psi$,

$$\int_V \nabla\cdot(\phi\nabla\psi)\,\mathrm{d}^3x = \oint_S \phi\nabla\psi\cdot\mathbf{n}\,\mathrm{d}a \implies \int_V \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\,\mathrm{d}^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\mathrm{d}a.$$

Taking the antisymmetric part, we obtain the second Green's theorem:

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\mathrm{d}^3x = \oint_S \left[\phi\frac{\mathrm{d}\psi}{\mathrm{d}n} - \psi\frac{\mathrm{d}\phi}{\mathrm{d}n}\right]\mathrm{d}a.$$

As an example, $\phi = \Phi$ and $\psi = 1/|\mathbf{x}-\mathbf{x}'|$. We get

$$\int_V \left[-4\pi\Phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{\epsilon_0 R}\rho(\mathbf{x}')\right]\mathrm{d}^3x' = \oint_S \left[\Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right]\mathrm{d}a'.$$

Therefore,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}')}{R}\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]\mathrm{d}a'.$$

2.2 Method of Images

Jackson Chapter 2-3

Conducting Sphere - Point Charge Consider a grounded conducting sphere of radius a centered at origin and a point charge q at $\mathbf{y}$. If we let the image charge be at $\mathbf{y}'$ with charge q', the potential at $\mathbf{x}$ is

$$\Phi(\mathbf{x}) = \frac{q/4\pi\epsilon_0}{a\left|\mathbf{n} - \frac{y}{a}\mathbf{n}'\right|} + \frac{q'/4\pi\epsilon_0}{y'\left|\mathbf{n}' - \frac{a}{y'}\mathbf{n}\right|},$$

where we let $\mathbf{y}$ and $\mathbf{y}'$ lie on $\mathbf{n}'$, a unit vector and $\mathbf{x}$ on $\mathbf{n}$. This has the solution

$$q' = -\frac{a}{y}q, \quad y' = \frac{a^2}{y}.$$

The force acting on charge q can be calculated via Coulomb's law or by integrating $(\sigma^2/2\epsilon_0)\mathrm{d}a$:

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2}.$$

If the total charge on the insulated sphere be Q. We then use linear superposition of induced charge, q' with the "added" charge $Q - q'$ as evenly spread over the surface. Hence,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x} - \mathbf{y}|} - \frac{aq}{y\left|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}\right|} + \frac{Q + \frac{a}{y}q}{|\mathbf{x}|}\right]$$

Conducting Sphere - Uniform Field One can think of uniform field as two opposite charges very far away. We get

$$\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \implies \sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 3\epsilon_0 E_0\cos\theta.$$

General Solution The green's function (for dirichlet boundary condition) is evidently

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x} - \frac{a^2}{x'^2}\mathbf{x}'\right|} = \frac{1}{(x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma\right)}.$$

Taking the derivative and applying Green's theorem, we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\int \Phi(a, \theta', \phi')\frac{a(x^2 - a^2)}{(x^2 + a^2 - 2ax\cos\gamma)^{3/2}}\mathrm{d}\Omega',$$

where γ is the angle between x and x'. Using this, one can find the potential of a conducting sphere with hemispheres at different potentials (Jackson 2.7).

2.3 Separation of Variables

If we let the solution of a PDE be a product of single variabled functions, we can reduce the PDE to a system of ODEs. This is particularly useful in solving Laplace's equation.

Cartesian We have

$$\Phi = XYZ \implies \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0.$$

We get

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\alpha^2, \quad \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = -\beta^2, \quad \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \gamma^2,$$

where $\alpha^2 + \beta^2 = \gamma^2$. The general solution is $\Phi = e^{\pm i\alpha x}e^{\pm i\beta y}e^{\pm\sqrt{\alpha^2+\beta^2}z}$. To find the specific solution, one must impose the boundary conditions. It is also useful to superpose such solutions, which orthonormality of e^{ix} comes in handy. (Jackson 2.9, 2.10, 2.11)

Spherical With $\Phi = \frac{U}{r} PQ$, (reminder that you have to divide by r!) we get the differential equations

$$\begin{aligned}
\frac{1}{Q}\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} = -m^2 \implies Q = e^{im\phi} \\
\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0 \implies P = P_l^m(\cos\theta) \\
\frac{\mathrm{d}^2 U}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2}U = 0 \implies U = Ar^{l+1} + Br^{-l},
\end{aligned}$$

where m, l are constants.

Spherical - Azimuthal Symmetry For $m = 0$, the solution for P are the legendre polynomials. The first few polynomials are

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= x \\
P_2 &= \frac{1}{2}(3x^2 - 1) \\
P_3 &= \frac{1}{2}(5x^3 - 3x) \\
P_4 &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5 &= \frac{1}{8}(63x^5 - 70x^3 + 15x).
\end{aligned}$$

Rodrigues' formula in particularly useful:

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2 - 1)^l.$$

Furthermore, the legendre polynomials are orthonormal in $(1, -1)$ if you let $U_l(x) = \sqrt{\frac{2l+1}{2}}P_l(x)$. Therefore, one can expand a function:

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x), \quad A_l = \frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)\mathrm{d}x.$$

In other words, given the general solution

$$\Phi(r, \theta) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}]P_l(\cos\theta),$$

one cam impose boundary condition since, on a sphere, for example, the boundary condition becomes $V(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) \implies A_l = \frac{2l+1}{2a^l}\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta\mathrm{d}\theta$.

If we have an azimuthally symmetric system, due to the uniqueness theorem, the potential on the symmetric axis gives the general potential, when multiplied by legendre polynomials; given $\Phi(\theta = 0) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}] \implies \Phi(r, \theta) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}]P_l(\cos\theta)$.

A useful application of this is the expansion of the $1/|\mathbf{x} - \mathbf{x}'|$:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma),$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$ and $r_>$ and $r_<$ are the greater and smaller of $\mathbf{x}$ and $\mathbf{x}'$, respectively.

Spherical - General For $m \in [-l, l]$, P has the formula

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l.$$

The associated legendre polynomials are again orthogonal in $\phi \in [0, 2\pi]$ and $\cos\theta \in [-1, 1]$. The normalized associated Legendre polynomials, called the spherical harmonics, are defined by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, \quad Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta).$$

The first few terms can be written out:

$$\begin{aligned}
Y_{00} &= \sqrt{\frac{1}{4\pi}} \\
Y_{1,\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_{2,\pm2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi}, \quad Y_{2,\pm1} = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{\pm i\phi}, \quad Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1) \\
Y_{3,\pm3} &= \mp\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta e^{\pm 3i\phi}, \quad Y_{3,\pm2} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{\pm 2i\phi}, \quad Y_{3,\pm1} = \mp\frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi} \\
Y_{3,0} &= \frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\theta - 3\cos\theta).
\end{aligned}$$

Its orthonormality and completeness relations state

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\mathrm{d}\theta Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm}$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta').$$

Evidently, an arbitrary function can be expanded into spherical harmonics:

$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\phi), \quad A_{lm} = \int\mathrm{d}\Omega Y^*_{lm}(\theta,\phi)g(\theta,\phi).$$

Lastly, the addition theorem for spherical harmonics expresses a Legendre polynomial of order l in the angle γ in products of the spherical harmonics of the angles θ,ϕ and θ',ϕ':

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi).$$

Therefore, $1/|\mathbf{x}-\mathbf{x}'|$ can be written

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi).$$

Cylindrical With $\Phi = RQZ$, we get the differential equations

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - k^2 Z = 0 \implies Z = e^{\pm kz}$$
$$\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} + \nu^2 Q = 0 \implies Q = e^{\pm i\nu\phi}$$
$$\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \implies R = AJ_\nu(k\rho) + BN_m(k\rho).$$

The radial equation, after a change of variable $x = k\rho$, becomes the Bessel equation ($\frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}R}{\mathrm{d}x} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0$). The two solutions are (Bessel functions of the first kind of order $\pm\nu$)

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}, \quad J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}.$$

The Bessel function can be written as an integral:

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi e^{iz\cos\theta}(n\theta)\mathrm{d}\theta.$$

For $\nu \neq \mathbb{Z}$, these two solutions are linearly dependent. It is then customary to use $J_\nu(x)$ and $N_\nu(x)$ - Bessel function of the second kind:

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}.$$

For boundary condition purposes, for $x \ll 1$, $J_\nu(x) \approx x^\nu$, $\quad N_\nu(x) \approx \ln(x/2)$, for $\nu = 0$, and $N_\nu(x) \approx x^{-\nu}$ for $\nu \neq 0$. For $x \gg 1$, $J_\nu(x) \approx \cos x$, $\quad N_\nu(x) \approx \sin x$.

A third set of solutions exist, called the Hankel functions, but are most commonly used in propagation of waves.

Lastly, only Bessel functions of the first kind are orthogonal on the interval $[0, 1]$: the Fourier-Bessel series states

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right), \quad A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho f(\rho) J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)\mathrm{d}\rho.$$

2.4 Green's Function

Green's function can be found by expanding both sides of $\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$ - left side can be normally expanded and the right side can be expanded using completeness of spherical harmonics/Bessel functions.

Spherical Green Function The Green function for a spherical shell bounded by $r = a$ and $r = b$ is

$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m-l}^{l} \frac{Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right).$$

Eigenfunction Expansions Consider an elliptic differential equation of the form

$$\nabla^2\psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n]\psi_n(\mathbf{x}) = 0.$$

Because $\nabla^2 + f(\mathbf{x})$ is a self-adjoint operator, its eigenfunctions are orthogonal:

$$\int_V \psi_m^*(\mathbf{x})\psi_n(\mathbf{x})\mathrm{d}^3x = \delta_{mn}.$$

Hence, if we want to solve the Green function for the equation

$$\nabla^2 G(\mathbf{x},\mathbf{x}') + [f(\mathbf{x}) + \lambda]G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'),$$

we can use $G(\mathbf{x},\mathbf{x}') = \sum_n a_n(\mathbf{x}')\psi_n(\mathbf{x})$. If we multiply both sides by $\psi_m^*(\mathbf{x}'$ and integrate, we get $G(\mathbf{x},\mathbf{x}') = 4\pi\sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n-\lambda}$.

2.5 Multipole Expansion

Comparing the separation of variables solution outside a sphere with the potential of a charge density distribution gives us the integral

$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0}\sum_{l,m}\frac{1}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}, \quad q_{lm} = \int Y_{lm}^*(\theta',\phi')r'^l\rho(\mathbf{x}')\mathrm{d}^3x'$$

We call these coefficients q_{lm} be called multipole moments. The first few terms are

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\sum_{i,j}Q_{ij}\frac{x_i x_j}{r^5} + \cdots\right],$$

where

$$q = \rho(\mathbf{x}')\mathrm{d}^3x', \quad \mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\mathrm{d}^3x', \quad Q_{ij} = \int(3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')\mathrm{d}^3x'.$$

Note,

1. There's $(l+1)(l+2)/2$ cartesian multipole moments versus $(2l+1)$ spherical moments. This is becuase Cartesian multipole moments are traceless symmetric tensors.

2. Multipole moments are dependent on the location on the origin.

For the dipole term, in specific,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{x}-\mathbf{x}_0|^3} - \frac{4\pi}{3}\mathbf{p}\delta(\mathbf{x}-\mathbf{x}_0)\right].$$

Furthermore, the energy of a multipole system takes the form

$$W = q\Phi(0) - \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_i\sum_j Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + \cdots.$$

2.6 Macroscopic Media

We define the macroscopic (electric) polarization $\mathbf{P}$ and charge density ρ as

$$\mathbf{P}(\mathbf{x}) = \sum_i N_i \langle \mathbf{p}_i \rangle , \quad \rho(\mathbf{x}) = \sum_i N_i \langle e_i \rangle + \rho_{\text{excess}},$$

where we're taking the weighted average of the dipole moment/molecular excess charge of the molecules in a small volume centered at $\mathbf{x}$ - N_i are the weights of the molecules. The potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \left[\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \mathbf{P}(\mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \right] = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} \left[\rho(\mathbf{x}') - \nabla' \cdot \mathbf{P}(\mathbf{x}') \right].$$

Defining the electric displacement $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$, we have $\nabla \cdot \mathbf{D} = \rho$.

For a linearly dielectric medium, the polarization is parallel to $\mathbf{E}$:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \implies \mathbf{E} = \epsilon \mathbf{E}, \quad \epsilon = \epsilon_0 (1 + \chi_e) \implies \nabla \cdot \mathbf{E} = \rho/\epsilon.$$

where χ_e is the susceptibility of the medium and $\epsilon/\epsilon_0 = 1 + \chi_e$ is the dielectric constant.

Boundary Problems The boundary conditions are

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{21} = \sigma, \quad (\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{21} = 0.$$

As an example, Jackson 4.4

3 Magnetostatics

3.1 Definitions

For magnetostatics, we have $\nabla \cdot \mathbf{J} = 0$.

Biot-Savart Law The Biot-Savart law is given as

$$\mathrm{d}\mathbf{F} = I_1 (\mathrm{d}\mathbf{I}_1 \times \mathbf{B}), \quad \mathrm{d}\mathbf{B} = kI \frac{(\mathrm{d}\mathbf{I} \times \mathbf{x})}{|\mathbf{x}|^3},$$

with $k = \mu_0 / 4\pi$ in SI units. From the force law,

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) \mathrm{d}^3x, \quad \mathbf{N} = \int \mathbf{x} \times (\mathbf{J} \times \mathbf{B}) \mathrm{d}^3x.$$

Ampere's law and potential(s) For a current density, we have

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathrm{d}^3x' = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3x'.$$

Evidently, $\nabla \cdot \mathbf{B} = 0$. If we take the curl of the magnetic field, we obtain

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \implies \oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I.$$

From the fact that $\nabla \cdot \mathbf{B} = 0$, we define the vector potential:

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) \implies \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3x' + \nabla\Psi(\mathbf{x}).$$

There's a degree of freedom in the vector potential and the related transformations are called gauge transformations - $\mathbf{A} \to \mathbf{A} + \nabla\Psi$. It is convenient to choose the gauge (Coulomb gauge) such that $\nabla \cdot \mathbf{A} = 0 \implies \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$. Ψ is constant for this gauge, because $\nabla \cdot {=}0 \implies \nabla^2\Psi = 0$.

However, in the case that $\mathbf{J} = 0$, we can define the magnetic scalar potential such that $\mathbf{H} = -\nabla\Phi_M$. If μ is piecewise constant, each region satisfies the Laplace equation: $\nabla^2\Phi_M = 0$.

3.2 Magnetic Moment

Recall the vector expansion $\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \cdots$. A given component of the vector potential is

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\mathbf{x}|}\int J_i(\mathbf{x}')\mathrm{d}^3x' + \frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\int J_i(\mathbf{x}')\mathbf{x}'\mathrm{d}^3x' + \cdots\right]$$

Using the identity $f\mathbf{J}\cdot\nabla' g + g\mathbf{J}\cdot\nabla' f + fg\nabla'\cdot\mathbf{J} = 0$ with $f = 1, g = x_i'$, we have $\int J_i(\mathbf{x}')\mathrm{d}^3x' = 0$. With $f = x_i', g = x_j'$, we have $\int(x_i'J_j + x_j'J_i)\mathrm{d}^3x' = 0$. Hence, we can write the dipole term as

$$\mathbf{x}\cdot\int\mathbf{x}'J_i\mathrm{d}^3x' = -\frac{1}{2}\sum_j x_j\int(x_i'J_j - x_j'J_i)\mathrm{d}^3x' = -\frac{1}{2}\left[\mathbf{x}\times\int(\mathbf{x}'\times\mathbf{J})\mathrm{d}^3x'\right]_i.$$

We can then define

$$\mathcal{M}(\mathbf{x}) = \frac{1}{2}(\mathbf{x}\times\mathbf{J}(\mathbf{x})), \quad \mathbf{m} = \frac{1}{2}\int\mathbf{x}'\times\mathbf{J}(\mathbf{x}')\mathrm{d}^3x' \implies \mathbf{A}_{\text{dipole}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3} \implies \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3}\right].$$

The magnitude of a planar current loop is evidently $|\mathbf{m}| = I\cdot(\text{Area})$. Lastly, at the origin, we write

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3} + \frac{8\pi}{3}\mathbf{m}\delta(0)\right],$$

similar to the electric dipole counterpart (Jackson 5.6).

In general, the force term is, with $B_k(\mathbf{x}) = B_k(0) + \mathbf{x}\cdot\nabla B_k(0) + \cdots$,

$$F_i = \sum_{jk}\epsilon_{ijk}\int J_j(\mathbf{x}')\mathbf{x}'\cdot\nabla B_k(0)\mathrm{d}^3x' + \cdots = \sum_{jk}\epsilon_{ijk}(\mathbf{m}\times\nabla)_j B_k(\mathbf{0}) \implies \mathbf{F} = (\mathbf{m}\times\nabla)\times\mathbf{B} = \nabla(\mathbf{m}\cdot\mathbf{B}).$$

Likewise,

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}(0), \quad U = -\mathbf{m}\cdot\mathbf{B}.$$

This has many effects in quantum mechanics; hyperfine splitting for example.

3.3 Macroscopic Media

Defining the magnetization $\mathbf{M}(\mathbf{x}) = \sum_i N_i\langle\mathbf{m}_i\rangle$, we have

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]\mathrm{d}^3x' = \frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}(\mathbf{x}') + \nabla'\times\mathbf{M}(\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'.$$

We define the magnetic field strength $\mathbf{H}$:

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \implies \nabla\times\mathbf{H} = \mathbf{J}.$$

The boundary conditions are such that

$$(\mathbf{B}_2 - \mathbf{B}_1)\cdot\mathbf{n} = 0, \quad \mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}.$$

Magnetization Problems With $\mathbf{M}$ given and $\mathbf{J}=0$, we have Poisson's equation;

$$\Phi_M(\mathbf{x}) = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x' = \frac{\mathbf{m}\cdot\mathbf{x}}{4\pi r^3}.$$

If there's a discontinuity in the magnetization at the surface, we write

$$\Phi_M(\mathbf{x}) = -\frac{1}{4\pi}\int_V\frac{\nabla'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S\frac{\mathbf{n}'\cdot\mathbf{M}(\mathbf{x}')\mathrm{d}a'}{|\mathbf{x}-\mathbf{x}'|},$$

with $n\cdot\mathbf{M}$ acting as the magnetic surface-charge density.

For example, Jackson 5.10, 5.11, 5.12

4 Electrodynamics

While the statics solution was valid for $\nabla\cdot\mathbf{J}=0$, by making the generalization $\mathbf{J}\to\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$, we get

$$\nabla\cdot\mathbf{D}=\rho,\quad \nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t},\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}.$$

4.1 Potentials

The magnetic Gauss' and Faraday's law can be solved with scalar and vector potentials:

$$\mathbf{B}=\nabla\times\mathbf{A},\quad \mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}.$$

We can make a gauge transformation $\mathbf{A}\to\mathbf{A}+\nabla\Lambda$ and $\Phi\to\Phi-\frac{\partial\Lambda}{\partial t}$. If we choose the gauge such that $\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0$, we get

$$\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-\rho/\epsilon_0,\quad \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}.$$

Furthermore, Λ that satisfies the Lorenz condition belongs to the Lorenz gauge.

4.2 Poynting's theorem

The rate of work (no B!) can be written as an integral of $\mathbf{J}\cdot\mathbf{E}$. If the medium is linear in μ,ϵ, we define the energy density and obtain

$$u=\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}) \implies \frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=-\mathbf{J}\cdot\mathbf{E}.$$

$\mathbf{S}$, the energy flow, is called the Poynting vector. Poynting's theorem expresses

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\oint_S\mathbf{n}\cdot\mathbf{S}\mathrm{d}a=\frac{\mathrm{d}}{\mathrm{d}t}(E_{\text{mech}}+E_{\text{field}}),\quad E_{\text{field}}=\frac{\epsilon_0}{2}\int_V(\mathbf{E}^2+c^2\mathbf{B}^2)\mathrm{d}^3x,\quad E_{\text{mech}}=\int_V\mathbf{J}\cdot\mathbf{E}\mathrm{d}^3x.$$

Similarly,

$$\sum_\beta\int_V\frac{\partial}{\partial x_\beta}T_{\alpha\beta}\mathrm{d}^3x=\oint_S\sum_\beta T_{\alpha\beta}n_\beta\mathrm{d}a=\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P}_{\text{mech}}+\mathbf{P}_{\text{field}})_\alpha,$$

where

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}=\int_V(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})\mathrm{d}^3x,\quad \mathbf{P}_{\text{field}}=\mu_0\epsilon_0\int_V\mathbf{E}\times\mathbf{H}\mathrm{d}^3x,\quad T_{\alpha\beta}=\epsilon_0\left[E_\alpha E_\beta+c^2B_\alpha B_\beta-\frac{1}{2}(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B})\delta_{\alpha\beta}\right].$$

Note that the momentum density is $\mathbf{g}=\frac{1}{c^2}\mathbf{E}\times\mathbf{H}=\frac{1}{c^2}\mathbf{S}$.

5 Lecture 24 (Nov. 19) - Radiation and Causality

Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu.$$

In Lorenz gauge,

$$\partial_\mu A^\mu = 0 \implies \partial_\mu F^{\mu\nu} = \Box A^\mu - \partial^\nu (\cancel{\partial_\mu A^\mu}) \implies \Box A^\nu = \frac{4\pi}{c} J^\nu.$$

This is the electromagnetic wave equation.

Green's Function Let us solve for the Green's function. The equation we'd like to solve is

$$\Box_z D(z) = \delta^{(4)}(z).$$

Working in the momentum space,

$$D(z) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 k \tilde{D}(k) e^{-ikz}, \quad \delta^{(4)} = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 k e^{-ikz}, \quad \Box = -k_\mu k^\mu,$$

we obtain

$$\tilde{D}(k) = -\frac{1}{k_\mu k^\mu} \implies D(z) = -\frac{1}{(2\pi)^4} \int \mathrm{d}^3 k_s e^{-i\mathbf{k}\cdot\mathbf{z}} \int_{\mathbb{R}} \mathrm{d}k_0 \frac{e^{-ikz_0}}{k_0^2 - k_s^2}.$$

For the advanced propagator prescription ($\mathrm{d}k_0$ integral runs above the real axis), we have,

$$\int_{\mathbb{R}} \mathrm{d}k_0 \frac{e^{-ikz_0}}{k_0^2 - k_s^2} = \begin{cases} 0 & z_0 < 0 \\ \frac{2\pi}{k_s} \sin(k_s z_0) & z_0 > 0 \end{cases}.$$

Note that $z_0 < 0$ corresponds to closing the contour on the upper half plane, and $z_0 > 0$ corresponds to closing the contour on the lower half plane. Therefore,

$$\begin{aligned} D(z) &= \frac{1}{(2\pi)^4} \int \mathrm{d}^3 k_s \Theta(z_0) \left[\frac{2\pi}{k_s} \sin(k_s z_0) \right] e^{-i\mathbf{k}_x\cdot\mathbf{z}} \\ &= \frac{\Theta(z_0)}{(2\pi)^3} \int \mathrm{d}k_s k_s^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi e^{-ik_s z\cos\theta} \frac{\sin(k_s z_0)}{k_s} \\ &= \frac{\Theta(z_0)}{2\pi^2 z} \int_{-\infty}^{\infty} \mathrm{d}k_s \sin(k_s z) \sin(k_s z_0) \\ &= \frac{\Theta(z_0)}{4\pi z} \delta(z_0 - z). \end{aligned}$$

Note that I've been using z as $|\mathbf{z}|$.

With $\delta(z_0^2 - z^2) = \frac{1}{2z}[\delta(z_0 + z) + \delta(z_0 - z)]$, we can finally write

$$D(z) = \frac{\Theta(z_0)\delta(z_\mu z^\mu)}{2\pi}.$$

Moving Charge, Causality Consider a moving charge. Its four-current can be written

$$J^\nu = e\int \mathrm{d}\tau \cdot cv^\nu(\tau)\delta^{(4)}(x - r(\tau)).$$

Using the Green's function, we obtain

$$\begin{aligned}
A^\nu(x) &= \frac{4\pi}{c}\int \mathrm{d}^4x' J^\nu(x')D(x-x') \\
&= \frac{4\pi}{c}\int \mathrm{d}^4x' \Theta(x_0 - x_0')\delta((x-x')^2)\int \mathrm{d}\tau \cdot cev^\alpha(\tau)\delta^{(4)}(x' - r(\tau)) \\
&= 2\int \mathrm{d}\tau \cdot cev^\nu(\tau) \cdot \Theta(x_0 - r_0(\tau)) \cdot \delta((x - r(\tau))^2)
\end{aligned}$$

We note that

- The delta function ensures that only light-like separation contributes to A;
- The theta function ensures causality.

Lienard-Wiechert Potential The potential of a moving charge is given

$$A^\mu = -e\frac{v^\nu}{v_\mu(x - r(\tau)^\mu}$$