Notes on Quantum Field Theory

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1 Scalar Field Theory

1.1 Motivation, Causality

Lec 1, P&S 2.1

The field viewpoint is needed because quantum mechanics breaks causality. This can be shown from:

1. Explicitly calculating the propagator between $\mathbf{x}$ and $\mathbf{x}_0$. For rational function of x and t, $U(t) \sim e^{-m\sqrt{x^2-t^2}}$ with no cutoff[1].
2. Recalling that all Hermitian operators are observable; in relativity, this cannot be true because 2 spatially separated observables cannot affect each other. In other words, $[O_1, O_2] \neq 0$ for spatially separated O_1, O_2.

With Lorentz transformations, we can prove that QFT is **causal** - in other words, if $(x-y)^2 < 0$, then $[O(x), O(y)] = 0$, or spacelike operators commute. To see this, we let $\phi(x) = \phi^-(x) + \phi^+(x)$, where each term corresponds to the creation and annihilation terms;

$$\phi^-(x) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} a_k e^{-ik\cdot x}, \quad \phi^+(x) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} a_k^\dagger e^{ik\cdot x}.$$

Then we can rewrite the commutator as

$$\begin{aligned}[\phi(x), \phi(y)] &= \left[\phi^-(x) + \phi^+(x), \phi^-(y) + \phi^+(y)\right] = \left[\phi^-(x), \phi^+(y)\right] + \left[\phi^+(x), \phi^-(y)\right] \\ &= \phi^-(x)\phi^+(y) - \phi^-(y)\phi^+(x).\end{aligned}$$

However, $\phi^-(x)\phi^+(y) = \phi^-(y)\phi^+(x)$ if $(x-y)^2 < 0$, because there's a proper Lorentz transformation between $x-y$ and $y-x$. One could think that causality is restored as the amplitude of the particle propagating from y to x is exactly canceled out by the antiparticle propagating from x to y.

1.2 Formulation

Lec 2, 3, 4, 5

In QFT, observables are operator valued fields. We start with scalar fields - fields that are invariant under Lorentz transformations; $\phi'(x') = \phi(x)$, where $x' = \Lambda x$.

We start by requiring that the action is

1. Lorentz invariant;
2. Causal - S is local, $S = \int \mathrm{d}^4 L(\phi, \partial_\mu \phi)$, with $\phi(x)$ only, no $\phi(y)$;
3. Corresponding to EoM with 2nd order in time derivative - only possible terms are ϕ^2 and $(\partial_\mu\phi)^2$ (note that higher order time derivatives would be one higher in length scale, $L \sim 10^{-35}$ m.)

This is sufficient to write down the general form of the action:

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right).$$

From the Euler-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0 \implies \partial_\mu\partial^\mu\phi + m^2\phi = 0.$$

This is known as the Klein-Gordon equation.

To solve the Klein-Gordon equation, we can try

$$\phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\phi}(\mathbf{k}, t) \implies \frac{\partial^2\tilde{\phi}}{\partial t^2} + \omega_k^2\tilde{\phi} = 0,$$

where $\omega_k^2 = k^2 + m^2$. Further imposing the realness conditions, we have $\tilde{\phi}(\mathbf{k}, t) = \tilde{\phi}^*(-\mathbf{k}, t)$, giving us

$$\phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 \cdot 2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^* e^{ik\cdot x}\right), \tag{1.1}$$

where we have defined $k^\mu = (\omega_k, \mathbf{k})$ and $x^\mu = (t, \mathbf{x})$. Note that the factors of $\sqrt{2\omega_k}$ and a_k, a_k^* are constants chosen for convention - this is called Harmonic normalization (this normalizes $\langle \mathbf{p}|\mathbf{q}\rangle$. Note that $\delta(\mathbf{p})$ has units $\mathbf{p}^{-1}$ so it's not Lorentz invariant)

For full quantinization, we obtain $\pi(x)$, the canonical momentum field, as $\partial_{\dot{\phi}}\mathcal{L} = \pi$. For the Klein-Gordon theory,

$$\pi(\mathbf{x}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 \cdot 2\omega_k}} \left(a_k(-i\omega_k)e^{-ik\cdot x} + a_k^\dagger(i\omega_k)e^{ik\cdot x}\right). \tag{1.2}$$

Imposing the canonical commutation relations[2] (add stuff):

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0, \quad [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y}), \tag{1.3}$$

we solve for the ladder operators[3]

$$a_k = i\int \frac{\mathrm{d}^3x}{\sqrt{(2\pi)^3 \cdot 2\omega_k}} \cdot e^{ik\cdot x}\left[\dot{\phi}(x) - i\omega_k\phi(x)\right], \quad a_k^\dagger = -i\int \frac{\mathrm{d}^3x}{\sqrt{(2\pi)^3 \cdot 2\omega_k}} \cdot e^{-ik\cdot x}\left[\dot{\phi}(x) + i\omega_k\phi(x)\right].$$

[1] https://physics.stackexchange.com/a/105049
[2] https://physics.stackexchange.com/a/573940
[3] https://physics.stackexchange.com/q/304539

We see that the ladder operators

$$\left[a_k, a_k^\dagger\right] = \delta(\mathbf{k} - \mathbf{k}'). \tag{1.4}$$

With the conjugate momentum defined, we can write the Hamiltonian:

$$H = \int \mathrm{d}^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi\right] = \int \mathrm{d}^3k \cdot \frac{1}{2}\omega_k(a_k a_k^\dagger + a_k^\dagger a_k).$$

Note that $a_k a_k^\dagger = \delta(0)$ blows up. This, analogous to summing the vacuum energy density over all space and modes, is known as the cosmological constant[4] and can be canceled in the absence of gravity. This operation of canceling out the vacuum energy is called the **normal ordering** - when acted on a product of annihilation and creation operators, all the creation operators come to the front - for example, $: aa^\dagger := a^\dagger a$, and so on.

The space of states in QFT is called the Fock space. The particle states $|k\rangle = a_k^\dagger |0\rangle$ are orthonormal and complete:

$$\langle \mathbf{k}'|\mathbf{k}\rangle = \delta(\mathbf{k} - \mathbf{k}'), \quad \mathbb{1} = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} |\mathbf{k}\rangle \langle \mathbf{k}|.$$

Addendum about translation operator, Lec. 4

Lorentz transformations The Lorentz transformations have a unitary representation $U(\Lambda)|k\rangle = |\Lambda k\rangle$ in the relativistic normalization of states - that is, there exists $U(\Lambda)$ such that $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$ and $U^\dagger U = \mathbb{1}$. We see that the following conditions are satisfied:

1. Representation condition: $U(\Lambda_1)(\Lambda_2)|k\rangle = U(\Lambda_1\Lambda_2)|k\rangle$, follows from the definition;
2. Unitarity: First, note that relativistic integrand is Lorentz invariant, as $\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} = \mathrm{d}^4k\delta(k^2 - m^2)$. Then, from the completeness relation,

$$UU^\dagger = \int \frac{\mathrm{d}^3}{(2\pi)^3 2\omega_k} |\Lambda k\rangle \langle \Lambda k| = \int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega_k} |k'\rangle \langle k'| = \mathbb{1},$$

where $k' = \Lambda k$.

Note that relativistic creation/annihilation operators and field operators follow

$$\alpha^\dagger(\Lambda k) = U(\Lambda)\alpha_k^\dagger U(\Lambda)^{-1}, \quad U\phi U^\dagger = \phi(\Lambda x).$$

In other words, $\phi(x)$ transforms like a scalar.

1.3 Symmetries and Conservation Laws

1.3.1 Continuous Symmetries

Lec 6,

As far as symmetries and conservation laws are concerned, we note that

1. Symmetries have unitary representations[5]; for $U(R)$ corresponding to rotation R, $U(R_1)U(R_2) = U(R_1R_2)$ and $U^\dagger U = \mathbb{1}$.
2. Symmetries should leave the dynamics unchanged; for $|\psi\rangle = U|\phi\rangle$, $e^{-iHt}|\psi\rangle = Ue^{-iHt}|\phi\rangle \implies U^\dagger HU = H$.

For a continuous symmetry, we can also write $R \sim \mathbb{1} + i\epsilon T$, where T is the infinitesimal generator of symmetry. We can also relate infinitesimal symmetry to a discrete one by exponentiating - $R = e^{i\epsilon T}$. The generators are a Lie algebra, closed under the Lie bracket; $\left[T^a, T^b\right] = if^{abc}T^c$, where f^{abc} is the structure constant; in SO(3), this is the Levi-Civita symbol. The generators for a unitary transformation are called charge operator:

$$U(\mathbb{1} + i\epsilon T) = 1 + i\epsilon Q, \quad U^{-1} = U^\dagger \implies Q = Q^\dagger.$$

Further imposing the dynamics condition, we have $[Q, H] = 0$.

Furthermore, the finite transformations $R = e^{i\epsilon T}$ form a Lie group - they are related to Lie algebra by the BCH formula. Lie groups are also closed under (matrix) multiplication; $R(h)R(g) = R(h \circ g) \implies e^{i\epsilon T^a}e^{i\epsilon T^b} = e^{i\epsilon T^c}$.

1.3.2 Conservation Laws

Recall that if the Lagrangian changes by a total derivative, the action doesn't change. Explicitly, we have

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\mu\delta\phi^a = \partial_\mu F^\mu \implies \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a - F^\mu\right) \equiv \partial_\mu J^\mu = 0.$$

We define J^μ to be the **Noether current**. The corresponding conserved charge can be found $Q = \int \mathrm{d}^3x J^0$.

As an example, we look at **translational symmetries**. If the action has no explicit x dependency, we can vary the fields

$$\phi^a(x^\mu + \epsilon a^\mu) \sim \phi^a(x^\mu) + \epsilon a^\mu\partial_\mu\phi^a, \quad \mathcal{L}(x^\mu + \epsilon a^\mu) \sim \mathcal{L}(x) + \epsilon a^\mu\partial_\mu\mathcal{L}.$$

Hence, $\delta\phi^a = a^\mu\partial_\mu\phi^a$, and $F^\mu = a^\mu\mathcal{L}$. The corresponding Noether current can be found to be

$$J^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^a)}\delta\phi^a - F^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^a)}a^\mu\partial_\mu\phi^a - a^\nu\mathcal{L} = a^\mu T^\nu_\mu, \quad T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi - \delta^\nu_\mu\mathcal{L}.$$

We define T^μ_ν to be the **stress-energy tensor**. From $\partial_\nu J^\nu = 0$, it holds that $\partial_\nu T^\mu_\nu = 0$ as well. The conserved charge, or the 4-momentum in this case, can be found

$$\int \mathrm{d}^3x T^{\mu 0} = p^\mu, \quad \partial_t p^\mu = 0.$$

As an example, for a real scalar field,

$$: H := \int \mathrm{d}^3k \cdot \omega_k a_k^\dagger a_k, \quad : p^\mu := \int \mathrm{d}^3k \cdot k^\mu a_k^\dagger a_k.$$

[4]Turns out, the observed cosmological constant is 10^{-120} times smaller - this is known as the cosmological constant/fine tuning problem.

[5]Stated by Wigner's theorem.

1.3.3 Discrete Symmetries

2nd half Lec 7

There also exists discrete symmetries associated with Lorentz transformations, defined $\Lambda^\intercal g \Lambda = g$. The first is the **parity reversal**, defined, for a scalar field, as a unitary, linear operator that satisfies

$$U_p : \phi(t, \mathbf{x}) \mapsto \phi(t, -\mathbf{x}) \implies U_p : a_k, a_k^\dagger \mapsto a_{-k}, a_{-k}^\dagger$$

where we let $\mathbf{k} \to -\mathbf{k}$ in the field integral.

Next, we have the **time reversal**, defined as a unitary, anti-linear operator that satisfies

$$U_t : \phi(t, \mathbf{x}) \mapsto \phi^*(-t, \mathbf{x}) \implies U_t : a_k, a_k^\dagger \mapsto a_{-k}, a_{-k}^\dagger,$$

where $k \sim \dot{x} \to -k$ and $i \to -i$ in this transformation.

Lastly, the **charge conjugation** is a unitary operator, defined for complex fields, that satisfies

$$U_c : \psi(t, \mathbf{x}) \mapsto \psi^\dagger(t, \mathbf{x}), \quad U_c : b_k, b_k^\dagger \mapsto c_k, c_k^\dagger.$$

For non-interacting theories, all combinations of C, P, and T are symmetries. However, for interacting theories, C, P, and CP can be broken, but never **CPT**.

1.4 Complex Scalar Fields

1st half Lec 7,

While we have found symmetries with x-dependence so far, another group of symmetries, called **internal symmetries** exist. As a motivation, consider a system consisting of two real (independent), identical scalar fields:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^1)^2 + \frac{1}{2}(\partial_\mu \phi^2)^2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2.$$

It is possible to "rotate" - $U(1) \sim SO(2)$ symmetry two fields into each other: $\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}$. The current and charge associated with this symmetry is

$$J^\mu = (\partial^\mu \phi^1)\phi^2 - (\partial^\mu \phi^2)\phi^1, \quad Q = -i \int \mathrm{d}^3k (a_k^{2\dagger} a_k^1 - a_k^{1\dagger} a_k^2).$$

Because $[H, Q] = 0$, it is possible to diagonalize the charge operator; let

$$b_k = \frac{1}{\sqrt{2}}(a_k^1 + i a_k^2), \quad c_k = \frac{1}{\sqrt{2}}(a_k^1 - i a_k^2).$$

whThe commutation relations still hold:

$$[b, b] = [c, c] = \left[b, c^\dagger\right] = 0,$$

and each operator corresponds to creating/annihilating a $+1$ charge $(b, b^\dagger)$ or a -1 charge $(c, c^\dagger)$ particle.

With these new definitions, we can write

$$Q = \int \mathrm{d}^3k (b_k^\dagger b_k - c_k^\dagger c_k), \quad : H := \int \mathrm{d}^3k \cdot \omega_k (b_k^\dagger b_k + c_k^\dagger c_k).$$

Furthermore, we can define the field operator as

$$\psi = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2), \quad \psi = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}}(b_k e^{ik\cdot x} + c_k^\dagger e^{-ik\cdot x}), \quad \psi^\dagger = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}}(b_k^\dagger e^{-ik\cdot x} + c_k e^{ik\cdot x}),$$

with commutation relations

$$[\psi, \psi] = \left[\psi^\dagger, \psi^\dagger\right] = 0, \quad \left[\psi(x,t), \dot{\psi}^\dagger(y,t)\right] = i\delta(x - y).$$

The ψ field creates an anti-particle $(c^\dagger)$ and annihilates particle (b), and the opposite for $\tilde{\psi}$.

As a side note, when written as a complex scalar, the "rotation internal symmetry" becomes a simple phase factor - they are $SO(2) \sim U(1)$ symmetries - $R\psi = \psi e^{-i\theta}$, where R is the motivating rotation matrix.

We can write the Lagrangian density and the rotation symmetry:

$$\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi, \quad J_\mu = i\left[\psi \partial_\mu \psi^\dagger - \psi^\dagger \partial_\mu \psi\right].$$

2 Interacting Fields

2.1 Formulation

Lec 8,9

For a system with an interaction, term, the Hamiltonian can be written $H = H_0 + H_{\text{int}}$. We define the interaction action field to be

$$\phi_I(t, \mathbf{x}) = e^{iH_0(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH_0(t-t_0)} = e^{iH_0(t-t_0)}\left(e^{-iH(t-t_0)}\phi(t, \mathbf{x})e^{iH(t-t_0)}\right)e^{-iH_0(t-t_0)}.$$

In other words, the interaction field is what the field would evolve (from t_0 to t) if it weren't for the interacting terms. Now, we define the time evolution operator, which gives a concise expression of the full field:

$$U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \implies \phi(t, \mathbf{x}) = U^\intercal(t, t_0)\phi_I(t, \mathbf{x})U(t, t_0).$$

Taking the derivative of the time evolution operator,

$$i\frac{\partial}{\partial t}U(t, t_0) = \left(e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}\right)U(t, t_0) = H_I(t)U(t, t_0),$$

we can see that $H_I(t)$ is the interaction Hamiltonian written in the interaction picture - evolving with the free Hamiltonian. The solution for this differential equation - with $U(t_0, t_0) = 1$ - is a power series:

$$\begin{aligned} U(t, t_0) =& 1 + (-i)\int_{t_0}^{t} \mathrm{d}t_1\, H_I(t_1) + (-i)^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2 H_I(t_1)H_I(t_2) \\ &+ (-i)^3\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\int_{t_0}^{t_2}\mathrm{d}t_3\, H_I(t_1)H_I(t_2)H_I(t_3) + \cdots. \end{aligned}$$

Note that the successive integration limits get smaller and the interaction Hamiltonians are time-ordered. To get rid of the time-ordering operators, we note that

$$\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\cdots\int_{t_0}^{t_{n-1}}\mathrm{d}t_n\, H_I(t_1)\cdots H_I(t_n) = \frac{1}{n!}\int_{t_0}^{t}\mathrm{d}t_1\cdots\mathrm{d}t_n T\left[H_I(t_1)\cdots H_I(t_n)\right].$$

Refer to Peskin 4.21 for a "proof"[6] - it is analogous to finding the volume of an n-simplex. With this, we can write $U(t, t_0)$ in a compact form:

$$\begin{aligned} U(t, t_0) &= 1 + (-i)\int_{t_0}^{t}\mathrm{d}t_1\, H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t}\mathrm{d}t_1\mathrm{d}t_2\, T[H_I(t_1)H_I(t_2)] + \cdots \\ &= T\exp\left(-i\int_{t_0}^{t}\mathrm{d}t' H_I(t')\right) = T\exp\left[i\int_{t_0}^{t}\int\mathrm{d}^3x'\mathcal{L}'(\phi_I(\lambda'))\right] \end{aligned}$$

Note that

- In this form, it is evident (from the limits of integrals) that $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$, and $U(t_1, t_3)U^\dagger(t_2, t_3) = U(t_1, t_2)$;
- The time ordering operator is invariant - order of events doesn't change for timelike events and all ϕ commute for spacelike events - the Lagrangian is invariant, and the integral $\int\mathrm{d}t\int\mathrm{d}x'$ is invariant if we let t, t_0 to $\pm\infty$.

We first claim that as $t \to \pm\infty$, interactions effectively turn off; asymptotically, we will have free field/particles, each state approaching the asymptotic state[7]. If we let $|\psi\rangle$ as the free states and $|\psi\rangle_{\text{in}}$ and $|\psi\rangle_{\text{out}}$ as the interacting states, we have

$$\lim_{t\to-\infty}e^{-iHt}\,|\psi(t=0)\rangle_{\text{in}} = e^{-iH_0t}\,|\psi(t=0)\rangle\,, \quad \lim_{t\to\infty}e^{-iHt}\,|\psi(t=0)\rangle_{\text{out}} = e^{-iH_0t}\,|\psi(t=0)\rangle\,.$$

In other words,

$$|\psi\rangle_{\text{in}} = \lim_{t\to-\infty}e^{iHt}e^{-iH_0t}\,|\psi\rangle\,, \quad |\psi\rangle_{\text{out}} = \lim_{t\to\infty}e^{iHt}e^{-iH_0t}\,|\psi\rangle\,.$$

We're interested in scattering amplitudes, defined as the probability of measuring $|\chi\rangle_{\text{out}}$ given an in state $|\psi\rangle_{\text{in}}$. We have

$$\langle\chi_{\text{out}}|\psi_{\text{in}}\rangle = \lim_{t'\to-\infty}\lim_{t\to\infty}\langle\chi|\,U(t, t')\,|\psi\rangle = S_{\chi\psi},$$

where we define $S = \lim_{t,t\to\pm\infty}U(t, t')$, which is also called the S or scattering matrix. The S-matrix is unitary; $S^\dagger S = \mathbb{1}$, and commutes with/preserves free energy; $[S, H_0] = 0$. Therefore,

$$S = T\exp\left[i\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(\Phi_I)\right]. \tag{2.1}$$

This is known as Dyson's equation.

Recall that Φ_I are fields that can be expressed in ladder operators, which are easier to work with in normal ordering. For this, we introduce **Wick's theorem**, a theorem that relates time ordering T to normal ordering ::. First, we define the **contraction**, defined

$$T(\phi^a(x)\phi^b(y)) =: \phi^a(x)\phi^b(y) : + \overbracket{\phi^a(x)\phi^b}(y).$$

We can find an explicit form for the contraction:

- For the $x^0 > y^0$ case, we have
$$T[\phi^a(x)\phi^b(y)] = \phi^a(x)\phi^b(y) = \left(\phi^{a+}(x) + \phi^{a-}(x)\right)\left(\phi^{b+}(x) + \phi^{b-}(y)\right) =: \phi^a\phi^b : + \delta_{ab}\Delta_+(x-y),$$
- For $x^0 < y^0$, we have
$$T[\phi^a(x)\phi^b(y)] = :\phi^a\phi^b: + \,\delta_{ab}\Delta_+(y-x).$$

[6] http://scipp.ucsc.edu/~haber/ph215/TimeOrderedExp.pdf

[7] This really isn't true because of self and vacuum interactions, but we will come back here for renormalization.

Hence, we obtain

$$\overbracket{\phi^a(x)\phi^b}(y) = \delta_{ab}\int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik(x-y)}\frac{i}{k^2-m^2+i\epsilon}.$$

Now, we can state Wick's theorem:

$$\begin{aligned} T(\phi_1\cdots\phi_n) = {} & :\phi_1\cdots\phi_n: \\ & + \overbracket{\phi_1\phi_2} :\phi_3\cdots\phi_n: + \overbracket{\phi_1\phi_3} :\phi_2\phi_4\cdots\phi_n: + \cdots \\ & + \overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4} :\phi_5\cdots\phi_n: \\ & \vdots \end{aligned} \tag{2.2}$$

We prove this by induction. Consider n fields $\phi_1,\cdots,\phi_n$ with times $\phi_1^0 \geq \phi_2^0 \geq,\cdots \phi_n^0$. If $\phi_2\cdots\phi_n$ were in Wick-expression - $T\phi_1\phi_2 =: \phi_1\phi_2 : + \overbracket{\phi_1\phi_2}$), which is the zeroth term, is in Wick ordering by definition - we can multiply by ϕ_1 on the left. Since all fields are time-ordered already,

$$T(\phi_1\cdots\phi_n) = \phi_1(:\phi_2\cdots\phi_n: + \overbracket{\phi_2\phi_3}:\phi_4\cdots\phi_n: + \cdots).$$

Now, let ϕ_1^+ and ϕ_1^- be the creation/annihilation parts of ϕ_1. Then, we have

$$T(\phi_1\cdots\phi_n) = \phi_1^+ :\phi_2\cdots\phi_n: + \phi_1^- :\phi_2\cdots\phi_n: + \phi_1 + (\phi_1^+ + \phi_1^-)\overbracket{\phi_2\phi_3}:\phi_4\cdots\phi_n: + \cdots.$$

Let's focus on the non-contraction terms for a minute. They can be written

$$\phi_1^+{:}\phi_2\cdots\phi_n{:} + \phi_1^-{:}\phi_2\cdots\phi_n{:} = {:}\phi_1^+\phi_2\cdots\phi_n{:} + {:}\phi_2\cdots\phi_n\phi_1^-{:} + \left[\phi_1^-, {:}\phi_2\cdots\phi_n{:}\right].$$

From definition, the first two terms are ${:}\phi_1\phi_2\cdots\phi_n{:}$. The commutator can be explicitly evaluated:

$$\begin{aligned} \left[\phi_1^-, {:}\phi_2\cdots\phi_n{:}\right] &= \left[\phi_1^-, \phi_2^+{:}\phi_3\cdots\phi_n{:}\right] + \left[\phi_1^-, \phi_2^-{:}\phi_3\cdots\phi_n{:}\right] \\ &= \left[\phi_1^-, \phi_2^+\right]{:}\phi_3\cdots\phi_n{:} + (\phi_2^+ + \phi_2^-)\left[\phi_1^-, {:}\phi_3\cdots\phi_n{:}\right] + \cancel{\left[\phi_1^-, \phi_2^-\right]{:}\phi_3\cdots\phi_n{:}}. \end{aligned}$$

Using the fact that $\left[\phi_1^-, \phi_2^+\right] = \overbracket{\phi_1\phi_2}$ for $x_1^0 > x_2^0$, we get

$$\left[\phi_1^-, {:}\phi_2\cdots\phi_n{:}\right] = \sum_{i=2}^{n} \phi_2\cdots\phi_{i-1}\overbracket{\phi_1\phi_i}\phi_{i+1}\cdots\phi_n.$$

Keeping in mind that all contractions are complex numbers, we can deduce that the contraction terms are

$$(\phi_1^+ + \phi_1^-)\overbracket{\phi_2\phi_3}{:}\phi_4\cdots\phi_n{:} + \cdots = \overbracket{\phi_2\phi_3}{:}\phi_1\phi_4\cdots\phi_n{:} + \overbracket{\phi_2\phi_3}\sum_{i=4}^{n}\phi_4\cdots\phi_{i-1}\overbracket{\phi_1\phi_i}\phi_{i+1}\cdots\phi_n + \cdots.$$

In other words, multiplying a normal ordered sequence by ϕ_1 is the normal ordering sequence with ϕ_1 and all single contractions with ϕ_1. Hence Wick's theorem is proved.

2.1.1 Interaction with External Field

Lec 9, 10

Let the interacting Hamiltonian of the external field be $H' = \lambda\phi(x)\rho(x)$. For boundary conditions, we let $\rho(x) \to 0$ for $t \pm \infty$. From Dyson's formula, we have

$$S = T\sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!}\left(\int \rho(x)\phi(x)\cdot \mathrm{d}^3x\right)^n.$$

From Wick's formula, we have to sum over every possible number of contractions:

$$\sum_{n,p}\frac{(-i\lambda)^n}{n!}\left(\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\rho(x_1)\rho(x_2)\overbracket{\phi(x_1)\phi(x_2)}\right)^p : \left(\int \mathrm{d}^4x_3\rho(x_3)\phi(x_3)\right)^{n-2p} : \frac{n!}{2^p(n-2p)!p!}.$$

Note that the combinatoric factor can be considered by choosing p pairs from n objects:

$$\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-(2p-2)}{2}\cdot\frac{1}{p!} = \frac{n!}{2^p(n-2p)!p!}.$$

If we redefine $k = n-2p$, we get

$$S = A{:}\exp\left(-i\lambda\int \mathrm{d}^4x\,\rho(x)\phi(x)\right){:}, \quad A = \exp\left(\frac{1}{2}(-i\lambda)^2\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\rho(x_1)\rho(x_2)\overbracket{\phi(x_1)\phi(x_2)}\right).$$

The Fourier transformation of the $\rho(x)$ is $\rho(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\rho}(k)e^{-ikx}$, we have

$$\int \mathrm{d}^4x\,\rho(x)\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}(\tilde{\rho}(-k)a_k + \tilde{\rho}(k)a_k^\dagger)$$

From this, we can calculate the probability of creating n particles: $\langle k_1\cdots k_n|\,S\,|0\rangle$; since the S matrix is acting on the vacuum state, we want to have n creation operators that exactly match the momenta $k_1\cdots k_n$;

$$\langle k_1\cdots k_n|\,S\,|0\rangle = A\tilde{\rho}(k_1)\cdots\rho(k_n)(-i\lambda)^n \implies \langle k_1|\,S\,|0\rangle = -i\lambda A\tilde{\rho}(k_1).$$

Recall that $\langle k_1 k_2 | S |0\rangle \sim \tilde{\rho}(k_1)$, which is the Fourier transform of the source. For a static source, no particles are created, but the ground/vacuum energy changes:

$$\langle 0| S |0\rangle = A = \exp\left(\frac{(-i\lambda)^2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\left|\tilde{f}(k^0)\right|^2 |\tilde{\rho}(k)|^2 \frac{i}{k^2 - m^2 + i\epsilon}\right) = \exp\left(-i(\gamma_{\text{on}} + \gamma_{\text{off}} + E_0 T)\right).$$

This can be interpreted as the phase from turning the source on/off plus the phase from $E_0 T$, where T is the duration of the source. We have

$$E_0 = \frac{(-i\lambda)^2}{2T}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\left|f(k^0)\right|^2 |\tilde{\rho}(\mathbf{k})|^2 \frac{i}{k^2 - m^2 + i\epsilon} + \cancel{\frac{-\gamma_{\text{on}} - \gamma_{\text{off}}}{T}}.$$

From Parseval's theorem[8], we have

$$E_0 = -\frac{\lambda^2}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^0} |\rho(\mathbf{k})|^2 \frac{1}{k^2 + m^2 - i\epsilon} = \frac{\lambda^2}{2}\int \rho(x_1)\rho(x_2) V(\mathbf{x}_1 - \mathbf{x}_2)\mathrm{d}^3 x_1 \mathrm{d}^3 x_2.$$

From the Fourier transform, we have

$$V(\mathbf{x}) = -\int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\mathbf{k}^2 + m^2 - i\epsilon} = -\int_{-\infty}^{\infty} = \frac{1}{(2\pi)^2}\frac{1}{ir}\frac{k\mathrm{d}k}{k^2 + m^2} e^{ikr} = -\frac{1}{4\pi r} e^{-mr}.$$

2.2 Feynman Diagrams

Lec. 11-12

We are often interested in the (nontrivial) elements of the S-matrix as it relates to the scattering amplitude of a specific process:

$$\langle f| S - \mathbb{1} |i\rangle = i(2\pi)^4 \delta(k_f - k_i) A_{fi},$$

where A_{fi} is the scattering amplitude; the momentum-conserving delta function follows from the translational invariance of the S-matrix.

Often, we represent scattering process with a Feynman diagram. To find the corresponding element of the S-matrix $\langle f| S - \mathbb{1} |i\rangle$,

1. Draw all topologically distinct diagrams;
2. For each vertex, add $(-i\lambda)\int \mathrm{d}^4 x$;
3. For each internal line, add $\overbracket{\phi(x_i)\phi(x_j)} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-ik(x_i - x_j)}}{k^2 - m^2 + i\epsilon}$;
4. For each incoming external line, add $e^{-ik_\alpha x_i}$; for each outgoing external line, add $e^{ik_\alpha x_i}$;
5. Divide by the symmetry factor.

In momentum space, the Feynman rules are much simpler; to find the scattering amplitude times i iA_{fi},

1. Draw all topologically distinct diagrams;
2. For each vertex, add $(-i\lambda)$;
3. For each internal line, add $\frac{i}{k^2 - m^2 + i\epsilon}$;
4. For each undetermined/loop momentum, integrate - $\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$;
5. Divide by the symmetry factor.

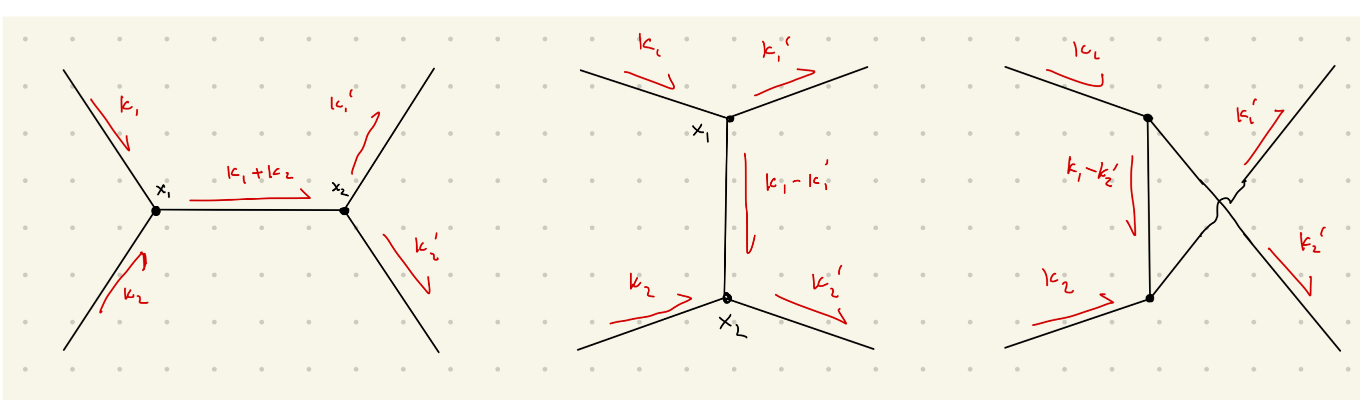

Figure 1: Feynman diagrams for the λ^2 order 2-2 scattering in $\lambda^3\phi^3/3!$ theory, also known as the S, T, and U channel

Example. Consider the λ^2 order 2-2 scattering in ϕ^3 theory with identical particles, as shown in figure 1. The total scattering amplitude is

$$A_{fi} = (-i\lambda)^2 \cdot i\left(\frac{1}{(k_1 + k_2)^2 - m^2 + i\epsilon} + \frac{1}{(k_1 - k_1')^2 - m^2 + i\epsilon} + \frac{1}{(k_1 - k_2')^2 - m^2 + i\epsilon}\right).$$

The scattering amplitude for these processes are often written in terms of the **Mandelstam variables**. To get an intuition, we consider an elastic collision in the center of mass frame. The momenta can be written

$$k_1 = (E, 0, 0, p), \quad k_1' = (E, p\sin\theta, 0, p\cos\theta), \quad k_2 = (E, 0, 0, -p), \quad k_2' = (E, -p\sin\theta, 0, -p\cos\theta),$$

[8] `https://mathworld.wolfram.com/ParsevalsTheorem.html`

where θ is the scattering angle. We now write the definition of the Mandelstam variables and write them in terms of E, p, and θ:

$$s = (k_1 + k_2)^2 = 4E^2 = E_T^2;$$
$$t = (k_1 - k_1')^2 = -4p^2 \sin^2\left(\frac{\theta}{2}\right) = -\mathbf{q}'^2$$
$$u = (k_1 - k_2')^2 = -4p^2 \cos^2\left(\frac{\theta}{2}\right) = -\mathbf{q}_c^2,$$

where E_T is the total energy, $\mathbf{q}$ is the momentum transfer from p_1 to p_1' and $\mathbf{q}_c$ is the momentum transfer in the cross channel. In terms of these variables, the scattering amplitude is

$$A_{fi} = -i\lambda^2\left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right) = -i\lambda^2\left(\frac{1}{E_T^2 - m^2} + \frac{1}{\mathbf{q}^2 - m^2} + \frac{1}{\mathbf{q}_c^2 - m^2}\right).$$

Note that

- The s-channel amplitude, while having no direct approximation in Born approximation, simplifies to the perturbation term.
- The t-channel scattering amplitude can be written

$$\frac{1}{\mathbf{q}^2 + m^2} = \int \mathrm{d}^3 x e^{i\mathbf{q}\cdot\mathbf{x}}\left(\frac{e^{-mr}}{4\pi r}\right);$$

  in other words, the t-channel amplitude is the Fourier transform (Born-Oppenheimer approximation) of the Yukawa potential;
- The u-channel amplitude is necessary for Bose symmetry;
- s,t,u are not independent; $s + t + u = 4m^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2$, in general;
- The scattering amplitude is invariant under $s \leftrightarrow t \leftrightarrow u$. This is called a **crossing symmetry**.

2.3 Physical Quantities

Lec. 13-14

Because the transition amplitude is not L^2, it is necessary to work in box normalization, characterized by

- Periodic boundary conditions: $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$;
- Box commutation relations: $\left[a_k^{\text{box}}, a_{k'}^{\dagger\text{box}}\right] = \delta_{kk'}$;
- State density $\mathrm{d}n = \frac{V}{(2\pi)^3}\mathrm{d}^3 k$.

Furthermore, the Feynman rules don't change in box normalization; only the external states are changed:

$$\langle 0|\,\phi(x)\,|k\rangle = e^{-ik\cdot x} \to \frac{1}{\sqrt{2\omega_k V}} e^{-ik\cdot x}.$$

Therefore,

$$\langle f|\, S - \mathbb{1}\,|i\rangle = i(2\pi)^4 A_{fi}\delta^{(4)}(k_{\text{in}} - k_{\text{out}})\prod_i \frac{1}{\sqrt{2E_i V}}$$
$$\implies |\langle f|\, S - \mathbb{1}\,|i\rangle|^2 = |A_{fi}|^2 (2\pi)^4 \delta(k_{\text{in}} - k_{\text{out}}) \cdot VT \cdot \prod_i \frac{1}{(2E_i V)}$$

where the index i is over all external states and we have used $\delta^2 k = \delta(k)\delta(0) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 x = \frac{VT}{(2\pi)^4}$. Furthermore, for a measurable quantity, we should cancel out V and T; the quantity we are interested in, the **differential transitional probability**, is obtained from the transitional probability by

- Dividing by T;
- Dividing by the flux of the initial particles;
- Multiplying by the density of the state factor for final states, $\mathrm{d}N = \frac{V}{(2\pi)^3}\mathrm{d}^3 k$,

which ensures that the quantities are intrinsic to the process.

$$V\prod_{\text{init.}} \frac{1}{(2E_i V)} |A_{fi}|^2 \cdot D^{(n)}, \quad D^{(n)} = \prod_i \left(\frac{\mathrm{d}^3 k_i}{(2\pi)^3 2E_i}\right) \cdot (2\pi)^4 \delta^{(4)}(k_{\text{in}} - k_{\text{out}}), \tag{2.3}$$

where $D^{(n)}$ is the **n-body phase space**, which is equal to the probability of scattering into a phase volume. We have that

1. The rate of a particle decaying into n is

$$\mathrm{d}\Gamma = \frac{1}{2E_i}|A_{fi}|^2 D^{(n)};$$

   from $\mathrm{d}\Gamma$, we obtain Γ, the decay width, from which we get $\tau = \frac{1}{\Gamma}$, the characteristic lifetime of a particle.
2. The scattering cross section of a two-particle initial state is

$$\mathrm{d}\sigma = \frac{1}{4E_T k_i}|A_{fi}|^2 D^{(n)},$$

   in the center of mass frame.

Example. Two-body phase space can be calculated (in the center of momentum frame),

$$D^{(2)} = \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(3)}(k_1 + k_2)\delta(E_1 + E_2 - E_T) = \frac{k^2 \mathrm{d}k \mathrm{d}\Omega}{(2\pi)^2 \cdot 2E_1 \cdot 2E_2} \delta(E_1 + E_2 - E_T).$$

Further writing $\delta(E)$, we have $\delta(E_1 + E_2 - E_T) = \frac{\delta(k - k_0)}{|\partial_k E_1 + \partial_k E_2|_{k=k_0}}$. With $\frac{\mathrm{d}E}{\mathrm{d}k} = \frac{k}{E}$, we have

$$D^{(2)} = \frac{k^2 \mathrm{d}\Omega}{k(E_1^{-1} + E_2^{-1}) \cdot 4\pi^2 \cdot 4E_1 E_2} = \frac{k \mathrm{d}\Omega}{16\pi^2 E_T}. \tag{2.4}$$

For a 1-2 decay and 2-2 scattering, we obtain

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{1}{32\pi^2 m_A^2} k_B |A_{fi}|^2, \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 E_T^2} \frac{k_f}{k_i} |A_{fi}|^2.$$

Example. Three-body phase space can be calculated (in the CoM frame),

$$D^{(3)} = \frac{1}{(2\pi)^3} \frac{1}{8E_1 E_2 E_3} k_1^2 \mathrm{d}k_1 \mathrm{d}\Omega_1 k_2^2 \mathrm{d}k_2 \mathrm{d}\Omega_{12} \delta(E_1 + E_2 + E_3 - E_T).$$

With $\mathrm{d}\Omega_{12} = \mathrm{d}(\cos\theta_{12})\, \mathrm{d}\phi_{12}$ and $dd(\cos\theta_{12}\delta(E_1 + E_2 + E_3 - E_T) = \frac{E_3}{k_1 k_2}$, we obtain

$$D^{(3)} = \frac{1}{256\pi^5} \mathrm{d}E_1 \mathrm{d}E_2 \mathrm{d}\Omega_1 \mathrm{d}\phi_{12}.$$

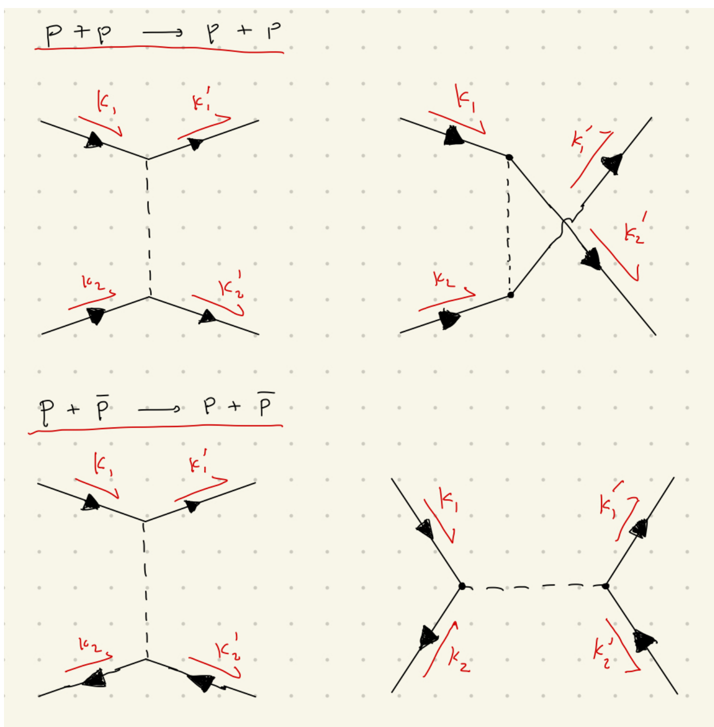

Figure 2: Feynman diagrams for the λ^2 order 2-2 scattering in $\lambda\psi\overline{\psi}\phi/3!$ theory.

Example. $2 \to 2$ Yukawa scattering. Consider an interaction Lagrangian $\mathcal{L} = \frac{\lambda}{3!}\psi^* \psi \phi$. The scattering amplitudes for $p + p \to p + p$ and $p + \overline{p} \to p + \overline{p}$ scattering, as shown in figure 2 are

$$A_{pp \to pp} = -i\lambda^2 \left(\frac{1}{t - m^2} + \frac{1}{u - m^2} \right), \quad A_{p\overline{p} \to p\overline{p}} = -i\lambda^2 \left(\frac{1}{t - m^2} + \frac{1}{s - m^2} \right).$$

Note that amplitudes should stay the same for $A_{if} = A_{\overline{fi}}$.

2.4 Unitarity

Lec. 16-17

One consequence of the unitarity of S is the **optical theorem**. We have

$$S = \mathbb{1} + iT \implies T - T^\dagger = iTT^\dagger = iT^\dagger T,$$

where $T = (2\pi)^4 \delta(k_f - k_i) A_{fi}$. Inserting the completeness identity and taking the transitional amplitude, we obtain

$$T_{fi} - T_{if}^* = i \sum_n T_{fn} T_{in}^* = i \sum_n T_{nf}^* T_{ni},$$

where $T_{fi}^\dagger = T_{if}^*$. With the delta identity $\delta(k_n - k_f)\delta(k_n - k_i) = \delta(k_i - k_f)\delta(k_i - k_n)$ and $i = f$ (this condition is known as forward scattering), we obtain

$$\frac{1}{2i}(A_{ii} - A_{ii}^*) = \operatorname{Im} A_{ii} = \frac{1}{2} \sum_n (2\pi)^4 \delta(k_i - k_f) |A_{in}|^2 = \frac{1}{2} \sum_n \int |A_{in}|^2 D^{(n)}. \tag{2.5}$$

Example. For a 2-2 scattering, $\operatorname{Im} A_{ii} = 2k_i E_{\text{tot}} D_{\text{tot}}$.

Using the optical theorem, we can examine **poles** in scattering amplitudes, which correspond to 1-particle intermediate state (hence, $n = 1$). Using the optical theorem,

$$A_{fi} - A_{if}^* = i \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta(k^2 - m^2)(2\pi)^4 \delta(p_i - k) \langle f| A |k\rangle \langle k| A^\dagger |i\rangle = 2\pi i \delta(p_i^2 - m^2) \langle f| A |p_i\rangle \langle p_i| A^\dagger |i\rangle ,$$

where we have used $\mathbb{1} = \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta(k^2 - m^2) \langle k|\rangle k$ for 1-particle states. Using $\lim_{\epsilon \to 0} \left(\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} \right) = -2\pi i \delta(x)$, we have

$$A_{fi} = \Delta(p_i) \langle f| A |p_i\rangle \langle p_i| A^\dagger |i\rangle \implies \Delta(k) = \pm \frac{1}{k^2 - m^2 + i\epsilon}.$$

Example. Consider a creation process $m_1 + m_2 \to m_3$. There is a pole at $s^2 - m_3^2$. This is known as **resonance production**.

Similarly, **branch cuts** correspond to 2-particle intermediate states. With the optical theorem,

$$A_{fi} - A_{if}^* = \frac{i}{64\pi^2} \sqrt{\frac{s - 4m^2}{s}} \theta(s - 4m^2) \int \mathrm{d}\Omega \langle f| A |k_1 k_2\rangle \langle k_1 k_2| A |k\rangle .$$

It is clear that there is a square root branch cut starting at $s = 4m^2$. To work around branch cuts, it is necessary to use analytic continuation - if A_{fi} and A_{if}^* are equal on $\mathbb{R} \bigcap [s < 4m^2]$, they are equal everywhere except on singularities.:

$$A_{fi}(s) = A_{if}^*(s), \quad s < 4m^2 \implies A_{fi}(s) = A_{if}^*(s^*), \quad s > 4m^2.$$

Example. For forward scattering, $A(s) = A^*(s^*)$. For $s = s + i\epsilon$, this states $\operatorname{Re} A(s + i\epsilon) + i \operatorname{Im} A(s + i\epsilon) = \operatorname{Re} A(s - i\epsilon) - i \operatorname{Im} A(s - i\epsilon)$ - the real part is continuous while the imaginary part has a sign discontinuity, $\operatorname{Disc} A = 2i \operatorname{Im} A(s + i\epsilon)$. Because of this discontinuity, it is useful to use Feynman prescription, writing m^2 ad $m^2 - i\epsilon$, which moves the branch cut to slightly under the real axis.

Method of dispersion relations makes use of poles and branch cuts on $A(s)$ along the real axis to reconstruct the amplitude. For a single branch cut (implying no crossing symmetries), by the Residue theorem, we get

$$\frac{1}{2\pi i} \oint \frac{f(x)}{x - z} \mathrm{d}z = f(z) = \frac{1}{2\pi i} \int_0^\infty \mathrm{d}x \frac{f(x + i\epsilon) - f(x - i\epsilon)}{x - z}.$$

Since the optical theorem relates Feynman diagrams of order λ^α to the imaginary part of $\lambda^{2\alpha}$, we can recursively use dispersion relation to get higher order terms.

2.5 Renormalization

Lec. 17-20 but will be using Burdman's notes[9].

We define the **one part irreducible** diagram as a diagram that cannot be disconnected by cutting a single internal line. Then, a full propagator can be written as a geometric series of one part irreducibles:

$$\frac{i}{p^2 - m^2 + i\epsilon} + (-i\Pi(p^2)) \left(\frac{i}{p^2 - m^2 + i\epsilon} \right)^2 + (-i\Pi(p^2))^2 \left(\frac{i}{p^2 - m^2 + i\epsilon} \right)^3 + \cdots = \frac{i}{p^2 - m^2 - \Pi(p^2) + i\epsilon}.$$

Example. Breit-Wigner resonance. First, consider the scattering amplitude $A(p^2) = \langle p| A |p\rangle$ of an unstable particle. It holds that

$$2i \operatorname{Im} A(p^2) = i \int |\langle k_1 k_2| A |p\rangle|^2 D^{(2)} \, \mathrm{d}(k_1 k_2) \implies \operatorname{Im} A(p^2) = m\Gamma.$$

Ignoring the real part of Π, we have $\operatorname{Im} \Pi(p^2 = m^2) = -\operatorname{Im} A(p^2 = m^2) = -m\Gamma$: the scattering amplitude can be written

$$A_{fi} \sim \frac{1}{2m(E_T - m) + im\Gamma} \implies \sigma \sim |A|^2 \sim \frac{1}{(E_T - m)^2 + \Gamma^2/4}$$

2.5.1 Mass Renormalization

We define the **physical mass** m to be where the pole is in the amplitude, as compared to the mass that appears in the Lagrangian, which we will call the bare mass m_0. Therefore, we have

$$p^2 - m_0^2 - \Pi(p^2)\big|_{p^2 = m^2} = p^2 - m_0^2 - \left[\lambda^2 \Pi^{(2)}(p^2) + \lambda^3 \Pi^{(3)}(p^2) + \lambda^4 \Pi^{(4)} \cdots \right]\Big|_{p^2 = m^2} = 0,$$

where $-i\Pi^{(n)}$ is the 1 part irreducible scattering amplitude to the $\mathcal{O}(\lambda^n)$ order.

Evidently, $m^2 = m^2(m_0^2, \lambda)$. If we write

$$m_0^2 = m^2 + \delta m^2 = m^2 + \lambda^2 (\delta m^{(2)})^2 + \lambda^3 (\delta m^{(3)})^2 + \lambda^4 (\delta m^{(4)})^2 + \cdots ,$$

where δm^2 is the mass correction, we obtain

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\delta m^2}{2} \phi^2 + \mathcal{L}_{\text{int}}.$$

Hence, the effective interaction Lagrangian has a $\delta m^2 / 2\phi^2$ term, which we will call the **mass counterterm**.

2.5.2 Field Renormalization

Writing the propagator,

$$\frac{i}{p^2 - m^2 - \Pi^{\text{full}}(p^2)} \sim \frac{i}{(p^2 - m^2)(1 - \Pi'(m^2))} \equiv \frac{iZ}{p^2 - m^2}, \quad z = \frac{1}{1 - \Pi'(m^2)} \sim 1 + \Pi'(m^2).$$

For the residue to stay constant, we need $\Pi'(m^2) = 0$.

From the optical theorem, we see that $\int \mathrm{d}^4 k Z |k\rangle \langle k| = \mathbb{1}$, requiring $\phi \to \frac{\phi_0}{\sqrt{Z}}$ for the external momenta to be properly normalized. For this, it is convenient to define $Z = 1 + \delta Z$.

[9] `http://fma.if.usp.br/\%7Eburdman/QFT1/lecture_21.pdf`

2.6 Regularization

2.6.1 Momentum Cutoff

It is possible to evaluate how much an interaction diverges by introducting a cutoff momentum Λ to the integrals. The superficial degree of divergence of a ϕ^n theory in d dimensions is

$$D = dL - 2I, \quad nV = E + 2I, \quad L = I - (V-1) \implies D = d\left(1 - \frac{1}{n}\right)E - 2\left(1 + \frac{d}{n}\right)I + d,$$

where there are E external lines, V vertices, I internal lines, and L loops. Given D, we say

- $D > 0$: Super-renormalizable. In EFT, relevant;
- $D = 0$: Renormalizable. In EFT, marginarlly relevant;
- $D < 0$: Non-renormalizable. In EFT, irrelevant.

In EFT, we let the interaction constant λ be dimensionless and match higher order dimensions with cutoff scale Λ. In EFT, non-renormalizable operators are p^2/Λ^2 suppressed.

However, in this perspective, bare mass is given $m^2 = \Lambda^2\lambda_2$ where λ_2 is dimensionless, which means m_0^2 is tuned to accuracy $m_{\text{phys}}^2/\Lambda^2$.

2.7 Loop Integrals

Lec. 21

2.7.1 Useful Formulae

Feynman parametrization is given as

$$\frac{1}{A_1 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \cdot \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + \cdots + x_n A_n)^n}. \tag{2.6}$$

d dimensional volume element is given as (assuming isotropy)

$$\mathrm{d}^d k = \Omega_{d-1} k^{d-1} \mathrm{d}k = \frac{2\pi^{d/2}}{\Gamma(d/2)} k^{d-1} \mathrm{d}k. \tag{2.7}$$

Example. ϕ^3 1-Loop Diagram. The one-loop integral for ϕ^3 theory is

$$\begin{aligned}
iA &= \frac{1}{2}(-i\lambda)^2 \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k-p)^2 - m^2 + i\epsilon} \\
&= \frac{\lambda^2}{2} \int_0^1 \mathrm{d}x \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{[(k^2 - m^2)(1-x) + x((k-p)^2 - m^2) + i\epsilon]^2} \\
&= \frac{\lambda^2}{2} \int_0^1 \mathrm{d}x \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{[k^2 - (m^2 - p^2 x(1-x)) + i\epsilon]^2} \\
&= \frac{i\lambda^2}{2} \int_0^1 \mathrm{d}x \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{[k^2 + a]^2}, \quad a = (m^2 - p^2 x(1-x)) \\
&= \frac{i\lambda^2}{16\pi^2} \int_0^1 \mathrm{d}x \int_0^\Lambda \mathrm{d}k \frac{k^2}{(k^2 + a - i\epsilon)^2} \\
&= \frac{i\lambda^2}{16\pi^2} \int_0^1 \mathrm{d}x \left(\log\left(\frac{\Lambda^2}{a}\right) - 1\right).
\end{aligned}$$

3 Spin 1/2 Field Theory

3.1 Groups

The **Poincare group** $P = \{(\Lambda, a)\}$ with multiplication $(\Lambda_1, a_1) \cdot (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$ is the group of isometry transformations (Lorentz + translations, $x' = \Lambda x + a$) of the Minkowski spacetime.

The basic subgroups include homogeneous Lorentz group $L = \{(\Lambda, 0)\}$ and translations $T_4 = \{(\mathbb{1}, a)\}$.

The **Lorentz Group** $\{\Lambda\}$ is a matrix group defined $\Lambda^\intercal g \Lambda = g$, where g is the Minkowski $+,-,-,-$ metric. The generators can be written

$$\Lambda^\intercal g\Lambda = (\mathbb{1} - \omega^\intercal)g(\mathbb{1} - \omega) = g \implies \omega g = -g\omega^\intercal, \quad \omega^{\mu\nu} = -\omega^{\nu\mu}.$$

From antisymmetry, there are 6 real parameters over $\mu\nu$. Per convention, the generators ω are typically written

$$-\omega = \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu},$$

where $S_{\mu\nu}$ are fixed matrices. Its lie algebra is

$$[S_{\mu\nu}, S_{\omega\rho}] = -i(g_{\mu\omega}S_{\nu\rho} + g_{\nu\rho}S_{\mu\omega} - g_{\mu\rho}S_{\nu\omega} - g_{\nu\omega}S_{\mu\rho}).$$

For intuition, the fundamental/vector (1/2,1/2) representation of $S_{\mu\nu}$ is

$$(S_{\mu\nu})^{\lambda\rho} = i(\delta^\lambda_\mu \delta^\rho_\nu - \delta^\lambda_\nu \delta^\rho_\mu).$$

If we define

- $S^{0i} = -S^{i0} = K^i$ generator of boost, $\omega^{i0} = -\omega^{0i}$ rapidity;
- $S^{ik} = \epsilon^{ikl}J^l$ generator of rotation, $\omega^{ik} = \epsilon^{ikl}\omega^l$ rotation angle,

then

$$\left[J^i, J^k\right] = i\epsilon^{ikl}J_l, \quad \left[J^i, K^i\right] = i\epsilon^{ikl}K_l, \quad \left[K^i, K^l\right] = i\epsilon^{ikl}J_l.$$

Note that $\{K\}$ isn't a subgroup.

Lastly, because representation preserves structure, we can simply take the matrix exponential to get the finite transformations:

$$\Lambda = \exp\left(\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\right) = \exp\left(i(\omega \cdot \mathbf{J} + \omega u \cdot \mathbf{K})\right).$$

3.2 Representations

First, we introudce fields and states:

- **Fields** are operators on Hilbert space. They are finite dimensional representations of the Lorentz group and transform per the Poincare group. We write

  $$U(\Lambda, a) \circ \phi_\alpha(x) \circ U^{-1}(\Lambda, a) = D_{\alpha\beta}\phi_\beta(\Lambda x + a).$$

  $D_{\alpha\beta}$ is the Lorentz transformation matrix, which isn't necessarily unitary.

- **States** are vectors in Hilbert space. They are representations of the Poincare group and transform per the Lorentz group (states don't necessarily depend on the position.) Therefore, we simply write

  $$\Lambda : |k\rangle \to U(\Lambda)|k\rangle.$$

3.2.1 Representations of the Poincare Group

We use Wigner's classification on irreducible representations of the Poincare group. Given a state's momentum, we can find orbit $\hat{p}$ for which there exists $L(p)$ such that $L(p)|\hat{p}\rangle = |p\rangle$. The little group denotes the isometry group of $\hat{p}$ (formally defined $\{\Lambda\}$ such that $\Lambda\hat{p} = \hat{p}$.)

Orbit/characteristic momentum	$\hat{p}$	Little group $H\hat{p}$
$p^2 = m^2$, $p^2 > 0$	$(m, 0)$	SO(3) $\sim$ SU(2)
$p^2 = 0$, $p^0 > 0$	$(1, \hat{e}_3)$	E_2
$p = 0$	$(0, 0)$	$L^\uparrow_+ \sim$ SL(2, $\mathbb{C}$)
$p^2 = -m^2$	$(0, m\hat{e}_3)$	SU(1,1)

Now, we define the representation of $H\hat{p}$ as

$$U(\Lambda)|\hat{p}, \alpha\rangle = D_{\alpha\alpha'}(\Lambda)|\hat{p}, \alpha'\rangle, \quad \Lambda \in H\hat{p}.$$

Given the representation of $H\hat{p}$, we can find the representation of the general $\Lambda \notin H\hat{p}$ as well. Given $|p, \alpha\rangle \equiv U(L(p))|\hat{p}, \alpha\rangle$,

$$\begin{aligned}
U(\Lambda)|p, \alpha\rangle &= \left[U(L(\Lambda p))U(L^{-1}(\Lambda p))\right]U(\Lambda)U(L(p))|\hat{p}, \alpha\rangle \\
&= U(L(\Lambda p))\left[U(L^{-1}(\Lambda p))U(\Lambda)U(L(p))\right]|\hat{p}, \alpha\rangle \\
&= U(L(\Lambda p))D_{\alpha\alpha'}(L^{-1}(\Lambda p)\Lambda L(p))|\hat{p}, \alpha\rangle \\
&= D_{\alpha\alpha'}(L^{-1}(\Lambda p)\Lambda L(p))|\Lambda p, \alpha'\rangle.
\end{aligned}$$

We have used the fact that $U(L^{-1}(\Lambda p))U(\Lambda)U(L(p))$ is a representation of the little group and $D_{\alpha\alpha'}$ is a matrix.

Case 1 - $\hat{p} = (m, 0)$, $H\hat{p} = \mathbf{SO}(3) \sim \mathbf{SU}(2)$ The SU(2) group can be represented by the particle's J^2 and J_3 values. We let $|\hat{p}, \alpha\rangle \to |\hat{p}, j\sigma\rangle$, where j is total spin and σ is the third component. The D matrix is written

$$U(\Lambda)\,|p, j\sigma\rangle = D^j_{\sigma\sigma'}(L^{-1}(\Lambda p)\Lambda L(p))\,|\Lambda p, j\sigma'\rangle\,.$$

For moving particles, $\left[J^i, p^k\right] = i\epsilon^{ikl}p^l \neq 0$, which necessitates the helicity, defined

$$\Sigma = \frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|} \implies \left[p^i, \Sigma\right] = \left[p^i, J^k\right]\frac{p^k}{|\mathbf{p}|} = 0.$$

Therefore, for moving particles, we write $|p, j\sigma\rangle \to |p, j\lambda\rangle$ with $\Sigma\,|p, j\lambda\rangle = \lambda\,|p, j\lambda\rangle\,.$

Case 2 - $\hat{p} = (1, \hat{e}_3)$, $H\hat{p} = E_2$ The E_2 group can be written

$$\Lambda\hat{p} = \hat{p} \implies \omega\hat{p} = 0, \qquad \begin{pmatrix} 0 & u^1 & u^2 & u^3 \\ -u^1 & 0 & \omega^{12} & \omega^{13} \\ -u^2 & -\omega^{12} & 0 & \omega^{23} \\ -u^3 & -\omega^{13} & -\omega^{23} & 0 \end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix} u_3 \\ -u_1 + \omega^{13} \\ -u_2 + \omega^{23} \\ -u_3 \end{pmatrix}.$$

Therefore there are only three generators, $J_3, J_1 + K_2, J_2 - K_1$. If we let $H^1 = J^1 + K^2, H^2 = J^2 - K^1$, the algebra becomes $\left[J^3, H^{1,2}\right] = \pm iH^{2,1}$ and $\left[H^1, H^2\right] = 0$, showing E_2 is a semidirect product of SO(2) and T_2: $E_2 = \mathrm{SO}(2) \rtimes T_2$

To find the irreducible representations of E_2, we use Wigner's classification again. For $\mathbf{H} = \begin{pmatrix}H^1\\H^2\end{pmatrix}$ with $\mathbf{H}\,|\Pi, \alpha\rangle = \Pi\,|\Pi, \alpha\rangle$,

Orbit	$\hat{\Pi}$	$H\hat{\Pi}$
$\Pi^2 = c^2$	$(1, 0)$	e; anything
$\Pi^2 = 0$	$(0, 0)$	SO(2)

$\hat{\Pi} = (1, 0)$ corresponds to continuous spin states, which we will neglect here.

For $\hat{\Pi} = (0, 0)$, the little group is SO(2), which has a one-dimensional representation $D(R(\omega)) = e^{i\omega\lambda}$ with $\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots$.

Therefore, for massless particles, we write $|p, \alpha\rangle \to |p, \lambda\rangle$ with

$$U(\Lambda)\,|p, \lambda\rangle = e^{i\lambda\omega}(L^{-1}(\Lambda p)\Lambda L(p))\,|\Lambda p, \lambda\rangle\,.$$

λ is called the helicity. Note that γ doesn't have spin.

Case 3 - $\hat{p} = (0, 0)$**,** $H\hat{p} = L_+^\uparrow$ This case corresponds to the vacuum states. For simplicity, we choose the trivial representation $|0\rangle$.

3.2.2 Weyl Spinor Representation of SL$(n, \mathbb{C}) \sim L_+^\uparrow$

Consider the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

We define the 4-pauli vectors as

$$\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma}) \implies \sigma_\mu = (\mathbb{1}, -\boldsymbol{\sigma}), \quad \overline{\sigma}^\mu = (\mathbb{1}, -\boldsymbol{\sigma}).$$

If we consider the 2x2 matrices $[x] = x^\mu\sigma_\mu$, for $A \in \mathrm{SL}(2, \mathbb{C})$, we have that $[x]_A = A[x]A^\dagger$ is a Lorentz transformation;

$$\det[x] = x^2, \quad x_A^2 = \det[x]_A = \det\bigl(A[x]A^\dagger\bigr) = x^2.$$

From this, we can find four different representations for $L_+^\uparrow$:

- Trivial representation; $\psi_\alpha \to A_\alpha{}^\beta\psi_\beta \sim A\psi$
- Conjugate representation; $\psi_{\dot{\alpha}} \to (A^*)_{\dot{\alpha}}{}^{\dot{\beta}}\psi_{\dot{\beta}} \sim \psi A^\dagger$
- Transpose representation $\psi^\alpha \to (A^{\intercal -1})^\alpha{}_\beta\psi^\beta \sim \psi A^{-1}$
- Hermite representation $\psi^{\dot{\alpha}} \to \bar{A}^{\dot{\alpha}}{}_{\dot{\beta}}\psi^{\dot{\beta}} \sim \bar{A}\psi$. We define $\bar{A} = (A^\dagger)^{-1}$.

Note that, because $L_+^\uparrow$ has $\det = 1$, $A^{-1} = A^\intercal$ and therefore we only have lower indices. If we had a unitary representation, $\bar{A} = A$, and would've only undotted indices.

Further, note that we use the levi-civita symbol to lower and raise indices:

$$\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

From the condition that $\det A = 1$, the generators must be such that $\mathrm{Tr}\,\omega = 0 \implies \omega = \frac{i}{2}(\boldsymbol{\alpha} + i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}$. Then,

$$A = \exp\left(\frac{i}{2}(\boldsymbol{\alpha} + i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}\right), \quad \bar{A} = \exp\left(\frac{i}{2}\,(\boldsymbol{\alpha} - i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}\right).$$

Therefore, the general representation of $\mathrm{SL}(2, \mathbb{C}) \sim L_+^\uparrow$ is

$$D^{m,n} = \varphi_{\alpha_1\cdots\alpha_m}{}^{\dot{\beta}_1\cdots\dot{\beta}_n} = \exp\left(\frac{i}{2}(\boldsymbol{\alpha} + i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}\right)\exp\left(\frac{i}{2}\,(\boldsymbol{\alpha} - i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}\right).$$

Another way to derive is to start by rewriting $\mathbf{M} = \frac{1}{2}(\mathbf{J} + i\mathbf{K})$, $\quad \mathbf{N} = \frac{1}{2}(\mathbf{J} - i\mathbf{K})$. By writing this, $\mathbf{M}$ and $\mathbf{N}$ commute with each other and have their own $\mathfrak{su}(2)$ algebra. Therefore,

$$\Lambda = e^{i(\omega\cdot\mathbf{J}+\mathbf{v}\cdot\mathbf{k})} = e^{i\mathbf{M}\cdot(\mathbf{w}-i\mathbf{v})}e^{i\mathbf{N}\cdot(\mathbf{w}+i\mathbf{v})}.$$

We now consider the (1/2,0) representation. We have

$$\mathbf{M} = \frac{\sigma}{2}, \quad \mathbf{N} = 0 \implies \mathbf{J} = \frac{\sigma}{2}, \quad \mathbf{K} = -\frac{i\sigma}{2}.$$

The objects transform under this representation are called left-handed spinors ψ_L. These transform

$$(\psi_L)_\alpha \to A_\alpha{}^\beta \psi_\beta, \quad A_\alpha{}^\beta = \exp\left(i\frac{w\cdot\sigma}{2} + \frac{u\cdot\sigma}{2}\right).$$

Similarly, the (0,1/2) representation has

$$M = \frac{\sigma}{2}, \quad N = 0 \implies J = \frac{\sigma}{2}, \quad K = \frac{i\sigma}{2}.$$

The objects that transform under this representation are called right-handed spinors ψ_R. These transform

$$(\psi_R)^{\dot\beta} \to \bar{A}^{\dot\alpha}{}_{\dot\beta}(\psi_R)^{\dot\beta}, \quad \bar{A}^{\dot\alpha}{}_{\dot\beta} = \exp\left(i\frac{w\cdot\sigma}{2} - \frac{u\cdot\sigma}{2}\right).$$

Lastly, the (1/2,1/2) representation can be written

$$(\psi_L)^\alpha (\sigma^\mu)_{\alpha\dot\beta}(\psi_R)^{\dot\beta}.$$

This is an ordinary 4-vector; recall that $x^\mu \sim [x] \to A[x]A^\dagger$.

3.2.3 Dirac Spinor Representation

For theories with parity symmetry, Dirac spinors are used to include both Left-handed spinors as well as Right-handed spinors. We let $\psi = \begin{pmatrix}\chi_\alpha \\ \bar\psi^{\dot\alpha}\end{pmatrix}$. The Lorentz transformation is

$$\begin{aligned} D(\Lambda) &= \begin{pmatrix} \exp\left(\frac{i}{2}\boldsymbol{\sigma}(\boldsymbol{\omega} - i\mathbf{u})\right) & 0 \\ 0 & \exp\left(\frac{i}{2}\boldsymbol{\sigma}(\boldsymbol{\omega} + i\mathbf{u})\right)\end{pmatrix} \\ &= \exp\left(i\boldsymbol{\omega}\cdot\boldsymbol{\Sigma} + \frac{1}{2}\mathbf{u}\cdot\boldsymbol{\alpha}\right), \quad \boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix}-\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\end{pmatrix}. \end{aligned}$$

$\boldsymbol{\Sigma}$ is a representation of $\mathbf{J}$, and $\boldsymbol{\alpha}$ is a representation of $\mathbf{K}$. The generator algebra is such that

$$\Sigma^i\Sigma^k = \alpha^i\alpha^k = \delta^{ik} + i\epsilon^{ikl}\Sigma^l, \quad \left[\Sigma^i, \gamma^0\right] = \left[\alpha^i, \gamma^0\right] = 0.$$

where

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}.$$

These are known as the Dirac gamma matrices. These have an algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{1}.$$

3.3 Formalism

We want a Lagrangian that:

- Is Lorentz invariant,
- Has at most 2 derivatives and 2 fields,
- Has a real action,
- Has a U(1) symmetry ($\chi \to e^{i\alpha}\chi$). Experimentally, every 1/2 has a conserved charge, for example, leptop/baryon number, electric charge.

The kind of left-handed objects we can have are

$$\chi_\alpha, \quad \chi^*_{\dot\alpha}, \quad \partial^\mu, \quad \sigma^\mu_{\alpha\dot\alpha}, \quad \bar\sigma^{\mu\dot\alpha\alpha}, \quad \epsilon_{\alpha\beta}.$$

For a bilinear $\mathcal{L}$, there are three possibilities:

1. $\mathcal{L} = c\chi^*_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\dot\beta}(\partial_\mu\chi_\beta)$. Taking the complex conjugate then integrating by parts, realness requires that c^* is purely imaginary; from convention, we choose $c = i$.
2. $\mathcal{L} = \chi_\alpha\epsilon^{\alpha\beta}\chi_\beta$. This is not U(1) invariant. This corresponds to the Majonara mass, for neutrinos without charge.
3. $(\partial_\mu\chi_\alpha)\epsilon^{\alpha\beta}(\partial^\mu\chi_\beta)$. This is not U(1) invariant either.

Therefore, we write the action

$$S = \int \mathrm{d}^4x\, i\chi^*_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}(\partial_\mu\chi_\alpha) = \int \mathrm{d}^4x\, i\bar\chi\bar{\partial\!\!\!/}\chi, \quad \bar{\partial\!\!\!/} \equiv \bar\sigma^\mu\partial_\mu. \tag{3.1}$$

From the Euler-Lagrange equation, we obtain

$$\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\chi_\beta = 0, \tag{3.2}$$

known as the (left-handed) Weyl equation.

This is consistent with the massless Klein-Gordon equation.

The plain wave solution is

$$\chi_L = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2\omega_p}} \left[a_{p-} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ipx} + b^\dagger_{p+} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ipx} \right]$$
$$\chi_L^\dagger = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2\omega_p}} \left[a^\dagger_{p-} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ipx} + b_{p+} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ipx} \right].$$

Similarly, the right-handed Lagrangian and the corresponding equation are

$$\mathcal{L} = \pm i\psi^{*\alpha} \sigma^\mu_{\alpha\dot{\beta}} (\partial_\mu \psi^{\dot{\beta}}) \implies \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \psi^{\dot{\beta}} = 0.$$

The plain wave solution is

$$\psi_R = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2\omega_p}} \left[a_{p-} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ipx} + b^\dagger_{p+} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ipx} \right].$$

3.3.1 Majorana Theory

To add mass to our theory, we consider the Majorana mass term. The corresponding action is

$$S = \int \mathrm{d}^4x \left[i\psi^*_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_\alpha - \frac{m}{2} \psi_\alpha \epsilon^{\alpha\beta} \psi_\beta + \text{h.c.} \right].$$

The equations of motion are

$$i\bar{\sigma}^{\mu\dot{\alpha}\beta} \partial_\mu \psi_\beta = m(\psi^*)^{\dot{\alpha}}, \quad -i\partial_\mu \psi^*_{\dot{\alpha}} \bar{\sigma}^{\mu\alpha\beta} = m\psi^\beta.$$

With the ansatz $\psi_\beta = x_\beta e^{-ipx} + y_\beta e^{ipx}$, we can reduce the equations to

$$(p \cdot \bar{\sigma})x = my^\dagger, \quad (p \cdot \bar{\sigma})y = -mx^\dagger, \iff (p \cdot \sigma)y^\dagger = mx, \quad (p \cdot \sigma)x^\dagger = -my.$$

The solutions are such that

$$x_\alpha(p,s) = \sqrt{p \cdot \sigma} \chi_s, \quad y_\alpha(p,s) = (2s)\sqrt{p \cdot \sigma} \chi_s,$$

giving us

$$\psi_\alpha(x) = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2\omega_p}} \sum_s \left(x_\alpha(p,s) a_{p,s} e^{-ipx} + y_\alpha(p,s) a^\dagger_{p,s} e^{ipx} \right).$$

3.4 Dirac Theory

3.4.1 Gamma Matrices

We define the Dirac gamma matrices (in **Weyl/Chiral basis**):

$$\gamma = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

The gamma matrices have a Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{1}.$$

3.4.2 Dirac Formalism

To add mass to the theory while keeping U(1) symmetry, we consider a Lagrangian of the form

$$\mathcal{L} = i(\psi_L^\dagger)_{\dot{\alpha}} \bar{\sigma}^{\mu\alpha\dot{\beta}} \psi_{L\beta} + (\psi_R^\dagger)^\alpha \sigma^\mu_{\dot{\alpha}\beta} \partial_\mu \psi_R^{\dot{\beta}} - m(\psi_R^\dagger)^\alpha (\psi_{L\beta}) + \text{h.c.}.$$

4 Path Integral Formulation

4.1 Derivation

We claim that the propagation amplitude can be written as a functional integral of the exponent of the action:

$$U(x_a, x_b; T) = \int \mathcal{D}x(t) e^{iS[x(t)]/\hbar}. \tag{4.1}$$

In the classical limit ($S[x] \gg \hbar$), this makes sense because $\frac{\delta S[x(t)]}{\delta x(t)} = 0$ by the method of stationary phase, agreeing with the principle of least action.

In a more practical form, (4.1) can be written as

$$\begin{aligned} U(q_0, q_N; T) &= \left(\prod_{i,k} \int \frac{\mathrm{d}q_k^i \mathrm{d}q_k^i}{2\pi}\right) \exp\left[i \sum_{i,k} p_k^i (q_{k+1}^i - q_k^i) - \epsilon H\left(\frac{q_{k+1} + q_k}{2}, p_k\right)\right] \\ &= \left(\prod_i \int \mathcal{D}q(t)\mathcal{D}p(t)\right) \exp\left[i \int_0^T \mathrm{d}t \left(\sum_i p^i \dot{q}^i - H(q,p)\right)\right]. \end{aligned} \tag{4.2}$$

4.2 Quantization of Scalar Fields

The correlation function, in terms of path integrals, is

$$\langle \Omega | T\phi_H(x_1)\phi_H(x_2) |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \phi(x_1)\phi(x_2) \exp\left[i \int_{-T}^T \mathrm{d}^4x \mathcal{L}\right]}{\int \mathcal{D}\phi \exp\left[i\int_{-T}^T \mathrm{d}^4x\mathcal{L}\right]} \tag{4.3}$$

Consider a non-interacting real scalar field ($\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$.) In the momentum space, defined

$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_n\cdot x_i}\phi(k_n),$$

where the $n > 0$ comes from $\phi^*(k) = \phi(-k)$. The denominator can be written

$$\int \mathcal{D}\phi e^{iS_0} = \left(\prod_{k^0>0}\int \mathrm{d}(\operatorname{Re}\phi_n)\,\mathrm{d}(\operatorname{Im}\phi_n)\right) \exp\left[-\frac{i}{V}\sum_{k^0>0}(m^2 - k_n^2)|\phi_n|^2\right] = \prod_{\text{all } k_n}\sqrt{\frac{i\pi V}{k_n^2 - m^2 + i\epsilon}}.$$

We can write our result using the **functional determinant**. Consider the integral

$$\left(\prod_k \int \mathrm{d}\xi_k\right) \exp\left[-\xi_i B_{ij}\xi_j\right] \propto \left[\det B\right]^{-1/2}, \tag{4.4}$$

which allows us to write

$$\int \mathcal{D}\phi e^{iS_0} \propto \left[\det(m^2 + \partial^2)\right]^{-1/2}. \tag{4.5}$$

Similarly, the numerator can be written

$$\begin{aligned} \int \mathcal{D}\phi\phi(x_1)\phi(x_2)e^{iS_0} =& \frac{1}{V^2}\sum_{m,l} e^{-i(k_m\cdot x_1 + k_l\cdot x_2)} \left(\prod_{k_n^0>0}\int \mathrm{d}(\operatorname{Re}\phi_n)\,\mathrm{d}(\operatorname{Im}\phi_n)\right) \cdot (\operatorname{Re}\phi_m + i\operatorname{Im}\phi_m)(\operatorname{Re}\phi_l + i\operatorname{Im}\phi_l) \\ &\cdot \exp\left[-\frac{i}{V}\sum_{k_n^0>0}(m^2 - k_n^2)\left[(\operatorname{Re}\phi_n)^2 + (\operatorname{Im}\phi_n)^2\right]\right] \\ =& \frac{1}{V^2}\sum_m e^{-ik_m\cdot(x_1 - x_2)}\left(\prod_{k_n^0>0}\frac{-i\pi V}{m^2 - k_n^2}\right)\frac{-iV}{m^2 - k_m^2 - i\epsilon}, \end{aligned}$$

where we have used the fact that unless $k_m = -k_l$, the integrand is odd ($k_m = k_l$ cancels Re^2 with Im^2.)

Taking the continuum limit of the fraction, we obtain

$$\langle 0| T\phi(x_1)\phi(x_2)|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(x_1-x_2)}}{k^2 - m^2 + i\epsilon}. \tag{4.6}$$

Similarly, the four-point function is nonzero for when two of its fields are equal but opposite in sign.

It is useful to consider the **generating functional** when computing the correlation functions. We define

$$Z[J] \equiv \int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x\left[\mathcal{L} + J(x)\phi(x)\right]\right]. \tag{4.7}$$

The two-point function is simply

$$\langle 0| T\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{Z[J=0]}\left(\frac{1}{i^2}\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\right)Z[J]\Big|_{J=0}, \tag{4.8}$$

from (4.3).

For the free scalar field theory, we can explicitly find the generating functional.

$$\begin{aligned} Z[J] &\equiv \int D\phi \exp\left[i\int \mathrm{d}^4x\left[\frac{1}{2}\phi(-\partial^2 - m^2 + i\epsilon)\phi + J(x)\phi(x)\right]\right] \\ &= \int \mathcal{D}\phi' \exp\left[i\int \mathrm{d}^4x\left[\frac{1}{2}\phi'(-\partial^2 - m^2 + i\epsilon)\phi' - \frac{1}{2}J(-\partial^2 - m^2 + i\epsilon)^{-1}J\right]\right] \\ &= Z_0 \exp\left[-\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4y J(y)D_F(y-x)J(x)\right] \end{aligned}$$

where $\phi' = \phi + (-\partial^2 - m^2 + i\epsilon)^{-1} J$. Note that to compute $(-\partial^2 - m^2 + i\epsilon)^{-1}$ we need to work in the momentum space.

The four-point function is then

$$\begin{aligned}
\langle 0| T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |0\rangle &= \frac{1}{Z_0} \prod_{j=1}^{4} \left(\frac{1}{i} \frac{\delta}{\delta J(x_j)} \right) e^{-\frac{1}{2} J_x D_{xy} J_y} \\
&= \frac{1}{Z_0} \prod_{j=1}^{3} \left(\frac{1}{i} \frac{\delta}{\delta J(x_j)} \right) \left(-J_x D_{x4} e^{-\frac{1}{2} J_x D_{xy} J_y} \right) \\
&= \frac{1}{Z_0} \prod_{i=1}^{2} \left(\frac{1}{i} \frac{\delta}{\delta J(x_j)} \right) \left((-D_{34} + J_x D_{x4} J_y D_{y3})\, e^{-\frac{1}{2} J_x D_{xy} J_y} \right) \\
&= \frac{1}{Z_0} \prod_{j=1}^{1} \left(\frac{1}{i} \frac{\delta}{\delta J(x_j)} \right) \left((J_x D_{x2} D_{34} + D_{24} J_y D_{y3} + J_x D_{x4} D_{23})\, e^{-\frac{1}{2} J_x D_{xy} J_y} \right) \\
&= (D_{12} D_{34} + D_{24} D_{13} + D_{14} D_{23}).
\end{aligned}$$

4.3 Quantization of EM Field

The functional integral for the electromagnetic theory is

$$\int \mathcal{D}A e^{iS[A]}, \quad S = \int \mathrm{d}^4 x \left[-\frac{1}{4} (F_{\mu\nu})^2 \right] = \frac{1}{2} \int \mathrm{d}^4 x A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu .$$

This expression vanishes if $\tilde{A}_\mu = k_\mu \alpha(k)$ for any scalar function $\alpha(k)$, meaning the path integral is badly divergent.

Let $G(A) = 0$ be a gauge-fixing condition; for example, $G(A) = \partial_\mu A^\mu$ for Lorentz gauge. We consider the identity

$$1 = \int \mathcal{D}\alpha(x) \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right), \quad A^\alpha_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x). \tag{4.9}$$

Note that $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{\partial^2}{e}\right)$ is constant. With this condition, the functional integral becomes

$$\int \mathcal{D}A e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(G(A)), \tag{4.10}$$

where we have used the gauge invariance of S and the fact that $A \to A^\alpha$ is a simple shift. Consider letting $G(A) = \partial^\mu A_\mu(x) - \omega(x)$ for a generalization of the Lorentz gauge condition and integrate over all $\omega(x)$ with a weighting function:

$$\begin{aligned}
\int \mathcal{D}A e^{iS[A]} &= N(\xi) \int \mathcal{D}\omega \exp\left[-i \int \mathrm{d}^4 x \frac{\omega^2}{2\xi} \right] \det\left(\frac{1}{e} \partial^2 \right) (\mathcal{D}\alpha) \int \mathcal{D}A e^{iS[A]} \delta(\partial^\mu A_\mu - \omega(x)) \\
&= N(\xi) \det\left(\frac{1}{e} \partial^2 \right) \left(\int \mathcal{D}\alpha \right) \int \mathcal{D}A e^{iS[A]} \exp\left[-i \int \mathrm{d}^4 x \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right].
\end{aligned}$$

Hence, the correlation function is

$$\langle \Omega | T\mathcal{O}(A) |\Omega\rangle = \lim_{T \to \inf(1 - i\epsilon)} \frac{\int \mathcal{D}A \mathcal{O}(A) \exp\left[i \int_{-T}^{T} \mathrm{d}^4 x \left[\mathcal{L} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right] \right]}{\int \mathcal{D}A \exp\left[i \int_{-T}^{T} \mathrm{d}^4 x \left[\mathcal{L} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right] \right]}. \tag{4.11}$$

The photon propagator is

$$\tilde{D}_F^{\mu\nu}(k) = -\frac{-i}{k^2 + i\epsilon} \left(g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right). \tag{4.12}$$