JEE Advanced 2015 Paper 2 $Offline$ $60Questions$

Question 001 Numerical

QUESTION

In dilute aqueous H_2SO_4, the complex diaquodioxalatoferrate II is oxidized by

$$MnO_4^-$$

. For this reaction, the ratio of the rate of change of $[H^+]$ to the rate of change of

$$[MnO_4^-]$$

is

SOURCE

Chemistry • chemical-kinetics-and-nuclear-chemistry

EXPLANATION

In complex,

$$\underset{Diaquodioxalatoferrate\ (II)}{[Fe(C_2O_4)_2(H_2O)_2]^{2-}}$$

,

Fe is in +2 oxidation state.

In acidic medium,

$$KMnO_4$$

oxidises

$$Fe^{2+}$$

to

$$Fe^{3+}$$

,

$$2MnO_4^- + 16H^+ + 10Fe^{2+} \rightarrow 2Mn^{2+} + 8H_2O + 10Fe^{3+}$$

or

$$MnO_4^- + 8H^+ + 5Fe^{2+} \rightarrow Mn^{2+} + 4H_2O + 5Fe^{3+}$$

$$\frac{Rate\,of\,change\,of\,[H^+]}{Rate\,of\,change\,of\,[MnO_4^-]} = \frac{8}{1} = 8$$

Question 002 Numerical

QUESTION

A closed vessel with rigid walls contains 1 mol of

$$^{238}_{92}U$$

and 1 mol of air at 298 K. Considering complete decay of

$$^{238}_{92}U$$

to

$$^{206}_{82}Pb$$

, the ratio of the final pressure to the initial pressure of the system at 298 K is

SOURCE

Chemistry • chemical-kinetics-and-nuclear-chemistry

EXPLANATION

$$_{92}U^{238} \rightarrow {}_{82}Pb^{206} + 8\,{}_2He^4(g) + 6\,{}_{-1}\beta^0$$

To calculate pressure, only gaseous products need to be considered.

Initially, only 1 mol of air is present and finally, after complete decay, 8 moles of

$$^4_2He$$

gas are produced and 1 mol of air will also remain in the mixture.

Ratio of the final pressure to the initial pressure

$$= \frac{8+1}{1} = 9$$

Question 003 Numerical

QUESTION

The molar conductivity of a solution of a weak acid HX $0.01M$ is 10 times smaller than the molar conductivity of a solution of a weak acid HY $0.10M$. If

$$\lambda^0_{x^-} \approx \lambda^0_{y^-}$$

the difference in their pK$_a$ values, pK$_a$ HX - pK$_a$ HY, is $considerdegreeofionizationofbothacidstobe << 1$

SOURCE

Chemistry • electrochemistry

EXPLANATION

Given :

$$\Lambda^c_{m(HX)} = \frac{\Lambda^c_{m(HY)}}{10}$$

$$\Lambda^o_{m(HX)} = \Lambda^o_{m(HY)}$$

$$ \because $$$$\lambda^o_{X-} \approx \lambda^o_{Y-}$$

$$K_{a(HX)} = \left(\frac{C\alpha^2}{1-\alpha}\right)_{HX}$$

$$K_{a(HX)} = 0.01(\alpha_{HX})^2$$

$$ \because $$$$\alpha <<< 1$$ i

Similarly,

$$K_{a(HY)} = 0.01(\alpha_{HY})^2$$

.... ii

On dividing equation i by ii, we get

$$\frac{K_{a(HX)}}{K_{a(HY)}} = \frac{0.01}{0.10}\left(\frac{\alpha_{HX}}{\alpha_{HY}}\right)^2$$

.... iii

$$\alpha = \frac{\Lambda^c_m}{\Lambda^o_m}$$

$$\frac{\alpha_{HX}}{\alpha_{HY}} = \frac{(\Lambda^c_m/\Lambda^o_m)_{HX}}{(\Lambda^c_m/\Lambda^o_m)_{HY}} = \left(\frac{1}{10}\Lambda^c_{m(HY)}\right) \times \frac{1}{\Lambda^c_{m(HY)}} = \frac{1}{10}$$

Substituting above value in equation iii,

$$\frac{K_{a(HX)}}{K_{a(HY)}} = \frac{0.01}{0.10}\left(\frac{1}{10}\right)^2 = 1 \times 10^{-3}$$

$$\log K_{a(HX)} - \log K_{a(HY)} = \log(1 \times 10^{-3})$$

$$-\log K_{a(HX)} - (-\log K_{a(HY)}) = -\log(1 \times 10^{-3})$$

$$pK_{a(HX)} - pK_{a(HY)} = 3$$

Question 004 MCQ

QUESTION

When O_2 is adsorbed on a metallic surface, electron transfer occurs from the metal to O_2. The true statement s regarding this adsorption is are

A O_2 is physisorbed

B Heat is released

C Occupancy of

π^*_{2p}

of O_2 is increased

D Bond length of O_2 is increased

CORRECT OPTION

C Occupancy of

π^*_{2p}

of O_2 is increased

SOURCE

Chemistry • surface-chemistry

EXPLANATION

Since, adsorption involves electron transfer from metal to O_2, it is chemical adsorption not physical adsorption, hence a is incorrect. Adsorption is spontaneous which involves some bonding between adsorbent and adsorbate, hence exothermic. The last occupied molecular orbital in O_2 is

$$\pi$$

*2p. Hence, electron transfer from metal to oxygen will increase occupancy of

$$\pi$$

*2p molecular orbitals. Also increase in occupancy of

$$\pi$$

*2p orbitals will decrease bond order and hence increase bond length of O_2.

Question 005 MCQ

QUESTION

Paragraph

When 100 mL of 1.0 M KCl was mixed with 100 mL of 1.0 M NaOH in an insulated beaker at constant pressure, a temperature increase of 5.7^{o} C was measured for the beaker and its contents $Expt.1$. Because the enthalpy of neutralization of a strong acid with a strong base is constant $-57.0kJ/mol$, this experiment could be used to measure the calorimeter constant. In a second experiment $Expt.2$ 100 mL of 2.0 M acetic acid (K_a = 2.0

$$\times$$

10^{-5}) was mixed with 100 mL of 1.0 M NaOH $underidenticalconditionstoExpt.1$ where a temperature rise of 5.6^{o} C was measured.

$Considerheatcapacityofallsolutionsas4.2J/gKanddensityofallsolutionsa$

Question

The pH of the solution after Expt. 2 is

A 2.8

B 4.7

C 5.0

D 7.0

CORRECT OPTION

B 4.7

SOURCE

Chemistry • ionic-equilibrium

EXPLANATION

In Expt. 2, the final solution is a buffer as it contains equimolar amounts of acid and salt.

$$pH = pK_a + \log \frac{(salt)}{(acid)}$$

..... 1

$$pK_a = -\log(2 \times 10^{-5})$$
$$= -0.3010 + 5$$
$$= 4.699 \approx 4.7$$

$$[Salt] = [CH_3COONa] = \frac{100}{200} \times 2 = 1\,M$$

$$[Acid) = [CH_3COOH] = \frac{200 - 100}{200} \times 2 = \frac{100}{200} \times 2 = 1\,M$$

Substituting the values in Eq. 1, we get

$$pH = 4.7 + \log \frac{1}{1} = 4.7$$

Question 006 MCQ

QUESTION

Paragraph

When 100 mL of 1.0 M HCl was mixed with 100 mL of 1.0 M NaOH in an insulated beaker at constant pressure, a temperature increase of 5.7° C was measured for the beaker and its contents $Expt. 1$. Because the enthalpy of neutralization of a strong acid with a strong base is constant $-57.0 kJ/mol$, this experiment could be used to measure the calorimeter constant. In a second experiment $Expt. 2$ 100 mL of 2.0 M acetic acid (K_a = 2.0

×

10^{-5}) was mixed with 100 mL of 1.0 M NaOH $under identical conditions to Expt. 1$ where a temperature rise of 5.6° C was measured.

$Consider heat capacity of all solutions as 4.2 J/gK and density of all solutions a$

Question

Enthalpy of dissociation $in kJ/mol$ of acetic acid obtained from the Expt. 2 is

A 1.0

B 10.0

C 24.5

D 51.4

CORRECT OPTION

A 1.0

SOURCE

Chemistry • ionic-equilibrium

EXPLANATION

Energy evolved on neutralization of HCl and NaOH is

0.1

$\times$

57 = 5.7 kJ = 5700 J

Energy utilized to rise the temperature of the solution is

ms .

Δ

T = 200

$\times$

1

$\times$

4.2

$\times$

5.7 = 4788 J

Energy used to increase temperature of calorimeter is

= 5700

$-$

4788 = 912 J

ms .

$$\Delta$$

T = 912

m

$$\times$$

s

$$\times$$

5.7 = 912

$$\rightarrow$$

ms = 160 J

$$\circ$$

C

$$-$$

1

$$Calorimeter constant$$

Energy evolved by neutralization of CH_3COOH and NaOH is

= 200

$$\times$$

4.2

$$\times$$

5.6 + 160

$$\times$$

5.6 = 5600 J

So, the energy used in dissociation of 0.1 mol CH_3COOH is

= 5700

$$-$$

5600 = 100 J

Thus, enthalpy of dissociation is 1 kJ mol

−

1.

Question 007 Numerical

QUESTION

The number of hydroxyl group s in Q is ___________.

SOURCE

Chemistry • alcohols-phenols-and-ethers

EXPLANATION

Therefore, the number of hydroxyl groups is 4.

Question 008 Numerical

QUESTION

Among the following the number of reactions that produces benzaldehyde is _________.

SOURCE

Chemistry • aldehydes-ketones-and-carboxylic-acids

EXPLANATION

Therefore, the number of reactions leading to the formation of benzaldehyde is 4.

Question 009 Numerical

QUESTION

In the complex acetylbromidodicarbonylbis$triethylphosphine$ ironII, the number of Fe-C bond$s$ is ___________.

SOURCE

Chemistry • coordination-compounds

EXPLANATION

The structure of the given complex is

From the above structure, we can conclude the number of Fe-C bonds is 3.

Question 010 Numerical

QUESTION

Among the complex ions, $[Co(NH_2\text{-}CH_2\text{-}CH_2\text{-}NH_2)_2Cl_2]^+$, $[CrCl_2(C_2O_4)_2]^3$

—

, $[Fe(H_2O)_4)OH)_2]^+$, $[Fe(NH_3)_2 CN_4]$

$-$

, $[Co(NH_2\text{-}CH_2\text{-}CH_2\text{-}NH_2)_2(NH_3)Cl]^{2+}$ and $[Co(NH_3)_4(H_2O)Cl]^{2+}$, the number of complex ions that shows cis-trans isomerism is ______________.

SOURCE

Chemistry • coordination-compounds

EXPLANATION

The complex is with MA_2B_4 or MA_4B_2 or $M AA_2B_2$ or $M AA_2BC$ structure can exhibit cis-trans isomerism. Therefore, number of given complex ions showing cis-trans isomerism is 6.

Question 011 Numerical

QUESTION

Three moles of B_2H_6 are completely reacted with methanol. The number of moles of boron containing product formed is ____________.

SOURCE

Chemistry • p-block-elements

EXPLANATION

The reaction is

$B_2H_6 + 6CH_3OH$

$\rightarrow$

$2B(OCH_3)_3 + 6H_2$

From the reaction, 1 mol of B_2H_6 reacts with 6 mol of CH_3OH to produce 2 mol of $B(OCH_3)_3$.

Therefore, 3 mol of B_2H_6 would react with 18 mol of CH_3OH to produce 6 mol of $B(OCH_3)_3$.

Question 012 MCQ

QUESTION

In the following reactions, the product S is

A

B

C

D

CORRECT OPTION

A

SOURCE

Chemistry • hydrocarbons

EXPLANATION

The product S is

In this reaction, NH_3 chooses to attack at aliphatic aldehyde group than a less reactive aromatic aldehyde group. Now,

Question 013 MCQ

QUESTION

The major product U in the following reactions is

A

B

C

D

CORRECT OPTION

B

SOURCE

Chemistry • hydrocarbons

EXPLANATION

The reaction is

Question 014 MCQ

QUESTION

In the following reactions, the major product W is

A

B

C

D

CORRECT OPTION

A

SOURCE

Chemistry • compounds-containing-nitrogen

EXPLANATION

The reaction is

Question 015 MCQ

QUESTION

The correct statements regarding i HClO, ii $HClO_2$, iii $HClO_3$ and iv $HClO_4$ is are

A The number of Cl=O bonds ii and iii together is two.

B The number of lone pairs of electrons on Cl in ii and iii together is three.

C The hybridization of Cl in iv is sp^3

D Amongst i to iv, the strongest acid is i.

CORRECT OPTION

B The number of lone pairs of electrons on Cl in ii and iii together is three.

SOURCE

Chemistry • p-block-elements

EXPLANATION

In all the oxyacids of chlorine : Cl undergoes sp^3 hybridization $HClO_4$ is the strongest acid among the given compounds.

Question 016 MCQ

QUESTION

The pairs of ions where BOTH the ions are precipitated upon passing H_2S gas in presence of dilute HCl, is are

A. Ba^{2+}, Zn^{2+}

B. Bi^{3+}, Fe^{3+}

C. Cu^{2+}, Pb^{2+}

D. Hg^{2+}, Bi^{3+}

CORRECT OPTION

C. Cu^{2+}, Pb^{2+}

SOURCE

Chemistry • d-and-f-block-elements

EXPLANATION

The pairs Cu^{2+}, Pb^{2+} and Hg^{2+}, Bi^{3+} are precipitated as sulphides upon passing H_2S in acidic medium.

Question 017 MCQ

QUESTION

Under hydrolytic conditions, the compounds used for preparation of linear polymer and for chain termination, respectively, are

A CH_3SiCl_3 and $Si(CH_3)_4$.

B $(CH_3)_2SiCl_2$ and $(CH_3)_3SiCl$.

C $(CH_3)_2SiCl_2$ and CH_3SiCl_3.

D $SiCl_4$ and $(CH_3)_3SiCl$.

CORRECT OPTION

B $(CH_3)_2SiCl_2$ and $(CH_3)_3SiCl$.

SOURCE

Chemistry • polymers

EXPLANATION

The reaction involved in the preparation of liner polymer is

Question 018 MCQ

QUESTION

One mole of a monoatomic real gas satisfies the equation p $V$$ - $$b$ = RT where b is a constant. The relationship of interatomic potential V r and interatomic distance r for the gas is given by

A

B

C

D

CORRECT OPTION

C

SOURCE

Chemistry • gaseous-state

EXPLANATION

The given equation is $\mathrm{P}(\mathrm{V}-b)=\mathrm{RT}$,

On comparing with van Der Waal's equation

$$\left[\mathrm{P}+\frac{a}{\mathrm{V}^2}\right][\mathrm{V}-b]=\mathrm{RT} \text{ we get } a=0$$

Hence, only repulsive forces are present which are contributive only at very close distance.

Thus, the potential energy will increase abruptly, so graph c is correct. The dominance of repulsive force can be shown by using the compressibility factor.

$$\mathrm{P}(\mathrm{V}-b)=\mathrm{RT}$$

$$\mathrm{PV}=\mathrm{Pb}+\mathrm{RT}$$

$$\frac{\mathrm{PV}}{\mathrm{RT}}=\frac{\mathrm{Pb}}{\mathrm{RT}}+1$$

$$Z=\frac{Pb}{\mathrm{RT}}+1, \text{ i.e., } \mathrm{Z}>1 \text{ (Repulsive forces)}$$

Question 019 MCQ

QUESTION

In the following reactions

Compound X is

A

B

C

D

CORRECT OPTION

C

SOURCE

Chemistry • aldehydes-ketones-and-carboxylic-acids

EXPLANATION

Question 020 MCQ

QUESTION

In the following reactions

The major compound Y is

A

B

C

CORRECT OPTION

D

SOURCE

Chemistry • aldehydes-ketones-and-carboxylic-acids

EXPLANATION

Question 021 MCQ

QUESTION

Let

$$S$$

be the set of all non-zero real numbers

$$\alpha$$

such that the quadratic equation

$$\alpha x^2 - x + \alpha = 0$$

has two distinct real roots

$$x_1$$

and

$$x_2$$

satisfying the inequality

$$|x_1 - x_2| < 1.$$

Which of the following intervals is are

$$a$$

subset s os

$$S$$

?

A

$$\left(-\frac{1}{2} - \frac{1}{\sqrt{5}}\right)$$

B

$$\left(-\frac{1}{\sqrt{5}}, 0\right)$$

C

$$\left(0, \frac{1}{\sqrt{5}}\right)$$

D

$$\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

CORRECT OPTION

A

$$\left(-\frac{1}{2} - \frac{1}{\sqrt{5}}\right)$$

SOURCE

Mathematics • quadratic-equation-and-inequalities

EXPLANATION

Given, x_1 and x_2 are roots of

$$\alpha x^2 - x + \alpha = 0$$

$$\therefore$$

$$x_1 + x_2 = \frac{1}{\alpha}$$

and

$$x_1 x_2 = 1$$

Also,

$$|x_1 - x_2| < 1$$

$$\Rightarrow |x_1 - x_2|^2 < 1 \Rightarrow (x_1 - x_2)^2 < 1$$

or,

$$(x_1 + x_2)^2 - 4x_1x_2 < 1$$

$$\Rightarrow \frac{1}{\alpha^2} - 4 < 1$$

or

$$\frac{1}{\alpha^2} < 5$$

$$\Rightarrow 5\alpha^2 - 1 > 0$$

or,

$$(\sqrt{5}\alpha - 1)(\sqrt{5}\alpha + 1) > 0$$

$$\therefore$$

$$\alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$$

.....i

Also,

$$D > 0$$

$$\Rightarrow 1 - 4\alpha^2 > 0$$

or

$$\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

...... ii

$$\alpha \in \left(-\frac{1}{2}, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

Question 022 Numerical

QUESTION

For any integer k, let

$$a_k = \cos\left(\frac{k\pi}{7}\right) + i\ \sin\left(\frac{k\pi}{7}\right)$$

, where

$$i = \sqrt{-1}$$

. The value of the expression

$$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - a_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$$

is

SOURCE

Mathematics • complex-numbers

EXPLANATION

Given,

$$a_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right) = e^{\frac{k\pi}{7}i}$$

We have to find

$$\frac{\sum_{k=1}^{12} |a_{k+1} - a_k|}{\sum_{k=1}^{3} |a_{4k-1} - a_{4k-2}|}$$

$$a_{k+1} = \cos\left(\frac{k+1}{7}\right)\pi + i\sin\left(\frac{k+1}{7}\right)\pi = e^{i\left(\frac{k+1}{7}\right)\pi}$$

$$\therefore$$

$$a_{k+1} - a_k = e^{\left(\frac{k+1}{7}\right)\pi i} - e^{\frac{k\pi}{7}i}$$

$$= e^{\frac{k\pi}{7}i} .\ e^{\frac{\pi}{7}i} - e^{\frac{k\pi}{7}i}$$

$$= e^{\frac{k\pi}{7}i}\left(e^{\frac{\pi}{7}i} - 1\right)$$

$$\therefore$$

$$|a_{k+1} - a_k| = \left|e^{\frac{k\pi}{7}i}\left(e^{\frac{\pi}{7}i} - 1\right)\right|$$

$$= \left|e^{\frac{k\pi}{7}i}\right|\left|e^{\frac{\pi}{7}i} - 1\right|$$

$$= \left|e^{\frac{\pi}{7}i} - 1\right|$$

If

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

then

$$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

that is why

$$\left|e^{\frac{k\pi}{7}i}\right| = 1$$

Now,

$$a_{4k-1} = e^{\left(\frac{4k-1}{7}\right)\pi i}$$

$$a_{4k-2} = e^{\left(\frac{4k-2}{7}\right)\pi i}$$

$$a_{4k-1} - a_{4k-2} = e^{\left(\frac{4k-1}{7}\right)\pi i} - e^{\left(\frac{4k-2}{7}\right)\pi i}$$

$$= e^{\frac{4k\pi}{7}i} \, . \, e^{-\frac{\pi}{7}i} - e^{\frac{4k\pi}{7}} \, . \, e^{-\frac{2\pi}{7}i}$$

$$= e^{\frac{4k\pi}{7}i}\left(e^{-\frac{\pi}{7}i} - e^{-\frac{2\pi}{7}i}\right)$$

$$\therefore$$

$$\left|a_{4k-1} - a_{4k-2}\right| = \left|e^{\frac{4k\pi}{7}i}\right|\left|e^{-\frac{\pi}{7}i} - e^{-\frac{2\pi}{7}i}\right|$$

$$= \left|e^{-\frac{\pi}{7}i} - e^{-\frac{2\pi}{7}i}\right|$$

Now,

$$\frac{\sum_{k=1}^{12}\left|a_{k+1} - a_k\right|}{\sum_{k=1}^{3}\left|a_{4k-1} - a_{4k-2}\right|}$$

$$= \frac{12\left|e^{\frac{\pi}{7}i} - 1\right|}{3\left|e^{-\frac{\pi}{7}i} - e^{-\frac{2\pi}{7}i}\right|}$$

$$= 4\, . \, \frac{\left|e^{\frac{\pi}{7}i} - 1\right|}{\left|\left(e^{-\frac{\pi}{7}i}1 - e^{-\frac{\pi}{7}i}\right)\right|}$$

$$= 4\, . \, \frac{\left|e^{\frac{\pi}{7}i} - 1\right|}{\left|1 - e^{-\frac{\pi}{7}i}\right|}$$

$$= 4\, . \, \frac{\left|e^{\frac{\pi}{7}i}\left(1 - e^{-\frac{\pi}{7}i}\right)\right|}{\left|1 - e^{-\frac{\pi}{7}i}\right|}$$

$$= 4\, . \, \left|e^{\frac{\pi}{7}i}\right|$$

= 4 as

$$\left|e^{\frac{\pi}{7}i}\right| = 1$$

Question 023 Numerical

QUESTION

Suppose that

$$\vec{p}, \vec{q}$$

and

$$\vec{r}$$

are three non-coplanar vectors in

$$R^3$$

. Let the components of a vector

$$\vec{s}$$

along

$$\vec{p},$$

$$\vec{q}$$

and

$$\vec{r}$$

be

$$4, 3$$

and

$$5,$$

respectively. If the components of this vector

$$\vec{s}$$

along

$$\left(-\overrightarrow{p}+\overrightarrow{q}+\overrightarrow{r}\right),\left(\overrightarrow{p}-\overrightarrow{q}+\overrightarrow{r}\right)$$

and

$$\left(-\overrightarrow{p}-\overrightarrow{q}+\overrightarrow{r}\right)$$

are

$$x, y$$

and

$$z,$$

respectively, then the value of

$$2x+y+z$$

is

SOURCE

Mathematics • vector-algebra

EXPLANATION

Here,

$$\overrightarrow{s}=4\overrightarrow{p}+3\overrightarrow{q}+5\overrightarrow{r}$$

.... i

and

$$\overrightarrow{s}=(-\overrightarrow{p}+\overrightarrow{q}+\overrightarrow{r})x+(\overrightarrow{p}-\overrightarrow{q}+\overrightarrow{r})y+(-\overrightarrow{p}-\overrightarrow{q}+\overrightarrow{r})z$$

...... ii

$$\therefore$$

$$4\overrightarrow{p}+3\overrightarrow{q}+5\overrightarrow{r}=\overrightarrow{p}(-x+y-z)+\overrightarrow{q}(x-y-z)+\overrightarrow{r}(x+y+z)$$

On comparing both sides, we get

$$-x + y - z = 4$$

,

$$x - y - z = 3$$

and

$$x + y + z = 5$$

On solving above equations, we get

$$x = 4$$

,

$$y = \frac{9}{2}$$

,

$$z = \frac{-7}{2}$$

$$\therefore$$

$$2x + y + z = 8 + \frac{9}{2} - \frac{7}{2} = 9$$

Question 024 MCQ

QUESTION

Let

$$n_1$$

and

$$n_2$$

be the number of red and black balls, respectively, in box

$$\mathrm{I}$$

. Let

$$n_3$$

and

$$n_4$$

be the number of red and black balls, respectively, in box

$$\mathrm{II.}$$

One of the two boxes, box

$$\mathrm{I}$$

and box

$$\mathrm{II},$$

was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box

$$\mathrm{II}$$

is

$$\frac{1}{3},$$

then the correct option s with the possible values of

$$n_1$$
$$n_2,$$
$$n_3$$

and

$$n_4$$

is are

A $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$

B $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

C $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$

D $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

CORRECT OPTION

A $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$

SOURCE

Mathematics • probability

EXPLANATION

Box I : red balls

$$\rightarrow$$

n_1

black balls

$$\rightarrow$$

n_2

Box II : red balls

$$\rightarrow$$

n_3

black balls

$$\rightarrow$$

n4

$$P(R)=\frac{1}{2}\cdot\frac{n_1}{n_1+n_2}+\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}$$

P $Box II/R$

$$=\frac{\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}}{\frac{1}{2}\cdot\frac{n_1}{n_1+n_2}+\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}}$$

$$=\frac{1}{1+\left(\frac{\frac{n_1}{n_1+n_2}}{\frac{n_3}{n_3+n_4}}\right)}=\frac{1}{3}$$

Thus,

$$\frac{n_1}{n_1+n_2}=2\frac{n_3}{n_3+n_4}$$

i.e.

$$2\left(1+\frac{n_2}{n_1}\right)=1+\frac{n_4}{n_3}$$

i.e.

$$\frac{n_4}{n_3}-2\frac{n_2}{n_1}=1$$

Question 025 MCQ

QUESTION

Let

$$n_1$$

and

$$n_2$$

be the number of red and black balls, respectively, in box

$$\mathrm{I}$$

. Let

$$n_3$$

and

$$n_4$$

be the number of red and black balls, respectively, in box

$$\mathrm{II.}$$

A ball is drawn at random from box

$$\mathrm{I}$$

and transferred to box

$$\mathrm{I}$$
$$\mathrm{I.}$$

If the probability of drawing a red ball from box

$$\mathrm{I},$$

after this transfer, is

$$\frac{1}{3},$$

then the correct option s with the possible values of

$$n_1$$

and

$$n_2$$

is are

$$n_1 = 4$$

A and

$$n_2 = 6$$

B and

$$n_1 = 2$$

$$n_2 = 3$$

C and

$$n_1 = 10$$

$$n_2 = 20$$

D and

$$n_1 = 3$$

$$n_2 = 6$$

CORRECT OPTION

C and

$$n_1 = 10$$

$$n_2 = 20$$

SOURCE

Mathematics • probability

EXPLANATION

$$\therefore$$

P (drawing red ball from B_1) =

$$\frac{1}{3}$$

$$\Rightarrow \left(\frac{n_1 - 1}{n_1 + n_2 - 1}\right)\left(\frac{n_1}{n_1 + n_2}\right) + \left(\frac{n_2}{n_1 + n_2}\right)\left(\frac{n_1}{n_1 + n_2 - 1}\right) = \frac{1}{3}$$

$$\Rightarrow \frac{n_1^2 + n_1 n_2 - n_1}{(n_1 + n_2)(n_1 + n_2 - 1)} = \frac{1}{3}$$

Clearly, options c and d satisfy.

Question 026 Numerical

QUESTION

Let

$$f : R \rightarrow R$$

be a continuous odd function, which vanishes exactly at one point and

$$f(1) = \frac{1}{2.}$$

Suppose that

$$F(x) = \int_{-1}^{x} f(t) dt$$

for all

$$x \in \ [-1, 2]$$

and

$$G(x) =$$

$$\int_{-1}^{x} t \, |f(f(t))| dt$$

for all

$$x \in \ [-1, 2].$$

If

$$\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14},$$

then the value of

$$f\left(\frac{1}{2}\right)$$

is

SOURCE

Mathematics • application-of-integration

EXPLANATION

Here,

$$\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$$

$$\Rightarrow \lim_{x \to 1} \frac{F'(x)}{G'(x)} = \frac{1}{14}$$

$$using L'Hospital's rule$$

.... i

As

$$F(x) = \int_{-1}^{x} f(t) dt$$

$$\Rightarrow F'(x) = f(x)$$

...... ii

and

$$G(x) = \int_{-1}^{x} t|f\{f(t)\}|dt$$

$$\Rightarrow G'(x) = x|f\{f(x)\}|$$

...... iii

$$\therefore$$

$$\lim_{x \to 1} \frac{F(x)}{G(x)} = \lim_{x \to 1} \frac{F'(x)}{G'(x)} = \lim_{x \to 1} \frac{f(x)}{x|f\{f(x)\}|}$$

$$= \frac{f(1)}{1|f\{f(1)\}|} = \frac{1/2}{|f(1/2)|}$$

.... iv

Given,

$$\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$$

$$\therefore$$

$$\frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14} \Rightarrow \left|f\left(\frac{1}{2}\right)\right| = 7$$

Question 027 Numerical

QUESTION

Suppose that all the terms of an arithmetic progression $A.P$ are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is

SOURCE

Mathematics • sequences-and-series

EXPLANATION

$$\frac{S_7}{S_{11}} = \frac{6}{11}$$

...... 1

$$130 \leq t_7 \leq 140$$

.... 2

$$\Rightarrow \frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11}$$

$$\Rightarrow \frac{a+3d}{a+5d} = \frac{6}{7}$$

.... 3

$$\Rightarrow \frac{t_4}{4_6} = \frac{6}{7}$$

Let

$$t_4 = 6k$$

,

$$t_6 = 7k$$

;

$$2d = k \Rightarrow d = k/2$$

and

$$a + 3d = 6k$$

$$\Rightarrow a = 6k - 3k/2 = 9k/2$$

Hence,

$$130 \leq t_7 \leq 140$$

.

$$\Rightarrow 130 \leq \frac{9k}{2} + 3k \leq 140$$

$$\Rightarrow 130 \leq \frac{15k}{2} \leq 140$$

$$\Rightarrow \frac{52}{3} \leq k \in \frac{56}{3}$$

Since,

$$k \in N \Rightarrow k = 18$$

.

$$\Rightarrow d = \frac{k}{2} = \frac{18}{2} = 9$$

Question 028 Numerical

QUESTION

The coefficient of

$$x^9$$

in the expansion of $1 + x \;\; 1 +$ $$x^2)$$(1 + $$x^3$$ ....

$$(1 + x^{100})$$

is

SOURCE

Mathematics • sequences-and-series

EXPLANATION

Given expression is

$$E = (1 + x)(1 + x^2)(1 + x^3)\ldots\ldots(1 + x^{100})$$

Coefficient of x^9 in E

= coefficient of x^9 in

$$(1+x)(1+x^2)(1+x^3)......(1+x^9)$$

$$\Rightarrow$$

Terms containing x^9

$$=(1.x^9+x^1.x^8+x^2.x^7+x^3.x^6+x^4.x^5+x^1.x^2.x^6+x^1.x^3.x^5+$$

$$\Rightarrow$$

Term containing x^9 is $8x^9$ in E

$$\Rightarrow$$

Coefficient of x^9 = 8.

Question 029 MCQ

QUESTION

Let

$$E_1$$

and

$$E_2$$

be two ellipses whose centres are at the origin. The major axes of

$$E_1$$

and

$$E_2$$

lie along the

$$x$$

-axis and the

$$y$$

-axis, respectively. Let

$$S$$

be the circle

$$x^2 + (y-1)^2 = 2$$

. The straight line

$$x + y = 3$$

touches the curves

$$S$$

,

$$E_1$$

and

$$E_2$$

at

$$P, Q$$

and

$$R$$

respectively. Suppose that

$$PQ = PR = \frac{2\sqrt{2}}{3}$$

. If

$$e_1$$

and

$$e_2$$

are the eccentricities of

$$E_1$$

and

$$E_2$$

, respectively, then the correct expression s is are

A

$$e_1^2 + e_2^2 = \frac{43}{40}$$

B

$$e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$$

C

$$\left|e_1^2 + e_2^2\right| = \frac{5}{8}$$

D

$$e_1 e_2 = \frac{\sqrt{3}}{4}$$

CORRECT OPTION

A

$$e_1^2 + e_2^2 = \frac{43}{40}$$

SOURCE

Mathematics • ellipse

EXPLANATION

Here,

$$E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ (a > b)$$

$$E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1,\ (c < d)$$

and

$$S : x^2 + (y-1)^2 = 2$$

as tangent to E_1, E_2 and S is

$$x + y = 3$$

.

Let the point of contact of tangent be

$$(x_1, y_1)$$

to S.

$$\therefore$$

$$x\,.\,x_1 + y\,.\,y_1 - (y + y_1) + 1 = 2$$

or

$$xx_1 + yy_1 - y = (1 + y_1)$$

, same as

$$x + y = 3$$

.

$$\Rightarrow \frac{x_1}{1} = \frac{y_1 - 1}{1} = \frac{1 + y_1}{3}$$

i.e.

$$x_1 = 1$$

and

$$y_1 = 2$$

$$\therefore$$

$$P = (1, 2)$$

Since,

$$PR = PQ = \frac{2\sqrt{2}}{3}$$

. Thus, by parametric form,

$$\frac{x-1}{-1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm\frac{2\sqrt{2}}{3}$$

$$\Rightarrow \left(x = \frac{5}{3}, y = \frac{4}{3}\right)$$

and

$$\left(x = \frac{1}{3}, y = \frac{8}{3}\right)$$

$$\therefore$$

$$Q = \left(\frac{5}{3}, \frac{4}{3}\right)$$

and

$$R = \left(\frac{1}{3}, \frac{8}{3}\right)$$

Now, equation of tangent at Q on ellipse E_1 is

$$\frac{x\,.\,5}{a^2\,.\,3} + \frac{y\,.\,4}{b^2\,.\,3} = 1$$

On comparing with x + y = 3, we get

$$a^2 = 5$$

and

$$b^2 = 4$$

$$\therefore$$

$$e_1^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{5} = \frac{1}{5}$$

..... i

Also, equation of tangent at R on ellipse E_2 is

$$\frac{x\,.\,1}{a^2\,.\,3} + \frac{y\,.\,8}{b^2\,.\,3} = 1$$

On comparing with x + y = 3, we get

$$a^2 = 1,\ b^2 = 8$$

$$\therefore$$

$$e_2^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{1}{8} = \frac{7}{8}$$

...... ii

Now,

$$e_1^2\,.\,e_2^2 = \frac{7}{40} \Rightarrow e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$$

and

$$e_1^2 + e_2^2 = \frac{1}{5} + \frac{7}{8} = \frac{43}{40}$$

Also,

$$\left|e_1^2 - e_2^2\right| = \left|\frac{1}{5} - \frac{7}{8}\right| = \frac{27}{40}$$

Question 030 MCQ

QUESTION

Consider the hyperbola

$$H : x^2 - y^2 = 1$$

and a circle

$$S$$

with center

$$N(x_2, 0)$$

. Suppose that

$$H$$

and

$$S$$

touch each other at a point

$$P(x_1, y_1)$$

with

$$x_1 > 1$$

and

$$y_1 > 0$$

. The common tangent to

$$H$$

and

$$S$$

at

$$P$$

intersects the

$$x$$

-axis at point

$$M$$

. If

$$(l, m)$$

is the centroid of the triangle

$$PMN$$

, then the correct expressions s is are

A

$$\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$$

for

$$x_1 > 1$$

B

$$\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$$

for

$$x_1 > 1$$

C

$$\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$$

for

$$x_1 > 1$$

$$\frac{dm}{dy_1} = \frac{1}{3}$$

for

$$y_1 > 0$$

CORRECT OPTION

$$\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$$

A for

$$x_1 > 1$$

SOURCE

Mathematics • hyperbola

EXPLANATION

Equation of family of circles touching hyperbola at (x_1, y_1) is

(x

$$-$$

$x_1)^2$ + (y

$$-$$

$y_1)^2$ +

$$\lambda$$

(xx_1

$$-$$

yy_1

$$-$$

1) =0

Now, its centre is (x_2, 0).

$$\therefore$$

$$\left[\frac{-(\lambda x_1 - 2x_1)}{2}, \frac{-(-2y_1 - \lambda y_1)}{2}\right] = (x_2, 0)$$

$$\Rightarrow 2y_1 + \lambda y_1 = 0 \Rightarrow \lambda = -2$$

and

$$2x_1 - \lambda x = 2x_2 \Rightarrow x_2 = 2x_1$$

$$\therefore$$

$$P(x_1, \sqrt{x_1^2 - 1})$$

and

$$N(x_2, 0) = (2x_1, 0)$$

As tangent intersect X-axis at

$$M\left(\frac{1}{x}, 0\right)$$

.

Centroid of

$$\Delta PWN = (l, m)$$

$$\Rightarrow \left(\frac{3x_1 + \frac{1}{x_1}}{3}, \frac{y_1 + 0 + 0}{3}\right) = (l, m)$$

$$\Rightarrow l = \frac{3x_1 + \frac{1}{x_1}}{3}$$

On differentiating w.r.t. x_1, we get

$$\frac{dl}{dx_1} = \frac{3 - \frac{1}{x_1^2}}{3}$$

$$\Rightarrow \frac{dl}{dx_1} = 1 - \frac{1}{3x_2^1}$$

, for $x_1 > 1$

and

$$m = \frac{\sqrt{x_1^2 - 1}}{3}$$

On differentiating w.r.t. x_1, we get

$$\frac{dm}{dx_1} = \frac{2x_1}{2 \times 3\sqrt{x_1^2 - 1}} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$$

, for $x_1 > 1$

Also,

$$m = \frac{y_1}{3}$$

On differentiating w.r.t. y_1, we get

$$\frac{dm}{dy_1} = \frac{1}{3}$$

, for $y_1 > 0$

Question 031 Numerical

QUESTION

Suppose that the foci of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

are

$$(f_1, 0)$$

and

$$(f_2, 0)$$

where

$$f_1 > 0$$

and

$$f_2 < 0$$

. Let

$$P_1$$

and

$$P_2$$

be two parabolas with a common vertex at

$$(0, 0)$$

and with foci at

$$(f_1, 0)$$

and

$$(2f_2, 0)$$

, respectively. Let

$$T_1$$

be a tangent to

$$P_1$$

which passes through

$$(2f_2, 0)$$

and

$$T_2$$

be a tangent to

$$P_2$$

which passes through

$$(f_1, 0)$$

. If

$$m_1$$

is the slope of

$$T_1$$

and

$$m_2$$

is the slope of

$$T_2$$

, then the value of

$$\left(\frac{1}{m_1^2} + m_2^2\right)$$

is

SOURCE

Mathematics • parabola

EXPLANATION

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{5}{9} = \frac{4}{9}$$

The foci are $\$\$ \pm \$\$ ae, 0$ i.e. $2, 0$ and $\$\$ - \$\$2, 0$.

The parabola P_1 is

$$y^2 = 8x$$

and P_2 is

$$y^2 = -16x$$

As tangent with slope m_1 to P_1 passes through $\$\$-\$\$4, 0$, we have

$$y = m_1 x + \frac{2}{m_1}$$

giving

$$0 = -4m_1 + \frac{2}{m_1}$$

i.e.

$$4m_1^2 = 2 \Rightarrow m_1^2 = \frac{1}{2}$$

Again for tangent with slope m_2 to P_2 passing through $2, 0$, we have

$$y = m_2 x - \frac{4}{m_2} \Rightarrow 0 = 2m_2 - \frac{4}{m_2}$$

$$\Rightarrow 2m_2^2 = 4$$

$$\therefore$$

$$m_2^2 = 2$$

Thus,

$$\frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$$

Question 032 MCQ

QUESTION

If

$$\alpha$$

$$= 3\sin^{-1}\left(\frac{6}{11}\right)$$

and

$$\beta = 3\cos^{-1}\left(\frac{4}{9}\right),$$

where the inverse trigonimetric functions take only the principal values, then the correct options s is are

A $cos\beta > 0$

B $\sin\beta < 0$

C $\cos(\alpha + \beta) > 0$

D $\cos\alpha < 0$

CORRECT OPTION

B $\sin\beta < 0$

SOURCE

Mathematics • inverse-trigonometric-functions

EXPLANATION

Here,

$$\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$$

and

$$\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$$

as

$$\frac{6}{11} > \frac{1}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{6}{11}\right) > \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore$$

$$\alpha = 3\sin^{-1}\left(\frac{6}{11}\right) > \frac{\pi}{2} \Rightarrow \cos\alpha < 0$$

Now,

$$\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$$

As

$$\frac{4}{9} < \frac{1}{2} \Rightarrow \cos^{-1}\left(\frac{4}{9}\right) > \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\therefore$$

$$\beta = 3\cos^{-1}\left(\frac{4}{9}\right) > \pi$$

$$\therefore$$

$$\cos\beta < 0$$

and

$$\sin\beta < 0$$

Now,

$$\alpha + \beta$$

is slightly greater than

$$\frac{3\pi}{2}$$

.

$$\therefore$$

$$\cos(\alpha+\beta)>0$$

Question 033 MCQ

QUESTION

Let

$$f, g:$$

$$[-1,2]\to R$$

be continuous functions which are twice differentiable on the interval

$$(-1,2)$$

. Let the values of f and g at the points

$$-1,0$$

and

$$2$$

be as given in the following table:

	X = -1	X = 0	X = 2
f x	3	6	0
g x	0	1	-1

In each of the intervals

$$(-1,0)$$

and

$$(0, 2)$$

the function

$$(f - 3g)''$$

never vanishes. Then the correct statement s is are

A

$$f'(x) - 3g'(x) = 0$$

has exactly three solutions in

$$(-1, 0) \cup (0, 2)$$

B

$$f'(x) - 3g'(x) = 0$$

has exactly one solution in

$$(-1, 0)$$

C

$$f'(x) - 3g'(x) = 0$$

has exactly one solution in

$$(0, 2)$$

D

$$f'(x) - 3g'(x) = 0$$

has exactly two solutions in

$$(-1, 0)$$

and exactly two solutions in

$$(0, 2)$$

CORRECT OPTION

$$f'(x) - 3g'(x) = 0$$

C has exactly one solution in

$$(0, 2)$$

SOURCE

Mathematics • application-of-derivatives

EXPLANATION

Let

$$F(x) = f(x) - 3g(x)$$

$$\therefore$$

$$F(-1) = 3$$

,

$$F(0) = 3$$

and

$$F(2) = 3$$

So,

$$F'(x)$$

will vanish at least twice in

$$(-1, 0) \cup (0, 2)$$

.

$$\because$$

$$F''(x) > 0$$

or

$$< 0$$

,

$$\forall x \in (-1, 0) \cup (0, 2)$$

Hence,

$$f'(x) - 3g'(x) = 0$$

has exactly one solution in

$$(-1, 0)$$

and one solution in

$$(0, 2)$$

.

Question 034 MCQ

QUESTION

Let

$$f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$$

for all

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Then the correct expression s is are

A

$$\int_0^{\pi/4} x f(x) dx = \frac{1}{12}$$

B $$\int_{0}^{\pi/4} f(x)dx = 0$$

C $$\int_{0}^{\pi/4} xf(x)dx = \frac{1}{6}$$

D $$\int_{0}^{\pi/4} f(x)dx = 1$$

CORRECT OPTION

A $$\int_{0}^{\pi/4} xf(x)dx = \frac{1}{12}$$

SOURCE

Mathematics • definite-integration

EXPLANATION

$$f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x \ \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$= 7\tan^6 x \,.\, \sec^2 x - 3\tan^2 x \,.\, \sec^2 x$$

$$= (7\tan^6 x - 3\tan^2 x) \,.\, \sec^2 x$$

$$\Rightarrow \int_0^{\pi/4} f(x)dx = \int_0^{\pi/4} (7\tan^6 x - 3\tan^2 x)\sec^2 dx = \int_0^1 (7 + 603t^2)dt = [t^7 - t^3]$$

Also,

$$I = \int_0^{\pi/4} xf(x)dx$$

$$= \left|x \,.\, \int (7\tan^6 x - 3\tan^2 x)\sec^2 x dx\right|_0^{\pi/4} - \int_0^{\pi/4} 1 \,.\, \int (7\tan^6 x - 3\tan^2 x)\sec^2 x$$

$$= \left|x \,.\, (\tan^7 x - \tan^3 x\right|_0^{\pi/4} - \int_0^{\pi/4} (\tan^7 x - \tan^3 x)dx$$

$$= 0 - \int_0^{\pi/4} \tan^3 x(\tan^4 x - 1)dx$$

$$= -\int_0^{\pi/4} \tan^3 x(\tan^2 x - 1)(\sec^2 x)dx$$

$$= -\int_0^1 (t^5 - t^3)dt$$

$$= -\left[\frac{t^6}{6} - \frac{t^4}{4}\right]_0^1 = \left[\frac{1}{4} - \frac{1}{6}\right] = \frac{1}{12}$$

Question 035 MCQ

QUESTION

The option s with the values of a and

L

that satisfy the following equation is are

$$\$\frac{\int\limits_0^{4\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt}{\int\limits_0^{\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt}=L?$$

\$

A

$$a=2, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$$

B

$$a=2, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$$

C

$$a=4, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$$

D

$$a=4, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$$

CORRECT OPTION

A

$$a=2, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$$

SOURCE

Mathematics • definite-integration

EXPLANATION

Let

$$I_1=\int_0^{4\pi} e^t(\sin^6 at+\cos^6 at)dt$$

$$=\int_0^{\pi} e^t(\sin^6 at+\cos^6 at)dt+\int_{\pi}^{2\pi} e^t(\sin^6 at+\cos^6 at)dt+\int_{2\pi}^{3\pi} e^t(\sin^6 at+c$$

$$\therefore$$

$$I_1=I_2+I_3+I_4+I_5$$

...... i

Now,

$$I_3=\int_{\pi}^{2\pi} e^t(\sin^6 at+\cos^6 at)dt$$

Put

$$t=\pi+t\Rightarrow dt=dt$$

$$\therefore$$

$$I_3=\int_0^{\pi} e^{\pi+t}.\ (\sin^6 at+\cos^6 at)dt$$

$$=e^t\ .\ I_2$$

..... ii

Now,

$$I_4=\int_{2\pi}^{3\pi} e^t(\sin^6 at+\cos^6 at)dt$$

Put

$$t=2\pi+t\Rightarrow dt=dt$$

$$\therefore$$

$$I_4=\int_0^{\pi} e^{t+2\pi}(\sin^6 at+\cos^6 at)dt$$

$$= e^{2\pi}. I_2$$

...... iii

and

$$I_5 = \int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^6 at)dt$$

Put

$$t = 3\pi + t$$

$$\therefore$$

$$I_5 = \int_0^{\pi} e^{3\pi+t}(\sin^6 at + \cos^6 at)dt$$

$$= e^{3\pi}.\ I_2$$

...... iv

From Eqs. i, ii, iii and iv, we get

$$I_1 = I_2 + e^{\pi}.\ I_2 + e^{2\pi}.\ I_2 + e^{3\pi}.\ I_2$$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi})I_2$$

$$\therefore$$

$$L = \frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^6 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^6 at)dt}$$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi})$$

$$= \frac{1.(e^{4\pi} - 1)}{e^{\pi} - 1}$$

for

$$a \in R$$

Question 036 MCQ

QUESTION

Let

$$f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$$

for all

$$x \in R$$

with

$$f\left(\frac{1}{2}\right) = 0$$

.

If

$$m \le \int_{1/2}^{1} f(x)dx \le M,$$

then the possible values of

$$m$$

and

$$M$$

are

A

$$m = 13,$$
$$M = 24$$

B

$$m = \frac{1}{4}, M = \frac{1}{2}$$

C

$$m = -11,$$
$$M = 0$$

D

$$m = 1,$$
$$M = 12$$

CORRECT OPTION

D

$$m = 1,$$
$$M = 12$$

SOURCE

Mathematics • definite-integration

EXPLANATION

We have,

$$f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$$

Given,

$$f\left(\frac{1}{2}\right) = 0$$

and

$$m \le \int_{1/2}^{1} f(x) dx \le m$$

$$\therefore$$

fx is increasing in

$$\left(\frac{1}{2}, 1\right)$$

.

$$\therefore$$

$$f'(x)_{\max} = \frac{192}{24} = 96$$

$$\Rightarrow 96 = \frac{f(1) - f(1/2)}{1/2} \Rightarrow f(1) = 96 \times \frac{1}{2} = 48$$

$$M = \frac{1}{2} \times \frac{1}{2} \times 48 = 12$$

$$f'(x)_{\text{min.}} = \frac{\left(\frac{192}{8}\right)}{3} = \frac{192}{24}$$

$$\Rightarrow \frac{192}{24} = \frac{f(1) - 0}{(1/2)} \Rightarrow f(1) = \frac{192}{24} \times \frac{1}{2}$$

$$\therefore$$

$$m = \frac{1}{2} \times \frac{1}{2} \times \frac{192}{24} \times \frac{1}{2} = 1$$

Question 037 MCQ

QUESTION

Let

$$F : R \rightarrow R$$

be a thrice differentiable function. Suppose that

$$F(1) = 0, F(3) = -4$$

and

$$F(x) < 0$$

for all

$$x \in \left(\frac{1}{2}, 3\right).$$

Let

$$f(x) = xF(x)$$

for all

$$x \in R.$$

If

$$\int_1^3 x^2 F'(x) dx = -12$$

and

$$\int_1^3 x^3 F''(x) dx = 40,$$

then the correct expression s is are

A $9f'(3) + f'(1) - 32 = 0$

B $\int_1^3 f(x) dx = 12$

C $9f'(3) - f'(1) + 32 = 0$

D $\int_1^3 f(x) dx = -12$

CORRECT OPTION

C $9f'(3) - f'(1) + 32 = 0$

SOURCE

Mathematics • application-of-integration

EXPLANATION

Given,

$$\int_1^3 x^2 F'(x)dx = -12$$

$$\Rightarrow [x^2 F(x)]_1^3 - \int_1^3 2x \,.\, F(x)dx = -12$$

$$\Rightarrow 9F(3) - F(1) - 2\int_1^3 f(x)dx = -12$$

$$\$\$ \because \$\$\$\$ xF(x) = f(x)\$\$ given$$

$$\Rightarrow -36 - 0 - 2\int_1^3 f(x)dx = -12$$

$$\therefore$$

$$\int_1^3 f(x)dx = -12$$

and

$$\int_1^3 x^3 F''(x)dx = 40$$

$$\Rightarrow [x^3 F'(x)]_1^3 - \int_1^3 3x^2 F'(x)dx = 40$$

$$\Rightarrow [x^2(xF'(x)]_1^3 - 3 \times (-12) = 40$$

$$\Rightarrow \{x^2 \,.\, [f'(x) - F(x)]\}_1^3 = 4$$

$$\Rightarrow 9[f'(3) - F(3)] - [f'(1) - F(1)] = 4$$

$$\Rightarrow 9[f'(3) + 4] - [f'(1) - 0] = 4$$

$$\Rightarrow 9f'(3) - f'(1) = -32$$

Question 038 MCQ

QUESTION

Let

$$F : R \to R$$

be a thrice differentiable function. Suppose that

$$F(1) = 0, F(3) = -4$$

and

$$F'(x) < 0$$

for all

$$x \in \left(\frac{1}{2}, 3\right).$$

Let

$$f(x) = xF(x)$$

for all

$$x \in R.$$

The correct statement s is are

A $$f'(1) < 0$$

B $f(2) < 0$

C for any $f'(x) \neq 0$

$x \in (1, 3)$

D for some $f'(x) = 0$

$x \in (1, 3)$

CORRECT OPTION

A $f'(1) < 0$

SOURCE

Mathematics • differentiation

EXPLANATION

Given,

F 1 = 0, F 3 = 4

F' x < 0 for all x

$\in$

$1, 3$

and f x = xF x

Now, f' x = F x + xF' x

$$\Rightarrow$$

f' 1 = F 1 + 1 . F' 1

$$\Rightarrow$$

f' 1 = 0 + F' 1 1

As F' x < 0 for all x

$$\in$$

$1, 3$

$$\therefore$$

F' 1 < 0

From 1, we get

f' 1 = F' 1 < 0

From Lagrange theorem

$$F'(2) = \frac{F(3) - F(1)}{3 - 1}$$

$$= \frac{-4 - 0}{2}$$

=

$$-$$

2

As f x = xF x

$$\therefore$$

f 3 = 3 . F 3 = 3 . $\$\$ - \$\$4$ =

$$-$$

12

and f 1 = 1 . F 1 = 0

$$\therefore$$

$$f'(2) = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{-12}{2} = -6$$

As, f' x = F x + x . F' x

$$\therefore$$

f' 2 = F 2 + 2 . F' 2

$$\Rightarrow -6 = \frac{f(2)}{2} + 2.(-2)$$

$$\Rightarrow \frac{f(2)}{2} = -2$$

$$\Rightarrow$$

f 2 =

$$-$$

4

$$\therefore$$

f 2 < 0

$$f'(x) = \frac{f(3) - f(1)}{3 - 1}$$

when x

$$\in$$

$1, 3$

f 3 =

$$-$$

12

and f 1 = 0

$$\therefore$$

$$f'(x) = \frac{-12}{2} = -6$$

$$\therefore$$

f' x

$$\neq$$

0 for any x

$$\in$$

$1, 3$

Question 039 Numerical

QUESTION

If

$$\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$$

where

$$\tan^{-1}x$$

takes only principal values, then the value of

$$\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$$

is

SOURCE

Mathematics • definite-integration

EXPLANATION

$$\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$$

Set

$$9x + 3\tan^{-1}x = t$$

so that

$$\frac{dt}{dx} = 9 + \frac{3}{1+x^2} = \frac{12+9x^2}{1+x^2}$$

We have,

$$\alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\therefore$$

$$\ln|\alpha+1| = 9 + \frac{3\pi}{4}$$

Thus

$$\ln|\alpha+1| - \frac{3\pi}{4} = 9$$

Question 040 Numerical

QUESTION

Let m and n be two positive integers greater than 1. If

$$\$ \lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$$

$then the value of$ $\frac{m}{n}$ \$ is __________.

SOURCE

Mathematics • limits-continuity-and-differentiability

EXPLANATION

Given,

$$\lim_{\alpha \to 0}\left[\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right] = -\frac{e}{2}$$

$$\Rightarrow \lim_{\alpha \to 0} \frac{e\{e^{\cos(\alpha^n)-1} - 1\}}{\cos(\alpha^n) - 1} \cdot \frac{\cos(\alpha^n) - 1}{\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow \lim_{\alpha \to 0} e\left\{\frac{e^{\cos(\alpha^n)-1} - 1}{\cos(\alpha^n) - 1}\right\} \cdot \lim_{\alpha \to 0} \frac{-2\sin^2\frac{\alpha^n}{2}}{\alpha^m} = -e/2$$

$$\Rightarrow e \times 1 \times (-2) \lim_{\alpha \to 0} \frac{\sin^2\left(\frac{\alpha^n}{2}\right)}{\frac{\alpha^{2n}}{4}} \cdot \frac{\alpha^{2n}}{4\alpha^m} = \frac{-e}{2}$$

$$\Rightarrow e \times 1 \times -2 \times 1 \times \lim_{\alpha \to 0} \frac{\alpha^{2n-m}}{4} = \frac{-e}{2}$$

For this to be exists,

$$2n - m = 0 \Rightarrow \frac{m}{n} = 2$$

Question 041 MCQ

QUESTION

Consider a uniform spherical charge distribution of radius

$$R_1$$

centred at the origin

$$O.$$

In this distribution, a spherical cavity of radius

$$R_2,$$

centred at

$$P$$

with distance

$$OP = a$$
$$= R_1 - R_2$$

$see figure$ is made. If the electric field inside the cavity at position

$$\vec{r}$$

is

$$\vec{E}(\vec{r}),$$

then the correct statement s is are

A

$$\vec{E}$$

is uniform, its magnitude is independent of

$$R_2$$

but its direction depends on

$$\vec{r}.$$

B

$$\vec{E}$$

is uniform, its magnitude depends on

$$R_2$$

and its direction depends on

$$\vec{r}.$$

$$\vec{E}$$

C is uniform, its magnitude is independent of a but its direction depends on

$$\vec{a}$$

$$\vec{E}$$

D is uniform and both its magnitude and direction depend on

$$\vec{a}$$

CORRECT OPTION

$$\vec{E}$$

D is uniform and both its magnitude and direction depend on

$$\vec{a}$$

SOURCE

Physics • electrostatics

EXPLANATION

Let

$$\rho$$

be the charge density of the spherical charge distribution of radius r_1 centred at the origin O. A spherical cavity of radius r_2 centred at P with distance OP = a = r_1

$$-$$

r_2 is made in the spherical charge distribution.

The sphere with cavity is equivalent to a sphere of uniform charge density

$$-$$

$$\rho$$

and radius r_2 centred at P embedded in the original sphere. Thus, the electric field at a point Q in the cavity is superposition of i electric field at Q due to the sphere of charge density

$$\rho$$

and radius r_1 centred at O (say

$$\vec{E}$$

$_1$), and ii electric field at Q due to the sphere of charge density

$$-$$

$$\rho$$

and radius r_2 centred at P (say

$$\vec{E}$$

$_2$). Let

$$\vec{a}$$

,

$$\vec{r}$$

, and

$$\vec{r}$$

$$-$$

$$\vec{a}$$

be the vectors as shown in the figure. The electric fields

$$\vec{E}$$

1,

$$\vec{E}$$

2, and their superposition

$$\vec{E}$$

12 are given by

$$\vec{E}_1 = \frac{1}{4\pi\in_0} \frac{\frac{4}{3}\pi|\vec{r}|^3\rho}{|\vec{r}|^2}\hat{r} = \frac{\rho}{3\in_0}\vec{r}$$

,

$$\vec{E}_2 = -\frac{\rho}{3\in_0}(\vec{r} - \vec{a})$$

,

$$\vec{E}_{12} = \vec{E}_1 + \vec{E}_2 = \frac{\rho}{3\in_0}\vec{a}$$

.

Thus, the electric field at a point within the cavity is uniform and its magnitude and direction both depend on

$$\vec{a}$$

.

Question 042 MCQ

QUESTION

A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If Pr is the pressure at r $r < R$, then the correct option s is are

A

$$P(r=0)=0$$

B

$$\frac{P(r=3R/4)}{P(r=2R/3)}=\frac{63}{80}$$

C

$$\frac{P(r=3R/5)}{P(r=2R/5)}=\frac{16}{21}$$

D

$$\frac{P(r=R/2)}{P(r=R/3)}=\frac{20}{27}$$

CORRECT OPTION

B

$$\frac{P(r=3R/4)}{P(r=2R/3)}=\frac{63}{80}$$

SOURCE

Physics • properties-of-matter

EXPLANATION

The acceleration due to gravity at a radial distance r $r < R$ from the centre of a sphere of constant mass density

$$\rho$$

is given by

$$g = \frac{G\left(\frac{4}{3}\pi r^3\right)\rho}{r^2} = \frac{4\pi\rho G}{3} r$$

.

Consider a spherical shell of radius r and thickness dr.

Let P and P + dP be the fluid pressures inside and outside the shell. Consider a small element of area A and mass

$$dm = \rho A dr$$

on the shell. The forces on this element are gravitational force dmg direted inwards, the force due to pressure of the fluid outside the shell

$$((P + dP)A)$$

directed inwards, and the force due to pressure of the fluid inside the shell PA directed outwards. In equilibrium, net force on the element is zero, i.e.,

$$(\rho A dr)\frac{4}{3}\pi\rho G r + (P + dP)A = PA$$

, or

$$dP = -\frac{4}{3}\pi G \rho^2 r\, dr$$

.

Note that the pressure decreases with increase in r. The pressure is zero at the surface of the sphere i.e., P = 0 at r = R. Integrate to get

$$\int_0^{P(r)} dP = P(r) = -\frac{4}{3}\pi G\rho^2 \int_R^r r dr$$

$$= \frac{2}{3}\pi G\rho^2 (R^2 - r^2)$$

.

Now,

$$P(r=0) = \frac{2\pi}{3}\rho^2 G R^2 \neq 0$$

$$\frac{P(r=3R/4)}{P(r=2R/3)} = \frac{R^2-(3R/4)^2}{R^2-(2R/3)^2} = \frac{7}{16} \times \frac{9}{5} = \frac{63}{80}$$

$$\frac{P(r=3R/5)}{P(r=2R/5)} = \frac{R^2-(3R/5)^2}{R^2-(2R/5)^2} = \frac{16}{25} \times \frac{25}{21} = \frac{16}{21}$$

$$\frac{P(r=R/2)}{P(r=R/3)} = \frac{R^2-(R/2)^2}{R^2-(R/3)^2} = \frac{3}{4} \times \frac{9}{8} = \frac{27}{32} \neq \frac{20}{27}$$

Question 043 Numerical

QUESTION

The densities of two solid spheres A and B of the same radii R vary with radial distance r as

$$\rho_A(r) = k\left(\frac{r}{R}\right)$$

and

$$\rho_B(r) = k\left(\frac{r}{R}\right)^5$$

, , respectively, where k is a constant. The moments of inertia of the individual spheres about axes passing through their centres are

$$I_A$$

and

$$I_B$$

, respectively. If,

$$\frac{I_B}{I_A} = \frac{n}{10}$$

, the value of n is

SOURCE

Physics • rotational-motion

EXPLANATION

Consider a spherical shell of radius r and small thickness dr.

The volume of the shell is

$$dV = 4\pi r^2 dr$$

and its mass is

$$dm = \rho dV = 4\pi\rho r^2 dr$$

.

The moment of inertia of the spherical shell of mass dm and radius r about an axis passing through its centre O is given by

$$dI = \frac{2}{3} dm\, r^2$$

. Substitute the expressions for dm and

$$\rho$$

and then integrate to get the moment of inertia of the two spheres.

$$I_A = \int_0^R \frac{2}{3}(4\pi\rho_A r^4) dr = \frac{8\pi k}{3R}\int_0^R r^5 dr = \frac{8\pi k R^5}{18}$$

,

$$I_B = \int_0^R \frac{2}{3}(4\pi\rho_B r^4) dr = \frac{8\pi k}{3R^5}\int_0^R r^9 dr = \frac{8\pi k R^5}{30}$$

.

Divide to get

$$I_B/I_A = 6/10$$

.

Question 044 Numerical

QUESTION

The energy of a system as a function of time t is given as E t =

$$A^2 \exp(-\alpha t)$$

, where

$$\alpha = 0.2\,s^{-1}$$

. The measurement of A has an error of 1.25 %. If the error in the measurement of time is 1.50 %, the percentage error in the value of E t at t = 5 s is

SOURCE

Physics • units-and-measurements

EXPLANATION

$$E(t) = A^2 e^{-\alpha t}$$

...... i

$$\alpha$$

= 0.2 s

$$-$$

1

$$\left(\frac{dA}{A}\right) \times 100 = 1.25\%$$

$$\left(\frac{dt}{t}\right) \times 100 = 1.50$$

$$\Rightarrow (dt \times 100) = 1.5t = 1.5 \times 5 = 7.5$$

Differentiating on both sides of equation i, we get

$$d\mathrm{E} = (2\ \mathrm{A}d\ \mathrm{A})e^{-\alpha t} + \mathrm{A}^2 e^{-\alpha t}(-\alpha dt)$$

Dividing throughout by $\mathrm{E} = \mathrm{A}^2 e^{-\alpha t}$

$$\frac{d\mathrm{E}}{\mathrm{E}} = \frac{2}{\mathrm{A}} d\ \mathrm{A} + \alpha dt$$

$$Consideringworstpossiblecase$$

$$\therefore$$

$$\left(\frac{dE}{E}\right) \times 100 = 2\left(\frac{dA}{A}\right) \times 100 + \alpha(dt \times 100)$$

$$= 2(1.25) + 0.2(7.5)$$

$$= 2.5 + 1.5$$

$$= 4\%$$

Question 045 MCQ

QUESTION

In terms of potential difference V, electric current I, permittivity

$$\varepsilon_0$$

, permeability

$$\mu_0$$

and speed of light c, the dimensionally correct equation s is are :

A

$$\mu_0 I^2 = \varepsilon_0 V^2$$

B

$$\varepsilon_0 I = \mu_0 V$$

C

$$I = \varepsilon_0 c V$$

D

$$\mu_0 c I = \varepsilon_0 V$$

CORRECT OPTION

A

$$\mu_0 I^2 = \varepsilon_0 V^2$$

SOURCE

Physics • electromagnetic-waves

EXPLANATION

The speed of light in vacuum is given by,

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

and The impedance $resistance$ of free space is defined a

$$R = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

. Using this check the dimensional correctness of equations.

a

$$\mu_0 I^2 = \varepsilon_0 V^2$$

$$\frac{\mu_0}{\varepsilon_0} = \frac{V^2}{I^2} = R^2$$

$$\therefore$$

$$R^2 = R^2$$

which is dimensionally correct.

b

$$\varepsilon_0 I = \mu_0 V \Rightarrow \frac{\varepsilon_0}{\mu_0} = \frac{V}{I} \Rightarrow \frac{1}{R^2} = R$$

which is dimensionally incorrect.

c

$$I = \varepsilon_0 c V \Rightarrow \frac{I}{V} = \varepsilon_0 C = \frac{\varepsilon_0}{\sqrt{\varepsilon_0 \mu_0}}$$

=

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}$$

=

$$\frac{1}{R}$$

$$\frac{1}{R} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \Rightarrow \frac{1}{R} = \frac{1}{R}$$

which is dimensionally correct.

d

$$\mu_0 c I = \varepsilon_0 V \Rightarrow \frac{\mu_0 c}{\varepsilon_0} = \frac{V}{I}$$

$$\frac{\mu_0}{\varepsilon_0} \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = R \Rightarrow \frac{1}{\varepsilon_0} \sqrt{\frac{\mu_0}{\varepsilon_0}} = R$$

$$\frac{R}{\varepsilon_0} = R$$

, which is dimensionally incorrect.

Question 046 Numerical

QUESTION

Four harmonic waves of equal frequencies and equal intensities I_0 have phase angles 0,

$$\frac{\pi}{3}, \frac{2\pi}{3}$$

and

$$\pi$$

. When they are superposed, the intensity of the resulting wave is nI_0. The value of n is

SOURCE

Physics • waves

EXPLANATION

The intensity of a wave is proportional to the square of its amplitude i.e., $I_0 = cA^2$, where c is a constant. The amplitudes of four harmonic waves are equal as their intensities are equal. Let these waves be travelling along the x direction with wave vector k and angular frequency

$$\omega$$

. The resultant displacement of these waves is given by

$$y = y_1 + y_2 + y_3 + y_4$$

$$= A\sin(\omega t - kx + 0) + A\sin(\omega t - kx + \pi/3) + Asin(\omega t - kx + 2\pi/3) + A\,s$$

$$= A\sin(\omega t - kx + \pi/3) + Asin(\omega t - kx + 2\pi/3)$$

$$= 2A\sin(\omega t - kx + \pi/2)\cos(\pi/6)$$

$$= \sqrt{3}A\cos(\omega t - kx)$$

.

The amplitude of the resultant wave is

$$A_r = \sqrt{3}A$$

and its intensity is

$$I_r = cA_r^2 = 3cA^2 = 3I_0$$

.

Question 047 Numerical

QUESTION

For a radioactive material, its activity A and rate of change of its activity R are defined as

$$A = -\frac{dN}{dt}$$

and

$$R = -\frac{dA}{dt}$$

, where Nt is the number of nuclei at time t. Two radioactive source P $meanlife$$\tau$$$ and Q $meanlife2$$\tau$$$ have the same activity at t = 0. Their rate of change of activities at t = 2

τ

are R_P and R_Q, respectively. If

$$\frac{R_P}{R_Q} = \frac{n}{e}$$

, then the value of n is

SOURCE

Physics • atoms-and-nuclei

EXPLANATION

Law of radioactivity :

$$N = N_0 e^{-\lambda t}$$

where

$$\lambda$$

= decay constant

Activity

$$|A| = \left|\frac{-dN}{dt}\right| = N_0 \lambda e^{-\lambda t}$$

Rate of activity

$$R = \frac{d|A|}{dt} = N_0 \lambda^2 e^{-\lambda t}$$

At t = 0, $A_1 = A_2$. Therefore,

$$N_{OP}\lambda_P = N_{OQ}\lambda_Q$$

At

$$t = 2\tau, \frac{R_P}{R_Q} = \left(\frac{\lambda_P}{\lambda_Q}\right)^2 \left(\frac{N_{OP}}{N_{OQ}}\right) \frac{e^{-\lambda P(25)}}{e^{-\lambda Q(25)}} = \frac{\lambda_P}{\lambda_Q} e^{(\lambda_Q - \lambda_P)25}$$

Since mean life is given by

$$\tau = \frac{1}{\lambda}$$

.

Therefore,

$$\frac{R_P}{R_Q} = \frac{\lambda_P}{\lambda_Q} e^{\left(\frac{1}{25} - \frac{1}{5}\right)^{25}} = \frac{\lambda_P}{\lambda_Q} e^{-1}$$

$$\frac{R_P}{R_Q} = \frac{\lambda_{\{}}{\lambda_Q} \frac{1}{e} = \frac{n}{e}$$

$$n = \frac{\lambda_P}{\lambda_Q} = \frac{2\tau}{\tau} = 2$$

Question 048 Numerical

QUESTION

A monochromatic beam of light is incident at 60

$$\circ$$

on one face of an equilateral prism of refractive index n and emerges from the opposite face making an angle

$$\theta$$

n with the normal $see figure$. For n =

$$\sqrt{3}$$

the value of

$$\theta$$

is 60

$$\circ$$

and

$$\frac{d\theta}{dn} = m$$

. The value of m is

SOURCE

Physics • geometrical-optics

EXPLANATION

Using Snell's law:

$$\sin 60^\circ = n \sin r_1$$

..... 1

$$\sin r_1 = \frac{\sqrt{3}}{2 \times \sqrt{3}} = \frac{1}{2}$$

$$r_1 = 30^\circ$$

Also,

$$n \sin r_2 = 1 \sin \theta$$

Also

$$r_1 + r_2 = A = 60^\circ$$

Therefore,

$$n \sin(60^\circ - r_1) = 1 \sin \theta$$

.... 2

Differentiating on both sides, we get

$$\sin(60^\circ - r_1) - n \cos(60^\circ - r_1) \frac{dr_1}{dn} = \cos \theta \frac{d\theta}{dn}$$

Differentiating Eq. 1 on both sides, we get

$$0 = \sin r_1 + n \cos r_1 \frac{dr_1}{dn}$$

$$0 = \frac{1}{2} + \sqrt{3}. \frac{\sqrt{3}}{2} \frac{dr_1}{dn}$$

Therefore,

$$\frac{dr_1}{dn} = \frac{-1}{3}$$

Hence, substituting

$$r_1 = 30^\circ$$

, we get

$$\frac{dr_1}{dn} = \frac{-1}{3}$$

Now,

$$\sin 30^\circ - \sqrt{3} \cos 30^\circ \left(-\frac{1}{3}\right) = \cos 60^\circ \frac{d\theta}{dn}$$

$$\frac{1}{2} + \frac{3}{2 \times 3} = \frac{1}{2} \frac{d\theta}{dn}$$

$$\frac{d\theta}{dn} = 2$$

Question 049 Numerical

QUESTION

In the following circuit, the current through the resistor R$=$ $2\$\$\Omega\$\$$ is I amperes. The value of I is :

SOURCE

Physics • current-electricity

EXPLANATION

Consider the following figure:

ACGA constitutes a Wheatstone bridge; hence, 8

$$\Omega$$

is redundant and hence can be removed. Therefore,

$$R_{AG} = \frac{3 \times 6}{9} = 2\,\Omega$$

AGDFA again constitutes a Wheatstone bridge 10

$$\Omega$$

which is redundant and hence can be removed.

$$R_{AB} = \frac{6 \times 18}{24} = 4.5\ \Omega$$

$$I = \frac{6.5}{6.5} = 1\,A$$

Question 050 Numerical

QUESTION

An electron in an excited state of Li^{2+} ion has angular momentum

$$\frac{3h}{2\pi}$$

. The de Broglie wavelength of the electron in this state is p

$$\pi$$

a_0 (where a_0 is the Bohr radius). The value of p is

SOURCE

Physics • dual-nature-of-radiation

EXPLANATION

Angular momentum

$$mvr = \frac{nh}{2\pi}$$

where

$$r = 3a_0$$

where

$$n = 3$$

, that is, electron in

$$Li^{2+}$$

is in second excited state

$$\lambda = \frac{h}{mv} = p\pi a_0$$

$$\Rightarrow n = p\pi(mva_0) = p\pi\left(\frac{mvr}{3}\right) = \frac{p\pi}{3}\left(\frac{3h}{2\pi}\right) = \frac{ph}{2}$$

Therefore,

$$p = 2$$

.

Question 051 Numerical

QUESTION

A large spherical mass M is fixed at one position and two identical masses m are kept on a line passing through the centre of M $see figure$. The point masses are connected by a rigid massless rod of length l and this assembly is free to move along the line connecting them.

All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance r = 3l from M the tension in the rod is zero for m =

$$k\left(\frac{M}{288}\right)$$

. The value of k is

SOURCE

Physics • gravitation

EXPLANATION

The acceleration $\vec{a}$ of the point masses are equal because they are connected by a massless rigid rod.

Consider the situation when tension in the rod is zero. The gravitational forces on the two point masses are shown in the figure. The forces $f_1 = \frac{GMm}{r^2}$ and $f_3 = \frac{GMm}{(r+l)^2}$ are due to the attraction by the larger mass M. The force $f_2 = \frac{Gmm}{l^2}$ is due to mutual attraction between the two point masses. Apply Newton's second law on the two point masses to get

$$\frac{GMm}{r^2} - \frac{Gmm}{l^2} = ma$$

.... 1

$$\frac{GMm}{(r+l)^2} + \frac{Gmm}{l^2} = ma$$

.... 2

From eqn. 1 and 2, we get

$$\frac{GM}{9l^2} - \frac{Gm}{l^2} = \frac{GM}{16l^2} + \frac{Gm}{l^2}$$

$$\frac{M}{9} - \frac{M}{16} = m + m \Rightarrow \frac{7M}{144} = 2m$$

$$m = \frac{7M}{288} = k\left(\frac{M}{288}\right)$$

$$\therefore$$

k = 7

Question 052 MCQ

QUESTION

In plotting stress versus strain curves for two materials P and Q, a student by mistake puts strain on the y-axis and stress on the x-axis as shown in the figure. Then, the correct statements is/are

A P has more tensile strength than Q

B P is more ductile than Q

C P is more brittle than Q

D The Young's modulus of P is more than that of Q

CORRECT OPTION

A P has more tensile strength than Q

SOURCE

Physics • properties-of-matter

EXPLANATION

We know,

$$Y = \frac{Stress}{Strain}$$

According to graph,

Slope of curve

$$= \frac{Change\, in\, strain}{Change\, in\, stress} = \frac{1}{Y}$$

$Slope_P > Slope_Q$

$$\therefore$$

$Y_P < Y_Q$

P has more tensile strength than Q as it sustains more stress after elastic limit.

There is large deformation between the elastic limit and the fracture point for material P as compared to material Q. Hence, P is more ductile than Q.

After the elastic limit, Q breaks soon as compared to P. So, Q is more brittle than P.

Question 053 MCQ

QUESTION

A parallel plate capacitor having plates of area S and plate separation d, has capacitance C_1 in air. When two dielectrics of different relative permittivities (

$$\varepsilon$$

$_1$ = 2 and

$$\varepsilon$$

$_2$ = 4) are introduced between the two plates as shown in the figure, the capacitance becomes C_2. The ratio

$$\frac{C_2}{C_1}$$

is

A $\frac{6}{5}$

B $\frac{5}{3}$

C $\frac{7}{5}$

D $\frac{7}{3}$

CORRECT OPTION

D

$$\frac{7}{3}$$

SOURCE

Physics • capacitor

EXPLANATION

We can think of this configuration to be made up of three parts:

a Capacitor of plate area s/2, plate separation d, and filled with a dielectric of relative permittivity

$$\in$$

$_1$ = 2 $lowerhalf$. The capacitance of this part is

$$C_a = \in_1 \in_0 \frac{s/2}{d} = \in_0 \frac{s}{d} = C_1$$

.

b Capacitor of plate area s/2, plate separation d/2, and filled with a dielectric of relative permittivity

$$\in$$

$_1$ = 2 $leftupperhalf$. The capacitance of this part is

$$C_b = \in_1 \in_0 \frac{s/2}{d/2} = 2 \in_0 \frac{s}{d} = 2C_1$$

.

c Capacitor of plate area s/2, plate separation d/2, and filled with a dielectric of relative permittivity

$$\in$$

$_2$ = 4 $rightupperhalf$. The capacitance of this part is

$$C_c = \in_2 \in_0 \frac{s/2}{d/2} = 4\in_0 \frac{s}{d} = 4C_1$$

.

The capacitors C_b and C_c are connected in series with equivalent capacitance

$$C_{bc} = \frac{C_b C_c}{C_b + C_c} = \frac{(2C_1)(4C_1)}{(2C_1) + (4C_1)} = \frac{4}{3}C_1$$

.

The capacitor C_{bc} is connected in parallel with C_a. The equivalent capacitance is

$$C_2 = C_{abc} = C_a || C_{bc} = C_1 + \frac{4}{3}C_1 = \frac{7}{3}C_1$$

.

Question 054 MCQ

QUESTION

An ideal monoatomic gas is confined in a horizontal cylinder by a spring loaded piston $as shown in the figure$. Initially the gas is at temperature T_1, pressure P_1 and volume V_1 and the spring is in its relaxed state. The gas is then heated very slowly to temperature T_2, pressure P_2 and volume V_2. During this process the piston moves out by a distance x.

Ignoring the friction between the piston and the cylinder, the correct statements is/are

A If $V_2 = 2V_1$ and $T_2 = 3T_l$, then the energy stored in the spring is

$$\frac{1}{4}P_1V_1$$

B If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the change in internal energy is

$$3P_1V_1$$

C If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the work done by the gas is

$$\frac{7}{3}P_1V_1$$

D If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the heat supplied to the gas is

$$\frac{17}{6}P_1V_1$$

CORRECT OPTION

B If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the change in internal energy is

$$3P_1V_1$$

SOURCE

Physics • heat-and-thermodynamics

EXPLANATION

Initially both the compartments has same pressure as they are in equilibrium.

Suppose spring is compressed by x on heating the gas.

Let A be the area of cross-section of piston. As gas is ideal monoatomic, so

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

...... i

Force on spring by gas = kx

$$\therefore$$

$$P_2 = P_1 + \frac{kx}{A}$$

...... ii

Case I : When

$$V_2 = 2V_1$$

,

$$T_2 = 3T_1$$

From Eqn. i

$$\frac{P_1V_1}{T_1} = \frac{P_2(2V_1)}{3T_1} \Rightarrow P_2 = \frac{3}{2}P_1$$

Putting this value in eqn. ii we get

$$\frac{3}{2}P_1 = P_1 + \frac{kx}{A} \Rightarrow kx = \frac{P_1A}{2}$$
$$x = \frac{V_2 - V_1}{A} = \frac{2V_1 - V_1}{A} = \frac{V_1}{A}$$

Energy stored in the spring

$$= \frac{1}{2}kx^2 = \frac{1}{2}(kx)(x) = \frac{P_1V_1}{4}$$

So, option a is correct.

Change in internal energy,

$$\Delta U = \frac{f}{2}(P_2V_2 - P_1V_1) = \frac{3}{2}\left(\frac{3}{2}P_1 \times 2V_1 - P_1V_1\right) = 3P_1V_1$$

So, option b is correct.

Case II : When

$$V_2 = 3V_1$$

and

$$T_2 = 4T_1$$

From Eqn. i,

$$\frac{P_1V_1}{T_1} = \frac{P_2(3V_1)}{4T_1} \Rightarrow P_2 = \frac{4}{3}P_1$$

$$x = \frac{V_2 - V_1}{A} = \frac{2V_1}{A}$$

From eqn. ii,

$$\frac{4}{3}P_1 = P_1 + \frac{kx}{A} \Rightarrow kx = \frac{P_1A}{3}$$

Gas is heated very slowly so pressure on the other compartment remains same.

Work done by gas = Work done by gas on atmosphere + Energy stored in spring.

$$W_g = P_1Ax + \frac{1}{2}kx^2 = P_1(2V_1) + \frac{1}{2}\left(\frac{P_1A}{3}\right)\left(\frac{2V_1}{A}\right)$$

$$= 2P_1V_1 + \frac{1}{3}P_1V_1 = \frac{7}{3}P_1V_1$$

So, option c is correct.

Heat supplied to the gas,

$$\Delta Q = W_g + \Delta U$$

$$= \frac{7}{3}P_1V_1 + \frac{3}{2}(P_2V_2 - P_1V_1)$$

$$= \frac{7}{3}P_1V_1 + \frac{3}{2}\left(\frac{4}{3}P_1 \times 3V_1 - P_1V_1\right)$$

$$= \frac{7}{3}P_1V_1 + \frac{9}{2}P_1V_1 = \frac{41}{6}P_1V_1$$

So, option d is incorrect.

Question 055 MCQ

QUESTION

A fission reaction is given by

$$^{236}_{92}U \rightarrow ^{140}_{54}Xe + ^{94}_{38}Sr + x + y$$

, where x and y are two particles. Considering

$$^{236}_{92}U$$

to be at rest, the kinetic energies of the products are denoted by

$$K_{Xe}, K_{Sr}, K_x(2MeV)$$

$$\text{and } \mathrm{K_y(2MeV)}$$

, respectively. Let the binding energies per nucleon of

$$^{236}_{92}U$$

,

$$^{140}_{54}Xe$$

and

$$^{94}_{38}Sr$$

be 7.5 MeV, 8.5 MeV and 8.5 MeV, respectively. Considering different conservation laws, the correct options is/are

A x = n, y = n, K_{sr} = 129 MeV, K_{Xe} = 86 MeV

B $x = p, y = e$

$-$

, K_{sr} = 129 MeV, K_{Xe} = 86 MeV

C $x = p, y = n$, K_{sr} = 129 MeV, K_{Xe} = 86 MeV

D $x = n, y = n$, K_{sr} = 86 MeV, K_{Xe} = 129 MeV

CORRECT OPTION

A $x = n, y = n$, K_{sr} = 129 MeV, K_{Xe} = 86 MeV

SOURCE

Physics • atoms-and-nuclei

EXPLANATION

$$^{236}_{92}U \to {}^{140}_{54}Xe + {}^{94}_{38}Sr + x + y$$

K_x = 2 MeV, K_y = 2 MeV, K_{Xe} = ?, K_{Sr} = ?

By conservation of charge number and mass number, x

$$\equiv$$

y

$$\equiv$$

n

B.E. per nucleon of

$$^{236}_{92}U = 7.5$$

MeV

B.E. per nucleon of

$$^{140}_{54}Xe$$

or

$$^{94}_{38}Sr = 8.5$$

MeV

Q value of reaction,

Q = Net kinetic energy gained in the process

$$= K_{Xe} + K_{Sr} + 2 + 2 - 0 = K_{Xe} + K_{Sr} + 4$$

..... 1

As number of nucleons is conserved in a reaction, so Q = Difference of binding energies of the nuclei

$$= 140 \times 8.5 + 94 \times 8.5 - 236 \times 7.5 = 219$$

MeV 2

From eqns. i and ii

$$K_{Xe} + K_{Sr} = 219 - 4 = 215$$

MeV 3

The linear momentum of a particle of mass m and kinetic energy K is given by

$$p = \sqrt{2mK}$$

. Since the masses and kinetic energies of x and y are very small in comparison to that of

$$^{140}_{54}Xe$$

and

$$^{94}_{38}Sr$$

, we can neglect the linear momentum of these particles. Initially, the linear momentum of

$$^{236}_{92}U$$

is zero $atrest$. Finally, the products

$$^{140}_{54}Xe$$

and

$$^{94}_{38}Sr$$

will move in opposite direction with equal linear momentum $byconservationoflinearmomentum$. Thus,

$$\sqrt{2M_{Xe}K_{Xe}} = \sqrt{2M_{Sr}K_{Sr}}$$

, i.e.,

$$140K_{Xe} = 94K_{Sr}$$

. ... 4

Solve equations 3 and 4 to get K_{Xe} = 86 MeV and K_{Sr} = 129 MeV.

Question 056 MCQ

QUESTION

Two spheres P and Q for equal radii have densities

$$\rho$$

$_1$ and

$$\rho$$

$_2$, respectively. The spheres are connected by a massless string and placed in liquids L_1 and L_2 of densities

$$\sigma$$

$_1$ and

$$\sigma$$

$_2$ and viscosities

$$\eta_1$$

and

$$\eta_2$$

, respectively. They float in equilibrium with the sphere P in L_1 and sphere Q in L_2 and the string being taut $see figure$ If sphere P along in L_2 has terminal velocity v_P and Q alone in L_1 ha terminal velocity v_Q, then

A

$$\frac{|v_P|}{|v_Q|} = \frac{\eta_1}{\eta_2}$$

B

$$\frac{|v_P|}{|v_Q|} = \frac{\eta_2}{\eta_1}$$

C $v_P \cdot v_Q > 0$

D $v_P \cdot v_Q < 0$

CORRECT OPTION

A

$$\frac{|v_P|}{|v_Q|} = \frac{\eta_1}{\eta_2}$$

SOURCE

Physics • properties-of-matter

EXPLANATION

Let $V=\frac{4}{3}\pi r^3$ be the volume of the spheres P and Q of equal radii r. The forces acting on the sphere P are its weight $\rho_1 Vg$, tension from the string T, and the buoyancy force $\sigma_1 Vg$.

Similarly, forces on the sphere Q are $\rho_2 Vg, T$, and $\sigma_1 Vg$. In equilibrium, the net force on the spheres P and Q are separately zero i.e.,

$$T+\rho_1 Vg=\sigma_1 Vg.........(1)$$

$$T+\sigma_2 Vg=\rho_2 Vg..........(2)$$

The tension $T>0$ because the string is taut. Thus, equation 1 gives $\rho_1<\sigma_1$ and equation 2 gives $\rho_2>\sigma_2$. Eliminate T from equations 1 and 2 to get

$$\sigma_1-\rho_1=\rho_2-\sigma_2$$

..........3

Now, consider the situation when the sphere P moves in liquid L_2 and the sphere Q moves in liquid L_1. These spheres will attain the terminal velocities $\vec{v}_{\mathrm{P}}$ and $\vec{v}_{\mathrm{Q}}$ after some time. The direction of the velocity $upwardsordownwards$ will depend on the density of the sphere in comparison to the density of the liquid. Let us consider the case when $\rho_1>\sigma_2$. In this case, the velocity of the sphere P is downwards. From equation 3, if $\rho_1>\sigma_2$ then $\rho_2<\sigma_1$. If the density of a sphere is less than the density of the liquid in which it is immersed, it will move up. Thus, the velocity of the sphere Q is upwards i.e., the directions of $\vec{v}_{\mathrm{P}}$ and $\vec{v}_{\mathrm{Q}}$ are opposite.

Hence, $\vec{v}_{\mathrm{P}}\cdot\vec{v}_{\mathrm{Q}}<0$.

The forces on the sphere P are its weight $\rho_1 V g$, buoyancy force $\sigma_2 V g$, and viscous drag $6\pi\eta_2 r v_{\mathrm{P}}$ $see figure$. Similarly, the forces on the sphere Q are $\rho_2 V g, \sigma_1 V g$ and $6\pi\eta_1 r v_{\mathrm{Q}}$. Net forces on the spheres are zero when they move with terminal velocities i.e.,

$$6\pi\eta_2 r v_{\mathrm{P}} + \rho_1 V g = \sigma_2 V g \ldots\ldots\ldots\ldots (4)$$

$$6\pi\eta_1 r v_{\mathrm{Q}} + \sigma_2 V g = \rho_1 V g \ldots\ldots\ldots (5)$$

Solving equations 4 and 5, we get

$$v_{\mathrm{P}} = \frac{2r^2\left(\rho_1 - \sigma_2\right)}{9\eta_2} \ldots\ldots\ldots (6)$$

$$v_{\mathrm{Q}} = \frac{2r^2\left(\sigma_1 - \rho_2\right)}{9\eta_1} \ldots\ldots\ldots (7)$$

By dividing equation 6 by 7, we get

$$\frac{\left|\vec{v}_{\mathrm{P}}\right|}{\left|\vec{v}_{\mathrm{Q}}\right|} = \frac{\eta_1}{\eta_2}\frac{\rho_1 - \sigma_2}{\sigma_2 - \rho_2} = \frac{\eta_1}{\eta_2}.$$

Question 057 MCQ

QUESTION

For two structures namely S_1 with n_1 =

$$\frac{\sqrt{45}}{4}$$

and n_2 =

$$\frac{3}{2}$$

, and S_2 with n_1 =

$$\frac{8}{5}$$

and n_2 =

$$\frac{7}{5}$$

and taking the refractive index of water to be

$$\frac{4}{3}$$

and that to air to be 1, the correct options is/are :

A

NA of S_1 immersed in water is the same as that of S_2 immersed in a liquid of refractive index

$$\frac{16}{3\sqrt{15}}$$

B

NA of S_1 immersed in liquid of refractive index

$$\frac{6}{\sqrt{15}}$$

is the same as that of S_2 immersed in water

C

NA of S_1 placed in air is the same as that S_2 immersed in liquid of refractive index

$$\frac{4}{\sqrt{15}}$$

D NA of S_1 placed in air is the same as that of S_2 placed in water

CORRECT OPTION

A NA of S_1 immersed in water is the same as that of S_2 immersed in a liquid of refractive index

$$\frac{16}{3\sqrt{15}}$$

SOURCE

Physics • geometrical-optics

EXPLANATION

Let the whole structure be placed in the medium of refractive index n_0. From geometry, if the angle of incidence at Q is

$$\theta$$

then the angle of refraction at P is 90

$$\circ$$

$$-$$

$$\theta$$

.

The ray will undergo total internal reflection at Q if the angle of incidence at Q is greater than or equal to the critical angle i.e.,

$$\theta$$

$$\geq$$

$$\theta$$

$_c$ = sin

$$-$$

$^1(n_2 / n_1)$. 1

Apply Snell's law for refraction at P to get

n_0 sin i = n_1 sin $90\$\$^\circ\$\$\$\$ - \$\$\$\$\theta\$\$$ = n_1 cos

$$\theta$$

=

$$n_1\sqrt{1-\sin^2\theta}$$

. ... 2

From equation 2, the angle of incidence is maximum (i = i_m) when

$$\theta$$

is minimum i.e., when

$$\theta$$

=

$$\theta$$

$_c$ $fromequation(1)$. Thus, the numerical aperture is given by

$$NA = \sin i_m = \frac{n_1\sqrt{1-\sin^2\theta_c}}{n_0} = \frac{\sqrt{n_1^2 - n_2^2}}{n_0}$$

.... 3

Substitute

$$n_1 = \sqrt{45}/4$$

and

$$n_2 = 3/2$$

in equation 3 to get the numerical aperture for the structure S_1 is

$$NA_1 = \frac{\sqrt{45/16 - 9/4}}{n_0} = \frac{3}{4n_0}$$

...... 4

Similarly, substitute n_1 = 8/4 and n_2 = 7/5 in equation 3 to get the numerical aperture for the structure S_2 as

$$NA_2 = \frac{\sqrt{64/25 - 49/25}}{n_0} = \frac{\sqrt{15}}{5n_0}$$

...... 5

In case A, substitute n_0 = 4/3 in equation 4 to get NA_1 = 9/16 and substitute n_0 = 16/3

$$\sqrt{15}$$

in equation 5 to get NA_2 = 9/16.

In case B, substitute n_0 = 6/

$$\sqrt{15}$$

in equation 4 to get NA_1 =

$$\sqrt{15}$$

/8 and substitute n_0 = 4/3 in equation 5 to get NA_2 = 3

$$\sqrt{15}$$

/20.

In case C, substitute n_0 = 1 in equation 4 to get NA_1 = 3/4 and substitute n_0 = 4/

$$\sqrt{15}$$

equation 5 to get NA_2 = 3/4.

In case D, substitute n_0 = 1 in equation 4 to get NA_1 = 3/4 and substitute n_0 = 4/3 equation 5 to get NA_2 = 3

$$\sqrt{15}$$

/20.

Question 058 MCQ

QUESTION

If two structures of same cross-sectional area, but different numeral apertures NA_1 and NA_2 ($NA_2 < NA_1$) are joined longitudinally, the numerical aperture of the combined structure is

A $$\frac{NA_1 NA_2}{NA_1 + NA_2}$$

B $$NA_1 + NA_2$$

C $$NA_1$$

D $$NA_2$$

CORRECT OPTION

D $$NA_2$$

SOURCE

Physics • geometrical-optics

EXPLANATION

$$\sin i_m = n_1 \sin(90 - \theta_c)$$

$$\Rightarrow \sin i_m = n_1 \cos\theta_c$$

$$\Rightarrow NA = n_1\sqrt{1 - \sin^2\theta_c}$$

$$= n_1\sqrt{1 - \frac{n_2^2}{n_1^2}} = \sqrt{n_1^2 - n_2^2}$$

Substituting the values we get,

$$NA_1 = \frac{3}{4}$$

$$\text{and } NA_2 = \frac{\sqrt{15}}{5} = \sqrt{\frac{3}{4}}$$

and

$$NA_2 < NA_1$$

Therefore, the numerical aperture of combined structure is equal to the lesser of the two numerical aperture, which is NA_2.

Question 059 MCQ

QUESTION

Consider two different metallic strips $1 and 2$ of the same material. Their lengths are the same, widths are w_1 and w_2 and thickness are d_1 and d_2, respectively. Two points K and M are symmetrically located on the opposite faces parallel to the x-y plane $see figure$. V_1 and V_2 are the potential differences between K and M in strips 1 and 2, respectively. Then, for a given current I flowing through them in a given magnetic field strength B, the correct statements is/are

A. If $w_1 = w_2$ and $d_1 = 2d_2$, then $V_2 = 2V_1$

B. If $w_1 = w_2$ and $d_1 = 2d_2$, then $V_2 = V_1$

C. If $w_1 = 2w_2$ and $d_1 = d_2$, then $V_2 = 2V_1$

D. If $w_1 = 2w_2$ and $d_1 = d_2$, then $V_2 = V_1$

CORRECT OPTION

A. If $w_1 = w_2$ and $d_1 = 2d_2$, then $V_2 = 2V_1$

SOURCE

Physics • magnetism

EXPLANATION

At equilibrium,

$$\mathrm{F_M} = \mathrm{F_E}$$

$$q\mathrm{vB} = q\mathrm{E}$$

$$\mathrm{E} = \mathrm{v} \cdot \mathrm{B}$$

$$i = ne\ \mathrm{A}v_d$$

where $v_d =$ drift speed of electrons and $n =$ electron density

$$\begin{aligned} \mathrm{E} &= \frac{i}{ne\ \mathrm{A}}\ \mathrm{B} \\ &= \frac{i}{ne(\omega d)} \end{aligned}$$

Area of cross section is always taken normal to the direction of flow of current.
Potential difference,

$\mathrm{V} = \mathrm{E}\omega = \frac{i\,\mathrm{B}}{ned}$

For the two strips

$\mathrm{V}_1 = \frac{i\,\mathrm{B}}{ne}\left(\frac{1}{d_1}\right)$

$$\mathrm{V}_2 = \frac{iB}{ne}\left(\frac{1}{d_2}\right)$$

For option A

$$\frac{V_1}{V_2} = \frac{d_2}{d_1} = \frac{1}{2}$$

For option B

$$\frac{V_1}{V_2} = 1, \frac{d_2}{d_1} = \frac{1}{2}$$

For option C

$$\frac{V_1}{V_2} = \frac{1}{2}, \frac{d_2}{d_1} = 1$$

For option D

$$\frac{V_1}{V_2} = \frac{d_2}{d_1} = 1$$

Question 060 MCQ

QUESTION

Consider two different metallic strips $1 and 2$ of same dimensions $length l, width w and thickness d$ with carrier densities n_1 and n_2, respectively. Strip 1 is placed in magnetic field B_1 and strip 2 is placed in magnetic field B_2, both along positive y-directions. Then V_1 and V_2 are the potential differences developed between K and M in strips 1 and 2, respectively. Assuming that the current I is the same for both the strips, the correct options is/are :

A If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = 2V_1$

B If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = V_1$

C If $B_1 = 2B_2$ and $n_1 = n_2$, then $V_2 = 0.5V_1$

D If $B_1 = 2B_2$ and $n_1 = n_2$, then $V_2 = V_1$

CORRECT OPTION

A If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = 2V_1$

SOURCE

Physics • magnetism

EXPLANATION

$$\mathrm{V}_1 = \frac{i\,\mathrm{B}_1}{n_1 e d}$$

$$\text{and} \qquad \mathrm{V}_2 = \frac{i\,\mathrm{B}_2}{n_2 e d}$$

$$\frac{\mathrm{V}_1}{\mathrm{V}_2} = \frac{\mathrm{B}_1}{\mathrm{B}_2}\frac{n_2}{n_1}$$

For option A :

$$\frac{\mathrm{V}_1}{\mathrm{V}_2} = \frac{1}{2}$$

and $\frac{\mathrm{B}_1}{\mathrm{B}_2} \frac{n_2}{n_1} = 1 \times \frac{1}{2} = \frac{1}{2}$

For option B :

$$\frac{\mathrm{V}_1}{\mathrm{V}_2} = 1$$

$$\text{and } \frac{\mathrm{B}_1}{\mathrm{B}_2} \frac{n_2}{n_1} = 1 \times \frac{1}{2} = \frac{1}{2}$$

For option C :

$$\frac{\mathrm{V}_1}{\mathrm{V}_2} = 2$$

$$\text{and } \frac{\mathrm{B}_1}{\mathrm{B}_2} \frac{n_2}{n_1} = 2 \times 1 = 2$$

For option D :

$$\frac{\mathrm{V}_1}{\mathrm{V}_2} = 1$$

$$\text{and } \frac{\mathrm{B}_1}{\mathrm{B}_2} \frac{n_2}{n_1} = 2 \times 1 = 2$$