JEE Advanced 2016 Paper 1 $Offline$ $54 Questions$

Question 001 MCQ

QUESTION

A plot of the number of neutrons N against the number of protons P of stable nuclei exhibits upward deviation from linearity for atomic number, Z > 20. For an unstable nucleus having N/P ratio less than 1, the possible mode s of decay is are

A. β^- -decay $$\beta$$ $emission$

B. orbital or K-electron capture

C. neutron emission

D. β^+ -decay $positronemission$

CORRECT OPTION

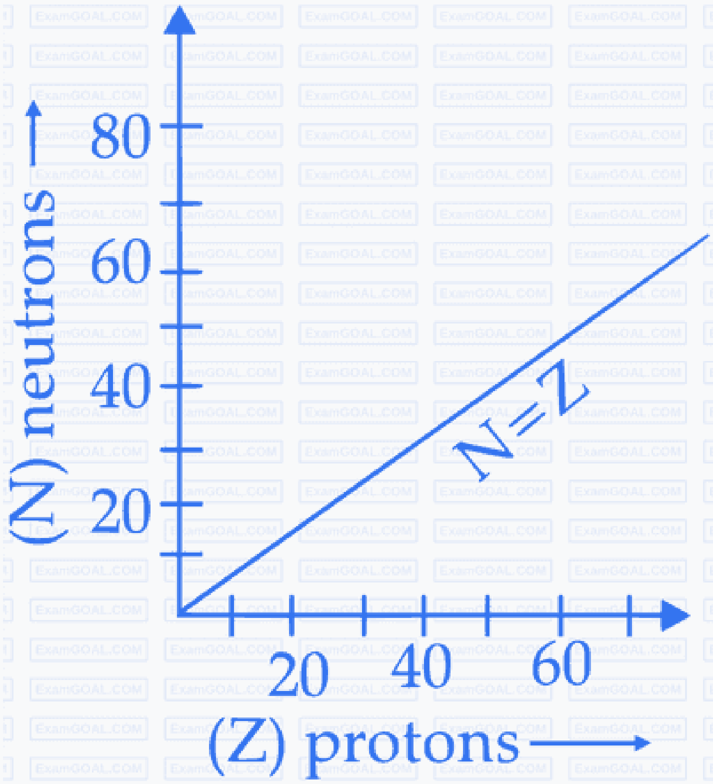
B. orbital or K-electron capture

SOURCE

Chemistry • chemical-kinetics-and-nuclear-chemistry

EXPLANATION

Plot of Z $numberofprotons$ verses N $numberofneutrons$ is represented as :



a For a stable nucleus $\mathrm{N/Z}\left(\frac{\text{no. of neutrons}}{\text{no. of protons}}\right)$ should be equal to 1.

This happens for elements till atomic number $(Z = 20)$. For such nucleus number of proton are equal to number of neutrons.

b For heavier nucleides with atomic number (Z) greater than $20(Z > 20)$, the number of neutron are more than proton as seen by curved $green$ appearance in the plot.

c For nucleids with $\mathrm{N/Z} < 1$, the number of neutrons are less than the protons making the nucleus unstable. This is due to increase in coulombic repulsions between positively charged protons.

Such nuclei become stable by K-electron capture and positron emission.

K-electron capture :

$$ {}_{1}^{1}p + {}_{-1}^{0}e \rightarrow {}_{0}^{1}n $$

The K-electron capture, proton is converted to the neutron as increases the $\mathrm{N/Z}$ ratio.

$\beta^{\pm}$ **decay** $positronemission$ **:**

$$ {}_{1}^{1}p \rightarrow \underset{neutron}{{}_{0}^{1}n} + \underset{positron}{e^{+}} + \underset{electron\,nutrino}{v_e} $$

Proton decays to a neutron, positron and electron nutrino. This decreases the number of protons and increases the number of neutrons, thereby increasing $\mathrm{N/Z}$ ratio.

Options B and D are correct.

Question 002 MCQ

QUESTION

According to the Arrhenius equation,

A a high activation energy usually implies a fast reaction

B rate constant increases with increase in temperature. This is due to a greater number of collisions whose energy exceeds the activation energy

C higher the magnitude of activation energy, stronger is the temperature dependence of the rate constant.

D the pre-exponential factor is a measure of the rate at which collisions occur, irrespective of their energy.

CORRECT OPTION

B rate constant increases with increase in temperature. This is due to a greater number of collisions whose energy exceeds the activation energy

SOURCE

Chemistry • chemical-kinetics-and-nuclear-chemistry

EXPLANATION

The Arrhenius equation is a mathematical model of the temperature dependence of reaction rates. It is expressed as follows:

$$k = \mathrm{Ae}^{-\mathrm{E}_a/\mathrm{RT}}$$

where :

- k is the rate constant

- A is the pre-exponential factor $alsoknownasthefrequencyfactor$

- E_a is the activation energy

- R is the universal gas constant

- T is the temperature $inKelvin$

- e is Euler's number, a mathematical constant approximately equal to 2.71828

The options provided correlate to the Arrhenius equation as follows:

Option A : Incorrect. A high activation energy usually implies a slower reaction, not a fast one. The activation energy is the energy barrier that needs to be overcome for a reaction to occur. The higher the activation energy, the fewer the number of molecules that have enough energy to surpass this barrier and react, resulting in a slower reaction rate.

Option B : Correct. Temperature T_2 is greater than T_1; hence, the fraction of molecules $shadedingreen$ having energy greater than activation energy increases with increase in temperature result, rate of reaction increases.

Higher temperature, more number of collisions $betweenextents$ having energy greater than activation energy happen increases speed of reaction.

Option C : Correct. A reaction at higher activation energy makes the reaction rate slow. An increase in temperature causes reactant molecules to cross the high energy barrier; i.e., E_a and form activated complex.

Option D : Correct. The pre-exponential factor A takes into account the probability of proper orientation of reactant molecules to form the product and the frequency at which reactant molecules collide to form the product.

$$\mathrm{A} = \mathrm{Z}_{\mathrm{AB}} \times \mathrm{P}$$

Z_{AB} : Collision frequency or frequency with which A collides with B.

P : Probability of reactant molecules oriented in right orientation with respect to each other to form the product.

Option D, where pre-exponential factor affects the rate reaction irrespective of the energy they possess.

Question 003 Numerical

QUESTION

The mole fraction of a solute in a solution is 0.1. At 298 K, molarity of this solution is the same as its molality. Density of this solution at 298 K is 2.0 g cm^{-3}. The ratio of the molecular weights of the solute and solvent,

$$\left(\frac{MW_{solute}}{MW_{solvent}}\right)$$

, is

SOURCE

Chemistry • solutions

EXPLANATION

It is given that the mole fraction of the solute (x_{solute}) is 0.1 , therefore mole fraction of the solvent (x_{solvent}) is 0.9 . So we have

$$x_{\text{solute}} = \frac{n_{\text{solute}}}{(n_{\text{solute}} + n_{\text{solvent}})} = 0.1$$

...... 1

$$x_{\text{solvent}} = \frac{n_{\text{solvent}}}{(n_{\text{solute}} + n_{\text{solvent}})} = 0.9$$

$......2$

where n_{solute} and n_{solvent} are the number of moles of solute and solvent, respectively. Dividing Eq. 1 by Eq. 2 gives

$$\frac{x_{\text{solute}}}{x_{\text{solvent}}} = \frac{n_{\text{solute}}}{n_{\text{solvent}}} = \frac{0.1}{0.9}$$

$.......3$

Given that the density of solution is 2 g cm^{-3}, we have

$$W_{\text{solution}} = \text{ density } \times V_{\text{solution}} = 2 \times V_{\text{solution}}$$

$$\therefore W_{\text{solute}} + W_{\text{solvent}} = 2 \times V_{\text{solution}} \ldots\ldots\ldots (4)$$

We know that molality is given by

$$m = \frac{n_{\text{solute}} \times 1000}{n_{\text{solvent}} \times W_{\text{solvent}}}$$

Substituting from Eq. 3, we have

$$m = \left(\frac{0.1}{0.9}\right) \times \frac{1000}{W_{\text{solvent}}}$$

$......5$

Molarity is given by

$$M = \frac{n_{\text{solute}} \times 1000}{n_{\text{solvent}} \times V_{\text{solution}}}$$

Substituting from Eq. 3, we have

$$M=\left(\frac{0.1}{0.9}\right)\times\frac{1000}{V_{\text{solution}}}$$

.... 6

Given that, Molarity $(M)=$ Molality (m);

Therefore, from Eqs. 5 and 6, we get

$$\frac{0.1\times1000}{0.9\times W_{\text{solvent}}}=\frac{0.1\times1000}{0.9\times V_{\text{solution}}}$$

$$W_{\text{solvent}}=V_{\text{solution}}$$

From Eq. 4, we have

$$\begin{aligned}W_{\text{solvent}}&=\frac{W_{\text{solute}}+W_{\text{solvent}}}{2}\\2W_{\text{solvent}}&=W_{\text{solute}}+W_{\text{solvent}}\\W_{\text{solvent}}&=W_{\text{solute}}\ldots\ldots\ldots(7)\end{aligned}$$

The molecular weight MW of the solute can be calculated by dividing the weight of the solute by the number of moles of solute. This can be written as :

$$MW_{\text{solute}}=\frac{W_{\text{solute}}}{n_{\text{solute}}}$$

2. Similarly, the molecular weight MW of the solvent can be calculated by dividing the weight of the solvent by the number of moles of solvent. This can be written as :

$$MW_{\text{solvent}} = \frac{W_{\text{solvent}}}{n_{\text{solvent}}}$$

In these formulas, MW_{solute} and MW_{solvent} represent the molecular weights of the solute and solvent respectively, W_{solute} and W_{solvent} represent their weights, and n_{solute} and n_{solvent} represent the number of moles of solute and solvent respectively.

From Eq. 3, we have

$$\frac{n_{\text{solute}}}{n_{\text{solvent}}} = \frac{W_{\text{solute}} / MW_{\text{solute}}}{W_{\text{solvent}} / MW_{\text{solvent}}} = \frac{0.1}{0.9}$$

Using Eq. 7, we get

$$\frac{MW_{\text{solute}}}{MW_{\text{solvent}}} = 9$$

Question 004 MCQ

QUESTION

One mole of an ideal gas at 300 K in thermal contact with surroundings expands isothermally from 1.0 L to 2.0 L against a constant pressure of 3.0 atm. In this process, the change in entropy of surrounding (

Δ

S_{surr})in JK^{-1} is $1 Latm = 101.3 J$

A 5.763

B 1.013

C – 1.013

D – 5.763

CORRECT OPTION

C – 1.013

SOURCE

Chemistry • thermodynamics

EXPLANATION

Work done in an irreversible isothermal process is given by :

$$W_{irr} = -P_{ext}(V_2 - V_1)$$

$$= -3.0 \text{ atm}(2.0 \text{ L} - 1.0 \text{ L})$$

$$= -3.0 \times 1.0 \text{ atm L}$$

$$W_{irr} = -3.0 \text{ L atm}$$

$$W_{irr} = -3.0 \times 101.3 \text{ J} \qquad [1 \text{ L atm} = 101.3]$$

$$= -303.9 \text{ J}$$

At a constant pressure,

$$\Delta U = q_p + W_{irr} = q_p - P_{ext}(V_2 - V_1)$$

$$\Delta U = \Delta H - P_{ext}(V_2 - V_1)$$

$$\Delta H = \Delta U + P_{ext}(V_2 - V_1)$$

Since, reaction takes place at constant temperature; hence, $\Delta U = 0$.

$$\Delta H = P_{ext}(V_2 - V_1) = -W_{irr} = 303.9 \text{ J}$$

$$\Delta H_{sys} = 303.9 \text{ J}$$

$$\Delta S_{sys} = \frac{\Delta H_{sys}}{T} \text{ and}$$

Finally, the change in entropy of the surroundings (ΔS_{surr}) is equal to the negative of the change in entropy of the system, because the system and its surroundings form an isolated system :

$\Delta S_{surr} = -\Delta S_{sys}$

$$\Delta \mathrm{S_{surr}} = -\frac{\Delta \mathrm{H_{sys}}}{\mathrm{T}} = -\frac{303.9\ \mathrm{J}}{300\ \mathrm{K}}$$

$$= -1.013\ \mathrm{J\ K^{-1}}$$

Question 005 Numerical

QUESTION

The diffusion coefficient of an ideal gas is proportional to its mean free path and mean speed. The absolute temperature of an ideal gas is increased 4 times and its pressure is increased 2 times. As a result, the diffusion coefficient of this gas increases x times. The value of x is ___________:

SOURCE

Chemistry • gaseous-state

EXPLANATION

The diffusion coefficient D is proportional to the mean free path λ and the mean speed (U_{mean}), so we can write

$$D \propto \lambda U_{\mathrm{mean}}$$

The mean free path λ is given by

$$\lambda = \frac{RT}{\sqrt{2} N_0 \sigma P}$$

hence

$$\lambda \propto \frac{T}{P}$$

The mean speed (U_{mean}) is given by

$$U_{\text{mean}} = \sqrt{\frac{8RT}{\pi M}}$$

hence

$$U_{\text{mean}} \propto \sqrt{T}$$

Therefore,

$$D \propto \frac{T^{3/2}}{P}$$

The change in the diffusion coefficient would then be given by :

$$\frac{(DC)_2}{(DC)_1} = \frac{P_1}{P_2} \cdot \left(\frac{T_2}{T_1}\right)^{3/2}$$

Substituting $P_2 = 2P_1$ and $T_2 = 4T_1$ into the equation, we get :

$$\frac{(D)_2}{(D)_1} = \frac{1}{2} \cdot (4)^{3/2} = 4$$

So, the diffusion coefficient of the gas increases 4 times. Hence, x = 4.

Question 006 MCQ

QUESTION

The compound s with TWO lone pairs of electrons on the central atom is are

A BrF_5

B ClF_3

C XeF_4

D SF_4

CORRECT OPTION

B ClF_3

SOURCE

Chemistry • chemical-bonding-and-molecular-structure

EXPLANATION

Structure of each of the compound is represented as :

A BrF_5

Structure : Square pyramidal

i One lone pair

ii Four bond pair

B ClF_3

Structure : T-Shaped

i Two lone pair

ii Three bond pair

C XeF_4

Structure : Square planar

i Two lone pair

ii Four bond pair

D SF_4

Structure : See-saw

i One lone pair

ii Four bond pair

Question 007 MCQ

QUESTION

P is the probability of finding the 1s electron of hydrogen atom in a spherical shell of infinitesimal thickness, dr, at a distance r from the nucleus. The volume of this shell is

$$4\pi r^2 dr$$

. The quantitative ketch of the dependence of P on r is

A

B

C

D

CORRECT OPTION

D

SOURCE

Chemistry • structure-of-atom

EXPLANATION

The probability of finding an electron of hydrogen atom in a spherical shell of infinitesimal thickness, dr, at a distance r from the nucleus, with volume $dV = 4\pi r^2 dr$ is

$$P = \psi^2 \cdot dV = \psi^2 \cdot 4\pi r^2 dr = R^2(r) 4\pi r^2 dr$$

Here, $R^2(r)$ is the radial density function.

For $1s$ subshell, $n = 1, l = 0$ and $n - l - 1 = 0$. Thus, the number of radial and angular nodes $= 0$. For 1s orbital, the probability of finding an electron will be maximum at near to the the nucleus, as 1s orbital is the nearest to the nucleus as it is the lowest orbital in terms of energy. Therefore, the plot of radial probability $P = R^2(r) 4\pi r^2$ versus r is as follows :

Question 008 MCQ

QUESTION

The product$s$ of the following reaction sequence is are

A

B

C

D

CORRECT OPTION

B

SOURCE

Chemistry • compounds-containing-nitrogen

EXPLANATION

Question 009 MCQ

QUESTION

Positive Tollen’s test is observed for

A

B

C

D

CORRECT OPTION

A

SOURCE

Chemistry • aldehydes-ketones-and-carboxylic-acids

EXPLANATION

Only aldehydes and

$$\alpha$$

-hydroxyketones show positive test with Tollen's reagent containing $[Ag(NH_3)_2]^+$ ions. It is a weak oxidising agent that oxidises aldehydes to carboxylate ions and gets reduce to metallic silver Ag, depositing on the walls of test tube as silver mirror.

Question 010 Numerical

QUESTION

In the following mono-bromination reaction, the number of possible chiral product s is are...

SOURCE

Chemistry • basics-of-organic-chemistry

EXPLANATION

The number of possible monobrominated chiral products for:

a Bromination at carbon atom 1 :

b Bromination at carbon atom 2 :

all carbon become a chiral.

c Bromination at carbon atom 3 :

d Bromination at carbon atom 4 :

e Bromination at carbon atom 5 :

Monobrominated products I, III, IV, VI and VII are chiral products.

Question 011 Numerical

QUESTION

In neutral or faintly alkaline solution, 8 moles of permanganate anion quantitative oxidise thiosulphate anions to produce X moles of a sulphur containing product. The magnitude of X is....

SOURCE

Chemistry • redox-reactions

EXPLANATION

In neutral or faintly alkaline solution, thiosulphate is oxidized to sulphate by permanganate,

$8MnO_4^{-} + 3S_2O_3^{2-} + H_2O$

$\rightarrow$

$8MnO_2 + 6SO_4^{2-} + 2OH^-$

Question 012 Numerical

QUESTION

The possible number of geometrical isomers for the complex $[CoL_2Cl_2]^-$ ($L = H_2NCH_2CH_2O^-$) is are...

SOURCE

Chemistry • coordination-compounds

EXPLANATION

Total five isomers are possible :

Question 013 MCQ

QUESTION

The reagent s that can selectively precipitate S^{2-} from a mixture of S^{2-} and SO_4^{2-} in aqueous solution is are :

A $CuCl_2$

B $BaCl_2$

C $Pb(OOCCH_3)_2$

D $Na_2[Fe\,CN_5NO]$

CORRECT OPTION

A $CuCl_2$

SOURCE

Chemistry • d-and-f-block-elements

EXPLANATION

Consider the reactions of the given compounds with S^{2-} and SO_4^{2-}.

i With $CuCl_2$:

$$CuCl_2 + S^{2-} \rightarrow \underset{\text{black ppt}}{CuS} + 2Cl^-$$

$$CuCl_2 + {SO_4}^{2-} \rightarrow \text{No precipitate}$$

ii With $BaCl_2$: Alkaline earth metal sulphides are sparingly soluble in water but pass into solution with time and hence do not give precipitates from their aqueous solutions.

$$BaCl_2 + S^{2-} \rightarrow \underset{\text{No ppt}}{BaS} + 2Cl^-$$

$$BaS + 2H_2O \rightleftharpoons \underset{\text{water soluble}}{Ba(OH)_2 + Ba(SH)_2}$$

Alkaline earth metal sulphates, $BaSO_4, CaSO_4$ and $SrSO_4$ are insoluble in water.

$$BaCl_2 + SO_4^{2-} \rightarrow \underset{\text{white ppt}}{BaSO_4} + 2Cl^-$$

iii With $Pb(OAc)_2$:

$$Pb(OAc)_2 + S^{2-} \rightarrow \underset{\text{black ppt}}{PbS} + 2CH_3COO^-$$

$$Pb(OAc)_2 + SO_4^{2-} \rightarrow \underset{\text{white ppt}}{PbSO_4} + 2CH_3COO^-$$

The K_{sp} of PbS is 3×10^{-28} and of $PbSO_4$ is $25\times\ 10^{-28}$. The difference is large, thus only PbS is selectively precipitated.

iv With $Na_2[Fe(CN)_5NO]$: When sodium nitroprusside is added to solution of sulphide ions, purple colouration is observed. This is a confirmative test for sulphides.

$$[Fe(CN)_5NO]^{2-} + S^{2-} \rightarrow \underset{\text{Purple colour}}{[Fe(CN)_5NOS]}$$

$$Na_2[Fe(CN)_5NO] + {SO_4}^{2-} \rightarrow \text{ No precipitate}$$

Question 014 MCQ

QUESTION

The crystalline form of borax has.

A tetranuclear $[B_4O_5OH_4]^{2-}$ unit

B all boron atoms in the same plane

C equal number of sp^2 and sp^3 hybridised boron atoms

D one terminal hydroxide per boron atom

CORRECT OPTION

A tetranuclear $[B_4O_5OH_4]^{2-}$ unit

SOURCE

Chemistry • p-block-elements

EXPLANATION

Structure of crystalline form of borax

a Borax contains four boron atom; hence, it is tetranuclear.

b Only 2 boron atoms lie in the same plane.

c Two boron atoms are sp^3 hybridised other two borons are sp^2 hybridised.

d Each of the boron has hydroxyl group attached to it with two hydroxyl groups present as hydroxyl salts.

Question 015 MCQ

QUESTION

The increasing order of atomic radii of the following group 13 elements is

A Al < Ga < In < Tl

B Ga < Al < In < Tl

C Al < In < Ga < Tl

D Al < Ga < Tl < In

CORRECT OPTION

B Ga < Al < In < Tl

SOURCE

Chemistry • p-block-elements

EXPLANATION

The increasing order of atomic radii is as follows:

Ga < Al < In < Tl

The atomic radius generally increases on moving down a group in the periodic table. As an anomaly, the atomic radius of Ga is less than that of aluminium because of poor shielding of nuclear charge by 10 number of 3d electrons. As a result, the outershell electrons are held more firmly by the nucleus and contraction of radius is observed. This contraction is also called d-block contraction. The size of Tl is similarly affected by 14 number of 4f electrons

$lanthanoid contraction$ and the atomic radius of Tl is almost similar in size to In.

Question 016 MCQ

QUESTION

On complete hydrogenation, natural rubber produces

A ethylene-propylene copolymer.

B vulcanized rubber.

C polypropylene.

D polybutylene.

CORRECT OPTION

A ethylene-propylene copolymer.

SOURCE

Chemistry • polymers

EXPLANATION

Hydrogenation of natural rubber leads to formation of ethylene-propylene copolymer.

Question 017 MCQ

QUESTION

Among [Ni CO $_4$], $[NiCl_4]^2$

—

, $[Co(NH_3)_4)Cl_2]Cl$, $Na_3[CoF_6]$, Na_2O_2 and CsO_2, the total number of paramagnetic compound is

A 2

B 3

C 4

D 5

CORRECT OPTION

B 3

SOURCE

Chemistry • coordination-compounds

EXPLANATION

For the given complexes:

In $[\mathrm{Ni(CO)_4}]$: Ni $at. no. 28$: $1s^2 2s^2 2p^6 3s^2 3p^6 3d^8 4s^2$

CO being a strong field ligand causes the electrons to pair; hence, the complex diamagnetic.

$$\text{In } [\mathrm{NiCl_4}]^{2-} : \mathrm{Ni}^{2+} \text{ (at. No. 26)}: 1s^2 2s^2 2p^6 3s^2 3p^6 3d^8$$

There are two unpaired electrons, so the complex is paramagnetic.

$$\text{In } [\mathrm{Co(NH_3)_4Cl_2}]\mathrm{Cl} : \mathrm{Co}^{3+} \text{(at. No. 24)}: 1s^2 2s^2 2p^6 3s^2 3p^6 3d^6$$

The ligand $\mathrm{NH_3}$ results in formation of low spin complex, which is diamagnetic.

$$\text{In } \mathrm{Na_3}[\mathrm{CoF_6}] : \mathrm{Co}^{3+} \text{(at. No. 24)}: 1s^2 2s^2 2p^6 3s^2 3p^6 3d^6$$

F is a weak field ligand and results in the formation of high spin complex with four unpaired electrons, which is paramagnetic.

In $\mathrm{Na_2O_2}$: The molecular orbital configuration of $\mathrm{O_2^{2-}}$ is

$$\sigma 1s^2, \sigma^* 1\,\mathrm{s}^2, \sigma 2s^2, \sigma^* 2s^2, \sigma 2p_z^2, \begin{cases} \pi 2p_x^2 \\ \pi 2p_y^2 \end{cases}, \begin{cases} \pi * 2p_x^2 \\ \pi * 2p_y^2 \end{cases}$$

There is no unpaired electron, so it is diamagnetic.

In $\mathrm{CsO_2}$: The molecular orbital configuration of $\mathrm{O_2^-}$ is

$$\sigma 1s^2, \sigma * 1\,\mathrm{s}^2, \sigma 2s^2, \sigma * 2s^2, \sigma 2p_z^2, \begin{cases} \pi 2p_x^2 \\ \pi 2p_y^2 \end{cases}, \begin{cases} \pi * 2p_x^2 \\ \pi * 2p_y^1 \end{cases}$$

There is one unpaired electron, so it is paramagnetic. Hence, there are three paramagnetic compounds.

Question 018 MCQ

QUESTION

The correct statements about of the following reaction sequence is are

Cumene (C_9H_{12})

$$\xrightarrow[(ii)\ H_3O^+]{(i)\ O_2}$$

P

$$\xrightarrow{CHCl_3/NaOH}$$

Q $major$ + R $minor$

Q

$$\xrightarrow[PhCH_2Br]{NaOH}$$

S

A R is steam volatile.

B Q gives dark violet coloration with 1% aqueous $FeCl_3$ solution.

C S gives yellow precipitate with 2, 4-dinitrophenylhydrazine.

D S gives dark violet coloration with 1% aqueous $FeCl_3$ solution.

CORRECT OPTION

B Q gives dark violet coloration with 1% aqueous $FeCl_3$ solution.

SOURCE

Chemistry • aldehydes-ketones-and-carboxylic-acids

EXPLANATION

'Q' is steam volatile due to intramolecular hydrogen bonding while 'R' undergoes intermolecular hydrogen bonding hence, has higher boiling point. 'Q' gives dark violet coloration with 1% aqueous $FeCl_3$ solution due to the presence of phenolic group while 'S' gives yellow precipitate with 2, 4-dinitrophenyl hydrazine due to the presence of aldehydic group $\$\$ - \$\$CHO$.

Question 019 MCQ

QUESTION

Let

$$S = \left\{ x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2} \right\}.$$

The sum of all distinct solutions of the equation

$$\sqrt{3}\sec x + \cos ec\, x + 2(\tan x - \cot x) = 0$$

in the set S is equal to

A $-\frac{7\pi}{9}$

B $-\frac{2\pi}{9}$

C 0

D $\frac{5\pi}{9}$

CORRECT OPTION

C 0

SOURCE

Mathematics • trigonometric-functions-and-equations

EXPLANATION

Let us consider

$$S = \left\{x \in (-\pi, \pi), x \neq 0, \pm\frac{\pi}{2}\right\}$$

The given equation is

$$\sqrt{3}\sec x+\operatorname{cosec} x+2(\tan x-\cot x)=0$$

$$\Rightarrow \frac{\sqrt{3}}{\cos x}+\frac{1}{\sin x}+2\left(\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}\right)=0$$

$$\Rightarrow \sqrt{3}\sin x+\cos x+2\left(\sin^2 x-\cos^2 x\right)=0$$

$$\Rightarrow \sqrt{3}\sin x+\cos x=2\cos 2x$$

$$\Rightarrow \frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x=\cos 2x$$

$$\Rightarrow \cos x\cos\frac{\pi}{3}+\sin x\sin\frac{\pi}{3}=\cos 2x$$

$$\Rightarrow \cos 2x=\cos\left(x-\frac{\pi}{3}\right)$$

$$\Rightarrow 2x=2n\pi\pm\left(x-\frac{\pi}{3}\right)\qquad(x\in I)$$

Case 1: When

$$2x=2n\pi+x-\frac{\pi}{3},$$

we have $x=2n\pi-\frac{\pi}{3}$.

If $n=0$, we get $x=-\frac{\pi}{3}$.

If $n=1$, we get $x=2\pi-\frac{\pi}{3}$.

If $n=-1, x=-2\pi-\frac{\pi}{3}$.

Case 2: When

$2x=2n\pi-x+\frac{\pi}{3}$, we get $x=\frac{2n\pi}{3}+\frac{\pi}{9}$.

If $n=0$, we get $x=\frac{\pi}{9}$.

If $n=1$, we get $x=\frac{2\pi}{3}+\frac{\pi}{9}$.

If $n=2$, we get $n=2x=\frac{4\pi}{3}+\frac{\pi}{9}$.

If $n=-1$, we get $x=\frac{-2\pi}{3}+\frac{\pi}{9}$.

Therefore, the sum of all distinct solutions of the given equation is

$$\frac{-\pi}{3}+\frac{\pi}{9}+\frac{2\pi}{3}+\frac{\pi}{9}-\frac{2\pi}{3}+\frac{\pi}{9}=0$$

Question 020 MCQ

QUESTION

Consider a pyramid

$$OPQRS$$

located in the first octant

$$(x \geq 0, y \geq 0, z \geq 0)$$

with

$$O$$

as origin, and

$$OP$$

and

$$OR$$

along the

$$x$$

-axis and the

$$y$$

-axis, respectively. The base

$$OPQR$$

of the pyramid is a square with

$$OP = 3.$$

The point

$$S$$

is directly above the mid-point,

$$T$$

of diagonal

$$OQ$$

such that

$$TS = 3.$$

Then

A

the acute angle between

$$OQ$$

and

$$OS$$

is

$$\frac{\pi}{3}$$

B

the equation of the plane containing the triangle

$$OQS$$

is

$$x - y = 0$$

C

the length of the perpendicular from

$$P$$

to the plane containing the triangle

$$OQS$$

is

$$\frac{3}{\sqrt{2}}$$

D

the perpendicular distance from

$$O$$

to the straight line containing

$$RS$$

is

$$\sqrt{\frac{15}{2}}$$

CORRECT OPTION

the equation of the plane containing the triangle

$$OQS$$

B is

$$x - y = 0$$

SOURCE

Mathematics • 3d-geometry

EXPLANATION

Let us consider a pyramid OPQRS in the first octant with O as origin as shown in the following figure:

Now, the midpoint of OQ is $T\left(\frac{3}{2}, \frac{3}{2}, 0\right)$.

Also, it is obvious that the point S is given by $S\left(\frac{3}{2}, \frac{3}{2}, 3\right)$.

Now, $OT = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 0} = \frac{3}{2}\sqrt{2}$

Therefore, $\cos\theta = \frac{3}{2}\frac{\sqrt{2}}{3} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

where θ is the angle between OQ and OS, which is calculated as follows:

The ratio of the plane containing triangle $\triangle OQS$ can be written as

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 3 & 0 \\ 3/2 & 3/2 & 3 \end{vmatrix} = i(9) - i(9) + k(0) = i(i - j)$$

Therefore, the ratio is $(1, -1, 0)$.

Now, the equation of the plane is

$$1(x-0)-1(y-0)+0(z-0)=0$$

$$x-y=0$$

Hence, option B is correct.

The coordinates of the length of perpendicular from point P to the plane $x-y=0$ is $(3,0,0)$.

Hence, the length of perpendicular from point P to the plane containing the triangle OQS is

$$\frac{z-0}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

Hence, option C is correct.

Now, points R and S, respectively, are $R(0,3,0)$ and

$$S\left(\frac{3}{2},\frac{3}{2},3\right).$$

The equation of line RS is

$$\bar{r}=3\hat{j}+\lambda\left(\frac{3}{2}\hat{i}+\left(\frac{3}{2}-3\right)\hat{j}+3\hat{k}\right)$$

$$=3\hat{j}+\lambda\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$$

Let point A be the feet of perpendicular from O to line RS:

$$A\left(3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)\right)$$

$$\text{Now, } \overrightarrow{OA}=3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$$

That is, $(\vec{b}-\vec{a})=\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$

Therefore,

$$\overrightarrow{OA}\cdot(\vec{b}-\vec{a})=0$$

$$\left(3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)\right)\cdot\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)=0$$

$$\frac{-9}{2}+\mu\left(\frac{9}{4}+\frac{9}{4}+9\right)=0$$

$$\frac{-9}{2}+\mu\left(\frac{9}{2}+9\right)=0$$

$$\Rightarrow -\frac{1}{2}+\mu\left(\frac{3}{2}\right)=0$$

$$\Rightarrow \mu=\frac{1}{3}$$

Therefore, $A\left(3\hat{j}+\frac{\hat{i}}{2}-\frac{\hat{j}}{2}+\hat{k}\right)=A\left(\frac{\hat{i}}{2}+\frac{5\hat{j}}{2}+\hat{k}\right)$ and hence the perpendicular distance from point O to the straight line containing the line RS is

$$|\overrightarrow{OA}|=\sqrt{\frac{1}{4}+\frac{25}{4}+\frac{4}{4}}=\sqrt{\frac{30}{4}}=\sqrt{\frac{15}{2}}$$

Hence, option D is correct.

Question 021 MCQ

QUESTION

A computer producing factory has only two plants

$$T_1$$

and

$$T_2.$$

Plant

$$T_1$$

produces

$$20$$

% and plant

$$T_2$$

produces

$$80$$

% of the total computers produced.

$$7$$

% of computers produced in the factory turn out to be defective. It is known that

$$P$$

$$computerturnsouttobedefectivegiventhatitisproducedinplant\$\$T_1\$\$$$

$$= 10P$$

$$computerturnsouttobedefectivegiventhatitisproducedinplant\$\$T_2\$\$,$$

where

$$P(E)$$

denotes the probability of an event

$$E$$

. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant

$$T_2$$

is

A $\frac{36}{73}$

B $\frac{47}{79}$

C $\frac{78}{93}$

D

$$\frac{75}{83}$$

CORRECT OPTION

C

$$\frac{78}{93}$$

SOURCE

Mathematics • probability

EXPLANATION

Let P_1 be the defective computers that are produced from plant T_1 and P_2 be that from plant T_2.

The total percentage of the defective computers produced is 7%.

Now, $P_1 = 10P$ and $P_2 = P$.

The computers produced that are defective:

$$\frac{20}{100} \times P_1 + \frac{80}{100} \times P_2 = \frac{7}{100}$$

$$20P_1 + 80P_2 = 7$$

$$200P + 80P = 7$$

$$P = \frac{7}{280} = \frac{1}{40}$$

Now, the probability of the defective products is calculated as follows:

$$\frac{20}{100}P_1 + \frac{80}{100}P_2 = \frac{20}{100} \times \frac{1}{4} + \frac{80}{100} \times \frac{1}{40} = \frac{28}{400} = \frac{7}{100}$$

The probability of producing NOT defective computers is

$$1 - \frac{7}{100} = \frac{93}{100}$$

The probability that plant T_2 produces NOT defective computers is calculated as follows:

$$\frac{(80/100) \times (39/40)}{(93/100)} = \frac{78}{93}$$

Question 022 Numerical

QUESTION

The total number of distinct

$$x \in [0, 1]$$

for which

$$\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$$

SOURCE

Mathematics • definite-integration

EXPLANATION

Let $f(x)=\int_0^x \frac{t^2}{1+t^4} dt-(2x-1)$

Differentiating both sides w.r.t. x

$$\therefore f^{\prime}(x)=\frac{x^2}{1+x^4}-2$$

$$\therefore f^{\prime}(x)<0(\because x \in[0,1])$$

$\Rightarrow f$ is strictly decreasing function

$$\text{Let I}=\frac{1}{2} \int_0^x \frac{2 t^2}{t^4+1} dt=\frac{1}{2} \int_0^x \frac{\left(t^2+1\right)+\left(t^2-1\right)}{t^4+1}$$

$$=\frac{1}{2} \int_0^x \frac{t^2+1}{t^4+1} dt+\frac{1}{2} \int_0^x \frac{t^2-1}{t^4+1} dt$$

$$= \frac{1}{2}\int_0^x \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt + \frac{1}{2}\int_0^x \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}dt$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)\right]_0^x + \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right)\right]_0^x$$

$$= \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{t^2-1}{t\sqrt{2}}\right)\right]_0^x + \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right)\right]_0^x$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) - \tan^{-1}(-\infty)$$

$$+ \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)\right] - 0$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}\right)$$

$$\because f(x) = I - (2x-1)$$

$$\therefore f(0) = \frac{1}{2\sqrt{2}}\tan^{-1}(-\infty) + \frac{1}{4\sqrt{2}}\ln(1) + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}\right) + 1$$

$$\Rightarrow f(0) = \frac{-\pi}{4\sqrt{2}} + 0 + \frac{\pi}{4\sqrt{2}} + 1$$

$$\Rightarrow f(0) = 1$$

$$f(1) = \frac{1}{2\sqrt{2}}\tan^{-1}(-0) + \frac{1}{4\sqrt{2}}\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) + \frac{\pi}{4\sqrt{2}} - 1$$

$$\Rightarrow f(1) = \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\pi\right] - 1$$

Here, $f(1) < 0$ and $f(0) = 1 > 0$

$$\Rightarrow f(0) \cdot f(1) < 0$$

By intermediate value theorem

$\Rightarrow$ There exists at least one $c \in (0, 1)$ such that $f(c) = 0$

Since, $f(x)$ is a decreasing function such that $f(x) = 0$ has exactly one root.

Question 023 MCQ

QUESTION

A solution curve of the differential equation

$$\left(x^2 + xy + 4x + 2y + 4\right)\frac{dy}{dx} - y^2 = 0,$$

$$x > 0,$$

passes through the

point

$$(1, 3)$$

. Then the solution curve

A intersects

$$y = x + 2$$

exactly at one point

B

intersects

$$y = x + 2$$

exactly at two points

C

intersects

$$y = (x + 2)^2$$

D

does NOT intersect

$$y = (x + 3)^2$$

CORRECT OPTION

A

intersects

$$y = x + 2$$

exactly at one point

SOURCE

Mathematics • differential-equations

EXPLANATION

Options A and B: The given differential equation is

$$\left[x^2 + 4x + 4 + y(x + 2)\right] \frac{dy}{dx} - y^2 = 0 \quad (x > 0)$$

which is further simplified as follows:

$$\left[(x+2)^2 + y(x+2)\right]\frac{dy}{dx} - y^2 = 0$$

Substituting $x + 2 = t$, we get

$$\frac{dx}{dy} = \frac{dt}{dy}$$

Now,

$$(x+2)^2 + y(x+2) - y^2\frac{dx}{dy} = 0$$

That is,

$$t^2 + yt - y^2\frac{dt}{dy} = 0$$

$$y^2\frac{dt}{dy} - yt - t^2 = 0$$

$$\frac{1}{t^2}\frac{dt}{dy} - \frac{1}{yt} = \frac{1}{y^2}$$

Let $\frac{1}{t} = z$; therefore,

$$\frac{dt}{dy}\left(-\frac{1}{t^2}\right)=\frac{dz}{dy}$$

Now,

$$\frac{-dz}{dy}-\frac{z}{y}=\frac{1}{y^2}\Rightarrow\frac{dz}{dy}+\frac{z}{y}=\frac{-1}{y^2}$$

$$\Rightarrow d(zy)=\int-\frac{1}{y}dy$$

$$\Rightarrow zy=-\ln|y|+\ln c$$

$$\Rightarrow\frac{y}{t}=-\ln|y|+\ln c$$

$$\Rightarrow\frac{y}{(x+2)}=-\ln|y|+\ln c\quad\ldots(1)$$

which passes through the point $(1,3)$. Therefore, from Eq. 1, we get

$$\frac{z}{3}=-\ln 3+c\Rightarrow c=\ln 3e$$

$$\frac{y}{x+2}=-\ln|y|+\ln 3e=\ln\left(\frac{3e}{|y|}\right)$$

$$\frac{3e}{|y|}=e^{y/(x+2)}$$

$$3e=|y|e^{y/(x+2)}$$

Substituting $y = (x+2)$, we get $3e = |x+2|e^1$

$$|x+2| = 3 \Rightarrow x+2 = -3, 3 \Rightarrow x = -5, 1$$

Therefore, $x = 1 \;\; since\$x \neq -5\$$.

That is, the solution curve intersects $y = (x+2)$ exactly at one point and not at two points. Therefore, option A is correct and option B is incorrect.

Option C: We have

$$\frac{3e}{|(x+2)^2|} = e^{(x+2)}$$

which meets at two points for $x < 0$ and for $x > 0$, there is no intersection point.

Hence, option (C) is incorrect.

Option D: We have

$$\frac{3e}{(x+3)^2} = e^{\frac{(x+3)^2}{(x+2)}} = e^{\frac{(x+2)^2+1+2(x+2)}{(x+2)}} = e^{2+\frac{1}{(x+2)}+(x+2)}$$

Therefore, there is no intersection point for $x > 0$. Hence option D is correct.

Question 024 MCQ

QUESTION

The least value of a

$$\in R$$

for which

$$4ax^2 + \frac{1}{x} \geq 1,$$

, for all

$$x > 0$$

. is

A $$\frac{1}{64}$$

B $$\frac{1}{32}$$

C $$\frac{1}{27}$$

D $$\frac{1}{25}$$

CORRECT OPTION

C

$$\frac{1}{27}$$

SOURCE

Mathematics • application-of-derivatives

EXPLANATION

It is given that

$$4\alpha x^2+\frac{1}{x}\geq 1\forall x>0$$

That is, $4\alpha x^2 \geq 1-\frac{1}{x}$

$$4\alpha\geq\left(\frac{1}{x^2}-\frac{1}{x^3}\right)\forall x>0$$

Now, let us consider that

$$f(x)=\frac{1}{x^2}-\frac{1}{x^3}$$

Therefore, $f'(x)=\frac{-2}{x^3}+\frac{3}{x^4}=0$

When $x=3/2$:

$$(4\alpha) \geq \left(\frac{1}{x^2} - \frac{1}{x^3}\right)$$

$$4\alpha \geq \left(\frac{4}{9} - \frac{8}{27}\right)$$

That is,

$$\alpha \geq \frac{1}{27}$$

and hence the least value of α is

$$\alpha_{\text{least}} = \frac{1}{27}$$

Question 025 MCQ

QUESTION

In a triangle

$$\Delta$$

$$XYZ$$

, let

$$x, y, z$$

be the lengths of sides opposite to the angles

$$X, Y, Z$$

respectively, and

$$2s = x + y + z$$

.

If

$$\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$$

and area of incircle of the triangle

$$XYZ$$

is

$$\frac{8\pi}{3}$$

, then

A. area of the triangle

$$XYZ$$

is

$$6\sqrt{6}$$

B. the radius of circumcircle of the triangle

$$XYZ$$

is

$$\frac{35}{6}\sqrt{6}$$

C. $$\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$$

D. $$\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$$

CORRECT OPTION

A

area of the triangle

$$XYZ$$

is

$$6\sqrt{6}$$

SOURCE

Mathematics • properties-of-triangle

EXPLANATION

It is given that

$$2s = x + y + z$$

Let us consider

$$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = k$$

That is, $s = 4k + x$

$$s = 3k + y$$

$$s = 2k + z$$

$$3s = 9k + (x + y + z) = 9k + 2s$$

$$s = 9k$$

$$x = 9k - 4k = 5k$$

$$y = 6k, z = 7k$$

Area of in-circle of the triangle XYZ is

$$\pi r^2 = \pi\left(\frac{\Delta}{s}\right)^2$$

$$= \frac{\pi}{s^2}\Delta^2 = \frac{\pi}{s^2}s(s-x)(s-y)(s-z)$$

$$= \frac{\pi}{81k^2} \times 9k \times 4k \times 3k \times 2k$$

$$= \frac{\pi}{9k} \times 24k^3 = \frac{\pi}{9}24k^2$$

$$\text{Therefore, } \frac{\pi}{9}24k^2 = \frac{8\pi}{3}$$

$$\Rightarrow \quad k^2 = \frac{8\pi}{3} \times \frac{9}{24\pi} = 1 \Rightarrow k = 1$$

The sides of the triangle is given by $x = 5, y = 6, z = 7$. Now, the area of

$\triangle XYZ$ is

$$\sqrt{s(s-x)(s-y)(s-z)} = \sqrt{9 \times 4 \times 3 \times 2} = 6\sqrt{6}$$

Hence, option (A) is correct.

Now, $R = \frac{xyz}{4\Delta} = \frac{5\times6\times7}{4\times6\sqrt{6}} = \frac{35}{4\sqrt{6}}$

$$\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$$

$$= \sqrt{\left[\frac{(s-y)(s-z)}{yz}\right]\left[\frac{(s-z)(s-z)}{xz}\right]\left[\frac{(s-y)(s-x)}{xy}\right]}$$

$$= \frac{(s-z)(s-y)(s-z)}{xyz}$$

$$\text{Now, } \frac{(s-z)(s-y)(s-z)}{xyz} = \frac{4\times3\times2}{5\times6\times7} = \frac{4}{25}$$

Hence, option C is correct.

Now, $\sin^2\left(\frac{X+Y}{2}\right) = \sin^2\left(\frac{\pi-Z}{2}\right) = \cos^2\frac{z}{2} = \frac{s(s-z)}{xy}$

$$= \frac{9\times2}{5\times6} = \frac{3}{5}$$

Hence, option D is correct.

Question 026 MCQ

QUESTION

Let

$$f:\mathbb{R}\to\mathbb{R},\ g:\mathbb{R}\to\mathbb{R}$$

and

$$h:\mathbb{R}\to\mathbb{R}$$

be differentiable functions such that

$$f(x)=x^3+3x+2,$$

$$g(f(x))=x$$

and

$$h(g(g(x)))=x$$

for all

$$x\in R$$

. Then

A $g'(2)=\frac{1}{15}$

B $h'(1)=666$

C $h(0)=16$

D $h(g(3))=36$

CORRECT OPTION

B $h'(1) = 666$

SOURCE

Mathematics • differentiation

EXPLANATION

Option A: It is given that $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are differentiable functions.

Now, $f(x) = x^3 + 3x + 2$

Differentiating w.r.t. to x, we get

$$f'(x) = 3x^2 + 3$$

Also,

$$g(f(x)) = x$$

Now,

$$g'(f(x)) \cdot f'(x) = 1$$

$$f(x) = 2 \Rightarrow x^3 + 3x + 2 = 2$$

$$\Rightarrow x^3 + 3x = 0 \Rightarrow x\left(x^2 + 3\right) = 0$$

$$\Rightarrow x = 0$$

Now,

$$g'(f(0)) = \frac{1}{f'(0)} \Rightarrow g'(2) = \frac{1}{3}$$

Hence, option A is incorrect.

Option B: For all $x \in \mathbb{R}$:

$$h(g(g(f(x))) = x$$

$$h(g(g(x))) = x$$

$$\text{Now,} \quad x \rightarrow f(x) \Rightarrow h(g(g(f(x))) = f(x)$$

$$\Rightarrow \quad h(g(x)) = f(x)$$

$$\Rightarrow \quad h'(g(x)) \cdot g'(x) = f'(x) = 3x^2 + 3 \ \ldots(1)$$

For all $x \in \mathbb{R}$: $g(f(x)) = x$

Now, $\quad x = 1 \Rightarrow g(f(1)) = 1 \Rightarrow g(6) = 1 \quad [\because f(1) = 6]$

Substituting $x = 6$ in Eq. 1, we get

$$h'(g(6)) \cdot g'(6) = 3\left(6^2\right) + 3 = 111$$

Therefore,

$$h'(1) = \frac{111}{g'(6)} \quad \left(g'(6) = \frac{1}{f'(1)}\right)$$

That is, $h'(1) = 111 \cdot f'(x) = 111 \times (3+3) = 666$

Hence, option B is correct.

Option C: $h(g(g(x))) = x$.

For $g(g(x)) = 0$, we have

$$
\begin{aligned}
& g(x) = g^{-1}(0) = 2 \\
\Rightarrow \quad & x = g^{-1}(2) = f(2) = 16 \\
\Rightarrow \quad & h(0) = 16
\end{aligned}
$$

Hence, option C is correct.

Option D: Here, $g(g(x)) = g(3)$ which implies that

$$g(x) = 3 \Rightarrow x = g^{-1}(3) = f(3) = 38$$

Hence, option D is incorrect.

Question 027 MCQ

QUESTION

The circle

$$C_1 : x^2 + y^2 = 3,$$

with centre at

$$O$$

, intersects the parabola

$$x^2 = 2y$$

at the point

$$P$$

in the first quadrant, Let the tangent to the circle

$$C_1$$

, at

$$P$$

touches other two circles

$$C_2$$

and

$$C_3$$

at

$$R_2$$

and

$$R_3$$

, respectively. Suppose

$$C_2$$

and

$$C_3$$

have equal radil

$$2\sqrt{3}$$

and centres

$$Q_2$$

and

$$Q_3$$

, respectively. If

$$Q_2$$

and

$$Q_3$$

lie on the

$$y$$

-axis, then

A

$$Q_2Q_3 = 12$$

B

$$R_2R_3 = 4\sqrt{6}$$

C

area of the triangle

$$OR_2R_3$$

is

$$6\sqrt{2}$$

D

area of the triangle

$$PQ_2Q_3$$

is

$$4\sqrt{2}$$

CORRECT OPTION

A

$$Q_2Q_3 = 12$$

SOURCE

Mathematics • parabola

EXPLANATION

In case of option A

$x^2 + y^2 = 3$ and $x^2 = 2y$

Solving we get

$$P(\sqrt{2}, 1)$$

Equation of tangent at P on the circle

$$x\sqrt{2} + y = 3$$

$$\therefore$$

$$\tan\alpha = -\sqrt{2}$$

Again,

$$\frac{Q_2R_2}{Q_2M} = \sin\left(\alpha - \frac{\pi}{2}\right)$$

or,

$$\frac{2\sqrt{3}}{Q_2M} = -\cos\alpha = -\frac{1}{\sqrt{3}}$$

$$\$\$ \because \$\$\$\$ \tan\alpha = -\sqrt{2}\$\$$$

or,

$$Q_2M = 6$$

Similarly,

$$Q_3M = 6$$

$$\therefore$$

$$Q_2Q_3 = 12$$

In case of option B

$$\frac{MR_2}{Q_2R_2} = \tan\left(\alpha - \frac{\pi}{2}\right)$$

or,

$$\frac{MR_2}{2\sqrt{3}} = -\tan\alpha = \sqrt{2}$$

or,

$$MR_2 = 2\sqrt{6}$$

Similarly,

$$MR_3 = 2\sqrt{6}$$

$$\therefore$$

$$R_2R_3 = 4\sqrt{6}$$

In case of option C

OP is the perpendicular drawn from O to R_2R_3

$$\therefore$$

area of

$$\Delta OR_2R_3 = \frac{1}{2} \times OP \times R_2R_3 = \frac{1}{2} \times \sqrt{3} \times 4\sqrt{6} = 6\sqrt{2}$$

In case of option D

PT is perpendicular drawn from P to Q_2Q_3

$$\therefore$$

area of

$$\Delta PQ_2Q_3 = \frac{1}{2} \times PT \times Q_2Q_3 = \frac{1}{2} \times \sqrt{2} \times 12 = 6\sqrt{2}$$

Therefore, A, B, C are correct options.

Question 028 MCQ

QUESTION

Let RS be the diameter of the circle

$$x^2 + y^2 = 1$$

, where S is the point $1, 0$. Let P be a variable point $other than R and S$ on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point s

A. $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$

B. $\left(\frac{1}{4}, \frac{1}{2}\right)$

C. $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$

D. $\left(\frac{1}{4}, -\frac{1}{2}\right)$

CORRECT OPTION

A

$$\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$$

SOURCE

Mathematics • circle

EXPLANATION

Let, P

$$\equiv$$

$cos$$\theta$$, sin$$\theta$$$

$$\therefore$$

equation of tangent and normal at P

x cos

$$\theta$$

+ y sin

$$\theta$$

= 1 1

and y = x tan

$$\theta$$

...... 2

Now, equation of tangent at S : x = 1 3

Solving 1 and 3, Q

$$\equiv$$

$1, cosec$$\theta$$$$ - $$cot$$\theta$$

$$\therefore$$

equation of straight line parallel to RS drawn from Q

y = cosec

$$\theta$$

$$-$$

cot

$$\theta$$

..... 4

Let, E

$$\equiv$$

h, k

$$\therefore$$

k = h tan

$$\theta$$

$$from (2)$$

or,

$$\tan\theta = \frac{k}{h}$$

Again, k = cosec

$$\theta$$

$$-$$

cot

$$\theta$$

$$from (4)$$

or,

$$k = \frac{1 - \cos\theta}{\sin\theta}$$

or,

$$k = \frac{1 - \frac{h}{\sqrt{h^2+k^2}}}{\frac{k}{\sqrt{h^2+k^2}}} = \frac{\sqrt{h^2+k^2} - h}{k}$$

or,

$$k^2 = \sqrt{h^2+k^2} - h$$

or,

$$h + k^2 = \sqrt{h^2+k^2}$$
$$\therefore$$

locus of E

$$x + y^2 = \sqrt{x^2+y^2}$$

Clearly, points

$$\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$$

and

$$\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$$

are on locus of E.

Therefore, A and C are the correct options.

Question 029 Numerical

QUESTION

Let

$$m$$

be the smallest positive integer such that the coefficient of

$$x^2$$

in the expansion of

$$(1+x)^2+(1+x)^3+\dots\dots\dots+(1+x)^{49}+(1+mx)^{50}$$

is

$$(3n+1)\,{}^{51}C_3$$

for some positive integer

$$n$$

. Then the value of

$$n$$

is

SOURCE

Mathematics • mathematical-induction-and-binomial-theorem

EXPLANATION

It is given that the coefficient of x^2 in $(1+x)^2+(1+x)^3+\dots+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)^{51}C_3$.

Now,

$$ {}^{2}C_2 + {}^{3}C_2 + {}^{4}C_2 + \cdots + {}^{49}C_2 + m^{2\,50}C_2 = (3n+1)^{51}C_3 $$

$$ \Rightarrow {}^{3}C_3 + {}^{3}C_2 + {}^{4}C_2 + \cdots + {}^{49}C_2 + m^{2\,50}C_2 = (3n+1)^{51}C_3 $$

$$ \left(\text{ as } nCr + nCr - 1 = n^{+1}Cr\right) $$

$$ \Rightarrow {}^{4}C_3 + {}^{4}C_2 + \cdots + {}^{49}C_2 + m^{250}C_2 = (3n+1)^{51}C_3 $$

$$ \Rightarrow {}^{49}C_3 + {}^{49}C_2 + m^{250}C_2 = (3n+1)^{51}C_3 $$

$$ \Rightarrow {}^{50}C_3 + m^{2\,50}C_2 = (3n+1)^{51}C_3 $$

$$ \Rightarrow {}^{50}C_3 + {}^{50}C_2 + m^{2\,50}C_2 = (3n+1)^{51}C_3 + {}^{50}C_2 $$

$$ \Rightarrow {}^{51}C_3 + m^{2\,50}C_2 = 3n^{51}C_3 + {}^{51}C_3 + {}^{50}C_2 $$

$$ \Rightarrow {}^{-50}C_2 + m^{250}C_2 = 3n^{51}C_3 $$

$$ \Rightarrow {}^{50}C_2\left(m^2-1\right) = 3n \cdot \frac{51}{3}{}^{50}C_2 \left({}^{n}C_r = \frac{n}{r} \quad {}^{n-1}C_{r-1}\right) $$

$$ \Rightarrow 2\left(m^2-1\right) = 51 $$

Therefore, $m^2 = 51n + 1$.

$$ \Rightarrow n = \frac{m^2-1}{51} $$

$$ \Rightarrow m = 16 \Rightarrow n = 5 \left(m, n \in \mathrm{I}^{+}\right) $$

Question 030 MCQ

QUESTION

A debate club consists of 6 girls and 4 boys. A team of 4 members is to be select from this club including the selection of a captain $from among these 4 members$ for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

A 380

B 320

C 260

D 95

CORRECT OPTION

A 380

SOURCE

Mathematics • permutations-and-combinations

EXPLANATION

We have 6 girls and 4 boys in the club and we have to select team of 4 members in which one is captain and at most one boy.

$\therefore$ We can have one boy and three girls in team or all four girls. So, selection of 1 boy from 4 boys $= {}^4C_1$ ways

Selection of 3 girls from 6 girls $= {}^6\mathrm{C}_3$ ways

Selection of 4 girls from 6 girls $= {}^6\mathrm{C}_4$ ways

$\therefore$ Total number of ways selecting the team

$$= \left({}^{4}\mathrm{C}_1 \cdot {}^{6}\mathrm{C}_3 + {}^{6}\mathrm{C}_4\right) \times 4$$

$$since, amonganyoftheselectionwehave4choicetoselectcaptain$$

$$\therefore \text{ Total number of ways } = (4 \times 20 + 15) \times 4$$

$$= 380$$

Question 031 MCQ

QUESTION

Let

$$-\frac{\pi}{6} < \theta < -\frac{\pi}{12}.$$

Suppose

$$\alpha_1$$

and

$$\beta_1$$

are the roots of the equation

$$x^2 - 2x \sec\theta + 1 = 0$$

and

$$\alpha_2$$

and

$$\beta_2$$

are the roots of the equation

$$x^2 + 2x\tan\theta - 1 = 0.$$

$$If\, \alpha_1 > \beta_1$$

and

$$\alpha_2 > \beta_2,$$

then

$$\alpha_1 + \beta_2$$

equals

A $2\left(\sec\theta - \tan\theta\right)$

B $2\ \sec\ \theta$

C $-2\tan\theta$

D 0

CORRECT OPTION

C $-2\tan\theta$

SOURCE

Mathematics • quadratic-equation-and-inequalities

EXPLANATION

Given, first equation $x^2 - 2x\sec\theta + 1 = 0$

Using quadratic equation formula we get,

$$x=\frac{-(-2\sec\theta)\pm\sqrt{(-2\sec\theta)^2-4}}{2}$$

$$\Rightarrow x=\frac{2\sec\theta\pm\sqrt{4\sec^2\theta-4}}{2}$$

$$\Rightarrow x=\frac{2\sec\theta\pm2\tan\theta}{2}$$

$$\Rightarrow x=\sec\theta\pm\tan\theta \text{ as } \theta\in\left(\frac{-\pi}{6},\frac{-\pi}{2}\right)$$

$$\Rightarrow \alpha_1=\sec\theta-\tan\theta \text{ and } \beta_1=\sec\theta+\tan\theta$$

Now, for the equation $x^2+2x\tan\theta-1=0$

$$x=\frac{-2\tan\theta\pm\sqrt{4\tan^2\theta+4}}{2}$$

$$\Rightarrow x=\frac{-2\tan\theta\pm2\sec\theta}{2}$$

$$\Rightarrow x=-\tan\theta\pm\sec\theta$$

$$\Rightarrow x=(\sec\theta-\tan\theta),-(\sec\theta+\tan\theta)$$

Given, that α_2 and β_2 are the roots of equation and $\alpha_2>\beta_2$

$$\Rightarrow \alpha_2 = \sec\theta - \tan\theta, \beta_2 = -(\sec\theta + \tan\theta)$$

$$\Rightarrow \alpha_1 + \beta_2 = (\sec\theta - \tan\theta) - (\sec\theta + \tan\theta)$$

$$= \sec\theta - \tan\theta - \sec\theta - \tan\theta$$

$$\Rightarrow \alpha_1 + \beta_2 = -2\tan\theta$$

Question 032 MCQ

QUESTION

Let

$$f : (0, \infty) \to R$$

be a differentiable function such that

$$f'(x) = 2 - \frac{f(x)}{x}$$

for all

$$x \in (0, \infty)$$

and

$$f(1) \neq 1$$

. Then

A

$$\lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = 1$$

B

$$\lim_{x \to 0^+} x f\left(\frac{1}{x}\right) = 2$$

C

$$\lim_{x \to 0^+} x^2 f'(x) = 0$$

D for all

$$|f(x)| \le 2$$

$$x \in (0, 2)$$

CORRECT OPTION

A

$$\lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = 1$$

SOURCE

Mathematics • differential-equations

EXPLANATION

$$f'(x) = 2 - \frac{f(x)}{x}$$

$$\Rightarrow f'(x) = \frac{1}{x} f(x) = 2$$

is linear differential equation.

Hence,

$$I.F. = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Thus the solution is given by

$$x\,.\,f(x)=\int 2x\,dx+\lambda$$

i.e.,

$$xf(x)=x^2+\lambda$$

As f1

$$\neq$$

1, we have

$$\lambda$$

$$\neq$$

0

$$\therefore$$

$$f(x)=x+\frac{\lambda}{x}$$

,

$$\lambda$$

$$\neq$$

0

Thus,

$$f'(x)=1-\frac{\lambda}{x^2}$$

,

$$\lambda$$

$$\neq$$

0

Now,

$$\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x\to 0^+}(1-\lambda x^2) = 1$$

$$\lim_{x\to 0^+} xf\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\left(\frac{1}{x}+\lambda x\right) = \lim_{x\to 0^+}(1+\lambda x^2) = 1$$

$$\lim_{x\to 0^+} x^2 f'(x) = \lim_{x\to 0^+} x^2\left(1-\frac{\lambda}{x^2}\right) = \lim_{x\to 0^+}(x^2-\lambda) = -\lambda$$

Again,

$$\lim_{x\to 0^+} f(x) \to \infty$$

Hence the function is not bounded.

Note that

$$\lambda$$

can be

$$-$$

ve or +ve.

Question 033 MCQ

QUESTION

Let

$$P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$$

, where

$$\alpha$$

$$\in$$

R. Suppose

$$Q = [q_{ij}]$$

is a matrix such that PQ = kI, where k

$$\in$$

R, k

$$\neq$$

0 and I is the identity matrix of order 3. If

$$q_{23} = -\frac{k}{8}$$

and

$$\det(Q) = \frac{k^2}{2}$$

, then

A

$$\alpha$$

= 0, k = 8

B

$$4\alpha - k + 8 = 0$$

C

$$\det(Padj(Q)) = 2^9$$

D

$$\det(Qadj(P)) = 2^{13}$$

CORRECT OPTION

B

$$4\alpha - k + 8 = 0$$

SOURCE

Mathematics • matrices-and-determinants

EXPLANATION

$$P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$$

Now,

$$|P| = 3(5\alpha) + 1(-3\alpha) - 2(-10)$$
$$= 12\alpha + 20$$

.... i

$$\therefore$$

$$(P) = \begin{bmatrix} 5\alpha & 2\alpha & -10 \\ -10 & 6 & 12 \\ -\alpha & -(3\alpha + 4) & 2 \end{bmatrix}^T$$
$$= \begin{bmatrix} 5\alpha & -10 & -\alpha \\ 2\alpha & 6 & -3\alpha - 4 \\ -10 & 12 & 2 \end{bmatrix}$$

...... ii

As,

$$PQ = kI$$
$$\Rightarrow |P|\,|Q| = |kI|$$
$$\Rightarrow |P|\,|Q| = k^3$$
$$\Rightarrow |P|\left(\frac{k^2}{2}\right) = k^3$$
$$given, \$\$\,|Q| = \frac{k^2}{2}\$\$$$
$$\Rightarrow |P| = 2k$$

...... iii

$$\because$$

$$PQ = ki$$

$$\therefore$$

$$Q = kp^{-1}I$$

$$= k\,.\ \frac{adj\,P}{|P|} = \frac{k(adj\,P)}{2k}$$

$$fromEq.\,(iii)$$

$$= \frac{adj\,P}{2}$$

$$= \frac{1}{2}\begin{bmatrix} 5\alpha & -10 & -\alpha \\ 2\alpha & 6 & -3\alpha-4 \\ -10 & 12 & 2 \end{bmatrix}$$

$$\therefore$$

$$q_{23} = \frac{-3\alpha-4}{2}$$

$$given, \$\$q_{23} = -\frac{k}{8}\$\$$$

$$\Rightarrow -\frac{(3\alpha+4)}{2} = -\frac{k}{8}$$

$$\Rightarrow (3\alpha+4)\times 4 = k$$

$$\Rightarrow 12\alpha + 16 = k$$

...... iv

From Eq. iii,

$$|P| = 2k$$

$$\Rightarrow 12\alpha + 20 = 2k$$

$$fromEq.\,(i)$$

...... v

On solving Eqs. iv and v, we get

$$\alpha$$

=

$$-$$

1 and k = 4 vi

$$\therefore$$

$$4\alpha - k + 8 = -4 - 4 + 8 = 0$$

$$\therefore$$

Option b is correct.

Now,

$$|P\,adj(Q)| = |P|\,|adj\,Q|$$

$$= 2k\left(\frac{k^2}{2}\right)^2 = \frac{k^5}{2} = \frac{2^{10}}{2} = 2^9$$

$$\therefore$$

Option c is correct.

Question 034 Numerical

QUESTION

The total number of distinct x

$$\in$$

R for which

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$$

is _______________.

SOURCE

Mathematics • matrices-and-determinants

EXPLANATION

Given,

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$$

$$\Rightarrow x\,.\,x^2 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$$

Apply R_2

$$\rightarrow$$

R_2

$$-$$

$2R_1$ and R_3

$$\rightarrow$$

R_3

$$-$$

$3R_1$, we get

$$x^3 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 0 & 2 & -1+6x^3 \\ 0 & 6 & -2+24x^3 \end{vmatrix} = 10$$

$$\Rightarrow x^3\,.\begin{vmatrix} 2 & 6x^2-1 \\ 6 & 24x^3-2 \end{vmatrix} = 10$$

$$\Rightarrow x^3(48x^3-4-36x^3+6) = 10$$

$$\Rightarrow 12x^6+2x^3 = 10$$

$$\Rightarrow 6x^6 + x^3 - 5 = 0$$

$$\Rightarrow 6(x^3)^2 + x^3 - 5 = 6$$

$$\Rightarrow 6(x^3)^2 + 6x^3 - 5x^3 - 5 = 0$$

$$\Rightarrow 6x^3(x^3 + 1) - 5(x^3 + 1) = 0$$

$$\Rightarrow (6x^3 - 5)(x^2 - x + 1)(x + 1) = 0$$

$$\therefore$$

$$x = \left(\frac{5}{6}\right)^{1/3}, -1$$

Hence, the number of real solutions is 2.

Question 035 Numerical

QUESTION

Let

$$z = \frac{-1 + \sqrt{3}i}{2}$$

, where

$$i = \sqrt{-1}$$

, and r, s

$$\in$$

{1, 2, 3}. Let

$$P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$$

and I be the identity matrix of order 2. Then the total number of ordered pairs r, s for which P^2 =

$$-$$

I is ______________.

SOURCE

Mathematics • matrices-and-determinants

EXPLANATION

Here,

$$z = \frac{-1 + i\sqrt{3}}{2} = \omega$$

$$\because$$

$$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$P^2 = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix} \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s}[(-1)^r + 1] \\ \omega^{r+2s}[(-1)^r + 1] & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

Given,

$$P^2 = -I$$

$$\therefore$$

$$\omega^{2r} + \omega^{4s} = -1$$

and

$$\omega^{r+2s}[(-1)^r + 1] = 0$$

Since, r

$$\in$$

{1, 2, 3} and $$ $-$ $$1 ^{r} + 1 = 0

$$\Rightarrow$$

r = {1, 3}

Also,

$$\omega^{2r} + \omega^{4s} = -1$$

If r = 1, then

$$\omega^{2} + \omega^{4s} = -1$$

which is only possible, when s = 1.

As,

$$\omega^{2} + \omega^{4} = -1$$
$$\therefore$$

r = 1, s = 1

Again, if r = 3, then

$$\omega^{6} + \omega^{4s} = -1$$
$$\Rightarrow \omega^{4s} = -2$$
$$never possible$$
$$\therefore$$

r

$$\neq$$

3

$$\Rightarrow$$

r, s = $1, 1$ is the only solution.

Hence, the total number of ordered pairs is 1.

Question 036 Numerical

QUESTION

Let

$$\alpha$$

,

$$\beta$$

$$\in$$

R be such that

$$\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$$

. Then 6 $$\alpha$$ + $$\beta$$ equals __________.

SOURCE

Mathematics • limits-continuity-and-differentiability

EXPLANATION

Here,

$$\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$$

$$\lim_{x \to 0} \frac{x^2 \left(\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \ldots\right)}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)} = 1$$

$$\Rightarrow \lim_{x \to 0} \frac{x^3 \left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \ldots\right)}{(\alpha - 1)x + \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots} = 1$$

Limit exists only, when

$$\alpha$$

$$-$$

1 = 0

$$\Rightarrow$$

α

= 1 i

$$\therefore$$

$$\Rightarrow \lim_{x \to 0} \frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \ldots.\right)}{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} - \ldots\right)} = 1$$

$$\Rightarrow$$

6

$$\beta$$

= 1 ii

From Eqs. i and ii, we get

6 $$\alpha$$ + $$\beta$$ = 6

$$\alpha$$

+ 6

$$\beta$$

= 6 + 1

= 7

Question 037 MCQ

QUESTION

A length-scale l depends on the permittivity $$\varepsilon$$ of a dielectric material, Boltzmann constant (k_B), the absolute temperature T, the number per unit volume n of certain charged particles, and the charge q carried by each of the particles. Which of the following expression s for I is are dimensionally correct?

A

$$l = \sqrt{\left(\frac{nq^2}{\varepsilon k_b T}\right)}$$

B

$$l = \sqrt{\left(\frac{\varepsilon k_b T}{nq^2}\right)}$$

C

$$l = \sqrt{\left(\frac{q^2}{\varepsilon n^{2/3} k_B T}\right)}$$

D

$$l = \sqrt{\left(\frac{q^2}{\varepsilon n^{1/3} k_B T}\right)}$$

CORRECT OPTION

B

$$l = \sqrt{\left(\frac{\varepsilon k_b T}{nq^2}\right)}$$

SOURCE

Physics • units-and-measurements

EXPLANATION

$$[n] = [L^{-3}]; [q] = [AT]$$

$$[\varepsilon] = [M^{-1}L^{-3}A^2T^4]$$

$$[T] = [L]$$

$$[l] = [L]$$

$$[k_B] = [M^1L^2T^{-2}K^{-1}]$$

a RHS

$$= \sqrt{\frac{[L^{-3}A^2T^2]}{[M^{-1}L^{-3}T^4A^2][M^1L^2T^{-2}K^{-1}][K]}}$$

$$= \sqrt{\frac{[L^{-3}A^2T^2]}{[L^{-1}A^2T^2]}} = \sqrt{[L^{-2}]} = [L^{-1}]$$

Wrong

b RHS

$$= \sqrt{\frac{[M^{-1}L^{-3}T^4A^2][M^1L^2T^{-2}K^{-1}][K]}{[L^{-3}A^2T^2]}}$$

$$= \sqrt{\frac{[L^{-1}A^2T^2]}{[L^{-3}A^2T^2]}} = L$$

Correct

c RHS

$$= \sqrt{\frac{[A^2T^2]}{[M^{-1}L^{-3}T^4A^2][L^{-2}][M^1L^2T^{-2}K^{-1}][K]}}$$

$$= \sqrt{[L^3]}$$

Wrong

d RHS

$$= \sqrt{\frac{[A^2T^2]}{[M^{-1}L^{-3}T^4A^2][L^{-1}][M^1L^2T^{-2}K^{-1}][K]}}$$

$$= \sqrt{\frac{[A^2T^2]}{[L^{-2}T^2A^2]}} = [L]$$

Correct

Question 038 Numerical

QUESTION

A metal is heated in a furnace where a sensor is kept above the metal surface to read the power radiated P by the metal. The sensor has a scale that displays

$$\log_2\left(\frac{P}{P_0}\right)$$

, where P_0 is a constant. When the metal surface is at a temperature of 487^{o}C, the sensor shows a value 1. Assume that the emissivity of the metallic surface remains constant. What is the value displayed by the sensor when the temperature of the metal surface is raised to 2767^{o}C?

SOURCE

Physics • heat-and-thermodynamics

EXPLANATION

$$\log_2 = \frac{p_1}{p_0} = 1$$

Therefore,

$$\frac{p_1}{p_0} = 2$$

According to Stefan's law, p

$$\propto$$

T^2

$$\Rightarrow \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^4 = \left(\frac{2767+273}{487+273}\right)^4 = 4^4$$

$$\frac{p_2}{p_1} = \frac{p_2}{2p_0} = 4^4 \Rightarrow \frac{p_2}{p_0} = 2 \times 4^4$$

$$\log_2 \frac{p_2}{p_0} = \log_2[2 \times 4^4]$$

$$= \log_2 2 + \log_2 4^4$$

$$= 1 + \log_2 2^8 = 1 + 8 = 9$$

Question 039 Numerical

QUESTION

Consider two solid spheres P and Q each of density 8 gm cm^{-3} and diameters 1 cm and 0.5 cm, respectively. Sphere P is dropped into a liquid of density 0.8 gm cm^{-3} and viscosity

$$\eta$$

= 3 poiseulles. Sphere Q is dropped into a liquid of density 1.6 gm cm^{-3} and viscosity

$$\eta$$

= 2 poiseulles. The ratio of the terminal velocities of P and Q is

SOURCE

Physics • properties-of-matter

EXPLANATION

Terminal velocity is given by

$$v_T = \frac{2}{9}\frac{r^2}{\eta}(d-\rho)g$$

$$\frac{v_P}{v_Q} = \frac{r_P^2}{r_Q^2} \times \frac{\eta_Q}{\eta_P} \times \frac{(d-\rho_P)}{(d-\rho_Q)}$$

$$= \left(\frac{1}{0.5}\right)^2 \times \left(\frac{2}{3}\right) \times \frac{(8-08)}{(8-1.6)} = 4 \times \frac{2}{3} \times \frac{7.2}{6.4} = 3$$

Question 040 MCQ

QUESTION

The position vector

$$\vec{r}$$

of a particle of mass m is given by the following equation

$$\$\vec{r}(t) = \alpha t^3 \hat{i} + \beta t^2 \hat{j},$$

$where\ \alpha = \frac{10}{3} ms^{-3}$, $\beta = 5\, ms^{-2}$\$ and m = 0.1 kg. At t = 1 s, which of the following statement$s$ isare true about the particle?

The velocity

$$\vec{v}$$

A is given by

$$\vec{v} = \left(10\hat{i} + 10\hat{j}\right)$$

ms^{-1}

The angular momentum

$$\vec{L}$$

B with respect to the origin is given by

$$\vec{L} = -\left(\frac{5}{3}\right)\widehat{k}\,N\,m\,s$$

The force

$$\vec{F}$$

C is given by

$$\vec{F} = \left(\hat{i} + 2\hat{j}\right)N$$

The torque

$$\vec{\tau}$$

D with respect to the origin is given by

$$\vec{\tau} = -\left(\frac{20}{3}\right)\widehat{k}\,N\,m$$

CORRECT OPTION

The velocity

$$\vec{v}$$

A is given by

$$\vec{v} = \left(10\hat{i} + 10\hat{j}\right)$$

ms^{-1}

SOURCE

Physics • rotational-motion

EXPLANATION

$$r = \alpha t^3 \hat{i} + \beta t^2 \hat{j}$$

$$v = \frac{dr}{dt} = 3\alpha t^2 \hat{i} + 2\beta t^2 \hat{j}$$

$$a = \frac{d^2 r}{dt^2} = 6\alpha t^2 \hat{i} + 2\beta t^2 \hat{j}$$

At t = 1 s,

a

$$v = 3 \times \frac{10}{3} \times 1\hat{i} + 2 \times 5 \times 1\hat{j}$$

$$= (10\hat{i} + 10\hat{j})$$

m/s

b

$$\widehat{L} = \hat{r} \times \widehat{p}$$

$$= \left(\frac{10}{3} \times 1\hat{i} + 5 \times 1\hat{j}\right) \times 0.1(10\hat{i} + 10\hat{j})$$

$$= \left(-\frac{5}{3}\widehat{k}\right)$$

N-ms

c

$$F = ma$$

$$= m \times \left(6 \times \frac{10}{3} \times 1\hat{i} + 2 \times 5\hat{j}\right) = (2\hat{i} + \hat{j})N$$

d

$$\tau = r \times F = \left(\frac{10}{3}\hat{i} + 5\hat{j}\right) \times (2\hat{i} + \hat{j})$$

$$= +\frac{10}{3}\widehat{k} + 10(-\widehat{k}) = \left(-\frac{20}{3}\widehat{k}\right)$$

N-m

Question 041 MCQ

QUESTION

A uniform wooden stick of mass 1.6 kg and length

$$l$$

rests in an inclined manner on a smooth, vertical wall of height h $<$ \$\$$l$\$\$ such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle of

$$30^{\circ}$$

with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio

$$\frac{h}{l}$$

and the frictional force f at the bottom of the stick are (g =10 ms^{-2})

A $$\frac{h}{l}=\frac{\sqrt{3}}{16}, f=\frac{16\sqrt{3}}{3}N$$

B $$\frac{h}{l}=\frac{3}{16}, f=\frac{16\sqrt{3}}{3}N$$

C

$$\frac{h}{l} = \frac{3\sqrt{3}}{16}, f = \frac{8\sqrt{3}}{3}N$$

D

$$\frac{h}{l} = \frac{3\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}N$$

CORRECT OPTION

D

$$\frac{h}{l} = \frac{3\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}N$$

SOURCE

Physics • rotational-motion

EXPLANATION

$$\sum F_x = 0$$

,

$$N_1 \cos 30^\circ - f = 0$$

.... i

$$\sum F_y = 0$$

,

$$N_1 \sin 30^\circ + N_2 - mg = 0$$

.... ii

$$\sum \tau_0 = 0$$

$$mg\frac{1}{2}\cos 60^\circ - N_1\frac{h}{\cos 30^\circ} = 0$$

.... iii

Also, given

$$N_1 = N_2$$

.... iv

Solving Eqs. i, ii, iii and iv we have

$$\frac{h}{l} = \frac{3\sqrt{3}}{16}$$

and

$$f = \frac{16\sqrt{3}}{3}$$

Question 042 MCQ

QUESTION

A parallel beam of light is incident from air at an anglea on the side PQ of a right angled triangular prism of refractive index n =

$$\sqrt{2}$$

. Light undergoes total internal reflection in the prism at the face PR when a has a minimum value of 45°. The angle q of the prism is :

A 15^{o}

B 22.5^{o}

C 30^{o}

D 45^{o}

CORRECT OPTION

A 15^{o}

SOURCE

Physics • geometrical-optics

EXPLANATION

Applying Snell's law at M,

$$n=\frac{\sin \alpha}{\sin r_1} \Rightarrow \sqrt{2}=\frac{\sin 45^{\circ}}{\sin r_1}$$

$$\Rightarrow \sin r_1=\frac{\sin 45^{\circ}}{\sqrt{2}}=\frac{1 / \sqrt{2}}{\sqrt{2}}=\frac{1}{2}$$

$$r_1 = 30^\circ$$

$$\sin\theta_c = \frac{1}{n} = \frac{1}{\sqrt{2}} \Rightarrow \theta_c = 45^\circ$$

Let us take

$$r_2 = \theta_c = 45^\circ$$

for just satisfying the condition of TIR.

In

$$\Delta PNM$$

,

$$\theta + 90 + r_1 + 90 - r_2 = 180^\circ$$

or

$$\theta = r_2 - r_1 = 45^\circ - 30^\circ = 15^\circ$$

Note If

$$\alpha$$

= 45

$$\circ$$

$thegivenvalue$. Then, r_1 > 30

$$\circ$$

$theobtainedvalue$

$$r_2 > \theta_c$$

(as r_2

$$-$$

r_1 =

$$\theta$$

or r_2 =

$$\theta$$

+ r_1)

or TIR will take place. So, for taking TIR under all conditions

$$\alpha$$

should be greater than 45

$$\circ$$

or this is the minimum value of

$$\alpha$$

.

Question 043 MCQ

QUESTION

In a historical experiment to determine Planck's constant, a metal surface was irradiated with light of different wavelengths. The emitted photoelectron energies were measured by applying a stopping potential. The relevant data for the wavelength $$\lambda$$ of incident light and the corresponding stopping potential

(V_0) are given below:

$\lambda\,(\mu m)$	$V_0\,Volt$
0.3	2.0
0.4	1.0
0.5	0.4

Given that c = 3

$\times$

10^8 ms^{-1} and e = 1.6

$\times$

10^{-19} C, Planck's constant $in units of J - s$ found from such an experiment is)
:

A 6.0

$\times$

10^{-34}

B 6.6

$\times$

10^{-34}

C 6.4

$\times$

10^{-34}

D

6.8

×

10^{-34}

CORRECT OPTION

C

6.4

×

10^{-34}

SOURCE

Physics • dual-nature-of-radiation

EXPLANATION

$$\frac{hc}{\lambda} - \phi = eV_0$$

$\$\$\phi\$\$ = workfunction$

$$\frac{hc}{0.3 \times 10^{-6}} - \phi = 2e$$

.... i

$$\frac{hc}{0.4 \times 10^{-6}} - \phi = 1e$$

.... ii

Subtracting Eq. ii from Eq. i

$$hc\left(\frac{1}{0.3} - \frac{1}{0.4}\right)10^6 = e$$

$$hc\left(\frac{0.1}{0.12}\times 10^{6}\right)=e$$

$$h=0.64\times 10^{-33}$$

$$=6.4\times 10^{-34}$$

J-s

Question 044 MCQ

QUESTION

A water cooler of storage capacity 120 litres can cool water at a constant rate of P watts. In a closed circulation system $as shown schematically in the figure$, the water from the cooler is used to cool an external device that generates constantly 3 kW of heat $thermal load$.
The temperature of water fed into the device cannot exceed 30°C and the entire stored 120 litres of water is initially cooled to 10°C. The entire system is thermally insulated. The minimum value of P $in watts$ for which the device can be operated for 3 hours is :

(Specific heat of water is 4.2 kJ kg^{-1} K^{-1} and the density of water is 1000 kg m^{-3})

A 1600

B 2067

C 2533

D 3933

CORRECT OPTION

B 2067

SOURCE

Physics • heat-and-thermodynamics

EXPLANATION

Heat generated in device in 3 h

= Time

$\times$

power

= 3

$\times$

3600

$\times$

3

$\times$

10^3 = 324

$\times$

10^5 J

Heat used to heat water = ms

Δ

θ

= 120

$\times$

1

$\times$

4.2

$\times$

10^3

$\times$

20J

Heat absorbed by coolant

Pt = 324

$\times$

10^5

$-$

120

$\times$

1

$\times$

4.2

$\times$

10^3

$$\times$$

20J

Pt = $325\$\$ - \$\$100.8$

$$\times$$

10^5 J = 223.2

$$\times$$

10^5 J

P =

$$\frac{223.2 \times 10^5}{3600}$$

= 2067 W

Question 045 MCQ

QUESTION

A plano-convex lens is made of material of refractive index n. When a small object is placed 30 cm away in front of the curved surface of the lens, an image of double the size of the object is produced. Due to reflection from the convex surface of the lens, another faint image is observed at a distance of 10 cm away form the lens. Which of the following statement s is are true?

A The refractive index of the lens is 2.5

B The radius of curvature of the convex surface is 45 cm

C The faint image is erect and real

D The focal length of the lens is 20 cm

CORRECT OPTION

A The refractive index of the lens is 2.5

SOURCE

Physics • geometrical-optics

EXPLANATION

For refraction through plano - Convex lens,

$$m=-2=\frac{v}{u}$$

$$\Rightarrow v=-2u$$

$$\text{As, } u=-30 \text{ cm}$$

$$\Rightarrow v=60 \text{ cm}$$

Using lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$

$$\frac{1}{60}-\frac{1}{-30}=\frac{1}{f}$$

$$\Rightarrow f=20 \text{ cm}$$

By lens maker's formula,

$$\frac{1}{f}=(n-1)\left(\frac{1}{\mathrm{R}_1}-\frac{1}{\mathrm{R}_2}\right)$$

$$\Rightarrow \frac{1}{20}=(n-1)\left(\frac{1}{\mathrm{R}}+\frac{1}{\infty}\right)=\frac{n-1}{\mathrm{R}}$$

$$\Rightarrow n=1+\left(\frac{\mathrm{R}}{20}\right)\ldots\ldots\ldots(i)$$

for poor reflection from convex surface,

$$u=-30 \mathrm{~cm},$$

$$v=10 \mathrm{~cm}, f=\mathrm{R}/2$$

Using mirror formula,

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}=\frac{2}{\mathrm{R}};$$

$$\Rightarrow \frac{1}{10}+\frac{1}{-30}=\frac{2}{\mathrm{R}}$$

$$\Rightarrow \frac{2}{\mathrm{R}} = \frac{2}{30}$$

$$\therefore \mathrm{R} = 30 \text{ cm}$$

Substitute $R = 30 \text{ cm}$ in equation i to get $n = 2.5$ for $f = 20 \text{ cm}$ $inverted image$ and $n = 1.5$ for $f = 60 \text{ cm}$ $erect image$.

Question 046 MCQ

QUESTION

A conducting loop in the shape of a right angled isosceles triangle of height 10 cm is kept such that the 90

$$\circ$$

vertex is very close to an infinitely long conducting wire $see the figure$. The wire is electrically insulated from the loop. The hypotenuse of the triangle is parallel to the wire. The current in the triangular loop is in counterclockwise direction and increased at a constant rate of 10 As

$$-$$

^{1}. Which of the following statement$s$ is are true?

A There is a repulsive force between the wire and the loop.

B If the loop is rotated at a constant angular speed about the wire, an additional emf of

$$\left(\frac{\mu_0}{\pi}\right)$$

volt is induced in the wire

C

The magnitude of induced emf in the wire is

$$\left(\frac{\mu_0}{\pi}\right)$$

volt

D

The induced current in the wire is in opposite direction to the current along the hypotenuse.

CORRECT OPTION

A

There is a repulsive force between the wire and the loop.

SOURCE

Physics • electromagnetic-induction

EXPLANATION

The flux passing through the triangular wire if ' i ' current flows through the infinitely long conducting wire.

$$\frac{d\mathrm{I}}{dt} = 10\mathrm{As}^{-1}$$

$$\phi = \int_0^{0.1} \frac{\mu_0 i}{2\pi x} \times 2x dx$$

$$\phi = \frac{\mu_0 i}{10\pi} = \mathrm{M}i$$

$$\therefore \mathrm{M} = \frac{\mu_0}{10\pi}$$

$$\text{Induced emf in the wire } = \frac{\mathrm{M}di}{dt} = \frac{\mu_0}{10\pi} \times 10 = \frac{\mu_0}{\pi}$$

According to Lenz's law, the induced current opposes the flux change that creates it. Since the flux is increasing, the induced emf opposes the existing magnetic field and the current flows from left to right in the straight wire, that is, along the hypotenuse current. That is, there is a repulsive force between the wire and the loop.

Also, if the loop is rotated about the wire, there is no change in the flux through the wire, so no extra induced emf is generated in the wire.

Question 047 MCQ

QUESTION

Two loudspeakers M and N are located 20m apart and emit sound at frequencies 118 Hz and 121 Hz, respectively. A car in initially at a point P, 1800 m away from the midpoint Q of the line MN and moves towards Q constantly at 60 km/h along the perpendicular bisector of MN. It crosses Q and eventually reaches a point R, 1800 m away from Q.

Let vt represent the beat frequency measured by a person sitting in the car at time t. Let v_P, v_Q and v_R be the beat frequencies measured at locations P, Q and R respectively. The speed of sound in air is 330 ms

$-$

[1]. Which of the following statement$s$ is are true regarding the sound heard by the person?

A The plot below represents schematically the variation of beat frequency with time

B The rate of change in beat frequency is maximum when the car passes through Q

C $v_P + v_R = 2v_Q$

D The plot below represents schematically the variations of beat frequency with time

CORRECT OPTION

B The rate of change in beat frequency is maximum when the car passes through Q

SOURCE

Physics • waves

EXPLANATION

Consider the time $t\,(\leq t_Q)$ when car is at S_1 between P and Q. The distance travelled by the car in time t is $\mathrm{PS}_1 = ut$. At this instant, the lines $\mathrm{S}_1 N$ and $\mathrm{S}_1\mathrm{M}$ both make angle θ with the velocity vector $\vec{u}$. The component of observer $personsittinginthecar$ velocity towards the sources N and M is $u_o = u\cos\theta$. The sources N and M are at rest i.e., $u_s = 0$. Apply Doppler's effect equation to get frequencies of the sources N and M heard by the observer as

$$\nu_N' = \frac{v+u_o}{v-u_s}\nu_N = \frac{v+u\cos\theta}{v}\nu_N$$

$$= \left(1+\frac{u}{v}\frac{D-ut}{\sqrt{d^2+(D-ut)^2}}\right)\nu_N,$$

$$\nu_M' = \frac{v+u_o}{v-u_s}\nu_M = \frac{v+u\cos\theta}{v}\nu_M$$

$$= \left(1+\frac{u}{v}\frac{D-ut}{\sqrt{d^2+(D-ut)^2}}\right)\nu_M.$$

The beat frequency heard by the observer at time $t(\leq\ t_Q)$ is

$$\nu(t) = \nu_N' - \nu_M'$$

$$= \left(1+\frac{u}{v}\frac{D-ut}{\sqrt{d^2+(D-ut)^2}}\right)(\nu_N-\nu_M)\ldots\ldots..(1)$$

Now, consider the time $t\,(\geq t_Q)$ when car is at S_2 between Q and R. The distance travelled by the car in time t is $\mathrm{PS}_2 = ut$. At this instant, the lines $\mathrm{S}_2\,\mathrm{N}$ and $\mathrm{S}_2\mathrm{M}$ both make angle $(180^\circ-\theta)$ with the velocity vector $\vec{u}$. The component of observer velocity towards the sources N and M is $u_o = -u\cos\theta$. Apply Doppler's effect equation to get

$$\nu'_N = \frac{v+u_o}{v-u_s}\nu_N = \frac{v - u\cos\theta}{v}\nu_N$$

$$= \left(1 - \frac{u}{v}\frac{ut-D}{\sqrt{d^2+(ut-D)^2}}\right)\nu_N,$$

$$\nu'_M = \frac{v+u_o}{v-u_s}\nu_M = \frac{v - u\cos\theta}{v}\nu_M$$

$$= \left(1 - \frac{u}{v}\frac{ut-D}{\sqrt{d^2+(ut-D)^2}}\right)\nu_M.$$

The beat frequency heard by the observer at time $t(\geq t_Q)$ is

$$\nu(t) = \nu'_N - \nu'_M$$

$$= \left(1 - \frac{u}{v}\frac{ut-D}{\sqrt{d^2+(ut-D)^2}}\right)(\nu_N - \nu_M) \ldots\ldots\ldots\ldots (2)$$

Substitute $t = 0$ and $t = t_Q = \frac{D}{u}$ in equation 1 to get ν_P and ν_Q and substitute $t = t_R = 2\frac{D}{u}$ in equation 2 to get ν_R i.e.,

$$\nu_P = \nu(t=0) = \left(1 + \frac{u}{v}\frac{D}{\sqrt{d^2+D^2}}\right)(\nu_N - \nu_M)$$

$$\nu_Q = \nu(t = D/u) = (\nu_N - \nu_M)$$

$$\nu_R = \nu(t = 2D/u) = \left(1 - \frac{u}{v}\frac{D}{\sqrt{d^2+D^2}}\right)(\nu_N - \nu_M)$$

$$\therefore$$

$$\nu_P + \nu_R = 2\nu_Q$$

Differentiate equations 1 w.r.t. time t to get rate of change of beat frequency

$$\frac{\mathrm{d}\nu(t)}{\mathrm{d}t} = -\left(\nu_N - \nu_M\right)\frac{u^2}{v}\frac{d^2}{\left(d^2 + (D-ut)^2\right)^{3/2}}$$

..........3

From equation 3, the slope is negative and its magnitude is maximum when $t = \frac{D}{u} = t_Q \ denominatorisminimum$. Thus the rate of change of beat frequency is maximum when car passes through Q.

Question 048 MCQ

QUESTION

A transparent slab of thickness d has a refractive index n z that increases with z. Here, z is the vertical distance inside the slab, measured from the top. The slab is placed between two media with uniform refractive indices n_1 and n_2 (> n_1), as shown in the figure. A ray of light is incident with angle

$$\theta$$

$_i$ from medium 1 and emerges in medium 2 with refraction angle

$$\theta$$

$_f$ with a lateral displacement l.

Which of the following statement$s$ is are true?

A l is dependent on n z

B

n_1 sin

θ

$_i$ = (n_2

−

n_1) sin

θ

f

C

n_1 sin

θ

$_i$ = n_2 sin

θ

f

D l is independent of n_2

CORRECT OPTION

A l is dependent on n z

SOURCE

Physics • geometrical-optics

EXPLANATION

From Snell's law,

$$n_1 \sin \theta_i = n\left(z_1\right) \sin \theta_{z_1} = n\left(z_2\right) \sin \theta_{z_2} = \ldots\ldots$$

$$\ldots = n(d) \sin \theta_d = n_2 \sin \theta_f$$

The lateral displacement of a ray by a parallel slab depends on the angle of incidence θ_i, refractive index n_1, slab thickness d and the refractive index of the slab n
$itisindependentof$n_2$because$l$ismeasuredbeforethestartofmedium2,t
. Consider the parallel slab shown in the figure.

By Snell's law, $n_1 \sin i_1 = n \sin r_1$ and $n \sin i_2 = n_2 \sin r_2$. By geometry, $i_2 = r_1$ and the lateral displacement is given by

$$l = d \frac{\sin r_1}{\cos r_1} = \frac{d n_1 \sin i_1}{\sqrt{n^2 - n_1^2 \sin^2 i_1}}.$$

Hence, l is independent of n_2 but it is dependent of $n(z)$.

Question 049 MCQ

QUESTION

Highly excited states for hydrogen-like atoms $alsocalledRydbergstates$ with nuclear charge Ze are defined by their principle quantum number n, where n >> 1. Which of the following statement$s$ isare true?

A Relative change in the radii of two consecutive orbitals does not depend on Z.

B Relative change in the radii of two consecutive orbitals varies as 1/n

C Relative change in the energy of two consecutive orbitals varies as $1/n^3$

D Relative change in the angular momenta of two consecutive orbitals varies as 1/n

CORRECT OPTION

A Relative change in the radii of two consecutive orbitals does not depend on Z.

SOURCE

Physics • atoms-and-nuclei

EXPLANATION

The radius of n^{th} orbital for a hydrogen-like atom of atomic number Z is given by $r_n = \frac{n^2 a_0}{Z}$, where $a_0 = 0.53$

$$\overset{o}{A}$$

is the Bohr's radius.

$$\therefore$$

$$\text{Radius of orbit, } r \propto \frac{n^2}{Z}$$

The relative change in the radii of two consecutive orbitals is

$$\frac{r_{n+1}-r_n}{r_n}=\frac{\frac{(n+1)^2}{Z}-\frac{n^2}{Z}}{\frac{n^2}{Z}}=\frac{2n+1}{n^2}\approx\frac{2}{n}\quad(\because n>>1)$$

The energy of the n^th orbital is given by

$$E_n=\frac{-13.6Z^2}{n^2}eV$$

$$\therefore$$

$$E_n\propto\frac{-Z^2}{n^2}$$

The relative change in the energy of two consecutive orbitals is

$$\frac{E_{n+1}-E_n}{E_n}=\frac{\frac{-Z^2}{(n+1)^2}+\frac{Z^2}{n^2}}{\frac{-Z^2}{n^2}}=-\frac{(2n+1)}{(n+1)^2}$$

$$\approx-\frac{2n}{n^2}=-\frac{2}{n}\quad(n>>1)$$

The angular momentum of the n^{th} orbital is given by $L_n=\frac{nh}{2\pi}$. The relative change in the angular momentum of two consecutive orbitals is

$$\frac{L_{n+1}-L_n}{L_n}=\frac{(n+1)\frac{h}{2\pi}-\frac{nh}{2\pi}}{\frac{nh}{2\pi}}=\frac{1}{n}$$

Question 050 MCQ

QUESTION

An incandescent bulb has a thin filament of tungsten that is heated to high temperature by passing an electric current. The hot filament emits black-body radiation. The filament is observed to break up at random locations after a sufficiently long time of operation due to non-uniform evaporation of tungsten from the filament. If the bulb is powered at constant voltage, which of the following statement s is are true?

- **A** The temperature distribution over the filament is uniform
- **B** The resistance over small sections of the filament decreases with time
- **C** The filament emits more light at higher band of frequencies before it breaks up
- **D** The filament consumes less electrical power towards the end of the life of the bulb

CORRECT OPTION

C The filament emits more light at higher band of frequencies before it breaks up

SOURCE

Physics • current-electricity

EXPLANATION

Because of non-uniform evaporation at different section, area of cross-section would be different at different sections.

Region of highest evaporation rate would have rapidly reduced area and would become break up cross-section.

Resistance of the wire as whole increases with time. Overall resistance increases hence power decreases.

$\$\$p = \frac{V^2}{R}\$\$or\$\$p \propto \frac{1}{R}\$\$asVisconstant.$

At break up junction temperature would be highest, thus light of highest band frequency would be emitted at those cross-section.

Question 051 Numerical

QUESTION

A hydrogen atom in its ground state is irradiated by light of wavelength 970

$$\overset{o}{A}$$

.

Taking hc = 1.237

$$\times$$

10

$$-$$

6 eVm and the ground state energy of hydrogen atom as

$$-$$

13.6 eV, the number of lines present in the emission spectrum is

SOURCE

Physics • atoms-and-nuclei

EXPLANATION

The energy of the incident photon of wavelength $\lambda = 970$

$$\overset{o}{A}$$

is

$$\Delta E = \frac{hc}{\lambda} = \frac{1.237 \times 10^{-6}}{970 \times 10^{-10}} = 12.75\mathrm{eV}$$

$$\text{Final energy of electron in } n^{\text{th}} \text{ state } = -\frac{13.6}{n^2}\mathrm{eV}$$

$$\Rightarrow -\frac{13.6}{n^2} = -13.6 + 12.75 = -0.85$$

$$\Rightarrow n^2 = 16 \Rightarrow n = 4$$

The hydrogen atom can make ${}^{n}\mathrm{C}_2 = 6$ transitions while returning to ground state. Thus, the emission spectrum will have six lines.

Question 052 Numerical

QUESTION

The isotope

$${}_{5}^{12}B$$

having a mass 12.014 u undergoes

$$\beta$$

-decay to

$$ {}_{6}^{12}C $$

.

$$ {}_{6}^{12}C $$

has an excited state of the nucleus $$${}_{6}^{12}C$$*$ at 4.041 MeV above its ground state. If

$$ {}_{5}^{12}B $$

decays to

$$ {}_{6}^{12}C $$

*, the maximum kinetic energy of the

$$ \beta $$

-particle in units of MeV is (1u = 931.5 MeV/c^2, where c is the speed of light in vacuum).

SOURCE

Physics • atoms-and-nuclei

EXPLANATION

As the ${}_{5}^{12}B$ is going under β-decay to form

$$ {}_{6}^{12}C $$

. We can write the balanced equation as follows

$$ {}_{5}^{12}\mathrm{B} \rightarrow {}_{6}^{12}\mathrm{C} + e^{-1} + \bar{\nu} $$

The Q-value of this reaction is given by

$$Q=\left[m\left({}_{5}^{12}\mathrm{B}\right)-m\left({}_{6}^{12}\mathrm{C}\right)\right]c^{2}$$

$$=[12.041-12.0]\times 931.5=13.041\mathrm{MeV}$$

The energy $Q=13.041\mathrm{MeV}$ is released in the reaction. Out of this energy, 4.041 MeV is used to excite ${}_{6}^{12}\mathrm{C}$ to its excited state ${}_{6}^{12}\mathrm{C}^{*}$. Thus, the kinetic energy available to the β-particle (K_β) and the antineutrino $(K_{\bar{\nu}})$ is $K_\beta+K_{\bar{\nu}}=13.041-4.041=9\mathrm{MeV}$. In $\beta-$ decay, the kinetic energy of the $\bar{\nu}$ can vary from zero to a maximum value. Hence, the maximum kinetic energy of the β-particle is $K_{\beta,\max}=9\mathrm{MeV}\ when\$K_{\bar{\nu}}=0\$$.

Question 053 MCQ

QUESTION

An infinite line charge of uniform electric charge density l lies along the axis of an electrically conducting infinite cylindrical shell of radius R. At time t = 0, the space inside the cylinder is filled with a material of permittivity e and electrical conductivity s. The electrical conduction in the material follows Ohm's law. Which one of the following graphs best describes the subsequent variation of the magnitude of current density j t at any point in the material?

A

B

C

D

CORRECT OPTION

D

SOURCE

Physics • current-electricity

EXPLANATION

Let the charge per unit length on the axis be

$$\lambda$$

t at a time t. The electric field due to this line charge at a radial distance r is given by

$$\overrightarrow{E}(r, t) = \frac{\lambda(t)}{2\pi \in r}\hat{r}$$

. ... 1

The free charges inside the conductor start moving radially due to the presence of electric field. The current density at a point is defined as the current flowing across a unit area placed perpendicular to the direction of current flow. The current density at a point is related to the electric field at that point by Ohm's law i.e.,

$$\overrightarrow{j}(r, t) = \sigma\overrightarrow{E}(r, t)$$

, ... 2

where

$$\sigma$$

is the electrical conductivity of the conductor. The current through a cylindrical shell of radius r and length l is given by

$$I = \int_{surface} \overrightarrow{j}(r, t) \, . \, d\overrightarrow{A} = j(r, t)(2\pi rl)$$

. ... 3

By conservation of charge, the current I is equal to the rate of decrease of charge q on axial line segment of length l i.e.,

$$I = -\frac{dq}{dt} = -l\frac{d\lambda(t)}{dt}$$

. ... 4

From equations 1 - 4

$$\frac{d\lambda(t)}{dt} = -\frac{\sigma\lambda(t)}{\in}$$

.

Integrate with initial condition

$$\lambda$$

$t = 0$ =

$$\lambda$$

$_0$ to get

$$\lambda(t) = \lambda_0 \exp e^{-\frac{\sigma t}{\in}}$$

,

Substitute

$$\lambda$$

t in equation 2 to get

$$\vec{j}(r,t) = \frac{\sigma\lambda_0}{2\pi \in r} e^{-\frac{\sigma t}{\in}}$$

.

Note that current density varies as 1/r with radial distance r. It decreases exponentially with time and becomes zero at t

$$\rightarrow$$

$$\infty$$

.

Question 054 Numerical

QUESTION

Two inductors L$_1$ $inductance 1 mH, internal resistance 3$$$\Omega$$ and L$_2$ $inductance 2 mH, internal resistance 4$$$\Omega$$, and a resistor R $resistance 12$$$\Omega$$ are all connected in parallel across a 5V battery. The circuit is switched on at time t = 0. The ratio of the maximum to the minimum current (I_{max} / I_{min}) drawn from the battery is

SOURCE

Physics • electromagnetic-induction

EXPLANATION

The circuit is shown in the figure

Initially, right after the circuit is switched on $\$t \rightarrow 0^+\$$, the impedance $effective resistance$ of inductors L_1 and L_2 is extremely high. Therefore, the inductors act as open circuits, and all the current flows through the resistor $R = 12\Omega$. According to Kirchhoff's law, the current through the battery at this moment is $i_{\min} = \frac{V}{R} = \frac{5}{12}\,\mathrm{A}$.

As the circuit reaches steady state $\$t \rightarrow \infty\$$, the impedance of the inductors drops to zero, and they function as resistors with their specified internal resistances.

The effective resistance of the circuit is $R_e = (12\Omega \|$

$4\Omega)||3\Omega = 3\Omega||3\Omega = 3/2\Omega$.

In the steady state, the current through the circuit is $i_{\text{max}} = \frac{V}{R_e} = \frac{5}{\frac{3}{2}} = \frac{10}{3}\,\text{A}$.

Therefore, the ratio of the maximum current to the minimum current is:

$\frac{i_{\text{max}}}{i_{\text{min}}} = \frac{\frac{10}{3}}{\frac{5}{12}} = \frac{10}{3} \times \frac{12}{5} = 8$