iit Jee 2010 Paper 1 Offline $84 Questions$

Question 001 Numerical

QUESTION

Based on VSEPR theory, the number of 90 degree F-Br-F angles in BrF_5 is

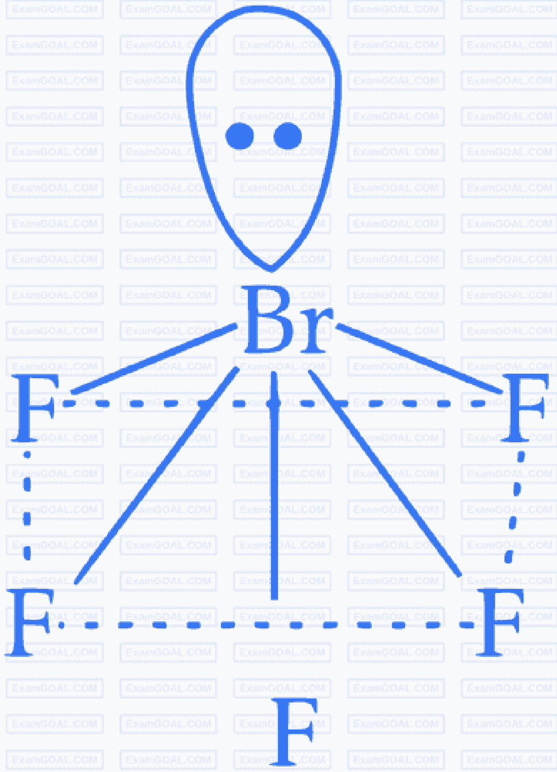
SOURCE

Chemistry • chemical-bonding-and-molecular-structure

EXPLANATION

According to VSEPR, BrF_5 has square pyramidal structure with axial plane containing a lone pair and fluorine. The other four fluorine are arranged in square planner configuration around central metal atom. Thus, BrF_5 assumes square pyramidal shape where the valence electron pairs surrounding an atom tend to repel each other and will, therefore, adopt an arrangement that minimises this repulsion, thus, determining the molecule's geometry. All four planar bonds $(\mathrm{F}-\mathrm{Br}-\mathrm{F})$ will reduce from 90° to 84.8° after lone pair - bond pair repulsion.

So, there are no 90 -degree $\mathrm{F}-\mathrm{Br}-\mathrm{F}$ angles in BrF_5.



Question 002 MCQ

QUESTION

The species which by definition has **ZERO** standard molar enthalpy of formation at 298 K is

A $Br_2\ g$

B $Cl_2\ g$

C $H_2O\ g$

D $CH_4\ g$

CORRECT OPTION

B $Cl_2\ g$

SOURCE

Chemistry • thermodynamics

EXPLANATION

The standard molar enthalpy of formation, denoted as

$$\Delta H_f^\circ$$

, is defined as the change in enthalpy when one mole of a compound is formed from its constituent elements in their standard states under standard conditions $298 K and 1 atm pressure$. By definition, the standard molar enthalpy of formation of a pure element in its most stable form at 298 K is zero.

Looking at the options:

- **Br_2** g - Bromine's standard state at room temperature is liquid ($Br_2 l$), not gas.
- **Cl_2** g - Chlorine's standard state at room temperature is a gas, which means the standard molar enthalpy of formation for $Cl_2 g$ is indeed zero.
- **H_2O** g - Water in gaseous form is not a basic element but a compound of hydrogen and oxygen, so its enthalpy of formation is not zero.
- **CH_4** g - Methane, represented here, is also a compound composed of carbon and hydrogen, thus its enthalpy of formation is also not zero.

Therefore, the correct option is **Option B**, $Cl_2\ g$, as chlorine gas is a diatomic molecule and an element in its standard state at 298 K, and as such, its standard molar enthalpy of formation is zero.

Question 003 MCQ

QUESTION

Among the following, the intensive property is $properties are$

A molar conductivity

B electromotive force

C resistance

D heat capacity

CORRECT OPTION

A molar conductivity

SOURCE

Chemistry • thermodynamics

EXPLANATION

An intensive property is a property that is independent of the amount of mass or size of the system. In contrast, an extensive property is dependent on the size or amount of mass in the system. Let's analyze each option based on this definition:

- **Molar Conductivity**: Molar conductivity, denoted as

$$\Lambda_m$$

, is a measure of the ionic conductivity of an electrolyte solution divided by the concentration of the electrolyte. Since it is normalized by the amount of substance, it does not depend on the total amount of the substance present in the solution. Thus, molar conductivity is an intensive property.

- **Electromotive Force** EMF: EMF is the voltage generated by a battery or by the magnetic force according to Faraday's Law of electromagnetic induction. It refers to a potential difference and does not depend on the quantity of material or size of the battery. Therefore, EMF is considered an intensive property.
- **Resistance**: Resistance, denoted by

$$R$$

, measures how much a material opposes the flow of electric current. It depends on the material's length, cross-sectional area, and resistivity. For instance, a longer wire has greater resistance. Therefore, resistance is an extensive property.
- **Heat Capacity**: Heat capacity is the amount of heat required to change the temperature of a substance by a certain temperature interval. It is dependent on the amount of substance present, and therefore, it is an extensive property. The specific heat capacity, however, is intensive since it is the heat capacity per unit mass.

From the options provided:

- Option A $MolarConductivity$ is intensive.
- Option B $ElectromotiveForce$ is intensive.
- Option C $Resistance$ is extensive.
- Option D $HeatCapacity$ is extensive.

Therefore, the intensive properties among the given options are Option A $MolarConductivity$ and Option B $ElectromotiveForce$.

Question 004 MCQ

QUESTION

Aqueous solution of HNO_3, KOH and CH_3COOH and CH_3COONa of identical concentrations are provided. The pairs of solutions which form a buffer upon mixing is are

A HNO_3 and CH_3COOH

B KOH and CH_3COONa

C HNO_3 and CH_3COONa

D CH_3COOH and CH_3COONa

CORRECT OPTION

C HNO_3 and CH_3COONa

SOURCE

Chemistry • ionic-equilibrium

Question 005 Numerical

QUESTION

Amongst the following the total number of compounds whose aqueous solution turns red litmus paper blue is

KCN, K_2SO_4, $(NH_4)_2C_2O_4$, $NaCl$, $Zn(NO_3)_2$, $FeCl_3$, K_2CO_3, NH_4NO_3 and $LiCN$

SOURCE

Chemistry • ionic-equilibrium

Question 006 MCQ

QUESTION

The reagent s used for softening the temporary hardness of water is are

A $Ca_3(PO_4)_2$

B CaOH_2

C Na_2CO_3

D NaOCl

CORRECT OPTION

B CaOH_2

SOURCE

Chemistry • hydrogen

Question 007 Numerical

QUESTION

The total number of cyclic isomers possible for a hydrocarbon with the molecular formula C_4H_6 is

SOURCE

Chemistry • basics-of-organic-chemistry

Question 008 MCQ

QUESTION

The synthesis of 3-octyne is achieved by adding a bromoalkane into a mixture of sodium amide and and alkyne

A

$BrCH_2CH_2CH_2CH_2CH_3$ and CH_2CH_2C

≡

CH

B

$BrCH_2CH_2CH_3$ and $CH_3CH_2CH_2C$

≡

CH

C

$BrCH_2CH_2CH_2CH_2CH_3$ and CH_3C

≡

CH

D

$BrCH_2CH_2CH_2CH_3$ and CH_3CH_2C

≡

CH

CORRECT OPTION

D

$BrCH_2CH_2CH_2CH_3$ and CH_3CH_2C

≡

CH

SOURCE

Chemistry • hydrocarbons

EXPLANATION

To get appropriate bromoalkane an alkyne that were involved in the synthesis, we have to cleave the molecule and we know that sodium amide is involved, so the synthesis of 3-octyne using the alkyne and bromoalkane was $\mathrm{S_N2}$ reaction. And as the sodium amide is involved, the alkyne involved must be terminal then only it could abstract the acidic hydrogen 3-octyne is a molecule that will have eight carbon atoms in their parent chain since the prefix-oct represents 8 and as in the name it is given that 3 octyne, ' 3 ' represents the position of triple bond. Thus suffix-one represents triple bond.

So, in the molecule triple bond will be at the third carbon. Structure of 3-octyne :

Now, cleave the molecule between $\mathrm{C_2-C_3}$ or $\mathrm{C_4}$ - C_5.

Now, see the products formed if cleavage takes place between C_2-C_3, the reactants will be

If cleavage takes place between C_4-C_5, the reactants will be :

Then correct option is D.

Question 009 MCQ

QUESTION

The bond energy (in **kcal mol^{-1}**) of a C-C single bond is approximately

A 1

B 10

C 100

D 1000

CORRECT OPTION

C 100

SOURCE

Chemistry • thermodynamics

EXPLANATION

The bond energy of a C-C $carbon - carbon$ single bond is a measure of the strength of that bond, or how much energy is required to break one mole of such bonds in a gaseous substance. This value is a fundamental concept in chemistry, particularly in the study of organic molecules and reactions.

For a C-C single bond, the bond energy is typically around 83 kcal/mol $kcalpermole$. This places the value closest to option C amongst the given choices.

Therefore, the correct answer is:

Option C: 100

Question 010 MCQ

QUESTION

The concentration of potassium ions inside a biological cell is at least twenty times higher than the outside. The resulting potential difference across the cell is important in several processes such as transmission of nerve impulses and maintaining the ion balance. A simple model for such a concentration cell involving a metal M is :

M s | M$^+$ $aq; 0.05 molar$ || M$^+$ $aq; 1 molar$ | M s

For the above electrolytic cell the magnitude of the cell potential | E_{cell} | = 70 mV.

For the above cell :

A

$E_{cell} < 0$;

$$\Delta G > 0$$

B

$E_{cell} > 0$;

$$\Delta G < 0$$

C

$E_{cell} < 0$;

$$\Delta G^o > 0$$

D

$E_{cell} > 0$;

$$\Delta G^o > 0$$

CORRECT OPTION

B

$E_{cell} > 0$;

$$\Delta G < 0$$

SOURCE

Chemistry • electrochemistry

EXPLANATION

To determine the right option for the given concentration cell, we must first understand the concepts of cell potential, Gibbs free energy, and the relationship between these quantities.

The cell potential,

$$E_{cell}$$

, for a concentration cell like the one described is directly related to the concentrations of the ions on both sides of the cell. It's calculated using the Nernst equation:

$$E_{cell} = E^o - \frac{RT}{nF} \ln \frac{[M^+](\text{low concentration})}{[M^+](\text{high concentration})}$$

Since the given cell involves the same metal on both sides in its standard state, the standard cell potential,

$$E^o$$

, is zero. Therefore, the equation simplifies to:

$$E_{cell} = -\frac{RT}{nF} \ln \frac{0.05}{1}$$

Given that

$$R$$

$theidealgasconstant$ is approximately 8.314 J/mol·K,

$$F$$

$theFaradayconstant$ is about 96485 C/mol,

$$T$$

is the temperature in Kelvin $assumingstandardtemperatureof298K$, and

$$n$$

$thenumberofmolesofelectronstransferredpermoleofreaction$ is 1, the equation further simplifies to:

$$E_{cell} = -\frac{8.314 \times 298}{96485} \ln\left(\frac{0.05}{1}\right)$$

Calculate the ln term:

$$\ln\left(\frac{0.05}{1}\right) = \ln(0.05) \approx -2.9957$$

Then the equation is:

$$E_{cell} = -\frac{2476.422}{96485} \times -2.9957 \approx 0.077 \text{ volts} = 77 \text{ mV}$$

Here, the calculated

$$E_{cell} > 0$$

, which matches the given cell potential of 70 mV. It differs slightly due to rounding and exact values used in constants. This cell potential being positive indicates spontaneous reaction $tendencytogofromhightolowconcentration$.

We now apply this to the relationship between the cell potential and Gibbs free energy, which is given by:

$$\Delta G = -nFE_{cell}$$

Since

$$E_{cell} > 0$$

,

$$\Delta G < 0$$

which indicates that the process is thermodynamically favorable $spontaneous$.

Looking at the provided options:

- Option A: $E_{cell} < 0$;

$$\Delta G > 0$$

$$Incorrect; as\$\$E_{cell}\$\$ispositiveand\$\$\Delta G\$\$isnegative$$

- Option B: $E_{cell} > 0$;

$$\Delta G < 0$$

$$Correct; matchesourcalculation$$

- Option C: $E_{cell} < 0$;

$$\Delta G^o > 0$$

$$Incorrect$$

- Option D: $E_{cell} > 0$;

$$\Delta G^o > 0$$

$$Incorrect, \$\$\Delta G^o\$\$notdirectlyrelevanthere, butcentralizedonthestar$$

Thus, the correct answer is **Option B**.

Question 011 MCQ

QUESTION

The concentration of potassium ions inside a biological cell is at least twenty times higher than the outside. The resulting potential difference across the cell is important in several processes such as transmission of nerve impulses and maintaining the ion balance. A simple model for such a concentration cell involving a metal M is :

M s | M^+ $aq; 0.05 molar$ || M^+ $aq; 1 molar$ | M s

For the above electrolytic cell the magnitude of the cell potential | E_{cell} | = 70 mV.

If the 0.05 molar solution of M^+ is replaced by a 0.0025 molar M^+ solution, then the magnitude of the cell potential would be :

A 35 mV

B 70 mV

C 140 mV

D 700 mV

CORRECT OPTION

C 140 mV

SOURCE

Chemistry • electrochemistry

EXPLANATION

To understand how the concentration of ions affects the cell potential, we will first use the Nernst equation. The Nernst equation gives us a way to calculate the potential of a cell under non-standard conditions and is represented as follows:

$$E = E^0 - \frac{RT}{nF} \ln \frac{a_{\mathrm{Red}}}{a_{\mathrm{Ox}}}$$

In our case, we're analyzing a concentration cell where the metal M is the same in both the anode and the cathode but with different concentrations of M^+. Here, the standard potential E^0 is zero because the same substance is used as both the anode and cathode.

The cell reaction becomes:

$$\mathrm{M}(s)|\mathrm{M}^+(0.05 \text{ M at anode}) \Longrightarrow \mathrm{M}(s)|\mathrm{M}^+(1 \text{ M at cathode})$$

Since $E^0 = 0$, the Nernst equation simplifies to:

$$E = -\frac{RT}{nF} \ln \frac{[\mathrm{M}^+ \text{ (cathode)}]}{[\mathrm{M}^+ \text{ (anode)}]}$$

Which turns into:

$$E = -\frac{RT}{nF} \ln \frac{1}{0.05},$$

and we can further simplify using $\ln(\frac{1}{0.05}) = -\ln(0.05)$. Assume the reaction involves the transfer of 1 mole of electrons $n = 1$, the value of F $Faradayconstant$ is approximately 96485 C/mol, and the temperature T is 298K $roomtemperature$. The gas constant R is 8.314 J/$mol \cdot K$. Plugging in these values:

$$E \approx -\frac{(8.314\,\mathrm{J/mol{\cdot}K})(298\,\mathrm{K})}{(1)(96485\,\mathrm{C/mol})} \ln(0.05)$$

This equation can be used to find the potential in volts when the concentration at anode was 0.05 M and was resulting in 70 mV.

Now, switching the anode concentration to 0.0025 M, we reapply the Nernst equation:

$$E = -\frac{RT}{nF} \ln \frac{1}{0.0025}$$

We can see the ratio of concentrations has changed, moving from 1/0.05 to 1/0.0025. Thus, the concentration difference across the membrane has increased, which should increase the voltage according to the Nernst equation. Specifically, the ratio changed by a factor of 0.05/0.0025 = 20 times.

Given that $\ln(\frac{1}{x})$ is proportional to the voltage, when the ratio of concentration changes by twenty-fold, and because $\ln(0.0025) = -\ln(400)$ which is twice the $\ln(20)$, the potential will double. Since the initial 70 mV doubles, the new cell potential will be:

$$E \approx 2 \times 70\,\mathrm{mV} = 140\,\mathrm{mV}.$$

Therefore, the correct answer is:

Option C: 140 mV

Question 012 Numerical

QUESTION

The concentration of R in the reaction R

$$\rightarrow$$

P was measured as a function of time and the following data is obtained

R **molar**	1.0	0.75	0.40	0.10
t $min.$	0.0	0.05	0.12	0.18

The order of reaction is

SOURCE

Chemistry • chemical-kinetics-and-nuclear-chemistry

Question 013 Numerical

QUESTION

The number of neutrons emitted when

$$^{235}_{92}U$$

undergoes controlled nuclear fission to

$$^{142}_{54}Xe$$

and

$$^{90}_{38}Sr$$

is

SOURCE

EXPLANATION

To determine the number of neutrons emitted during the nuclear fission of

$$^{235}_{92}U$$

resulting in the products

$$^{142}_{54}Xe$$

and

$$^{90}_{38}Sr$$

, we need to ensure the conservation of mass number and atomic number.

The mass number A and atomic number Z have to be conserved. This means that the sum of the mass numbers and atomic numbers of the products $includinganyneutronsemitted$ must equal those of the uranium nucleus undergoing fission.

Let's start by writing down the conservation of mass number and atomic number:

1. Conservation of Mass Number:

$$235 = 142 + 90 + n \times 1$$

Here,

$$n$$

represents the number of neutrons released. We can now calculate

$$n$$

:

$$n = 235 - (142 + 90) = 235 - 232 = 3$$

1. Conservation of Atomic Number:

$$92 = 54 + 38 + 0 \times n$$

Neutrons do not contribute to the atomic number as they have no charge.

This calculation shows that three neutrons are needed to satisfy the conservation of mass number. The atomic number conservation also coincides, as neutrons do not alter it. Therefore, the number of neutrons emitted during this fission process is

$$3$$

.

Question 014 Numerical

QUESTION

A student performs a titration with different burettes and finds titre values of 25.2 mL, 25.25 mL, and 25.0 mL. The number of significant figures in the average titre value is

SOURCE

Chemistry • some-basic-concepts-of-chemistry

EXPLANATION

To find the average titre value, first add up the three measurements provided and then divide by the number of measurements.

The sum of the measurements is:

$$25.2 \, \text{mL} + 25.25 \, \text{mL} + 25.0 \, \text{mL} = 75.45 \, \text{mL}$$

Since there are three measurements, divide this sum by 3 to calculate the average:

$$\text{Average titre value} = \frac{75.45 \, \text{mL}}{3} = 25.15 \, \text{mL}$$

When reporting the average, we must consider the significant figures of the original measurements. The number of significant figures is determined by the least precise measurement, which in this case is 25.0 mL with three significant figures. Therefore, we should report the average value to three significant figures as well.

The average value of 25.15 mL has four significant figures, so we need to round it to three significant figures. However, this is slightly tricky since 25.15 already appears to be rounded to four significant figures. We should consult the original measurements to decide on the best course of action.

Looking at the individual measurements $25.2, 25.25, and 25.0$, we should consider the lowest decimal place which they all have in common, which is the first decimal place. The third measurement has no second decimal place, indicating its level of precision. Thus, the number of significant figures for the average titre value should be in line with this level of precision. Since the average calculated is 25.15, when we adjust to the first decimal place for consistent significant figures, the average is 25.1 mL with three significant figures.

$$\text{Corrected Average titre value} = 25.1\,\text{mL}$$

Therefore, the number of significant figures in the average titre value is three: 25.1 mL.

Question 015 MCQ

QUESTION

The correct statement about the following disaccharide is :

A Ring a is pyranonse with

$$\alpha$$

-glycosidic link.

Ring a is furanonse with

B

$$\alpha$$

-glycosidic link.

Ring b is furanonse with

C

$$\alpha$$

-glycosidic link.

Ring b is pyranonse with

D

$$\alpha$$

-glycosidic link.

CORRECT OPTION

Ring a is pyranonse with

A

$$\alpha$$

-glycosidic link.

SOURCE

Chemistry • biomolecules

EXPLANATION

Pyranonse structure is a six-membered heterocyclic ring containing five carbon atoms and one oxygen atom. A glycosidic bond is formed between the hemiacetal group of a saccharide and the hydroxyl group of some organic compound such as an alcohol. C-1 of ring a has an

$$\alpha$$

-glycosidic linkage.

Ring b has 5-members and hence is a furanose.

Question 016 MCQ

QUESTION

In the reaction

the products are :

A

B

C

D

CORRECT OPTION

D

SOURCE

Chemistry • alcohols-phenols-and-ethers

EXPLANATION

Alkyl aryl ethers are cleaved at the alkyl-oxygen bond due to stronger aryl-oxygen bond. The reaction yields phenol and alkyl halide. Thus,

The reaction is

Question 017 MCQ

QUESTION

Plots showing the variation of the rate constant $$k$$ with temperature $$T$$ are given below. The point that follows Arrhenius equation is

A

B

C

D

CORRECT OPTION

A

SOURCE

Chemistry • chemical-kinetics-and-nuclear-chemistry

EXPLANATION

According to the Arrhenius equation

$$k = Ae^{-E_a/RT}$$

where, k = rate constant, E_a = activation energy and T = temperature

From the expression, we have that as temperature increases, the rate constant $$k$$ increases exponentially.

Question 018 MCQ

QUESTION

The correct structure of ethylenediaminetetraacetic acid $EDTA$ is

A

B

C

D

CORRECT OPTION

C

SOURCE

Chemistry • coordination-compounds

EXPLANATION

EDTA $ethylenediaminetetraceticacid$ is a hexadentate ligand. It is also called the amino poly-carboxylic acid.

It is structure consists of the ethane group which is attached to tertiary amine groups at the terminal positions forming an ethylene diamine. Tertiary amine group is situated by the acetic acid groups $(-\mathrm{CH_2COOH})$.

Question 019 MCQ

QUESTION

The ionisation isomer of

$$[\mathrm{Cr(H_2O)_4Cl(NO_2)]Cl}$$

is

A $[Cr(H_2O)_4(O_2N)]Cl_2$

B $[Cr(H_2O)_4Cl_2](NO_2)$

C $[Cr(H_2O)_4Cl(ONO)]Cl$

D $[Cr(H_2O)_4Cl_2(NO_2)]H_2O$

CORRECT OPTION

B $[Cr(H_2O)_4Cl_2](NO_2)$

SOURCE

Chemistry • coordination-compounds

EXPLANATION

The ionisation isomers give different ions in solution. In the complex given in option B, Cl

$-$

is replaced by NO

$\frac{-}{2}$

in ionisation sphere.

$$[\mathrm{Cr(H_2O)_4Cl(NO_2)]Cl}$$

and

$$[Cr(H_2O)_4Cl_2](NO_2)$$

have different ions inside and outside the coordinate sphere and they are isomers. Therefore, they are ionisation isomers.

Question 020 MCQ

QUESTION

In the Newman projection for 2,2-dimethylbutane, X and Y can, respectively, be

A H and H

H and C

2

H

5

C

2

C H

5

and H

CH

3

D and CH

3

CORRECT OPTION

H and C

2

B H

5

SOURCE

Chemistry • basics-of-organic-chemistry

EXPLANATION

In the structure of 2,2-dimethylbutane:

On C

2

—

C

3

bond axis X = CH

3

, Y = CH

3

.

On C

1

—

C

2

bond axis X = H, Y = C

2

H

5

.

Question 021 MCQ

QUESTION

In the reaction

The intermediate$s$ isare

A

B

C

D

CORRECT OPTION

A

SOURCE

Chemistry • alcohols-phenols-and-ethers

EXPLANATION

In presence of base, phenol forms phenoxide ion which is

$$ortho$$

-

$$para$$

directing because in the resonating structures, the electron density increases at ortho and para positions. The phenoxide ion further undergoes electrophilic substitution with Br

$$2$$

at

$$ortho$$

and

$$para$$

positions.

Question 022 MCQ

QUESTION

Partial roasting of chalcopyrite produces

A. Cu_2S and FeO

B. Cu_2O and FeO

C. CuS and Fe_2O_3

D. Cu_2O and Fe_2O_3

CORRECT OPTION

A Cu_2S and FeO

SOURCE

Chemistry • isolation-of-elements

EXPLANATION

Partial roasting of chalcopyrite gives

$2CuFeS_2 + O_2 \rightarrow Cu_2S + 2FeS + SO_2$

$2CuFeS_2 + 4O_2 \rightarrow Cu_2$

$S + 2FeO + 3SO_2$

Since iron is more electropositive than copper, so its sulphide is oxidised in preference. Cu_2S remains mostly unaffected and the small amount of Cu_2O formed on oxidation, reacts with FeS to give back Cu_2S.

$$2Cu_2S + 3O_2 \rightarrow 2Cu_2O + 2SO_2$$

$$Cu_2O + FeS \rightarrow Cu_2S + FeO$$

Question 023 MCQ

QUESTION

Iron is removed from chalcopyrite as

A. FeO

B. FeS

C. Fe_2O_3

D. $FeSiO_3$

CORRECT OPTION

D. $FeSiO_3$

SOURCE

Chemistry • isolation-of-elements

EXPLANATION

Iron obtained in form of its oxide is removed by slag formation with SiO

$_2$

.

$FeO + SiO_2 \rightarrow FeSiO_3$ $slag$

Question 024 MCQ

QUESTION

In self-reduction, the reducing species is

A. S

B. O^{2-}

C. S^{2-}

D. SO_2

CORRECT OPTION

C

S^{2-}

SOURCE

Chemistry • isolation-of-elements

EXPLANATION

In the self-reduction step,

$Cu_2S + \frac{3}{2}O_2 \rightarrow Cu_2O + SO_2$

$Cu_2S + 2Cu_2O \rightarrow$

$$\underset{\text{Blister copper}}{6Cu(l)}$$

+ SO

$$_2$$

In the reaction, the oxidation number of sulphur $S$$^{2-}$$ changes to $S$$^{4+}$$ so, S

$$^{2-}$$

is the reducing agent.

Question 025 Numerical

QUESTION

The total number of basic groups in the following form of lysine is

SOURCE

Chemistry • biomolecules

EXPLANATION

There are two basic groups in lysine

Question 026 Numerical

QUESTION

In the scheme given below, the total number of intra molecular aldol condensation products formed from Y is ____________.

SOURCE

Chemistry • aldehydes-ketones-and-carboxylic-acids

EXPLANATION

There is only one product formed in intramolecular aldol condensation, which is explained as follows:

Question 027 Numerical

QUESTION

Amongst the following, the total number of compounds soluble in aqueous NaOH is _________.

SOURCE

Chemistry • aldehydes-ketones-and-carboxylic-acids

EXPLANATION

Aromatic alcohols and carboxylic acids forms salt with NaOH, will dissolve in aqueous NaOH :

Benzylic alcohol is less acidic than water and hence does dissolve in aqueous NaOH.

Question 028 Numerical

QUESTION

The value of

$$n$$

in the molecular formula

$$\mathrm{Be_nAl_2Si_6O_{18}}$$

is ___________.

SOURCE

Chemistry • p-block-elements

EXPLANATION

In the given molecular formula

$$\mathrm{Be_nAl_2Si_6O_{18}}$$

, according to charge balance in a molecule, we get

$$2n + 2(+3) + 6(+4) - 18(2) = 0$$

$$\Rightarrow n = 3$$

So, the formula is

$$\mathrm{Be_3Al_2Si_6O_{18}}$$

.

Question 029 Numerical

QUESTION

The number of values of

$$\theta$$

in the interval,

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

such that

$$\theta \neq \frac{n\pi}{5}$$

for

$$n = 0, \pm 1, \pm 2$$

and

$$\tan\ \theta = \cot\ 5\theta$$

as well as

$$\sin\ 2\theta = \cos\ 4\theta$$

is

SOURCE

Mathematics • trigonometric-functions-and-equations

EXPLANATION

Given,

$$\tan\theta = \cot 5\theta$$

$$\Rightarrow \tan\theta = \tan\left(\frac{\pi}{2} - 5\theta\right)$$

$$\Rightarrow \frac{\pi}{2} - 5\theta = n\pi + \theta$$

$$\Rightarrow 6\theta = \frac{\pi}{2} - \frac{n\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{12} - \frac{n\pi}{6}$$

Also

$$\cos 4\theta = \sin 2\theta = \cos\left(\frac{\pi}{2} - 2\theta\right)$$

$$\Rightarrow 4\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right)$$

Taking positive

$$6\theta = 2n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{12}$$

Taking negative

$$2\theta = 2n\pi - \frac{\pi}{2} \Rightarrow \theta = n\pi - \frac{\pi}{4}$$

Above values of

$$\theta$$

suggests that there are only 3 common solutions.

Question 030 Numerical

QUESTION

The number of all possible values of

$$\theta$$

where

$$0 < \theta < \pi,$$

for which the system of equations

$$\$ \, (y+z)\cos 3\theta = (xyz)\sin 3\theta$$

$$x \sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$

$$(xyz)\sin 3\theta = (y+2z)\cos 3\theta + ysin3\theta$$

$

have a solution

$$(x_0, y_0, z_0)$$

with

$$y_0 z_0 \neq 0,$$

is

SOURCE

Mathematics • trigonometric-functions-and-equations

EXPLANATION

View the equation in xyz, y and t.

We have,

$$(xyz)\sin 3\theta - y\cos 3\theta - z\cos 3\theta = 0$$

$$(xyz)\sin 3\theta - 2y\sin 3\theta - 2z\cos 3\theta = 0$$

$$(xyz)\sin 3\theta - y(\cos 3\theta + \sin 3\theta) - 2z\cos 3\theta = 0$$

$$xyz \neq 0$$

Hence, the equation has non-trivial solution which gives

$$\begin{vmatrix} \sin 3\theta & -\cos 3\theta & -\cos 3\theta \\ \sin 3\theta & -2\sin 3\theta & -2\cos 3\theta \\ \sin 3\theta & -(\cos 3\theta + \sin 3\theta) & -2\cos 3\theta \end{vmatrix} = 0$$

$$\Rightarrow \sin 3\theta \cos 3\theta (\sin 3\theta - \cos 3\theta) = 0$$

$$\Rightarrow \sin 3\theta = 0$$

then

$$xyz = 0$$

$notpossible$

$$\cos 3\theta = 0$$

not possible

$$\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1$$

$$3\theta = n\pi + \frac{\pi}{4}, n \in z$$

$$\theta = \frac{n\pi}{3} + \frac{\pi}{12}$$

;

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$$

Thus there are 3 solutions.

Question 031 Numerical

QUESTION

The maximum value of the expression

$$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$$

is

SOURCE

Mathematics • trigonometric-functions-and-equations

EXPLANATION

Let

$$f(\theta) = \frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$$

Again let

$$g(\theta) = \sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta$$

$$= \frac{1-\cos 2\theta}{2} + 5\left(\frac{1+\cos 2\theta}{2}\right) + \frac{3}{2}\sin 2\theta$$

$$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$$

$$\therefore$$

$$g(\theta)_{\min} = 3 - \sqrt{4 + \frac{9}{4}} = 3 - \frac{5}{2} = \frac{1}{2}$$

$$\therefore$$

$$f(\theta) = \frac{1}{g(\theta)_{\min}} = 2$$

Question 032 MCQ

QUESTION

Let

$$z_1$$

and

$$z_2$$

be two distinct complex number and let z = $1-t$

$$z_1$$

+ t

$$z_2$$

for some real number t with 0 < t < 1. IfArg w denote the principal argument of a non-zero complex number w, then

A

$$|z-z_1|+|z-z_2|=|z_1-z_2|$$

B

Arg

$$(z-z_1)$$

= Arg

$$(z-z_2)$$

C

$$\begin{vmatrix} z-z_1 & \overline{z}-\overline{z}_1 \\ z_2-z_1 & \overline{z}_2-\overline{z}_1 \end{vmatrix}$$

= 0

D

Arg

$$(z-z_1)$$

= Arg

$$(z_2-z_1)$$

CORRECT OPTION

D

Arg

$$(z - z_1)$$

= Arg

$$(z_2 - z_1)$$

SOURCE

Mathematics • complex-numbers

EXPLANATION

Given,

$$z = \frac{(1-t)z_1 + t\,z_2}{(1-t)+t}$$

Clearly, z divides z_1 and z_2 in the ratio of t : $1\$\$ - \$\$t$, 0 < t < 1

$$\Rightarrow$$

AP + BP = AB

i.e.,

$$|z - z_1| + |z - z_2| = |z_1 - z_2|$$

$$\Rightarrow$$

Option a is true.

and

$$\arg(z - z_1) = \arg(z_2 - z) = \arg(z_2 - z_1)$$

$$\Rightarrow$$

b is false and d is true.

Also,

$$\arg(z - z_1) = \arg(z_2 - z_1)$$

$$\Rightarrow \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$$

$$\therefore$$

$$\frac{z - z_1}{z_2 - z_1}$$

is purely real.

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\overline{z} - \overline{z}_1}{\overline{z}_2 - \overline{z}_1}$$

or,

$$\begin{vmatrix} z - z_1 & \overline{z} - \overline{z}_1 \\ z_2 - z_1 & \overline{z}_2 - \overline{z}_1 \end{vmatrix} = 0$$

$$\therefore$$

Option c is correct.

Hence, a, c, d is the correct option.

Question 033 MCQ

QUESTION

Let

$$p$$

and

$$q$$

be real numbers such that

$$p \neq 0,\, p^3 \neq q$$

and

$$p^3 \neq -q.$$

If

$$p^3 \neq -q.$$

and

$$\beta$$

are nonzero complex numbers satisfying

$$\alpha + \beta = -p$$

and

$$\alpha^3 + \beta^3 = q,$$

then a quadratic equation having

$$\frac{\alpha}{\beta}$$

and

$$\frac{\beta}{\alpha}$$

as its roots is

A $$(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$$

B $$(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$$

C $$(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$$

D $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

CORRECT OPTION

B $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

SOURCE

Mathematics • quadratic-equation-and-inequalities

EXPLANATION

Sum of roots =

$$\frac{\alpha^2 + \beta^2}{\alpha\beta}$$

and product = 1

Given,

$$\alpha$$

+

$$\beta$$

=

$$-$$

p and

$$\alpha$$

3 +

$$\beta$$

3 = q

$$\Rightarrow (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)=q$$

$$\therefore$$

$$\alpha^2+\beta^2-\alpha\beta=\frac{-q}{p}$$

..... i

and

$$(\alpha+\beta)^2=p^2$$

$$\Rightarrow \alpha^2+\beta^2+2\alpha\beta=p^2$$

..... ii

From Eq. i and ii, we get

$$\alpha^2+\beta^2=\frac{p^3-2q}{3p}$$

and

$$\alpha\beta=\frac{p^3+q}{3p}$$

$$\therefore$$

Required equation is

$$x^2-\frac{(p^2-2q)x}{(p^3+q)}+1=0$$

$$\Rightarrow (p^3+q)x^2-(p^3-2q)x+(p^3+q)=0$$

Question 034 Numerical

QUESTION

Let

$$S_k$$

= 1, 2,....., 100, denote the sum of the infinite geometric series whose first term is

$$\frac{k-1}{k\,!}$$

and the common ratio is

$$\frac{1}{k}$$

. Then the value of

$$\frac{100^2}{100!} + \sum_{k=1}^{100} \left|(k^2 - 3k + 1)\ S_k\right|$$

is

SOURCE

Mathematics • sequences-and-series

EXPLANATION

Using $S_\infty = \dfrac{a}{1-r}$, we get

$$S_k = \begin{cases} 0, & k = 1 \\ \frac{1}{(k-1)!}, & k \geq 2 \end{cases}$$

$$\text{Now } \sum_{k=1}^{100} \left|\left(k^2 - 3k + 1\right) S_k\right| = \sum_{k=2}^{100} \left|\left(k^2 - 3k + 1\right)\right| \frac{1}{(k-1)!}$$

$$= |-1| + \sum_{k=3}^{100} \frac{\left(k^2 - 1\right) + 1 - 3(k-1) - 2}{(k-1)!}$$

$$\text{as } k^2 - 3k + 1 > 0 \forall k \geq 3$$

$$= 1 + \sum_{k=3}^{100}\left(\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right)$$

$$= 1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{1!} - \frac{1}{3!}\right) + \left(\frac{1}{2!} - \frac{1}{4!}\right) + \ldots +$$

$$\left(\frac{1}{96!} - \frac{1}{98!}\right) + \left(\frac{1}{97!} - \frac{1}{99!}\right)$$

$$= 3 - \frac{1}{98!} - \frac{1}{99!} = 3 - \frac{9900}{100!} - \frac{100}{100!}$$

$$= 3 - \frac{10000}{100!} = 3 - \frac{(100)^2}{100!}$$

$$\therefore \frac{100^2}{100!} + \sum_{k=1}^{100}\left|\left(k^2 - 3k + 1\right)S_k\right| = 3$$

Question 035 MCQ

QUESTION

Let

$$A$$

and

$$B$$

be two distinct points on the parabola

$$y^2 = 4x$$

. If the axis of the parabola touches a circle of radius

$$r$$

having

$$AB$$

as its diameter, then the slope of the line joining

$$A$$

and

$$B$$

can be

A

$$-\frac{1}{r}$$

B

$$\frac{1}{r}$$

C

$$\frac{2}{r}$$

D

$$-\frac{2}{r}$$

CORRECT OPTION

C

$$\frac{2}{r}$$

SOURCE

Mathematics • parabola

EXPLANATION

Let A

$$\equiv$$

(t

$$\frac{2}{1}$$

, $2t_1$) and B

$$\equiv$$

(t

$$\frac{2}{2}$$

, $2t_2$)

The centre of the circle =

$$\left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$$

As the circle touches the x-axis thus

$$t_1 + t_2 = \pm r$$

Slope of

$$AB = \frac{2}{t_1 + t_2} = \pm \frac{2}{r}$$

Question 036 MCQ

QUESTION

The circle

$$x^2 + y^2 - 8x = 0$$

and hyperbola

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

intersect at the points

$$A$$

and

$$B$$

.

Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

A $$2x - \sqrt{5y} - 20 = 0$$

B $$2x - \sqrt{5y} + 4 = 0$$

C $$3x - 4y + 8 = 0$$

D $$4x - 3y + 4 = 0$$

CORRECT OPTION

B $$2x - \sqrt{5y} + 4 = 0$$

SOURCE

Mathematics • hyperbola

EXPLANATION

A tangent to

$$\frac{x^2}{9}-\frac{y^2}{4}=1$$

is

$$y=mx+\sqrt{9m^2-4},\ m>0$$

.... 1

A tangent to

$$(x-4)^2+y^2=16$$

is

$$xy=m(x-4)+4\sqrt{1+m^2}$$

..... 2

Comparing 1 and 2,

$$\sqrt{9m^2-4}=-4m+4\sqrt{1+m^2}\Rightarrow\sqrt{9-\frac{4}{m^2}}=-4+4\sqrt{1+\frac{1}{m^2}}$$

Let

$$\frac{1}{m^2}=t$$

, we have

$$\sqrt{9-4t}=-4+4\sqrt{1+t}$$

Squaring, we have

$$\Rightarrow 9-4t=16+16(1+t)-32\sqrt{1+t}\Rightarrow 32\sqrt{1+t}=23+20t$$

Again squaring

$$1024(1+t)=529+920t+400t^2$$

$$\Rightarrow 400t^2 - 104t - 495 = 0 \Rightarrow t = \frac{5}{4}$$

Thus

$$m^2 = \frac{4}{5}, m = \frac{2}{\sqrt{5}}$$

The tangent is

$$y = \frac{2}{\sqrt{5}}x + \frac{4}{\sqrt{5}}$$

i.e.

$$2x - \sqrt{5}y + 4 = 0$$

Question 037 MCQ

QUESTION

The circle

$$x^2 + y^2 - 8x = 0$$

and hyperbola

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

intersect at the points

$$A$$

and

$$B$$

.

Equation of the circle with

$$AB$$

as its diameter is

A

$$x^2 + y^2 - 12x + 24 = 0$$

B

$$x^2 + y^2 + 12x + 24 = 0$$

C

$$x^2 + y^2 + 24x - 12 = 0$$

D

$$x^2 + y^2 - 24x - 12 = 0$$

CORRECT OPTION

A

$$x^2 + y^2 - 12x + 24 = 0$$

SOURCE

Mathematics • hyperbola

EXPLANATION

A point on hyperbola is $3sec$$\theta$$, 2tan$$\theta$$$.

It lies on the circle, so

$$9\sec^2\theta + 4\tan^2\theta - 24\sec\theta = 0$$

.

$$\Rightarrow 13\sec^2\theta - 24\sec\theta - 4 = 0 \Rightarrow \sec\theta = 2, -\frac{2}{13}$$

Therefore,

$$\sec\theta = 2 \Rightarrow \tan\theta = \sqrt{3}$$

The point of intersection are

$$A(6, 2\sqrt{3})$$

and

$$B(6, -2\sqrt{3})$$

Hence, The circle with AB as diameter is

$$(x-6)^2 + y^2 = (2\sqrt{3})^2 \Rightarrow x^2 + y^2 - 12x + 24 = 0$$

Question 038 Numerical

QUESTION

The line

$$2x + y = 1$$

is tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

.

If this line passes through the point of intersection of the nearest directrix and the

$$x$$

-axis, then the eccentricity of the hyperbola is

SOURCE

Mathematics • hyperbola

EXPLANATION

On substituting

$$\left(\frac{a}{e}, 0\right)$$

in

$$y = -2x + 1$$

,

we get

$$0 = -\frac{2a}{e} + 1$$

$$\Rightarrow \frac{a}{e} = \frac{1}{2}$$

Also,

$$y = -2x + 1$$

is tangent to hyperbola

$$\therefore$$

$$1 = 4a^2 - b^2$$

$$\Rightarrow \frac{1}{a^2} = 4 - (e^2 - 1)$$

$$\Rightarrow \frac{4}{e^2} = 5 - e^2$$

$$\Rightarrow e^4 - 5e^2 + 4 = 0$$

$$\Rightarrow (e^2 - 4)(e^2 - 1) = 0$$

$$\Rightarrow$$

e = 2, e = 1

e = 1 gives the conic as parabola. But conic is given as hyperbola, hence e = 2.

Question 039 MCQ

QUESTION

If the angles

$$A, B$$

and

$$C$$

of a triangle are in an arithmetic progression and if

$$a, b$$

and

$$c$$

denote the lengths of the sides opposite to

$$A, B$$

and

$$C$$

respectively, then the value of the expression

$$\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$$

is

A

$$\frac{1}{2}$$

B $\frac{\sqrt{3}}{2}$

C 1

D $\sqrt{3}$

CORRECT OPTION

D $\sqrt{3}$

SOURCE

Mathematics • properties-of-triangle

EXPLANATION

Since, A, B, C are in AP

$$\Rightarrow$$

2B = A + C i.e.,

$$\angle$$

B = 60

$$\circ$$

$$\therefore$$

$$\frac{a}{c}$$

$2sinCcosC$ +

$$\frac{c}{a}$$

$2sinAcosA$

= 2k $acosC + ccosA$

$$using, \$\$ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{k} \$\$$$

= 2k b

= 2 sin B

$$using, b = acosC + ccosA$$

=

$$\sqrt{3}$$

Question 040 Numerical

QUESTION

Let

$$f$$

be a real-valued differentiable function on

$$R$$

$thesetofallrealnumbers$ such that

$$f(1) = 1$$

. If the

$$y$$

-intercept of the tangent at any point

$$P(x, y)$$

on the curve

$$y = f(x)$$

is equal to the cube of the abscissa of

$$P$$

, then find the value of

$$f(-3)$$

SOURCE

Mathematics • application-of-derivatives

EXPLANATION

The equation of the tangent at x, y to the given curve y = fx is

$$Y - y = \frac{dy}{dx}(X - x)$$

Y-intercept

$$= y - x\frac{dy}{dx}$$

According to the question

$$x^3 = y - x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -x^2$$

which is linear in x.

$$IF = e^{\int \frac{-1}{x} dx} = \frac{1}{x}$$

$$\therefore$$

Required solution is

$$y \, . \, \frac{1}{x} = \int -x^2 \, . \, \frac{1}{x} dx$$

$$\Rightarrow \frac{y}{x} = \frac{-x^2}{2} + c$$

$$\Rightarrow y = \frac{-x^3}{2} + cx$$

At x = 1, y = 1,

$$1 = \frac{-1}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

Now,

$$f(-3) = \frac{27}{2} + \frac{3}{2}(-3)$$

$$= \frac{27 - 9}{2} = 9$$

Question 041 Numerical

QUESTION

If the distance between the plane

$$Ax - 2y + z = d$$

and the plane containing the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

is

$$\sqrt{6}\ ,$$

then

$$|d|$$

is ___________.

SOURCE

Mathematics • 3d-geometry

EXPLANATION

We have a plane

$$Ax - 2y + z = d$$

and another plane that contains the two lines

$$\text{Line 1:} \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4},$$

$$\text{Line 2:} \quad \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}.$$

We know the distance between these two planes is $\sqrt{6}$, and we want to find $|d|$.

---

1. Find the equation of the plane containing the two given lines

Step 1a: Parametric forms of the lines

Line 1: Let the parameter be t. Then

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = t \implies \begin{cases} x = 1 + 2t, \\ y = 2 + 3t, \\ z = 3 + 4t. \end{cases}$$

A direction vector for Line 1 is $\mathbf{v}_1 = (2, 3, 4)$.

A point on Line 1 is $\mathbf{P}_1 = (1, 2, 3)$.

Line 2: Let the parameter be s. Then

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} = s \implies \begin{cases} x = 2 + 3s, \\ y = 3 + 4s, \\ z = 4 + 5s. \end{cases}$$

A direction vector for Line 2 is $\mathbf{v}_2 = (3,\ 4,\ 5)$.

A point on Line 2 is $\mathbf{P}_2 = (2, 3, 4)$.

Step 1b: Normal to the plane containing these lines

A plane that contains both lines must contain their direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Therefore, a normal to this plane is given by the cross product $\mathbf{v}_1 \times \mathbf{v}_2$.

$$\mathbf{v}_1 = (2,\ 3,\ 4), \quad \mathbf{v}_2 = (3,\ 4,\ 5).$$

Compute the cross product:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} = \big(3 \cdot 5 - 4 \cdot 4,\ 4 \cdot 3 - 2 \cdot 5,\ 2 \cdot 4 - 3 \cdot 3\big) = (15$$

Hence a normal vector to the plane is $\mathbf{n} = (-1,\ 2,\ -1)$. Equivalently, we can multiply by $-1\ whichdoesnotchangetheplane$ to get $\mathbf{n} = (1, -2, 1)$.

Thus the plane containing the two lines has the form

$$1 \cdot x \ -\ 2 \cdot y \ +\ 1 \cdot z \ =\ K,$$

i.e.

$$x \ -\ 2y \ +\ z \ =\ K.$$

Step 1c: Find the constant K

To find K, just plug in any point on either line. For instance, the point $\mathbf{P}_1 = (1, 2, 3)$ on Line 1:

$$1(1) \ -\ 2(2) \ +\ 1(3) \ =\ 1 - 4 + 3 = 0.$$

So $K = 0$.

Check also with $\mathbf{P}_2 = (2, 3, 4)$ from Line 2:

$$2 - 2 \cdot 3 + 4 = 2 - 6 + 4 = 0.$$

That also gives 0. So indeed the plane containing both lines is

$$\boxed{x - 2y + z = 0}.$$

2. Determine A so that the planes can be parallel

We are given the plane

$$Ax - 2y + z = d$$

and have found that the plane containing the lines is

$$x - 2y + z = 0.$$

For these two planes to have a finite, nonzero distance between them, they must be parallel. Two planes are parallel precisely when their normal vectors are scalar multiples of each other.

The normal to $Ax - 2y + z = d$ is $(A,\, -2,\, 1)$.

The normal to $x - 2y + z = 0$ is $(1,\, -2,\, 1)$.

Set

$$(A,\, -2,\, 1) \;=\; \lambda\,\bigl(1,\, -2,\, 1\bigr).$$

Matching components:

$$A = \lambda \cdot 1 = \lambda.$$

$$-2 = \lambda \cdot (-2) \implies \lambda = 1.$$

$$1 = \lambda \cdot (1) \implies \lambda = 1.$$

Hence $\lambda = 1$ and $A = 1$.

Therefore, the given plane must be

$$\boxed{x - 2y + z = d}.$$

---

3. Use the formula for the distance between two parallel planes

Now we have two parallel planes:

$x - 2y + z = 0$,

$x - 2y + z = d$.

The normal vector to both is $\mathbf{n} = (1, -2, 1)$. Its magnitude is

$$\|\mathbf{n}\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1 + 4 + 1} = \sqrt{6}.$$

The distance D between two parallel planes

$$\alpha_1 : \quad \mathbf{n} \cdot \mathbf{x} = k_1, \quad \alpha_2 : \quad \mathbf{n} \cdot \mathbf{x} = k_2$$

is given by

$$D = \frac{|k_1 - k_2|}{\|\mathbf{n}\|}.$$

In our case:

For the plane $x - 2y + z = 0$, we have $k_1 = 0$.

For the plane $x - 2y + z = d$, we have $k_2 = d$.

The distance is given to be $\sqrt{6}$.

Thus

$$\sqrt{6} = \frac{|0 - d|}{\sqrt{6}} = \frac{|d|}{\sqrt{6}} \implies |d| = 6.$$

---

4. Conclusion

$$\boxed{|d| = 6}.$$

Question 042 Numerical

QUESTION

If

$$\vec{a}$$

and

$$\overrightarrow{b}$$

are vectors in space given by

$$\overrightarrow{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$$

and

$$\overrightarrow{b} = \frac{2\hat{i} + \hat{j} + 3\widehat{k}}{\sqrt{14}},$$

then find the value of

$$\left(2\overrightarrow{a} + \overrightarrow{b}\right) . \left[\left(\overrightarrow{a} \times \overrightarrow{b}\right) \times \left(\overrightarrow{a} - 2\overrightarrow{b}\right)\right].$$

SOURCE

Mathematics • vector-algebra

EXPLANATION

$$(2\vec{a} + \vec{b})[(\vec{a} \times \vec{b}) \times (\vec{a} - 2\vec{b})]$$

$$= 4(\vec{a} \cdot \vec{a}) + \vec{b} \cdot \vec{b} + 4\vec{a} \cdot \vec{b} \text{ where}$$

$$\vec{a} \cdot \vec{b} = \frac{2 - 2}{\sqrt{70}} = 0 \quad |\vec{a}| = 1 \text{ and } |\vec{b}| = 1$$

$$= 5$$

Question 043 MCQ

QUESTION

Equation of the plane containing the straight line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

and perpendicular to the plane containing the straight lines

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$$

and

$$\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$$

is

A $x + 2y - 2z = 0$

B $3x + 2y - 2z = 0$

C $x - 2y + z = 0$

D $5x + 2y - 4z = 0$

CORRECT OPTION

C $x - 2y + z = 0$

SOURCE

EXPLANATION

Plane 1 :

$$ax + by + cz = 0$$

contains line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

. Therefore,

$$2a + 3b + 4c = 0$$

..... 1

Plane 2 :

$$a'x + b'y + c'z = 0$$

is perpendicular to plane containing line

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$$

and

$$\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$$

.

Hence,

$$3a' + 4b' + 2c' = 0$$

and

$$4a' + 2b' + 3c' = 0$$

.

$$\Rightarrow \frac{a'}{12 - 4} = \frac{b'}{8 - 9} = \frac{c'}{6 - 16}$$

$$\Rightarrow 8a - b - 10c = 0$$

.... 2

From Eq. 1 and 2, we get

$$\frac{a}{-30+4} = \frac{b}{32+20} = \frac{c}{-2-24}$$

$$\Rightarrow$$

Equation of plane is

$$x - 2y + z = 0$$

Question 044 MCQ

QUESTION

Let

$$P, Q, R$$

and

$$S$$

be the points on the plane with position vectors

$$-2\hat{i} - \hat{j}, 4\hat{i}, 3\hat{i} + 3\hat{j}$$

and

$$-3\hat{i} + 2\hat{j}$$

respectively. The quadrilateral

$$PQRS$$

must be a

A parallelogram, which is neither a rhombus nor a rectangle

B square

C rectangle, but not a square

D rhombus, but not a square

CORRECT OPTION

A parallelogram, which is neither a rhombus nor a rectangle

SOURCE

Mathematics • vector-algebra

EXPLANATION

We have

$$PS = \sqrt{1^2 + 3^2} = \sqrt{10}$$
$$SR = \sqrt{6^2 + 1^2} = \sqrt{37}$$

,

$$RQ = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$QP = \sqrt{6^2 + 1^2} = \sqrt{37}$$

Then PQRS is a parallelogram. It can be a rectangle, so let's check the product of slopes of PS and SR

$$(m_{PS}) \, . \, (m_{SR}) = \frac{3}{-1} \times \frac{1}{6} = -\frac{1}{2} \neq -1$$

Thus PS and SR are not perpendicular. So, it's not a rectangle.

Also,

$$m_{PR} \cdot m_{QS} = \frac{4}{5} \times \frac{2}{-7} = -\frac{8}{35} \neq -1$$

Thus, PR and QS are also not perpendicular. So it's not a rhombus either.

Question 045 MCQ

QUESTION

Let

$$\omega$$

be a complex cube root of unity with

$$\omega \neq 1.$$

A fair die is thrown three times. If

$$r_1,$$

$$r_2$$

and

$$r_3$$

are the numbers obtained on the die, then the probability that

$$\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$$

is

A

$$\frac{1}{18}$$

B $\frac{1}{9}$

C $\frac{2}{9}$

D $\frac{1}{36}$

CORRECT OPTION

C $\frac{2}{9}$

SOURCE

Mathematics • probability

EXPLANATION

Sample space A dice is thrown thrice,

$$n(s) = 6 \times 6 \times 6$$

.

Favorable events

$$\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$$

i.e.

$$(r_1, r_2, r_3)$$

are ordered 3-triples which can take values,

$$\begin{matrix} (1,2,3), & (1,5,3), & (4,2,3), & (4,5,3) \\ (1,2,6), & (1,5,6), & (4,2,6), & (4,5,6) \end{matrix}\Big\}$$

i.e. 8 ordered pairs and each can be arranged in 3! ways = 6

$$\therefore$$

$$n(E) = 8 \times 6$$

$$\Rightarrow P(E) = \frac{8 \times 6}{6 \times 6 \times 6} = \frac{2}{9}$$

Question 046 Numerical

QUESTION

For any real number

$$x,$$

let

$$[x]$$

denote the largest integer less than or equal to

$$x.$$

Let

$$f$$

be a real valued function defined on the interval

$$[-10, 10]$$

by

$$\$f(x) = \begin{cases} x - [x] & if\,[x] is\,odd, \\ 1 + [x] - x & if\,[x] is\,even \end{cases}$$

$

Then the value of

$$\frac{\pi^2}{10}\int\limits_{-10}^{10} f\left(x\right)\cos\,\pi x\,dx$$

is

SOURCE

Mathematics • definite-integration

EXPLANATION

Case 1 :

Let

$$0 \leq x < 1$$

then

$$[x] = 0$$

, which is even

$$\therefore$$

$$\begin{aligned} f(x) &= 1 + [x] - x \\ &= 1 + 0 - x \\ &= 1 - x \end{aligned}$$

Case 2 :

Let

$$1 \leq x < 2$$

then

$$[x] = 1$$

, which is odd

$$\therefore$$

$$f(x) = x - [x]$$

$$= x - 1$$

Case 3 :

Let

$$2 \leq x < 3$$

then

$$[x] = 2$$

, which is even

$$\therefore$$

$$f(x) = 1 + [x] - x$$

$$= 1 + 2 - x$$

$$= 3 - x$$

Case 4 :

Let

$$3 \leq x < 4$$

then

$$[x] = 3$$

, which is odd

$$\therefore$$

$$f(x) = x - [x]$$

$$= x - 3$$

$$\therefore$$

$$f(x) = \begin{cases} 1 - x & ; \quad 0 \leq x < 1 \\ x - 1 & ; \quad 1 \leq x < 2 \\ 3 - x & ; \quad 2 \leq x < 3 \\ x - 3 & ; \quad 3 \leq x < 4 \end{cases}$$

$$\therefore$$

$$f(x)$$

is periodic and period of

$$f(x) = 2$$

And period of

$$\cos \pi x = \frac{2\pi}{\pi} = 2$$

$$\therefore$$

Period of

$$f(x) \cos \pi x = 2$$

Now,

$$I = \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$$

$$= \frac{\pi^2}{10} \int_{-10}^{-10+10\times 2} f(x) \cos \pi x \, dx$$

$$= \frac{\pi^2}{10} \int_{0}^{10\times 2} f(x) \cos \pi x \, dx$$

$$= \frac{\pi^2}{10} \times 10 \int_{0}^{2} f(x) \cos \pi x \, dx$$

$$= \pi^2 \int_{0}^{2} f(x) \cos \pi x \, dx$$

$$\therefore$$

$$I = \pi^2 \left[\int_0^1 f(x) \cos \pi x \, dx + \int_1^2 f(x) \cos \pi x \, dx \right]$$

$$= \pi^2 \left[\int_0^1 (1-x) \cos \pi x \, dx + \int_1^2 (x-1) \cos \pi x \, dx \right]$$

$$= \pi^2 \left[\int_0^1 \cos \pi x \, dx - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - \int_1^2 \cos \pi x \, dx \right]$$

$$= \pi^2 \left[\frac{1}{\pi} [\sin \pi x]_0^1 - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - \frac{1}{\pi} [\sin \pi x]_1^2 \right]$$

$$= \pi^2 \left[0 - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - 0\right]$$

$$= \pi^2 \left[-\left[x \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x\right]_0^1 + \left[x \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x\right]_1^2\right]$$

$$\Bigg[\text{As} \int x \cos \pi x \, dx = x \, . \int \cos \pi x - \int \left(1 \, . \, \frac{\sin \pi x}{\pi}\right) dx = x \, . \, \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \text{c}$$

$$= \pi^2 \left[-\left[\left(1 \, . \, \frac{\sin \pi}{\pi} + \frac{1}{\pi^2} \, . \, \cos \pi\right) - \left(0 + \frac{1}{\pi^2} \, . \, \cos 0\right)\right] + \left[\left(2 \, . \, \frac{\sin 2\pi}{\pi} + \right.\right.\right.$$

$$= \pi^2 \left[-\left\{\left(-\frac{1}{\pi^2}\right) - \left(\frac{1}{\pi^2}\right)\right\} + \left(\left(+\frac{1}{\pi^2}\right) - \left(-\frac{1}{\pi^2}\right)\right]\right.$$

$$= \pi^2 \left[-\left(-\frac{2}{\pi^2}\right) + \frac{2}{\pi^2}\right]$$

$$= \pi^2 \left[\frac{2}{\pi^2} + \frac{2}{\pi^2}\right]$$

$$= \pi^2 \times \frac{4}{\pi^2}$$

$$= 4$$

Question 047 MCQ

QUESTION

Let

$$f$$

be a real-valued function defined on the interval

$$(0, \infty)$$

by

$$f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt.$$

then which of the following statement$s$ is are true?

A exists for all

$$f''(x)$$

$$x \in (0, \infty)$$

B

$$f'(x)$$

exists for all

$$x \in (0, \infty)$$

and

$$f'$$

is continuous on

$$(0, \infty)$$

, but not differentiable on

$$(0, \infty)$$

C there exists

$$\alpha > 1$$

such that

$$|f'(x)| < |f(x)|$$

for all

$$x \in (\alpha, \infty)$$

D

there exists

$$\beta > 0$$

such that

$$|f(x)| + |f'(x)| \leq \beta$$

for all

$$x \in (0, \infty)$$

CORRECT OPTION

C

there exists

$$\alpha > 1$$

such that

$$|f'(x)| < |f(x)|$$

for all

$$x \in (\alpha, \infty)$$

SOURCE

Mathematics • application-of-integration

EXPLANATION

$$f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$$

$$f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}$$

f' is not differentiable at sin x =

$$-$$

1

i.e.

$$x = 2n\pi - \frac{\pi}{2}, n \in N$$

in the interval $0, \$\$\infty\$\$$

$$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$$

f'' does not exist for all x

$$\in$$

$0, \$\$\infty\$\$$

f' exist for x > 0

we have

$$\frac{1}{x} + \sqrt{1+\sin x} < \ln x + \int_0^x \sqrt{1+\sin x}dx$$

because L.H.S. is bounded and R.H.S. is not bounded so

$$\exists$$

some

$$\alpha$$

beyond which R.H.S. is greater than L.H.S.

i.e.

$$|f'(x)| < |f(x)|$$

for all x

$$\in$$

$\$\$\alpha\$\$$, $\$\$\infty\$\$$

$$|f| + |f'| \leq \beta$$

is wrong as f is unbounded while f' is bounded.

Question 048 MCQ

QUESTION

The value of

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$$

is are

A $\frac{22}{7} - \pi$

B $\frac{2}{105}$

C 0

D $\frac{71}{15} - \frac{3\pi}{2}$

CORRECT OPTION

A

$$\frac{22}{7} - \pi$$

SOURCE

Mathematics • definite-integration

EXPLANATION

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \frac{x^4\{(1+x^2)-2x\}^2}{1+x^2}dx$$

$$= \int_0^1 x^4 \frac{(1+x^2)^2 - 4x(1+x^2) + 4x^2}{1+x^2}dx$$

$$= \int_0^1 x^4 \left[1 + x^2 - 4x + \frac{4\{1+x^2-1\}}{1+x^2}\right]dx$$

$$= \int_0^1 x^4 \left[1 + x^2 - 4x + 4 - \frac{4}{1+x^2}\right]dx$$

$$= \int_0^1 \left[x^6 - 4x^5 + 5x^4 - 4\frac{x^4-1+1}{1+x^2}\right]dx$$

$$= \int_0^1 (x^6 - 4x^5 + 5x^4 - 4x^2 + 4)dx - 4\int_0^1 \frac{dx}{1+x^2}$$

$$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x\right]_0^1 - 4[\tan^{-1}x]_0^1$$

$$= \left[\frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4\right] - \pi = \frac{22}{7} - \pi$$

Question 049 MCQ

QUESTION

The value of

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t \ln(1+t)}{t^4+4} dt$$

is

A 0

B $\frac{1}{12}$

C $\frac{1}{24}$

D $\frac{1}{64}$

CORRECT OPTION

B $\frac{1}{12}$

SOURCE

Mathematics • definite-integration

EXPLANATION

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t \log(1+t)}{4+t^4} dt$$

Using L' Hospital's rule,

$$= \lim_{x \to 0} \frac{\frac{x \log(1+x)}{4+x^4}}{3x^2}$$

$$= \lim_{x \to 0} \frac{\log(1+x)}{3x} \cdot \frac{1}{4+x^4}$$

$$= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$using, \$\$ \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\$\$$$

Question 050 MCQ

QUESTION

Let

$$ABC$$

be a triangle such that

$$\angle ACB = \frac{\pi}{6}$$

and let

$$a, b$$

and

$$c$$

denote the lengths of the sides opposite to

$$A$$

,

$$B$$

and

$$C$$

respectively. The value s of

$$x$$

for which

$$a = x^2 + x + 1, \;\; b = x^2 - 1$$

and

$$c = 2x + 1$$

is are

A $-\left(2+\sqrt{3}\right)$

B $1+\sqrt{3}$

C $2+\sqrt{3}$

D $4\sqrt{3}$

CORRECT OPTION

B

$$1 + \sqrt{3}$$

SOURCE

Mathematics • properties-of-triangle

EXPLANATION

Using,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(x^2 + x + 1)^2 + (x^2 - 1)^2 - (2x + 1)^2}{2(x^2 + x + 1)(x^2 - 1)}$$

$$\Rightarrow (x + 2)(x + 1)(x - 1)x + (x^2 - 1)^2 = \sqrt{3}(x^2 + x + 1)(x^2 - 1)$$

$$\Rightarrow x^2 + 2x + (x^2 - 1) = \sqrt{3}(x^2 + x + 1)$$

$$\Rightarrow (2 - \sqrt{3})x^2 + (2 - \sqrt{3})x - (\sqrt{3} + 1) = 0$$

$$\Rightarrow x = -(2 + \sqrt{3})$$

and

$$x = 1 + \sqrt{3}$$

But,

$$x = -(2 + \sqrt{3}) \Rightarrow c$$

is negative.

$$\therefore$$

$$x = 1 + \sqrt{3}$$

is the only solution.

Hence, b is the correct option.

Question 051 MCQ

QUESTION

The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system

$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

A 0

B $2^9 - 1$

C 168

D 2

CORRECT OPTION

A 0

SOURCE

Mathematics • matrices-and-determinants

EXPLANATION

Given,

A $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has two distinct solution we know that three Planes cannot intersect at two distinct point.

Hence, number of 3×3 matrix A is zero.

Question 052 MCQ

QUESTION

Let f, g and h be real valued functions defined on the interval $[0,1]$ by

$f(x) = e^{x^2} + e^{-x^2}$,

$g(x) = xe^{x^2} + e^{-x^2}$

and $h(x) = x^2e^{x^2} + e^{-x^2}$.

If a, b and c denote, respectively, the absolute maximum of f, g and h on $[0,1]$, then :

A. $a = b$ and $c \neq b$

B. $a = c$ and $a \neq b$

C. $a \neq b$ and $c \neq b$

D $a = b = c$

CORRECT OPTION

D $a = b = c$

SOURCE

Mathematics • functions

EXPLANATION

$$f(x) = e^{x^2} + e^{-x^2}$$

$$f'(x) = e^{x^2}\frac{d}{dx}(x^2) + e^{-x^2}\frac{d}{dx}(-x^2)$$

$$= e^{x^2}(2x) + e^{-x^2}(-2x)$$

$$= 2x\left(e^{x^2} - e^{-x^2}\right) \geq 0 \forall x \in [0, 1]$$

$$g(x) = xe^{x^2} + e^{-x^2}$$

$$h(x) = x^2e^{x^2} + e^{-x^2}$$

Clearly for $0 \leq x \leq 1 \quad f(x) \geq g(x) \geq h(x)$

$\because f(1) = g(1) = h(1) = e + \frac{1}{e}$ and $f(1)$ is greatest

$$\because a = b = c = e + \frac{1}{e}$$

Question 053 MCQ

QUESTION

Let z_1 and z_2 be two distinct complex numbers let $z=(1-t)z_1+tz_2$ for some real number t with $0<t<1$.

If $\mathrm{Arg}(w)$ denotes the principal argument of a nonzero complex number w, then :

A $|z-z_1|+|z-z_2|=|z_1-z_2|$

B $\mathrm{Arg}\,(z-z_1)=\mathrm{Arg}\,(z-z_2)$

C $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix}=0$

D $\mathrm{Arg}\,(z-z_1)=\mathrm{Arg}\,(z_2-z_1)$

CORRECT OPTION

A $|z-z_1|+|z-z_2|=|z_1-z_2|$

SOURCE

Mathematics • complex-numbers

EXPLANATION

$$\text{Given, } z=(1-t)z_1+tz_2$$

$$\Rightarrow \frac{z-z_1}{z_2-z_1}=t$$

$$\Rightarrow \arg\left(\frac{z-z_1}{z_2-z_1}\right)=0.\ldots\ldots(i)$$

$$\Rightarrow \arg(z - z_1) = \arg(z_2 - z_1)$$

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

$$\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$$

$$\mathrm{AP} + \mathrm{PB} = \mathrm{AB}$$

$$\Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$$

Question 054 MCQ

QUESTION

The number of A in T_p such that A is either symmetric or skew-symmetric or both, and $\det(\mathrm{A})$ divisible by p is :

A $(p-1)^2$

B $2(p-1)$

C $(p-1)^2+1$

D $2p-1$

CORRECT OPTION

D $2p-1$

SOURCE

Mathematics • matrices-and-determinants

EXPLANATION

We must have $a^2 - b^2 = 1 < p$

$$(a+b)(a-b) = 1 < p$$

Either $a - b = 0$ or $a + b$ is a multiple of p when $a = b$ number of matrices is p and when $a + b$ $=$ multiple of p.

$$\Rightarrow a, b \text{ has } p - 1$$

$$\text{Total number of matrices } = p + p - 1$$

$$= 2p - 1$$

Question 055 MCQ

QUESTION

The number of A in T_p such that the trace of A is not divisible by p but $\det(\mathrm{A})$ is divisible by p is

[**Note :** The trace of a matrix is the sum of its diagonal entries.]

A $(p-1)\left(p^2 - p + 1\right)$

B $p^3 - (p-1)^2$

C $(p-1)^2$

D $(p-1)\left(p^2-2\right)$

CORRECT OPTION

C $(p-1)^2$

SOURCE

Mathematics • matrices-and-determinants

EXPLANATION

We have an odd prime p. Consider the set

$$T_p = \left\{A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\}\right\}.$$

We want to count the number of such matrices A that satisfy:

$$\operatorname{trace}(A) \not\equiv 0 \pmod{p}.$$

Since $\operatorname{trace}(A) = a + a = 2a$, this means

$$2a \not\equiv 0 \pmod{p}.$$

Because p is an odd prime, 2 is invertible $\bmod p$.

Hence $2a \equiv 0 \pmod{p}$ if and only if $a \equiv 0 \pmod{p}$.

Thus the condition

$$\operatorname{trace}(A) \not\equiv 0 \pmod{p}$$

is equivalent to

$$a \neq 0 \pmod{p}.$$

In other words, a must be a nonzero element modulo p; so

$$a \in \{1, 2, \ldots, p-1\}.$$

$\det(A) \equiv 0 \pmod{p}$.

For the matrix

$\begin{pmatrix} a & b \\ c & a \end{pmatrix}$, we have

$\det(A) \ = \ a^2 - bc.$

The condition $\det(A) \equiv 0 \pmod{p}$ means

$a^2 \ - \ bc \ \equiv \ 0 \pmod{p} \quad \iff \quad bc \ \equiv \ a^2 \pmod{p}.$

Putting these two pieces together:

a is nonzero \$$a \in \{1, \ldots, p-1\}$\$.

$bc \equiv a^2 \pmod{p}.$

Counting the solutions

We must count the number of triples (a, b, c) with $a \neq 0$ and $bc \equiv a^2$.

Range for a

Since $a \in \{1, 2, \ldots, p-1\}$, there are $p - 1$ possible values of a.

Condition $bc \equiv a^2$ for given a

Since $a \neq 0$, a^2 is also nonzero $\mod p$.

If $b = 0$, then $bc \equiv 0$, which would only be equal to a^2 if $a^2 \equiv 0$. But $a^2 \neq 0$ since $a \neq 0$. So $b = 0$ gives no solutions.

Similarly, if $c = 0$, then $bc = 0$, which again cannot equal the nonzero a^2. So $c = 0$ gives no solutions either.

Therefore b and c must both be nonzero modulo p.

Once b is chosen to be any nonzero element in $\{1, \ldots, p-1\}$, there is a unique $c \equiv a^2 b^{-1} \pmod{p}$. Hence:

For each nonzero b $thatis, \$b \in \{1, \dots, p-1\}\$$, there is exactly **one** $c \in \{1, \dots, p-1\}$ satisfying $bc \equiv a^2$.

Consequently, for each fixed nonzero a, the number of (b, c)-pairs is $p-1$.

Putting it all together

We have $p-1$ choices for $a \neq 0$.

For each such a, there are $p-1$ pairs (b, c) satisfying $bc \equiv a^2$.

Therefore, in total, the number of matrices is

$$(p-1) \times (p-1) = (p-1)^2.$$

Hence the correct answer matches option **C**, which is

$$\boxed{(p-1)^2.}$$

Question 056 MCQ

QUESTION

The number of A in T_p such that $\det(\mathrm{A})$ is not divisible by p is :

A $2p^2$

B $p^3 - 5p$

C $p^3 - 3p$

D $p^3 - p^2$

CORRECT OPTION

D $p^3 - p^2$

SOURCE

Mathematics • matrices-and-determinants

EXPLANATION

Let p be an odd prime number, and consider the set

$$T_p = \left\{ A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\} \right\}.$$

We want to count the number of matrices $A \in T_p$ whose determinant is **not** divisible by p.

Recall that for

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix},$$

the determinant is

$$\det(A) = a^2 - bc.$$

Hence $\det(A)$ is **not** divisible by p precisely when

$$a^2 - bc \not\equiv 0 \pmod{p}, \quad \text{i.e.,} \quad a^2 \not\equiv bc \pmod{p}.$$

Total number of matrices

Since a, b, c each range over $\{0, 1, \ldots, p-1\}$, there are

$$p^3$$

total possible matrices in T_p.

Step 1: Count how many (a, b, c) do satisfy $a^2 \equiv bc \pmod{p}$

It will be easier to first count the number of triples (a, b, c) for which

$$a^2 \;\equiv\; b\,c \pmod{p},$$

and then subtract that from p^3.

Case A: $a = 0$

Then $a^2 = 0$. We need

$$b\,c \;\equiv\; 0 \pmod{p}.$$

Over the field $\mathbf{F}_p$, the product $b\,c \equiv 0$ if and only if at least one of b or c is 0 $since$p$isprime$.

If $b = 0$, then c can be anything in $\{0, \ldots, p-1\}$.

This gives p possibilities.

If $c = 0$, then b can be anything in $\{0, \ldots, p-1\}$.

This also gives p possibilities.

However, the pair $(b = 0, c = 0)$ is counted in both cases, so we must subtract it out once.

Hence, for $a = 0$, the number of (b, c) such that $b\,c \equiv 0$ is

$$p \;+\; p \;-\; 1 \;=\; 2p \;-\; 1.$$

Case B: $a \neq 0$

Then a can be any nonzero element in $\{1, 2, \ldots, p-1\}$; there are $p-1$ such choices. In that scenario, $a^2 \neq 0 \pmod{p}$. We want

$$b\,c \;\equiv\; a^2 \pmod{p}.$$

Over $\mathbf{F}_p$, a nonzero right-hand side a^2 can be written as $b\,c$ in the following way:

If $b = 0$, then $b\,c = 0$, which cannot equal $a^2 \neq 0$.

No solutions in that subcase.

If $b \neq 0$, then for each nonzero b there is **exactly one** c satisfying $b\,c \equiv a^2 \pmod p$, namely $c \equiv a^2 b^{-1} \pmod p$.

Since b can be any of the $p-1$ nonzero elements, we get $p-1$ solutions (b, c) for each nonzero a.

Thus, for each $a \neq 0$, there are $p-1$ pairs (b, c). Hence, for $p-1$ values of a, the total number of solutions in this case is

$$(p-1) \times (p-1) = (p-1)^2.$$

Total number of solutions to $a^2 = b\,c$

Summing these two cases:

Case A $(a = 0)$: $2p-1$ solutions.

Case B $(a \neq 0)$: $(p-1)^2$ solutions.

Hence the total count of (a, b, c) satisfying $a^2 \equiv b\,c$ is

$$(2p-1) + (p-1)^2 = (2p-1) + \left(p^2 - 2p + 1\right) = p^2.$$

$A classic result: there are exactly \$p^2\$ solutions to \$b\,c = a^2\$ over \$\mathbf{F}_p^3\$.$

Step 2: Count how many (a, b, c) satisfy $a^2 \neq b\,c$

Since there are p^3 total triples (a, b, c), the number that satisfy

$$a^2 \neq b\,c \pmod p$$

is simply

$$p^3 - p^2.$$

This directly corresponds to matrices A whose determinant $a^2 - b\,c$ is $\neq 0 \pmod p$.

Hence, the number of matrices in T_p with $\det(A)$ **not** divisible by p is

$\boxed{p^3 - p^2}$.

The expression $p^3 - p^2$ matches **Option D**.

Question 057 Numerical

QUESTION

A 0.1 kg mass is suspended from a wire of negligible mass. The length of the wire is 1 m and its crosssectional area is 4.9

$$\times$$

10^{-7} m^2. If the mass is pulled a little in the vertically downward direction and released, it performs simple harmonic motion of angular frequency 140 rad s^{-1}. If the Young's modulus of the material of the wire is n

$$\times$$

10^9 Nm^{-2}, the value of n is

SOURCE

Physics • simple-harmonic-motion

EXPLANATION

When mass m is pulled by a force F, the wire elongation x, length l, cross-sectional area A, and Young's modulus of wire material Y are related by

$$Y = \frac{F/A}{x/l}$$

i.e.,

$$F = (Y\,A/l)x$$

.

The restoring force by the wire is equal but opposite to F i.e., F_r =

$$-$$

F. Apply Newton's second law to get

$$md^2x/dt^2 = -(YA/l)x = -\omega^2 x$$

.

This equation represents SHM with an angular frequency

$$\omega = \sqrt{YA/(lm)}$$

. Substitute the values to get

$$Y = \omega^2 lm/A = 4 \times 10^9$$

N/m^2.

Question 058 MCQ

QUESTION

A block of mass m is on an inclined plane of angle θ. The coefficient of friction between the block and the plane is μ and tan θ > μ. The block is held stationary by applying a force P parallel to the plane. The direction of force pointing up the plane is taken to be positive. As P is varied from P_1 = mg $sin\theta - \mu cos\theta$ to P_2 = mg $sin\theta + \mu cos\theta$, the frictional force f versus P graph will look like

A

B

C

D

CORRECT OPTION

A

SOURCE

Physics • laws-of-motion

EXPLANATION

The forces acting on the block are its weight mg, normal reaction N, applied force P and frictional force f.

Resolve mg along and normal to the inclined plane and apply Newton's second law to get

$$0 = P + f - mg\sin\theta$$

,

which gives

$$f = -P + mg\sin\theta$$

. ... 1

This is a straight line with slope

$$-$$

1. Substitute the values of P_1 and P_2 in equation 1 to get the frictional force at those points i.e.,

$$f_1 = \mu mg\cos\theta$$

and

$$f_2 = -\mu mg\cos\theta$$

.

Question 059 MCQ

QUESTION

A point mass of 1 kg collides elastically with a stationary point mass of 5 kg. After their collision, the 1 kg mass reverses its direction and moves with a speed of 2 ms^{-1}. Which of the following statement$s$ is are correct for the system of these two masses?

A Total momentum of the system is 3 kg ms^{-1}

B Momentum of 5 kg mass after collision is 4 kg ms^{-1}

C Kinetic energy of the centre of mass is 0.75 J

D Total kinetic energy of the system is 4 J

CORRECT OPTION

A Total momentum of the system is 3 kg ms^{-1}

SOURCE

Physics • impulse-and-momentum

EXPLANATION

Here, m_1 = 1 kg, m_2 = 5 kg

u_1 = u, u_2 = 0

v_1 =

$$-$$

2 m s

$$-$$

1, v_2 = v

By the law of conservation of linear momentum, we get

$$m_1u_1 + m_2u_2 = m_1u_1 + m_2u_2$$

$$1 \times u + 5 \times 0 = 1 \times (-2) + 5 \times v$$

$$u = 5v - 2$$

...... i

By the definition of coefficient of restitution,

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

For a perfectly elastic collision, e = 1

$$\therefore$$

$$1 = \frac{v + 2}{u}$$

or

$$u = v + 2$$

..... ii

Solving equations i and ii, we get

u = 3 m s

$$-$$

1, v = 1 m s

—

1

Before collision,

Total momentum of the system = 1

×

3 + 5

×

0 = 3 kg m s

—

1

After collision,

Total momentum of the system = 1

×

$$ − $$2 + 5

×

1 = 3 kg m s

—

1

Hence, option a is correct.

Momentum of 5 kg mass after collision = 5

×

1 = 5 kg m s

—

1.

Hence, option b is incorrect.

Velocity of centre mass is

$$v_{cm} = \frac{1 \times 3 + 5 \times 0}{1 + 5} = \frac{1}{2}$$

m s

$-$

1

Kinetic energy of the centre of mass

$$= \frac{1}{2} m_{system} v_{CM}^2 = \frac{1}{2} \times 6 \times \left(\frac{1}{2}\right)^2 = 0.75$$

J

Hence, option c is correct.

Before collision,

Total kinetic energy of the system

$$= \frac{1}{2} \times 1 \times 3^2 + \frac{1}{2} \times 5 \times 0^2 = 4.5$$

J

After collision,

Total kinetic energy of the system

$$= \frac{1}{2} \times 1 \times (-2)^2 + \frac{1}{2} \times 5 \times (1)^2 = 4.5$$

J

Hence, option d is incorrect.

Question 060 Numerical

QUESTION

A binary star consists of two stars A (mass $2.2M_S$) and B (mass $11M_S$), where M_S is the mass of the sun. They are separated by distance d and are rotating about their centre of mass, which is stationary. The ratio of the total angular momentum of the binary star to the angular momentum of star B about the centre of mass is

SOURCE

Physics • gravitation

EXPLANATION

Let stars A and B are rotating about their centre of mass with angular velocity

$$\omega$$

.

Let distance of stars A and B from the centre of mass be r_A and r_B respectively as shown in the figure.

Total angular momentum of the binary stars about the centre of mass is

$$L = M_A r_A^2 \omega + M_B r_B^2 \omega$$

Angular momentum of the star B about centre of mass is

$$L_B = M_B r_B^2 \omega$$

$$\therefore$$

$$\frac{L}{L_B} = \frac{(M_A r_A^2 + M_B r_B^2)\omega}{M_B r_B^2 \omega} = \left(\frac{M_A}{M_B}\right)\left(\frac{r_A}{r_B}\right)^2 + 1$$

Since

$$M_A r_A = M_B r_B$$

or,

$$\frac{r_A}{r_B} = \frac{M_B}{M_A}$$

$$\therefore$$

$$\frac{L}{L_B} = \frac{M_B}{M_A} + 1 = \frac{11M_S}{2.2M_S} + 1 = \frac{11 + 2.2}{2.2} = 6$$

Question 061 Numerical

QUESTION

Gravitational acceleration on the surface of a planet is

$$\frac{\sqrt{6}}{11}g$$

, where

$$g$$

is the gravitational acceleration on the surface of the earth. The average mass density of the planet is

$$\frac{2}{3}$$

times that of the earth. If the escape speed on the surface of the earth is taken to be 11 kms^{-1}, the escape speed on the surface of the planet in kms^{-1} will be

SOURCE

Physics • gravitation

EXPLANATION

On the planet,

$$g_p = \frac{GM_p}{R_p^2} = \frac{G}{R_p^2}\left(\frac{4}{3}\pi R_p^3 \rho_p\right) = \frac{4}{3}G\pi R_p \rho_p$$

On the earth,

$$g_e = \frac{GM_e}{R_e^2} = \frac{G}{R_e^2}\left(\frac{4}{3}\pi R_e^3 \rho_e\right) = \frac{4}{3}G\pi R_e \rho_e$$

$$\therefore$$

$$\frac{g_p}{g_e} = \frac{R_p \rho_p}{R_e \rho_e}$$

or

$$\frac{R_p}{R_e} = \frac{g_p \rho_p}{g_e \rho_e}$$

..... i

On the planet,

$$v_p = \sqrt{2 g_p R_p}$$

On the earth,

$$v_e = \sqrt{2 g_e R_e}$$

$$\therefore$$

$$\frac{v_p}{v_e} = \sqrt{\frac{g_p R_p}{g_e R_e}} = \frac{g_p}{g_e}\sqrt{\frac{\rho_e}{\rho_p}}$$

$Using(i)$

Here,

$$\rho_p = \frac{2}{3}\rho_e$$

,

$$g_p = \frac{\sqrt{6}}{11} g_e$$

$$\therefore$$

$$\frac{v_p}{v_e} = \frac{\sqrt{6}}{11}\sqrt{\frac{3}{2}}$$

or,

$$v_p = 11 \times \frac{\sqrt{6}}{11} \times \sqrt{\frac{3}{2}}$$

(

$\because$

v_e = 11 km s

−

1 $Given$)

= 3 km s

−

1

Question 062 MCQ

QUESTION

A real gas behaves like an ideal gas if its

- **A** pressure and temperature are both high
- **B** pressure and temperature are both low
- **C** pressure is high and temperature is low
- **D** pressure is low and temperature is high

CORRECT OPTION

D pressure is low and temperature is high

SOURCE

Physics • heat-and-thermodynamics

EXPLANATION

In an ideal gas, the average force of attraction between the molecules and volume of the molecules $incomparisontovolumeofthegas$ are negligibly small. These conditions are satisfied for a real gas when pressure is low and temperature is high.

Question 063 Numerical

QUESTION

Two spherical bodies A $radius6cm$ and B $radius18cm$ are at temperature T_1 and T_2, respectively. The maximum intensity in the emission spectrum of A is at 500 nm and in that of B is at 1500 nm. Considering them to be black bodies, what will be the ratio of the rate of total energy radiated by A to that of B?

SOURCE

Physics • heat-and-thermodynamics

EXPLANATION

According to Wien's displacement law,

$$\lambda$$

$_m$T = constant

$$\therefore$$

$$(\lambda_m)_A T_A = (\lambda_m)_B T_B$$

or,

$$\frac{T_A}{T_B} = \frac{(\lambda_m)_B}{(\lambda_m)_A} = \frac{1500\,nm}{500\,nm}$$

or

$$\frac{T_A}{T_B} = 3$$

...... i

According to Stefan Boltzmann law, rate of energy radiated by a black body

$$E = \sigma A T^4 = \sigma 4\pi R^2 T^4$$

$$Here, \$\$A = 4\pi R^2\$\$$$

$$\therefore$$

$$\frac{E_A}{E_B} = \left(\frac{R_A}{R_B}\right)^2 \left(\frac{T_A}{T_B}\right)^4 = \left(\frac{6\,cm}{18\,cm}\right)^2 (3)^4$$

$Using(i)$

$$= 9$$

Question 064 Numerical

QUESTION

A piece of ice (heat capacity = 2100 J kg^{-1} ^{o}C^{-1} and latent heat = 3.36

$$\times$$

10^{5} J kg^{-1}) of mass m grams is at - 5 ^{o}C at atmospheric pressure. It is given 420 J of heat so that the ice starts melting. Finally when the ice-water mixture is in equilibrium, it is found that 1 gm of ice has melted. Assuming there is no other heat exchange in the process, the value of m is

SOURCE

Physics • heat-and-thermodynamics

EXPLANATION

In the final state, ice-water mixture is in equilibrium. Thus, the temperature of m grams of ice is raised from

$-$

5

$\circ$

C to 0

$\circ$

C. The heat absorbed in this process is

Q_1 = mS

Δ

T. 1

The state of m_1 = 1 g of ice is changed from solid to liquid. The heat absorbed in the melting process is

Q_2 = m_1L. 2

The heat supplied is Q = 420 J. By energy conservation, Q = Q_1 + Q_2. Substitute Q_1 and Q_2 from equations 1 and 2 to get

$$m = \frac{Q - m_1 L}{S\Delta T} = \frac{420 - (10^{-3})(3.36 \times 10^5)}{(2100)(5)}$$

$$= 8 \times 10^{-3}$$

kg = 8 g.

Question 065 MCQ

QUESTION

A few electric field lines for a system of two charges

$$Q_1$$

and

$$Q_2$$

fixed at two different points on the

$$x$$

-axis are shown in the figure. These lines suggest that

A
$$|Q_1| > |Q_2|$$

B
$$|Q_1| < |Q_2|$$

C
at a finite distance to the left of

$$Q_1$$

the electric field is zero

D
at a finite distance to the right of

$$Q_2$$

the electric field is zero

CORRECT OPTION

$$|Q_1| > |Q_2|$$

A

SOURCE

Physics • electrostatics

EXPLANATION

Number of electric field lines originating from Q_1 is more than terminating at Q_2.

$$\therefore$$

$$|Q_1| > |Q_2|$$

Here, Q_1 is positive while Q_2 is negative.

The electric field at a distance x towards the right of Q_2 is given by

$$|\vec{E}| = \frac{1}{4\pi\in_0}\frac{Q_1}{(d+x)^2} - \frac{1}{4\pi\in_0}\frac{Q_2}{x^2}$$

,

where d is the separation between Q_1 and Q_2. Since

$$Q_1 > Q_2$$

,

$$|\vec{E}|$$

becomes zero for some finite x.

Question 066 MCQ

QUESTION

A student uses a simple pendulum of exactly 1m length to determine g, the acceleration due to gravity. He uses a stop watch with the least count of 1 sec for this and records 40 seconds for 20 oscillations. For this observation, which of the following statement$s$ is are true?

A Error ΔT in measuring T, the time period, is 0.05 seconds

B Error ΔT in measuring T, the time period, is 1 second

C Percentage error in the determination of g is 5%

D Percentage error in the determination of g is 2.5%

CORRECT OPTION

C Percentage error in the determination of g is 5%

SOURCE

Physics • units-and-measurements

EXPLANATION

Relative error in measurement of time,

$$\frac{\Delta t}{t}=\frac{1\,s}{40\,s}=\frac{1}{40}$$

Time period,

$$T=\frac{40\,s}{20}=2\,s$$

Error in measurement of time period,

$$\Delta T=T\times\frac{\Delta t}{t}=2\,s\times\frac{1}{40}=0.05\,s$$

The time period of simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

or,

$$T^2 = \frac{4\pi^2 l}{g}$$

or,

$$g = \frac{4\pi^2 l}{T^2}$$

$$\therefore$$

$$\frac{\Delta g}{g} = \frac{2\Delta T}{T} = 2 \times \frac{1}{40} = \frac{1}{20}$$

$$\because \frac{\Delta T}{T} = \frac{\Delta t}{t}$$

Percentage error in determination of g is

$$\frac{\Delta g}{g} \times 100 = \frac{1}{20} \times 100 = 5\%$$

Question 067 MCQ

QUESTION

Incandescent bulbs are designed by keeping in mind that the resistance of their filament increases with the increase in temperature. If at room temperature, 100, 60 and 40 W bulbs have filament resistances R_{100}, R_{60} and R_{40} respectively, the relation between these resistances is

A $$\frac{1}{R_{100}} = \frac{1}{R_{40}} + \frac{1}{R_{60}}$$

B $$R_{100} = R_{40} + R_{60}$$

C $$R_{100} > R_{60} > R_{40}$$

D $$\frac{1}{R_{100}} > \frac{1}{R_{60}} > \frac{1}{R_{40}}$$

CORRECT OPTION

D $$\frac{1}{R_{100}} > \frac{1}{R_{60}} > \frac{1}{R_{40}}$$

SOURCE

Physics • current-electricity

EXPLANATION

The power of the bulb is

$$P = \frac{V^2}{R}$$

Therefore,

$$100 = \frac{V^2}{R_{100}} \Rightarrow \frac{1}{R_{100}} = \frac{100}{V^2}$$

where R_{100} is the resistance $at any temperature$ corresponds to 100 W.

Similarly,

$$60 = \frac{V^2}{R_{60}} \Rightarrow \frac{1}{R_{60}} = \frac{60}{V^2}$$

and

$$40 = \frac{V^2}{R_{40}} \Rightarrow \frac{1}{R_{40}} = \frac{40}{V^2}$$

From these equations, we get

$$P_{100} > P_{60} > P_{40} \Rightarrow \frac{1}{R_{100}} > \frac{1}{R_{60}} > \frac{1}{R_{40}}$$

Question 068 MCQ

QUESTION

To verify Ohm's law, a student is provided with a test resitor R_T, a high resistance R_1, a small resistance R_2, two identical galvanometers G_1 and G_2, and a variable voltage source V. The correct circuit to carry out the experiment is

A

B

C

D

CORRECT OPTION

C

SOURCE

Physics • current-electricity

EXPLANATION

To verify Ohm's law, we need to measure the voltage across the test resistance R_T and current passing through it. The voltage can be measured by connecting a high resistance R_1 in series with galvanometer. This combination becomes a voltmeter and should be connected in parallel to R_T. The current can be measured by connecting a low resistance R_2 $shunt$ in parallel with galvanometer. This combination becomes an ammeter and should be connected in series to measure the current through R_T.

Question 069 MCQ

QUESTION

An AC voltage source of variable angular frequency

$$\omega$$

and fixed amplitude V_0 is connected in series with a capacitance C and an electric bulb of resistance R $inductancezero$. When

$$\omega$$

is increased

A the bulb glows dimmer.

B the bulb glows brighter.

C total impedance of the circuit is unchanged.

D total impedance of the circuit increases.

CORRECT OPTION

B the bulb glows brighter.

SOURCE

Physics • alternating-current

EXPLANATION

Impedance of the circuit,

$$Z=\sqrt{\left(X_C\right)^2+(R)^2}=\sqrt{\left(\frac{1}{\omega C}\right)^2+R^2}$$

As

$$\omega$$

increases, Z decreases.

Current in the circuit,

$$I=\frac{V_0}{Z}$$

When

$$\omega$$

is increased, the impedance of the circuit decreases and the current through the bulb increases. Therefore the bulb glows brighter.

Question 070 MCQ

QUESTION

A thin flexible wire of length L is connected to two adjacent fixed points and carries a current I in the clockwise direction, as shown in the figure. When the system is put in a uniform magnetic field of strength B going into the plane of the paper, the wire takes the shape of a circle. The tension in the wire is

A IBL

B $$\frac{IBL}{\pi}$$

C $$\frac{IBL}{2\pi}$$

D $$\frac{IBL}{4\pi}$$

CORRECT OPTION

C $$\frac{IBL}{2\pi}$$

SOURCE

Physics • magnetism

EXPLANATION

Consider an small element AB of length dl of the circle of radius R subtending an angle

$$\theta$$

at the centre O.

If T is the tension in the wire, then force towards the centre will be equal to

$$2T\sin\left(\frac{\theta}{2}\right)$$

which is balanced by outward magnetic force on the current carrying element

$$(= IdlB)$$

$$2T\sin\left(\frac{\theta}{2}\right) = IdlB$$

For small angle

$$\theta$$

,

$$\sin\frac{\theta}{2} \approx \frac{\theta}{2}$$

or,

$$T = \frac{IBdl}{\theta} = IBR$$

$$\because$$ $$\theta = \frac{dl}{R}$$

$$= \frac{IBL}{2\pi}$$

$$\because$$ $$R = \frac{L}{2\pi}$$

Question 071 MCQ

QUESTION

A thin uniform annular disc $see figure$ of mass M has outer radius 4R and inner radius 3R. The work required to take a unit mass from point P on its axis to infinity is

A

$$\frac{2GM}{7R}(4\sqrt{2}-5)$$

B

$$-\frac{2GM}{7R}(4\sqrt{2}-5)$$

C

$$\frac{GM}{4R}$$

D

$$\frac{2GM}{5R}(\sqrt{2}-1)$$

CORRECT OPTION

A

$$\frac{2GM}{7R}(4\sqrt{2}-5)$$

SOURCE

Physics • gravitation

EXPLANATION

We need to find gravitational potential energy of a unit mass placed at the point P.

The surface mass density of the annular disc is

$$\sigma = M/(16\pi R^2 - 9\pi R^2) = M/(7\pi R^2)$$

.

Consider a small ring of radius r and thickness dr. The mass of the ring is dm = 2

$$\pi$$

r

$$\sigma$$

dr. As distance of any point of the ring from P is same, the potential at P due to the ring is

$$V_P = -\frac{G(2\pi r\sigma dr)}{\sqrt{16R^2 + r^2}}$$

Integrate from r = 3R to r = 4R to get the potential energy of the unit mass placed at P

$$V_P = \int_{3R}^{4R} -\frac{GdM}{\sqrt{(4R)^2 + (r)^2}} = -\frac{GM2\pi}{7\pi R^2}\int_{3R}^{4R}\frac{rdr}{\sqrt{16R^2 + r^2}}$$

Solving, we get

$$V_P = -\frac{GM2\pi}{7\pi R^2}\left[\sqrt{16R^2 + r^2}\right]_{3R}^{4R} = -\frac{2GM}{7R}\left(4\sqrt{2} - 5\right)$$

Work done in moving a unit mass from P to

$$\infty$$

= V

$$\infty$$

$$-$$

V_P

$$= 0 - \left(\frac{-2GM}{7R} \left(4\sqrt{2} - 5 \right) \right) = \frac{2GM}{7R} \left(4\sqrt{2} - 5 \right)$$

Question 072 MCQ

QUESTION

Consider a thin square sheet of side L and thickness, made of a material of resistivity

$$\rho$$

. The resistance between two opposite faces, shown by the shaded areas in the figure is

A. directly proportional to L.

B. directly proportional to t.

C. independent to L.

D. independent of t.

CORRECT OPTION

C. independent to L.

SOURCE

Physics • current-electricity

EXPLANATION

The sheet is of square shape with thickness t, width

$$\omega$$

= L, and length l = L. The resistance between the two opposite faces is given by

$$R = \frac{\rho l}{A} = \frac{\rho l}{wt} = \frac{\rho L}{Lt} = \frac{\rho}{t}$$

.

Question 073 MCQ

QUESTION

One mole of an ideal gas in initial state A undergoes a cyclic process ABCA, as shown in the figure. Its pressure at A is P_0. Choose the correct option s from the following:

A. Internal energies at A and B are the same.

B. Work done by the gas in process AB is P_0V_0 ln4.

C. Pressure at C is $P_0/4$.

D. Temperature at C is $T_0/4$.

CORRECT OPTION

A Internal energies at A and B are the same.

SOURCE

Physics • heat-and-thermodynamics

EXPLANATION

The internal energy of one mole of an ideal gas at temperature T is given by U = 3RT/2. The process AB is isothermal i.e., $T_A = T_B = T_0$, which makes $U_A = U_B$.

The work done in the isothermal process AB is

$W_{AB} = nRT \ln(V_B/V_A)$

$= nRT_0 \ln(4V_0/V_0) = p_0V_0 \ln 4$.

Information regarding p and T at C can not be obtained from the given graph. Unless it is mentioned that line BC passes through origin or not.

Hence, the correct options are a and b.

Note:

If we assume that the line BC pass through the origin. In the process BC, the slope V/T is constant. Thus, BC is an isobaric process $asV/T = nR/p, bytheidealgasequation$. Thus,

$p_C = p_B = RT_B / V_B$

$= RT_0/(4V_0) = (RT_0/V_0)4 = p_0/4$.

Apply the ideal gas equation for the states A and C to get

$$T_C = \frac{p_C V_C}{p_0 V_0} T_0 = \frac{(p_0/4)V_0}{p_0 V_0} T_0 = T_0/4$$

.

Question 074 MCQ

QUESTION

A ray OP of monochromatic light is incident on the face AB of prism ABCD near vertex B at an incident angle of 60

$$\circ$$

$see figure$. If the refractive index of the material of the prism is

$$\sqrt{3}$$

, which of the following is are correct?

A The ray gets totally internally reflected at face CD.

B The ray comes out through face AD.

C The angle between the incident ray and the emergent ray is 90

$$\circ$$

.

D The angle between the incident ray and the emergent ray is 120

$$\circ$$

.

CORRECT OPTION

A The ray gets totally internally reflected at face CD.

SOURCE

EXPLANATION

Consider the refraction at the face AB. Snell's law,

$$\sin i / \sin r = \sin 60^\circ / \sin r = \sqrt{3}$$

,

gives

$$r = 30^\circ$$

.

The geometry in BCQP gives

$$\angle BPQ = 30^\circ + 90^\circ = 120^\circ$$
$$\angle CQP = 360^\circ - (135^\circ + 60^\circ + 120^\circ) = 45^\circ$$

.

Thus, the angle of incidence at Q is

$$i_1 = 45^\circ$$

. The critical angle for prism to air refraction is given by

$$\sin i_c = 1/\sqrt{3}$$

. Since

$$\sin i_1 = 1/\sqrt{2} > 1\sqrt{3}$$

, we get

$$i_1 > i_c$$

i.e., the angle of incidence is greater than the critical angle. Thus, the ray undergoes total internal reflection at Q. The laws of reflection gives

$$r_1 = i_1 = 45^\circ$$

. In triangle QRD,

$$\angle QRD = 60^{\circ}$$

and hence the angle of incidence at R is

$$i_2 = 30^{\circ}$$

.

Applying Snell's law at face AD, we get

$$\sqrt{3} \times \sin 30^{\circ} = 1 \times \sin e$$

or,

$$\sqrt{3} \times \frac{1}{2} = \sin e$$

$$\sin e = \frac{\sqrt{3}}{2}$$

or,

$$e = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^{\circ}$$

From figure,

The angle between the incident ray and the emergent ray is 90

$$\circ$$

.

Hence, option c is correct and option d is incorrect.

Note : Angle between incident and emergent rays is the same as the angle between the two faces = 90

$$\circ$$

.

Question 075 MCQ

QUESTION

In the graph below, the resistance R of a superconductor is shown as a friction of its temperature T for two different magnetic fields B_1 $solidline$ and B_2 $dashedline$. If B_2 is larger than B_1 which of the following graphs shows the correct variation of R with T in these fields?

A

B

C

D

CORRECT OPTION

A

SOURCE

Physics • magnetism

EXPLANATION

From the given figure, $T_c B$ is a monotonically decreasing function of B. Thus, $B_2 > B_1$ implies $T_c(B_2) < T_c(B_1)$. Hence resistance with B_2 will become zero at lower temperature in comparison to B_1.

Question 076 MCQ

QUESTION

A superconductor has $T_c(0)$ = 100 K. When a magnetic field of 7.5 T is applied, its T_c decreases to 75 K. For this material, one can definitely say that when

A. B = 5 T, $T_c(B)$ = 80 K

B. B = 5 T, 75 K < $T_c(B)$ < 100 K

C. B = 10 T, 75 K < T_c < 100 K

D. B = 10, T_c = 70 K

CORRECT OPTION

B. B = 5 T, 75 K < $T_c(B)$ < 100 K

SOURCE

Physics • magnetism

EXPLANATION

It is given that $T_c(0)$ = 100 K and $T_c(7.5)$ = 75 K. Since $T_c(B)$ is a monotonically decreasing function of B, $T_c(5) < T_c(0)$ and $T_c(5) > T_c(7.5)$. Thus, 75 K < $T_c(5)$ < 100 K.

Question 077 MCQ

QUESTION

If the total energy of the particle is E, it will perform periodic motion only if

A. $E < 0$

B. $E > 0$

C. $V_0 > E > 0$

D. $E > V_0$

CORRECT OPTION

C. $V_0 > E > 0$

SOURCE

Physics • simple-harmonic-motion

EXPLANATION

The kinetic energy of the particle cannot be negative. The total energy E is the sum of the kinetic energy Kx and potential energy Vx i.e.,

E = Kx + Vx 1

From the given figure, Vx

$\geq$

0 for all x. If E

$\leq$

0 for some x then kinetic energy Kx = E

$-$

Vx

$\leq$

0 for those x and hence the motion of the particle is not allowed for those x. On the other hand, if E

$\geq$

V_0 then Kx = E

$-$

Vx

$\geq$

0 for all x and particle is allowed to move for all x, including infinity $inthiscasetheparticlewillescapetoinfinity$. Thus, for the particle to have a periodic motion, V_0 > E > 0. In this case, particle is allowed at points where

Kx = E

$-$

Vx = E

$-$

α

x^4

$\geq$

0 i.e.,

| x |

$\leq$

$E/$$\alpha$$^{1/4}$.

Question 078 MCQ

QUESTION

For periodic motion of small amplitude A, the time period T of this particle is proportional to

A $A\sqrt{m/\alpha}$

B $\frac{1}{A}\sqrt{m/\alpha}$

C $A\sqrt{\alpha/m}$

D $\frac{1}{A}\sqrt{\alpha/m}$

CORRECT OPTION

B $\frac{1}{A}\sqrt{m/\alpha}$

SOURCE

Physics • simple-harmonic-motion

EXPLANATION

As

$$V = ax^4$$

$$[\alpha] = \frac{[V]}{[x^4]} = \frac{[ML^2T^{-2}]}{[L^4]} = [ML^{-2}T^{-2}]$$

By method of dimensions,

$$\left[\frac{1}{A}\sqrt{\frac{m}{\alpha}}\right] = \frac{[M]^{1/2}}{[L][ML^{-2}T^{-2}]^{1/2}} = [T]$$

Only option b has the dimensions of time.

Question 079 MCQ

QUESTION

The acceleration of this particle for

$$|x| > X_0$$

is

A. proportional to V_0.

B. proportional to V_0/mX_0.

proportional to

C

$$\sqrt{V_0/mX_0}$$

.

D zero.

CORRECT OPTION

D zero.

SOURCE

Physics • simple-harmonic-motion

EXPLANATION

For

$$|x| > X_0$$

,

$$V = V_0$$

= constant

Force

$$= -\frac{dV}{dx} = 0$$

Hence, acceleration of the particle is zero for

$$|x| > X_0$$

.

Question 080 Numerical

QUESTION

A stationary source is emitting sound at a fixed frequency f_0, which is reflected by two cars approaching the source. The difference between the frequencies of sound reflected from the cars is 1.2% of f_0. What is the difference in the speeds of the cars $inkmperhour$ to the nearest integer? The cars are moving at constant speeds much smaller than the speed of sound which is 330 ms

–

1.

SOURCE

Physics • waves

EXPLANATION

Let car B be the observer $movingtowardsS$.

The frequency observed is

$$f_1 = f_0 \left(\frac{c+v}{c} \right)$$

When sound gets reflected, the frequency observed by source S is

$$f_2 = f_1 \left(\frac{c}{c-v} \right)$$

where v is the speed of car and c is the speed of sound. Therefore,

$$f_2 = f_0 \left(\frac{c+v}{c-v} \right)$$

Now,

$$df_x = f_0\left[\frac{(c-v)dv-(c+v)(-dv)}{(c-v)^2}\right]$$

$$= \frac{2f_0 c\,dv}{(c-v)^2}$$

That is,

$$\frac{2f_0 c\,dv}{(c-v)^2} = \left(\frac{1.2}{100}\right)f_0$$

$$\Rightarrow dv = \frac{1.2}{100}\times\frac{(c-v)^2}{2c}$$

Since, v << c, we get c

$$-$$

v

$$\simeq$$

c.

Therefore,

$$dv = \frac{1.2}{100}\times\frac{c}{2} = 1.98$$

m/s

$$= 1.98\times\frac{18}{5}$$

km/h = 7 km/h.

Question 081 Numerical

QUESTION

The focal length of a thin biconvex lens is 20 cm. When an object is moved from a distance of 25 cm in front of it to 50 cm, the magnification of its image changes from m_{25} to m_{50}. The ratio

$$\frac{m_{25}}{m_{50}}$$

is __________.

SOURCE

Physics • geometrical-optics

EXPLANATION

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

or,

$$\frac{u}{v}-1=\frac{u}{f}$$

or,

$$\frac{u}{v}=\left(\frac{u+f}{f}\right)$$

$$\therefore$$

$$m=\frac{v}{u}=\left(\frac{f}{u+f}\right)$$

$$\frac{m_{25}}{m_{50}}=\frac{\left(\frac{20}{-25+20}\right)}{\left(\frac{20}{-50+20}\right)}=6$$

Question 082 Numerical

QUESTION

An

$$\alpha$$

-particle and a proton are accelerated from the rest by a potential difference of 100 V. After this, their de Broglie wavelengths are

$$\lambda$$
$$\alpha$$

and

$$\lambda$$

p, respectively. The ratio

$$\frac{\lambda_p}{\lambda_\alpha}$$

, to the nearest integer, is _____________.

SOURCE

Physics • dual-nature-of-radiation

EXPLANATION

The de Broglie wavelength of a particle with momentum p is given by

$$\lambda$$

= h/p.

The momentum and kinetic energy of a particle of mass m are related by

$$p = \sqrt{2mK}$$

.

The kinetic energy of a charge q, accelerated through potential V, is given by K = qV. Thus,

$$\lambda = h/\sqrt{2mK} = h/\sqrt{2mqV}$$

,

which gives

$$\frac{\lambda_p}{\lambda_\alpha} = \sqrt{\frac{2m_\alpha q_\alpha V}{2m_p q_p V}} = \sqrt{\frac{2\,.\,4u\,.\,2e\,.\,100}{2\,.\,1u\,.\,1e\,.\,100}}$$

$$= \sqrt{8} = 2.8 \approx 3$$

Question 083 Numerical

QUESTION

When two identical batteries of internal resistance 1

$$\Omega$$

each are connected in series across a resistor R, the rate of heat produced in R is J_1. When the same batteries are connected in parallel across R, the rate is J_2. If J_1 = 2.25 J_2, then the value of R in

$$\Omega$$

is __________.

SOURCE

Physics • current-electricity

EXPLANATION

In series : When the batteries are connected in series, we have

$$J_1 = \left(\frac{2E}{R+2}\right)^2 R$$

In parallel : When the batteries are connected in parallel, we have

$$J_2 = \left(\frac{E}{R+(1/2)}\right)^2 R$$

It is given that,

$$\frac{J_1}{J_2} = 2.25$$

$$\Rightarrow \frac{4}{(R+2)^2} \times \frac{(2R+1)^2}{4} = 2.25 \Rightarrow \frac{2R+1}{R+2} = 1.5$$

$$\Rightarrow 2R+1 = 1.5R+3 \Rightarrow 0.5R = 2$$

Therefore,

$$R = 4\Omega$$

.

Question 084 Numerical

QUESTION

When two progressive waves

$$y_1 = 4\sin(2x - 6t)$$

and

$$y_2 = 3\sin\left(2x - 6t - \frac{\pi}{2}\right)$$

are superimposed, the amplitude of the resultant wave is ___________.

SOURCE

Physics • waves

EXPLANATION

Here,

$$y_1 = 4\sin(2x - 6t)$$

$$y_2 = 3\sin\left(2x - 6t - \frac{\pi}{2}\right)$$

The phase difference between two waves is

$$\phi = \frac{\pi}{2}$$

The amplitude of the resultant wave is

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

$$= \sqrt{4^2 + 3^2 + 2 \times 4 \times 3 \times \cos\frac{\pi}{2}} = 5$$