1 Analytical derivation of Lensing rates

Note: I will interchangeably use term like observable events and detectable events.

Define the all the parameters involved.

- Source parameters: $\theta \in \{m_1, m_2, D_l, \iota, \phi, \psi, ra, dec\}$
- Lens parameters: $\theta_L \in \{\sigma$(velocity-dispersion), q(axis-ratio), ψ(axis-rotation), γ(spectral-index), $[\gamma_1, \gamma_2]$(external-shear)$\}$
- z_L : red-shift of the galaxy lens
- image param: $\{\beta$(source position), μ(magnification), dt(time-delay)$\}$.

Given $dN^L_{obs}(z_s)$ is the number of lensed GW detectable events from sources at red-shift z_s in a spherical shell of thickness dz_s, then, let rate of lensing (number of lensed events happening per unit time) is given by,

$$\mathcal{R}_L = \int_{z_{min}}^{z_{max}} \frac{dN^L_{obs}(z_s)}{dt_{obs}} \tag{1a}$$

$$= \int_{z_{min}}^{z_{max}} \frac{dN^L_{obs}(z_s)}{dt\ dV_c} \frac{dV_c}{dz_s} dz_s \tag{1b}$$

$\frac{dN^L_{obs}(z_s)}{dt\ dV_c}$ is the merger rate density at detector-frame, and $\frac{dV_c}{dz_s}$ is the deferential co-moving volume at red-shift z_s. After taking care of time-dilation, the expression looks,

$$\mathcal{R}_L = \int_{z_{min}}^{z_{max}} \frac{dN^L_{obs}(z_s)}{dt\ dV_c} \frac{1}{1+z_s} \frac{dV_c}{dz_s} dz_s \tag{2a}$$

$$= \int_{z_{min}}^{z_{max}} R^L_{obs}(z_s) dz_s \tag{2b}$$

Observed rate of lensed events at source red-shift z_s (source-frame) is given by $R^L_{obs}(z_s) = \frac{dN^L_{obs}(z_s)}{dt\ dV_c} \frac{1}{1+z_s} \frac{dV_c}{dz_s}$. And, let $R(z_s)$ be the rate of the merger of unlensed events (source frame), regardless of whether it is detectable or not.

$$\mathcal{R}_L = \int_{z_{min}}^{z_{max}} R^L_{obs}(z_s) dz_s \tag{3a}$$

$$= \int_{z_{min}}^{z_{max}} R(z_s) P(obs, SL|z_s) dz_s \tag{3b}$$

$P(obs, SL|z_s)$ is the probability of observing strong lensing event at red-shift z_s.

$$\mathcal{R}_L = \int_{z_{min}}^{z_{max}} R(z_s) P(obs|z_s, SL) P(SL|z_s) dz_s \tag{4a}$$

Probability of observing an event given that it is located at redshift z_s and it's strongly lensed: $P(obs|z_s, SL)$. Strong lensing probability with source at redshift z_s (optical depth): $P(SL|z_s)$. Now, using bayes theorem,

$$P(SL|z_s) = \frac{P(z_s|SL)P(SL)}{P(z_s)} \tag{5a}$$

$$P(z_s)P(SL|z_s) = P(z_s|SL)P(SL) \tag{5b}$$

$$\frac{R(z_s)}{N_1} P(SL|z_s) = P(z_s|SL)P(SL) \tag{5c}$$

Normalizing factor: $N_1 = \int_{z_{min}}^{z_{max}} R(z_s) dz_s$. Similarly, when lensing condition applied, let $N_2 = \int_{z_{min}}^{z_{max}} R(z_s) P(SL|z_s) dz_s$.

$$P(SL) = \int_{z_{min}}^{z_{max}} P(SL|z_s) P(z_s) dz_s \tag{6a}$$

$$= \int_{z_{min}}^{z_{max}} P(SL|z_s) \frac{R(z_s)}{N_1} dz_s \tag{6b}$$

$$= \frac{N_2}{N_1} \tag{6c}$$

Now putting together, equation 5c becomes,

$$\frac{R(z_s)}{N_1} P(SL|z_s) = P(z_s|SL) \frac{N_2}{N_1} \tag{7a}$$

$$R(z_s) P(SL|z_s) = N_2 P(z_s|SL) \tag{7b}$$

Replace the above result in the integrand of 4a. This also take cares of the normalizing factor. Note that $P(z_s|SL)$ is a normalised pdf of source red-shifts, z_s, conditioned on strong lensing.

$$\mathcal{R}_L = N_2 \int_{z_{min}}^{z_{max}} P(z_s|SL) P(obs|z_s, SL) dz_s \tag{8a}$$

$$\text{consider } \int \rightarrow \int_{zl} \int_{\beta} \int_{\theta} \int_{\theta_L}$$

$$\mathcal{R}_L = N_2 \int P(z_s|SL) P(obs|\theta, \theta_L, \beta, z_s, SL)$$
$$P(\beta|\theta_L, z_s, SL) P(\theta_L|z_s, SL) P(\theta) d\beta d\theta d\theta_L dz_s \tag{8b}$$

For $P(z_s|SL) = \frac{R(z_s)}{N_2} P(SL|z_s)$, from equation 7b, I have considered 'optical depth' ($\tau(z_s) = P(SL|z_s)$) is a function of z_s only. Otherwise, we need to considered the cross-section ($P(SL|z_s, \theta_L)$) which will be discussed in another section. Below shows how to get $P(SL|z_s)$, i.e. probability of strong lensing of source at z_s. $dN(z_l)$ is the number of galaxy lens at red-shift z_l (in dz_l).

$$P(SL|z_s) = \int_0^{z_s} \frac{P(SL|\theta_L, z_s)}{4\pi} dN(z_l) \tag{9a}$$

$$P(SL|z_s) = \int \frac{P(SL|z_s, z_l, \sigma, q)}{4\pi} \frac{dN(z_l)}{dz_l d\sigma dq} dz_l d\sigma dq \tag{9b}$$

$$P(SL|z_s) = \int \frac{P(SL|z_s, z_l, \sigma, q)}{4\pi} \frac{dN(z_l)}{dV_c d\sigma dq} \frac{dV_c}{dz_l} dz_l d\sigma dq \tag{9c}$$

writing the cross-section $P(SL|z_s, z_l, \sigma, q)$ as ϕ

$$P(SL|z_s) = \int \frac{\phi}{4\pi} \frac{dN(z_l)}{dV_c d\sigma dq} \frac{dV_c}{dz_l} dz_l d\sigma dq \tag{9d}$$

$$\tag{9e}$$

Consider SIS case (Ref Haris et al. 2018). Take ϕ as ϕ_{SIS}.

$$P(SL|z_s) = \int \frac{\phi_{SIS}}{4\pi} \frac{dN(z_l)}{dV_c d\sigma} \frac{dV_c}{dz_l} dz_l d\sigma \tag{10a}$$

$$P(SL|z_s) = \int \frac{\pi\theta_E^2}{4\pi} < n >_{\sigma\in P(\sigma)} P(\sigma) \frac{dV_c}{dz_l} dz_l d\sigma \tag{10b}$$

Cross-section of SIS lens is $\pi\theta_E^2$, where θ_E is the Einstein radius. Haris have considered number density of lens, $< n >_{\sigma\in P(\sigma)}$ and pdf of velocity dispersion, $P(\sigma)$ is independent of z_l. Take $< n >_{\sigma\in P(\sigma)} = n_o = 8 \times 10^{-3} h^3 Mpc^{-3}$.

$$P(SL|z_s) = \int \frac{\theta_E^2}{4} n_o P(\sigma) \frac{dV_c}{dz_l} dz_l d\sigma \tag{11a}$$

$$P(SL|z_s) = \int_0^{z_s} \Phi_{SIS}(z_l) dz_l \tag{11b}$$

$$\text{where, } \Phi_{SIS}(z_l) = \int \frac{\theta_E^2}{4} n_o P(\sigma) \frac{dV_c}{dz_l} d\sigma \ . \tag{11c}$$

$$\Phi_{SIS}(z_l) = \left\langle \frac{\theta_E^2}{4} n_o \frac{dV_c}{dz_l} \right\rangle_{\sigma\in P(\sigma)} \tag{11d}$$

$$\text{Note: } \theta_E \text{ and } \frac{dV_c}{dz_l} \text{ are functions of } z_l.$$

Consider SIE case with σ distribution dependent z_l. The expression for optical depth reads,

$$P(SL|z_s) = \int \frac{\phi_{SIE}}{4\pi} \frac{dN(z_l)}{dV_c d\sigma dq} \frac{dV_c}{dz_l} dz_l d\sigma dq \tag{12a}$$

$$P(SL|z_s) = \int \frac{\phi_{SIS}}{4\pi} \frac{\phi^{SIE}_{CUT}(q)}{\pi} < n >_{\sigma, q\in P(\sigma,q)} P(q|\sigma, z_l) P(\sigma|z_l) \frac{dV_c}{dz_l} dz_l d\sigma dq \tag{12b}$$

$$P(SL|z_s) = \int \frac{\phi_{SIS}}{4\pi} \frac{\phi^{SIE}_{CUT}(q)}{\pi} n_o P(q|\sigma, z_l) P(\sigma|z_l) \frac{dV_c}{dz_l} dz_l d\sigma dq \tag{12c}$$

$\frac{\phi^{SIE}_{CUT}(q)}{\pi}$ will be found through interpolation.

$$P(SL|z_s) = \int_0^{z_s} \Phi_{SIE}(z_l) dz_l \tag{12d}$$

$$\text{where, } \Phi_{SIE}(z_l) = \int \frac{\phi_{SIS}}{4\pi} \frac{\phi^{SIE}_{CUT}(q)}{\pi} n_o P(q|\sigma, z_l) P(\sigma|z_l) \frac{dV_c}{dz_l} d\sigma dq \;. \tag{12e}$$

$$\Phi_{SIE}(z_l) = \left\langle \frac{\phi_{SIS}}{4\pi} \frac{\phi^{SIE}_{CUT}(q)}{\pi} n_o \frac{dV_c}{dz_l} \right\rangle_{q\in P(q|\sigma,z_l), \sigma\in P(\sigma|z_l)} \tag{12f}$$

$$\text{If } \sigma \text{ is independent of } z_l\text{, then} \tag{12g}$$

$$\Phi_{SIE}(z_l) = \left\langle \frac{\phi_{SIS}}{4\pi} \frac{\phi^{SIE}_{CUT}(q)}{\pi} n_o \frac{dV_c}{dz_l} \right\rangle_{q\in P(q|\sigma), \sigma\in P(\sigma)} \tag{12h}$$

Final equation of the observed rate of lensed events is shown below. Note that, z_s sampled from it's prior distribution and then rejection sampled wrt to optical depth.

$$\mathcal{R}_L = N_2 \left\langle P(obs|\theta, \theta_L, \beta, z_s, SL) \right\rangle_{z_s\in P(z_s|SL), \theta\in P(\theta), \theta_L\in P(\theta_L|z_s,SL), \beta\in P(\beta|\theta_L, z_s, SL)} \tag{13a}$$

But in LeR implementation,

$$\mathcal{R}_L = N_2 \left\langle P(obs|\theta, \theta_L, \beta, z_s, SL) \right\rangle_{z_s\in P_1;\; \theta\in P_2;\; z_l,\sigma,q\in P_3;\; \beta,e_1,e_2,\gamma_1,\gamma_2,\gamma\in P_4} \tag{13b}$$

Prior distributions are given below.

$$P_1 = P(z_s|SL) \tag{14a}$$
$$P_2 = P(\theta) \tag{14b}$$
$$P_3 = P(z_l, \sigma, q|z_s, SL) \tag{14c}$$
$$P_4 = P(\beta, e_1, e_2, \gamma_1, \gamma_2, \gamma|\theta_L, z_s, SL) \tag{14d}$$

Where the sampling priors can be further simplify as follows,

$$P(z_s|SL) = P(SL|z_s)\, P(z_s) \tag{15a}$$
$$P(\theta_L|z_s, SL) = P(SL|z_s, \theta_L)\, P(\theta_L|z_s) \tag{15b}$$
$$P(\beta|z_s, \theta_L, SL) = P(SL|z_s, \theta_L, \beta)\, P(\beta|z_s, \theta_L) \tag{15c}$$

This allows z_s to sample from astrophysical prior, $P(z_s)$, and then later rejection sample wrt optical depth, $P(SL|z_s)$. Same is the case for θ_L (z_l, σ, q). Strong lensing condition is applied through rejection sampling wrt to $\phi^{SIE}_{CUT}(q)$ $(\propto \theta_E^2\, \phi^{SIE}_{CUT})$. For the source position, β, it is sample within the caustic and then check whether it has 2 or more images or not.

Order of sampling in LeR is listed below.

1. z_s from $\frac{R(z_s)}{N_1}$. And Apply rejection sample with optical depth, $P(SL|z_s)$. Other source parameters are sampled separately, $P(\theta)$.
2. z_l from $P(z_l|z_s)$.
3. σ together from $P(\sigma|z_l, SL)$.
4. q from $P(q|\sigma)$.
5. Calculation of Einstein radius and apply lensing condition to the sampled lens parameters, $P(SL|z_s, z_l, \sigma, q) \propto \theta_E^2\, \phi^{SIE}_{CUT}$.
6. Other lens parameters (e_1, e_2, γ_1, γ_2, , γ) are sampled independent to the SL condition, $P(e_1, e_2, \gamma_1, \gamma_2, \gamma)$. But, this will be rejection sampled later along with the image position.
7. Draw image position, β, from within the caustic boundary and solve lens equation. Accept it if it results in 2 or more images, otherwise resample β. Sometimes (once in 2-3 million), 2 or more images condition cannot be satisfied, so resample $e_1, e_2, \gamma_1, \gamma_2, \gamma$ again and repeat the process.