archived for easy access and future analysis. Additionally, all interpolators used in the process are preserved for future applications.
- Most cosmology-related calculations within the *ler* package are performed using the *astropy* library (Astropy Collaboration 2013). The default cosmological model is LambdaCDM ($H_0 = 70$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$); however, users have the flexibility to employ any cosmology available in *astropy*. All internal calculations in *ler* will then be based on the user-selected cosmological model.

Equations

Detectable Unlensed rates:

$$R_U = \int dz_s \frac{dV_c}{dz_s} \frac{R_m(z_s)}{1+z_s} \{\Theta[\rho(z_s, \theta) - \rho_{th}] P(\theta) d\theta\}$$

z_s: GW source redshift, $\frac{dV_c}{dz_s}$: Differential co-moving volume, $\frac{1}{1+z_s}$: Time dilation correction factor, $R_m(z_s)$: source frame merger rate density, θ: GW source parameters, P: probability distribution, ρ: SNR, ρ_{th}: SNR threshold, Θ: Heaviside function to select detectable GW events.

Detectable Lensed rates:

$$R_L = \int dz_s \frac{dV_c}{dz_s} \tau(z_s) \frac{R_m(z_s)}{1+z_s} \mathcal{O}_{images}(z_s, \theta, \mu_i, \Delta t_i, \rho_{th}) \\ P(\theta) P(\theta_L | \mathrm{SL}, z_s) P(\beta | \mathrm{SL}) d\theta d\beta d\theta_L dz_s$$

$\tau(z_s)$: Optical-depth of strong lensing, θ_L: lens parameters, β: source position, μ: image magnification, Δt: image time delay, $\mathcal{O}$: operator to select dectectable lensed GW events, i: index of images of a lensed event, SL: strong lensing condition.

Acknowledgements

The authors express their sincere appreciation for the significant contributions that have been instrumental in completing this research. Special thanks are extended to the academic advisors for their invaluable guidance and steadfast support. The collaborative efforts and enriching discussions with research colleagues significantly enhanced the study's quality. Acknowledgement is given to the Department of Physics, The Chinese University of Hong Kong, for the Postgraduate Studentship that facilitated this research. Further gratitude is extended to the Netherlands Organisation for Scientific Research (NWO) for their support. The authors also recognize the contributions of individuals who added empirical depth to this work. Appreciation is conveyed for the computational resources provided by the LIGO Laboratory, supported by National Science Foundation Grants No. PHY-0757058 and No. PHY-0823459.

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