- Simulated events and rate results, along with input configurations, are systematically archived for easy access and future analysis. Additionally, all interpolators used in the process are preserved for future applications.
- Most cosmology-related calculations within the *ler* package are performed using the *astropy* library (Astropy Collaboration 2013). The default cosmological model is LambdaCDM ($H_0 = 70$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$); however, users have the flexibility to employ any cosmology available in *astropy*. All internal calculations in *ler* will then be based on the user-selected cosmological model.

## Equations

**Detectable Unlensed rates:**

$$R_U = \int dz_s \frac{dV_c}{dz_s} \frac{R_m(z_s)}{1+z_s} \{\Theta[\rho(z_s,\theta) - \rho_{th}]P(\theta)d\theta\}$$

$z_s$: GW source redshift, $\frac{dV_c}{dz_s}$: Differential co-moving volume, $\frac{1}{1+z_s}$: Time dilation correction factor, $R_m(z_s)$: source frame merger rate density, $\theta$: GW source parameters, $P$: probability distribution, $\rho$: SNR, $\rho_{th}$: SNR threshold, $\Theta$: Heaviside function to select detectable GW events.

**Detectable Lensed rates:**

$$R_L = \int dz_s \frac{dV_c}{dz_s} \tau(z_s) \frac{R_m(z_s)}{1+z_s} \mathcal{O}_{images}(z_s, \theta, \mu_i, \Delta t_i, \rho_{th}) \\ P(\theta)P(\theta_L|\mathsf{SL}, z_s)P(\beta|\mathsf{SL})d\theta d\beta d\theta_L dz_s$$

$\tau(z_s)$: Optical-depth of strong lensing, $\theta_L$: lens parameters, $\beta$: source position, $\mu$: image magnification, $\Delta t$: image time delay, $\mathcal{O}$: operator to select dectectable lensed GW events, $i$: index of images of a lensed event, SL: strong lensing condition.

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## References

[]ref-VIRGO2015 Acernese, F, M Agathos, K Agatsuma, D Aisa, N Allemandou, A Allocca, J Amarni, et al. 2014. "Advanced Virgo: A Second-Generation Interferometric Gravitational Wave Detector." *Classical and Quantum Gravity* 32 (2): 024001. https://doi.org/10.1088/0264-9381/32/2/024001.

Akutsu, T, M Ando, K Arai, Y Arai, S Araki, A Araya, N Aritomi, et al. 2020. "Overview of KAGRA: Detector design and construction history." *Progress of Theoretical and Experimental Physics* 2021 (5): 05A101. https://doi.org/10.1093/ptep/ptaa125.

Ashton, Gregory, Moritz Hübner, Paul D. Lasky, Colm Talbot, Kendall Ackley, Sylvia Biscoveanu, Qi Chu, et al. 2019. "Bilby: A User-Friendly Bayesian Inference Library for Gravitational-Wave Astronomy." *The Astrophysical Journal Supplement Series* 241 (2): 27. https://doi.org/10.3847/1538-4365/ab06fc.

Astropy Collaboration. 2013. "Astropy: A community Python package for astronomy." *Astronomy and Astrophysics* 558 (October). https://doi.org/10.1051/0004-6361/201322068.

Birrer, Simon, Anowar J. Shajib, Daniel Gilman, Aymeric Galan, Jelle Aalbers, Martin Millon, Robert Morgan, et al. 2021. "Lenstronomy II: A Gravitational Lensing Software Ecosystem." *Journal of Open Source Software* 6 (62): 3283. https://doi.org/10.21105/joss.03283.

gwcosmo Contributors. 2022. "gwcosmo: A Python package for gravitational-wave cosmology." *GitHub Repository*. GitHub. https://github.com/gwcosmo/gwcosmo.

Janquart, J, M Wright, S Goyal, J C L Chan, A Ganguly, Á Garrón, D Keitel, et al. 2023. "Follow-up analyses to the O3 LIGO–Virgo–KAGRA lensing searches." *Monthly Notices of the Royal Astronomical Society* 526 (3): 3832–60. https://doi.org/10.1093/mnras/stad2909.

Lam, Stan, Stéphane Pitrou, and Mark Seibert. 2022. "Numba: A High Performance Python Compiler." *Numba Documentation*. Anaconda, Inc. https://numba.pydata.org/.

NumPy Community. 2022. "NumPy: A Fundamental Package for Scientific Computing with Python." *NumPy Website*. NumPy. https://numpy.org/.

The LIGO Scientific Collaboration, J Aasi, B P Abbott, R Abbott, T Abbott, M R Abernathy, K Ackley, et al. 2015. "Advanced LIGO." *Classical and Quantum Gravity* 32 (7): 074001. https://doi.org/10.1088/0264-9381/32/7/074001.

Virtanen, Pauli, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, et al. 2020. "SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python." *Nature Methods*. SciPy. https://www.scipy.org/.

Wempe, Ewoud, Léon V. E. Koopmans, A. Renske A. C. Wierda, Otto Akseli Hannuksela, Alberto Agnello, Cyril Bonvin, Bendetta Bucciarelli, et al. 2022. "A Lensing Multi-Messenger Channel: Combining LIGO-Virgo-Kagra Lensed Gravitational-Wave Measurements with Euclid Observations." https://arxiv.org/abs/2204.08732.

Wierda, A. Renske A. C., Ewoud Wempe, Otto A. Hannuksela, Léon V. E. Koopmans, Alberto Agnello, Cyril Bonvin, Bendetta Bucciarelli, et al. 2021. "Beyond the Detector Horizon: Forecasting Gravitational-Wave Strong Lensing." *The Astrophysical Journal* 921 (1): 154. https://doi.org/10.3847/1538-4357/ac1bb4.