– Simulated events and rate results, along with input configurations, are systematically archived for easy access and future analysis. Additionally, all interpolators used in the process are preserved for future applications.

## Equations

**Detectable Unlensed rates:**

$$R_U = \int dz_s \frac{dV_c}{dz_s} \frac{R_m(z_s)}{1+z_s} \{\Theta[\rho(z_s,\theta) - \rho_{th}]P(\theta)d\theta\}$$

$z_s$: GW source redshift, $\frac{dV_c}{dz_s}$: Differential co-moving volume, $\frac{1}{1+z_s}$: Time dilation correction factor, $R_m(z_s)$: source frame merger rate density, $\theta$: GW source parameters, $P$: probability distribution, $\rho$: SNR, $\rho_{th}$: SNR threshold, $\Theta$: Heaviside function to select detectable GW events.

**Detectable Lensed rates:**

$$R_L = \int dz_s \frac{dV_c}{dz_s} \tau(z_s) \frac{R_m(z_s)}{1+z_s} \mathcal{O}_{images}(z_s,\theta,\mu_i,\Delta t_i,\rho_{th})$$
$$P(\theta)P(\theta_L|\mathrm{SL},z_s)P(\beta|\mathrm{SL})d\theta d\beta d\theta_L dz_s$$

$\tau(z_s)$: Optical-depth of strong lensing, $\theta_L$: lens parameters, $\beta$: source position, $\mu$: image magnification, $\Delta t$: image time delay, $\mathcal{O}$: operator to select dectectable lensed GW events, $i$: index of images of a lensed event, SL: strong lensing condition.

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