archived for easy access and future analysis. Additionally, all interpolators used in the process are preserved for future applications.
- Most cosmology-related calculations within the *ler* package are performed using the *astropy* library (Astropy Collaboration 2013). The default cosmological model is LambdaCDM ($H_0 = 70$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$); however, users have the flexibility to employ any cosmology available in *astropy*. All internal calculations in *ler* will then be based on the user-selected cosmological model.

Equations

Detectable Unlensed rates:

$$R_U = \int dz_s \frac{dV_c}{dz_s} \frac{R_m(z_s)}{1+z_s} \{\Theta[\rho(z_s,\theta) - \rho_{th}] P(\theta) d\theta\}$$

z_s: GW source redshift, $\frac{dV_c}{dz_s}$: Differential co-moving volume, $\frac{1}{1+z_s}$: Time dilation correction factor, $R_m(z_s)$: source frame merger rate density, θ: GW source parameters, P: probability distribution, ρ: SNR, ρ_{th}: SNR threshold, Θ: Heaviside function to select detectable GW events.

Detectable Lensed rates:

$$R_L = \int dz_s \frac{dV_c}{dz_s} \tau(z_s) \frac{R_m(z_s)}{1+z_s} \mathcal{O}_{images}(z_s, \theta, \mu_i, \Delta t_i, \rho_{th}) \\ P(\theta) P(\theta_L|\mathsf{SL}, z_s) P(\beta|\mathsf{SL}) d\theta d\beta d\theta_L dz_s$$

$\tau(z_s)$: Optical-depth of strong lensing, θ_L: lens parameters, β: source position, μ: image magnification, Δt: image time delay, $\mathcal{O}$: operator to select dectectable lensed GW events, i: index of images of a lensed event, SL: strong lensing condition.

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