UVA CS 4501: Machine Learning

Lecture 11: Probability Review

Dr. Yanjun Qi

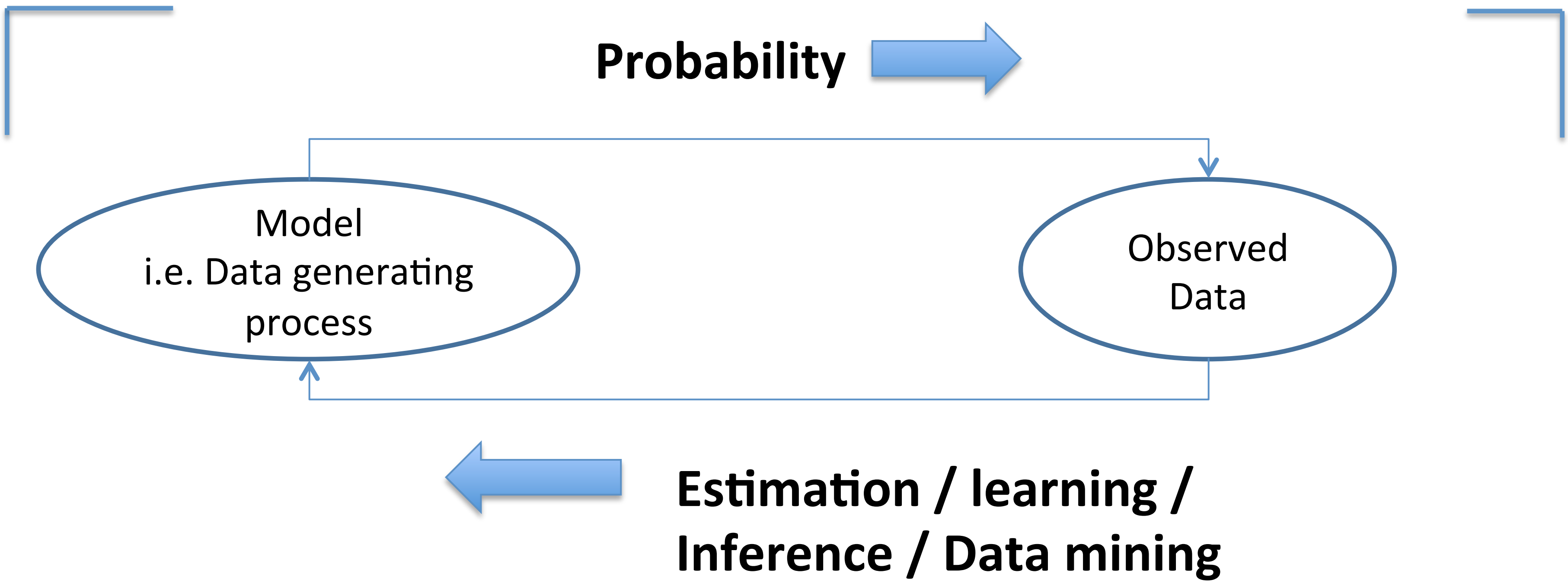
University of Virginia

Department of
Computer Science

Today : Probability Review

- The big picture
- Events and Event spaces
- Random variables
- Joint probability, Marginalization, conditioning, chain rule, Bayes Rule, law of total probability, etc.
- Structural properties, e.g., Independence, conditional independence
- Maximum Likelihood Estimation

The Big Picture



Probability

- Counting
- Basics of probability
- Conditional probability
- Random variables
- Discrete and continuous distributions
- Expectation and variance
- Tail bounds and central limit theorem
- ……

Statistics

- Maximum likelihood estimation
- Bayesian estimation
- Hypothesis testing
- Linear regression
- [Machine learning]
- ……

Probability as frequency

- Consider the following questions:
  - 1. What is the probability that when I flip a coin it is "heads"?

    We can count ➔ ~1/2
  - 2. why ?
  - 3. What is the probability of Blue Ridge Mountains to have an erupting volcano in the near future ?

    ➔ could not count

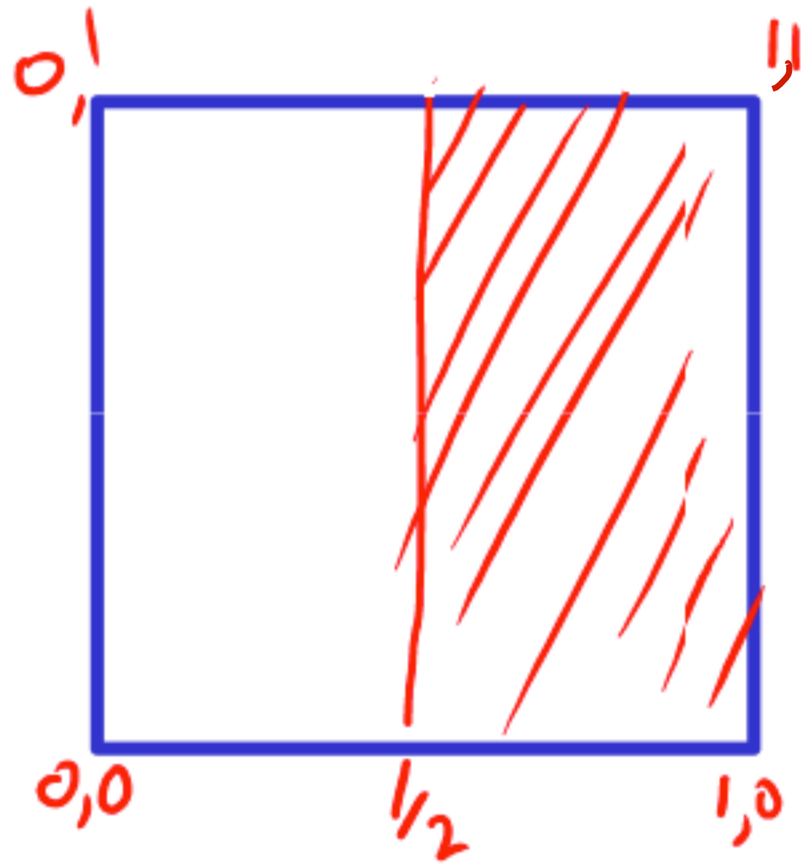
Message: *The frequentist view is very useful, but it seems that we can also use domain knowledge to come up with probabilities.*

Adapt from Prof. Nando de Freitas's review slides

Probability as a measure of uncertainty

- Imagine we are throwing darts at a wall of size 1x1 and that all darts are guaranteed to fall within this 1x1 wall.

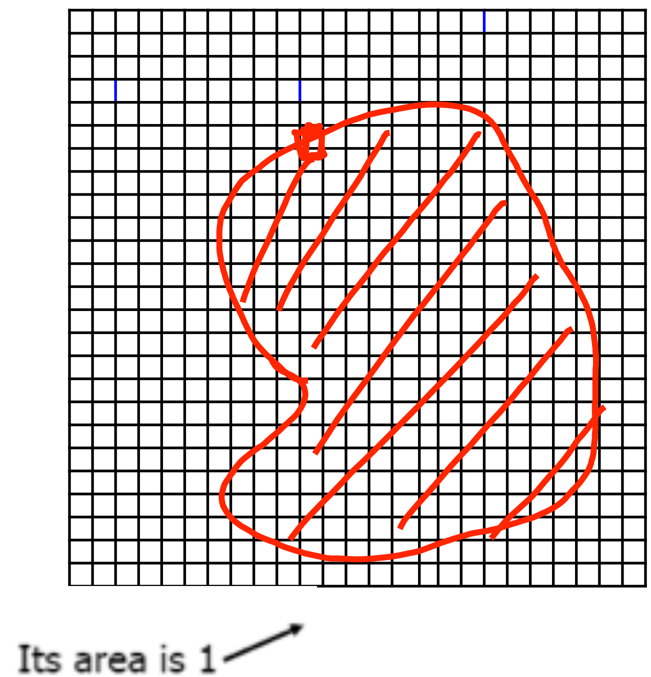
- What is the probability that a dart will hit the shaded area?



Adapt from Prof. Nando de Freitas's review slides

Probability as a measure of uncertainty

- *Probability is a measure of certainty of an event taking place.*
- *i.e. in the example, we were measuring the chances of hitting the shaded area.*



$$prob = \frac{\#RedBoxes}{\#Boxes}$$

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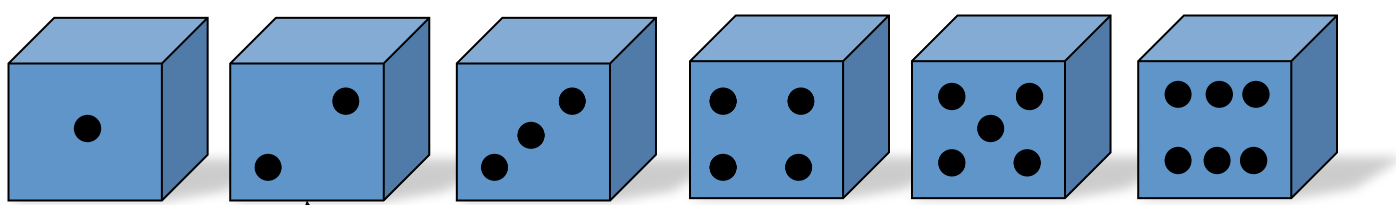
Probability

***Probability** is the formal study of the laws of chance. Probability allows us to **manage uncertainty**.*

*The **sample space** is the set of all **outcomes**. For example, for a die we have 6 outcomes:*

$$O_{die} = \{1,2,3,4,5,6\}$$

O:



Elementary Event “Throw 2”

The elements of O are called *elementary events.*

Probability

- *Probability allows us to measure many **events**.*
- ***The events are subsets of the sample space O**. For example, for a die we may consider the following events: e.g.,*

GREATER = {5, 6}

EVEN = {2, 4, 6}

- ***Assign probabilities to these events: e.g.,***

$$P(\text{EVEN}) = 1/2$$

Adapt from Prof. Nando de Freitas's review slides

Sample space and Events

- O : Sample Space,
  - result of an experiment / set of all outcomes
  - If you toss a coin twice O = {HH,HT,TH,TT}
- Event: a subset of O
  - First toss is head = {HH,HT}
- S: event space, a set of events:
  - Contains the empty event and O

Axioms for Probability

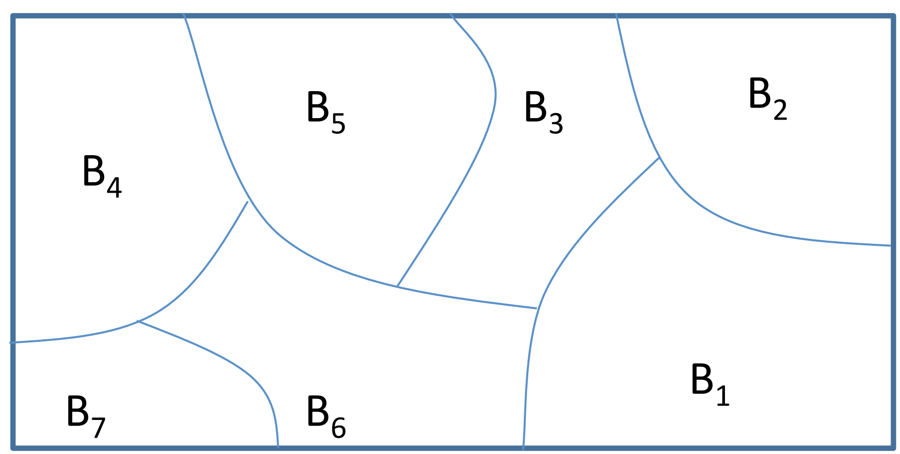
sample space

event space

- Defined over (O,S) s.t.
  - 1 >= P(a) >= 0 for all a in S
  - P(O) = 1
  - If A, B are disjoint, then
    - P(A U B) = p(A) + p(B)

Axioms for Probability

- $P(O) = \sum P(B_i)$

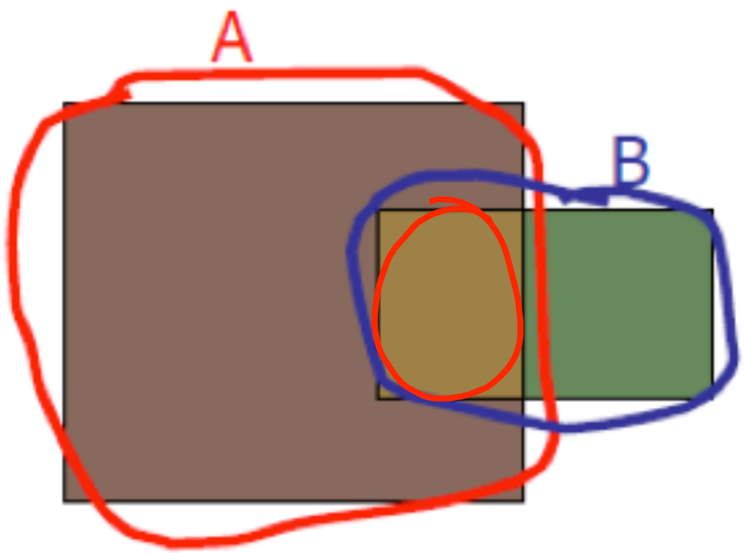


OR operation for Probability

- We can deduce other axioms from the above ones
  - Ex: P(A U B) for non-disjoint events

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

P(Union of A set and B set)



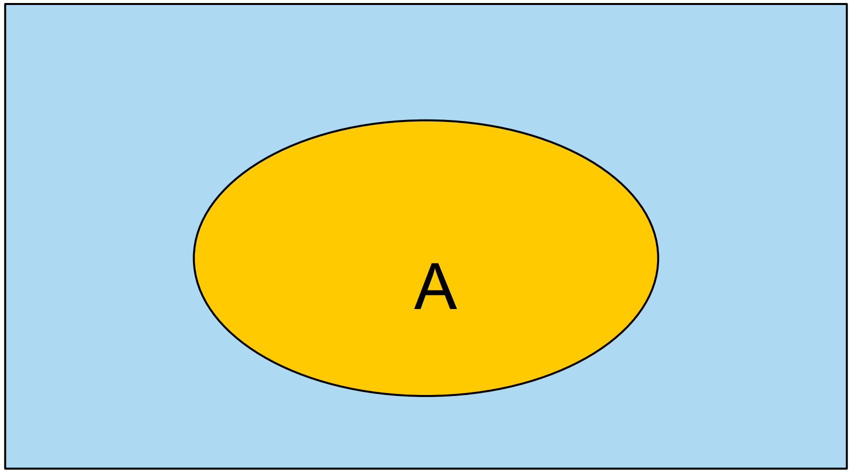
Theorems from the Axioms

- 0 <= P(A) <= 1,
- P(A or B) = P(A) + P(B) - P(A and B)

From these we can prove:

P(n~~o~~t A) = P(~A) = 1-P(A)

Complement



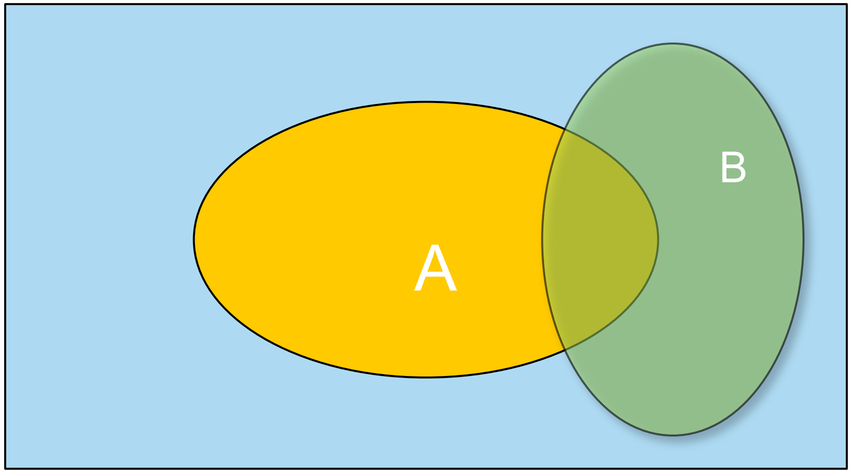
Another important theorem

- 0 <= P(A) <= 1,
- P(A or B) = P(A) + P(B) - P(A and B)

From these we can prove:

$$P(A) = P(A \wedge B) + P(A \wedge \sim B)$$

P(Intersection of A and B)



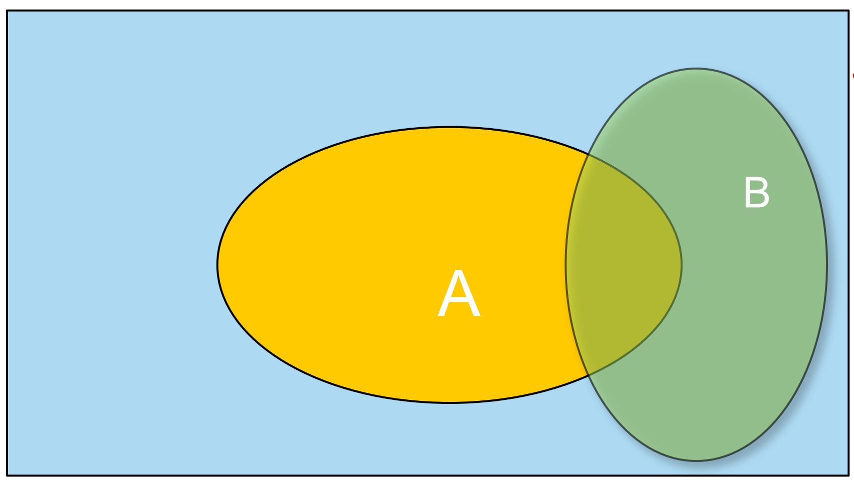
Another important theorem

- $0 <= P(A) <= 1$,
- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

From these we can prove:

$$P(A) = P(A \wedge B) + P(A \wedge \sim B)$$

$$P(A) = P(A \cap \Omega) = P(A \cap (B \cup \neg B)) = P((A \cap B) \cup (A \cap \neg B)) = P(A \cap B) + P(A \cap \neg B)$$



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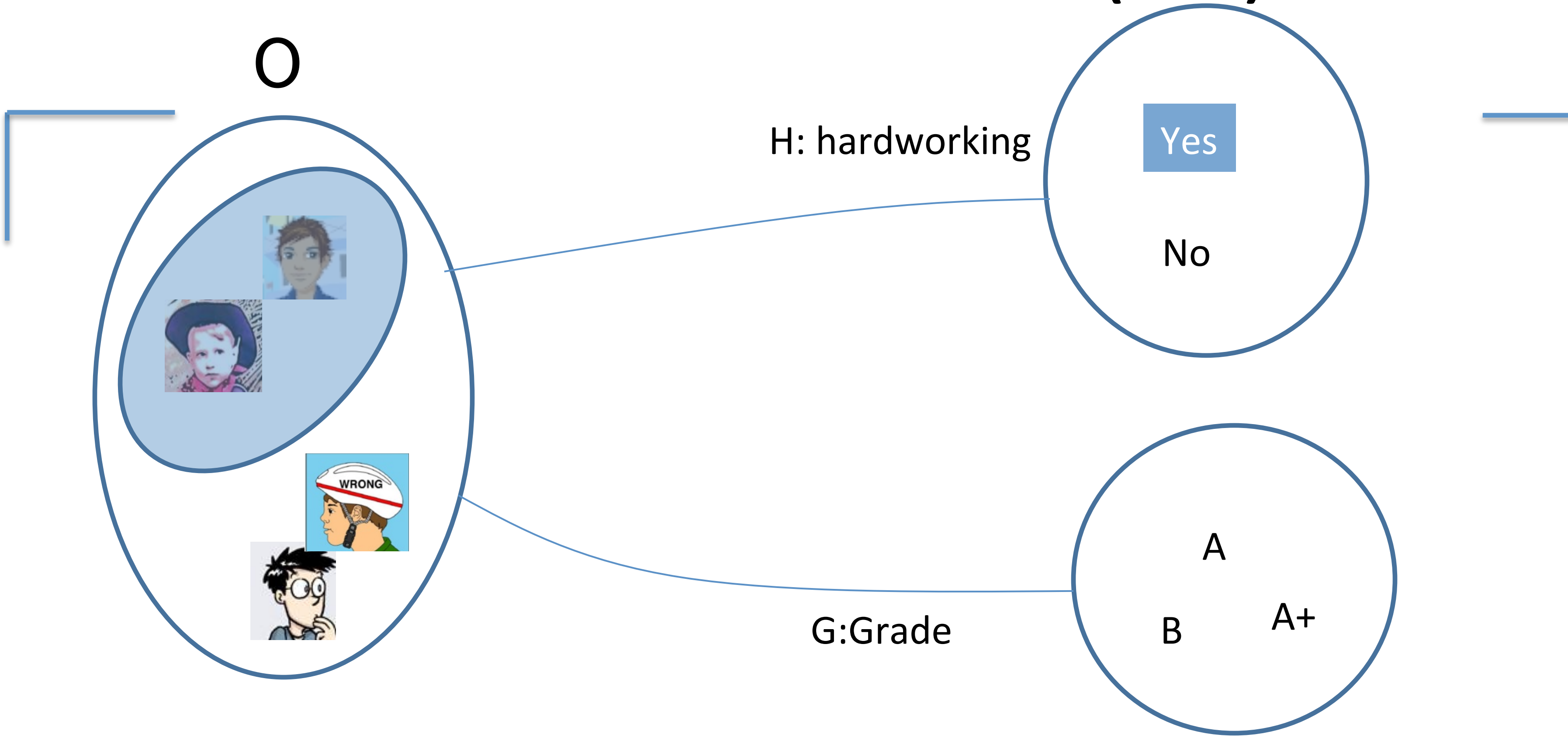
From Events to Random Variable

- Concise way of specifying attributes of outcomes
- Modeling students (Grade and Intelligence):
  - O = all possible students (sample space)
  - What are events (subset of sample space)
    - Grade_A = all students with grade A
    - Grade_B = all students with grade B
    - HardWorking_Yes = … who works hard
  - Very cumbersome

- Need "functions" that maps from O to an attribute space T.
- P(H = YES) = P({student ϵ O : H(student) = YES})

Random Variables (RV)



P(H = Yes) = P({all students who is working hard on the course})

- "functions" that maps from O to an attribute space T.

Notations

- P(A) is shorthand for P(A=true)
- P(~A) is shorthand for P(A=false)
- Same notation applies to other binary RVs: P(Gender=M), P(Gender=F)
- Same notation applies to *multivalued* RVs: P(Major=history), P(Age=19), P(Q=c)
- Note: upper case letters/names for *variables*, lower case letters/names for *values*

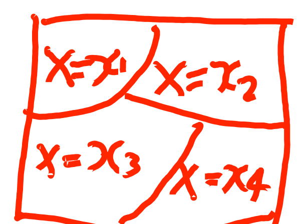
Discrete Random Variables

- Random variables (RVs) which may take on only a **countable** number of **distinct** values

- X is a RV with arity *k* if it can take on exactly one value out of $\{x_1, \ldots, x_k\}$

Probability of Discrete RV

- Probability mass function (pmf): $P(X = x_i)$
- Easy facts about pmf
  - $\Sigma_i P(X = x_i) = 1$
  - $P(X = x_i \cap X = x_j) = 0$ if $i \neq j$
  - $P(X = x_i \cup X = x_j) = P(X = x_i) + P(X = x_j)$ if $i \neq j$
  - $P(X = x_1 \cup X = x_2 \cup \ldots \cup X = x_k) = 1$



e.g. Coin Flips

- You flip a coin
  - Head with probability p, e.g. =0.5

- You flip a coin for *k*, e.g., =100 times
  - How many heads would you expect

e.g. Coin Flips cont.

$\{H, T\}$

- You flip a coin
  - Head with probability *p*
  - Binary random variable
  - Bernoulli trial with success probability *p*
- You flip *a* coin for *k* times
  - How many heads would you expect
  - Number of heads X is a discrete random variable
  - Binomial distribution with parameters *k* and *p*

Discrete Random Variables

- Random variables (RVs) which may take on only a **countable** number of **distinct** values
  - E.g. the total number of heads X you get if you flip 100 coins

- X is a RV with arity *k* if it can take on exactly one value out of $\left\{x_1,\ldots,x_k\right\}$
  - E.g. the possible values that X can take on are 0, 1, 2,…, 100

e.g., two Common Distributions

- Uniform $X \sim U[1,\ldots,N]$
  - X takes values 1, 2, ..., *N*
  - $P(X=i)=1/N$
  - E.g. picking balls of different colors from a box

- Binomial $X \sim Bin(k,p)$
  - X takes values 0, 1, ..., *k*
  - $P(X=i)=\binom{k}{i}p^{i}(1-p)^{k-i}$
  - E.g. coin flips k times

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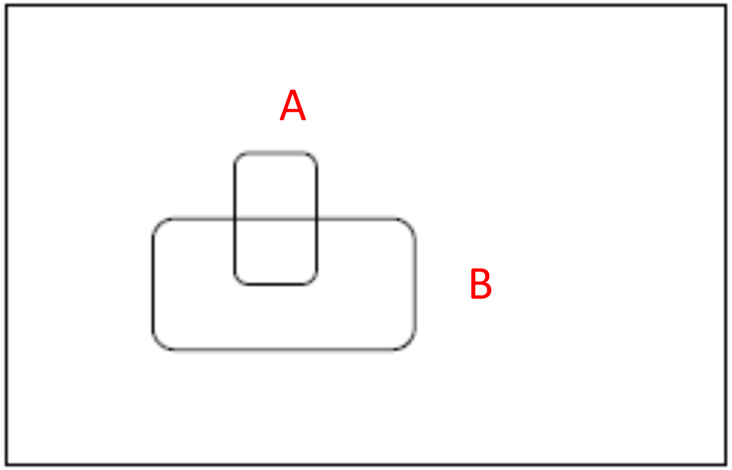
  - Independence, conditional independence

Conditional / Joint / Marginal Probability



P(A given B) = P(A and B) / P(B)

That is, in the frequentist interpretation, we calculate the ratio of the number of times both A and B occurred and divide it by the number of times B occurred.

Chain rule

For short we write: $P(A|B) = P(AB)/P(B)$; *or* $P(AB)=P(A|B)P(B)$, *where P(A|B) is the* ***conditional*** *probability, P(AB) is the* ***joint****, and P(B) is the* ***marginal****.*

If we have more events, we use the chain rule:

$$P(ABC) = P(A|BC)\,P(B|C)\,P(C)$$

from Prof. Nando de Freitas's review

If hard to directly estimate from data, most likely we can estimate

- 1. Joint probability
  - Use Chain Rule
- 2. Marginal probability
  - Use the total law of probability
- 3. Conditional probability
  - Use the Bayes Rule

(1). To calculate Joint Probability: Use Chain Rule

- Two ways to use chain rules on joint probability

joint

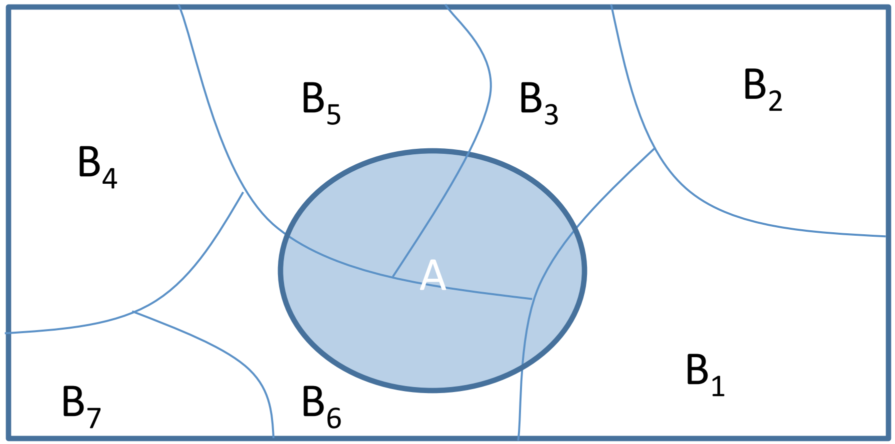
conditional

marginal

$$P(A,B) = p(B|A)p(A)$$

$$P(A,B) = p(A|B)p(B)$$

(2). To calculate Marginal Probability: Use Rule of total probability (Event)



$P(A) =$
$P(A \wedge B) +$
$P(A \wedge \sim B)$

$$\Rightarrow p(A) = \sum \overbrace{P(B_i)P(A|B_i)}^{P(B_i \wedge A)}$$

WHY ???

$P(A) = P(A \cap \Omega)$
$= P(A \cap (B_1 \cup B_2 \cdots \cup B_k))$
$= \sum P(A \cap B_i)$

(2). To calculate Marginal Probability: Use Rule of total probability (RV)

- Given two discrete RVs X and Y, which take values in $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_m\}$, We have

$$P(X = x_i) = \sum_j P(X = x_i \cap Y = y_j)$$

$$= \sum_j P(X = x_i | Y = y_j) P(Y = y_j)$$

$$P(A) = P(A \wedge B) + P(A \wedge \sim B)$$

(3). To calculate Conditional Probability: Use Bayes Rule

- $\mathrm{P}\left(\mathrm{X}=x \middle| \mathrm{Y}=y\right)$ is the probability of $\mathrm{X}=x$, given the occurrence of $\mathrm{Y}=y$

$$\mathrm{P}\left(\mathrm{X}=x \middle| \mathrm{Y}=y\right)=\frac{\mathrm{P}\left(\mathrm{X}=x \cap \mathrm{Y}=y\right)}{\mathrm{P}\left(\mathrm{Y}=y\right)}$$

Bayes Rule

- X and Y are discrete RVs…

$$P\left(X=x \middle| Y=y\right)=\frac{P\left(X=x\cap Y=y\right)}{P\left(Y=y\right)}$$

$$\Downarrow$$

$$P\left(X=x_i \middle| Y=y_j\right)=\frac{P\left(Y=y_j \middle| X=x_i\right)P\left(X=x_i\right)}{\sum_{k} P\left(Y=y_j \middle| X=x_k\right)P\left(X=x_k\right)}$$

$$\left\{x_1,\ldots,x_k\right\}$$

Bayes Rule

$$P(B|A) = \frac{P(A \wedge B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)}$$

This is Bayes Rule

Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London,* **53:370-418**

More General Forms of Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

$$P(A|B \wedge X) = \frac{P(B|A \wedge X)P(A \wedge X)}{P(B \wedge X)}$$

$$P(A = a_1 | B) = \frac{P(B | A = a_1)P(A = a_1)}{\sum_i P(B | A = a_i)P(A = a_i)}$$

One Example: Joint

*Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the **set** {r,r,r,b}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$P(B_1 = r, B_2 = r) \ =$$

Adapt from Prof. Nando de Freitas's review slides

One Example: Joint

Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the ***set*** *{r,r,r,b}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$\mathbf{P(B_1 = r, B_2 = r)} = P(B_1 = r)\,\underbrace{P(B_2 = r \mid B_1 = r)}_{\frac{2}{3}}$$

$$P(B_1 = r) = \frac{3}{4}$$

$$P(B_1 = b) = \frac{1}{4}$$

Adapt from Prof. Nando de Freitas's review slides

One Example: Joint

*Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the **set** {r,r,r,b}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$P(B_1 = r, B_2 = r) = P(B_1 = r)\, P(B_2 = r \mid B_1 = r)$$

$$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

Adapt from Prof. Nando de Freitas's review slides

One Example: Marginal

What is the probability that the 2^{nd} *ball drawn from the* **set** **{r,r,r,b}** *will be red?*

Using marginalization, $\boldsymbol{P(B_2 = r)}$ $=$ $P(B_2 = r, B_1 = r)$

$+ P(B_2 = r, B_1 = b)$

One Example: Marginal

What is the probability that the 2nd ball drawn from the **set** *{r,r,r,b} will be red?*

Using marginalization, $\boldsymbol{P(B_2 = r)}$ $= P(B_2 = r \wedge B_1 = r) + P(B_2 = r \wedge B_1 = b)$

$= P(B_1 = r)\, P(B_2 = r \mid B_1 = r) + P(B_1 = b)\, P(B_2 = r \mid B_1 = b)$

$= \frac{3}{4} \times \frac{2}{3} + \frac{1}{4} \times 1$

One Example: Conditional

$$P(B_1 = r \mid B_2 = r)$$

$$= \frac{P(B_2 = r \mid B_1 = r)\, P(B_1 = r)}{P(B_2 = r)} \Rightarrow \text{lost last page}$$

$\Downarrow$ Last paper

$=$

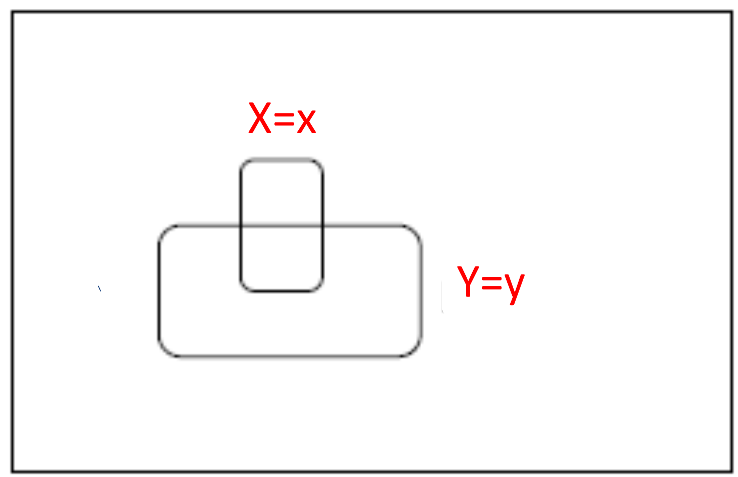
Simplify Notation: Conditional Probability

events

$$P\left(X = x \middle| Y = y\right) = \frac{P\left(X = x \cap Y = y\right)}{P\left(Y = y\right)}$$

But we will always write it this way:

$$P(x \mid y) = \frac{p(x, y)}{p(y)}$$



P(X=x true) -> P(X=x) -> P(x)

$P(x) \Leftarrow P(X=x) \Leftarrow P(X=x \text{ true})$

value RV event

Simplify Notation: Marginal

- We know p(X, Y), what is P(X=x)?
- We can use the law of total probability, why?

$$p(x) = \sum_{y} P(x,y)$$

$$= \sum_{y} P(y) P(x \mid y)$$

all possible Y values

$$\{y_1, \ldots, y_m\}$$

$$p(x) = \sum_{y,z} P(x,y,z)$$

$$= \sum_{z,y} P(y,z) P(x \mid y,z)$$

$$\sum_{y} \sum_{z} p(y,z) = 1$$

Simplify Notation: Conditional

- Bayes Rule

$$P(x \mid y) = \frac{P(x)P(y \mid x)}{P(y)}$$

- You can condition on more variables

$$P(x \mid y, z) = \frac{P(x \mid z)P(y \mid x, z)}{P(y \mid z)}$$

Simplify Notation: An Example

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

$$P(rain \mid wet) = \frac{P(rain)P(wet \mid rain)}{P(wet)}$$

Simplify Notation: An Example

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

W= G=

$$P(rain \mid wet) = \frac{P(rain)P(wet \mid rain)}{P(wet)}$$

P(W=S | wet)

$$P(x \mid y) = \frac{P(x)P(y \mid x)}{P(y)} = \frac{p(x,y)}{p(y)}$$

Simplify Notation: An Example

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

0.5 1

$$P(rain \mid wet) = \frac{P(rain)P(wet \mid rain)}{P(wet)}$$

P(wet, rain)

P(wet, rain) + P(wet, sunny)

Weather Grass

{rain, sunny} {wet, dry}

P(rain) P(Wet | rain) + P(Sunny) P(Wet | Sunny)

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Independent RVs

- Intuition: X and Y are independent means that $X = x$ **neither** makes it **more or less** probable that $Y = y$
- Definition: X and Y are independent ***iff***

$$P(X = x \cap Y = y) = P(X = x)P(Y = y)$$

More on Independence

$$P(X = x \cap Y = y) = P(X = x)P(Y = y)$$

$$P(X = x | Y = y) = P(X = x)$$

$$P(Y = y | X = x) = P(Y = y)$$

- E.g. no matter how many heads you get, your friend will not be affected, and vice versa

More on Independence

- X is independent of Y means that knowing Y does not change our belief about X. The following forms are equivalent:
  - $P(X=x, Y=y) = P(X=x)\,P(Y=y)$
  - $P(X=x|Y=y) = P(X=x)$
  - The above should hold for all x_i, y_j
  - It is symmetric and written as $X \perp Y$

Conditionally Independent RVs

- Intuition: X and Y are conditionally independent given Z means that once Z is **known**, the value of X does not add any **additional** information about Y
- Definition: X and Y are conditionally independent given Z ***iff***

$$P\left(X = x \cap Y = y \middle| Z = z\right) = P\left(X = x \middle| Z = z\right) P\left(Y = y \middle| Z = z\right)$$

If holding for all x_i, y_j, z_k $\qquad X \perp Y \mid Z$

More on Conditional Independence

$$P\left(X = x \cap Y = y \middle| Z = z\right) = P\left(X = x \middle| Z = z\right) P\left(Y = y \middle| Z = z\right)$$

$$P\left(X = x \middle| Y = y, Z = z\right) = P\left(X = x \middle| Z = z\right)$$

$$P\left(Y = y \middle| X = x, Z = z\right) = P\left(Y = y \middle| Z = z\right)$$

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- Structural properties, e.g., Independence, conditional independence
- Maximum Likelihood Estimation (next class)

References

- Prof. Andrew Moore's review tutorial
- Prof. Nando de Freitas's review slides
- Prof. Carlos Guestrin recitation slides