Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2897:

$(2, 1, 0)^{1\times3}$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}^{2\times1}$ rows x columns

dimensions

A **vector** is a list of elements.
A **matrix** is a table of elements.

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{2\times2}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{2\times2}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^{3\times2}$

Rule: you can add two matrices or two vectors **only if** they have the same dimensions.

Rule: you can **multiply** a matrix or a vector **by a constant**.

A^T

To take the **transpose** of a matrix, think of its columns as column vectors, and then write them as row vectors. The first column becomes the first row.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad \left(B^{4\times9}\right)^T = {B^T}^{9\times4}$$

Recap: multiplying two vectors

Rule: we can multiply a **row vector** by a **column vector** provided that they have the same number of elements.

Formula: Step **across the row vector** and **down the column vector**, multiplying each pair of elements. Then **add the products**.

1x3 3x1

$$\begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ 10 \end{pmatrix} = 2a + 1b + 0 \cdot 10 = 2a + 1b$$

$$\begin{pmatrix} 4 & 7 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = 4 \cdot 10 + 7 \cdot 10 = 110$$

Recap: multiplying a matrix and a vector

Suppose we have a **NxN matrix** and a **Nx1 vector**.

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Multiply the j^{th} row of the matrix by the vector, etc.
4. Stack the answers in a new vector.

Example:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{matrix} 2\cdot 3 + 1\cdot 2 + 0\cdot 1 \\ 1\cdot 3 + 1\cdot 2 + 1\cdot 1 \\ \underbrace{-1\cdot 3}_{-3} + \underbrace{3\cdot 2}_{6} + \underbrace{-1\cdot 1}_{-1} \end{matrix} = \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}$$

Multiplying two matrices

On the prev. slide, we took the idea of multiplying two vectors and expanded it:
We treated a **matrix on the left** as a set of **stacked row vectors**.

" inner "
dimensions match!

To multiply two matrices, what should we do with the **matrix on the right**?

treat it like two column vectors!

$\boxed{2}\times 2 \quad 2\times\boxed{2}$

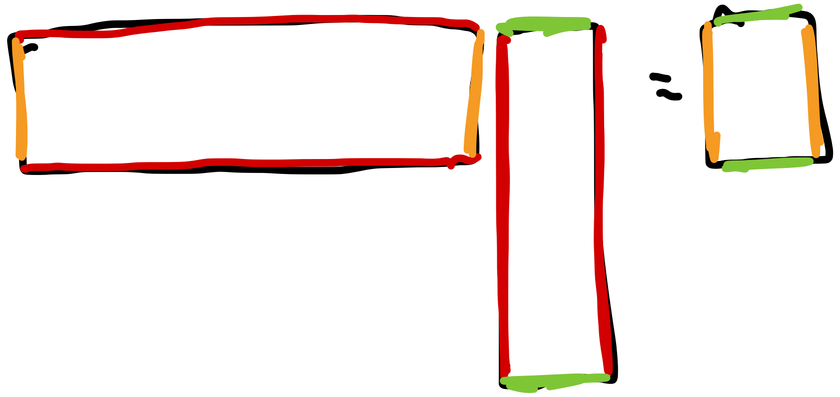
outer dims for product 2×2

i,j

$$i \begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&1\\5&2\end{pmatrix}_{j} = \begin{pmatrix}1\cdot0+2\cdot5 = 10 & 1\cdot1+2\cdot2 = 5\\ 3\cdot0+4\cdot5 = 20 & 3\cdot1+2\cdot4 = 11\end{pmatrix} = \begin{pmatrix}10&5\\20&11\end{pmatrix}$$

✓

$(3\times 9)\times(9\times 2) = (3\times 2)$



Key:
1. inner dims must match
2. resulting matrix has dims = "outer" dims
3. mult row i (left matrix) column j (right matrix) $\rightarrow$ entry i,j

Does matrix multiplication *commute*?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 0+2\cdot 5 & 1\cdot 1+2\cdot 2 \\ 3\cdot 0+4\cdot 5 & 3\cdot 1+2\cdot 4 \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 20 & 11 \end{pmatrix}$$

$$\ne$$

$$\begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0\cdot 1+3\cdot 1 & 0\cdot 2+1\cdot 4 \\ 5\cdot 1+2\cdot 3 & 5\cdot 2+2\cdot 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 11 & 18 \end{pmatrix}$$

✓ $a \times b = b \times a$

✓ $a + b = b + a$

✗ $b - a \neq a - b$

nope!

$$AB \neq BA$$

in general

Algebra -vs- Linear Algebra

Associative Law

$$(AB)C = A(BC)$$

sequence of multiplication doesn't matter, as long as left-to-right order is preserved.

Left Distributive Law

$$(A+B)C = AC + BC$$

Right Distributive Law

$$A(B+C) = AB + AC$$

Commutative Law for Scalars

$$k(AB) = (kA)B = A(kB) = (AB)k$$

const, scalar

Scalars → "spies"... can pass through undetected.

Tricks of the Transpose

We already learned one cute transpose trick: $\left(A^T\right)^T = A$

Here's another one: $(A+B)^T = A^T + B^T$

In other words, the transpose of the sum = the sum of the transposes.

Here's the *tricky* one: $(AB)^T = B^T A^T$ ⊛

Rule: To transpose a product, you can only "distribute" the transpose if you reverse the order of the product!

$$(ABC)^T = C^T B^T A^T$$

$$(A(BC))^T \overset{⊛}{=} (BC)^T A^T \overset{⊛\text{ again}}{=} (C^T B^T) A^T = C^T B^T A^T$$

$$(CDK + RTL)^T$$

$$= (CDK)^T + (RTL)^T$$

$$= K^T D^T C^T + L^T T^T R^T$$

The Zero Matrix & the Identity Matrix

In **algebra**, **zero** is the number that doesn't do anything in addition: $5 + 0 = 5$

In **algebra**, **one** is the number that doesn't do anything in multiplication: $9 \times 1 = 9$

In **linear algebra**, the **zero matrix** is doesn't do anything in addition:

$$\begin{pmatrix} 5 & 9 \\ 12 & 29 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5+0 & 9+0 \\ 12+0 & 29+0 \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 12 & 29 \end{pmatrix}$$

$$\overset{n\times m}{A} + \overset{n\times m}{O} = \overset{n\times m}{A}$$

In **linear algebra**, the **identity matrix** doesn't do anything in multiplication:

I

$$AI = A \qquad IA = A$$

I want both!

The Identity Matrix - what does it look like?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} = \begin{pmatrix} a\underset{1}{I_{11}} + b\underset{0}{I_{21}} & a\underset{0}{I_{12}} + b\underset{1}{I_{22}} \\ c\underset{1}{I_{11}} + d\underset{0}{I_{21}} & c\underset{0}{I_{12}} + d\underset{1}{I_{22}} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

want!

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: the fact that $IA = A = AI \Rightarrow I$ and A commute!

diagonal matrix

all zeros except the diagonal

Def'n

I = special diagonal matrix:

ones on diagonal, zeros elsewhere.

$$I^{n\times n} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

works for any $n\times n$ matrix!

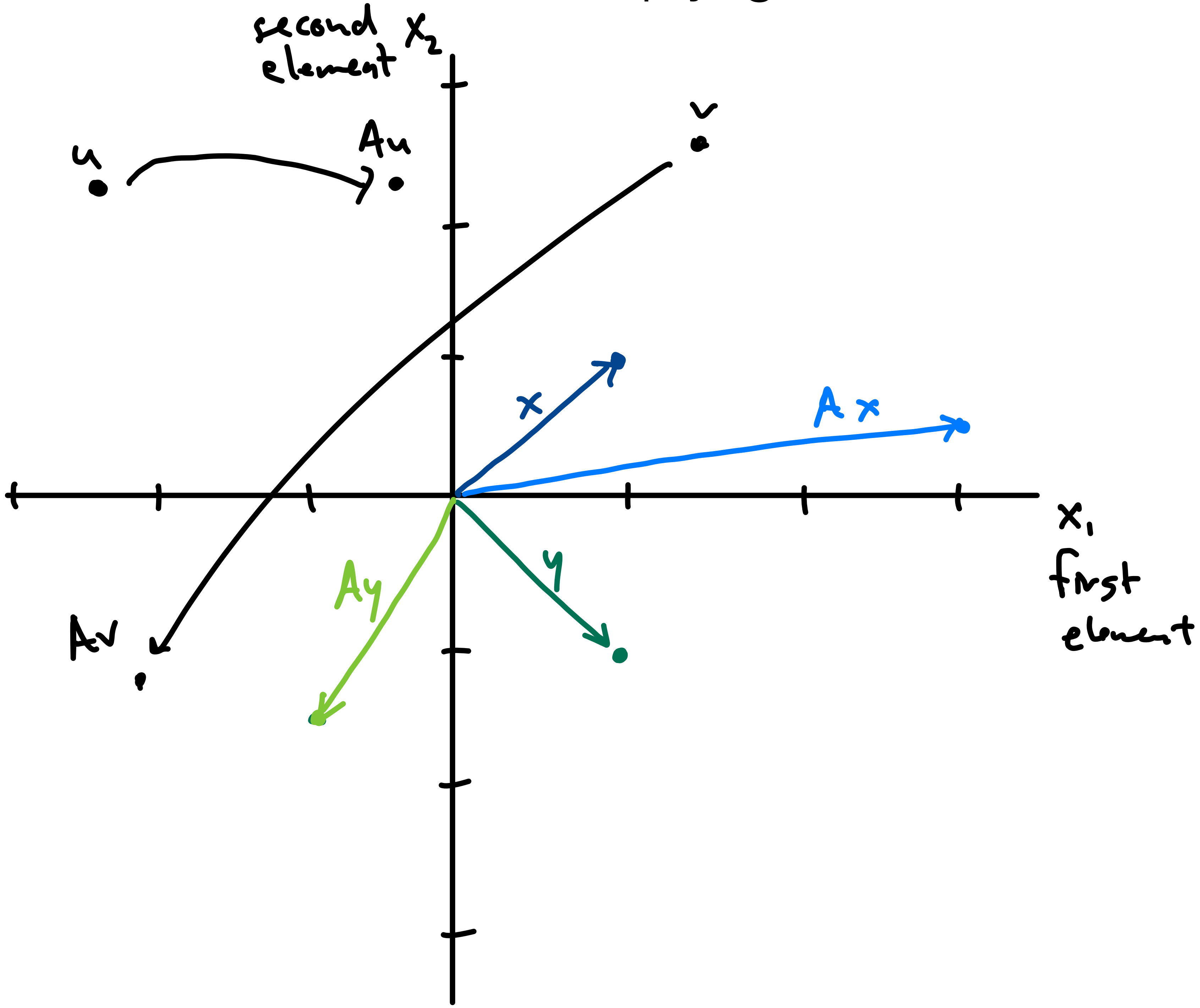
Matrices as Machines "operator"

A **matrix** is a machine that does stuff to vectors. Take a vector, multiply, get a new vector.

$$A = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad Ax = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}$$

$$y = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad Ay = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{3}{2} \end{pmatrix}$$



Trace

The **trace** of a matrix is the sum of the diagonal elements. The trace is a scalar.

$$A = \begin{pmatrix} 1 & 2 \\ 9 & 3 \end{pmatrix} \qquad tr(A) = 1+3 = 4 \qquad tr\left(I^{n \times n}\right) = tr\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}^{n \times n} = n$$

Practice: Suppose that the row i column j element of a matrix A is given by A_{ij}. How can we express the trace using summation notation?

$$\boxed{tr(A) = \sum_{i=1}^{n} A_{ii}}$$

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & & & & n \end{array} \\ \begin{array}{c} 1 \\ \\ \\ \\ n \end{array} & \begin{pmatrix} A_{11} & & & & \\ & A_{22} & & & \\ & & A_{33} & & \\ & & & \ddots & \\ & & & & A_{n-1,n-1} & \\ & & & & & A_{nn} \end{pmatrix} \end{array}$$

Note: the trace has no intuitive meaning, but it turns out to be rather convenient later.

Determinant (2x2 matrix)

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - cb$$

The **determinant** of a matrix is also a scalar. It has a rather peculiar formula:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Practice:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

$$A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix} = 1 \cdot 1 - 2 \cdot 0.5 = 1 - 1 = 0$$

Notes:

1. $|A|$ can be negative!
2. $|A|$ can be zero.
3. $|A| = |A^T|$ for any square matrix

Square matrix: #rows = #columns.

Note: the determinant of a matrix is the same as the determinant of its transpose.

Matrices as Machines II

when a matrix has det. = 0,
Its abilities as a machine
are diminished.
"singular"

$$A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad Ax = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad Ay = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}$$