Calculating Biological Quantities

CSCI 2897

- HW 3 Due today
- see Slack for a few notes on HW

Prof. Daniel Larremore

daniel.larremore@colorado.edu
@danlarremore

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Today:

1. Linear models with more than one variable
2. Matrices and vectors

Models with more than one dynamic variable

Let's go back to exponential growth in continuous time.

$$\frac{dn}{dt} = rn \longrightarrow \int \frac{dn}{n} = \int r\,dt \longrightarrow \ln n = rt + \overbrace{c}^{k}$$

$$\longrightarrow n = e^{rt+c} = e^{rt}e^{c} = ke^{rt}$$

We know by now that this is called exponential growth because

$$\boxed{n(t) = ke^{rt}}$$

where $k = n(0)$ is the initial condition.

Models with more than one dynamic variable

Now let's imagine that we have two populations, n_1 and n_2

$$\frac{dn_1}{dt} = r_1 n_1 \qquad \longrightarrow \qquad n_1(t) = k_1 e^{r_1 t}$$

no n_2

$$\frac{dn_2}{dt} = r_2 n_2 \qquad \longrightarrow \qquad n_2(t) = k_2 e^{r_2 t}$$

no n_1

This one is easy too: the populations are totally independent of each other, so we can solve each equation by itself.

Models with more than one dynamic variable

What are the equilibrium solutions for this set of equations?

set all derivs = 0

$$\frac{dn_1}{dt} = r_1 n_1 = 0 \quad \longrightarrow \quad n_1 = 0$$

$$\boxed{\text{equil: } (n_1, n_2) = (0, 0)}$$

$$\frac{dn_2}{dt} = r_2 n_2 = 0 \quad \longrightarrow \quad n_2 = 0$$

What can we say about stability of the equilibrium solution(s)?

look to the vector field!

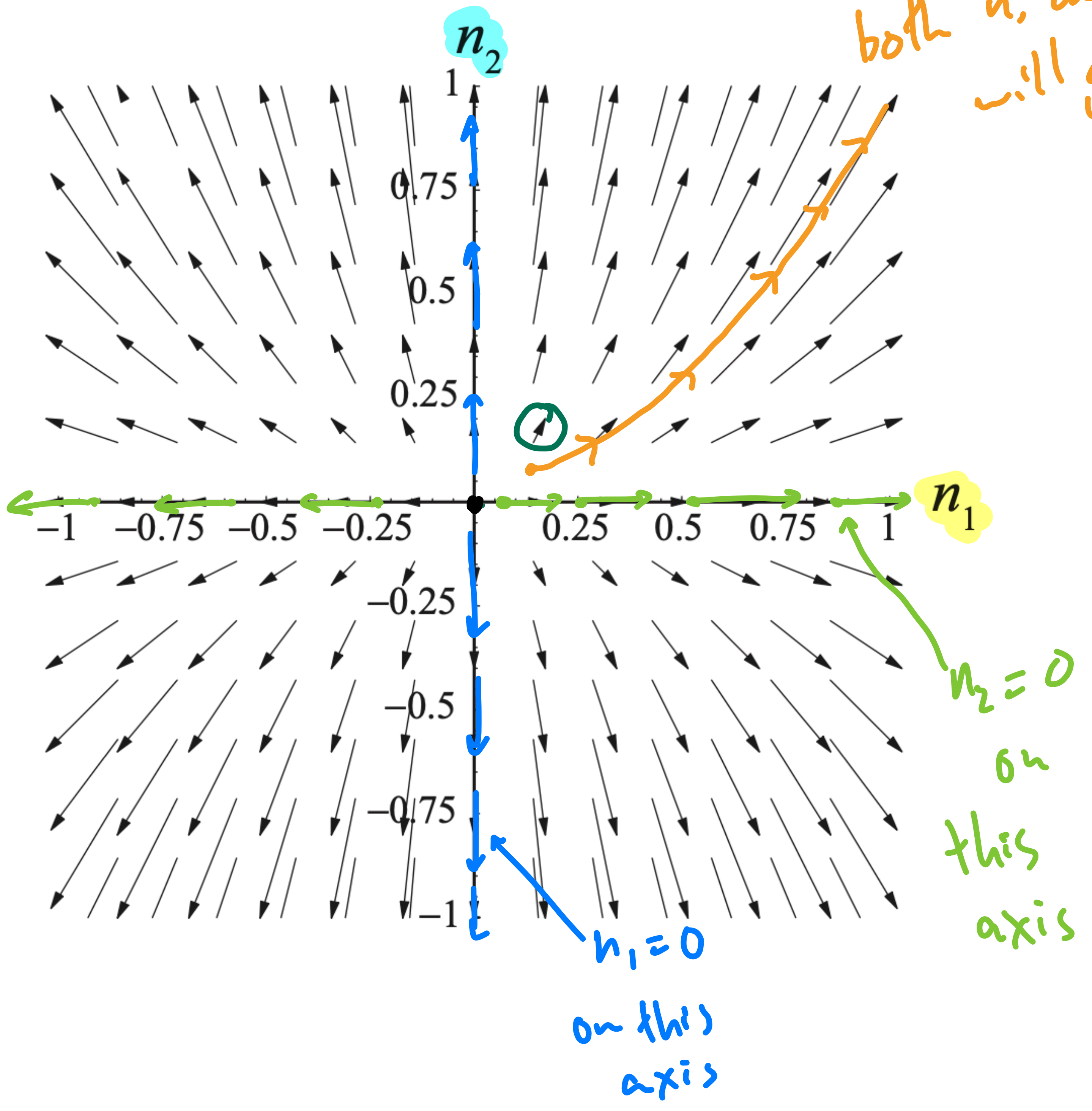
equations

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$

phase plane

vector field



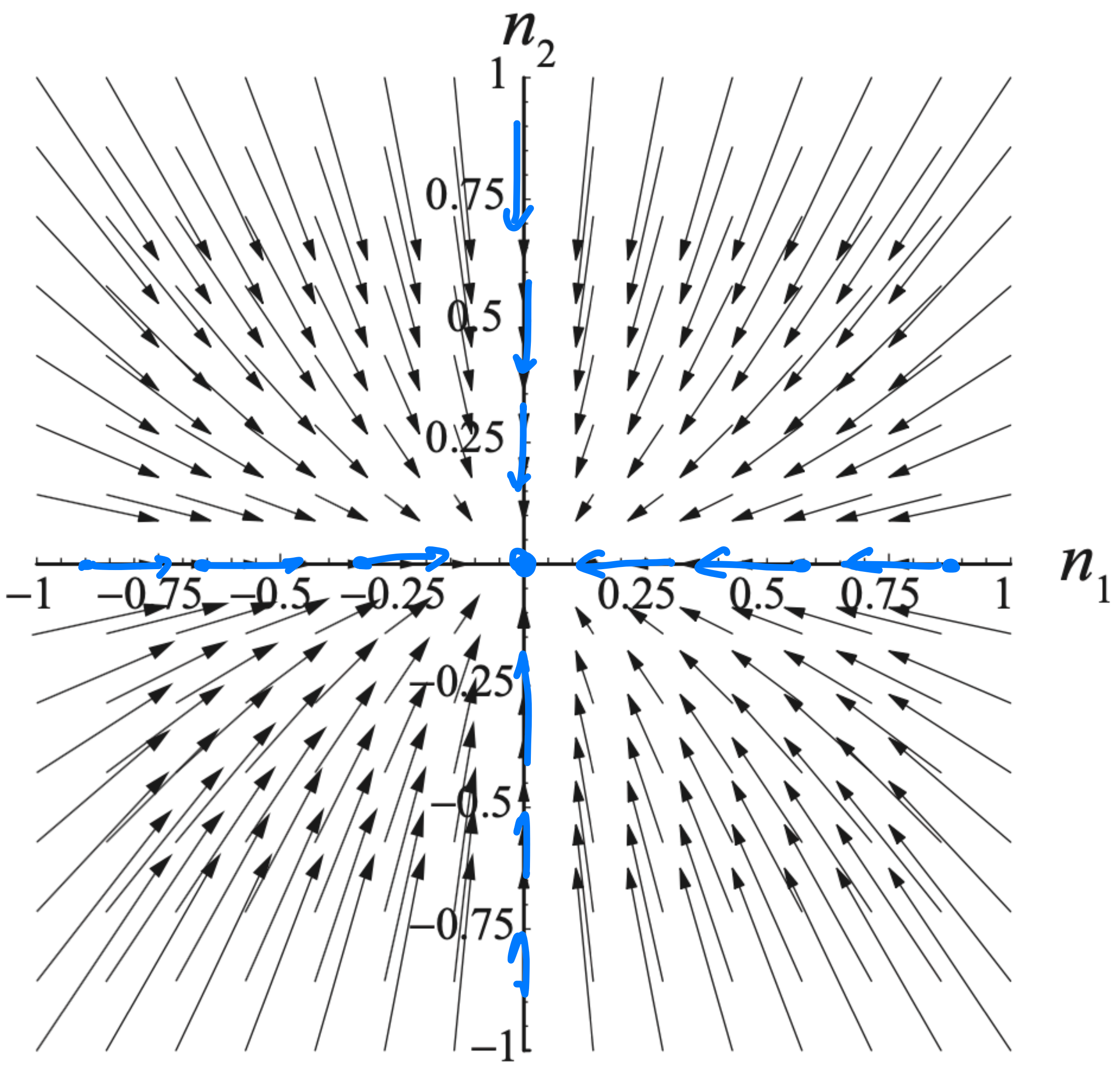
- $r_2 > r_1$
- $r_1 > 0$
- $r_2 > 0$

unstable

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$

stable

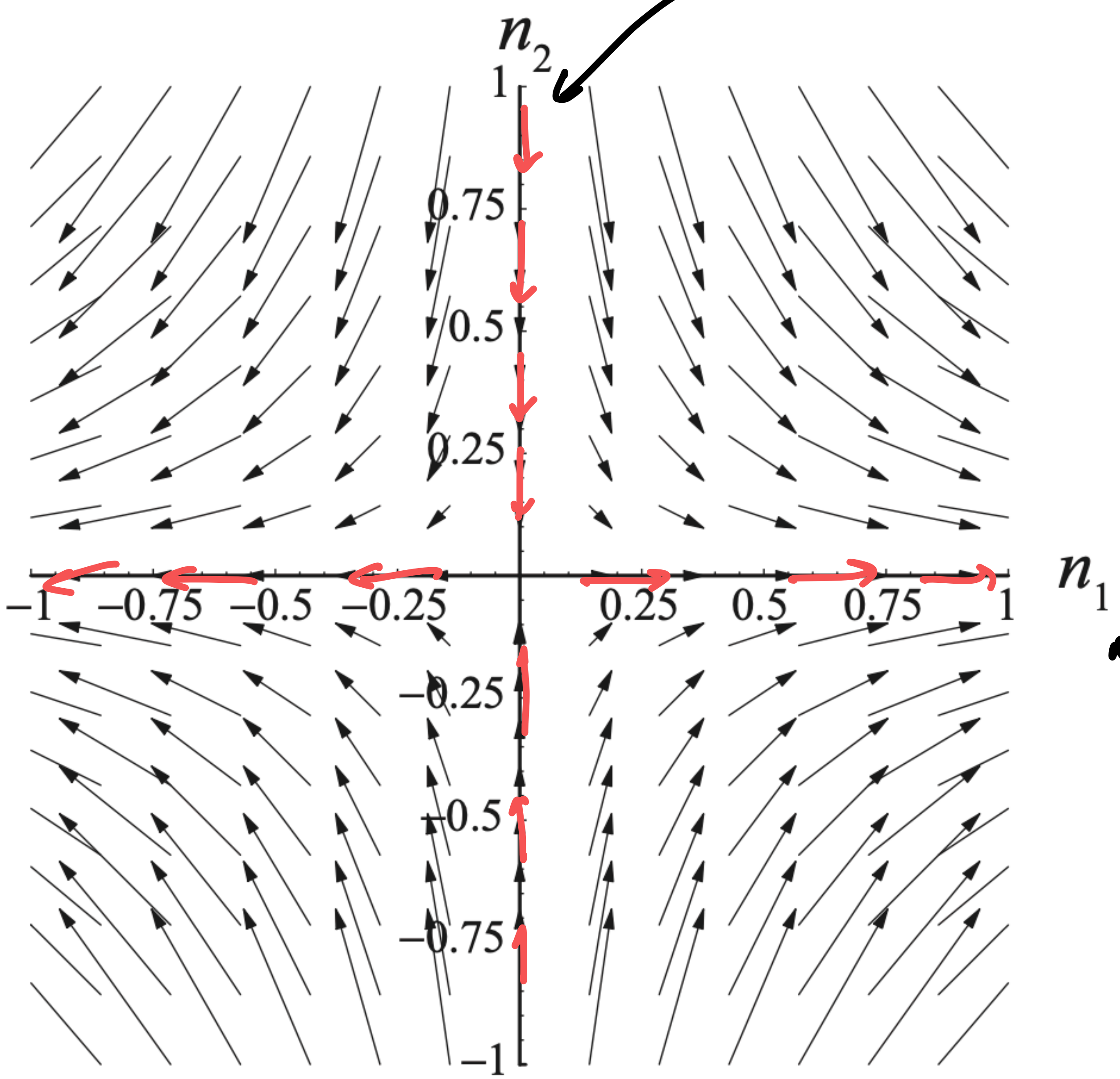


- $r_1 < 0$
- $r_2 < 0$
- even though we changed the sign of r_i the "axis directions" are special.

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$

if on y axis line, move toward the center.



• $r_1 > 0$

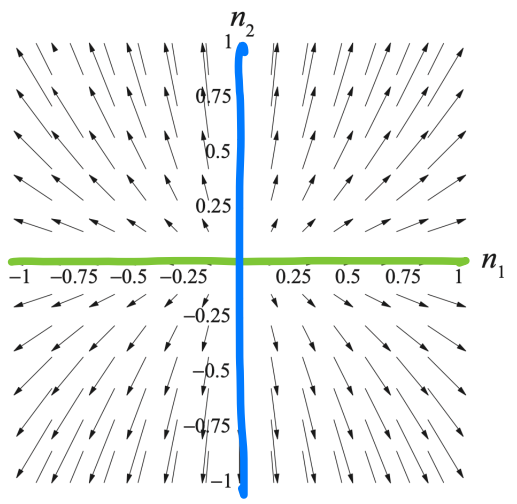
• $r_2 < 0$

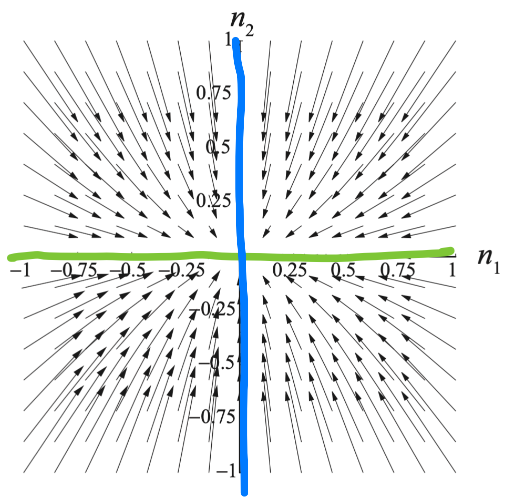
if on the x-axis line, move <u>away</u> from center.

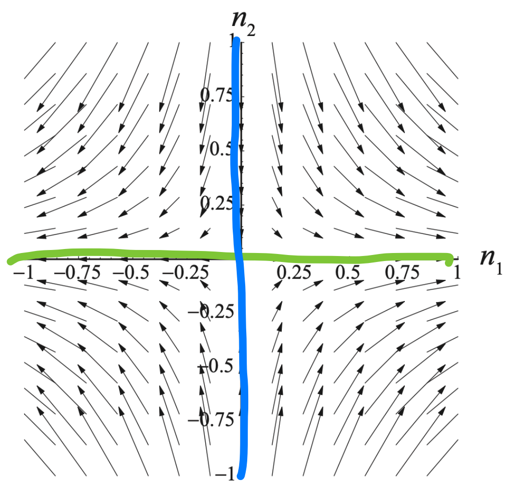
unstable

Characteristic directions

Eigen vectors







$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$

Note: for these equations, if you're on either axis, you never leave.

These directions are therefore special:

$(c,0)$ the horizontal axis

$(0,c)$ the vertical axis

for any arbitrary value of c.

Models with more than one dynamic variable - Part 2

Imagine that our 2 populations correspond to 2 strains of bacteria. Suppose that

- a is the rate at which **strain 1** produces **strain 1** daughter cells
- b is the rate at which **strain 2** produces **strain 1** daughter cells by mutation
- c is the rate at which **strain 1** produces **strain 2** daughter cells by mutation
- d is the rate at which **strain 2** produces **strain 2** daughter cells

self

mutation $n_2 \to n_1$

$$\frac{dn_1}{dt} = a n_1 + b n_2$$

$$\frac{dn_2}{dt} = c n_1 + d n_2$$

mutation $n_1 \to n_2$

self

Models with more than one dynamic variable - Part 2

We are going to rewrite this in a miraculous way

$$\begin{bmatrix} \dfrac{dn_1}{dt} = an_1 + bn_2 \\ \\ \dfrac{dn_2}{dt} = cn_1 + dn_2 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} \dfrac{dn_1}{dt} \\ \\ \dfrac{dn_2}{dt} \end{bmatrix}}_{\text{vector}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{matrix}} \underbrace{\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}_{\text{vector}} \longrightarrow \frac{d\vec{n}}{dt} = M\vec{n}$$

<u>linear</u> system of equations ⟶ matrix-vector notation

Vectors and Matrices

A **vector** is a list of elements.

$(1,1)$ $\quad \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\quad (3, 9, 4, 7, 2, \pi, -\pi, 0, 19, 2021)$

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

- some written vertically
- some written horizontally

stay tuned!

A **matrix** is a table of elements.

$\begin{pmatrix} 2 & 0 \\ 1 & 9 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{bmatrix} a & b \\ c & d \\ 19 & 20 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 5 \\ 9 & 12 & 21 & 999 \end{bmatrix}$

A vector is a matrix w/ only one row <u>or</u> one column.

NB: the plural of "matrix" is "matrices."

Vectors in the x-y plane

Remember those characteristic directions from before, $(0,c)$ and $(c,0)$? Those, too, are vectors!

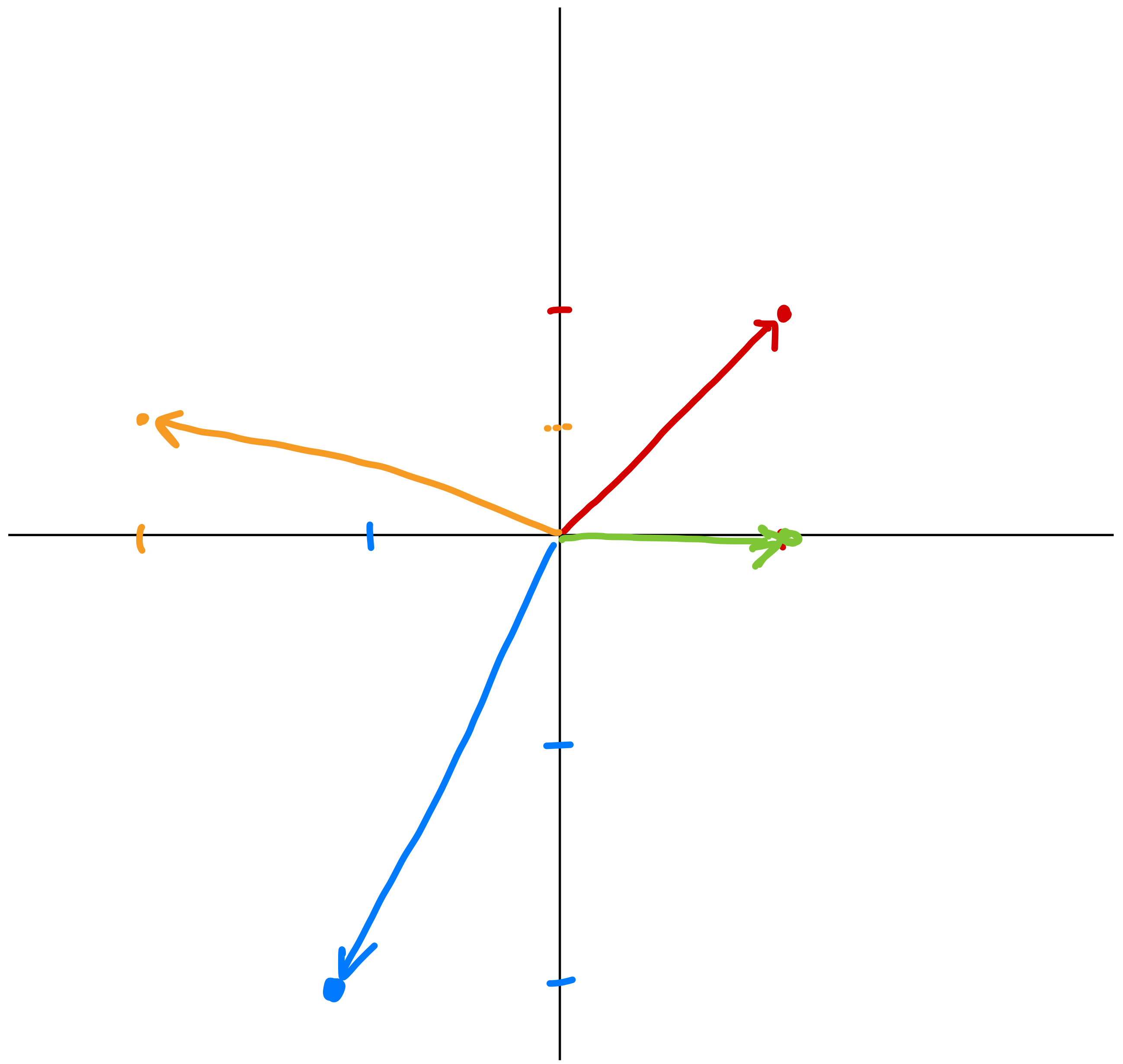
It turns out that points in the x-y plane are also vectors. Why?
Because a **vector** is a list of elements.

$(1,1)$

$(-1,-2)$

$(1,0)$

$(-2,\frac{1}{2})$



(length)

NB: Because of their use in modeling, we draw vectors as **arrows**, which point in a particular direction, and have a particular magnitude.

Vectors in the x-y-z plane

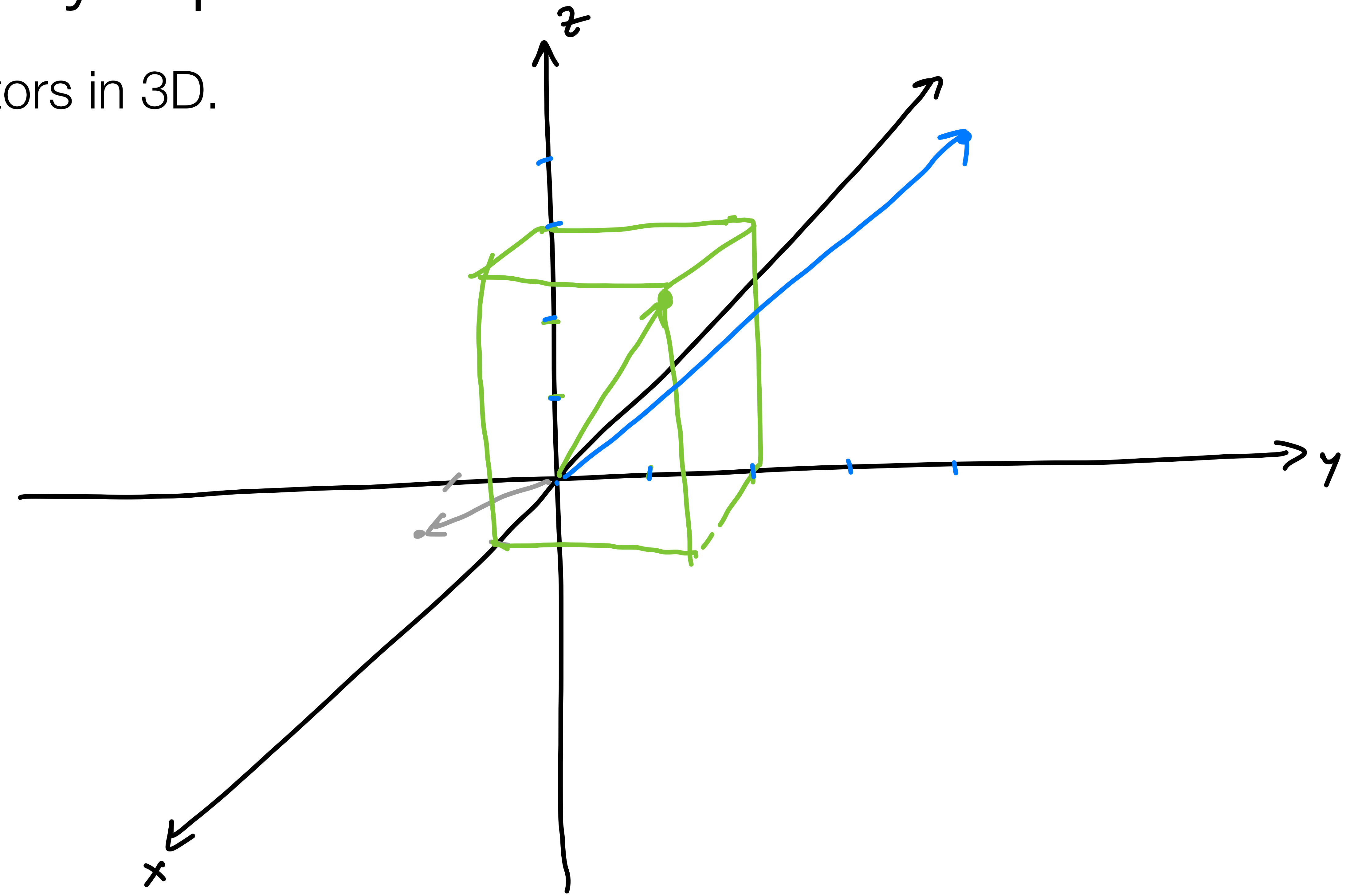
We can also draw vectors in 3D.

$(1, -1, 0)$

$(1, 2, 3)$

$(0, 4, 4)$

points
$\updownarrow$
vectors.



LaTeX: \vec{v} $\longrightarrow$ $\vec{v}$

Row vectors and Column vectors

In general, the **dimension** of a vector is: or matrix

(# of rows) × (# of columns)

height width

Because a vector has either 1 row or 1 column, we get two definitions for free:

1. a vector that's a row is called a **row vector**.
2. a vector that's a column is called a **column vector**.

Examples:

row vector: $[2, 1, 3]$ 1×3

column vector: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ 4×1

To denote a vector, we sometimes write a little arrow $\vec{v}$

LaTeX: \vec{v}

Matrix and Vector addition

Rule: you can add two matrices or two vectors **only if** they have the same dimensions.

Examples:

1. $$\underset{2\times 1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \underset{2\times 1}{\begin{bmatrix} 19 \\ 20 \end{bmatrix}} = \begin{bmatrix} 2+19 \\ 1+20 \end{bmatrix} = \underset{2\times 1}{\begin{bmatrix} 21 \\ 21 \end{bmatrix}}$$

2. $$\underset{1\times 3}{(3, 2, -1)} + \underset{1\times 3}{(12, 12, 12)} = \underset{1\times 3}{(15, 14, 11)}$$

3. $$\underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underset{2\times 2}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}}$$

4. $$\underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underset{2\times 1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \text{\underline{nope}.}$$

Matrix and Vector scalar multiplication

Rule: you can **multiply** a matrix or a vector~~s~~ **by a constant**.

Examples:

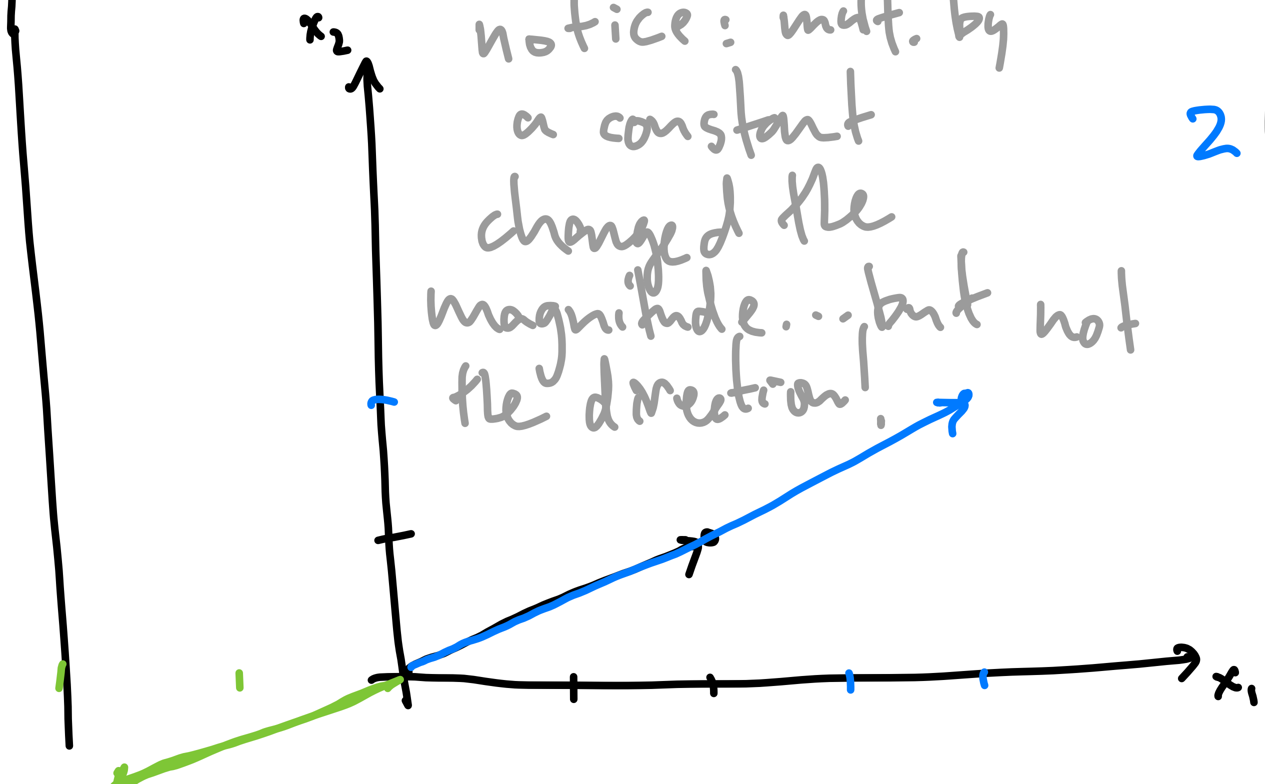
$$2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

$$\pi\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}2\pi & 0\\0 & \pi\end{pmatrix}$$

$$(-1)\cdot(2, 9, x) = (-2, -9, -x)$$

$\Rightarrow$ can factor stuff out!

$$\begin{pmatrix}2\\4\\6\end{pmatrix} = 2\begin{pmatrix}1\\2\\3\end{pmatrix}$$



notice: mult. by a constant changed the magnitude...but not the direction!

$$2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$$

$$-1\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}$$

Matrix and Vector subtraction

Because we can

1. multiply a matrix or vector by -1, and
2. add it to another matrix or vector,

this means that we we can do subtraction too*!

Ex:

① $\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-2 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

② $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}$

*Reason: $b - a = b + (-1)a$

Vector-vector multiplication

Rule: we can multiply a **row vector** by a **column vector** provided that they have the same number of elements.

Formula: Step **across the row vector** and **down the column vector**, multiplying each pair of elements. Then **add the products**.

1. $\underset{1\times 2}{(2 \quad 4)} \underset{2\times 1}{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} = 2\cdot 3 + 4\cdot 1 = 6+4 = 10$

2. $(1 \quad 0)\begin{pmatrix} 5 \\ 9 \end{pmatrix} = 1\cdot 5 + 0\cdot 9 = 5+0 = 5$

3. $\underset{1\times 3}{(2 \quad 3 \quad 1)} \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = 2\cdot x + 3\cdot y + 1\cdot z = 2x + 3y + z$

4. $(a \quad b)\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = an_1 + bn_2$

NB: This kind of vector-vector multiplication **produces a scalar**.

Vector-vector multiplication

This kind of vector-vector multiplication **produces a scalar**.

Question: how can we **multiply two column vectors?**

$$\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Flip it!

$$\underset{1\times 3}{\begin{pmatrix} 3 & 1 & 7 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} = 3\cdot 1 + 1\cdot 0 + 7\cdot 1 = 10$$

The Transpose^T

To take the **transpose** of a **row-vector**, flip it & write it as a column vector.
To take the **transpose** of a **column-vector**, flip it & write it as a row vector.

$$\underset{2\times 1}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}^T} = \underset{1\times 2}{\begin{pmatrix} 2 & 1 \end{pmatrix}}$$

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}^T\right)^T = \begin{pmatrix} a & b \end{pmatrix}^T = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\underset{1\times 4}{\begin{pmatrix} 5 & 19 & 12 & \pi \end{pmatrix}^T} = \underset{4\times 1}{\begin{pmatrix} 5 \\ 19 \\ 12 \\ \pi \end{pmatrix}}$$

Question: what happens to the *dimensions* of a vector when we take its transpose?

(rows, columns) $\longleftrightarrow$ (columns, rows)

The Transpose - Part 2

To take the **transpose** of a matrix, think of its columns as column vectors, and then write them as row vectors. The first column becomes the first row.

① $\begin{pmatrix} 1 & 0 \\ x & y \\ \pi & e \end{pmatrix}^T = \begin{pmatrix} 1 & x & \pi \\ 0 & y & e \end{pmatrix}$

3×2 $\qquad$ 2×3

② $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The digonal of a matrix is the elements on diagonal.

Question: what happens to the *dimensions* of a matrix when we take its transpose?

swap rows, columns.

Matrix-vector multiplication

Suppose we have a **2x2 matrix** and a **2x1 vector**.
We can define matrix-vector multiplication as follows:

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Stack the answers in a new vector.

! ! ! !

Example:

①
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{matrix} 1\cdot 5 + 2\cdot 10 \\ 3\cdot 5 + 4\cdot 10 \end{matrix} = \begin{pmatrix} 25 \\ 55 \end{pmatrix}$$

②
$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{matrix} 1\cdot 5 + 0\cdot 5 \\ 2\cdot 5 + (-1)5 \end{matrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

③
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{matrix} an_1 + bn_2 \\ cn_1 + dn_2 \end{matrix}$$

Matrix-vector multiplication

Suppose we have a **3x3 matrix** and a **3x1 vector**.

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Multiply the 3rd row of the matrix by the vector.
4. Stack the answers in a new vector.

Example:

$$\begin{pmatrix}2&1&4\\0&1&3\\1&5&7\end{pmatrix}\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{matrix}2\cdot1+1\cdot0+4\cdot2\\0\cdot1+1\cdot0+3\cdot2\\1\cdot1+5\cdot0+7\cdot2\end{matrix} \begin{matrix}=\\=\\=\end{matrix} \begin{pmatrix}10\\6\\15\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 2\end{pmatrix}$$

Matrix-vector multiplication

Suppose we have a **NxN matrix** M and a **Nx1 vector** $\overrightarrow{x}$.

Let's write a formula for the ith element of the resulting vector, $\overrightarrow{v} = M\overrightarrow{x}$

multiply the ith row of matrix with the vector.

$$V_i = \left(\overrightarrow{M_{row\ i}}\right)\vec{X} = M_{i1} \cdot X_1 + M_{i2} \cdot X_2 + M_{i3} \cdot X_3 + \ldots + M_{iN} \cdot X_N$$

ith entry of v vector

$$\boxed{V_i = \sum_{j=1}^{N} M_{ij} X_j}$$

columns of M = # rows in x.

Rule: To multiply a matrix and a vector, *what must be true of their dimensions?*

Models with more than one dynamic variable - Part 2

We are going to rewrite this in a miraculous way

$$\boxed{\begin{aligned} \frac{dn_1}{dt} &= an_1 + bn_2 \\ \frac{dn_2}{dt} &= cn_1 + dn_2 \end{aligned}}$$

$$\boxed{\begin{pmatrix} \frac{dn_1}{dt} \\ \frac{dn_2}{dt} \end{pmatrix} = \begin{pmatrix} an_1 + bn_2 \\ cn_1 + dn_2 \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$\frac{d\vec{n}}{dt} = M\vec{n}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{d}{dt}\vec{n} = M\vec{n} \qquad \Big|\Big| \qquad \frac{d}{dt}y = ry$$

$$y(t) = ke^{rt}$$