4/27/21

Lec 23

CBQ

- Final
- Review
- L. Sq.

Final

- 5/1 7:30-10:00
- goal: Δt 1:15
- Format/Coverage
  - covers everything
  - no 1st half bio model, e.g. no in-depth Q. about logistic growth
- see outline on github.
- mix of written answers, math answers, mult. choice.

Review

- Bring your questions to Thursday's class!
- see github list.

HW5 Due Th.

- Bonus Ex. Cr. (Newton's Law of Cooling)

FCQs

- Write something
  - ⊕ good stuff
  - ⊖ not good stuff.

"Least Squares"

"Regression"

Have: Data (x_i, y_i)

Want: "Best-Fit" line $y = m_0 + m_1 x$

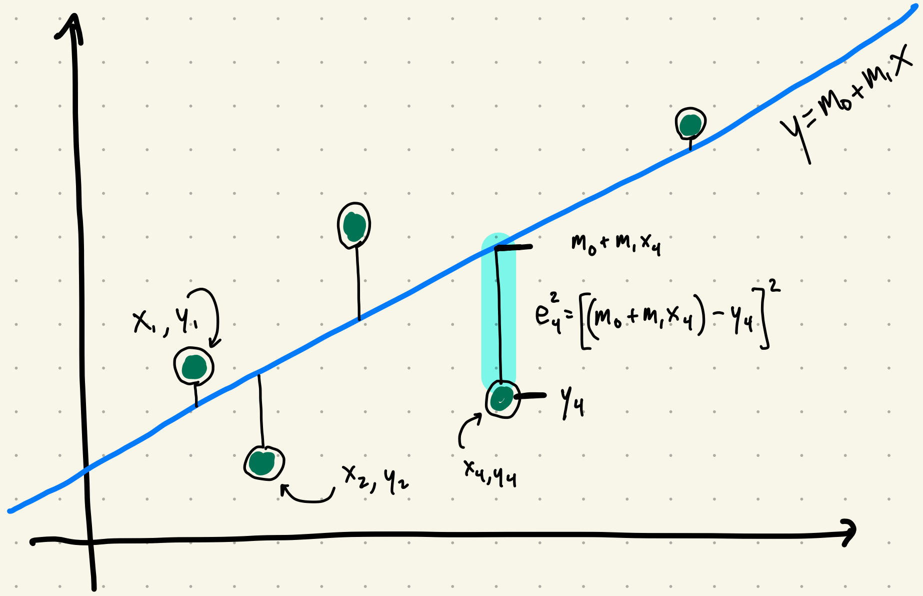
$$E(m_0, m_1) = \sum_{i=1}^{N} \left[(m_0 + m_1 x_i) - y_i \right]^2$$

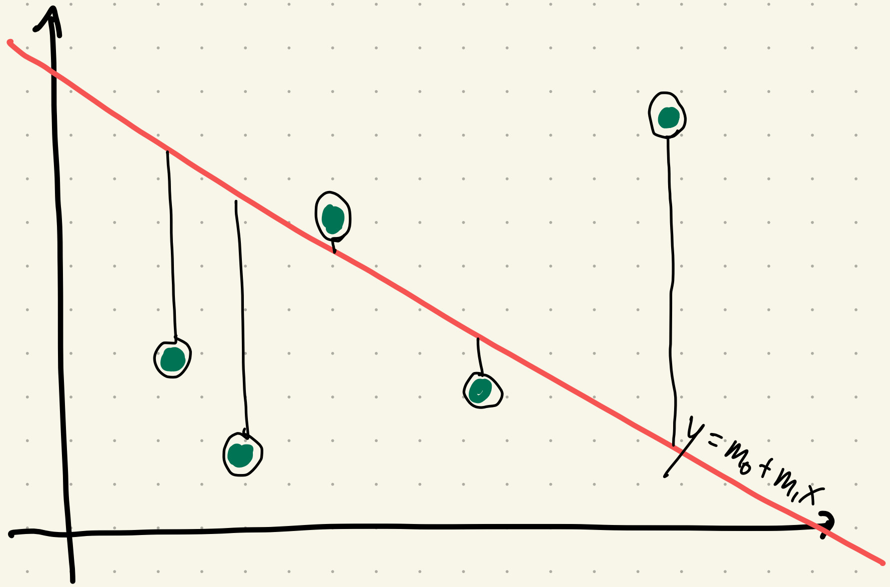
Goal: $\min\limits_{m_0, m_1} E(m_0, m_1)$

Find "best" slope m_1, intercept m_0.

$\downarrow$

minimizes Error.





$$E(m_0, m_1) = \sum_{i=1}^{N} \left[(m_0 + m_1 x_i) - y_i \right]^2$$

$$\frac{\partial E}{\partial m_0} = \sum_{i=1}^{N} 2 \left[(m_0 + m_1 x_i) - y_i \right] = 0$$

$$\cancel{2} \left[\sum_{i=1}^{N} m_0 + m_1 \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i \right] = 0$$

$$N m_0 + \left[\sum_{i=1}^{N} x_i \right] m_1 = \sum_{i=1}^{N} y_i$$

$$\frac{\partial E}{\partial m_1} = \sum_{i=1}^{N} 2 \left[(m_0 + m_1 x_i) - y_i \right] x_i = 0$$

$$\cancel{2} \left[\sum_{i=1}^{N} m_0 x_i + \sum_{i=1}^{N} m_1 x_i^2 - \sum_{i=1}^{N} x_i y_i \right] = 0$$

$$\left[\sum_{i=1}^{N} x_i \right] m_0 + \left[\sum_{i=1}^{N} x_i^2 \right] m_1 = \sum_{i=1}^{N} x_i y_i$$

Find the minimum of E in terms of m_0, m_1.

- Take derivative
- Set $= 0$
- [2^{nd} Deriv. test $\cap$ $\cup$]

$$N m_0 + \left[\sum_{i=1}^{N} x_i\right] m_1 = \sum_{i=1}^{N} y_i$$

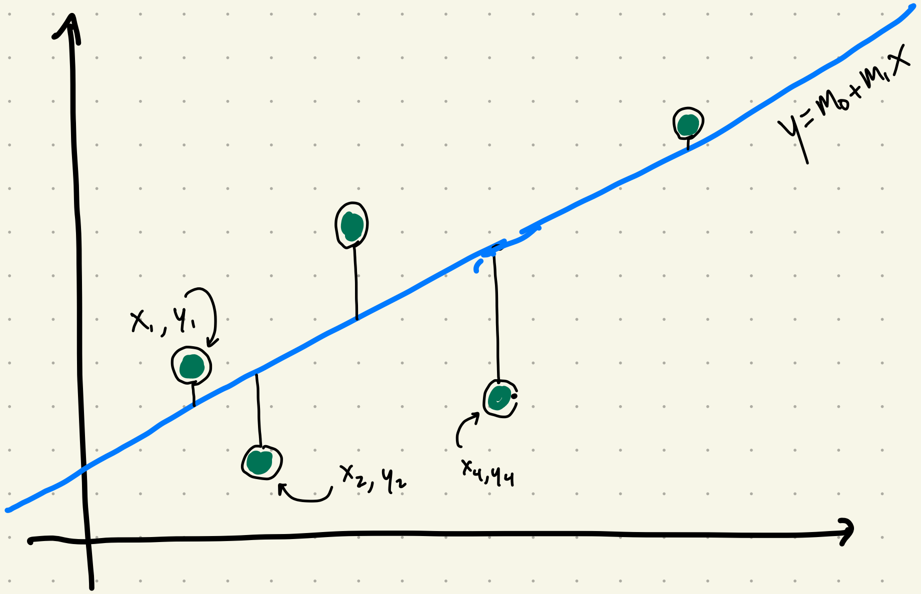
$$\left[\sum_{i=1}^{N} x_i\right] m_0 + \left[\sum_{i=1}^{N} x_i^2\right] m_1 = \sum_{i=1}^{N} x_i y_i$$

$$\boxed{\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} = \frac{1}{N\sum x_i^2 - \left(\sum x_i\right)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & N \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}}$$

$$\text{“} A \vec{m} = \tilde{b} \text{”}$$

$$\boxed{\begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}}$$

$$\overbrace{\cancel{A^{-1} A}}^{I} m = A^{-1} b \longrightarrow m = A^{-1} b$$



$Am = y$ "overdetermined" (more equations than unknowns.)

Normal Equations

mult both sides by A^T

$$
\begin{aligned}
y_1 &= m_0 + m_1 x_1 \\
y_2 &= m_0 + m_1 x_2 \\
y_3 &= m_0 + m_1 x_3 \\
&\vdots \\
y_n &= m_0 + m_1 x_n
\end{aligned}
\quad \rightarrow \quad
\begin{array}{cccc}
n \times 2 & 2 \times 1 & & n \times 1 \\
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_i \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} &
\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} & = &
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_n \end{bmatrix}
\end{array}
$$

$$\overset{\substack{\color{red}{2\times 2}\\ 2\times\cancel{n}\ \ \cancel{n}\times 2}}{\begin{bmatrix} 1 & \cdots & 1 & \cdots & 1 & 1 & 1 \\ x_n & \cdots & x_i & \cdots & x_3 & x_2 & x_1 \end{bmatrix}\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_i \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}} \overset{\substack{\color{red}{2\times 1}\\ 2\times 1}}{\begin{bmatrix} m_0 \\ m_1 \end{bmatrix}} = \overset{\substack{\color{red}{2\times 1}\\ 2\times\cancel{n}\ \ \cancel{n}\times 1}}{\begin{bmatrix} 1 & \cdots & 1 & \cdots & 1 & 1 & 1 \\ x_n & \cdots & x_i & \cdots & x_3 & x_2 & x_1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_n \end{bmatrix}}$$

$$\begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$X\vec{m} = \vec{y}$$

$$(X^T X)\vec{m} = X^T y$$

Normal

$$\boxed{\vec{m} = (X^T X)^{-1} X^T y}$$

What is in X?

$$\begin{array}{c} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ x_i^0 & x_i^1 \end{matrix} \end{array}$$

$$y = m_0 + m_1 x + m_2 x^2$$

parabola

$$(x_i, y_i)$$

$$y_1 = m_0 + m_1 x_1 + m_2 x_1^2$$

$$y_2 = m_0 + m_1 x_2 + m_2 x_2^2$$

$$y_3 = m_0 + m_1 x_3 + m_2 x_3^2$$

$$\vdots$$

$$y_i = m_0 + m x_i + m_2 x_i^2$$

$$X m = y$$

$$\begin{array}{c} \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & & \vdots \\ 1 & x_i & x_i^2 \\ 1 & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} \\ \begin{matrix} \uparrow & \quad\uparrow & \quad\uparrow \\ x_i^0 & \quad x_i^1 & \quad x_i^2 \end{matrix} \end{array} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$