4/29/21 Final Review

Equilibria:

1. general def'n
2. how to find
   - SIR, SEIR
   - linear/affine systems
3. stability def'n
4. how to find
5. Rel'ship to vector fields/nullclines

Recipe for finding equilibria

1. Is your problem in cont. or discrete t?
2. either set $x(t) = x(t+1) = x$ → solve for x.
   or $\dot{x} = 0$

Definition (Continuous time / Discrete time):

A system is at equilibrium when...
- no variables are changing over time
- all variable values are constant

How to find an equilibrium (generic):

Feels like a local min/max?

$$x(t) = c \qquad y(t) = d \qquad z(t) = f$$

$\Rightarrow$
$$\left.\begin{aligned} \dot{x}(t) &= 0 \\ \dot{y}(t) &= 0 \\ \dot{z}(t) &= 0 \end{aligned}\right] \text{ continuous time}$$

$\Rightarrow$
$$\left.\begin{aligned} x(t+1) &= x(t) \\ y(t+1) &= y(t) \\ z(t+1) &= z(t) \end{aligned}\right] \text{ discrete time}$$

Is my system linear?

$$\frac{d\vec{n}}{dt} = M\vec{n} \quad \text{(linear)}$$

$$\frac{d\vec{n}}{dt} = M\vec{n} + \vec{c} \quad \text{(affine)}$$

[subset of linear]

$\downarrow$

$$\dot{n}_1 = 3n_1 + 4n_2 - n_3$$
$$\dot{n}_2 = 5n_1 + 2n_2 - n_3$$
$$\dot{n}_3 = n_2 + n_3$$

no sin, cos, tan, e^x

no x^2, x^3, $\sqrt{x}$

$$\dot{n}_1 = 3n_1 + 4n_2$$
$$\dot{n}_2 = 3n_1 + n_1 n_2$$

nonlinear term

$$\frac{d\vec{n}}{dt} = M\vec{n}$$

matrix is not full of scalars — have variables!

$$\begin{pmatrix} \dot{n}_1 \\ \dot{n}_2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3+n_2 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{n}_1 \\ \dot{n}_2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & n_1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Linear Case, Continuous

① get system in form

$$\frac{d\vec{n}}{dt} = M\vec{n} + \vec{c}$$

② Apply def'n of Eq.

$$\frac{d\vec{n}}{dt} = \vec{0} \Rightarrow M\vec{n} + \vec{c} = \vec{0}$$

solve for $\vec{n}$

big assumption.

$$M^{-1}(M\vec{n} + \vec{c}) = M^{-1}\vec{0}$$

left distributive rule

any matrix times $\vec{0}$ gives $\vec{0}$.

$$\underbrace{M^{-1}M}_{I}\vec{n} + M^{-1}\vec{c} = \vec{0}$$

$$\vec{n} + M^{-1}\vec{c} = \vec{0}$$

$$\boxed{\vec{n} = -M^{-1}\vec{c}}$$

bonus result: if $\vec{c} = \vec{0}$ then $\vec{n} = \vec{0}$

When is M invertible?

① if M^{-1} exists, M is invertible.

② $\det(M) \neq 0$

③ $M\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for each different $\vec{b}$.

④ $M\vec{x} = 0 \Rightarrow \vec{x} = 0$

⑤ No $\lambda = 0$ eigenvalues. (All eigs $\lambda \neq 0$)

Eigen:

$$Ax = \lambda x$$

What if $\lambda = 0$ is an eig?

$$\Rightarrow \quad A\vec{x} = 0\,\vec{x}$$

$$A\vec{x} = \vec{0}$$

[By def'n, $\vec{x}$ eig.vector is non zero. $A\vec{x} = 0$ but $\vec{x} \neq 0$]

incompatible.

[to be invertible, must have $A\vec{x} = 0 \Rightarrow \vec{x} = 0$]

Stability Linear, Cont time

$$\frac{d\vec{n}}{dt} = M\vec{n} + \vec{c}$$

- if M invertible, then equilibrium is at $\vec{n} = -M^{-1}\vec{c}$
- stability:

  Equilibrium is stable when:

  ① λ Real: all $\lambda < 0$

  ② λ Complex: $\mathrm{Re}(\lambda) < 0$ for all λ.

Recipe:

(i) find eigenvalues

(ii) write only Real parts

(iii) if all $\mathrm{Re}(\lambda) < 0 \Rightarrow$ stable. Else: Not stable

SIR, SEIR

No growth. No decay.
Always same pop size!

(nice flow out, flow in!)

$$S \xrightarrow{\beta S I} E \xrightarrow{\alpha E} I \xrightarrow{\gamma I} R$$

nonlinear term

$$\dot{S} = -\beta S I$$
$$\dot{E} = \beta S I - \alpha E$$
$$\dot{I} = \alpha E - \gamma I$$
$$\dot{R} = \gamma I$$

- Is this system linear or $\boxed{\text{nonlinear?}}$
  (can I write this as $\dot{x} = Mx$? $x = \begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix}$)
- Set derivs to 0.

$$-\beta S I = 0 \qquad -\beta \cancel{S} \cdot 0 = 0 \ \checkmark \qquad 0 = 0 \ \checkmark$$
$$\beta S I - \alpha E = 0 \ \xrightarrow{I=0} \ \beta \cancel{S}\, 0 - \alpha E = 0 \qquad -\alpha E = 0 \qquad 0 = 0 \ \checkmark$$
$$\alpha E - \gamma I = 0 \qquad \alpha E - \cancel{\gamma\, 0} = 0 \qquad \alpha E = 0 \qquad 0 = 0 \ \checkmark$$
$$\gamma I = 0$$

easiest $I = 0$

$E = 0$

Equilibria

$I = 0,\ E = 0$

$S + \cancel{E} + \cancel{I} + R = 1$

$$\boxed{S + R = 1,\ I = 0,\ E = 0}$$

$$\boxed{\begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \\ 1-x \end{pmatrix} \quad 0 \le x \le 1}$$

Vector Fields

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

sketch V.F. for this.

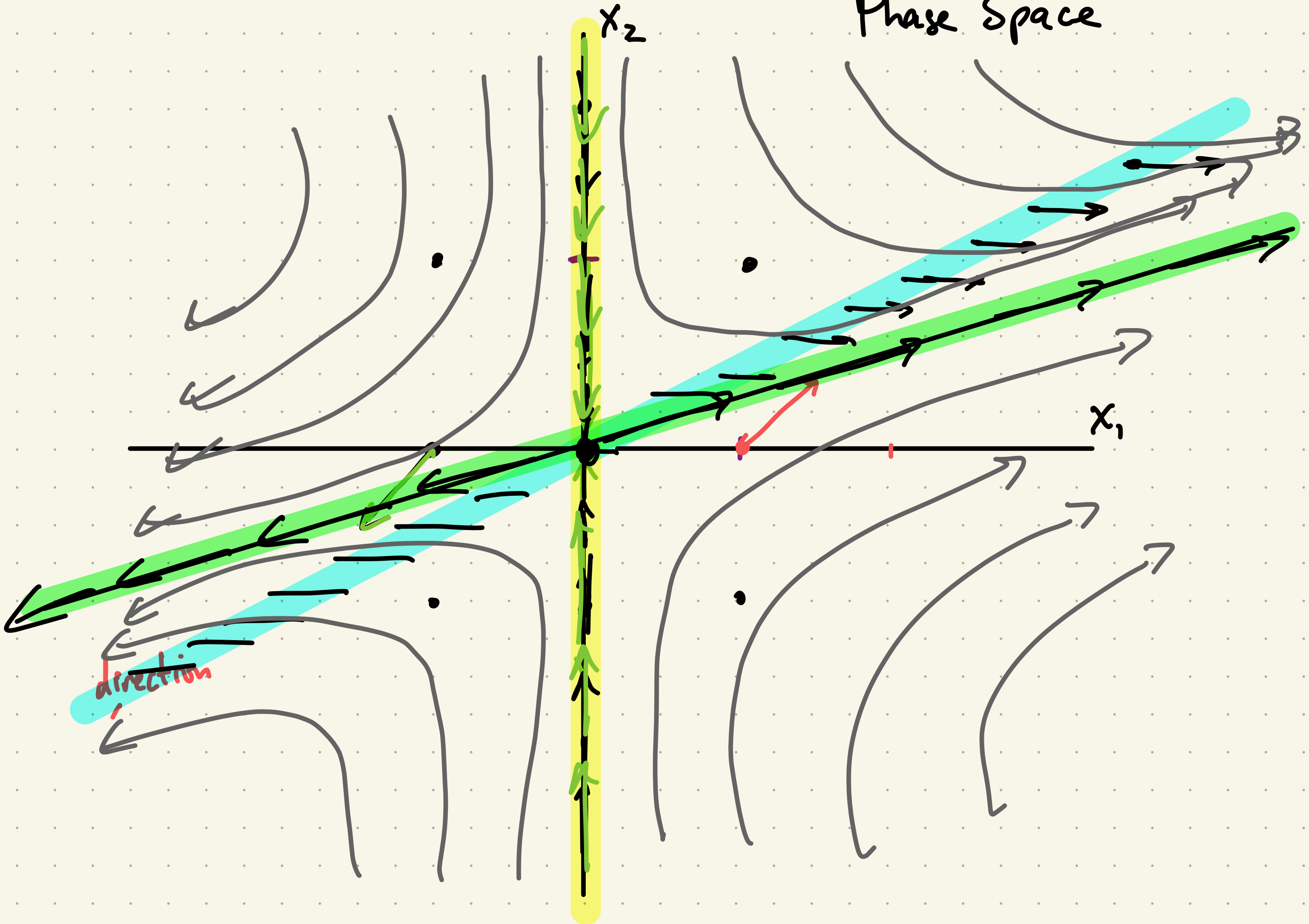
$\dot{x}_1 = x_1$

$\dot{x}_2 = x_1 - 2x_2$

Set $\dot{x}_1 = 0$ no change in x_1
no left/right.

$\dot{x}_2 = 0$ no change in x_2
no up/down.

Phase Space



Nullclines?

Two lines in phase space.

$\dot{x}_1 = 0$ $\quad x_1 = 0$ ONLY VERTICAL VECTORS.

$\dot{x}_2 = 0$ $\quad x_1 - 2x_2 = 0$ so $x_2 = \frac{1}{2}x_1$
ONLY HORIZ. MVMT

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

evals: $1, -2$

evecs?

① Pick one λ $(\lambda = 1)$

② Form $(A - \lambda I)\vec{x} = 0$

$$\begin{pmatrix} 1-1 & 0 \\ 1 & -2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

③ Solve for relationship btwn x_1, x_2.

$$\begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0x_1 + 0x_2 = 0$$

$$x_1 - 3x_2 = 0 \longrightarrow x_1 = 3x_2 \qquad \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

① $\lambda = -2$

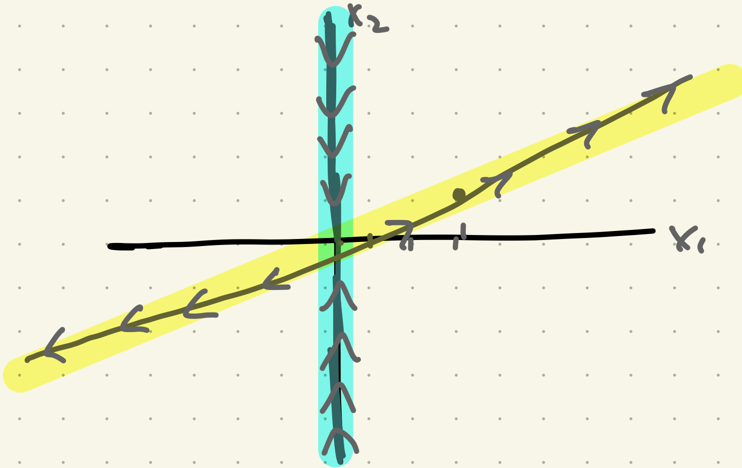
② $$\begin{pmatrix} 1 - {-2} & 0 \\ 1 & -2 - {-2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

③ $3x_1 = 0 \Rightarrow x_1 = 0$

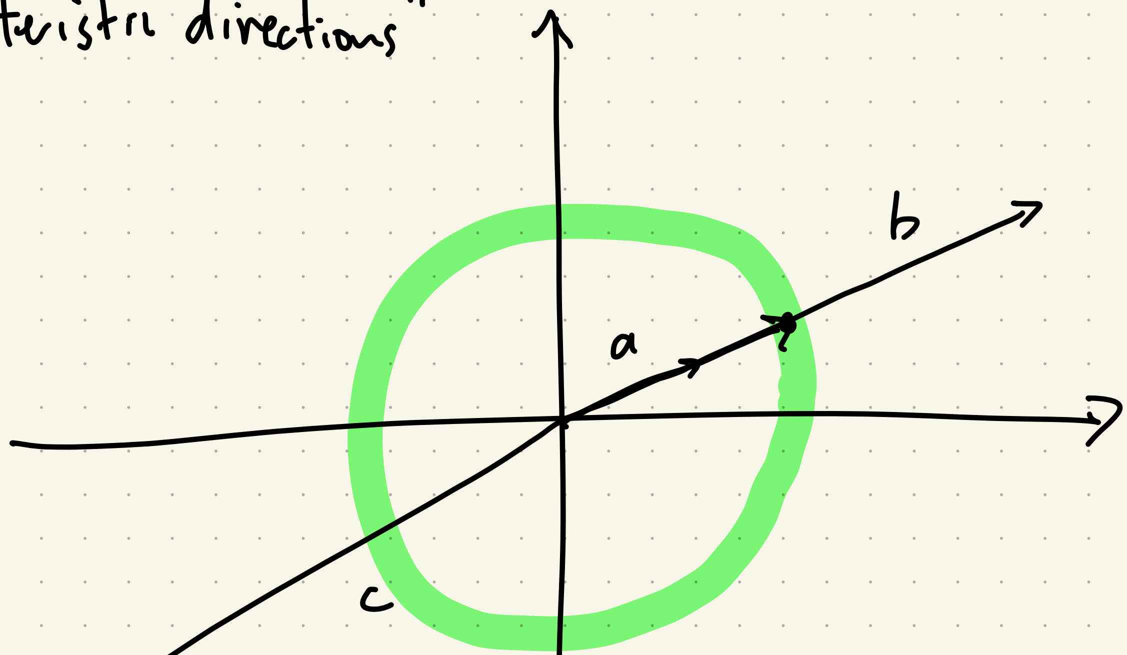
$x_1 = 0 \Rightarrow x_1 = 0$

$\longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



More on E. vectors

"characteristic directions"



$$\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$$

$\lambda = 1$

$(A - \lambda I)x = 0$

$$\begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\downarrow$

$x_1 - 3x_2 = 0$

$x_1 = 3x_2$

$\begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$ a

$\begin{pmatrix} -30 \\ -10 \end{pmatrix}$ c

$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ b

All of these are eigenvectors.

$$\boxed{A\vec{x} = \lambda \vec{x}}$$

$$\boxed{15\vec{y} = \vec{x}}$$

my ans

yours.

$$A(15\vec{y}) = \lambda(15\vec{y})$$

$$\cancel{15} A\vec{y} = \cancel{15}\lambda\vec{y}$$

$$\boxed{A\vec{y} = \lambda\vec{y}}$$

• Can I use row reduction to solve $A\vec{x} = \vec{b}$ (on the final) ?

NO.

$A\vec{x} = \vec{b} \rightsquigarrow$ form A^{-1}

$\vec{x} = A^{-1}\vec{b}$

You may check using

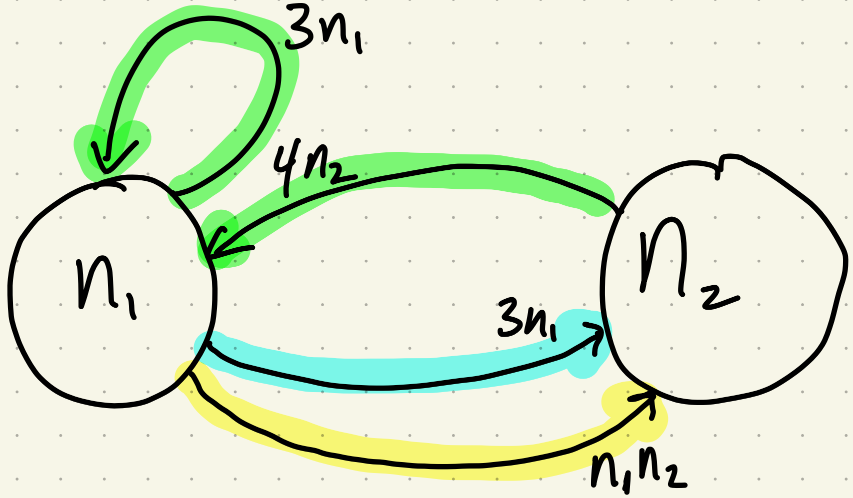
1) regular algebra

2) Row Reduction (not covered in class)

Flow Diagram Ex?

$$\dot{n}_1 = 3n_1 + 4n_2$$

$$\dot{n}_2 = 3n_1 + n_1 n_2$$



$$\frac{d\vec{n}}{dt} = M\vec{n}$$

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

<u>NOTE</u>: unlike SIR-type model, where total pop size is const, <u>here</u> we have no such guarantee.

→ We're not tracking individuals.

↓

An arrow in flow diagram doesn't mean flow out AND in

Ex: flow out of E, flow in to I

$$E \xrightarrow{\alpha E} I$$

Exponential Growth/Decay

$$\frac{dx}{dt} = \alpha x$$

$$\int \frac{dx}{x} = \int \alpha \, dt$$

$$\ln x = \alpha t + c$$

$$x = e^{\alpha t + c}$$

$$x = e^{\alpha t} \underbrace{e^{c}}_{k}$$

$$x = k e^{\alpha t}$$

IC: $x(t=0) = x_0$

$$x_0 = k e^{\alpha \cdot 0}$$

$$x_0 = k$$

Solution:

$$x(t) = x_0 e^{\alpha t}$$

growth: $\alpha > 0$

decay: $\alpha < 0$

Newton's Law of Cooling

$$\frac{dT}{dt} = -k\left(T - T_{ext}\right)$$

$k > 0$

also not super clear in notes!

notes were confusing, in which I wrote this as T_0!

Bad choice, Daniel!