Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2897

1. What is the **long-term growth rate** of a population?
2. What is the **long-term class structure** of a population?
3. Which **classes contribute most** to the long-term growth rate of a population.

Population **transition matrices** have interesting properties:
1. All entries are ≥ 0.
2. The matrix is square.

When these conditions are met, the **Perron-Frobenius Theorem** tells us that:
1. The eigenvalue with largest magnitude λ_1 will *never be negative*.
2. This eigenvalue will also *always be real*.
3. The eigenvector $\overrightarrow{x}_1$ associated with this eigenvalue will *also* be non-negative and real.

Revisiting the classics

Carbon dating allows us to estimate when an organism died because:
1. C14 and C12 are in a stable ratio in the environment.
2. C14 is radioactive, and decays with a half-life of 5730 years.
3. Thus measuring the ratio of C14 to C12 in an organism can tell us when the organism died.

Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings.

Buying a house involves taking out a loan that you pay back over 30 years of constant payments—but the amount that you owe grows in between payments.

Carbon Dating

$$X(t) = X_0 e^{-\left(\frac{\ln 2}{H.L.}\right)t}$$

Carbon dating allows us to estimate when an organism died because:

1. C14 and C12 are in a stable ratio in the environment.
2. C14 is radioactive*, and decays (to N14) with a half-life of 5730 years.
3. Thus measuring the ratio of C14 to C12 in an organism can tell us when the organism died.

Write this as a differential equation. A "half life" is the amount of time that it takes for the mass to decrease by a factor of 2. How should this show up in the ODE?

Radioactive Decay

$$\frac{dC}{dt} = -kC$$

placeholder. Initial amt.

$$C(0) = C_0$$

$$C(5730) = \frac{1}{2}C_0$$

statement about half life.

S.o.V.

$$\frac{dC}{C} = -k\,dt$$

$$\ln C = -kt + \text{const}$$

$$C(t) = C_0 e^{-kt}$$

$$C(5730) = \frac{1}{2}C_0$$

$$\frac{1}{2}\cancel{C_0} = \cancel{C_0} e^{-k\,5730}$$

$$\ln\frac{1}{2} = -k \cdot 5730$$

$$\boxed{\frac{\ln 2}{5730} = k}$$

* The radioactive decay means that the total mass of the radioactive isotope decreases at a rate directly proportional to that mass.

Carbon Dating

$-\ln x = \ln x^{-1} = \ln \frac{1}{x}$

You are wandering around Boulder looking for a COVID vaccine when you come across the frozen body of a woolly mammoth. Noting C14's half life of 5730 years, you observe that the mammoth has only **20%** of the C14 that it'd have if it were freshly deceased. How old is the mammoth? Solve for t, but... under what conditions?

$$C(t) = C_0 e^{-\left(\frac{\ln 2}{5730}\right)t}$$

$$0.2\cancel{C_0} = \cancel{C_0} e^{-\left(\frac{\ln 2}{5730}\right)t}$$

$$\ln \frac{1}{5} = -\frac{\ln 2}{5730} t$$

$$\ln 5 = \frac{\ln 2}{5730} t$$

$$5730 \cdot \frac{\ln 5}{\ln 2} = t$$

$$\boxed{13{,}304 = t}$$

100% $t=0$

50% $t=5730$

25% $t=2*5730$

20% 2.32

12.5% $t=3*5730$

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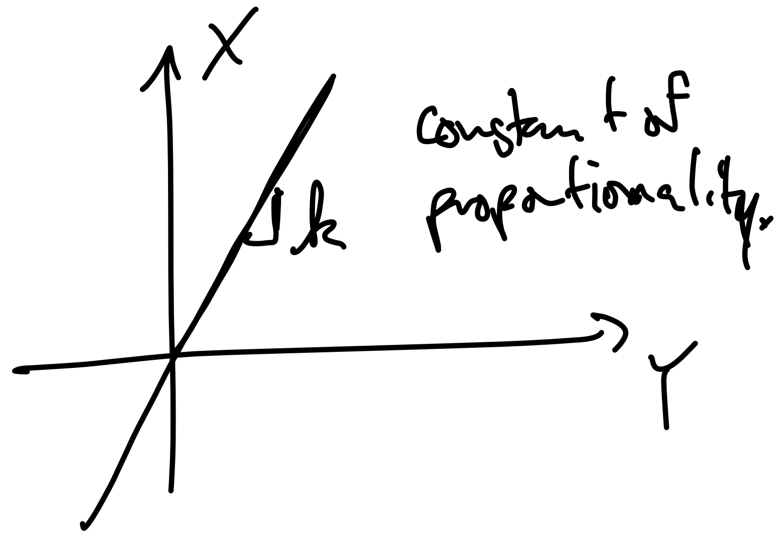
Newton's law of cooling

$\frac{dT}{dt}$ negative

Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings. **Write this an an ordinary differential equation**.

X is directly proportional to Y $\longrightarrow$ $X = kY$



$$\frac{dT}{dt} = -k\,(T - T_0)$$

$$\boxed{\frac{dT}{dt} = -k(T - T_0) \quad \text{solve!}}$$

using Initial Condition $T(0) = T_{initial}$.

two variables: $T_{surroundings}$ (T_0), T_{body} (T)

Newton's law of cooling

S.O.V. Also totally reasonable!

Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings. **Write this an an ordinary differential equation**.

$$\frac{dT}{dt} = -k(T - T_0)$$

$$\frac{dT}{dt} = -kT + kT_0$$

into standard form ↴

$$\frac{dT}{dt} + kT = kT_0$$

integrating factor

$$\mu = e^{kt}$$

$$e^{kt}\left(\frac{dT}{dt} + kT\right) = kT_0 e^{kt}$$

$$\int \frac{d}{dt}\left(Te^{kt}\right) dt = \int kT_0 e^{kt} dt$$

after integration...

$$Te^{kt} = kT_0\left(\frac{e^{kt}}{k} + c\right)$$

now solve for T

$$Te^{kt} = e^{kt}T_0 + c$$

$$T(t) = e^{-kt}e^{kt}T_0 + ce^{-kt}$$

$$T(t) = T_0 + ce^{kt}$$

plug in initial cond'n

$$T(0) = T_{init} = T_0 + ce^{0}$$

$$c = T_{init} - T_0$$

Thus:

$$T(t) = T_0 + (T_{init} - T_0)e^{kt}$$

Newton's law of cooling

$$\frac{dT}{dt} = k\left(T_{0} - T\right)$$

You show up to IRL office hours for CSCI 2897, but there is no one in the office. There is just a French Press.

You touch the French Press. Hmmm. Still warm. You estimate that it is 50 degrees. $\longrightarrow t = t_1 \quad T = 50$

You wait 10 minutes. Now the French Press is 40 degrees. $\longrightarrow t = t_1 + 10 \quad T = 40$

Given that room temperature is 20 degrees, **when was the boiling water** (100 degrees) **poured** into the French Press? $\longrightarrow T_0 = 20$

If $T = 100$, what is t?

$$T(t) = T_0 + (T_{init} - T_0)e^{kt}$$

Let $t_1 = 0$, $T = 50$

$t = 10$, $T = 40$

solve for k

$\longrightarrow$ $T = 100$, what is t?

E.C.

$\boxed{+10 \text{ to HW5}}$

Newton’s law of cooling

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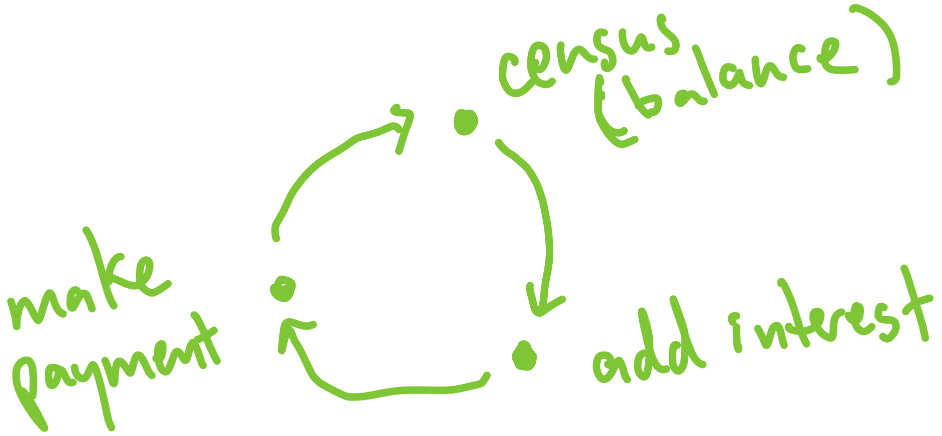
Home mortgages.

Buying a house involves taking out a loan that you pay back over 30 years of constant payments—but the amount that you owe grows in between payments.

Suppose that you take out a loan of $200,000, to be paid back after 30 years of monthly payments. Each month you pay down the loan by P, but you accumulate interest at an annual rate of 3%.

1. Write a **life-cycle diagram** for your loan balance. (One cycle = one month)
2. Write a **recursion** for your loan balance. L
3. What is the **initial condition** of the recursion? What is the **final condition**?
4. What is your **monthly payment P**?



$$L(t+1) = L(t)\left[1 + \frac{r}{12}\right] - P$$

$$L(0) = 200000 = L_0$$

$$L(360) = 0$$

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$$\boxed{P = L_0 \frac{R^{360}(1-R)}{1-R^{360}}}$$

$$L(t+1) = L(t)\left[1+\frac{r}{12}\right] - P$$

$$L(0) = L_0$$

$$L(360) = 0$$

Let $R = \left[1+\frac{r}{12}\right]$

so

$$L(t+1) = L(t)R - P$$

$$L(0) = L_0$$

$$L(1) = L_0 R - P$$

$$L(2) = (L_0 R - P)R - P$$

$$L(3) = ((L_0 R - P)R - P)R - P$$

$$L(n) = L_0 R^n - PR^{n-1} - PR^{n-2} - \dots - PR - P$$

$$L(n) = L_0 R^n - P\left[\sum_{i=0}^{n-1} R^i\right]$$ partial sum of a geometric series.

$$L(n) = L_0 R^n - P\frac{1-R^n}{1-R}$$

$$0 = L_0 R^{360} - P\frac{1-R^{360}}{1-R}$$

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