Calculating Biological Quantities

CSCI 2897

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- HW3 posted — due March 30.

Last time on CSCI 2897:

$$\dot{S} = -\beta SI$$

$$\dot{E} = \beta SI - \alpha E$$

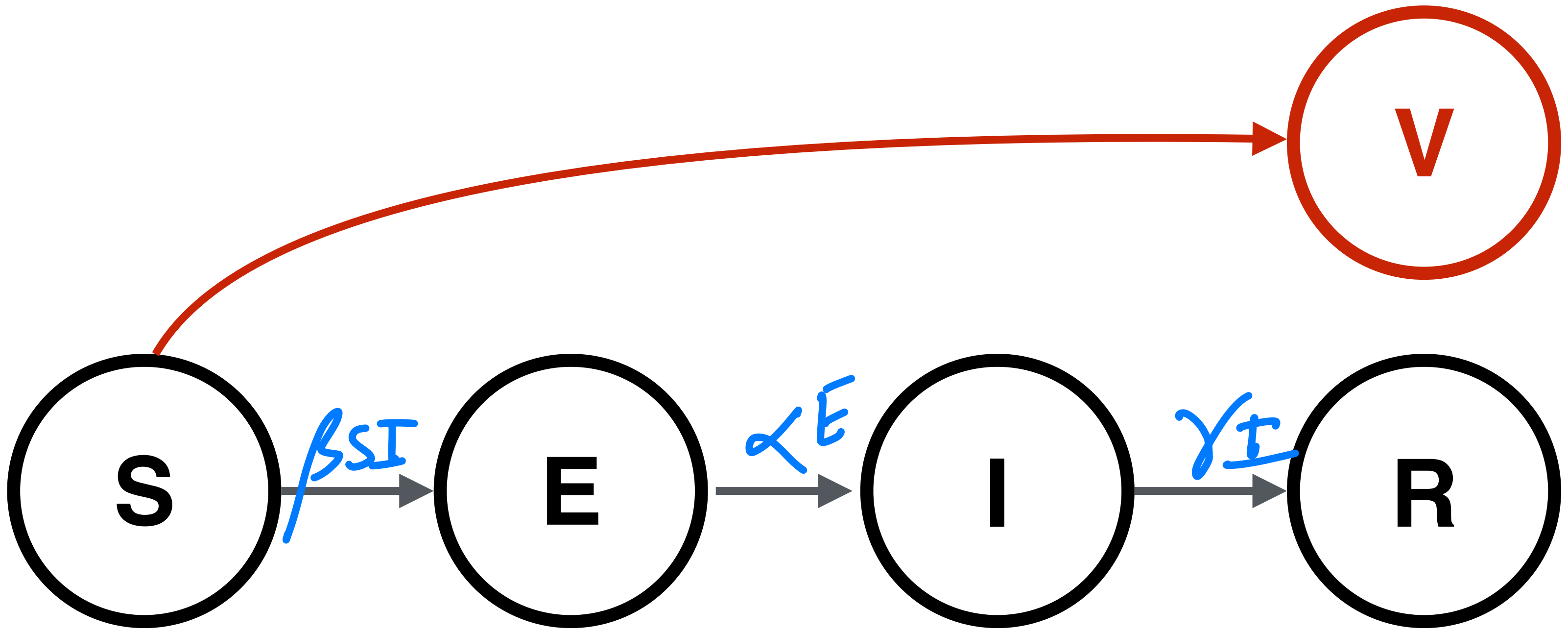
$$\dot{I} = \alpha E - \gamma I$$

$$\dot{R} = \gamma I$$

same

where $S + E + I + R + V = 1$

only diff.



This is a model for a vaccine with $ve = 1$.

vaccine efficacy

Model 2: The All-or-Nothing vaccine model

If $ve = 1 \rightarrow$ reduces to previous model.

An **all-or-nothing** vaccine completely protects ve and leaves $1 - ve$ unprotected.

infectiousness from unvaccinated

infectiousness from vaccinated.

$$\dot{S} = -\beta S I - \beta S I_v = -\beta S (I + I_v)$$

$$\dot{I} = \beta S I + \beta S I_v - \gamma I$$

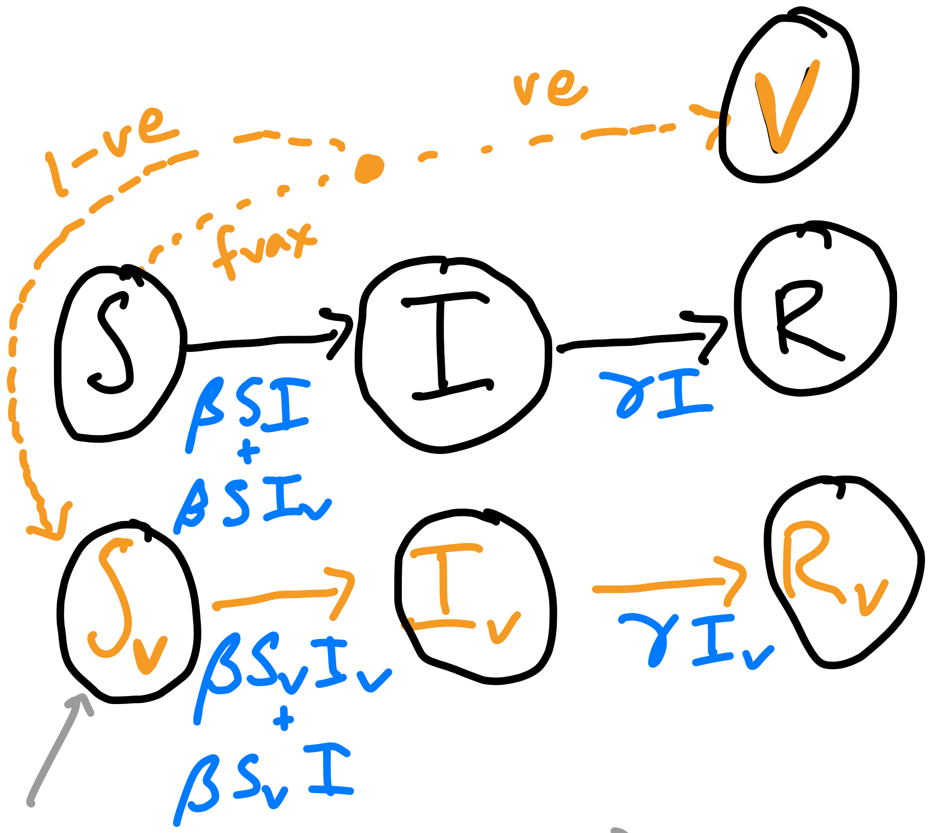
$$\dot{R} = \gamma I$$

$$\dot{S}_v = -\beta S_v I_v - \beta S_v I$$

$$\dot{I}_v = \beta S_v I_v + \beta S_v I - \gamma I_v$$

$$\dot{R}_v = \gamma I_v$$

$$\dot{V} = 0$$



S_v = susceptible (still) after vaccine

All of the vaccination takes place as an <u>initial condition</u>.

vaccinate a fraction f_{vax} of total pop.

$$S_0 \leftarrow S_0 - f_{vax} \qquad S_{v0} = f_{vax} \cdot (1 - ve)$$

$$I_0 \leftarrow I_0 \qquad V_0 = f_{vax} \cdot ve$$

$$R_0 \leftarrow R_0$$

Model 3: The Leaky Vaccine model

A **leaky** vaccine provides ve partial protection to everyone.

If we set $ve = 1$, this model reduces to Model 1.

$$\dot{S} = -\beta S (I + I_v)$$

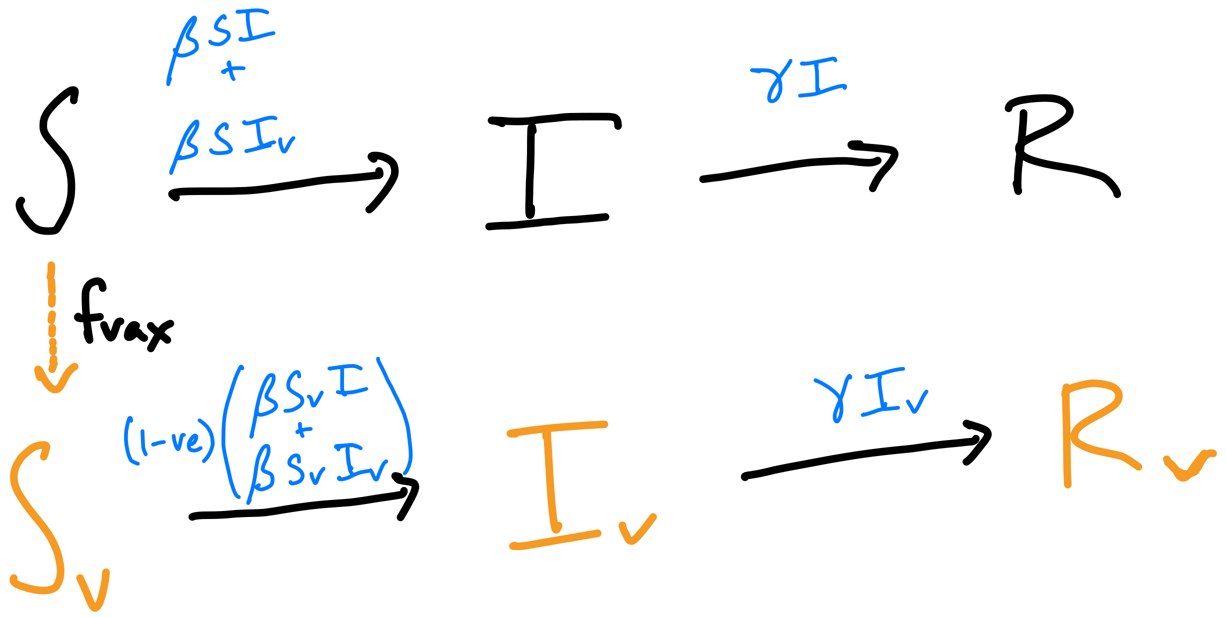
$$\dot{I} = -\gamma I + \beta S (I + I_v)$$

$$\dot{R} = \gamma I$$

$$\dot{S}_v = -\beta S_v (I + I_v)(1 - ve)$$

$$\dot{I}_v = -\gamma I_v + \beta S_v (I + I_v)(1 - ve)$$

$$\dot{R}_v = \gamma I_v$$



In general, if you take a basic model, and add a new "feature" with a parameter that governs the strength of that feature... then setting strength to 0 should reduce to previous model.

Model 4: The Three-Factor Vaccine model

A three-factor vaccine considers ve_s, ve_I and ve_p…

protects susceptibles from infection

decreases one's outward infectiousness

protects against symptomatic disease.

If I set $ve_s = 1$, this model reduces to previous (perfect vax) model.

IFR = Infection Fatality Rate

$$\dot{S} = -\beta S I - \beta S I_v (1 - ve_I)$$

$$\dot{I} = \beta S I + \beta S I_v (1 - ve_I) - \gamma I$$

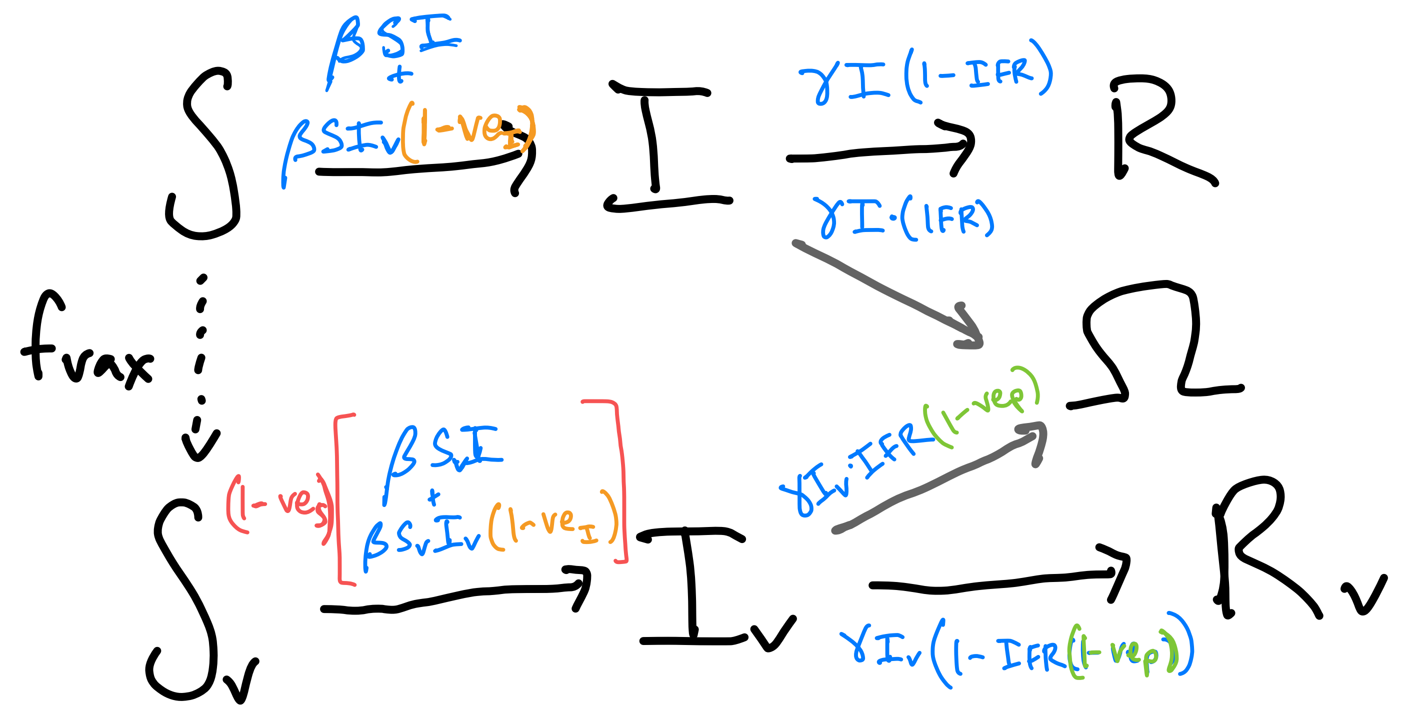
$$\dot{R} = \gamma I (1 - IFR)$$

$$\dot{S}_v = \left[-\beta S_v I - \beta S_v I_v (1 - ve_I)\right](1 - ve_s)$$

$$\dot{I}_v = \left[\beta S_v I + \beta S_v I_v (1 - ve_I)\right](1 - ve_s) - \gamma I_v$$

$$\dot{R}_v = \gamma I_v \left(1 - IFR(1 - ve_p)\right)$$

$$\dot{\Omega} = \gamma I (IFR) + \gamma I_v IFR (1 - ve_p)$$



Pr protected overall = not (not protected from infection × not protected from symptoms)

$$ve_{\text{clinical trial}} = 1 - (1 - ve_s)(1 - ve_p)$$

COVID-19 vaccines: measuring severe/symptomatic COVID-19.

Initial conditions or vaccine rollout?

How should we include vaccination itself?

Key: Is vaccination happening at the same time as transmission?

Initial Cond'n	Continuous Rollout
• Heroku Game "Vax"	• USA COVID-19 (rest of world-ish)
• Childhood vaccines	• Reactive vaccination campaign. (outbreak → vax)
• Annual Flu shot (before flu season)	• Ebola
• COVID-19 New Zealand Taiwan, S. Korea	• Polio

$$\begin{array}{c} V \\ \uparrow \\ S \longrightarrow I \longrightarrow R \end{array}$$

vaccination

$$\dot{S} = -\beta S I - \boxed{\quad}$$

$$\dot{V} = +\boxed{\quad}$$

vaccination

vax supply limited case.

$\boxed{\quad} = \theta$ const number of susc.

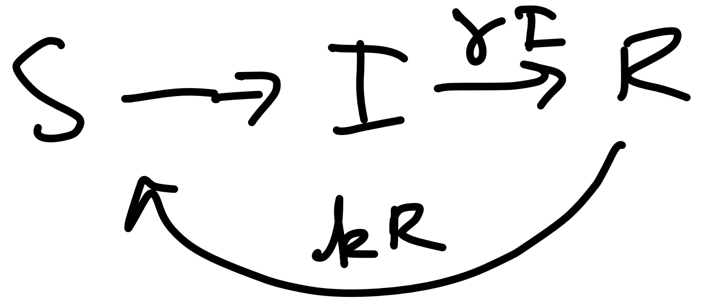
$\boxed{\quad} = \xi S$ const proportion of susc.

The durability of immunity: SIRS model (homework)

Suppose that immunity lasts only 5 years, on average.
How can we model this scenario?

$$\begin{tikzcd} S \arrow[r, "\beta S I"] & I \arrow[r, "\gamma I"] & R \arrow[ll, bend left, "??"] \end{tikzcd}$$

The durability of immunity: SIRS model



Suppose that immunity lasts only 5 years, on average.
How can we model this scenario?

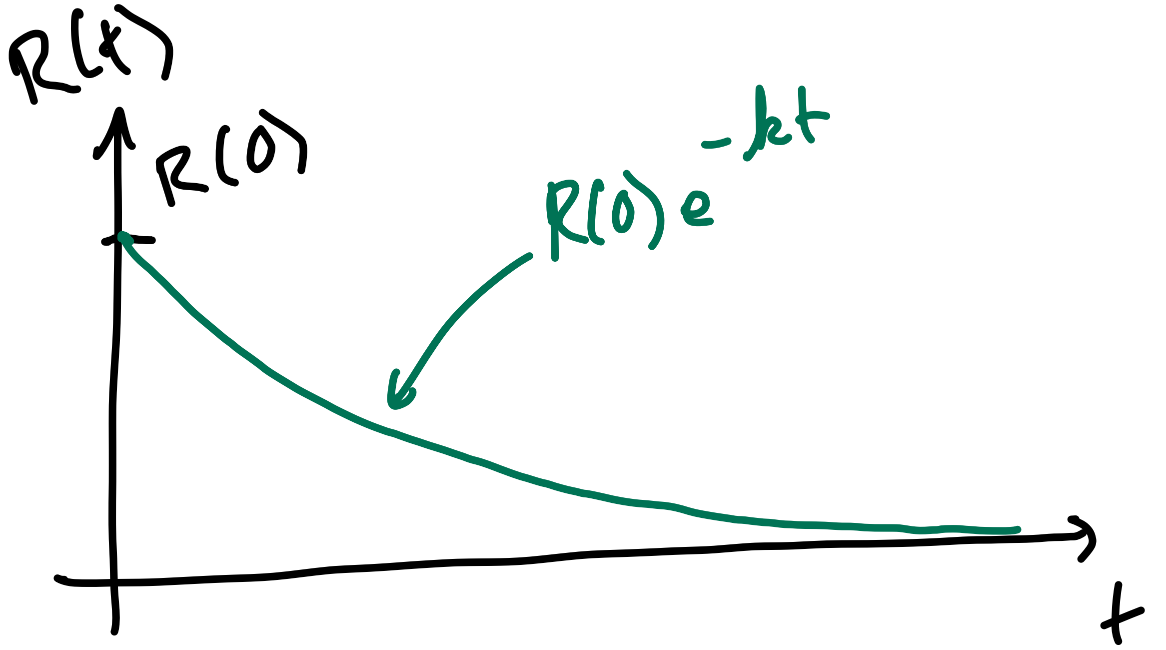
Point #1: Constant per-capita outflows are exponential.

In absence of I, $\dot{R} = \cancelto{0}{\gamma I} - kR$

$$\dot{R} = -kR$$

s.o.v.

$$\boxed{R(t) = R(0)\, e^{-kt}}$$



How long does the avg person in R stay there?

The durability of immunity: SIRS model

Suppose that immunity lasts only 5 years, on average.
How can we model this scenario?

I want avg to be 5 years...

$\Rightarrow \frac{1}{k} = 5 \text{ yrs}$

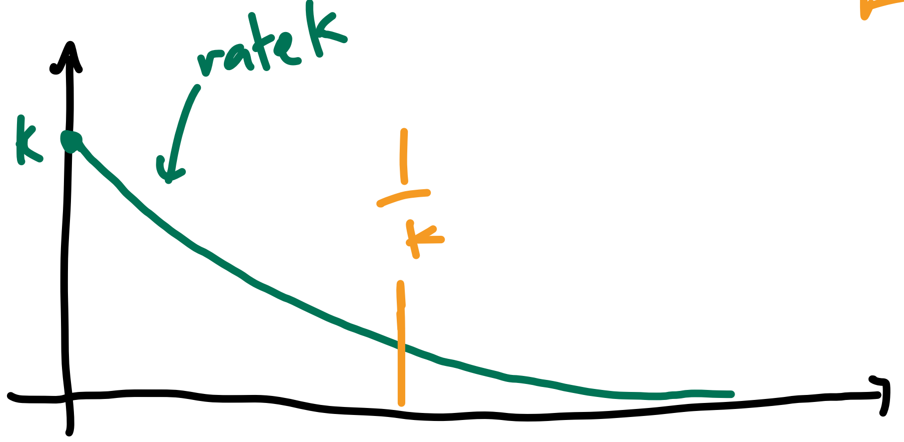
$k = \frac{1}{5 \text{ yrs}}.$

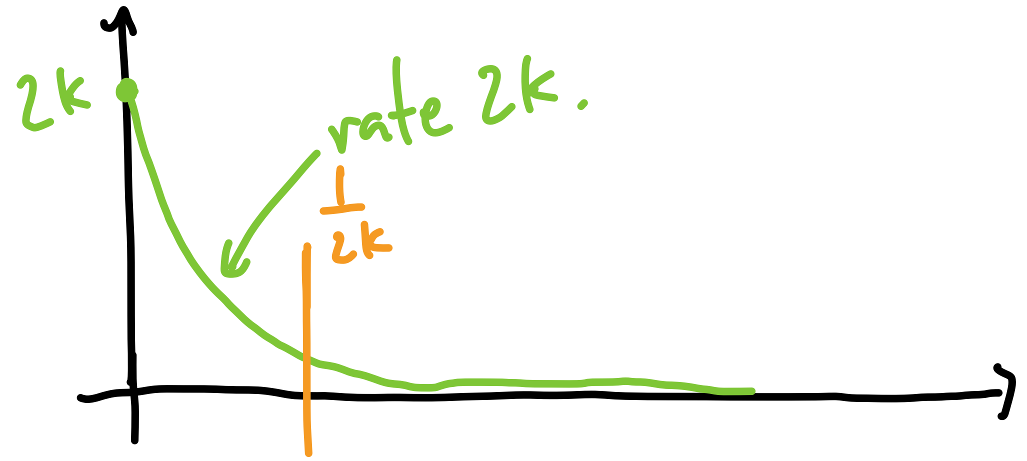
Point #1: Constant per-capita outflows are exponential.
Point #2: Typical waiting time = 1 / exponential rate.

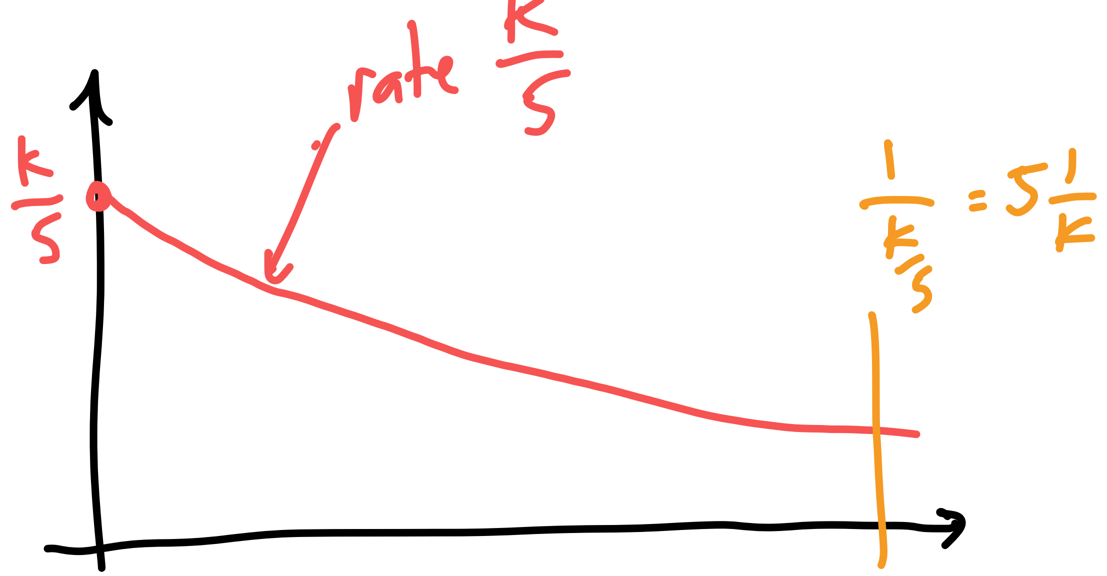
CSCI 3022:

Exponential Distr:

$P(\text{wait } x) = k e^{-kx} \longrightarrow \boxed{\text{avg (expected value)} = \frac{1}{k}}$







The durability of immunity: SIRS model

Suppose that immunity lasts only 5 years, on average.
How can we model this scenario?

$$
\begin{tikzcd}
S \arrow[r, "\beta S I"] & I \arrow[r, "\gamma I"] & R \arrow[ll, bend left, "kR"]
\end{tikzcd}
$$

Aside

$$I \xrightarrow{d_I^{-1}} R$$

$$\gamma = \frac{1}{d_I}$$

$$\dot{S} = -\beta S I + kR$$

$$\dot{I} = \beta S I - \gamma I$$

$$\dot{R} = \gamma I - kR$$

If γ is telling us about the rate of recovery on a timescale of days... $\longrightarrow$ k must also be in days.

~~$k = \frac{1}{5}$ (years timescale!)~~

$$k = \frac{1}{5 \cdot 365}$$

what we want

correct timescale.

The durability of immunity: SIRS model

Suppose that immunity lasts only 5 years, on average.
How can we model this scenario?

$$\dot{S} = -\beta S I + \frac{1}{5 \cdot 365} R$$

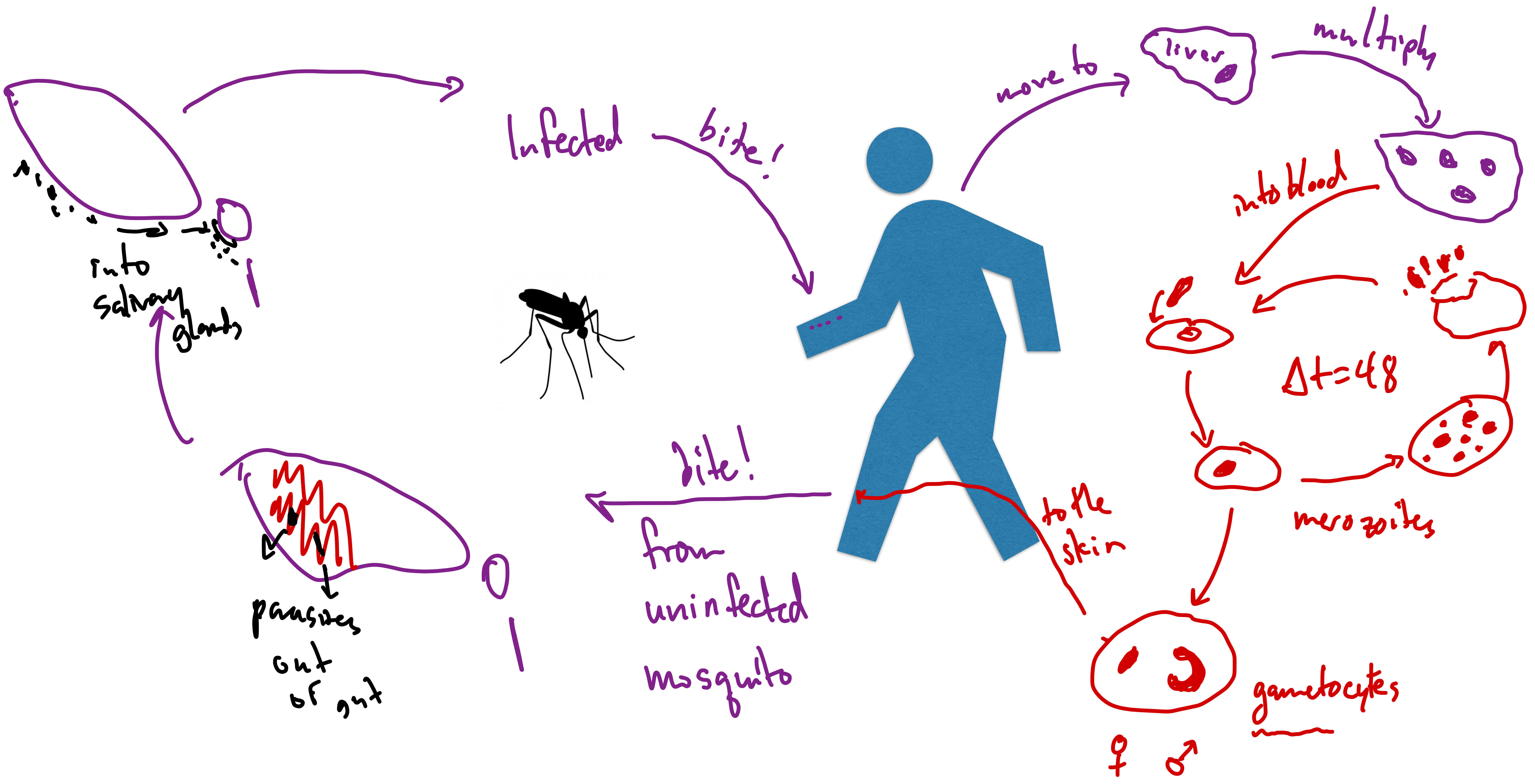
$$\dot{I} = \beta S I - \gamma I$$

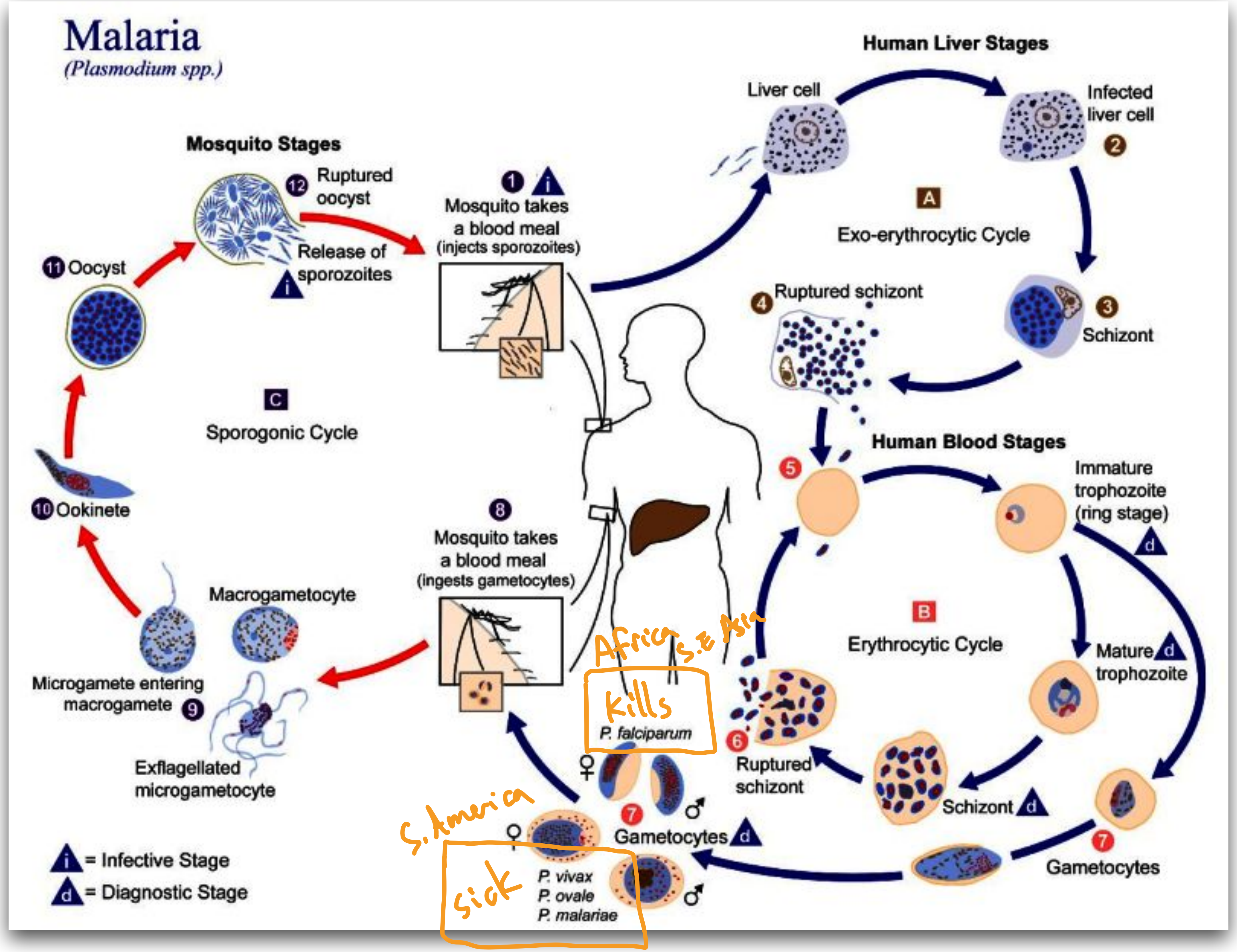
$$\dot{R} = \gamma I - \frac{1}{5 \cdot 365} R$$

where $S + I + R = 1$

The Malaria Parasite Life Cycle

- Two hosts: mosquito, human




Malaria
(Plasmodium spp.)
Mosquito Stages
Human Liver Stages
Liver cell
Infected liver cell
2
A
Exo-erythrocytic Cycle
3 Schizont
4 Ruptured schizont
12 Ruptured oocyst
Release of sporozoites
11 Oocyst
C
Sporogonic Cycle
10 Ookinete
1 Mosquito takes a blood meal (injects sporozoites)
8 Mosquito takes a blood meal (ingests gametocytes)
Macrogametocyte
Microgamete entering macrogamete 9
Exflagellated microgametocyte
Human Blood Stages
5
Immature trophozoite (ring stage)
B
Erythrocytic Cycle
Mature trophozoite
6 Ruptured schizont
Schizont
7 Gametocytes
7 Gametocytes
Africa
S.E Asia
kills
P. falciparum
S. America
sick
P. vivax
P. ovale
P. malariae
i = Infective Stage
d = Diagnostic Stage

NIH

Ross & MacDonald - Malaria

(contemporaries of Lotka)

PLOS Biology.

Two infected populations!

Let H be pop density of Humans.

Let M be pop density of Mosquitos.

Prevalence among mosq: $z = \dfrac{Z}{M}$

Prevalence among humans: $x = \dfrac{X}{H}$

actually measurable!

Mosquito-to-human ratio: $\dfrac{M}{H}$.

Human $S \longrightarrow X \longrightarrow R$

Mosquito $S \longrightarrow Z \longrightarrow \text{Dead}$

factors that affect the rates of new infections.

- fractions of inf humans, x mosquitoes, z
- biting rate ma, a
- effectiveness of each bite
  - acquire parasites c
  - unload parasites b

Ross & MacDonald - Malaria

Prev: $x = \frac{X}{H}$, $z = \frac{Z}{M}$ $\quad m = \frac{M}{H}$

Humans $\quad$ Mosq. $\quad$ Mosq. to Human Ratio.

$$S \underset{r}{\overset{mabz}{\rightleftarrows}} X$$

$$\begin{array}{ccc} T & \xrightarrow{acx} & Z \\ g\downarrow & & \downarrow g \\ \text{dead} & & \text{dead} \end{array}$$

Entomological Innoculation Rate : EIR

$m \cdot a \cdot z$

Pr (mosq. becomes inf | bites inf. human) = c

Rate of $S \rightarrow X$

$m \cdot a \cdot b \cdot z$

Proportion of inf mosq. bites that pass the infection to human = b

Rate of $T \rightarrow Z$

$a \cdot c \cdot x$

Proportion of mosquitos that feed on humans per day. = a

$$\boxed{\begin{aligned} \dot{x} &= mabz(1-x) - rx \\ \dot{z} &= acx(1-z) - gz \end{aligned}}$$

Ross & MacDonald - Malaria

$$\dot{x} = mabz(1 - x) - rx$$

$$\dot{z} = acx(1 - z) - gz$$

$x = \frac{X}{H}$ prevalence in humans

$z = \frac{Z}{M}$ prevalence in mosquitoes

$m = \frac{M}{H}$ mosquito-to-human ratio

$a =$ Proportion of mosquitoes that feed on humans per day

$b =$ Proportion of infectious mosquito bites that infect a human

$c =$ Probability that a mosquito becomes infected after biting an infected human.

$g =$ mosquito death rate

$r =$ human recovery rate