Calculating Biological Quantities

CSCI 2897

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Exam 1 Recap

- More practice
  narrative → eqn.

- mean 80, σ 16
- grade cutoffs will shift a bit

Money in Bank

Discrete	Continuous
• monthly - paycheck - bills - card bill • weekly - similar	• invested/interest ↓ continuous process • transactions need not take place on a monthly cadence.

Flow Diagram

no - sign

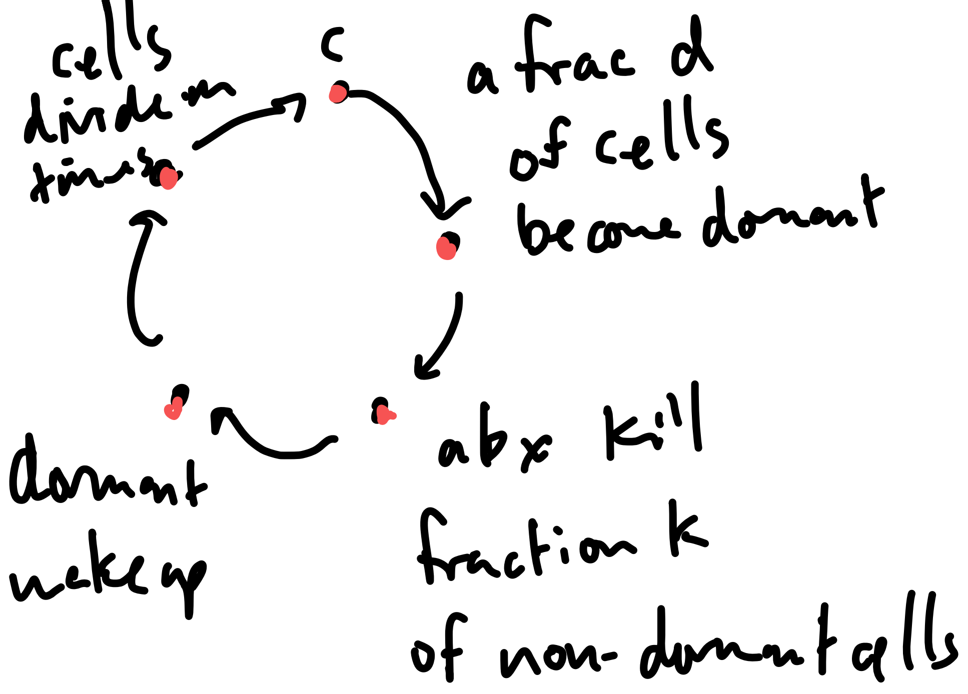
$$\xrightarrow{1} \boxed{w} \xrightarrow{0.05w}$$

included

$$\frac{dw}{dt} = \text{in} - \text{out}$$

$$\frac{dw}{dt} = 1 - 0.05w$$

14.



$$n' = (1-d)n$$

$$n'' = (1-k)n'$$

$$n''' = n'' + dn$$

dormant wake up

$$n'''' = mn''' \qquad n(t+1) = n''''$$

$$n(t+1) = m\big((1-k)(1-d)n + dn\big)$$

B

Last time on CSCI 2987: The SIR Model

β: infectiousness
rate at which SI contacts actually create new infections

γ: rate at which I recover to R

$$\dot{S} = -\beta SI$$

$$\dot{I} = \beta SI - \gamma I$$

$$\dot{R} = \gamma I$$

where $S + I + R = 1$

$$\implies \dot{S} + \dot{I} + \dot{R} = 0$$

Equilibrium when:

$$I = 0$$

Epidemic peak:

$$S^* = \frac{\gamma}{\beta}$$

Herd Immunity (vaccination)

$$v > 1 - \frac{\gamma}{\beta} \qquad v > 1 - \frac{1}{R_0}$$

- higher R_0 => need more vaccine.

COVID: $R_0 \sim 3$ $\quad v > 1 - \frac{1}{3} = \frac{2}{3}$

Measles: $R_0 \sim 18$ $\quad v > 1 - \frac{1}{18} = 0.95$

Basic Reproduction Number

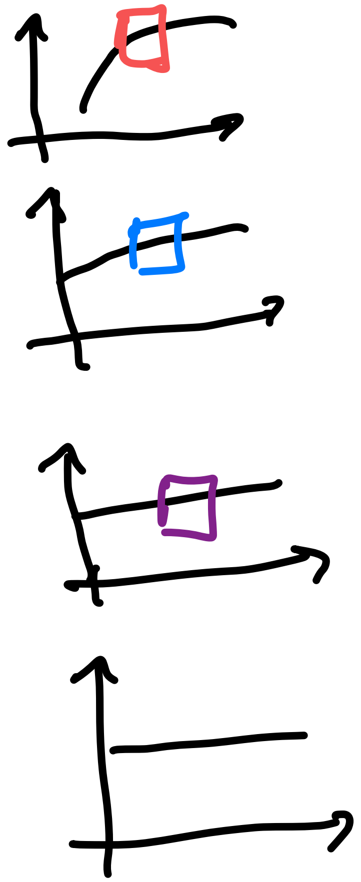
$$R_0 = \frac{\beta}{\gamma}$$

R_0 larger when...
- β large (more infections)
- γ smaller (slower recovery)

How many new infections per 1 infection in a susceptible population.

Linearization and Stability

The big question with an infectious agent is: **will we get an epidemic?**



$I = 0$ equilibrium.

Is this stable?

"nudge" the variable just a tiny bit away from equil.

→ see what happens.

① nudge could grow...

unstable equil.

② nudge could disappear, and we go back.

stable equil.

$$\dot{I} = \beta S I - \gamma I$$

Let's imagine that I is tiny, $S = 1 - I$

$$\dot{I} = \beta(1-I)I - \gamma I$$

$$\dot{I} = \beta I - \beta I^2 - \gamma I$$

$$\dot{I} = \underbrace{\beta I - \gamma I}_{\text{powers of } I^1} \underbrace{- \beta I^2}_{I^2}$$

If I is small $\Rightarrow$ I^2 is super tiny.

$\Rightarrow$ discard I^2 term.

(Hint: let $I = \frac{1}{10^6}$ $\Rightarrow I^2 = \frac{1}{10^{12}}$)

$$\dot{I} = \beta I - \gamma I \cancel{- \beta I^2} \to 0$$

note: linear! → Linearization

Linearization and Stability

The big question with an infectious agent is: **will we get an epidemic?**

$$\frac{dX}{dt} = aX \rightarrow X = ke^{at}$$

nearby... so I small!

$$\dot{I} = \beta I - \gamma I$$ (linearized dynamics around equilibrium $I = 0$)

$$\frac{dI}{dt} = (\beta - \gamma) I$$

$$\int \frac{dI}{I} = \int (\beta - \gamma) dt$$

$$\ln I = (\beta - \gamma)t + c$$

$$I = e^{(\beta-\gamma)t + c}$$

$$I = e^{(\beta-\gamma)t} e^{c}$$

$$\boxed{I = k e^{(\beta-\gamma)t}}$$

exp. growth or decay?

growth: $\beta > \gamma$

$\frac{\beta}{\gamma} > 1$ ... $R_0 > 1$

decay: $\beta < \gamma$

Conclusion:

- No outbreak if $R_0 < 1$.
  STABLE
- Outbreak if $R_0 > 1$
  UNSTABLE

The S**E**IR model—exposure without infectiousness

Some diseases have a **latent period** in which a person is infected but not yet infectious to others.

Delay between becoming infected (E) and being infectious (I) to others.

Let's consider a **new** compartment: **E**xposed, with a transition rate α.

$$S \xrightarrow{\beta S I} E \xrightarrow{\alpha E} I \xrightarrow{\gamma I} R$$

$$S + E + I + R = 1$$

$$\dot{S} + \dot{E} + \dot{I} + \dot{R} = 0$$

$$\dot{S} = -\beta S I$$

$$\dot{E} = \beta S I - \alpha E$$

$$\dot{I} = \alpha E - \gamma I$$

$$\dot{R} = \gamma I$$

SEIR Model — Equilibrium

$\dot{S} = -\beta SI \quad = 0$

$\dot{E} = \beta SI - \alpha E \quad = 0$

$\dot{I} = \alpha E - \gamma I \quad = 0$

$\dot{R} = \gamma I \quad = 0 \longrightarrow I = 0$

where $S + E + I + R = 1$

$\dot{S} = -\beta S(0) = 0$ ✓

$\dot{E} = \beta \cancel{S(0)}^{0} - \alpha E$

$\dot{I} = \alpha E - \gamma \cancel{(0)}^{0}$

$\uparrow$ plugin

$\xrightarrow{I=0}$

$\dot{E} = -\alpha E = 0$

$\dot{I} = \alpha E = 0$

$\Rightarrow E = 0$

$$\boxed{\text{Equilibrium: } I = E = 0}$$

(R and S can be anything as long as $R + S = 1$)

SEIR Model — Out of Equilibrium?

$$\dot{S} = -\beta SI$$

$$\dot{E} = \beta SI - \alpha E$$

$$\dot{I} = \alpha E - \gamma I$$

Let's hop into Jupyter Notebooks to explore a bit.

$$\dot{R} = \gamma I$$

where $S + E + I + R = 1$