Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2897:

We learned that **linear algebra** (matrices and vectors) is like regular algebra, but with a few twists:

- Laws:
  - Associative Law
  - Left Distributive Law
  - Right Distributive Law
  - Commutative Law for Scalars
  - Commutative Law for matrix multiplication? No. Does not usually commute!
- Transposes:
  - $(A+B)^T = A^T + B^T$
  - $\left(A^T\right)^T = A$
  - $(AB)^T = B^T A^T$
- Matrices are machines that turn vectors into other vectors.
  - The identity matrix (ones on the diagonal, zeros elsewhere) reproduces the same vector.
- Trace: sum the diagonals.
- Determinant (2x2): $ad - bc$

Solving a system of equations

Let's solve these two equations (Matrix)

$6x + 4y = 12$

$3x - 2y = 0$

$Mx = b$

$$\underset{2\times 2}{\begin{pmatrix} 6 & 4 \\ 3 & -2 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 6x + 4y \\ 3x - 2y \end{pmatrix}} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 6 & 4 \\ 3 & -2 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} \underset{=}{=} \underbrace{\begin{pmatrix} 12 \\ 0 \end{pmatrix}}_{\vec{b}}$$

$$\boxed{M\vec{x} = \vec{b}}$$

generic form

Solving a system of equations

~~We can also write these equations in "matrix-vector notation."~~

$6x + 4y = 12$

$3x - 2y = 0$

$6x - 4y = 0 \quad +$

$12x + 0y = 12 \longrightarrow x = 1$

$6x + 4y = 12$

$(3x - 2y = 0)(-2)$

$-6x + 4y = 0$

$0x + 8y = 12 \longrightarrow y = \frac{2}{3}$

1. solve for x. plug in, get y get x
2. solve for y plug in, get x get y
3. operate on "lines"

Solving a system of equations

We can also write these equations in "matrix-vector notation."

$6x + 4y = 12$

$3x - 2y = 0$

$$\frac{M\vec{x}}{M} = \frac{\vec{b}}{M}$$

$$\vec{x} = \frac{\vec{b}}{M}$$

illegal

alternative

$$\frac{1}{\alpha} \cdot \alpha x = 5 \frac{1}{\alpha}$$

$$\alpha^{-1} \alpha x = 5 \alpha^{-1}$$

α^{-1} and α are inverses of eachother

$$\alpha^{-1} \alpha = 1$$

$$M \vec{x} = \vec{b}$$

$$M^{-1} M \vec{x} = M^{-1} \vec{b}$$

$$I \vec{x} = M^{-1} \vec{b}$$

$$\vec{x} = M^{-1} \vec{b}$$

The Inverse Matrix

The **inverse** of a matrix A, denoted A^{-1} is a matrix such that $A^{-1}A = I$

What does our *trick of the transpose* tell us for free?

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

bc I is diagonal

If A^{-1} is the inv mtx of A, then A^T is the inverse of $(A^{-1})^T$

What else can we get "for free" from this equation?

$$\text{If } A^{-1}A = I$$

$$\Rightarrow AA^{-1} = I$$

Whoah! Remember: the *inverse* of a number a, denoted a^{-1}, is a number such that $a \cdot a^{-1} = 1$

The Inverse Matrix (2x2)

There turns out to be a nice formula for an inverse matrix.

What are $w\ x\ y\ z$ in terms of $a\ b\ c\ d$?

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let's say that generically, $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

If $A^{-1}A = I$ then:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$wa + xc = 1$$
$$wb + xd = 0$$
$$ya + zc = 0$$
$$yb + zd = 1$$

$$wa + xc = 1$$
$$(wb + xd = 0)\left(-\frac{c}{d}\right)$$
$$+ \quad w\left(-\frac{bc}{d}\right) + x\left(-\frac{c\cancel{d}}{\cancel{d}}\right) = 0$$
$$w\left(a - \frac{bc}{d}\right) + 0 = 1$$
$$w = \frac{1}{a - \frac{bc}{d}}\,\frac{d}{d} = \frac{d}{ad-bc}$$

$$\boxed{w = \frac{d}{ad-bc}}$$

The Inverse Matrix (2x2)

There turns out to be a nice formula for an inverse matrix.

$$wa + xc = 1$$
$$wb + xd = 0$$
$$ya + zc = 0$$
$$yb + zd = 1$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $A^{-1}A = I$ then: $\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$wb + xd = 0$$

plug in w

$$\frac{d}{ad-bc} \cdot b + xd = 0$$

$$xd = -\frac{bd}{ad-bc}$$

$$\boxed{x = \frac{-b}{ad-bc}}$$

$$\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}$$

$$\boxed{w = \frac{d}{ad-bc}}$$

What is the inverse matrix of $A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}$

$\det(A) = 1 \cdot 1 - (2 \cdot 0.5)$

$= 1 - 1$

$= 0$

so $A^{-1} = \frac{1}{0} \begin{pmatrix} 1 & 2 \\ -0.5 & 1 \end{pmatrix}$

illegal! can't div by 0.

$\det(A) = 0$

$=$

A^{-1} does not exist!

$\det(A^T) = 0$

$A^{T^{-1}}$ does not exist!

$$\boxed{A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

What is the inverse matrix of $A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}$

$$x + 2y = 9$$

$$\left(\frac{1}{2}x + y = 3\right)(-2)$$

$$-x - 2y = -6$$

$$0x + 0y = 3$$

$$x + 2y = 9$$

$$x + 2y = 6$$

contradict!

$$\frac{1}{2}x + y = 4.5$$

$$0 = 0$$