Calculating Biological Quantities

CSCI 2897

Prof. Daniel Larremore

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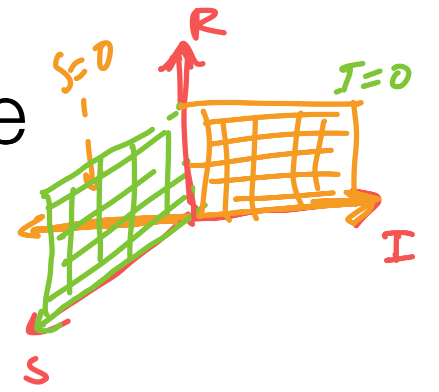
daniel.larremore@colorado.edu
@danlarremore

Last time on CSCI 2897

A **nullcline** is a curve (or surface) in phase space on which one of the variables' rate of change is zero, $\dot{n}_i = 0$. An **equilibrium** is therefore a point where all the nullclines intersect.

$\dot{S} = -\beta S I = 0$ $SI = 0$ a nullcline $I = 0$ or $S = 0$



Linear model $\dfrac{d\vec{n}}{dt} = M\vec{n}$

$$\hat{\vec{n}} = 0$$

Affine model $\dfrac{d\vec{n}}{dt} = M\vec{n} + \vec{c}$

$$\hat{\vec{n}} = -M^{-1}\vec{c}$$

Rules:

- A linear or affine model in continuous time has only one equilibrium regardless of the number of variables, provided that the determinant of M is not zero.
- If $\det(M) = 0$, there are an *infinite* number of equilibria.

Stability of equilibria (real eigenvalues):

- If all eigenvalues are negative, the system is stable.
- If one or more eigenvalues are positive, the system is unstable.

Otto & Day — Adapted from Glass & Kaplan 1995

Metastasis of Malignant Tumors

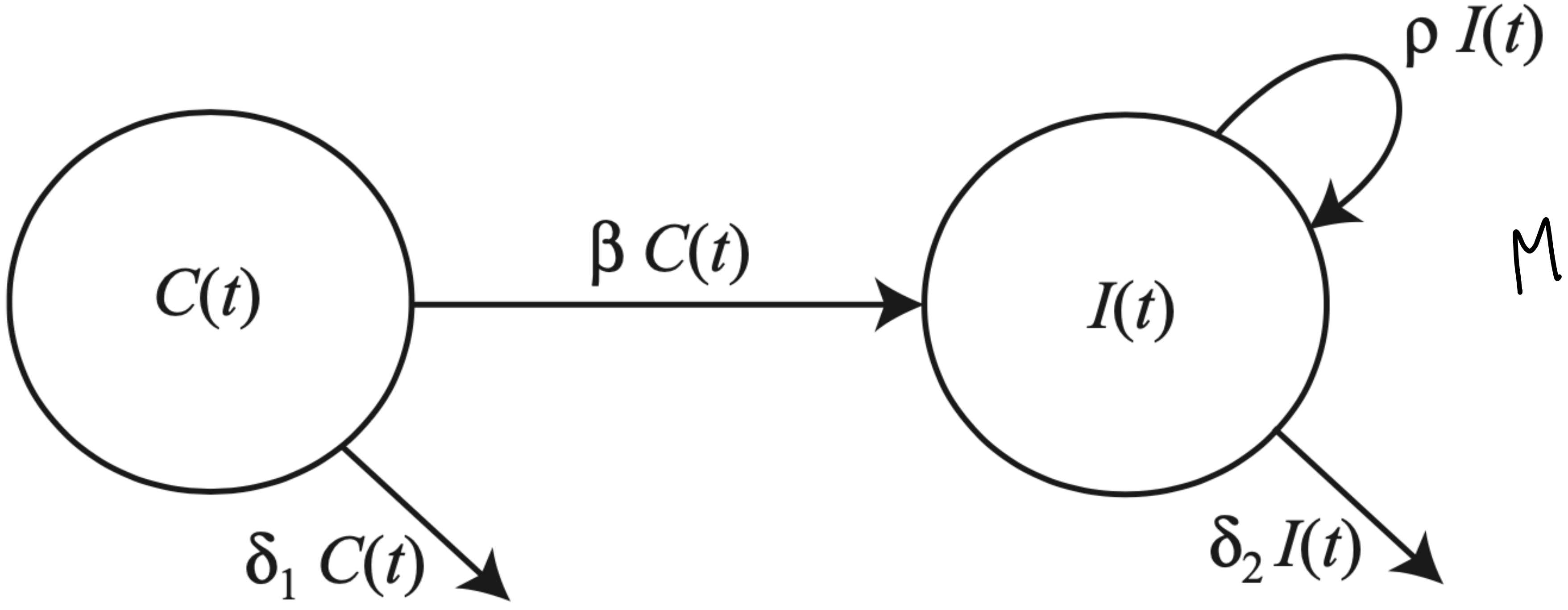
$$\begin{pmatrix} \frac{dC}{dt} \\ \frac{dI}{dt} \end{pmatrix} = M \begin{pmatrix} C \\ I \end{pmatrix} \qquad M = \begin{pmatrix} -(\delta_1 + \beta) & 0 \\ \beta & \rho - \delta_2 \end{pmatrix}$$

Suppose that cells are lost from the capillaries by dislodgement or death at a per capita rate δ_1 and that they invade the organ from the capillaries at a per capita rate β. Once cells are in the organ they die at a per capita rate δ_2, and the cancer cells replicate at a per capita rate ρ.

$\lambda_1 = -(\delta_1 + \beta) \qquad \lambda_2 = \rho - \delta_2$

1. Identify the equilibrium or equilibria.
2. Determine the stability.

$$\lambda_1, \lambda_2 = \frac{tr(A) \pm \sqrt{tr^2(A) - 4det(A)}}{2}$$



$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad det = ad - bc \quad b = 0$

$\Rightarrow det = ad$

$$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4ad}}{2}$$

no more bc!

Definitions & Tricks: Diagonal and Triangular Matrices

Eigenvalues are *especially* easy to find when a matrix is diagonal or triangular.

Definition: a **diagonal matrix** : Matrix M such that $M_{ij} = 0$ whenever $i \neq j$.

$$D \quad \begin{pmatrix} M_{11} & & & 0 \\ & M_{22} & & \\ & & M_{33} & \\ 0 & & & \ddots \; M_{nn} \end{pmatrix}$$

Everything off diagonal is zero.

Ex: Identity Matrix

Definition: a **triangular matrix** Matrix M such that $M_{ij} = 0$ w never $i < j$ (upper) or $i > j$ (lower).

$$\begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ & M_{22} & & \vdots \\ & & M_{33} & \vdots \\ 0 & & \ddots & M_{nn} \end{pmatrix} \qquad \begin{pmatrix} M_{11} & & & 0 \\ \vdots & M_{22} & & \\ \vdots & & M_{33} & \\ M_{n1} & \cdots & \ddots & M_{nn} \end{pmatrix}$$

upper U lower L

- if D is diagonal. $D = D^T$
- if U is upper triangular, U^T is lower triang.
- if L lower L^T upper
- if X is lower triangular <u>and</u> upper triangular, X is diag.

Definitions & Tricks: Diagonal and Triangular Matrices

Eigenvalues are *especially* easy to find when a matrix is diagonal or triangular.

Definition: a **diagonal matrix**

(prev)

If M is diagonal, lower or upper triangular, the eigenvalues are the elements of the diagonal.

Definition: a **triangular matrix**

(prev)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_1, \lambda_2, \lambda_3 = 1, 1, 1$$

$$X = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$\lambda_1, \lambda_2 = 1, 3$$

$$X^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$1, 3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 2 & 0 & 0 \\ \beta & x & 9 & 0 \\ 2 & \pi & e & 7 \end{pmatrix}$$

$$1, 2, 9, 7$$

Complex Eigenvalues

Unreal! Sometimes we can have eigenvalues which are *complex numbers*.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightarrow \quad \lambda_1, \lambda_2 = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\det(A)}}{2}$$

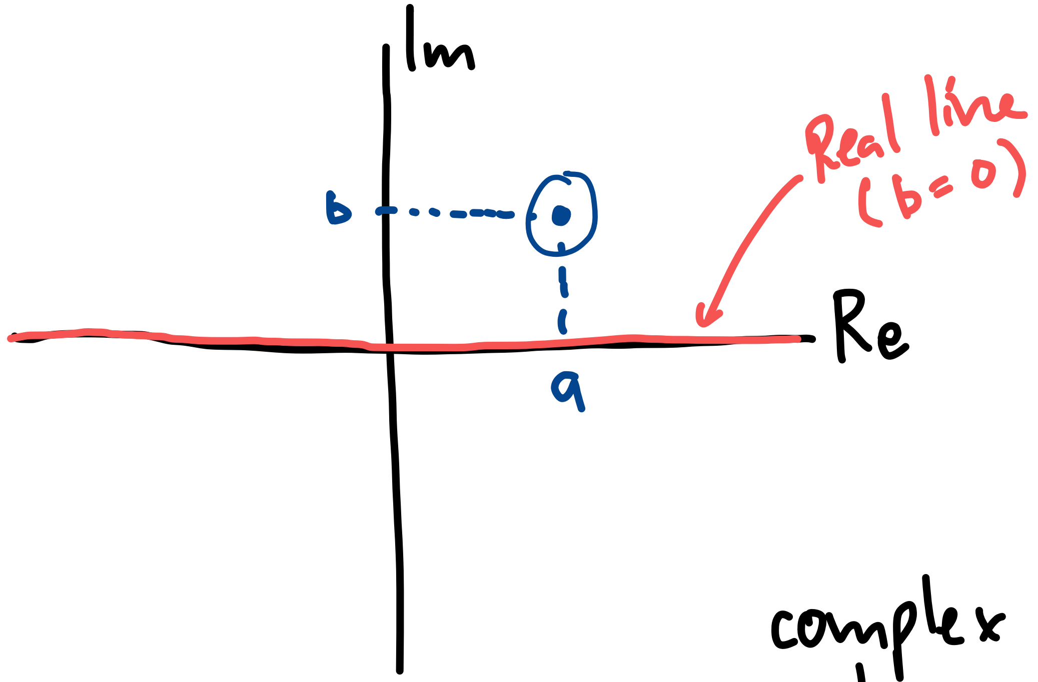
different c, a, b than the matrix.

If $\mathrm{tr}^2(A) - 4\det(A) < 0$, then λ_1, λ_2 will be **complex numbers**.

A **complex number** is a number $c = a + bi$, where a and b are real and $i = \sqrt{-1}$ is "imaginary."

In our formula above, what's the **real part**? And the **imaginary part**?

$$c = \underset{\substack{\uparrow \\ \text{real}}}{a} + \underbrace{b}_{\text{imag.}} i$$



complex plane

Real numbers are "complex", but with b = 0.
(so... purely real, no imaginary part).
(so... not <u>really</u> complex.)

Complex Eigenvalues

Unreal! Sometimes we can have eigenvalues which are *complex numbers*.

$$\begin{pmatrix} a^{1} & b^{5} \\ c_{-2} & d^{1} \end{pmatrix} \quad \rightarrow \quad \lambda_1, \lambda_2 = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\det(A)}}{2}$$

If $\mathrm{tr}^2(A) - 4\det(A) < 0$, then λ_1, λ_2 will be **complex numbers**.

A complex number is a number $c = a + bi$, where a and b are real and $i = \sqrt{-1}$ is "imaginary."

$(a+d)^2 - 4(ad - bc) < 0$

$(a+d)^2 - 4ad + 4bc < 0$

$(a+d)^2$: same sign.

$\underbrace{-4ad \; + 4bc}_{\text{negative term.}}$ with $+$ and $-$

$$a = \frac{\mathrm{tr}(A)}{2}, \quad \text{and} \quad b = \frac{\sqrt{\mathrm{tr}^2(A) - 4\det(A)}}{2}$$

and therefore $\lambda_1 = a + bi, \quad \lambda_2 = a - bi$

λ_1 and λ_2 are complex conjugates.

Fundamental Theorem of Algebra.

Notice: either both eigenvalues are complex, or both are real.

Euler's Equation

Recall: eigs show up in our solutions

Where? $\vec{n}(t) = k_1 \vec{x}_1 e^{\lambda_1 t} + k_2 \vec{x}_2 e^{\lambda_2 t}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We will not derive this miraculous equation, but come to office hours if you are excited or puzzled by this!

For extra magic, set $\theta = \pi$...

What happens when you put something imaginary up in an exponent.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$i\theta$ is purely imaginary.

$$e^{a+bi} = e^a e^{bi}$$

$$= \underset{\text{real part}}{e^a} \underbrace{(\cos b + i \sin b)}_{\text{imaginary part... trig functions?}}$$

$$e^{i\pi} = \underset{-1}{\cos\pi} + i \underset{0}{\sin\pi} \rightarrow \boxed{e^{i\pi} + 1 = 0}$$

FIVE fundamental numbers. All related?!

Solutions to linear systems

$\cos\theta = \cos(-\theta)$ even

odd

$\sin(-\theta) = -\sin\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d\vec{n}}{dt} = A\vec{n} \qquad \text{Solution: } \vec{n}(t) = k_1\vec{x}_1 e^{\lambda_1 t} + k_2\vec{x}_2 e^{\lambda_2 t}$$

e^{a+bi} $\quad$ e^{a-bi}

So what's going to happen when λ_1 and λ_2 are complex?

$$n(t) = k_1\vec{x}_1 e^{(a+bi)t} + k_2\vec{x}_2 e^{(a-bi)t}$$

$$= k_1\vec{x}_1 e^{at}e^{bit} + k_2\vec{x}_2 e^{at}e^{-bit}$$

$$= e^{at}\left(k_1\vec{x}_1 e^{bit} + k_2\vec{x}_2 e^{-bit}\right)$$

$\theta = bt$

overall

$a>0$ growth

$a<0$ decay

$$= e^{at}\Big[k_1\vec{x}_1\left(\cos bt + i\sin bt\right) + k_2\vec{x}_2\left(\cos(\cancel{-}bt) \cancel{+}{-}\, i\sin(-bt)\right)\Big]$$

delete

$$n(t) = e^{at}\Big[k_1\vec{x}_1\left(\cos bt + i\sin bt\right) + k_2\vec{x}_2\left(\cos bt - i\sin bt\right)\Big]$$

growth decay

rotation.

Solutions to linear systems

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d\overrightarrow{n}}{dt} = A\overrightarrow{n}$$

Solution: $\overrightarrow{n}(t) = k_1\overrightarrow{x_1}e^{\lambda_1 t} + k_2\overrightarrow{x_2}e^{\lambda_2 t}$

When λ_1 and λ_2 are complex, we can separate growth/decay from rotation.

Solutions to linear systems

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d\overrightarrow{n}}{dt} = A\overrightarrow{n}$$

Solution: $\overrightarrow{n}(t) = k_1\overrightarrow{x_1}e^{\lambda_1 t} + k_2\overrightarrow{x_2}e^{\lambda_2 t}$

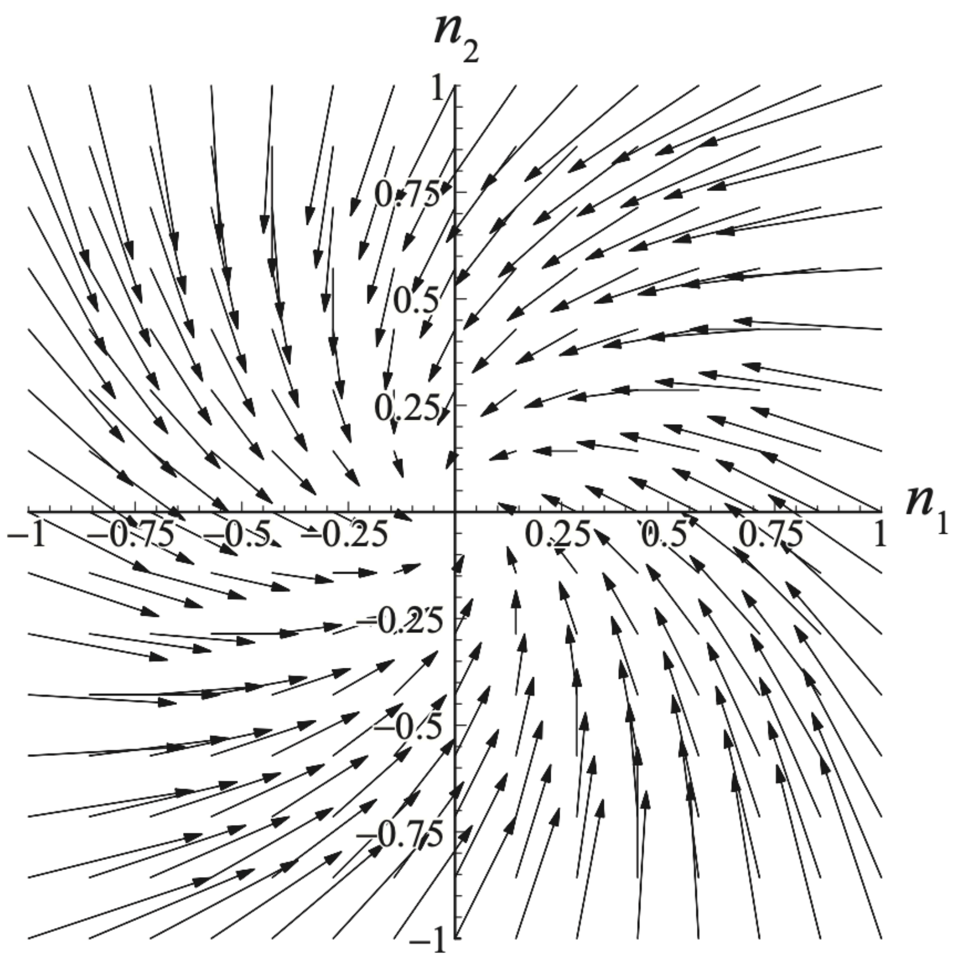
What do solutions look like if eigenvalues are complex?

Solutions to linear systems

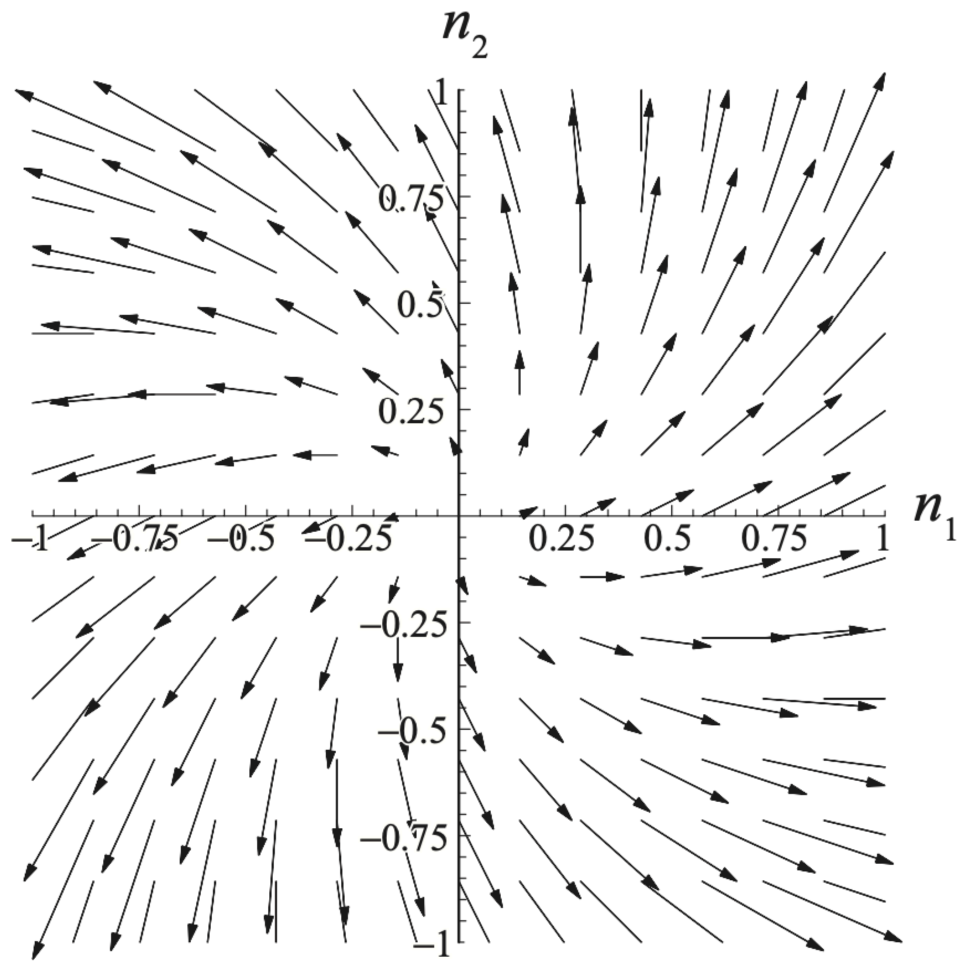
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d\overrightarrow{n}}{dt} = A\overrightarrow{n}$$

Solution: $\overrightarrow{n}(t) = k_1\overrightarrow{x_1}e^{\lambda_1 t} + k_2\overrightarrow{x_2}e^{\lambda_2 t}$



$\lambda = -2 \pm i$



$\lambda = 2 \pm i$

Solutions to linear systems

$\lambda = a \pm bi$