Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2897:

The inverse of square matrix A is a matrix called A^{-1} such that

$$A^{-1}A = I$$

and

$$AA^{-1} = I.$$

Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $A^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Things you can do with an inverse matrix.

Let's solve these two equations

$$6x + 4y = 12$$

$$3x - 2y = 0$$

find x, find y

① Write out as $Ax = b$

$$\begin{pmatrix} 6 & 4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

② If $Ax = b$

$$\underbrace{A^{-1}A}x = A^{-1}b$$

$$x = A^{-1}b$$

"solve for x"

③ $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\dfrac{1}{\det(A)}$

$$A^{-1} = \begin{pmatrix} -2 & -4 \\ -3 & 6 \end{pmatrix}\frac{1}{6(-2) - 3(4)}$$

so $x = \begin{pmatrix} -2 & -4 \\ -3 & 6 \end{pmatrix}\dfrac{1}{-24}\begin{pmatrix} 12 \\ 0 \end{pmatrix}$

$$= \frac{1}{-24}(-1)\begin{pmatrix} 2 & 4 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} 2 & 4 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$$

Things you can do with an inverse matrix.

Let's think about the Matrix as Machine idea.

$y = Ax$

What happens if I multiply $A^{-1}y$?

you get x.

$A^{-1}y = A^{-1}Ax$

$A^{-1}y = x$

(ctrl-z cmd-z) undo matrix!

Try this in the jupyter notebook.

1. take matrix
2. np.linalg.inv(matrix) to get inverse.
3. show $y = Ax$ then $A^{-1}y = x$

Equivalent statements:

1. The matrix A is invertible.
2. A^{-1} exists.
3. For an arbitrary b, $Ax = b$ does ~~has~~ have a unique solution x.
4. If $Ax = 0$, this means that $x = 0$.
5. $\text{Det}(A) \neq 0$.

There is one and only one x that solves each $Ax = b$.

1. The matrix A is not invertible.
2. A^{-1} does not exist.
3. For an arbitrary b, $Ax = b$ does not have a unique solution x.
4. There exists a *nonzero* vector x such that $Ax = 0$.
5. $\text{Det}(A) = 0$.

x is in the "nullspace" of A.

Characteristic Directions

For any matrix, there are some vectors which are special. For one of these special vectors $\overrightarrow{x}$, computing $\overrightarrow{y} = A\overrightarrow{x}$, produces a $\overrightarrow{y}$ that is just a rescaled version of $\overrightarrow{x}$.

In other words, $\overrightarrow{y} = \lambda\overrightarrow{x}$. This means that $A\overrightarrow{x} = \lambda\overrightarrow{x}$.

Example 1: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Ax = \underbrace{\begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x} = \begin{pmatrix} -5+2 \\ -9+6 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = \underbrace{-3}_{-3}\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x}$$

Characteristic Directions

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Example 2: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}, x = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

$$Ax = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} -10+18 \\ -18+54 \end{pmatrix} = \begin{pmatrix} 8 \\ 36 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 9 \end{pmatrix} = 4x$$

$$Ax = \quad . \quad . \quad . \quad . \quad - \quad - \quad . \quad = 4x$$

Let's pop over into our Matrix Machines notebook to see this in action.

Characteristic Directions

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Example 2: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}, x = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

Let's pop over into our Matrix Machines notebook to see this in action.

Characteristic Directions

Example 1: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}, x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $Ax_1 = -3x_1$

Example 2: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$. $Ax_2 = 4x_2$

Definitions: An **Eigenvector** of a square matrix A is a vector x such that $Ax = \lambda x$ for some scalar λ. An **Eigenvalue** is that scalar, λ.

There can be at most n eigenvectors and n eigenvalues for an $n \times n$ matrix.

Finding Eigenvectors and Eigenvalues

What if I give you the matrix $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$ and ask you for its eigenvalues?

$Ax = \lambda x$ solve for λ.

$Ax - \lambda x = 0$

$(A - \lambda) x = 0$

matrix scalar

oops!

$(A - \lambda I) x = 0$

4. If $Ax = 0$, and $x \neq 0$, then $\det(A) = 0$

If $(A - \lambda I)x = 0$, and $x \neq 0$, then $\det(A - \lambda I) = 0$.

Use $\det(A - \lambda I) = 0$ to get λ!

$$A - \lambda I = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Finding Eigenvectors and Eigenvalues

Recall: $\det(A) = ad - bc$

$$= \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{pmatrix}$$

$$\det(\overbrace{A - \lambda I}) = 0$$

$$\boxed{(-5-\lambda)(6-\lambda) - 2(-9) = 0}$$

$$-(5+\lambda)(6-\lambda) + 18 = 0$$

$$-(30 + 6\lambda - 5\lambda - \lambda^2) + 18 = 0$$

$$\lambda^2 - \lambda - 30 + 18 = 0$$

characteristic equation.

$$\boxed{\lambda^2 - \lambda - 12 = 0}$$

polynomial equation for eigenvalues.

$$(\lambda - 4)(\lambda + 3) = 0$$

$$\Rightarrow \boxed{\lambda = 4, -3}$$

$$\boxed{\begin{gathered} a\lambda^2 + b\lambda + c = 0 \\ \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{gathered}}$$

general quadratic solution.

Finding Eigenvalues

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

To compute eigenvalues, we:

1. Write $Ax = \lambda x$ as $Ax - \lambda x = 0$ and then as $(A - \lambda I)x = 0$.
2. If $(A - \lambda I)x = 0$ but $x \neq 0$, this means that $\det(A - \lambda I) = 0$. ← typical starting pt.
3. Write out the characteristic equation: $(a - \lambda)(d - \lambda) - bc = 0$ ← same char. eqn...
4. Solve for λ.

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - a\lambda - d\lambda + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\mathrm{tr}(A)}\lambda + \underbrace{ad - bc}_{\det(A)} = 0$$

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

$$\boxed{\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\det(A)}}{2}}$$

What if we also want eigenvectors?

Finding Eigenvectors and Eigenvalues

Tinder for matrices

↓

Define a nice relationship

Given the matrix $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$ and $\lambda = 4$, what's the matching eigenvector?

Plug in λ to $(A - \lambda I)x = 0$.

Solve for x.

↓

$$\begin{pmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{pmatrix} \overset{\lambda = 4}{=} \begin{pmatrix} -9 & 2 \\ -9 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 2 \\ -9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-9x_1 + 2x_2 = 0$$

$$\boxed{-9x_1 + 2x_2 = 0}$$

defines a relationship between x_1 and x_2

$$9x_1 = 2x_2$$

$$x_1 = \frac{2}{9}x_2$$

let $x_2 = 9$

then $x_1 = 2$

$$\vec{x} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

Finding Eigenvectors and Eigenvalues

Given the matrix $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$ and $\lambda = -3$, what's the matching eigenvector?

$$A - \lambda I = \begin{pmatrix} -5-(-3) & 2 \\ -9 & 6-(-3) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 \\ -9 & 9 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ -9 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + 2x_2 \\ -9x + 9x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 2x_2 = 0$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

Let $x_1 = 1$

$\Rightarrow x_2 = 1$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finding Eigenvalues & Eigenvectors

To compute eigenvalues, we:

1. Write $Ax = \lambda x$ as $Ax - \lambda x = 0$ and then as $(A - \lambda I)x = 0$.
2. If $(A - \lambda I)x = 0$ but $x \neq 0$, this means that $\det(A - \lambda I) = 0$.
3. Write out the characteristic equation: $(a - \lambda)(d - \lambda) - bc = 0$
4. Solve for λ.

To compute the eigenvectors, for each eigenvalue, we

1. Plug in the λ to $(A - \lambda I)x = 0$, and write out the equations.
2. The equations *should* be redundant. Pick one and determine the relationship between x_1 and x_2. That's your eigenvector!

Practice. Find the eigenvalues & eigenvectors of $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

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1. Write $Ax = \lambda x$ as $Ax - \lambda x = 0$ and then as $(A - \lambda I)x = 0$.
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To compute the eigenvectors, for each eigenvalue, we

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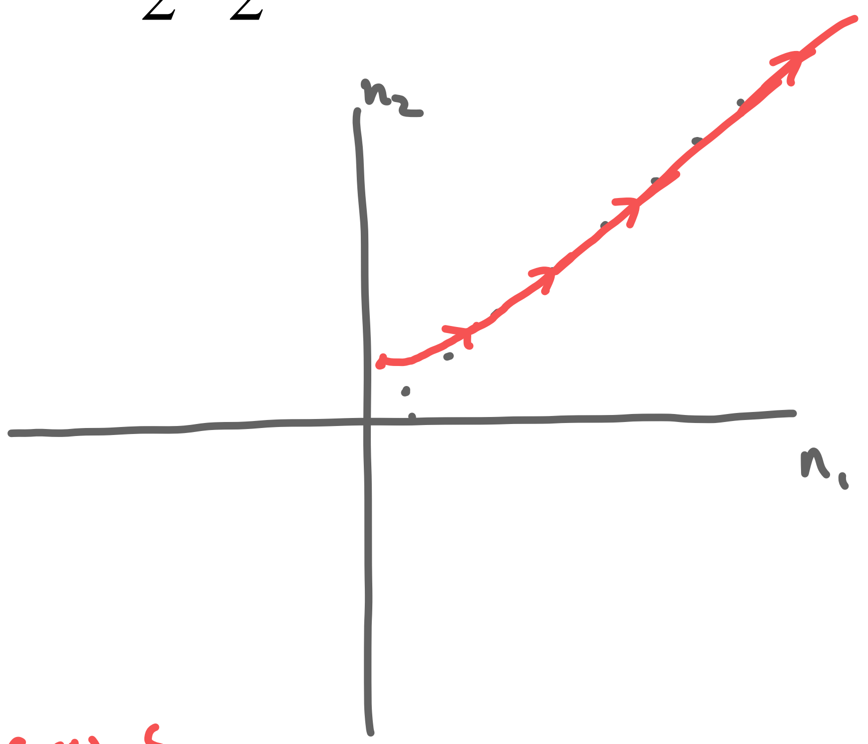
   That's your eigenvector!

Why do we care though?

k_1 and k_2 come from initial conditions.

$$\frac{d\vec{n}}{dt} = A\vec{n}$$

It turns out the answer is $\vec{n}(t) = k_1\vec{x_1}e^{\lambda_1 t} + k_2\vec{x_2}e^{\lambda_2 t}$



$$\begin{cases} \dfrac{dn_1}{dt} = 2n_1 + 3n_2 \\[2ex] \dfrac{dn_2}{dt} = 2n_1 + n_2 \end{cases}$$

$\lambda_1 = -1 \qquad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 4 \qquad x_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \xrightarrow{\text{solve}} \boxed{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \cancel{k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}} + k_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}}$$

as t grows → 0

① growth in n_1, n_2