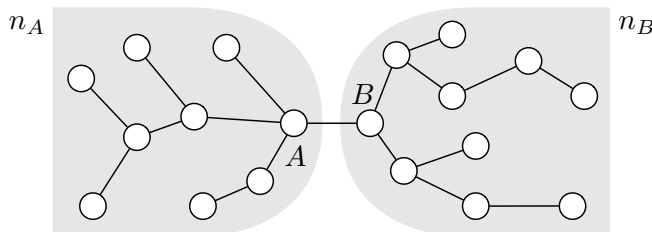1. (15 pts) Consider an undirected, unweighted network of n vertices that contains exactly two subnetworks of size n_A and n_B, which are connected by a single edge (A, B), as sketched here:

   Show that the closeness centralities C_A and C_B of vertices A and B, as defined by Eq. (7.21) in *Networks* [v1: 7.29], are related by

   $$\frac{1}{C_A} + \frac{n_A}{n} = \frac{1}{C_B} + \frac{n_B}{n} \ .$$

2. (15 pts) Consider an undirected (connected) tree of n vertices. Suppose that a particular vertex in the tree has degree k, so that its removal would divide the tree into k disjoint regions, and suppose that the sizes of those regions are $n_1, \ldots, n_k$. Show that the betweenness centrality b of the vertex is

   $$b = n^2 - \sum_{i=1}^{k} n_i^2 \ .$$

3. (30 pts) The Medici family was a powerful political dynasty and banking family in 15th century Florence. The classic network-based explanation of their power, offered by Padgett and Ansell in 1993, claims that they established themselves as the most central players within the network of prominent Florentine families (shown in Fig. 1 below).

   Visit the *Index of Complex Networks* (ICON) at `icon.colorado.edu` and obtain a copy of the `Medici network` data file, under the "Padgett Florentine families" ICON entry.

   Conduct the following two tests of their structural importance hypothesis:

   (a) Construct a table with four columns, each containing a list of (family, score) pairs for the following centrality score functions:
      - degree centrality,
      - harmonic centrality (Eq. 7.22 in *Networks* [v1: 7.30]),
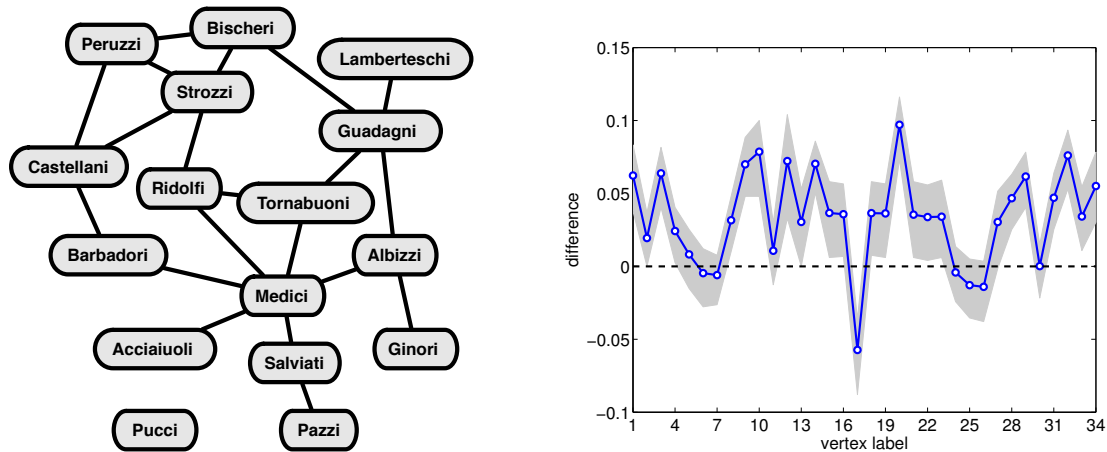   - eigenvector centrality (Eq. 7.2 in *Networks* [v1: 7.6]),

Figure 1: (left) The Medici family alliance network, from Padgett and Ansell (1993). (right) The difference between the observed centrality and the mean centrality under the configuration model, for the karate club network.

- betweenness centrality[1] (Eq. 7.25 in *Networks* [v1: 7.38]).

  Within each centrality, sort the pairs in decreasing order of importance; three decimal places is sufficient detail.

  Discuss (i) the degree to which these scores agree with Padgett and Ansell's claim that the Medicis occupied a structurally important position in this network, and (ii) what the scores say about the second most important family.

(b) Determine whether the relative structural importance of the Medicis can be explained solely by the degree sequence $\{k_i\}$ of the network G. To do this, use the configuration model to produce an ensemble of 1000 random graphs with the same degree sequence $\{k_i\}$ as G, and extract a distribution of harmonic centrality scores for each node.

  Make a figure showing the difference between a node's harmonic centrality on G and its average harmonic centrality in the ensemble. On this figure, include lines showing the 25% and 75% quantiles of the distribution around the mean for all vertices (as in the figure below, for the Karate club).

  Finally, (i) discuss your results here in terms of Padgett and Ansell's story of the Medici family, and (ii) comment on whether your results from part (3a) have changed.

4. (25 pts *extra credit*) Consider the task of designing a network in which distinct vertices hold the status of highest centrality for some set $\mathcal{S}$ of centrality measures. (Ties are prohibited.) If the network is relatively small compared to the number of centrality measures, this can be a very difficult task.

   - As a warm up, design by hand a small network ($n \leq 7$) that has this property with respect to $\mathcal{S} = \{$ degree, betweenness $\}$ centralities. Include a visualization of the

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[1]To properly compute betweenness, you must handle multiple geodesics with equal length between a pair i and j. See Lectures 2 nodes, footnote 16 for one way to do this.

network (label all the vertices, and indicate which ones win at which centrality measure) and produce a table in which the columns show the ranked vertices and scores for these centrality measures.

- Now, write a program to search the space of all possible connected, non-isomorphic networks of a given size n to find a network with this property for $\mathcal{S} = \{$ degree, harmonic, betweenness, eigenvector $\}$ centralities, or report that no such network exists for that size. Run this program for $n = 3, 4, 5, \ldots$ and report the smallest value of n for which at least one such network exists. Visualize that network, label all the vertices, and indicate which ones win at which centrality measure; and produce a table in which the columns show the ranked vertices and scores for these centrality measures. Comment briefly about what insights you gained about how to build networks that have distinct winners for different measures of centrality.