9/28/2020 (week 6)

- [ ] HW 5+6 posted → 10/9
- [ ] Project Proposal → 10/9
    1) Slack Channel!
    2) 1, 2, 3 people
    3) See Canvas/Slack Description.

Stochastic Block Model

Advantages:

1. Makes our assumptions explicit.
2. Interpretability of parameters.
3. Comparison via Likelihood Ratios, e.g.
4. Create synthetic data (stay tuned)

SBM: "tuple" = (B, b, w)

B: # groups, comms., blocks

b: vector. Assignment of node i to a group. $b_i \in [1, B]$

w: $B \times B$ "block" matrix. w_{rs} = prob that a node in group r connects to a node in group s.

w : $\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$ $\begin{bmatrix} 0 & q & 0 \\ q & 0 & q \\ 0 & q & 0 \end{bmatrix}$ $\begin{bmatrix} p & q & \\ q & p & q \\ & q & p \end{bmatrix}$

(see slides)

Rule:

$$A_{ij} = \begin{cases} 1 & \text{w.p. } \omega_{b_i b_j} \\ 0 & \text{else.} \end{cases}$$

1) get group $i \to b_i$
2) get group $j \to b_j$
3) look up entry $\omega_{b_i b_j}$
4) RNG
5) $\{1, 0\} \to A_{ij}$

generative.

MLE:

$L(X \mid \theta)$

data $\quad$ params

1) Imagine a generative process, whose probabilities I can grapple with.
2) Likelihood: Pr(data | parameters)

e.g. $\Pr(6H, 1T \mid \text{bias} = 0.59)$

$$= \binom{7}{6} (0.59)^6 (1-0.59)^1$$

$\Pr(6H, 1T \mid \text{bias} = p)$

$$= \binom{7}{6} p^6 (1-p)^1$$

$\Pr(H, T \mid \text{bias } p)$

$$L(X \mid \theta) = \binom{H+T}{H} p^H (1-p)^T$$

3) Max likelihood over all possible params.

Calculus!

$$\frac{\partial L(X \mid \theta)}{\partial \theta} = 0$$

often easier

$$\frac{\partial \log L(X \mid \theta)}{\partial \theta} = 0$$

SBM Likelihood

$$\log(xyz) = \log x + \log y + \log z$$

$$L(\text{observe mtx } A \mid \vec{b}, \omega)$$

Prob (likelihood) of any one edge, $i \to j$.

$$L(A \mid \vec{b}, \omega) = \prod_{i,j} \Pr(i \to j \mid \vec{b}, \omega)$$

uses independence of edges in SBM.

all possible edges

$$= \prod_{\substack{(i,j) \in E \\ \text{"ones"}}} \Pr(i \to j \mid \vec{b}, \omega) \prod_{\substack{(i,j) \notin E \\ \text{"zeros"}}} \left(1 - \Pr(i \to j \mid \vec{b}, \omega)\right)$$

$$= \prod_{(i,j) \in E} \omega_{b_i b_j} \prod_{(i,j) \notin E} (1 - \omega_{b_i b_j})$$

$$= \prod_{i,j} \omega_{b_i b_j}^{A_{ij}} (1 - \omega_{b_i b_j})^{1 - A_{ij}}$$

$$\log L(A \mid \vec{b}, \omega) = \sum_{ij} \log \left[\omega_{b_i b_j}^{A_{ij}} (1 - \omega_{b_i b_j})^{1 - A_{ij}} \right] = \sum_{ij} A_{ij} \log \omega_{b_i b_j} + (1 - A_{ij}) \log(1 - \omega_{b_i b_j})$$

$$\sum_{ij} A_{ij} \log w_{b_i b_j} + (1 - A_{ij}) \log(1 - w_{b_i b_j})$$

can't take a derivative

$$\boxed{\sum_{rs} e_{rs} \log w_{rs} + (n_{rs} - e_{rs}) \log(1 - w_{rs})} = \log L(\hat{b}, w)$$

can take a derivative!

$$e_{rs} = \sum_{ij} A_{ij}\, \delta(b_i, r)\, \delta(b_j, s)$$

not a fraction. # edges $r \to s$.

$$n_{rs} = \sum_{ij} 1\, \delta(b_i, r)\, \delta(b_j, s)$$

$$\frac{\partial \log L}{\partial w_{rs}} = \frac{e_{rs}}{w_{rs}} + (n_{rs} - e_{rs}) \frac{1}{1 - w_{rs}} (-1) = 0$$

find max

algebra

$$\boxed{\hat{w}_{rs} = \frac{e_{rs}}{n_{rs}}} = \frac{\text{\# observed edges } r \to s}{\text{\# possible edges.}}$$

heads

flips

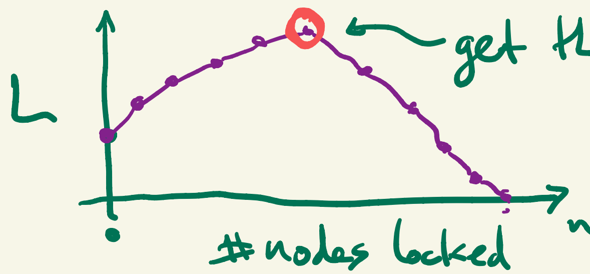
$$\log L = \sum_{rs} e_{rs} \log \frac{e_{rs}}{n_{rs}} + (n_{rs} - e_{rs}) \log\left(1 - \frac{e_{rs}}{n_{rs}}\right)$$

$$\log L(\vec{b}) = \underbrace{\sum_{rs} \boxed{e_{rs} \log \frac{e_{rs}}{n_r n_s}} + \boxed{(n_r n_s - e_{rs}) \log\left(1 - \frac{e_{rs}}{n_r n_s}\right)}}_{\text{function } (A, \vec{b}) \to \log L.}$$

search problem over partitions!

Local Search

assumes B.

1. Start with random $\vec{b}_0$
2. For every node, try moving it, hypothetically to every group available. ($n \times (B-1)$ proposals)
3. Choose the move that most ↑ L. "Lock it in."
4. Repeat 2, 3, but only considering nodes that are not yet locked in.
5. Repeat 4 until all nodes have moved.
6. 

get the best!

7. Go back to 2 with all nodes "released". $b_0 =$ ⊙

Poisson SBMs

$$\Pr(k \mid \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Let A_{ij} = # edges $i \leftrightarrow j$

$= \text{Pois}(\omega_{rs})$

Which terms depend only on the data, and can therefore be ignored?

#cancelculture

$$\Pr(A_{ij} \mid \omega, \vec{b}) \begin{cases} = \dfrac{(\omega_{b_i b_j})^{A_{ij}}}{A_{ij}!} e^{-\omega_{b_i b_j}} & i<j \\[2ex] = \dfrac{\left(\frac{1}{2}\omega_{b_i b_j}\right)^{A_{ii}/2}}{A_{ii}/2\,!} e^{-\frac{1}{2}\omega_{b_i b_j}} & i=j \end{cases}$$

$A_{ii} = 2$ 

A_{ii} = 2x #edges i to itself.

$$\Pr(A \mid \omega, b) = \prod_{i<j} \frac{\omega_{b_i b_j}^{A_{ij}}}{A_{ij}!} e^{-\omega_{b_i b_j}} \prod_i \frac{\left(\frac{1}{2}\omega_{b_i b_i}\right)^{A_{ii}/2}}{A_{ii}/2\,!} e^{-\frac{1}{2}\omega_{b_i b_i}}$$

$$\log L = \sum_{i<j} A_{ij} \log \omega_{b_i b_j} - \cancel{\log A_{ij}!} - \omega_{b_i b_j} + \sum_i \frac{A_{ii}}{2} \log\left(\frac{1}{2}\omega_{b_i b_i}\right) - \cancel{\log A_{ii}/2} - \frac{1}{2}\omega_{b_i b_i}$$

$$\log L = \sum_{i<j} A_{ij} \log \omega_{b_i b_j} - \omega_{b_i b_j} + \sum_i \frac{A_{ii}}{2} \log\left(\tfrac{1}{2}\omega_{b_i b_i}\right) - \tfrac{1}{2}\omega_{b_i b_i}$$

$$= \sum_{rs} e_{rs} \log w_{rs} - w_{rs} n_r n_s$$

$$\frac{\partial L}{\partial w_{rs}} = 0 \quad \cdots\cdots \Rightarrow \quad \hat{w}_{rs} = \frac{e_{rs}}{n_r n_s}$$

$$\log L = \sum_{rs} e_{rs} \log \frac{e_{rs}}{n_r n_s} - \frac{e_{rs}}{\cancel{n_r n_s}} \cancel{n_r n_s}$$

$$\log L(b) = \boxed{\sum_{rs} e_{rs} \log \frac{e_{rs}}{n_r n_s}} \quad \cancel{-2m}$$

☆ Karrer + Newman 2011 g instead of b. ☆

① How to choose B?

② $X_1, X_2, \ldots$ Pois

$X = X_1 + X_2 + \ldots$

$\Rightarrow$ X is also poisson.

$\downarrow$

every node has a Poisson deg. distrib!.

DCSBM

Let $A_{ij} = \text{Pois}\left(\theta_i \theta_j \omega_{b_i b_j}\right) = \text{Pois}\left(\underline{k_i k_j}\, \omega_{b_i b_j}\right)$

DCSBM is like configmodel.

+ block model

how hungry are nodes i and j for edges?

block structure.

$P_{ij} \simeq \frac{k_i k_j}{2m}$ Chung Lu!

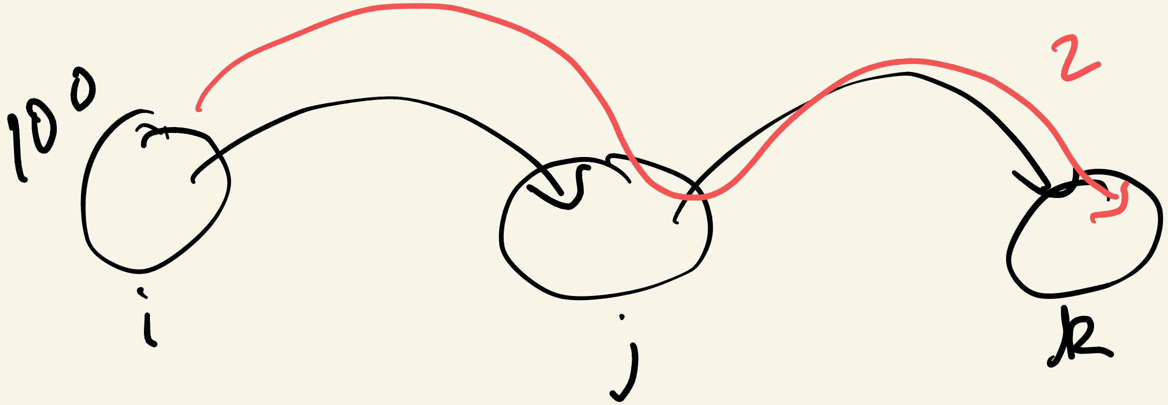
$$\boxed{\log L(b) = \sum_{rs} e_{rs} \log \frac{e_{rs}}{K_r K_s}}$$

local search.

$K_r = \sum_s e_{rs}$

"degree" of group r

$\hat{\theta}_i = k_i$



$$\text{Model}_1 = M^1 = P_{ijk} = P_{ij} P_{jk}$$

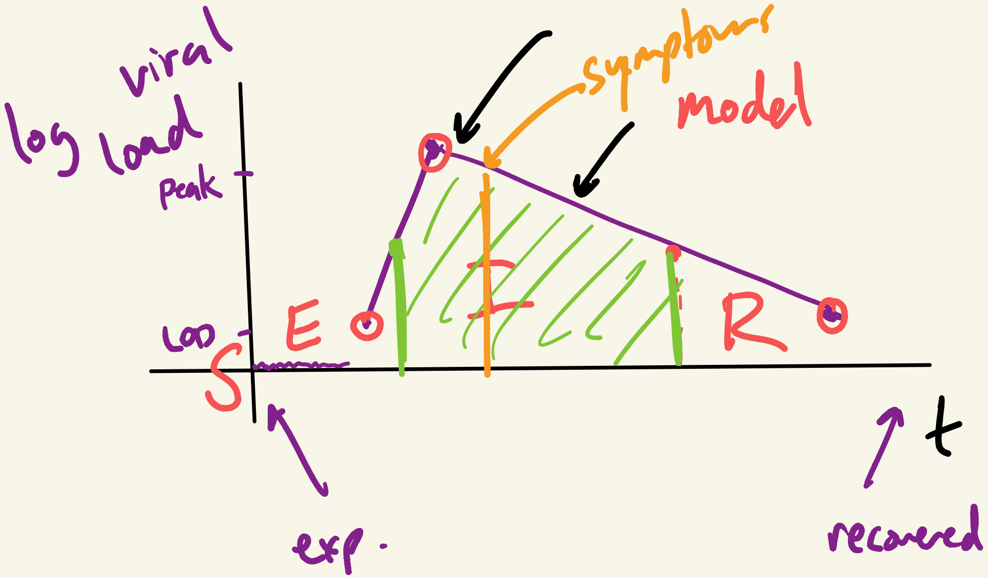
$$\text{Model}_2 = M^2 = P_{ijk} = P_{ijk}$$

$$\text{var}(\hat{p}_{ijk}) = \frac{\hat{p}_{ijk}(1-\hat{p}_{ijk})}{n_i} \quad ?$$

$$\hat{p}_{ijk} = \frac{\# \, i \to j \to k}{\# \, i}$$

? ? ?

$$\hat{p}_{ijk} = \hat{p}_{ij} \, \hat{p}_{jk} = \frac{\# i \to j}{\# i} \cdot \frac{\# j \to k}{\# j} \leftarrow \text{var}(\hat{p}_{ijk}) = \frac{\hat{p}_{ij}(1-\hat{p}_{ij})}{n_i} \; \frac{\hat{p}_{jk}(1-\hat{p}_{jk})}{n_j}$$



1) well-mixed

2) P(S becomes inf) same for all ppl in pop.

infectiousness $\propto f$(viral load)

f is monotonic ?

?

① Recipe for a A(t) ?

?

② viral loads

?

behavior: Δ in link formation.

A(t)

- heterogeneous structure. densities/comms.
- times of day when links ↑ or ↓
- behavior. sympt? asympt?