9/14/2020 Week 4.

Objection to ER: $G(n, p)$

unrealistic deg seq!

What if we fix degrees?

(but make it otherwise random?) $G(n, \vec{k})$

where $\vec{k} = \{k_i\}_{i=1}^{n}$ degrees.

How much of some observed pattern is driven by (or could be explained by) the degrees alone?

p_{ij} = Prob that $i \leftrightarrow j$.

(ER: $p_{ij} = p \;\; \forall\, i,j$)

Let p_{ij} depends on k_i, k_j

target. goal.

From i's perspective:

$\frac{k_j}{2m}$ prob that one edge from i goes to j

that match to j

total # slots

$$p_{ij} = k_i \left(\frac{k_j}{2m} \right) = \frac{k_i k_j}{\sum_i k_i}$$

Chung-Lu

$$p_{ij} = \frac{k_i k_j}{2m}$$

$\forall_{i>j} \quad A_{ij} = A_{ji} = \begin{cases} 1 & \text{w.p. } p_{ij} \\ 0 & \text{else.} \end{cases}$

- simple graph: no self loops
  no multi-edges.
- Requires $\Theta(n^2)$ rand. $\left[\binom{n}{2} \text{ calls}\right]$

Generalize:

- Directed Chung Lu? $G(n, k^{in}, k^{out})$

  $p_{ij} = \frac{k_i^{out} k_j^{in}}{m}$, $\forall\, i,j$ (both orders)

Problem:

$$p_{ij} \leq 1$$

$$\Rightarrow \frac{k_i k_j}{2m} \leq 1$$

Let $k_{max} \geq k_i$ $\forall i$

$$\Rightarrow \frac{k^{2}_{max}}{2m} \leq 1$$

$$k^{max} \leq \sqrt{2m}$$

$$k^{max} \ll n$$

Fix

$$p_{ij} = \min\left\{1, \frac{k_i k_j}{2m}\right\}$$

New Problem: Hubs are too small!

Consistency?

$$\underline{E[k_i]} = E\left[\sum_j A_{ij}\right]$$

$$= \sum_j E[A_{ij}]$$

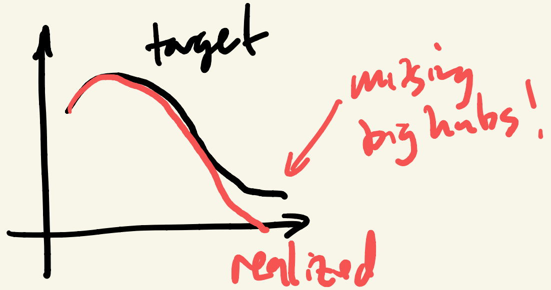
$$\boxed{A_{ij} = \begin{cases} 1 & \text{w.p. } p_{ij} \\ 0 & \text{else} \end{cases} \qquad \mathrm{Bern}(p_{ij})}$$

if I sub in $\min\{p_{ij}, 1\}$

$$= \sum_j p_{ij}$$

$$= \sum_j \frac{k_i k_j}{2m}$$

$$= \frac{k_i}{2m} \sum_j k_j = \frac{k_i}{\cancel{2m}} \cancel{2m} = k_i$$

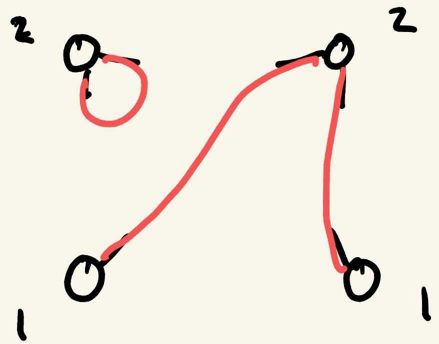


- Chung Lu gets degrees right <u>only</u> in expectation.

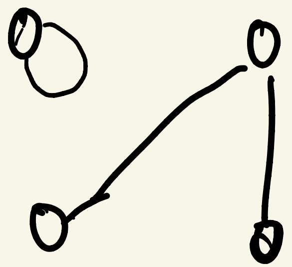
Configuration Model.

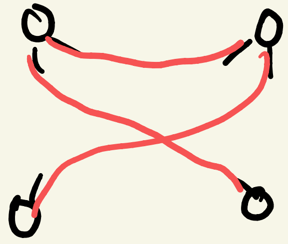
"stub matching"

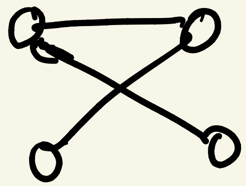
$\vec{k} = \{1, 2, 2, 1\}$



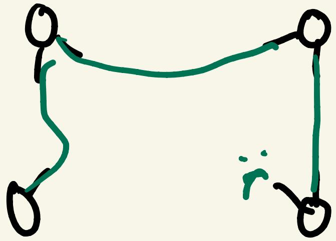
choose a pair, wire up!







$k = \{1, 2, 2, 2\}$



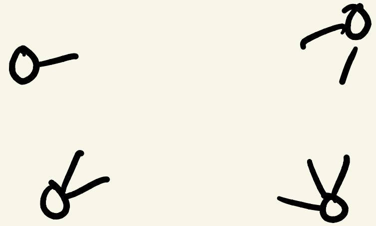
obviously even

Config Model:

$$2m = \sum_i k_i = \text{even}.$$

$k = \{1, 3, 2, 2\}$

1 2 3 4

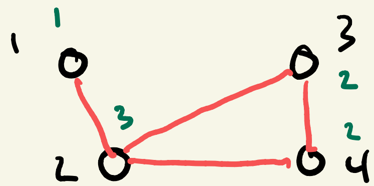


$1\ 2\ 2\ 2\ 3\ 3\ 4\ 4 = \vec{q}$

each node is in the list k_i times.

q.shuffle.

4 2 3 2 1 2 4 3



Problems! Next Time.