Week 7.

Spreading Processes on Networks.

- Diseases
- Gossip
- Memes
- Avalanches of neuronal excitations
- Research topics
- Blackouts

[Not Dynamics of networks]

- Continuous vs. Discrete Time (just like other dynamical systems)

- Simple vs. Complex contagions.

Simple → A single exposure may be sufficient for spread.

- Inf. Disease
- Song
- Meme

Complex → Multiple exposures required.

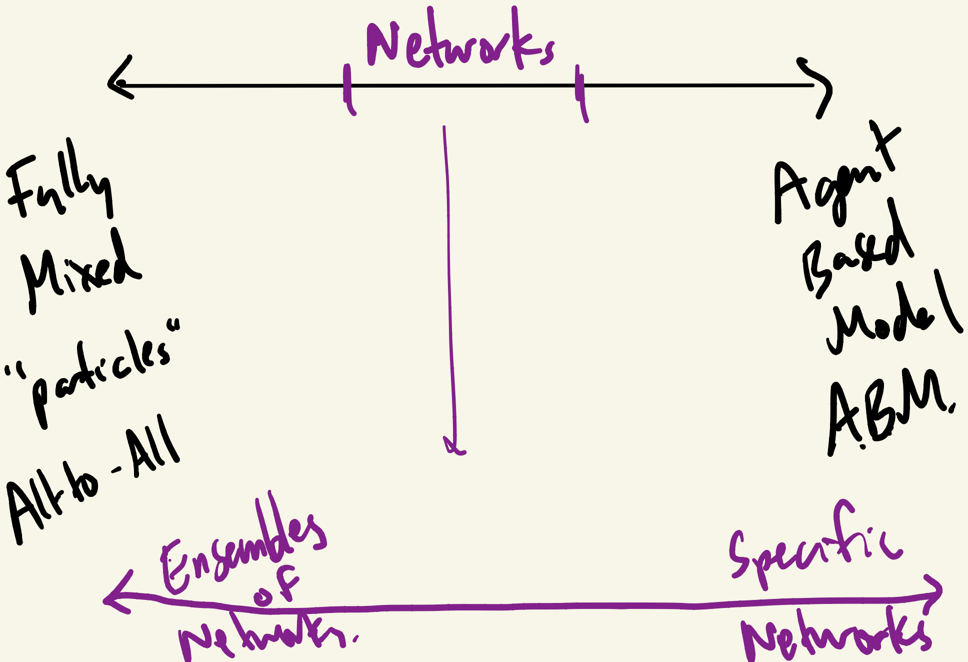
- Art/songs "acquired taste" e.g. Greatest Showman
- Candidate → Opinion.
- Fashion.

Two nodes are linked if they interact in a way that the disease can spread.

Mechanism Matters

↓

Links are a function of behavior.



Network + Dynamics

SIX dynamics

Compartmental Models.

↓

Indiv. are in compartments.

Each can be in only one at a time.

① Susceptible $S \to I$

Infected

② SIS $S \to I \to S$

③ SIR $S \to I \to R$

↓ vax.

Recovered

"Removed."

4. $S \rightarrow I \rightarrow R \rightarrow S$

5. $S \rightarrow E \rightarrow I \rightarrow R \dashrightarrow S$

??? ???

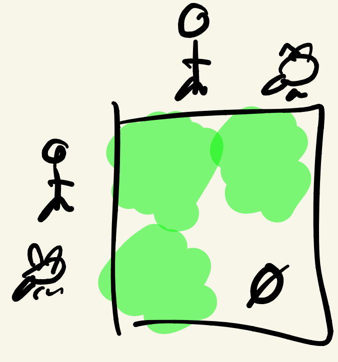
Unknown for SARS2

Exposed

6. $S \rightarrow I \rightarrow R \rightarrow S$

$S \rightarrow I \rightarrow R$

Dengue
Zika
Malaria.



The Fully Mixed SIS Model.

- Assumptions: everyone interacts w/ everyone else, in the same way.

Let N be # ppl/nodes.

- An I has a prob of $\bar{\beta}$ per unit time of transmitting to a S.

- An I has a prob γ per unit time to heal and become S again.

Bookkeeping.

$S(t) = \#$ susc.

$I(t) = \#$ infecteds.

$N = S(t) + I(t)$ const.

$$\frac{I(t)}{N} = \frac{N}{N} - \frac{S(t)}{N}$$

How does # of S change in a little slice of time?

$$S(t+\Delta t) = S(t) - S(t)\cdot I(t)\,\bar{\beta}\,\Delta t + I(t)\,\gamma\,\Delta t$$

a little bit later

$$s = \frac{S}{N} \quad i = \frac{I}{N} \quad \beta = \bar{\beta} N$$

1. sub in s, i, β
2. $s = 1 - i$
3. $\lim \Delta t \to 0$

New interpretation of β

Exp. # of ppl. that a single infected individual infects per unit time. (if that whole pop is S)

$$\boxed{\frac{di}{dt} = \beta i(1-i) - \gamma i} = \beta(i - i^2) - \gamma i$$

Key Q: Will there be an epidemic at all?

- Assume that i is very small.
- i^2 is so small that we don't care. Neglect it!

$\downarrow$ linearize

$$\frac{di}{dt} = \beta i - \gamma i$$

$$\frac{di}{dt} = (\beta - \gamma) i$$

When will i grow over time?

$$\beta - \gamma > 0$$

$$\beta > \gamma \qquad \frac{\beta}{\gamma} > 1$$

$\frac{\beta}{\gamma}$: epidemic threshold

$\frac{\beta}{\gamma} < 1$ no epi.

R_0

$\frac{\beta}{\gamma} > 1$ epi.

Key Q: What will be the steady state of $i(t)$?

$\frac{di}{dt} = 0$ no change

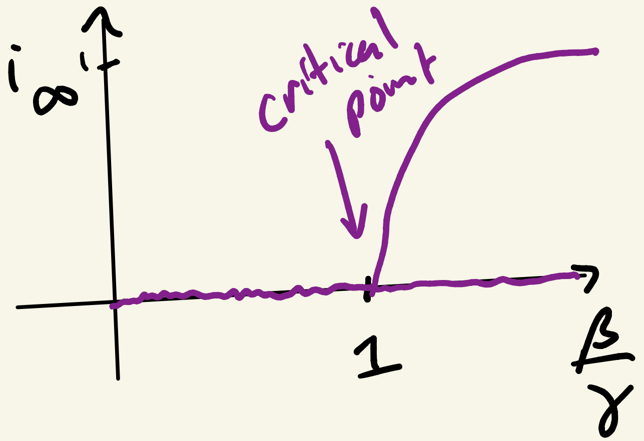
$$\beta(i - i^2) - \gamma i = 0$$

① $i = 0$

② $i \neq 0$ $\beta(1-i) - \gamma = 0$

$$i_\infty = 1 - \frac{\gamma}{\beta}$$

still a fraction. But it's only > 0 when $\frac{\beta}{\gamma} > 1$



2nd order phase transition
(c.f. Percolation)

All of this ↑ fully mixed

No Network (unless $A_{ij} = 1 \ \forall (i,j)$)

Let $i_n(t)$ = Pr that node n is infected at time t.

$i_n(t+\Delta t)$ = ① n infected at t and did not heal
OR ② n susc. at t and got infected.

$$i_n(t+\Delta t) = i_n(t)\left[1-\gamma\Delta t\right] + (1-i_n(t))\,\bar{\beta} \sum_{m=1}^{N} A_{nm}\, i_m$$

$\lim_{\Delta t \to 0}$

Now, let $A_{nm} = 1$ if and only if n,m interact in a disease-transmitting way.

$\bar{\beta}\Delta t$ = Pr(inf node passes to an uninfected neighbor)

$\gamma\Delta t$ = Pr(recover)

$$\frac{di_n}{dt} = -\gamma i_n + (1-i_n)\bar{\beta}\sum_{m=1}^{N} A_{nm} i_m$$

overall, i small $\Rightarrow$ $i_n i_m$ very small.

$$\frac{di_n}{dt} = -\gamma i_n + \bar{\beta}\sum_{m=1}^{N} A_{nm} i_m$$

$\downarrow$ matrix-vector.

$$\frac{di_1}{dt} = -\gamma i_1 + \bar{\beta}\sum_m A_{1m} i_m$$

$$\frac{di_2}{dt} = -\gamma i_2 + \bar{\beta}\sum_m A_{2m} i_m$$

$$\vdots$$

$$\frac{di_N}{dt} = -\gamma i_N + \beta\sum_m A_{3m} i_m$$

$$\rightarrow \frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = -\gamma \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} + \bar{\beta}\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} i \end{bmatrix}$$

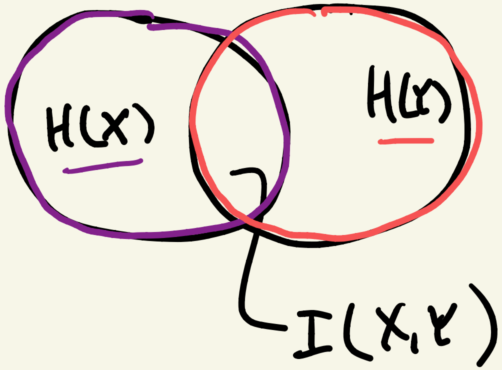
$$\frac{d\mathbf{i}}{dt} = -\gamma \mathbf{i} + \bar{\beta} A \mathbf{i}$$

$\uparrow$

$$\boxed{\frac{d\mathbf{i}}{dt} = (-\gamma I + \bar{\beta} A)\mathbf{i}}$$

$$i(t) = \vec{u} e^{(\bar{\beta}\lambda - \gamma)t}$$????

$$NMI(X,Y) = \frac{\overbrace{2\,I(X,Y)}^{\text{Mutual Information}}}{\underbrace{H(X) + H(Y)}_{\text{entropy}}}$$

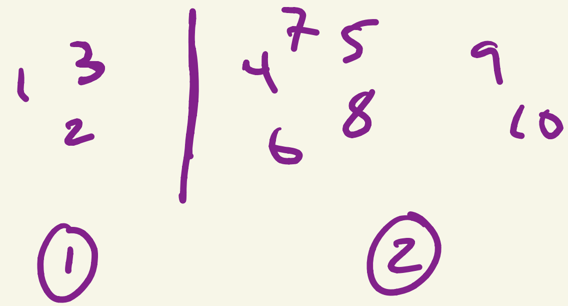


Imagine that a partition into communities is a random variable X. (another partition is Y).

X communities: group 1: 1, 2, 3

group 2: 4, 5, 6 ..., 10.

What is the entropy of this partition?



$$Pr(1) = \frac{3}{10} \qquad Pr(2) = \frac{7}{10}$$

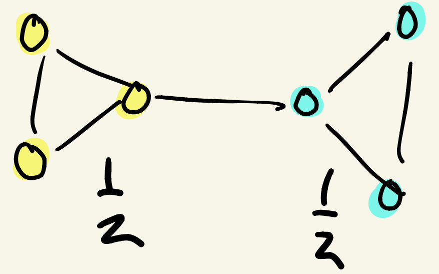
$$X = \left[\frac{3}{10}, \frac{7}{10}\right]$$

categorical distribution

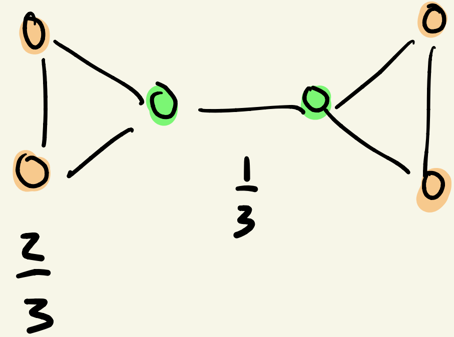
$$H(X) = -\sum_{k=1}^{n \text{ outcomes}} p_k \log p_k$$

$$= -\frac{3}{10}\log\frac{3}{10} - \frac{7}{10}\log\frac{7}{10}$$

$$I(X,Y) = -\sum_x \sum_y p_{x,y} \log \frac{p_x \, p_y}{p_{x,y}}$$



X



Y

$1/6$	$1/6$	$= \frac{1}{3}$
$1/3$	$1/3$	$= \frac{2}{3}$
$\frac{1}{2}$	$\frac{1}{2}$	

If X and Y are totally unrelated. (indep.)

$\Rightarrow p_{xy} = p_x \, p_y$

$\Rightarrow I = 0 \Rightarrow NMI = 0$

$$I(X,Y) = -\left(\frac{1}{6} \log \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{6}} + \frac{1}{3} \log \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{3}} \cdots \right)$$

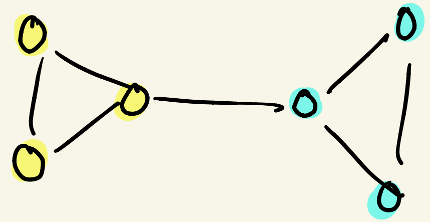
confusion matrix

$P(X,Y)$

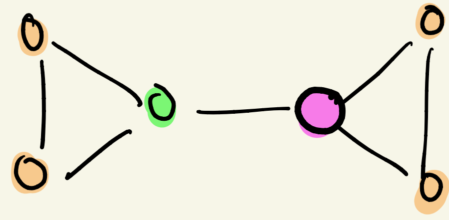
joint distrib.

$$NMI(X,Y) = \frac{2\,I(X,Y)}{H(X) + H(Y)}$$

$$I(X,Y) = -\sum_{x} \sum_{y} p_{x,y} \log \frac{p_x \, p_y}{p_{x,y}}$$



X



Y

$$\begin{bmatrix} 1/6 & 0 \\ 1/3 & 1/3 \\ 0 & 1/6 \end{bmatrix} \begin{array}{l} \to \frac{1}{6} \\ \to \frac{2}{3} \\ \to \frac{1}{6} \end{array}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ \frac{1}{2} & \frac{1}{2} \end{array}$$

When $X = Y$, what does the confusion matrix look like?

→ diagonal matrix.

$$I(X,X) = -\sum_{x} \cancel{p_{x,x}}^{\,p_x} \log \frac{p_x \, p_x}{\cancel{p_{x,x}}_{\,p_x}}$$

$$= -\sum_{x} p_x \log p_x = H(X)$$

$$\text{NMI} = \frac{2\, I(X,X)}{H(X) + H(X)} = 1$$

$i(t) = \vec{u}\, e^{(\bar{\beta}\lambda - \gamma)t}$???? $\longleftarrow$

$$\boxed{\frac{di}{dt} = (-\gamma I + \bar{\beta} A)\, i}$$

$$\boxed{i(t) = \vec{u}\, e^{(\bar{\beta}\lambda - \gamma)t}}$$

prob inf. $\propto$ eigenvector centrality!

growth/decay depends on λ, P.F. eval.

$$\vec{u}\,(\bar{\beta}\lambda - \gamma)\,\cancel{e^{(\bar{\beta}\lambda-\gamma)t}}\,) = (-\gamma I + \bar{\beta} A)\,\vec{u}\,\cancel{e^{(\bar{\beta}\lambda-\gamma)t}}$$

$$\underbrace{\vec{u}}_{\text{vector}}\,\underbrace{(\bar{\beta}\lambda - \gamma)}_{\text{scalar}} = (-\gamma \underbrace{I}_{\text{matrices}} + \bar{\beta} \underbrace{A}_{\text{matrices}})\,\underbrace{\vec{u}}_{\text{vector}}$$

$$\bar{\beta}\lambda\vec{u} - \cancel{\gamma\vec{u}} = -\cancel{\gamma\vec{u}} + \bar{\beta}A\vec{u}$$

$$\cancel{\bar{\beta}}\lambda\vec{u} = \cancel{\bar{\beta}}A\vec{u} \qquad \bar{\beta} > 0$$

$$\boxed{\lambda\vec{u} = A\vec{u}}$$

eval.

$M\vec{x} = k\vec{x}$

eigenvector

$$i(t) = \vec{u}\, e^{(\bar{\beta}\lambda - \gamma)t}$$

growth when $\bar{\beta}\lambda - \gamma > 0$

$$\Rightarrow \quad \bar{\beta}\lambda - \gamma > 0$$

$$\bar{\beta}\lambda > \gamma$$

$$\frac{\bar{\beta}}{\gamma} > \frac{1}{\lambda}$$

disease charact. $\quad$ structure of network.

epidemic threshold,

$$\frac{\bar{\beta}}{\gamma} = \frac{1}{\lambda}$$

All-to-All $\quad \beta = \bar{\beta} N$

Now $\quad \beta = \bar{\beta}\langle k \rangle$

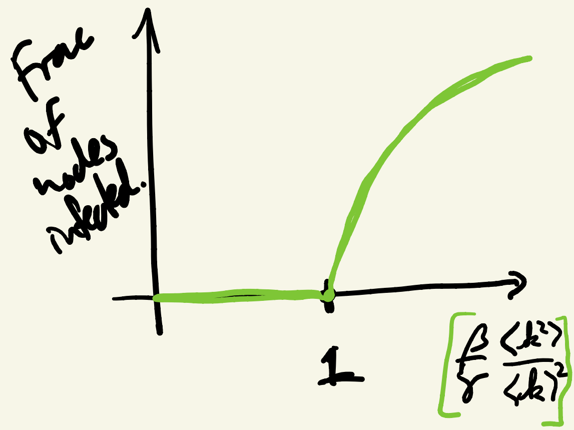
β ~ typical # of nodes to become infected at start of epidemic, per infected node, per timestep.

e.t. $$\frac{\beta}{\gamma} = \frac{\langle k \rangle}{\lambda}$$

For a wide class of networks.

$$\lambda \approx \frac{\langle k^2 \rangle}{\langle k \rangle}$$

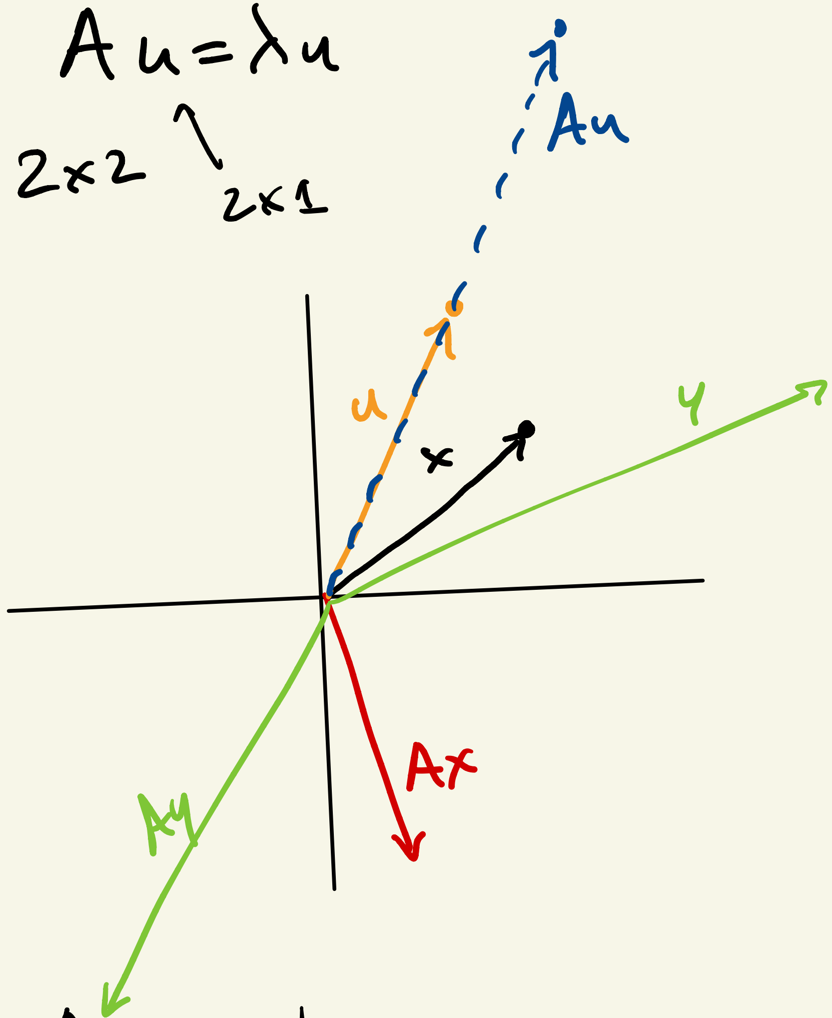
$$\frac{\beta}{\gamma} = \frac{\langle k \rangle^2}{\langle k^2 \rangle}$$



Eigenvalues / Eigenvectors

$$\underset{2\times 2}{A}\,\underset{2\times 1}{u} = \lambda u$$

$$\frac{|Au|}{|u|} = \lambda$$



Ax is also 2×1

If u represents the vector of infected nodes.

$\Rightarrow$ $Au = \lambda u$ represents stretched version of those probabilities.

"Direction of growth" for infections.

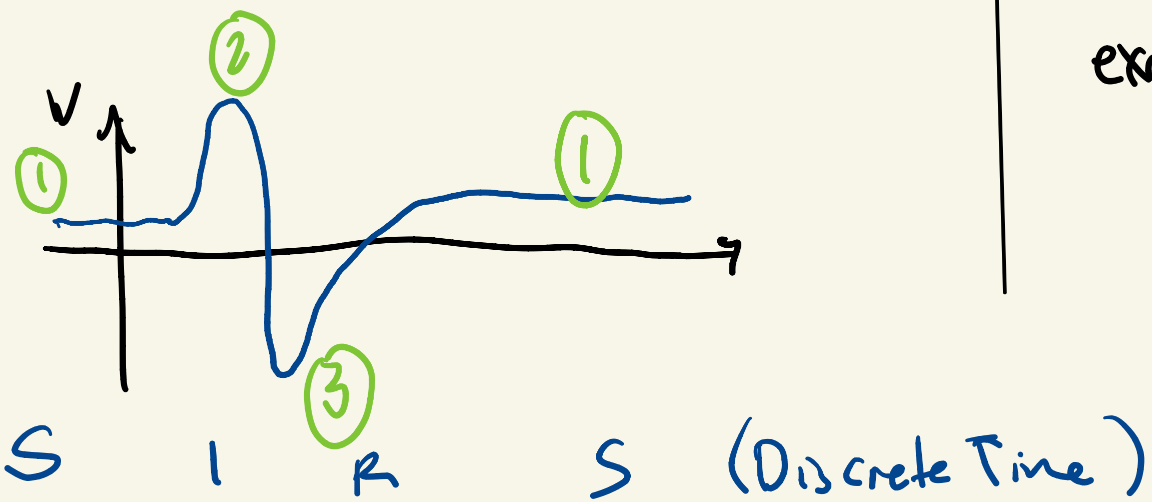
$$i(t) = \vec{u}\, e^{(\bar{\beta}\lambda - \gamma)t}$$

Excitable Dynamics

Greenberg Hastings Cellular Automaton.

$$\begin{array}{l} \text{① Resting} \\ \quad \downarrow \text{ stimulus!} \\ \text{② Excited} \\ \quad \downarrow \text{ deterministic} \\ \text{③ Refractory} \end{array}$$

(recovering – neither excited nor excitable)



Let there be 0 refractory states.

$$\begin{tikzcd} \text{① Rest} \arrow[r, bend left, "\text{stim}, \Delta t"] & \text{② Excited} \arrow[l, bend left, "\Delta t"] \end{tikzcd}$$

Consider a weighted network.

A_{nm} = Pr that excited n transmits stimulus to resting m

A is a weighted, directed adj. mtx.

Let $p_m(t)$ = Pr. that node m is excited at time t.

$$p_m(t+1) = \underline{(1-p_m(t))}\underline{\left[1-\prod_n\left(1-A_{nm}\,p_n(t)\right)\right]}$$

- must have been at resting t
- <u>And</u> got a stimulus from a nbr.

What are the fixed points of this dynamics?

$\vec{p} = 0$ Check $p_m = 0 \ \forall m$

$$0 = (1-0)\left[1-\prod_n \underline{(1-A_{nm}\cdot 0)}\right]$$

$$\underline{\underline{\underline{1}}}$$

$$\underline{1} \quad \underline{0}$$

$$0$$

Pr(stim by a net. nbr)

= 1 − Pr(not stim by any of its nbrs)

$= 1 - \prod_{\text{nbrs } n}$ Pr(not stim by n)

$= 1 - \prod_{\text{nbrs } n}$ (1 − Pr <u>was</u> stim by n)

$$p_m(t+1) = (1-p_m(t))\left[1-\prod_n (1-A_{nm}p_n(t))\right]$$

$(1-x)(1-x)(1-x)\dots$

$1-3x+\mathcal{O}(x^2)$

p_m small. $\longrightarrow$ $p_m p_n$ too small to care about!

$\mathcal{O}(p^2)$

$$p_m(t+1) = \left[1-\prod_n (1-A_{nm}p_n(t))\right] - p_m(t)\left[1-\prod_n (1-A_{nm}p_n(t))\right]$$

c $\qquad \mathcal{O}(p)$ $\qquad \mathcal{O}(p)$

$(1-Ap_1)(1-Ap_2)(1-Ap_3)\dots(1-Ap_N)$

$1-\sum A_{nm}p_n(t) + \mathcal{O}(p^2)$

$\downarrow$

Linearize

$\downarrow$

$$p_m(t+1) = \sum_n p_n(t) A_{nm}$$

Ansatz. Let $p_m(t) = u_m\lambda^t$ $\Longrightarrow$

$$u_m\lambda^{t+1} = \sum_n u_n\lambda^t A_{nm}$$

$$u_m\lambda = \sum_n A_{nm}u_n$$

$$\lambda u = A^T u$$

Eigenvalue Equation.

$Au = \lambda u$

$p_m(t) = u_m \lambda^t$

① Pr node m is excited is prop. to. eigenvector centrality! u_m

② <u>Stability.</u>

Growth: $\lambda > 1$

Decay: $\lambda < 1$

Dynamical Importance.

Juan G. APPM.

↓

Restrepo, Ott, Hunt.

<u>G. H. C. A.</u>

- discrete time
- excitable dynamics

$\Rightarrow$ growth dep. on λ.

$\Rightarrow$ node heterogeneity in Pr(excited) goes like eigen. centrality.

<u>SIS</u>

- cont. time
- inf. dynamics

$\Rightarrow$ growth dep. on λ.

$\Rightarrow$ node heterogeneity in Pr(infected) goes like eigen. centrality.

$$\underbrace{\frac{\beta}{\gamma}}_{\text{disease}}\ \underbrace{\frac{\langle k^2\rangle}{\langle k\rangle^2}}_{\text{contact network.}}$$

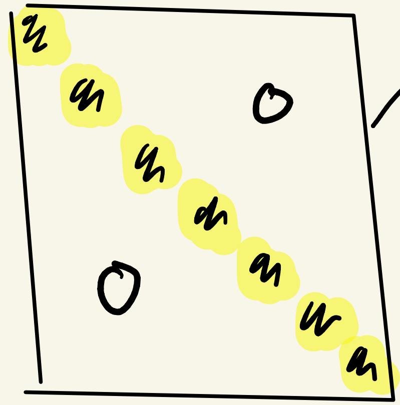
$$I(X,Y) = -\sum_x \sum_y P_{x,y} \log \frac{P_x P_y}{P_{x,y}}$$

What if $X = Y$?

$$I(X,X) = -\sum_x \sum_{x'} P_{x,x'} \log \frac{P_x P_{x'}}{P_{x,x'}}$$

States of x'

states of x



~ square.

$$NMI = \frac{2I_{XY}}{H_X + H_Y} = \frac{2H_X}{H_X + H_X} = 1$$

Greedy Mod Max!

1) Consider all possible Merges.

2) compute Q (or ΔQ) that would result.

3) Choose Best.

4) Implement that best merge

$$I(X,X) = -\sum_x P_{xx} \log \frac{P_x P_x}{P_{xx}}$$

$P_{xx} = P_x$

$$= -\sum_x P_x \log \frac{P_x^{\cancel{2}}}{\cancel{P_x}} = H(X)$$

October 9, 2020

- HW due tonight!
- Project Proposals due tonight!
  - 1 per group.
  - All names on proposal.

Granovetter Threshold Model. 1978

- N people in a crowd
- Each person i has a threshold θ_i above which they will participate.

  # people in crowd participating

- $f(x)$ # of people w/ threshold $= x$ PMF
- $F(x)$ # of people w/ threshold $\leq x$ CDF
- $s(t) =$ # of people participating at time t.

$S(0) = 0$

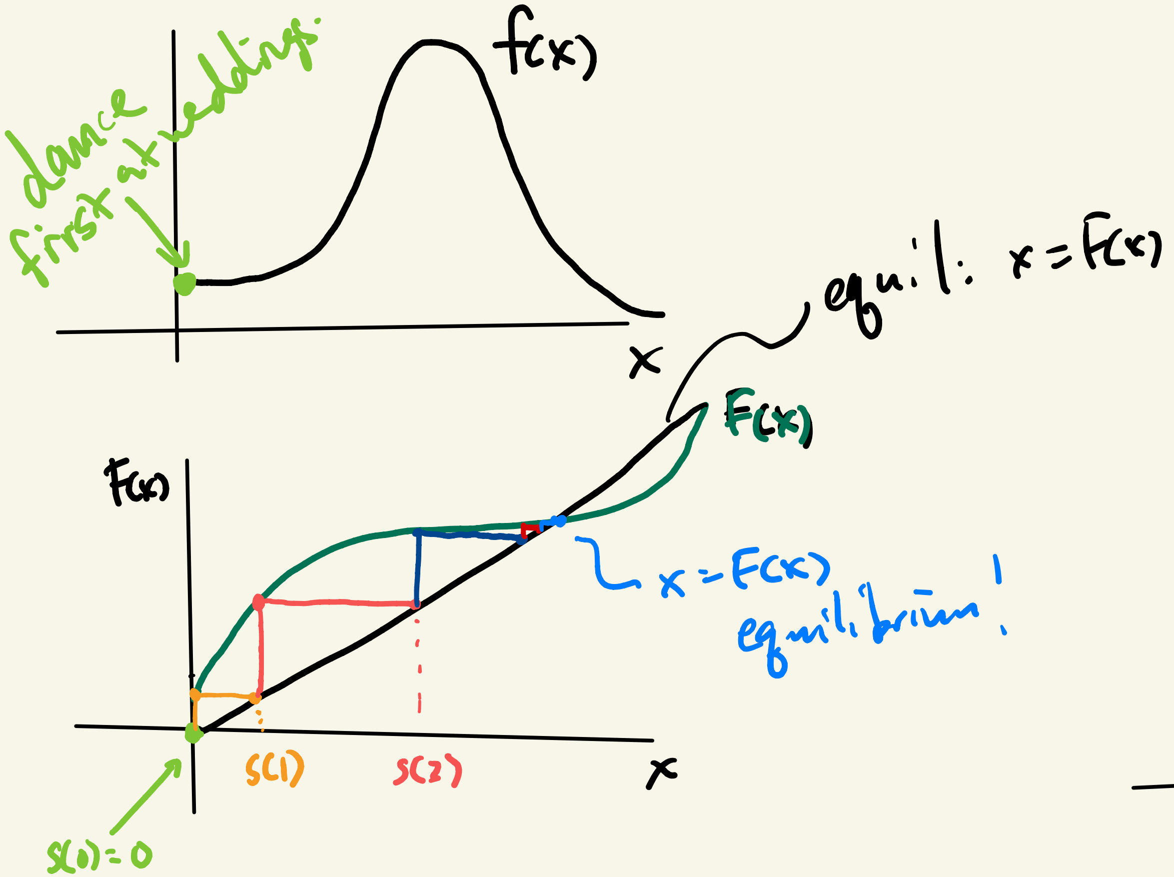
$s(1) =$ # ppl. w/ threshold ≤ 0

$= F(0)$

$s(2) = F(s(1)) = F(F(0))$

$s(t+1) = F(s(t))$

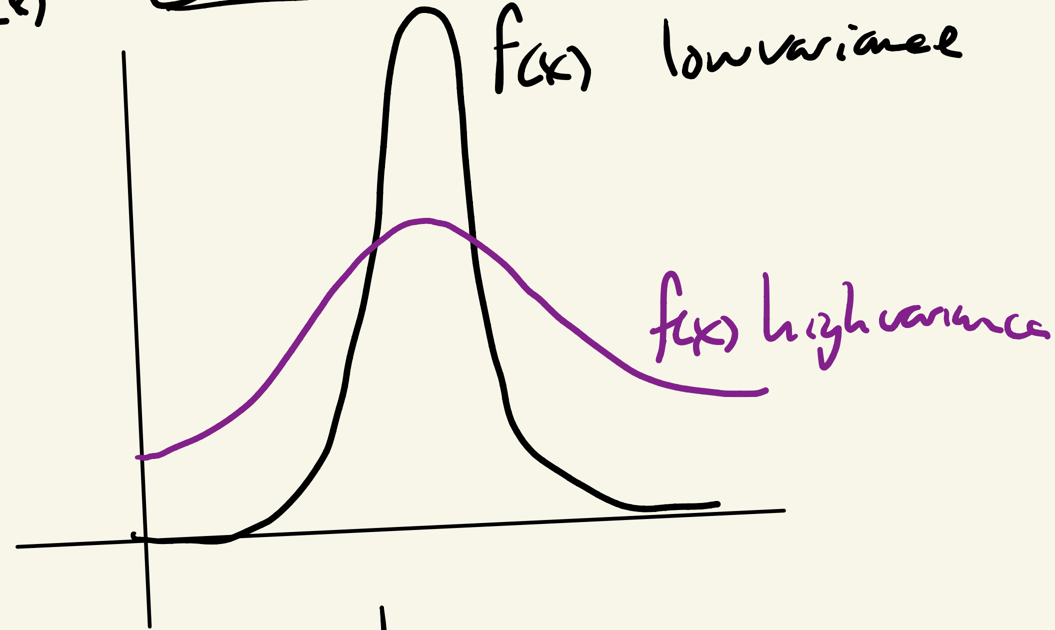
$\Rightarrow$ steady state: $s(t) = F(s(t))$



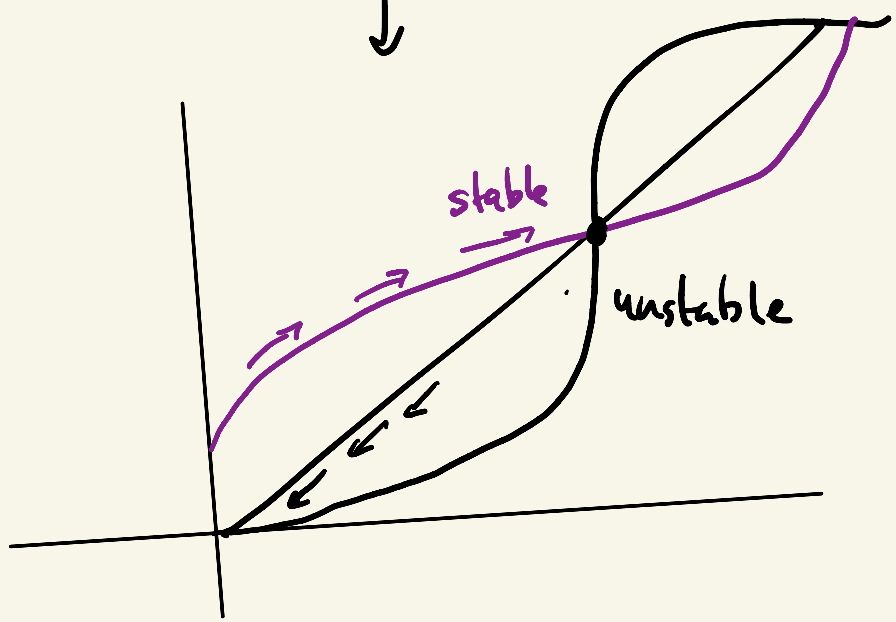
equil: $x = F(x)$ —

When do we get a riot?

↑ Variance of thresholds ⇒ Growth in participation.



↓



What about Cascade Models on a network?

(Kleinberg)

Networked Coordination Games

Behaviors A B
"old" "new"

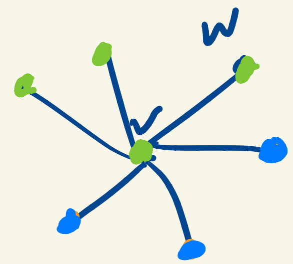
In the network, nodes have an incentive to have behaviors match.

$0 < q < 1$

If v, w both A $\rightarrow q$

v, w both B $\rightarrow 1-q$

v, w opposite $\rightarrow 0$



$A_{vw} = 1$ (neighbors)

What should v do?
(to max its payout)

If node v has deg d_v,
and d_v^A nbrs are A
d_v^B nbrs are B

- Payoff from choosing A is $d_v^A q$
- Payoff from choosing B is $d_v^B(1-q)$

$d_v^A = d_v - d_v^B$

$$\underline{(d_v - \cancel{d_v^B})q} < \underline{d_v^B(1-\cancel{q})}$$

$$d_v q < d_v^B \qquad q < \frac{d_v^B}{d_v}$$

] fraction of nbrs w/ B

A node should convert A → B when at least a fraction q of its nbrs have converted!

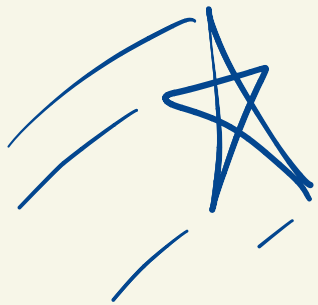
When does B spread?

S initial set of converted.

- After one round, w/ threshold q, $h_q(S)$ are now converted.
- $h_q^k(S)$ is the set of converted nodes after k rounds.

NB: nodes can switch A → B <u>AND</u> B → A.

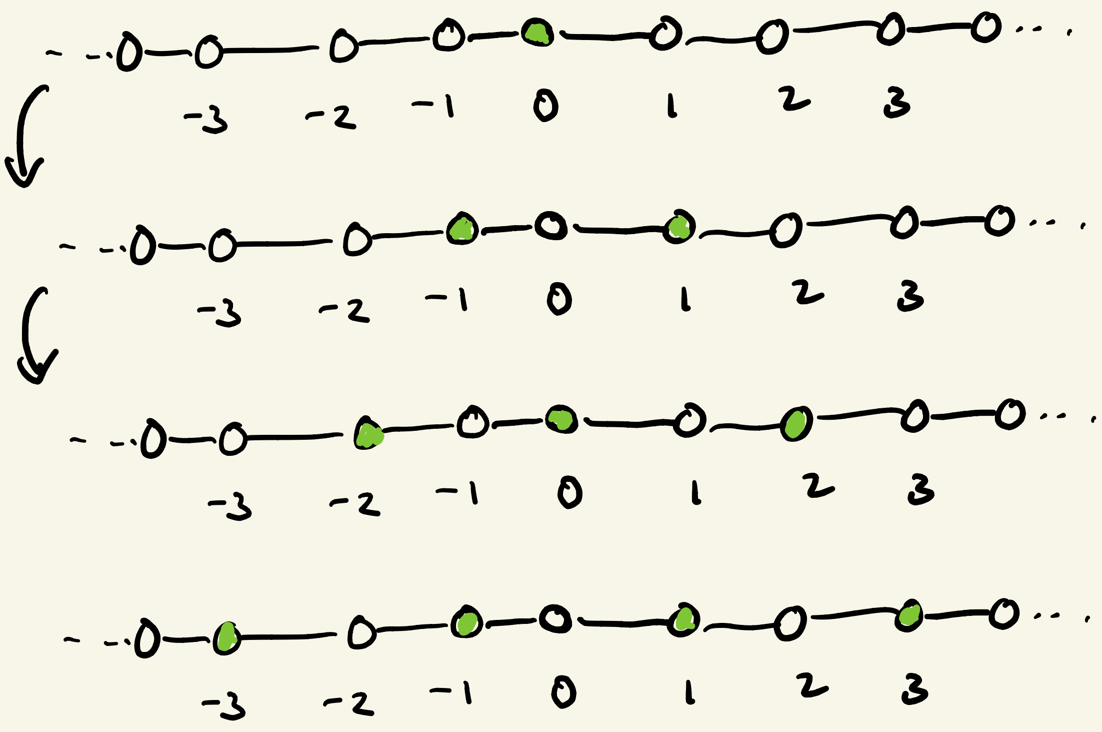
- A node w is converted by a set S <u>if</u> for some k^*, we have $w \in h_q^k(S)$ $\forall$ $k \geq k^*$.
  (w gets converted and never goes back)
- A set S is contagious w.r.t. h_q if every node is converted by S.
- What is the <u>maximum</u> q for which there exists a contagious set?

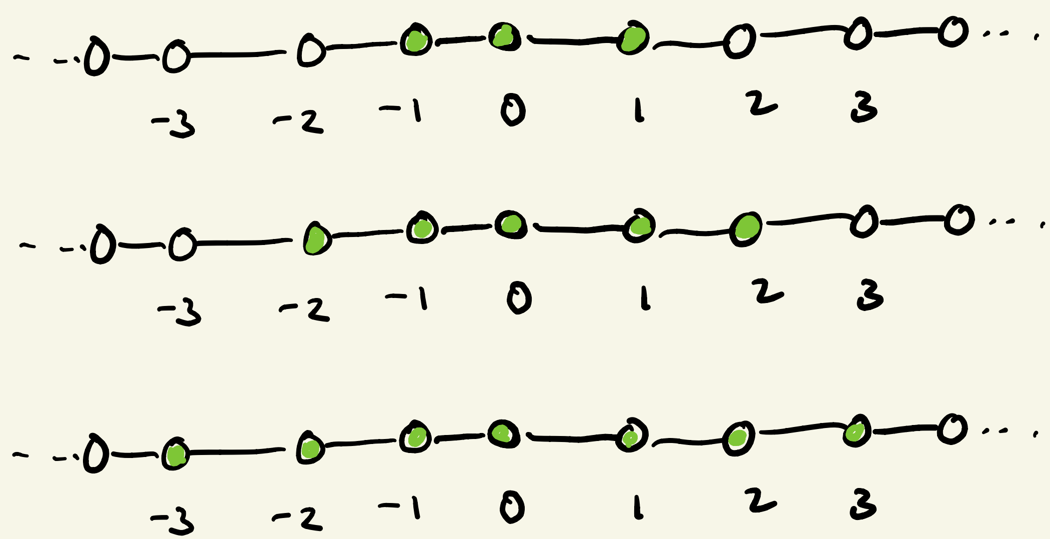


Kleinberg.
↓
Textbook:
. Networks, Crowds, Markets.

G: New thang! B $q = \frac{1}{2}$

adopt if: $q \leq \frac{d_{vB}}{d_v}$





$\{-1, 0, 1\}$ is contagious at $q = \frac{1}{2}$

Alternating, on/off, A/B, ... forever.

In fact, $\frac{1}{2}$ is this graph's contagion threshold.

No Node is converted! Oscillate forever.

Any graph, contagious threshold must be <u>at least</u> $\frac{1}{2}$.

$\{0\}$ not contagious for $q = \frac{1}{2}$

Does there exist any graph w/ a contagion threshold $> q$?

(1) JK describes "progressive" and "non-progressive" versions of the dynamics.

$A \to B$

$A \to B$, $B \to A$

(2) Progressive dynamics and non-progressive dynamics must have the same threshold!

magic!

(3) Show that contagion threshold is, at most, $\frac{1}{2}$, for any progressive dynamics . . .

By (2) $\Rightarrow$ same threshold hold for non-progressive dynamics.

$\boxed{\text{Cascading Behavior in Networks}}$