Approximating the Zero Lower Bound with a Hyperbola*

Kerk L. Phillips† Daryl B. Larsen‡

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Abstract

This paper illustrates the usefulness of the hyperbola in solving and simulating DSGE models with a zero lower bound. Rather than relying on an occasionally-binding constraint, a modeler can impose a hyperbola transformation on the interest rate target which guarantees it never drops below zero. The transition from a normal interest rate rule to a zero lower bound can be made a smooth as desired by altering a single parameter in the transformation.

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†Congressional Budget Office
‡Federal Reserve Board of Governors

1 Introduction

Guerrieri and Iacoviello (2015) show how to impose occasionally-binding constraints using Dynare.

2 A Hyperbola Transformation

In this section we use the formula for a hyperbola to transform the interest rate target from a targeting rule (x) into a truncated form that cannot be negative (y).

A cutoff function would do this easily, but it is not differentiable at x = 0 and this can lead to computational problems.

$$y = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise—} \end{cases}$$

A hyperbola transformation like that illustrated in Figure 1 would also ensure that y is never below zero, but does so with a smooth transition which make computation much more straightforward.

The formula for a hyperbola in conic form is given below.

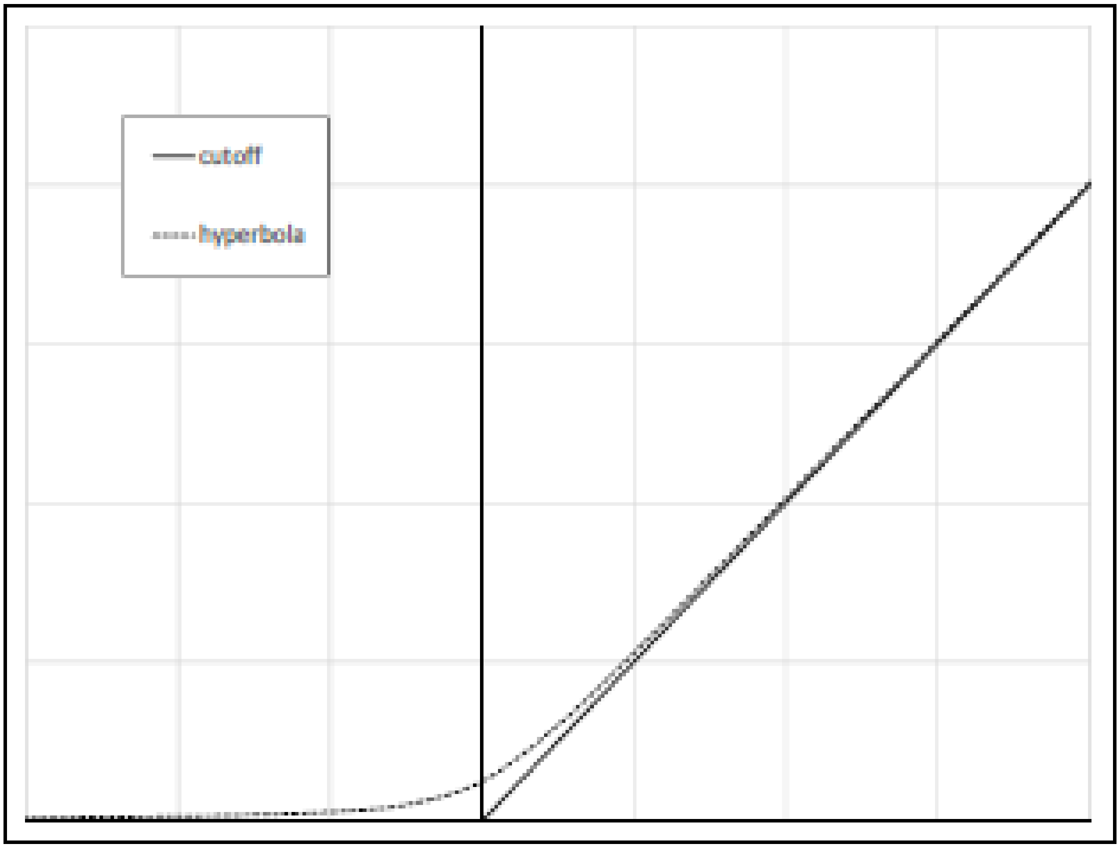
$$\left(\frac{y-k}{a}\right)^2 - \left(\frac{x-h}{b}\right)^2 = c^2$$

Since our cutoff occurs at the origin we set k = h = 0 and solve for y.

$$y = \sqrt{\frac{a^2}{b^2}x^2 + a^2c^2}$$

We will need to rotate our hyperbola and want to have asymptotic slopes of zero and one. This means that the unrotated hyperbole will have slopes of $\pm\frac{1}{2}$, which implies that

Figure 1: Comparison of Cutoff with Hyperbola Transformation



$\frac{a}{b} = \pm\frac{1}{2}$. By a normalization we set a = 1 and choose the positive square root. This gives us the following.

$$y = \sqrt{\frac{1}{4}x^2 + c^2}$$

To rotate the hyperbola we subtract the negative asymptote y = $-\frac{1}{2}x$ to get our final hyperbola transform function.

$$y = \sqrt{\frac{1}{4}x^2 + c^2} + \frac{1}{2}x$$

This transform has only one parameter, c, which determines how close the transform lies to the cutoff function. Lower values of c put it closer, with c = 0 yielding the cutoff function itself.

3 A Simple DSGE Model

3.1 The Model

Consider the following household problem.

$$
\begin{aligned}
V(k_t, m_t; \Omega_t) &= \max_{k_{t+1}, m_{t+1}} u(c_t, \tfrac{m_t}{P_t}) + \beta E\{V(k_{t+1}, m_{t+1}; \Omega_{t+1})\} \\
c_t &= w_t + (1 + r_t - \delta)k_t - k_{t+1} + \frac{m_t}{P_t} + \frac{\Delta M_t}{P_t} - \frac{m_{t+1}}{P_t} \qquad (3.1) \\
u(c, \tfrac{m_t}{P_t}) &= \frac{c^{1-\gamma} - 1}{1-\gamma} + \chi \frac{(\frac{m_t}{P_t})^{1-\theta} - 1}{1-\theta}
\end{aligned}
$$

The Euler equations from this problem are:

$$
c_t^{-\gamma} = \beta E\left\{c_{t+1}^{-\gamma}(1 + r_{t+1} - \delta)\right\} \qquad (3.2)
$$

$$
c_t^{-\gamma} = \beta E\left\{\left[c_{t+1}^{-\gamma} + \chi\left(\frac{m_{t+1}}{P_{t+1}}\right)^{-\theta}\right]\frac{P_t}{P_{t+1}}\right\} \qquad (3.3)
$$

Solving the firms problem with a Cobb-Douglas production function gives:

$$
Y_t = k_t^{\alpha} \exp z_t^{1-\alpha} \qquad (3.4)
$$

$$
r_t = \alpha \frac{Y_t}{k_t} \qquad (3.5)
$$

$$
w_t = (1-\alpha)Y_t \qquad (3.6)
$$

The central bank chooses an interest rate according to a Taylor rule.

$$i_{t+1}^* = (1+\bar{i}) \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y} \left(\frac{1+\pi_t}{1+\bar{\pi}}\right)^{\phi_\pi} e^{x_t} - 1 \tag{3.7}$$

$$i_t = \sqrt{\frac{1}{4}(i_t^*)^2 + \xi^2} + \frac{1}{2} i_t^* \tag{3.8}$$

The Fisher equation gives:

$$\pi_t = \frac{1+i_t}{1+r_t} - 1 \tag{3.9}$$

The labor and capital markets clear by choice of notation. Money market clearing gives the following:

$$\Delta M_t = m_{t+1} - mt \tag{3.10}$$

The inflation rate is defined by:

$$\pi_{t+1} = \frac{P_{t+1}}{P_t} - 1 \tag{3.11}$$

The exogenous laws of motion are:

$$z_{t+1} = \rho z_t + \varepsilon_{t+1};\ \ \varepsilon_{t+1} \sim iid(0, \sigma^2) \tag{3.12}$$

$$x_{t+1} = \psi x_t + \nu_{t+1};\ \ \nu_{t+1} \sim iid(0, \omega^2) \tag{3.13}$$

Equations (3.1) - (3.13) are a system of 13 equations and 13 unknowns: $c, k, m, Y, r, w, i^*, i, \pi, \Delta M, P, z, x$.

We can simplify this system by substituting (3.10) into (3.1) to eliminate Δ M from the system. We can also rewrite (3.9) and (3.11) in more convenient forms and combine (3.7) and (3.8) into a single equation. To ensure stationarity we define the following: $\hat{m}_t \equiv m_t(1+\bar{\pi})^{-t}$ and $\hat{P}_t \equiv P_t(1+\bar{\pi})^{-t}$

$$c_t = w_t + (1 + r_t - \delta)k_t - k_{t+1} \tag{3.14}$$

$$c_t^{-\gamma} = \beta E\left\{c_{t+1}^{-\gamma}(1 + r_{t+1} - \delta)\right\} \tag{3.15}$$

$$c_t^{-\gamma} = \beta E\left\{\left[c_{t+1}^{-\gamma} + \chi\left(\frac{m_{t+1}}{P_{t+1}}\right)^{-\gamma}\right]\frac{1 + r_t}{1 + i_t}\right\} \tag{3.16}$$

$$Y_t = k_t^{\alpha} \exp z_t{}^{1-\alpha} \tag{3.17}$$

$$r_t = \alpha\frac{Y_t}{k_t} \tag{3.18}$$

$$w_t = (1 - \alpha)Y_t \tag{3.19}$$

$$i_{t+1} = \sqrt{\frac{1}{4}\Gamma^2 + \xi^2} + \frac{1}{2}\Gamma \tag{3.20}$$

$$\Gamma \equiv (\bar{r} + \bar{\pi}) + \phi_y ln\left(\frac{Y_t}{\bar{Y}}\right) + \phi_\pi(\pi_t - \bar{\pi}) + x_t$$

$$\pi_t = \frac{1 + i_t}{1 + r_t - \delta} \tag{3.21}$$

$$\hat{P}_t = \hat{P}_{t-1}\frac{1 + \pi_t}{1 + \bar{\pi}} \tag{3.22}$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1};\ \varepsilon_{t+1} \sim iid(0, \sigma^2) \tag{3.23}$$

$$x_{t+1} = \psi x_t + \nu_{t+1};\ \nu_{t+1} \sim iid(0, \omega^2) \tag{3.24}$$

In the 11 equation system above the endogenous state variables are $\{k_t, m_t, i_t, P_t\}$, the exogenous state variables are $\{z_t, x_t\}$, and $\{Y_t, r_t, w_t, c_t, \pi_t\}$ are given as definitions. The parameters are $\{\delta, \beta, \gamma, \chi, \theta, \alpha, \xi, \phi_y, \phi_\pi, \bar{\pi}, \rho, \psi, \sigma, \omega\}$.

Recognizing that $\bar{z}$ and $\bar{x}$ are zero and normalizing $\bar{P}$ to one we can solve for the steady state of this system.

The steady state value for $\bar{r}$ comes from equation (3.15).

$$\bar{r} = \beta^{\frac{1}{\gamma}} + \delta - 1 \tag{3.25}$$

Equations (3.17), (3.18), (3.19), and (3.14) then give $\bar{Y}, \bar{k}, \bar{w}, and \bar{c}$.

$$\bar{k} = \frac{\alpha}{\bar{r}}^{\frac{1}{1-\alpha}} \tag{3.26}$$

$$\bar{Y} = \bar{k}^{\alpha} \tag{3.27}$$

$$\bar{w} = (1 - \alpha)\bar{Y} \tag{3.28}$$

$$\bar{c} = \bar{w} + (\bar{r} - \delta)\bar{k} \tag{3.29}$$

Equation (3.21 gives $\bar{i}$.

$$\bar{i} = (1 + \bar{r})(1 + \bar{\pi}) - 1 \tag{3.30}$$

And finally, (3.16) can be solved for $\bar{m}$.

$$\bar{m} = \left[\frac{\bar{c}^{-\gamma}}{\chi} \left(\frac{1 + \bar{\pi}}{\beta} - 1 \right) \right]^{\frac{1}{\theta}} \tag{3.31}$$

3.2 Simulation Results

Cannot rely on Dynare, since even 3rd-order approximations don't match the cutoff very well. Instead, we use the method from Judd et al. (2011)

4 Results

5 Discussions

6 Conclusion

References

Guerrieri, Luca and Matteo Iacoviello (2015) "OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily," *Journal of Monetary Economics*, Vol. 70, pp. 22–38.

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Tables

Figures

Appendix A. Placeholder