Approximating the Zero Lower Bound with a Hyperbola *

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Abstract

This paper illustrates the usefulness of the hyperbola in solving and simulating DSGE models with a zero lower bound. Rather than relying on an occasionally-binding constraint, a modeler can impose a hyperbola transformation on the interest rate target which guarantees it never drops below zero. The transition from a normal interest rate rule to a zero lower bound can be made a smooth as desired by altering a single parameter in the transformation

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1 Introduction

Guerrieri and Iacoviello (2015) show how to impose occasionally-binding constraints using Dynare.

2 A Hyperbola Transformation

In this section we use the formula for a hyperbola to transform the interest rate target from a targeting rule (x) into a truncated form that cannot be negative (y).

A cutoff function would to this easily, but it is not differentiable at $x = 0$ and this can lead to computational problems.

$$y = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

A hyperbola transformation like that illustrated in Figure 1 would also ensure that y is never below zero, but does so with a smooth transition which make computation much more straightforward.

The formula for a hyperbola in conic form is given below.

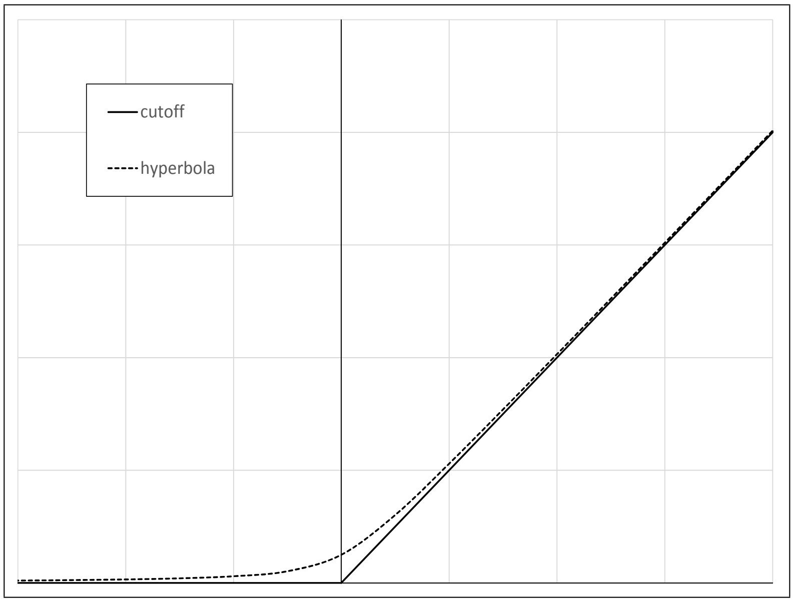
$$\left(\frac{y-k}{a}\right)^2 - \left(\frac{x-h}{b}\right)^2 = \xi^2$$

Since our cutoff occurs at the origin we set $k = h = 0$ and solve for y.

$$y = \sqrt{\frac{a^2}{b^2}x^2 + a^2\xi^2}$$

We will need to rotate our hyperbola and want to have asymptotic slopes of zero and one. This means that the unrotated hyperbole will have slopes

Figure 1: Comparison of Cutoff with Hyperbola Transform



of $\pm\frac{1}{2}$, which implies that $\frac{a}{b} = \pm\frac{1}{2}$. By a normalization we set $a = 1$ and choose the positive square root. This gives us the following.

$$y = \sqrt{\frac{1}{4}x^2 + \xi^2}$$

To rotate the hyperbola we subtract the negative asymptote $y = -\frac{1}{2}x$ to get our final hyperbola transform function.

$$y = \sqrt{\frac{1}{4}x^2 + \xi^2} + \frac{1}{2}x$$

This transform has only one parameter, c, which determines how close the transform lies to the cutoff function. Lower values of c put it closer, with $c = 0$ yielding the cutoff function itself.

3 A Simple DSGE Model

3.1 The Model

Consider the following household problem.

$$V(k_t, m_t; \Omega_t) = \max_{k_{t+1}, m_{t+1}} u(c_t, \tfrac{m_t}{P_t}) + \beta E \left\{ V(k_{t+1}, m_{t+1}; \Omega_{t+1}) \right\}$$

$$c_t = w_t + (1 + r_t - \delta)k_t - k_{t+1} + \frac{m_t}{P_t} + \frac{\Delta M_t}{P_t} + \frac{m_{t+1}}{P_t} \tag{3.1}$$

$$u(c, m) = \frac{c^{1-\gamma} - 1}{1-\gamma} + \chi \frac{m^{1-\theta} - 1}{1-\theta}$$

The Euler equations from this problem are:

$$c_t^{-\gamma} = \beta E \left\{ c_{t+1}^{-\gamma}(1 + r_{t+1} - \delta) \right\} \tag{3.2}$$

$$c_t^{-\gamma} = \beta E \left\{ \left[c_{t+1}^{-\gamma} + \chi \left(\tfrac{m_{t+1}}{P_{t+1}} \right)^{-\theta} \right] \frac{P_t}{P_{t+1}} \right\} \tag{3.3}$$

Solving the firms problem with a Cobb-Douglas production function gives:

$$Y_t = k_t^{\alpha} e^{(1-\alpha)z_t} \tag{3.4}$$

$$r_t = \alpha \frac{Y_t}{k_t} \tag{3.5}$$

$$w_t = (1-\alpha)Y_t \tag{3.6}$$

The central bank chooses an interest rate target according to a Taylor rule.

$$i^*_{t+1} = (1+\bar{i})\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}\left(\frac{1+\pi_t}{1+\bar{\pi}}\right)^{\phi_\pi} e^{x_t} - 1 \tag{3.7}$$

$$i_t = \sqrt{\frac{1}{4}(i^*_t)^2 + \xi^2} + \frac{1}{2}i^*_t \tag{3.8}$$

The Fisher equation gives:

$$\pi_t = \frac{1 + r_t - \delta}{1 + i_t} - 1 \tag{3.9}$$

The labor and captial markets clear by choice of notation. Money market clearing gives the following:

$$\Delta M_t = m_{t+1} - m_t \tag{3.10}$$

The inflation rate is defined by:

$$\pi_{t+1} = \frac{P_{t+1}}{P_t} - 1 \tag{3.11}$$

The exogenous laws of motion are:

$$z_{t+1} = \rho z_t + \varepsilon_{t+1};\ \varepsilon_{t+1} \sim iid(0, \sigma^2) \tag{3.12}$$

$$x_{t+1} = \psi z_t + \eta_t;\ \eta_{t+1} \sim iid(0, \omega^2) \tag{3.13}$$

Equations (3.1) - (3.11) are a system of 13 equations in the 13 unknowns, $c, k, m, Y, r, w, i^*, i, \pi, \Delta M, P, z, x$.

We can simplify this system by substituting (3.10) into (3.1) to eliminate ΔM from the system. We can also rewrite (3.9) and (3.11) in more convenient forms and combine (3.7) and (3.8) into a single equation. To ensure stationarity we define the following: $\hat{m}_t \equiv m_t(1+\bar{\pi})^{-t}$ and $\hat{P}_t \equiv P_t(1+\bar{\pi})^{-t}$.

$$c_t = w_t + (1 + r_t - \delta)k_t - k_{t+1} \tag{3.14}$$

$$c_t^{-\gamma} = \beta E\left\{c_{t+1}^{-\gamma}(1 + r_{t+1} - \delta)\right\} \tag{3.15}$$

$$c_t^{-\gamma} = \beta E\left\{\left[c_{t+1}^{-\gamma} + \chi\left(\tfrac{\hat{m}_{t+1}}{\hat{P}_{t+1}}\right)^{-\theta}\right]\frac{1 + r_{t+1} - \delta}{1 + i_{t+1}}\right\} \tag{3.16}$$

$$Y_t = k_t^{\alpha} e^{(1-\alpha)z_t} \tag{3.17}$$

$$r_t = \alpha\frac{Y_t}{k_t} \tag{3.18}$$

$$w_t = (1 - \alpha)Y_t \tag{3.19}$$

$$\begin{gathered} i_{t+1} = \sqrt{\frac{1}{4}\Gamma^2 + \xi^2} + \frac{1}{2}\Gamma; \\ \Gamma \equiv (\bar{r} + \bar{\pi}) + \phi_y \ln\left(\frac{Y_t}{\bar{Y}}\right) + \phi_\pi(\pi_t - \bar{\pi}) + x_t \end{gathered} \tag{3.20}$$

$$\pi_t = \frac{1 + r_t - \delta}{1 + i_t} - 1 \tag{3.21}$$

$$\hat{P}_t = \hat{P}_{t-1}\frac{1 + \pi_t}{1 + \bar{\pi}} \tag{3.22}$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1};\ \varepsilon_{t+1} \sim iid(0, \sigma^2) \tag{3.23}$$

$$x_{t+1} = \psi z_t + \eta_t;\ \eta_{t+1} \sim iid(0, \omega^2) \tag{3.24}$$

In the 11 equation system above the endogenous state variables are $\{k_t, m_t, i_t, P_t\}$, the exogenous state variables are $\{z_t, x_t\}$, and $\{Y_t, r_t, w_t, c_t, \pi_t\}$ are given as definitions. The parameters are $\{\delta, \beta, \gamma, \chi, \theta, \alpha, \xi, \phi_y, \phi_\pi, \bar{\pi}, \rho, \psi, \sigma, \omega\}$.

Recognizing that $\bar{z}$ and $\bar{x}$ are zero and normalizing $\bar{P}$ to one we can solve for the steady state of this system.

The steady state value for $\bar{r}$, comes from equation (3.15).

$$\bar{r} = \beta^{\frac{1}{\gamma}} + \delta - 1 \tag{3.25}$$

Equations (3.17), (3.18), (3.19) and (3.14) then give $\bar{Y}$, $\bar{k}$, $\bar{w}$ and $\bar{c}$.

$$\bar{k} = \left(\frac{\alpha}{\bar{r}}\right)^{\frac{1}{1-\alpha}} \tag{3.26}$$

$$\bar{Y} = \bar{k}^{\alpha} \tag{3.27}$$

$$\bar{w} = (1-\alpha)\bar{Y} \tag{3.28}$$

$$\bar{c} = \bar{w} + (\bar{r} - \delta)\bar{k} \tag{3.29}$$

Equation (3.21) gives $\bar{i}$.

$$\bar{i} = (1+\bar{r})(1+\bar{\pi}) - 1 \tag{3.30}$$

And finally, (3.16) can be solved for $\bar{m}$

$$\bar{m} = \left[\frac{\bar{c}^{-\gamma}}{\chi}\left(\frac{1+\bar{\pi}}{\beta} - 1\right)\right]^{\frac{1}{\theta}} \tag{3.31}$$

3.2 Simulation Results

Cannot rely on Dynare, since even 3rd-order approximations don't match the cutoff very well. Instead we use the method from Judd et al. (2011).

4 Conclusion

References

Guerrieri, Luca and Matteo Iacoviello, "OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily," *Journal of Monetary Economics*, 2015, *70* (C), 22–38.

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