

# THE PICARD NUMBER OF A KUMMER SURFACE

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## 1. INTRODUCTION

Let  $k$  be a separably closed field of characteristic not 2, and  $A/k$  an abelian surface. Then it is a basic fact (e.g. see [Huy16, Example 1.3 (iii)]) that one can make a K3 surface out of  $A$ . The construction is as follows. Consider the involution  $\iota : A \rightarrow A$  given by  $x \mapsto -x$ . The fixed locus of this involution is exactly  $A[2]$ , a finite constant closed  $k$ -subgroup scheme of  $A$  with 16  $k$ -points. Let  $Z := A[2]$ , and consider the blow-up  $p : \mathrm{Bl}_Z A \rightarrow A$ . By the universal property of the blow-up, the involution  $\iota$  lifts to a map  $\tilde{\iota} : \mathrm{Bl}_Z A \rightarrow \mathrm{Bl}_Z A$  such that the diagram

$$\begin{array}{ccc} \mathrm{Bl}_Z A & \xrightarrow{\tilde{\iota}} & \mathrm{Bl}_Z A \\ \downarrow p & & \downarrow p \\ A & \xrightarrow{\iota} & A \end{array}$$

commutes. Furthermore, by the uniqueness part of the statement of the universal property of  $\mathrm{Bl}_Z A$ , we deduce easily that  $\tilde{\iota}$  is also an involution.

Now the blow-up  $\mathrm{Bl}_Z A$  is a projective variety over  $k$  with an action of  $\mathbf{Z}/2\mathbf{Z}$  via the involution  $\tilde{\iota}$ . Therefore, the *categorical quotient*  $X := \mathrm{Bl}_Z A / (\mathbf{Z}/2\mathbf{Z})$  exists in the category of schemes. The scheme  $X$  constructed in this way is called the *Kummer surface associated to  $A$* , and turns out to be a K3 surface. In particular,  $H^1(X, \mathcal{O}_X) = 0$ , so the Picard scheme  $\mathrm{Pic}_{X/k}$  of  $X$  is étale, and  $\mathrm{Pic}(X)$  is therefore a finitely-generated abelian group.

For an arbitrary proper scheme  $Y/k$ , recall that the Néron–Severi group of  $Y$  is a finitely generated abelian group [SGA6, Exposé XIII, Théorème 5.1]. Therefore, we may consider the *Picard number*  $\rho(Y)$ , defined as the rank of the Néron–Severi group of  $Y$  (which for  $X$  is just the rank of the Picard group). It is proven in [Shi79, Proposition 3.1] that the Picard number of  $X$  is given by the formula  $\rho(X) = 16 + \rho(A)$ . However, a crucial step in Shioda’s proof relies on a calculation in [Shi75], of which we are not able to access a copy online. In this note, we give an explicit proof of this fact that is entirely self-contained. We do not use any Hodge theory, e.g. we do not study the complement of  $\mathrm{Pic}(X)$  in  $H^2(X, \mathbf{Z})$ , i.e. the transcendental lattice  $T(X)$ .

**Theorem 1.1.** *Let  $k$  be a separably closed field of characteristic not 2, and  $A/k$  an abelian surface. Let  $X$  denote the Kummer surface associated to  $A$ . Then the Picard number of  $X$  is given by*

$$\rho(X) = 16 + \rho(A).$$

## 2. PRELIMINARIES

In this section, we record a crucial result about abelian varieties that we will need. Let  $k$  be a separably closed field of characteristic not 2, and let  $A/k$  be an abelian variety (of arbitrary dimension). The group  $\mathbf{Z}/2\mathbf{Z}$  acts on  $\mathrm{Pic}(A)$  by  $\mathcal{L} \mapsto [-1]^*\mathcal{L}$ , and this action descends to one on the subgroup of numerically trivial line bundles  $\mathrm{Pic}^0(A)$ . In particular, the sequence

$$(1) \quad 0 \rightarrow \mathrm{Pic}^0(A) \rightarrow \mathrm{Pic}(A) \rightarrow \mathrm{NS}(A) \rightarrow 0$$

is an exact sequence of  $\mathbf{Z}/2\mathbf{Z}$ -modules.

**Proposition 2.1.** *Taking  $\mathbf{Z}/2\mathbf{Z}$ -invariants in (1), we obtain an exact sequence*

$$0 \rightarrow \hat{A}[2](k) \rightarrow \mathrm{Pic}(A)^{\mathbf{Z}/2\mathbf{Z}} \rightarrow \mathrm{NS}(A) \rightarrow 0,$$

where  $\widehat{A}$  is the dual abelian variety of  $A$ . In particular,  $\text{Pic}(A)^{\mathbf{Z}/2\mathbf{Z}}$  is a finitely generated abelian group of rank equal to the Picard number  $\rho(A)$  of  $A$ .

*Proof.* Let us first recall several facts about multiplication by  $n$  on  $A$ :

- (a) For  $\mathcal{L} \in \text{Pic}^0(A)$ ,  $[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n}$  [Con15, Proof of Lemma 5.2.5].
- (b) For  $\mathcal{L} \in \text{Pic}(A)$ , we have

$$[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n^2} \pmod{\text{Pic}^0(A)}.$$

In other words,  $[n]^*$  has the effect of multiplication by  $n^2$  on  $\text{NS}(A)$  [Con15, Lemma 7.5.2].

Granting these facts, let us first compute  $\text{Pic}^0(A)^{\mathbf{Z}/2\mathbf{Z}}$ . By (a), a line bundle  $\mathcal{L} \in \text{Pic}^0(A)$  satisfies  $[-1]^*\mathcal{L} = \mathcal{L}$  precisely when  $\mathcal{L}^{\otimes 2} \simeq 0$ . In other words,

$$\text{Pic}^0(A)^{\mathbf{Z}/2\mathbf{Z}} = \widehat{A}[2](k).$$

Next, by (b) above, the  $\mathbf{Z}/2\mathbf{Z}$ -action on  $\text{NS}(A)$  is trivial, so  $\text{NS}(A)^{\mathbf{Z}/2\mathbf{Z}} = \text{NS}(A)$ . Finally, we show that  $H^1(\mathbf{Z}/2\mathbf{Z}, \text{Pic}^0(A)) = 0$ , which will complete the proof of the proposition. Write  $\sigma$  for the generator of  $\mathbf{Z}/2\mathbf{Z}$ , and let  $N$  denote the “norm” map

$$\begin{aligned} N : \text{Pic}^0(A) &\rightarrow \text{Pic}^0(A) \\ \mathcal{L} &\mapsto \mathcal{L} \otimes [-1]^*\mathcal{L}. \end{aligned}$$

By the calculation of the cohomology of finite cyclic groups,

$$H^1(\mathbf{Z}/2\mathbf{Z}, \text{Pic}^0(A)) \simeq \ker N / (\sigma - 1) \text{Pic}^0(A).$$

By (a) above, we have  $\ker N = \text{Pic}^0(A)$ . On the other hand, for  $\mathcal{L} \in \text{Pic}^0(A)$ ,

$$(\sigma - 1)(\mathcal{L}) = [-1]^*\mathcal{L} \otimes \mathcal{L}^\vee \simeq (\mathcal{L}^\vee)^{\otimes 2}.$$

In other words,  $(\sigma - 1)$  has the effect of multiplication by  $-2$  on  $\text{Pic}^0(A)$ . But now recall that  $\text{Pic}^0(A) = \widehat{A}(k)$ , and  $[-2] : \widehat{A} \rightarrow \widehat{A}$  is surjective étale. Since  $k$  is separably closed, the map on  $k$ -points is surjective, and therefore  $(\sigma - 1) \text{Pic}^0(A) = \text{Pic}^0(A)$ , from which the vanishing of the cohomology group in question follows.  $\square$

### 3. PROOF OF THEOREM 1.1

Let  $p_1, \dots, p_{16}$  be the points in  $A[2](k)$ , let  $E_i$  be the preimage of  $p_i$  in  $\text{Bl}_Z A$  (the exceptional divisors), and let  $\widetilde{E}_i$  denote the image of  $E_i$  in the quotient  $X$ . Define

$$\begin{aligned} \widetilde{E} &:= \bigcup_{i=1}^{16} \widetilde{E}_i. \\ E &:= \bigcup_{i=1}^{16} E_i. \end{aligned}$$

It is proven in [Ba01, Theorem 10.6] that the  $\widetilde{E}_i$ 's are irreducible divisors in  $X$ , and are furthermore  $\mathbf{Z}$ -linearly independent in  $\text{Pic}(X)$ . Therefore, identifying the Weil class group of  $X$  with its Picard group (by smoothness), we obtain the exact sequence

$$(2) \quad 0 \rightarrow \bigoplus_{i=1}^{16} \mathbf{Z}[\widetilde{E}_i] \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(X - \widetilde{E}) \rightarrow 0$$

with the group on the left isomorphic in the obvious way to  $\mathbf{Z}^{16}$ . Now the formation of the quotient  $\pi : \text{Bl}_Z A \rightarrow X$  commutes with open immersions. More precisely, for any open subscheme  $\widetilde{V} \subset X$ , if we let  $V := \pi^{-1}(\widetilde{V})$ , then the map  $\pi : V \rightarrow \widetilde{V}$  is the categorical quotient of  $V$  by  $\mathbf{Z}/2\mathbf{Z}$ . Therefore,

$$X - \widetilde{E} \simeq \pi^{-1}(X - \widetilde{E}) / (\mathbf{Z}/2\mathbf{Z}).$$

But now observe that

$$\pi^{-1}(X - \widetilde{E}) \simeq \text{Bl}_Z A - E \simeq A - A[2],$$

so

$$(A - A[2]) / (\mathbf{Z}/2\mathbf{Z}) \simeq X - \widetilde{E}.$$

Therefore, the result

$$\rho(X) = 16 + \rho(A)$$

will follow from (2) if we can show that  $\text{Pic}((A - A[2])/(\mathbf{Z}/2\mathbf{Z}))$  is a finitely generated abelian group of rank equal to  $\rho(A)$ . To this end, define  $U := A - A[2]$  and  $G := \mathbf{Z}/2\mathbf{Z}$ . Since the action of  $G$  on  $U$  is free, the quotient map

$$U \rightarrow U/G$$

is a *Galois cover* [SGA3, Exposé V, Théorème 4.1(iii) and (iv)], and we have an associated Hochschild-Serre spectral sequence

$$H^i(G, H^j(U, \mathbf{G}_m)) \implies H^{i+j}(U/G, \mathbf{G}_m).$$

The low-degree terms of this spectral sequence are

$$0 \rightarrow H^1(G, H^0(U, \mathbf{G}_m)) \rightarrow \text{Pic}(U/G) \rightarrow \text{Pic}(U)^G \rightarrow H^2(G, H^0(U, \mathbf{G}_m)).$$

Now before we calculate any cohomology, we make the observation that

$$H^0(U, \mathbf{G}_m) = H^0(A, \mathbf{G}_m) = k^\times.$$

Indeed, this is true by Hartogs' Lemma since  $A$  is smooth (a fortiori normal!) and  $A \setminus U$  is codimension 2 in  $A$ . Also, observe that the Galois action of  $G$  on  $k^\times$  is trivial.

We now compute  $H^1(G, H^0(U, \mathbf{G}_m))$ . By the discussion above, this is isomorphic to  $\text{Hom}(G, k^\times) = \mu_2(k)$ . On the other hand, by the calculation of the cohomology of finite cyclic groups,  $H^2(G, k^\times) \simeq k^\times / (k^\times)^2$ . Since  $k$  is separably closed, this is zero and so  $\text{Pic}(U/G)$  sits in an exact sequence

$$(3) \quad 0 \rightarrow \mu_2(k) \rightarrow \text{Pic}(U/G) \rightarrow \text{Pic}(U)^G \rightarrow 0.$$

By the equivalence of the Picard group with the Weil divisor class group for regular schemes, and because  $A \setminus U$  has codimension 2 in  $A$ ,  $\text{Pic}(U) = \text{Pic}(A)$ . By Proposition 2.1,

$$\text{rk Pic}(A)^G = \rho(A).$$

Combining this with (3) yields the equality

$$\text{rk Pic}((A - A[2])/(\mathbf{Z}/2\mathbf{Z})) = \rho(A),$$

as desired.

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