THE PETER-WEYL THEOREM

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1. Introduction

A deep result in the representation theory of compact Lie groups is the theorem of the highest weight which asserts the following. Given a compact Lie group G, there is a bijective correspondence between irreducible, finite-dimensional complex representations of G and dominant integral elements of the weight lattice. The hardest part in the proof the theorem is the construction of an irreducible representation corresponding to some dominant integral element.

Three approaches to this construction are possible. The first is pure algebra and uses something called a Verma module; one obtains the desired irreducible representation as a quotient of some infinite-dimensional space. The second approach is via the Borel-Weil Theorem and the rough idea is like this. For a maximal torus T of G, one forms the quotient G/T and considers the twisted line bundle $G \times_{\rho_n} \mathbf{C}$ over G/T. Here ρ_n is a character of T corresponding to some dominant integral element μ. The space of global sections is then the irreducible representation of G corresponding to μ with the action of G given by $h \cdot f(g) = f(h^{-1}g)$. The third approach is to obtain the desired irreducible representation as a certain finite dimensional subspace of $L^2(G)$.

The goal of this expository essay will be to understand the Peter-Weyl Theorem, a key ingredient needed in the third approach above. There are several versions of this theorem including one which gives a decomposition of $L^2(G)$ as a $G \times G$-bimodule. For our purposes we would like a version from the point of view of functional analysis and not representation theory. In view of this, we will be concerned with the following statement that holds for more general compact Hausdorff groups.

Theorem 1.1 (Peter-Weyl)**.** *Let G be a compact Hausdorff group. The matrix coefficients of G are dense in $L^2(G)$, the space of all square-integrable functions on G.*

2. Preliminaries

2.1. **Haar Measures.** Given a compact (not necessarily Hausdorff) group G, there is a regular Borel measure μ_L that is left-invariant and unique up to multiplication by a constant. This is called the *left Haar measure* on G. By left-invariance we mean that $\mu_L(gX) = \mu_L(X)$ for any $g \in G$ and measurable subset $X \subseteq G$. Similarly, there is a *right Haar measure* μ_R on G that is unique up to multiplication by a constant. It turns out these measures have the property that any compact set has finite measure, in particular the whole of G has finite measure. We will not prove these properties or the existence of Haar measures; for this we refer the reader to Chapter 11 of [Fol99]

The hypothesis that G is compact is important in that it ensures the left and right-invariant measures coincide. If G is not compact, these may not agree as the following example shows: Take G to be the semidirect product $\mathbf{R} \rtimes \mathbf{R}^{>0}$ where $\mathbf{R}^{>0}$ is the multiplicative group of positive reals. Topologically, this may be identified with the upper half plane. Using the usual change of variables formula, we see for any Lebesgue measurable set E that

$$\mu_L(E) = \int_E y^{-2}\, dx\, dy, \qquad \mu_R(E) = \int_E y^{-1}\, dx\, dy$$

are left and right-invariant measures respectively on G that do not agree. Thus from now on, we work *only* with compact groups G on which we fix a left-invariant Haar measure μ_L. For convenience, we normalize μ_L so that the measure of G is 1.

Like any measure space, we may define analogously the concepts of Haar measurability and Haar integrability. For integration, we will often write $\int_G f(g)\, dg$ in place of $\int_G f(g)\, d\mu_L(g)$ for brevity. Though, if the need to emphasize the measure μ_L arises we will write the latter. In summary, the existence of a Haar measure allows us to do analysis on compact groups.

2.2. **Some Representation Theory.** Recall that a representation of G is a pair (π, V) where V is a finite-dimensional $\mathbf{C}$-vector space and $\pi : G \to \mathrm{GL}(V)$ a continuous group homomorphism. For an example of a representation, take G to be the unitary group $U(n)$ and (π, V) the representation that views an element of $U(n)$ as an invertible transformation of $\mathbf{C}^n$. This is sometimes known as the standard representation.

Associated to any representation (π, V) is something called the character χ. It is a complex valued function G that sends an element $g \in G$ to $\mathrm{Tr}(\pi(g))$. In the case of compact groups, it is an amazing fact that the character completely determines the representation. That is, if V and W are representations with characters χ_V and χ_W respectively, then $\chi_V = \chi_W$ if and only if $V \simeq W$. We refer the reader to Chapter 2 of [Bum13] for details.

Let us remark that because the trace is continuous and G is compact, the character of any representation is in $L^2(G)$. Furthermore, if $e_1, \ldots, e_n$ is a basis for V and $L_1, \ldots, L_n$ the associated dual basis, we may write $\chi(g) = \sum_{i=1}^n L(\pi(g)e_i)$. This motivates the following definition.

Definition 2.1. A matrix coefficient is a function $\Pi : G \to \mathbf{C}$ such that $\Pi(g) = L(\pi(g)v)$ for some representation (π, V) of G, $v \in V$ and $L \in V^*$.

From this definition it is immediate that a matrix coefficient is a continuous function. One may also prove that the sum and product of matrix coefficients is a matrix coefficient. There is not much to say about the proof of this last fact except to notice the following. Given matrix coefficients $\Pi_1(g) = L_1(\pi_1(g)v_1)$ and $\Pi_2 = L_2(\pi_2(g)v_2)$, the function $(L_1 \oplus L_2)(w_1, w_2) \coloneqq L_1(w_1) + L_2(w_2)$ is a linear functional on $V_1 \oplus V_2$. Similarly $(L_1 \otimes L_2)(w_1 \otimes w_2) \coloneqq L_1(w_1)L_2(w_2)$ is a linear functional on $V_1 \otimes V_2$.

3. Proof of the Peter-Weyl theorem

Having discussed the required preliminaries we are now ready to prove Theorem (3). To do this we will first discuss convolution, norms on a compact group and prove Propositions 3.2 and 3.3. These propositions will then be used to prove Proposition 3.4. Theorem (3) will then be a corollary of this proposition. Since the statement of Theorem (3) requires G to be Hausdorff, we will assume this for the rest of the section. In reality this is a mild assumption because the Peter-Weyl theorem is usually applied to compact Lie groups which are always Hausdorff. This is because any Lie group is a smooth manifold, and every smooth manifold in particular is a locally Euclidean, second-countable Hausdorff space (see Chapter 1 of [Lee13]).

Let $C(G)$ denote the space of all continuous, complex-valued functions on G. With respect to the infinity norm $\|f\|_\infty \coloneqq \sup_{g\in G} |f(g)|$, $C(G)$ is a Banach space. Given two functions $f_1, f_2 \in C(G)$, we may define their convolution

$$(f_1 * f_2)(g) \coloneqq \int_G f_1(gh^{-1})\, f_2(h)\, dh$$

Lemma 3.1. *The convolution $(f_1 * f_2)(g)$ is also equal to*

$$\int_G f_1(h)\, f_2(h^{-1}g)\, dh.$$

Proof. Fix $g \in G$. We may write $(f_1 * f_2)(g) = \int_G f_1(gh^{-1})\, f_2(h)\, d\mu_L(h)$ as

$$\int_G (f_1 \circ \phi_g)(h)\, (f_2 \circ \phi_g)(h^{-1}g)\, d\mu_L(h)$$

where $\phi_g : G \to G$ is the group homomorphism $h \mapsto gh^{-1}$. Applying [Mor13, Proposition 9.9] to the real and imaginary parts shows that

$$\int_G (f_1 \circ \phi_g)(h)\, (f_2 \circ \phi_g)(h^{-1}g)\, d\mu_L(h) = \int_G f_1(h)\, f_2(h^{-1}g)\, d(\phi_g)_*\mu_L(h),$$

where $(\phi_g)_*\mu_L$ is the pushforward measure defined by $(\phi_g)_*\mu_L(X) = \mu_L(X^{-1}g)$. Here X is any measurable set and $X^{-1} = \{x^{-1} : x \in X\}$. The key point now is that $(\phi_g)_*\mu_L(X) = \mu_L(X)$. Indeed, we have

$$\begin{aligned}\mu_L(X^{-1}g) &= \mu_L((gX)^{-1}) \\ &= \mu_L(gX) \\ &= \mu_L(X).\end{aligned}$$

We passed from the first to second line using [Bum13, Proposition 1.3]. This completes the proof of the lemma. □

In a similar fashion we may define the convolution of two functions $f_1, f_2 \in L^2(G)$ that are not necessarily continuous. Next, because we have normalized the Haar measure of G to be 1, we have the inequalities

$$\|f\|_1 \le \|f\|_2 \le \|f\|_\infty. \tag{1}$$

The first of these follows from the usual trick of applying Cauchy-Schwarz to $f = f \cdot 1$, while the second inequality is also trivial since

$$\|f\|_2^2 = \int_G \|f(g)|^2\, dg \le \sup_{g\in G} |f(g)|^2 \int_G dg = \|f\|_\infty^2.$$

Now let ϕ be any function in $L^2(G)$ and consider the linear operator $T_\phi : L^2(G) \to L^2(G)$ given by $T_\phi f \coloneqq \phi * f$. The first proposition of this section concerns important finiteness properties of this operator.

Proposition 3.2. *If $\phi \in L^2(G)$, then T_ϕ is a bounded operator on $L^2(G)$. Furthermore, T_ϕ is compact if and only if $\phi(g^{-1}) = \overline{\phi(g)}$ is self-adjoint.*

Proof. First we prove that T_ϕ maps $L^2(G)$ itself. Take any $f \in L^2(G)$; by Cauchy-Schwarz we have

$$\begin{aligned}|T_\phi f|^2 &\le \left(\int_G |\phi(gh^{-1})|^2\, dh\right)\left(\int_G |f(h)|^2\, dh\right) \\ &= \left(\int_G |\phi(gh^{-1})|^2\, dh\right)\|f\|_2^2\end{aligned}$$

by left-invariance. Hence

$$\begin{aligned}\|T_\phi f\|_2^2 &= \int_G \left|\int_G \phi(gh^{-1})\, f(h)\, dh\right|^2 dg \\ &\le \|f\|_2^2 \int_G \int_G |\phi(gh^{-1})|^2\, dh\, dg \\ &= \|f\|_2^2 \int_G \int_G |\phi(f)|^2 dh\, dg \\ &= \|f\|_2^2 \cdot \|\phi\|_2^2 \\ &< \infty,\end{aligned}$$

where the third line follows from the reasoning similarly as the proof of Lemma 3.1. This shows that T_ϕ maps $L^2(G)$ to itself; in fact the reader will also notice the inequality $\|T_\phi f\|_2^2 \le \|f\|_2^2 \cdot \|\phi\|_2^2$ also shows that T_ϕ is bounded. Furthermore, the fact that T_ϕ is compact comes from T_ϕ being a Hilbert-Schmidt operator with $L^2(G \times G)$ kernel $\phi(hg^{-1})$. Lastly, if $\phi(g^{-1}) = \overline{\phi(g)}$ we have

$$\begin{aligned}\langle T_\phi(f_1), f_2\rangle &= \int_G \left(\int_G \phi(gh^{-1})\, f_1(h)\, dh\right)\overline{f_2(g)}\, dg \\ &= \int_G \int_G \phi(gh^{-1})\, f_1(h)\overline{f_2(g)}\, dg\, dh && \text{(Fubini's theorem)} \\ &= \int_G f_1(h) \int_G \overline{f_2(g)\, \phi(hg^{-1})}\, dg\, dh && \text{(since } \phi(gh^{-1}) = \overline{\phi(hg^{-1})}) \\ &= \int_G f_1(h) \overline{\int_G f_2(g)\, \phi(hg^{-1})\, dg}\, dh && \text{(conjugate symmetry).}\end{aligned}$$

The use of Fubini's theorem is justified since any L^2-function is also in L^1 in view of (1). □

Proposition 3.3. *Suppose $\phi \in L^2(G)$ and $\lambda \in \mathbf{C}$. The λ-eigenspace $V(\lambda)$ of T_ϕ is invariant under right-translations $\phi(g)$ for all $g \in G$. That is, if $f(x) \in V(\lambda)$ then so is $\phi(g)f(x) = f(xg)$.*

Proof. If $T_\phi f = \lambda f$ then $(T_\phi \phi(g) f)(x) = \int_G \phi(xh^{-1})\, f(hg)\, dh$. Applying the change of variables $h \mapsto hg^{-1}$ and reasoning similarly as the proof of Lemma 3.1 shows that

$$\int_G \phi(xgh^{-1})\, f(h)\, dh = \rho(g)(T_\phi f)(x) = \lambda \phi(g) f(x).$$

□

We are now ready to use Propositions 3.2 and 3.3 to prove Proposition 3.4. Namely, that the space of matrix coefficients is dense in $C(G)$:

Proposition 3.4. *Let G be a compact Hausdorff group. The space of matrix coefficients is dense in $C(G)$.*

Our proof follows [Tao11] and is very beautiful. By the Stone–Weierstrass theorem we just need to show the matrix coefficients separate points. In fact, using right-translation it is enough to show for any $g \in G \backslash \{e\}$, there is a matrix coefficient Π so that $\Pi(g) \neq \Pi(e)$. We will prove there exists a finite-dimensional subspace $V \subseteq L^2(G)$ on which $\rho(g)$ does not act by the identity using Proposition 3.2 and the spectral theorem. The subspace V will then be a representation of G, and using this we produce a matrix coefficient Π such that $\pi(g) \neq \Pi(e)$.

Proof. First we show for all $g \in G$, there is *some* $\phi \in L^2(G)$ such that $\phi(h^{-1}) = \overline{\phi(h)}$ (for all $h \in G$) for which $\rho(g)$ is not the identity on at least one, non-zero eigenspace of T_ϕ. Suppose otherwise; then there is $g \in G$ such that for any ϕ satisfying $\phi(h^{-1}) = \overline{\phi(h)}$, $\rho(g)$ is the identity on every non-zero eigenspace of T_ϕ. Now we know that T_ϕ is compact and self-adjoint by Proposition 3.2. Thus by the spectral theorem,

$$L^2(G) = \ker T_\phi \oplus \bigoplus_{\lambda \neq 0} V_\lambda$$

with each $V(\lambda)$ finite-dimensional. Since $\rho(g)$ is the identity on every $V(\lambda)$, it follows that

$$\operatorname{im}(\rho(g) - 1) = (\rho(g) - 1)(\ker T_\phi).$$

Furthermore, the subspace on the right is contained in $\ker T_\phi$ by Proposition 3.3. Thus

$$T_\phi(\rho(g) - 1) f = 0$$

for every $f, \phi \in L^2(G)$ with ϕ satisfying $\phi(h^{-1} = \overline{\phi(h)})$ for all $h \in G$. In other words,

$$\rho(g)(\phi * f) = \phi * f. \tag{2}$$

However, a contradiction arises from this as we will produce functions $f, \phi \in L^2(G)$ for which (2) does not hold: Choose an open neighborhood U about the identity with $g \notin U^2$ (here we use that G is Hausdorff). Take $f = \phi = \chi_U$. Then $f * g$ is non-zero at $x = e$ but vanishes at $x = g$. We conclude that for any $g \in G \backslash \{e\}$, there is $\phi \in L^2(G)$ such that $\rho(g)$ is not the identity on a finite-dimensional, non-zero eigenspace $V(\lambda)$ of $L^2(G)$. To this end, choose $f \in V(\lambda)$ for which $\rho(g) f \neq f$. Then there is a linear functional $L \in V(\lambda)^*$ such that $L(\phi(g)f) \neq L(f)$. Now note that

$$\begin{aligned} \rho : \quad G &\to V(\lambda) \\ g &\mapsto \rho(g) \end{aligned}$$

defines a representation of G by finite-dimensionality of $V(\lambda)$ and Proposition 3.3. Hence we may consider the matrix coefficient $\Pi(g) := L(\rho(g)f)$ which by construction satisfies $\Pi(g) \neq \Pi(e)$. Now observe that the complex conjugate of a matrix coefficient is also a matrix coefficient. Thus we may apply the Stone–Weierstrass theorem to finish the proof. □

All the hard work is done and the proof of Theorem now comes naturally. Since $C(G)$ is dense in $L^2(G)$, Proposition 3.4 completes the proof of Theorem 3.

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