Geometric classes

Geometric classes represent objects in three-dimensional space. There are three board geometric classes: `Point`, `Vector`, and `Shape`. The first is pretty self-explanatory; it represents a point. `Vector` and `Shape` are more detailed classes and are explained below.

`Vector`

The `Vector` class represents any 2-D or 3-D vector quantity. It also supports vector arithmetics, including dot and cross products. The `dot()` function is also used to define the L2-norm. All `Vector`s are explicitly convertible to `bool`s - if a `Vector` does not have any `NaN` value, then it can be converted to `true`. This can be used to check whether a `Vector` has any `NaN` value, similar to how a pointer can be checked if it's a `nullptr`. A derived class called `MutableVector` can be used to modify any of the coordinates, though it's not sure whether it will be useful. The more important derived class is the `Direction` class, which dictates that the L2-norm be 1. Any non-zero `Vector` can be converted into a non-`NaN` `Direction`.

`Shape`

The abstract `Shape` class provides a template for volumetric objects. All classes derived from `Shape` must implement the following functions in order to be instantiated:

1. `xMin()`, `xMax()`, and so on: locate the extremum points in each of the three coordinates.
2. `surfaceArea()`
3. `volume()`
4. `surfaceContains(const Point&)`: checks whether the Point is strictly on the surface.
5. `encloses(const Point&)`: checks whether the `Point` is strictly inside `*this` (i.e. not on the surface or outside).
6. `encloses(const Shape&)`: checks whether the other `Shape` is completely contained within `*this`. This function evaluates to `true` even if the two `Shape`s intersect on the surface of `*this`.
7. `overlaps(const Shape&)`: checks whether the two `Shape`s overlap (i.e. the region of intersection is non-zero in volume).
8. `contentsOverlap(const Shape&)`: checks whether the other `Shape` overlap with any `Shape` contained within `*this`. This function may only differ from `overlaps()` if `*this` is of type `BoundingBox` (see below).

9. `distanceToSurface(const Point&, const Direction&)`: calculates the smallest, positive distance that a point has to travel to reach the surface.
   - If the point is inside the shape, the distance equation will have positive and negative roots, and only the smallest positive root is returned.
   - If the point is on the surface, returns 0 if it leaves the surface, and the distance to the other end if it enters the surface.
   - If the point is outside the shape, returns NAN if it never enters the shape, else the smallest distance to the surface.
10. `normal(const Point&)`: checks if the point is on the surface, and returns the outward normal vector if it is.
11. `print(std::ostream& os)`: outputs the shape into `std::cout`.

Additionally, `Shape` also has two member variables that represent the index of refraction and the total macroscopic cross section, respectively. They are both `double`s and therefore assumed to be constant. In the future, it might make sense to have them as a separate class (let's say, `Property`), in order to captures their respective dependencies on various parametes.

Right now, there are two implemented derived classes of `Shape`: `Sphere` and `Box`. `Sphere` requires a `Point` representing the origin and a radius. `Box` requires two `Point` representing the two opposing vertices. The `Sphere` class is our current representation of droplet particles, which has the advantage of easy implementations of the aforementioned functions. A more accurate geometric class may be implemented in the future to account for the affects of surface tension and gravity. The `Box` class probably will never be used to represent physical objects, but rather to construct a derived `BoundingBox` class. In this context, a `Box` is not a *terminal shape*, but a `Sphere` is.

Tree classes

The motive for using a Tree data structure is to quickly locate which —Shape— object (potentially out of thousands or millions) that a ray of photon will enter next, which is our version of the nearest neighbor search algorithm. A brute-force approach will have a time-complexity of order $\mathcal{O}(N)$, which in itself is not ideal. Furthermore, distance calculations are expensive because of the use of the square-root function (in case of the `Sphere` class) or some other functions that are potentially more expensive. A tree is a hierarchical data structure that groups objects based on certain common features. In this case, an Octree is used to group 3D shapes into each of the 8 octants of equal volume. Using a tree structure, the average time-complexity is down to $\mathcal{O}(\log N)$, which scales extremely well for large datasets. The majority of distance calculations involving those octants are to a surface of a box, and therefore are linear in nature, leaving a very few expensive calculations at the end.

`BoundingBox`

The `BoundingBox` class is derived from `Box` and represents an axis-aligned bounding box (AABB). It is the building block of an `Octree` and therefore is constrained to have at most 8 children. A `Node` is defined as a pointer to a `Shape` - more specifically, a `std::unique_ptr<Shape>` - that is contained within a `BoundingBox`. The pointed-to `Shape` itself can be either another `BoundingBox` (which means there are further subdivisions within that particular octant) or a terminal shape (which means they are leaf nodes).

`Octree`

Propogation of light in matter

Light is a manifestation of travelling electric and magnetic field. The two fields oscillate perpendicular to each other and to the direction at which light travels. They both have wavenumber k and frequency ω. Let x be the direction of propogation of the eletric field, the field can be expressed as

$$\mathbf{E}(x,t) = \mathbf{E}_0 \Re \left[e^{i(kx-\omega t)} \right] \tag{1}$$

To be precise, the term "wavenumber" refers to the angular wavenumber and is defined as the number of radians per unit length. In a vacuum with no attenuation, it is related to the phase velocity v_p as

$$k = \frac{\omega}{v_p}. \tag{2}$$

Since the phase velocity is a fraction n lower than the speed of light in vacuum c, the relation between wavenumber and refractive index is

$$k = \frac{\omega n}{c} \tag{3}$$

In an unattenuated medium, k is a real number, and the variable n is known as the refractive index. This only holds in vacuum; in any other material, k is a complex number, with its imaginary part accounting for the extent of attenuation. n therefore must also be a complex number. From now on, the variable n refers the real part of $n + i\kappa$, and the imaginary part κ is called the extinction coefficient. In other words,

$$k = \frac{\omega}{c}(n + i\kappa) \tag{4}$$

To show that attenuation is accounted for by the imaginary component, Equation 1 can be expanded as

$$\begin{aligned} \mathbf{E}(x,t) &= \mathbf{E}_0 \Re \left[\exp \left[-\frac{\omega}{c}\kappa x + i \left(\frac{\omega}{c} n x - \omega t \right) \right] \right] \\ &= \mathbf{E}_0 \exp \left(-\frac{\omega}{c}\kappa x \right) \cos \left(\frac{\omega}{c} n x - \omega t \right) \end{aligned} \tag{5}$$

Here, the electric field strength falls off exponentially with distance travelled, and the decay extent is characterized by the extinction coefficient. This resembles the linear attenuation of light intensity

$$I(x) = I_0 e^{-\sigma x}, \tag{6}$$

and the two equations are in fact related. Since the medium of interest is non-magnetic, $\mathbf{B} \approx \mu \mathbf{H}$, and the magnetic permeability μ can be readily approximated as its vacuum value $\mu_0 = 4\pi \times 10^{-7}$ H/m. The Poynting vector, defined as

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \tag{7}$$

can be written in terms of the $\mathbf{B}$ field:

$$\mathbf{S} = \frac{1}{\mu} \mathbf{E} \times \mathbf{B}. \tag{8}$$

Thanks to Maxwell's Equations, the $\mathbf{E}$ and $\mathbf{B}$ are in phase in both spatial and temporal domains. In terms of magnitude, the two fields are proportional by a factor of c/n, or the wave speed. As a result,

$$\mathbf{S}(x,t) = \frac{n}{\mu c} \|\mathbf{E}_0\|^2 \exp\left(-\frac{2\omega}{c}\kappa x\right) \cos^2\left(\frac{\omega}{c} n x - \omega t\right) \hat{\mathbf{x}}. \tag{9}$$

The light intensity is defined as the time-averaged magnitude of the Poynting vector:

$$\begin{aligned} I(x) &= \langle \|\mathbf{S}(x,t)\| \rangle & (10) \\ &= \frac{n}{\mu c} \|\mathbf{E}_0\|^2 \exp\left[-\frac{2\omega}{c}\kappa x\right] \cdot \frac{1}{T} \int_0^T \cos^2\left(\frac{\omega}{c} n x - \omega t\right) dt. & (11) \end{aligned}$$

Assume the integrating time T is much larger than the wave period - i.e. $T \gg \frac{2\pi}{\omega}$, then the averaging integral in Equation 11 will approach the average value over one single period, which is $\frac{1}{2}$. Therefore,

$$I(x) = \frac{n}{2\mu c} \|\mathbf{E}_0\|^2 \exp\left[-\frac{2\omega}{c}\kappa x\right]. \tag{12}$$

The relation between the attenuation coefficient σ and the extinction coefficient κ is then

$$\boxed{\sigma = \frac{2\omega\kappa}{c}}. \tag{13}$$

Snell's Law and Orientation of the Transmitted Ray

Let $\hat{\boldsymbol{\imath}}$ be the incident vector, $\hat{\boldsymbol{n}}$ be the (outward) normal vector, and $\hat{\boldsymbol{t}}$ be the transmitted vector. Also the index of refraction of a medium is labeled n.

Our job is to find $\hat{\boldsymbol{t}}$, given $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{n}}$, n_1 and n_2. The vectors are related through Snell's Law, and in vector notation it reads

$$n_1 \left(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{n}} \right) = n_2 \left(\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}} \right), \tag{14}$$

or

$$\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}} = \boldsymbol{c}, \tag{15}$$

for the new vector $\boldsymbol{c}$ defined as $\boldsymbol{c} = \dfrac{n_1}{n_2} \left(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{n}} \right)$. Note that $\boldsymbol{c}$ and $\hat{\boldsymbol{n}}$ are orthogonal.

We start solving for $\hat{\boldsymbol{t}}$ by decomposing it into two components: one parallel and one orthogonal to $\hat{\boldsymbol{n}}$:

$$\hat{\boldsymbol{t}} = t_{\parallel} \hat{\boldsymbol{n}} + t_{\perp} \hat{\boldsymbol{p}} \tag{16}$$

Since the cross product of the parallel component and $\hat{\boldsymbol{n}}$ is the zero vector, we require the cross product of the orthogonal component and $\hat{\boldsymbol{n}}$ be equal to $\boldsymbol{c}$.

$$t_{\perp} \hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} = \boldsymbol{c} \tag{17}$$

Since $\boldsymbol{c}$ and $\hat{\boldsymbol{n}}$ are already orthogonal, one of the possible values for the unit vector $\hat{\boldsymbol{p}}$ is $\hat{\boldsymbol{p}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{c}}$ (the other possibility is its additive inverse). Note that the unit vector $\hat{\boldsymbol{c}}$ is $\boldsymbol{c}$ normalized to unit length. Using the vector triple product identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$, it can be showed that

$$\begin{aligned} \hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} &= (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{c}}) \times \hat{\boldsymbol{n}} \\ &= \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{n}}) \\ &= (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{c}} - (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{c}})\hat{\boldsymbol{n}} \\ &= 1\hat{\boldsymbol{c}} - 0\hat{\boldsymbol{n}} \\ \hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} &= \hat{\boldsymbol{c}} \end{aligned} \tag{18}$$

Substituting the above expression into Equation 17, we arrive at the expression for $t_{\perp}$:

$$\begin{aligned} t_{\perp} \hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} &= \boldsymbol{c} \\ t_{\perp} \hat{\boldsymbol{c}} &= \boldsymbol{c} \\ t_{\perp} &= \|\boldsymbol{c}\| \end{aligned} \tag{19}$$

The parallel component of $\hat{\boldsymbol{t}}$ does not contribute to the cross product, so theoretically $t_{\parallel}$ can assume any value and Equation 15 will hold true. However, since $\hat{\boldsymbol{t}}$ is a unit vector, we require that

$$\begin{aligned} t_{\parallel}^2 + t_{\perp}^2 &= 1 \\ t_{\parallel} &= \pm\sqrt{1 - t_{\perp}^2} \end{aligned}$$

The choice of sign on $t_{\parallel}$ now depends on the orientation of the incident vector $\hat{\boldsymbol{\imath}}$. When the incident ray **enters** a surface, the transmitted vector $\hat{\boldsymbol{t}}$ will point inwards, away from the normal vector. Conversely, when the incident ray **exits** the surface, $\hat{\boldsymbol{t}}$ will point outwards. Therefore,

$$t_{\parallel} = \begin{cases} -\sqrt{1-t_{\perp}^2} & ,\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}} < 0 \\ \sqrt{1-t_{\perp}^2} & ,\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}} > 0 \end{cases}$$

Therefore, the final form of the transmitted vector $\hat{\boldsymbol{t}}$ is

$$\begin{aligned} \hat{\boldsymbol{t}} &= \|\boldsymbol{c}\| \left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{c}}\right) + \operatorname{sgn}\left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}}\right) \sqrt{1 - \|\boldsymbol{c}\|^2}\hat{\boldsymbol{n}} \qquad (20) \\ \boldsymbol{c} &= \frac{n_1}{n_2}\left(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{n}}\right) \end{aligned}$$

There are two special considerations. First, if $\|\boldsymbol{c}\| > 1$, the radical in Equation 20 is complex-valued, and no refraction occurs. This phenomenon is known as total internal reflection. Second, if the incident ray to tangent to the surface it intersects with ($\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}} = 0$), it neither enters or exits the surface. As a result, there is no refraction or reflection.