Let $\hat{\boldsymbol{\imath}}$ be the incident vector, $\hat{\boldsymbol{n}}$ be the (outward) normal vector, and $\hat{\boldsymbol{t}}$ be the transmitted vector. Also the index of refraction of a medium is labeled n.

Our job is to find $\hat{\boldsymbol{t}}$, given $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{n}}$, n_1 and n_2. The vectors are related through Snell's Law, and in vector notation it reads

$$n_1 \left(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{n}}\right) = n_2 \left(\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}\right), \tag{1}$$

or

$$\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}} = \boldsymbol{c}, \tag{2}$$

for the new vector $\boldsymbol{c}$ defined as $\boldsymbol{c} = \frac{n_1}{n_2} \left(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{n}}\right)$. Note that $\boldsymbol{c}$ and $\hat{\boldsymbol{n}}$ are orthogonal.

We start solving for $\hat{\boldsymbol{t}}$ by decomposing it into two components: one parallel and one orthogonal to $\hat{\boldsymbol{n}}$:

$$\hat{\boldsymbol{t}} = t_{\parallel} \hat{\boldsymbol{n}} + t_{\perp} \hat{\boldsymbol{p}} \tag{3}$$

Since the cross product of the parallel component and $\hat{\boldsymbol{n}}$ is the zero vector, we require the cross product of the orthogonal component and $\hat{\boldsymbol{n}}$ be equal to $\boldsymbol{c}$.

$$t_{\perp} \hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} = \boldsymbol{c} \tag{4}$$

Since $\boldsymbol{c}$ and $\hat{\boldsymbol{n}}$ are already orthogonal, one of the possible values for the unit vector $\hat{\boldsymbol{p}}$ is $\hat{\boldsymbol{p}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{c}}$ (the other possibility is its additive inverse). Note that the unit vector $\hat{\boldsymbol{c}}$ is $\boldsymbol{c}$ normalized to unit length. Using the vector triple product identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$, it can be showed that

$$\begin{aligned}
\hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} &= (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{c}}) \times \hat{\boldsymbol{n}} \\
&= \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{n}}) \\
&= (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{c}} - (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{c}})\hat{\boldsymbol{n}} \\
&= 1\hat{\boldsymbol{c}} - 0\hat{\boldsymbol{n}} \\
\hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} &= \hat{\boldsymbol{c}} \qquad (5)
\end{aligned}$$

Substituting the above expression into Equation 4, we arrive at the expression for $t_{\perp}$:

$$\begin{aligned}
t_{\perp} \hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}} &= \boldsymbol{c} \\
t_{\perp} \hat{\boldsymbol{c}} &= \boldsymbol{c} \\
t_{\perp} &= \|\boldsymbol{c}\| \qquad (6)
\end{aligned}$$

The parallel component of $\hat{\boldsymbol{t}}$ does not contribute to the cross product, so theoretically $t_{\parallel}$ can assume any value and Equation 2 will hold true. However, since $\hat{\boldsymbol{t}}$ is a unit vector, we require that

$$\begin{aligned}
t_{\parallel}^2 + t_{\perp}^2 &= 1 \\
t_{\parallel} &= \pm\sqrt{1 - t_{\perp}^2}
\end{aligned}$$

The choice of sign on $t_{\parallel}$ now depends on the orientation of the incident vector $\boldsymbol{\hat{\imath}}$. When the incident ray **enters** a surface, the transmitted vector $\boldsymbol{\hat{t}}$ will point inwards, away from the normal vector. Conversely, when the incident ray **exits** the surface, $\boldsymbol{\hat{t}}$ will point outwards. Therefore,

$$t_{\parallel} = \begin{cases} -\sqrt{1 - t_{\perp}^2} & , \boldsymbol{\hat{\imath}} \cdot \boldsymbol{\hat{n}} < 0 \\ \sqrt{1 - t_{\perp}^2} & , \boldsymbol{\hat{\imath}} \cdot \boldsymbol{\hat{n}} > 0 \end{cases}$$

Therefore, the final form of the transmitted vector $\boldsymbol{\hat{t}}$ is

$$\begin{aligned} \boldsymbol{\hat{t}} &= \|\boldsymbol{c}\| \left(\boldsymbol{\hat{n}} \times \boldsymbol{\hat{c}} \right) + \operatorname{sgn} \left(\boldsymbol{\hat{\imath}} \cdot \boldsymbol{\hat{n}} \right) \sqrt{1 - \|\boldsymbol{c}\|^2} \boldsymbol{\hat{n}} \\ \boldsymbol{c} &= \frac{n_1}{n_2} \left(\boldsymbol{\hat{\imath}} \times \boldsymbol{\hat{n}} \right) \end{aligned} \tag{7}$$

There are two special considerations. First, if $\|\boldsymbol{c}\| > 1$, the radical in Equation 7 is complex-valued, and no refraction occurs. This phenomenon is known as total internal reflection. Second, if the incident ray to tangent to the surface it intersects with ($\boldsymbol{\hat{\imath}} \cdot \boldsymbol{\hat{n}} = 0$), it neither enters or exits the surface. As a result, there is no refraction either.