CMPT 435
Assignment 1

Dan Blossom

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Problem 1A $\max(f(n), g(n)) = \theta(f(n) + g(n))$

Solution: True

$$\begin{aligned} max(f(n), g(n)) &\leq f(n) + g(n) \\ &\leq 2 \\ &\leq max(f(n), g(n))\, for\, n = 1 \end{aligned}$$

Problem 1B $\min(f(n), g(n)) = \theta(f(n) + g(n))$

Solution: False

$$\begin{aligned} min(f(n), g(n)) &\nleq f(n) + g(n) \\ &\nleq 2 \\ &\nleq max(f(n), g(n))\, for\, n = 2 \end{aligned}$$

Problem 2 $T(n) = 2T(\frac{n}{2}) + n^4$

Solution:

$$\begin{aligned} &= 2[2T(\frac{n}{4}) + (\frac{n}{2})^4] + n^4 \\ &= 4[2T(\frac{n}{8}) + (\frac{n}{4})^4] + 2(\frac{n}{2})^4 + n^4 \\ &= 8[2T(\frac{n}{16}) + (\frac{n}{8})^4] + 4(\frac{n}{2})^4 + 2(\frac{n}{2})^4 + n^4 \\ &= 2^r T(\frac{n}{2^r}) + n^4(1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + ... + \frac{1}{2^{(k-1)^3}} \\ &\therefore\ \theta(n^4) \end{aligned}$$

Problem 3A $T(1) = 1, T(n) = T(n-1) + n$

Solution: We will show that $T(n) = O(n^2), T(n) \leq cn^2$

Basis:

$T(2) = 3, 4 \leq c * 2 \, for \, c \geq 2$

$T(3) = 4, 9 \leq c * 3 \, for \, c \geq 3$

Inductive Hypothesis: Assume for $2 \leq k < n$ that

$T(k) \, is \, O(n^2), T(k) \leq cn^2$

Inductive Step: Prove $k = n$

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&\leq c * (n-1)^2 + n \\
&\leq c * n^2 - 2n + 1 + n \\
&\leq c(n^2 - n + 1) \\
&\leq c * n^2 - c * n + c \\
&\leq c * n^2 - n(c-1) \\
&\leq c * n^2 \, for \, c \geq 1 \\
&O(n^2)
\end{aligned}
$$

Problem 3B $T(1) = 1, T(n) = 3T(\frac{n}{3}) + 7n$

Solution: We will show that $T(n) = O(n), T(n) \leq cn$

Basis:

$T(2) = 16, 16 \leq c * 2 \, for \, c \geq 2$

$T(3) = 27, 27 \leq c * 3 \, for \, c \geq 3$

Inductive Hypothesis: Assume for $2 \leq k < n$ that

$T(k) \, is \, O(n), T(k) \leq cn$

Inductive Step: Prove $k = n$

$$\begin{aligned}
T(n) &= 3T(\frac{n}{3}) + 7n \\
&\leq 3(\frac{n}{3} + 7(\frac{n}{3}) \\
&\leq c * \frac{n}{3} + 7\frac{n}{3} \\
&\leq \frac{n(c+7)}{3} \\
&\leq 3n(c+7) \\
&\leq 3cn + 21 \\
&\leq cn\, for\, c \geq 3 \\
&O(n)
\end{aligned}$$

Problem 4A $T(n) = 2T(\frac{n}{2}) + n^3$

Solution: A=2, B=2, $f(n) = n^3$, $\log_b a = \log_2 2 = 1$

$$\begin{aligned}
f(n) = n^3 &= \Omega(n^{\log_b a+\epsilon}) \\
&= \Omega(n^{1+\epsilon})\; \epsilon = 1.
\end{aligned}$$

So, part 3 of the Master Theorem applies.

$af(n/b) \leq cf(n)$ for some constant $c < 1$

$$\begin{aligned}
2f(n/2) &= 2(\frac{n}{2})^3 \\
&= \frac{2}{8}n^3 \\
&= \frac{n^3}{4}
\end{aligned}$$

$\frac{n^3}{4} \leq cn^3$ for $c = \frac{1}{4}$ So, $T(n) = \theta(n^3)$

Problem 4B $T(n) = 16T(\frac{n}{2}) + n^2$

Solution: A=16, B=4, $f(n) = n^2$, $\log_b a = \log_4 16 = 2$

$$\begin{aligned}
f(n) = n^2 &= \theta(n^{\log_b a}) \\
&= \theta(n^2)
\end{aligned}$$

So, part 2 of the Master Theorem applies.

$\theta(n^{\log_b a} \log n) = \theta(n^2 \log n)$

Problem 4C $T(n) = 16T(\frac{n}{3}) + n^2$

Solution: A=7, B=3, $f(n) = n^2$, $\log_b a = \log_3 7 \geq 1\, AND \leq 2$ Approximately: 1.77

$$\begin{aligned} f(n) = n^2 &= \Omega(n^{\log_b a+\epsilon}) \\ &= \Omega(n^{\log_3 7+\epsilon})\, \epsilon = 1 - \log_3 7 \end{aligned}$$

So, part 3 of the Master Theorem applies.

$af(n/b) \leq cf(n)$ for some constant $c < 1$

$$\begin{aligned} 7f(n/3) &= 7(\frac{n}{3})^2 \\ &= \frac{7}{9}n^2 \end{aligned}$$

$\frac{7}{9}n^2 \leq cn^2$ for $c = \frac{7}{9}$ So, $T(n) = \theta(n^2)$

Problem 4D $T(n) = 7T(\frac{n}{2}) + n^2$

Solution: A=7, B=2, $f(n) = n^2$, $\log_b a = \log_2 7 \geq 2\, AND \leq 3$ Approximately: 2.8

$$\begin{aligned} f(n) = n^2 &= O(n^{\log_b a-\epsilon}) \\ &= O(n^{\log_2 7-\epsilon})\, \epsilon = \log_2 7 - 3 \end{aligned}$$

So, part 1 of the Master Theorem applies.

$\theta(n^{\log_b a}) = \theta(n^{\,log_2 7})$