The **bias** of an estimator $\hat{\theta}$ in estimating θ is defined as

Bias = $E(\hat{\theta}) - \theta$.

The **variance** of an estimator $\hat{\theta}$ θ is defined as

Var = $E(\hat{\theta} - E(\hat{\theta}))^2$.

Definition:

An estimator is **consistent** if its bias goes to zero as $n \to \infty$.

Criteria for a good estimator

A good estimator should have low bias and low variance. We can combine these two criteria into one: the **mean squared error**.

$$
\begin{aligned}
&MSE(\hat{\theta}) = E\left((\hat{\theta} - \theta)^2\right) \\
&= E\left((\hat{\theta} - E(\hat{\theta}) + E(\hat{\theta}) - \theta)^2\right) \\
&= E\left((\hat{\theta} - E(\hat{\theta}))^2\right) + 2E\left((\hat{\theta} - E(\hat{\theta}))(E(\hat{\theta}) - \theta)\right) + E\left((E(\hat{\theta}) - \theta)^2\right) \\
&= \text{var}(\hat{\theta}) + 2 \times 0 \times (E(\hat{\theta}) - \theta) + \left(E(\hat{\theta}) - \theta\right)^2 \\
&= \text{var}(\hat{\theta}) + bias(\hat{\theta})^2
\end{aligned}
$$

This is a **very important result**. *MSE* = var + $bias^2$

MSE is an example of *expected loss* (in this case, loss = squared error).

Neither variance nor bias alone can be interpreted as an expected loss.

Examples

The two main frequentist ways to estimate:

- Maximum Likelihood Estimation
- Moment Estimation

Maximum Likelihood Estimator (MLE):

For a model family $\{f_\theta : \theta \in \Theta\}$ and an observation x_{obs}, the likelihood function $\ell : \Theta \to \mathbb{R}$ is defined by

$$\ell(\theta) = f_\theta(x_{obs}).$$

A maximum likelihood estimator $\hat{\theta}$ satisfies

$$f_{\hat{\theta}}(x_{obs}) = \max_{\theta \in \Theta} \left\{ f_\theta(x_{obs}) \right\}.$$

Often $\hat{\theta}$ is unique.

Example If $X \sim binom(n,p)$, and we observe $X=x_{obs}$, then

$$f_p(x_{obs}) = \begin{pmatrix} n \\ x_{obs} \end{pmatrix} p^{x_{obs}} (1-p)^{n-x_{obs}}, \text{ therefore } \ell(p) \propto p^{x_{obs}} (1-p)^{n-x_{obs}},$$

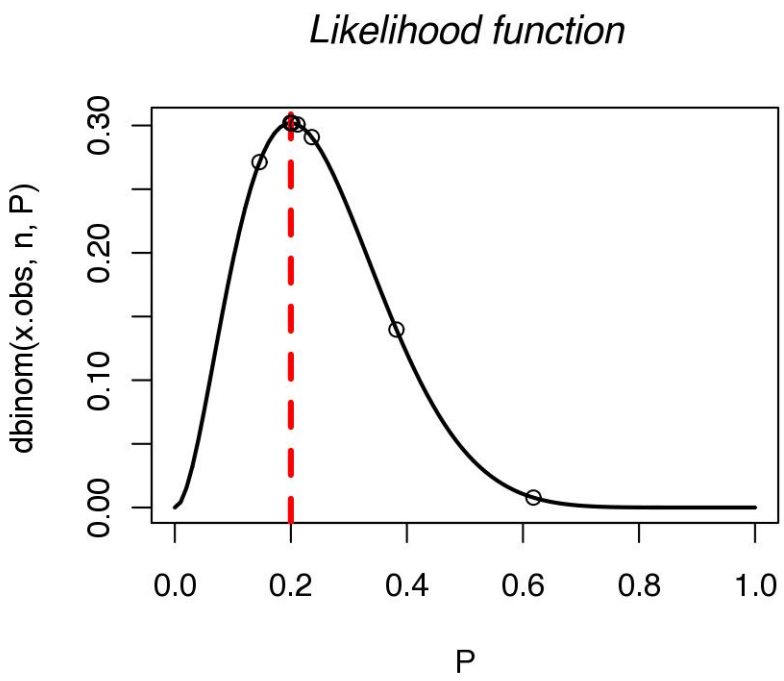
and to maximize, we can differentiate and set the slope to zero:

$$\frac{d}{dp}\ell(p) = \frac{d}{dp}\left(p^{x_{obs}} (1-p)^{n-x_{obs}} \right)$$
$$= \left(x_{obs} p^{x_{obs}-1} (1-p)^{n-x_{obs}} \right) + \left(p^{x_{obs}} (-(n-x_{obs})(1-p)^{n-x_{obs}-1} \right)$$
$$\propto x_{obs} p^{-1} - (n-x_{obs})(1-p)^{-1}$$

This derivative is zero when

$$x_{obs} p^{-1} = (n-x_{obs})(1-p)^{-1}$$
$$x_{obs}(1-p) = (n-x_{obs})p$$
$$x_{obs} = (n-x_{obs})p + x_{obs}p = np$$
$$p = x_{obs} / n$$

So $\hat{p} = x_{obs} / n$.




```
optimize(
   function(arg) {
      result=dbinom(x=2,
          size=10, prob=arg)
      points(arg, result)
      return(result)
    },
      lower=1e-10,
      upper=1-1e-10,
      maximum=TRUE
)
```

(When the parameter has dimension $D > 1$, you can do maximum likelihood estimation on the vector parameter, for example

- with algebra (set the vector of partial derivatives equal to the zero vector, and solve the D equations simultaneously),
- by searching in the D-dimensional space where the parameter lives,
- with the EM algorithm.

Moment estimators

A moment estimator is obtained by setting an observed value to its expected value (as a function of the parameter) and solving for the parameter.

Example For the binomial, $E(X) = np$. So we solve $x_{obs} = n\hat{p}$, to get $\hat{p} = x_{obs}/n$.

So for the binomial, MLE and moment estimator are the same, $\hat{p} = x_{obs}/n$.

Example: estimating a normal distribution's variance: MLE $\neq$ Moment estimator

Suppose we observe i.i.d. data $(x_1,...,x_n) \sim N(0,\sigma^2)$. We know that the mean is zero, but we don't know the variance. Goal: to estimate σ^2.

First, let's try the MLE, the maximum likelihood estimator:

$$f(x_1,...,x_n \mid \sigma^2) = \prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x_i^2}{2\sigma^2}\right)\right)$$

$$\ell(\sigma^2) = \left(\frac{1}{\sqrt{\sigma^2}}\right)^n \exp\left(-\frac{\sum x_i^2}{2\sigma^2}\right) = \left(\sigma^2\right)^{-n/2}\exp\left(-\frac{\sum x_i^2}{2\sigma^2}\right)$$

Maximizing the likelihood is the same as maximizing the log-likelihood. Differentiate the log-likelihood to find the maximizer:

$$\frac{d}{d\sigma^2}\log\ell(\sigma^2) = \frac{d}{d\sigma^2}\left(-\frac{n}{2}\log(2\pi\sigma^2) - \frac{\sum x_i^2}{2\sigma^2}\right)$$

$$= -\frac{n}{2}\frac{1}{\sigma^2} - \frac{\sum x_i^2}{2}\left(-(\sigma^2)^{-2}\right)$$

$$\propto -n\sigma^2 + \sum x_i^2$$

Setting this equal to zero, we get the MLE is $\widehat{\sigma^2} = n^{-1}\sum x_i^2$. That's logical!!

Now, how about the moment estimator?

$\sigma^{-2}\sum x_i^2$ has a "chi-square distribution on *n* degrees of freedom":

$$\frac{x_1^2 + ... + x_n^2}{\sigma^2} \sim \chi_n^2.$$

(Aside: χ_n^2 is the same as a gamma distribution $G(n/2, 1/2)$.)

The mean of χ_n^2 equals *n.* ← **important fact!**

So the moment estimator comes from setting $\sigma^{-2}\sum x_i^2 = n$:

$\widehat{\sigma^2} = n^{-1}\sum x_i^2$ just like the MLE.

But wait!
What if we don't know the true population *mean* μ either?

$$(x_1,...,x_n) \sim N(\mu,\sigma^2)$$

How about trying **MLE**? Start with

$$\ell(\mu,\sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{\sum(x_i-\mu)^2}{2\sigma^2}\right).$$

Maximize the likelihood over μ for fixed σ^2: $\hat{\mu} = \overline{x} = n^{-1}\sum x_i$, the sample mean. Then maximizing over σ^2 gives $\widehat{\sigma^2} = n^{-1}\sum(x_i-\overline{x})^2$, like before,
but we replace the unknown mean μ by its estimate $\hat{\mu} = \overline{x}$.

How about **Method of Moments**? Things get different.
$\frac{1}{\sigma^2}\sum(x_i-\overline{x})^2$ has a chi-square distribution, but this time on *n* – 1 degrees of freedom: χ^2_{n-1}. You can think of it as "*losing a degree of freedom"* (losing a chunk of information) because we have to use the extra information to estimate μ.

So the moment estimator comes from setting $\sigma^{-2}\sum(x_i-\overline{x})^2 = n-1$, to get

$$\widehat{\sigma^2} = (n-1)^{-1}\sum\left(x_i-\overline{x}\right)^2.$$

This is BIGGER than the MLE, by a factor of $n/(n-1) = 1 + 1/(n-1)$.

We say that the MLE in this case is **biassed**.
The **bias** of an estimator $\hat{\theta}$ in estimating θ is defined as

Bias = $E(\hat{\theta}) - \theta$.

$$\text{Bias (MLE)} = E(\widehat{\sigma^2}) - \theta = E\left(n^{-1}\sum(x_i - \bar{x})^2\right) - \theta$$

$$= \frac{n-1}{n} E\left((n-1)^{-1}\sum(x_i - \bar{x})^2\right) - \theta$$

$$= \left(\frac{n-1}{n}\right)\theta - \theta = -n^{-1}\theta$$

This is **overfitting**; because we can tinker with a free parameter (μ), we can be fooled into thinking the noise (variance) is less than it is, and thinking that the parameter estimates are more accurate than they are. Notice that the bias goes to zero as *n* goes to ∞.