**The Problem**: A new treatment is given to 100 patients. Of them, only 8 respond. But there is a subgroup of 5 in which 3 patients respond, yielding a response rate of 60%! Should the treatment be recommended (or at least developed further) for people in the subgroup?

	Dark Hair (D)	Light Hair (L)	TOTAL
Responder (R)	3	5	8
Nonresponder (N)	2	90	92
TOTAL	5	95	100

We want to express prior knowledge or belief concerning how related  $P_{R.D}$  and  $P_{R.L}$  are.

Dr. Lump is sure that they are identical; Dr. Lump lumps together the *D* and *L* groups.

Dr. Split is sure that they are totally unrelated.

Others may have intermediate positions.

Or, how close to splitting or lumping seem appropriate will depend on the nature of X.

Let's transform the two conditional probabilities with the log odds function.

For X=D or L,

$$z_{R.X} = \operatorname{logit}(\Pr(R \mid X)) = \operatorname{log}\left(\frac{\Pr(R \mid X)}{1 - \Pr(R \mid X)}\right)$$

Then a broad family of useful priors is bivariate normal in the (logit( $P_{R.D}$ ), logit( $P_{R.D}$ )) plane:

$$(z_{R.D}, z_{R.L})^T \sim N(\mu_z, \Sigma_z) \text{ where } \mu_z = (\mu_{R.D}, \mu_{R.L})^T \text{ and } \Sigma_z = \begin{pmatrix} \tau + \phi & \tau \\ \tau & \tau + \phi \end{pmatrix}.$$

The prior means for the logits are  $\mu_{R.D}$ ,  $\mu_{R.L}$ . The superscript "T" means "transpose"; customarily these vectors are written as column vectors. (We usually set the prior means of  $z_{R.D}$  and  $z_{R.L}$  to be the same, so  $\mu_{R.D} = \mu_{R.L}$ .) Any factors that might affect both response probabilities are represented by the covariance  $\tau$ . Any that are not shared are represented by  $\phi = (\tau + \phi) - \tau$ . The prior correlation between  $z_{R.D}$  and  $z_{R.L}$  is  $\tau/(\tau + \phi)$ .

So the joint density of  $(logit(P_{R.D}), logit(P_{R.L}))$  is

$$f_z(z_{R.D}, z_{R.L}) = (2\pi)^{-1/2} \det(\Sigma_z)^{-1/2} \exp\left(-\frac{1}{2}(z - \mu_z)^T \Sigma_z^{-1} (z - \mu_z)\right)$$

which can be converted to a density on the original scale by multiplying by the Jacobean of the logit transformation:

$$\begin{split} f_P(P_{R.D}, P_{R.L}) &= f_z(z_{R.D}, z_{R.L}) \times \det \begin{pmatrix} \partial z_{R.D} / \partial P_{R.D} & 0 \\ 0 & \partial z_{R.L} / \partial P_{R.L} \end{pmatrix}^{-1} \\ &= f_z(\operatorname{logit} P_{R.D}, \operatorname{logit} P_{R.L}) \times P_{R.D} (1 - P_{R.D}) \times P_{R.L} (1 - P_{R.L}) \end{split}$$

because the Jacobean of the transformation is

$$\frac{d}{dP}z = \frac{d}{dP}\left(\log((P/(1-P)))\right) = \frac{1}{P/(1-P)}\frac{d}{dP}(1-1/(1-P))$$
$$= \frac{1-P}{P}(1-P)^{-2} = P^{-1}(1-P)^{-1} = P^{-1} + (1-P)^{-1}$$

We use  $f_P(P_{R.D}, P_{R.L})$  to draw the contours of prior and posterior distributions for **P**.

## The model (conditional distribution of the data):

The observed proportions are also converted to logits:  $x_{R,X} = \operatorname{logit} \hat{P}_{R,X}$ , and the variances of these logits is estimated with the delta method applied to the logit function.

$$\operatorname{var}(\operatorname{logit}(\hat{\boldsymbol{P}})) \doteq \operatorname{var}(\hat{\boldsymbol{P}}) \left( \frac{\boldsymbol{d}}{\boldsymbol{dP}} \operatorname{logit} \boldsymbol{P} \right)^{2}$$

$$= \boldsymbol{n}^{-1} \boldsymbol{P} (1 - \boldsymbol{P}) \left( \boldsymbol{P}^{-1} (1 - \boldsymbol{P})^{-1} \right)^{2} = \boldsymbol{n}^{-1} \left( \boldsymbol{P}^{-1} (1 - \boldsymbol{P})^{-1} \right)$$

$$\doteq \boldsymbol{n}^{-1} \left( \hat{\boldsymbol{P}}^{-1} (1 - \hat{\boldsymbol{P}})^{-1} \right)$$

When there is a zero count, the variance formula yields infinity.

A small "fudge factor" can be added to each count, to produce a finite variance.

We model the data, conditional on the unknown  $z_{R.D}$ ,  $z_{R.L}$  (which is what we are interested in knowing), with a normal approximation for the logits:

$$\begin{pmatrix} x_{R.D} \\ x_{R.L} \end{pmatrix} | \begin{pmatrix} z_{R.D} \\ z_{R.L} \end{pmatrix} \sim N \begin{pmatrix} z_{R.D} \\ z_{R.L} \end{pmatrix}, \Sigma_{x|z}$$
where  $\Sigma_{x|z} = \begin{pmatrix} n^{-1}P_{R.D}^{-1}(1-P_{R.D})^{-1} & 0 \\ 0 & n^{-1}P_{R.L}^{-1}(1-P_{R.L})^{-1} \end{pmatrix}.$ 

(Notice the very tall "|" indicating conditioning on the z 's.) Conditionally, the data are independent, as we see from the off-diagonal zeros.

## The joint distribution

The joint distribution of the unknown z's and the observed x's is the product of the prior distribution and the model. This is a good spot to simplify our notation.

$$Y_{1} = \begin{pmatrix} z_{R.D} \\ z_{R.L} \end{pmatrix}, \quad Y_{2} = \begin{pmatrix} x_{R.D} \\ x_{R.L} \end{pmatrix}, \quad \mu_{1} = E(Y_{1}) = \mu_{2} = E(Y_{2}) = \begin{pmatrix} \mu_{R.D} \\ \mu_{R.L} \end{pmatrix}$$

$$\Sigma_{11} = \operatorname{var}(Y_{1}) = \Sigma_{z} = \begin{pmatrix} \tau + \phi & \tau \\ \tau & \tau + \phi \end{pmatrix}, \quad \Sigma_{22} = \operatorname{var}(Y_{2}) = \Sigma_{x} = \Sigma_{z} + \Sigma_{x|z}$$

$$\Sigma_{12} = \Sigma_{21} = \operatorname{cov}(Y_{1}, Y_{2}) = \Sigma_{z}$$

Then the joint distribution of the Y's (in our case, z and x) is

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

and the beautiful thing is that the posterior distribution we want is the conditional distribution of the unknown  $Y_1$  (which is z) conditional on the observed  $Y_2$  which is (x). We get this from the standard formulas for conditional mean and variance for the multivariate normal distribution:

$$Y_2 | Y_1 = y_1 \sim N(\mu_{2.1}, \Sigma_{22.1})$$

where

$$\mu_{2.1} = E(Y_2 \mid Y_1 = y_1) = \mu_2 + \sum_{21} \sum_{11}^{-1} (y_1 - \mu_1)$$

and

$$\sum_{22.1} = Var(Y_2 \mid Y_1 = y_1) = \sum_{22} -\sum_{21} \sum_{11}^{-1} \sum_{12}.$$

Here it is in code. It's not hard to do. Th function solve() does the matrix inversion.

```
varhat = apply(DLdata, 1, function(r)sum(1/r))
sig11 = sig12 = sig21 = matrix(c(tau+phi,tau,tau,tau+phi),nrow=2)
sig22 = sig11 + diag(varhat)  ## marginal variance of the data
logit.prior.mean = c(mu0, mu0)
## always use fudged here:
postmean.logit = logit.prior.mean +
    sig12%*%solve(sig22) %*% (logit.hat.fudged-logit.prior.mean)
postmean.p = antilogit(postmean.logit)
postvar.logit = sig11 - sig12%*%solve(sig22)%*%sig21
```

To draw the contour plots, here is the code:

 $\underline{\texttt{https://github.com/professorbeautiful/T15lumpsplit/blob/master/inst/T15lumpsplit/Plight-Pdark-posterior-new.Reserved to the professor of the professor of$