The Problem: A new treatment is given to 100 patients. Of them, only 8 respond. But there is a subgroup of 5 in which 3 patients respond, yielding a response rate of 60%! Should the treatment be recommended (or at least developed further) for people in the subgroup?

	Dark Hair (D)	**Light Hair (L)**	TOTAL
Responder (R)	**3**	**5**	8
Nonresponder (N)	**2**	**90**	92
TOTAL	5	95	100

We want to express prior knowledge or belief concerning how related $P_{R.D}$ and $P_{R.L}$ are.
Dr. Lump is sure that they are identical; Dr. Lump lumps together the *D* and *L* groups.
Dr. Split is sure that they are totally unrelated.
Others may have intermediate positions.
Or, how close to splitting or lumping seem appropriate will depend on the nature of *X*.

Let's transform the two conditional probabilities with the log odds function.
For *X*= *D* or *L*,

$$z_{R.X} = \text{logit}(\Pr(R\,|\,X)) = \log\left(\frac{\Pr(R\,|\,X)}{1-\Pr(R\,|\,X)}\right)$$

Then a broad family of useful priors is bivariate normal in the (logit($P_{R.D}$), logit($P_{R.L}$)) plane:

$$(z_{R.D}, z_{R.L})^T \sim N(\mu,\Sigma) \text{ where } \mu = (\mu_{R.D},\mu_{R.L})^T \text{ and } \Sigma = \begin{pmatrix} \tau+\phi & \tau \\ \tau & \tau+\phi \end{pmatrix}.$$

The prior means for the logits are $\mu_{R.D},\mu_{R.L}$. We ordinarily assume equal prior information about $z_{R.D}$ and $z_{R.L}$, so $\mu_{R.D}=\mu_{R.L}$, which we notate as μ_0 . Any factors that might affect both response probabilities are represented by the covariance τ . Any that are not shared are represented by $\phi=(\tau+\phi)-\tau$. The prior correlation between $z_{R.D}$ and $z_{R.L}$ is $\tau/(\tau+\phi)$.

So the joint density of (logit($P_{R.D}$), logit($P_{R.L}$)) is

$$f_z(z_{R.D},z_{R.L}) = (2\pi)^{-1/2}\det(\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(z-\mu)^T\Sigma^{-1}(z-\mu)\right)$$

which can be converted to a density on the original scale by multiplying by the Jacobean of the logit transformation:

$$f_P(P_{R.D},P_{R.L}) = f_z(z_{R.D},z_{R.L})\times\det\begin{pmatrix} \partial z_{R.D}/\partial P_{R.D} & 0 \\ 0 & \partial z_{R.L}/\partial P_{R.L} \end{pmatrix}^{-1}$$

$$= f_z(\text{logit}\,P_{R.D},\text{logit}\,P_{R.L})\times P_{R.D}(1-P_{R.D})\times P_{R.L}(1-P_{R.L})$$

because

$$\frac{d}{dP}z = \frac{d}{dP}\left(\log((P/(1-P))\right) = \frac{1}{P/(1-P)}\frac{d}{dP}(1-1/(1-P))$$
$$= \frac{1-P}{P}(1-P)^{-2} = P^{-1}(1-P)^{-1} = P^{-1}+(1-P)^{-1}$$

(We use $f_P(P_{R.D}, P_{R.L})$ to draw the contours of prior and posterior.)

The observed proportions are also converted to logits: $x_{R.X} = \operatorname{logit}\hat{P}_{R.X}$, and the variances of these logits is estimated with the delta method applied to the logit function.

$$\operatorname{var}(\operatorname{logit}(\hat{\boldsymbol{P}})) \doteq \operatorname{var}(\hat{\boldsymbol{P}})\left(\frac{\boldsymbol{d}}{\boldsymbol{dP}}\operatorname{logit}\boldsymbol{P}\right)^2$$
$$= \boldsymbol{n}^{-1}\boldsymbol{P}(1-\boldsymbol{P})\left(\boldsymbol{P}^{-1}(1-\boldsymbol{P})^{-1}\right)^2 = \boldsymbol{n}^{-1}\left(\boldsymbol{P}^{-1}+(1-\boldsymbol{P})^{-1}\right)$$
$$\doteq \boldsymbol{n}^{-1}\left(\hat{\boldsymbol{P}}^{-1}+(1-\hat{\boldsymbol{P}})^{-1}\right)$$

When there is a zero count, the variance formula yields infinity.
A small "fudge factor" can be added to each count, to produce a finite variance.
We model the data with a normal approximation for the logits:

$$(x_{R.D}, x_{R.L})^T \sim N\left((\operatorname{logit}\hat{P}_{R.D}, \operatorname{logit}\hat{P}_{R.L})^T, \begin{pmatrix} n^{-1}\hat{P}_{R.D}^{-1}(1-\hat{P}_{R.D})^{-1} & 0 \\ 0 & n^{-1}\hat{P}_{R.L}^{-1}(1-\hat{P}_{R.L})^{-1} \end{pmatrix}\right)$$

The posterior is calculated on the logit-logit space, using the standard
the formulas for conditional mean and variance for the multivariate normal:

```
varhat = apply(DLdata, 1, function(r)sum(1/r))
sig11 = sig12 = sig21 = matrix(c(tau+phi,tau,tau,tau+phi),nrow=2)
sig22 = sig11 + diag(varhat)   ## marginal variance of the data
logit.prior.mean = c(mu0, mu0)
## always use fudged here:
postmean.logit = logit.prior.mean +
  sig12%*%solve(sig22) %*% (logit.hat.fudged-logit.prior.mean)
postmean.p = antilogit(postmean.logit)
postvar.logit = sig11 - sig12%*%solve(sig22)%*%sig21
```