The following is an example of a proof from the book that follows the rules of Chapter 4, but is written in a more readable form: it uses complete sentences, and it is modular, separating certain pieces of the proof into separate lemmas and claims.

Let us try to prove the following theorem.

Domain of discourse: numbers (which we will denote by lowercase letters) and sets of numbers (which we will denote by uppercase letters).

Hypothesis 1: There is a number x greater than everything in A such that there is a number less than everything in B and greater than x:

$$1.\exists x(\forall y(y \in A \to x > y) \land \exists z(\forall y(y \in B \to y > z) \land z > x))$$

Hypothesis 2: $>$ is a transitive relation:

$$2.\forall x \forall y \forall z((x > y \land y > z) \to x > z)$$

It's reasonable to ask why this is a hypothesis. The answer is that we can prove it from other basic facts about numbers, but we will not do so here.

Goal: Prove that everything in B is greater than everything in A:

$$\forall x \forall y(x \in B \land y \in A) \to x > y)$$

Proof. By existential instantiation of Hypothesis 1, for a particular variable s,

$$3.\forall y(y \in A \to s > y) \land \exists z(\forall y(y \in B \to y > z) \land s > x)$$

By simplification from line 3 above, we get two lines:

$$4.\forall y(y \in A \to s > y)$$

$$5.\exists z(\forall y(y \in B \to y > z) \land z > s)$$

By existential instantiation from line 5, letting ℓ be a particular variable, we get

$$6.\forall y(y \in B \to y > \ell) \land \ell > s$$

By simplification from line 6, we get the following two lines:

$$7.\forall y(y \in B \to y > \ell)$$

$$8.\ell > s$$

By universal instantiation from line 4, letting a be an arbitrary variable, we have

$$9.a \in A \to s > a$$

By universal instantiation from line 6, letting b be an arbitrary variable, we have

$$10.b \in B \to b > \ell$$

Using Lemma 2 (proven below) applied to lines 9 and 10, we have

$$11. a \in A \wedge b \in B \rightarrow s > a \wedge b > \ell$$

Using Lemma 3 (proven below) applied to line 11 and 8, and again to the result and line 2, we have

$$12. a \in A \wedge b \in B \rightarrow s > a \wedge b > \ell \wedge \ell > s \wedge \forall x \forall y \forall z((x > y \wedge y > z) \rightarrow x > z)$$

Note that our formula has gotten quite long, and at this point continuing the proof in the same way would be cumbersome. Instead, we'll prove an intermediate step and then come back to finish the proof. (Essentially, we are switching from horizontal notation to vertical notation by separating the propostions that are connected by and.)

Lemma 1. *If*

1.1. $s > a$
1.2. $b > \ell$
1.3. $\ell > s$
1.5. $\forall x \forall y \forall z((x > y \wedge y > z) \rightarrow x > z)$

then

$b > a$

Proof. Apply universal instantiation three times to line 1.4 to instantiate x with ℓ, y with s, and z with a, to get

1.6. $(\ell > s \wedge s > a) \rightarrow \ell > a$

Apply modus ponens to line 1.4 and the conjunction of lines 1.3 and 1.1

1.7. $\ell > a$

Apply universal instantiation three times to line 1.4 to instantiate x with b, y with ℓ, and z with a, to get

1.8. $(b > \ell \wedge \ell > a) \rightarrow b > a$

Apply modus ponens to line 1.8 and the conjunction of lines 1.2 and 1.7 to get

1.9. $b > a$

This concludes the proof of Lemma 1. ☐

We are now back to the proof of our main result. We can write the result of Lemma 1.9 as

$$13. s > a \wedge b > \ell \wedge \ell > s \wedge \forall x \forall y \forall z((x > y \wedge y > z) \rightarrow x > z) \rightarrow a > b$$

Applying hypothetical syllogism to lines 12 and 13, we get

$$14. a \in a \wedge b \in B \rightarrow a > b$$

Now applying universal generalization twice (because a and b were arbitrary variables), we get the result of our theorem:

$$15. \forall x \forall y(x \in A \wedge y \in B \rightarrow x > y)$$ ☐

We are done with the proof of theorem. However, we used two Lemmas in the proof of the theorem that we have not yet proven. We now state and prove them. We mix boolean algebra and propositional logic notation depending on what notation we feel makes the formula more readable.

Lemma 2. *If $p \rightarrow r$ and $q \rightarrow s$ then $pq \rightarrow rs$.*

Proof. We prove this lemma by putting the following two claims together.

Claim 1. *If $p \rightarrow r$ and $p \rightarrow s$ then $p \rightarrow rs$.*

Proof. $(\bar{p} + r)(\bar{p} + s) \equiv \bar{p} + \bar{p}r + \bar{p}s + rs \equiv \bar{p} + rs$ (distributive, idempotent, absorption) ☐

Claim 2. *If* $p \to r$ *then* $pq \to r$

Proof. If $\bar{p} + r$ then $\bar{p} + \bar{q} + r$ (addition) and hence $\overline{pq} + r$ (De Morgan's), which is the same as $pq \to r$. $\square$

Thus, applying Claim 2, we have $pq \to r$. Applying it again to $q \to s$ and p, we have and $qp \to s$, which, by commutativity, is equivalent $pq \to s$. Now applying Claim 1 to $pq \to r$ and $pq \to s$, we have $pq \to rs$. This concludes the proof of Lemma 2 $\square$

Lemma 3. *If* $p \to r$ *and* s *then* $p \to rs$.

Proof. If s, then $F \vee s$ (identity law) and therefore $T \to s$ by definition of $\to$. Now apply Lemma 2 to get $p \wedge T \to rs$ which is equivalent to $p \to rs$ by the identity law. $\square$