ANGULAR MOMENTUM
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APPENDIX III

TABLES OF FORMULAS

In this appendix we tabulate formulas that are primarily of practical use in angular momentum calculations rather than being key formulas for understanding the subject. You can therefore use the tables for ready reference. If you wish to understand the origins and uses of a formula in more detail, look for it in the text under the equation number given here. Also, the formulas given here appear in the text with a box around them, but not all boxed formulas appear here. The order of presentation of formulas is by increasing equation number.

T1 LEGENDRE FUNCTIONS AND SPHERICAL HARMONICS

Tables of Formulas

Legendre functions $\ell \leq 4$, Table 4.1, p. 131.
Spherical harmonics $\ell \leq 4$, Table 4.2, p. 139.
Solid harmonics $\ell \leq 4$, Tables 4.4, 4.5, pp. 145, 146.

Legendre Functions

Integral property:

$$\int_0^{\pi} P_{\ell'}^m(\cos\theta) P_\ell^m(\cos\theta)\sin\theta\, d\theta = \frac{2(\ell+m)!\,\delta_{\ell'\ell}}{(2\ell+1)(\ell-m)!} \tag{4.6}$$

Parity:

$$P_\ell^m(-\cos\theta) = (-1)^{\ell-m} P_\ell^m(\cos\theta) \tag{4.8}$$

Spherical Harmonics

Stretched-m value:

$$Y_{\ell\ell}(\theta,\phi) = (-1)^{\ell} \sqrt{\frac{(2\ell+1)!}{2^{2\ell+2}(\ell!)^2 \pi}} \sin^{\ell}\theta \, e^{i\ell\phi} \tag{4.14}$$

General formula ($m \geq 0$):

$$\begin{aligned} Y_{\ell m}(\theta\phi) &= (-1)^m \sqrt{\frac{(2\ell+1)(\ell+m)!(\ell-m)!}{4\pi}} e^{im\phi} \\ &\times \sum_x \frac{(-1)^x (\sin\theta)^{2x+m} (\cos\theta)^{\ell-2x-m}}{2^{2x+m}(m+x)!\, x!\, (\ell-m-2x)!} \end{aligned} \tag{4.15}$$

Parity:

$$PY_{\ell m}(\theta\phi) = Y_{\ell m}(\pi-\theta, \pi+\phi) = (-1)^{\ell} Y_{\ell m}(\theta\phi) \tag{4.16}$$

Relation to Legendre functions:

$$Y_{\ell m}(\theta\phi) = (-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} P_{\ell}^{m}(\cos\theta) e^{im\phi} \qquad m \geq 0 \tag{4.18}$$

Spherical harmonic for $m < 0$:

$$Y_{\ell,-|m|}(\theta\phi) = (-1)^m Y_{\ell,|m|}^{*}(\theta\phi) \tag{4.19}$$

Relation to Legendre polynomials for $m = 0$:

$$Y_{\ell 0}(\theta\phi) = Y_{\ell 0}(\theta,0) = \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\cos\theta) \tag{4.20}$$

Values at $\theta = 0, \pi$:

$$Y_{\ell m}(0,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} \delta_{m,0} \qquad Y_{\ell m}(\pi,\phi) = (-1)^{\ell} \sqrt{\frac{2\ell+1}{4\pi}} \delta_{m,0} \tag{4.21}$$

Spherical harmonic addition theorem:

$$\sum_m Y_{\ell m}^{*}(\theta'\phi') Y_{\ell m}(\theta\phi) = \frac{2\ell+1}{4\pi} P_{\ell}(\cos\omega) \tag{4.23}$$

Solid harmonic formulas:

$$\begin{aligned}
\mathcal{Y}_{\ell m}(\mathbf{r}) &\equiv r^{\ell} Y_{\ell m}(\theta\phi) \\
&= (-1)^m \sqrt{\frac{(2\ell+1)(\ell+m)!\,(\ell-m)!}{4\pi}} \\
&\quad \times \sum_{x'} \frac{(-1)^{x'} (x+iy)^{x'+m} (x-iy)^{x'} z^{\ell-2x'-m}}{2^{2x'+m}(m+x')!\,x'!\,(\ell-m-2x')!}
\end{aligned} \tag{4.28}$$

Real forms of spherical harmonics:

$$\begin{aligned}
Z_{\ell}(\theta\phi) &\equiv \sqrt{\frac{2\ell+1}{4\pi}}\, P_{\ell}(\cos\theta) \\
Z_{\ell m}^{c}(\theta\phi) &\equiv \sqrt{\frac{(2\ell+1)(\ell-m)!}{2\pi(\ell+m)!}}\, P_{\ell}^{m}(\cos\theta)\cos m\phi \qquad m \geq 0 \\
Z_{\ell m}^{s}(\theta\phi) &\equiv \sqrt{\frac{(2\ell+1)(\ell-m)!}{2\pi(\ell+m)!}}\, P_{\ell}^{m}(\cos\theta)\sin m\phi \qquad m \geq 0
\end{aligned} \tag{4.32}$$

Real spherical harmonic addition theorem:

$$\sum_{\kappa=0,c,s} \sum_{m=0}^{\ell} Z_{\ell m}^{\kappa}(\theta'\phi') Z_{\ell m}^{\kappa}(\theta\phi) = \frac{2\ell+1}{4\pi} P_{\ell}(\cos\omega) \tag{4.34}$$

T2 ROTATION MATRIX ELEMENTS

Tables of Formulas

Reduced rotation matrices $j \leq 2$, Tables 6.1 – 6.3, pp. 223, 224.
Rotation matrix elements for special parameters, Table 6.4, p. 240.

Reduced Rotation Matrix Elements

General formula:

$$\begin{aligned}
d_{m'm}^{j}(\beta) &= \sqrt{(j+m')!(j-m')!(j+m)!(j-m)!} \\
&\quad \times \sum_{x} \frac{(-1)^x [\cos(\beta/2)]^{2j+m'-m-2x} [\sin(\beta/2)]^{2x+m-m'}}{(j+m'-x)!(j-m-x)!\,x!\,(x+m-m')!}
\end{aligned} \tag{6.9}$$

Reduced rotation matrix element with both projection numbers zero:

$$d_{00}^{\ell}(\beta) = P_{\ell}(\cos\beta) \tag{6.13}$$

Reduced rotation matrix element with first projection number zero:

$$d_{m'0}^{\ell}(\beta) = (-1)^{m'} \sqrt{\frac{(\ell - m')!}{(\ell + m')!}} P_{\ell}^{m'}(\cos\beta) \qquad m' \geq 0 \tag{6.14}$$

Rotated state ket:

$$|jm,\alpha\beta\gamma\rangle = \sum_{m'=-j}^{j} |jm',000\rangle D_{m'm}^{j}(\alpha\beta\gamma) \tag{6.19}$$

Transformed amplitudes after rotation:

$$a_{m'}(\alpha\beta\gamma) = \sum_{m=-j}^{j} D_{m'm}^{j}(\alpha\beta\gamma) a_m(000) \tag{6.33}$$

Symmetries of reduced rotation matrix elements:

$$d_{-m',-m}^{j}(\beta) = d_{m'm}^{j}(-\beta) \tag{6.40}$$

$$d_{m'm}^{j}(-\beta) = d_{mm'}^{j}(\beta) \tag{6.41}$$

$$d_{mm'}^{j}(\beta) = (-1)^{m-m'} d_{m'm}^{j}(\beta) \tag{6.42}$$

$$d_{m'm}^{j}(\pi - \beta) = (-1)^{j+m'} d_{m',-m}^{j}(\beta) \tag{6.43}$$

Relation of reduced and full rotation matrix elements:

$$D_{m'm}^{j}(\alpha\beta\gamma) = e^{-im'\alpha} d_{m'm}^{j}(\beta) e^{-im\gamma} \tag{6.44}$$

Symmetries of full rotation matrix elements:

$$D_{-m',-m}^{j}(\alpha,\beta,\gamma) = D_{m'm}^{j}(-\alpha,-\beta,-\gamma) = \left[D_{m'm}^{j}(\alpha,-\beta,\gamma)\right]^{*} \tag{6.45}$$

$$D_{m'm}^{j}(-\alpha,-\beta,-\gamma) = D_{mm'}^{j}(\gamma,\beta,\alpha) = \left[D_{m'm}^{j}(-\gamma,-\beta,-\alpha)\right]^{*} \tag{6.46}$$

$$D_{mm'}^{j}(\alpha,\beta,\gamma) = (-1)^{m-m'} D_{m'm}^{j}(\gamma,\beta,\alpha) \tag{6.47}$$

$$D_{m'm}^{j}(\pi - \beta) = (-1)^{j+m'} D_{m',-m}^{j}(\alpha,\beta,-\gamma) \tag{6.48}$$

Unitarity sum for full rotation matrix elements:

$$\sum_{m''} D_{m'm''}^{j}(\alpha\beta\gamma)\left[D_{mm''}^{j}(\alpha\beta\gamma)\right]^{*} = \delta_{m'm} \tag{6.50}$$

Orthogonality sum for reduced rotation matrix elements:

$$\sum_{m''} d^j_{m'm''}(\beta) d^j_{mm''}(\beta) = \delta_{m'm} \tag{6.51}$$

Orthogonality integral for rotation matrix elements:

$$\begin{aligned} &\int_R \left[D^{j_1}_{m_1'm_1}(\alpha\beta\gamma)\right]^* D^{j_2}_{m_2'm_2}(\alpha\beta\gamma)\, d\alpha \sin\beta\, d\beta\, d\gamma \\ &\qquad = \frac{16\pi^2}{2j_1+1}\delta_{j_1 j_2}\,\delta_{m_1'm_2'}\,\delta_{m_1 m_2} \\ &R: \qquad 0 \le \alpha \le 4\pi \quad 0 \le \beta \le \pi \quad 0 \le \gamma \le 2\pi \end{aligned} \tag{6.56}$$

Spherical harmonics and reduced rotation matrix elements:

$$\begin{aligned} Y_{\ell m}(\theta,\phi) &= \sqrt{\frac{2\ell+1}{4\pi}}\, d^\ell_{m0}(\theta) e^{im\phi} \\ Y_{\ell,-m}(\theta,\phi) &= (-1)^m \left[Y_{\ell m}(\theta,\phi)\right]^* \end{aligned} \qquad m \ge 0 \tag{6.60}$$

T3 THE 3-j COEFFICIENTS

Tables of Formulas

3-j with all m zero and smallest $j \le 3$ (numerical), Table 7.2, p. 275.
3-j coefficients with smallest $j \le 1$ (algebraic), Table 7.3, p. 281.

Clebsch-Gordan Coefficients

Clebsch-Gordan transformation between coupled and direct-product representations:

$$|j_1 j_2 JM\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 JM\rangle \tag{7.35}$$

J-sum unitarity for Clebsch-Gordan coefficients:

$$\sum_J \langle j_1 j_2 m_1' m_2' | j_1 j_2 JM\rangle \langle j_1 j_2 JM | j_1 j_2 m_1 m_2\rangle = \delta_{m_1 m_1'}\,\delta_{m_2 m_2'} \tag{7.39}$$

M-sum unitarity for Clebsch-Gordan coefficients:

$$\sum_{m_1 m_2} \langle j_1 j_2 J'M' | j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 JM\rangle = \delta_{J'J}\,\delta_{M'M} \tag{7.41}$$

The 3-j Coefficients

Definition in terms of Clebsch-Gordan coefficients:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv (-1)^{j_1-j_2-m_3} \frac{\langle j_1 j_2 m_1 m_2 | j_1 j_2 j_3, -m_3 \rangle}{\sqrt{2j_3+1}} \tag{7.57}$$

$$m_1 + m_2 + m_3 = 0 \tag{7.58}$$

General formula for 3-j coefficient:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \delta_{m_1+m_2+m_3,0} (-1)^{j_1-j_2-m_3} \\ &\times \sqrt{\frac{(j_3+j_1-j_2)!(j_3-j_1+j_2)!(j_1+j_2-j_3)!(j_3-m_3)!(j_3+m_3)!}{(j_1+j_2+j_3+1)!(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!}} \\ &\times \sum_k \frac{(-1)^{k+j_2+m_2}(j_2+j_3+m_1-k)!(j_1-m_1+k)!}{k!(j_3-j_1+j_2-k)!(j_3-m_3-k)!(k+j_1-j_2+m_3)!} \end{aligned} \tag{7.59}$$

3-j transformation between coupled and direct-product representations:

$$\begin{aligned} |j_1 j_2 JM\rangle &= \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1\rangle |j_2 m_2\rangle \delta_{m_1+m_2,M} \\ &\times (-1)^{j_1-j_2+M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \end{aligned} \tag{7.60}$$

J-sum unitarity for 3-j coefficients:

$$\sum_{j_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \delta_{m_1+m_2,-m_3} \tag{7.61}$$

M-sum unitarity for 3-j coefficients:

$$\sum_{m_1,m_2} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{j_3' j_3} \delta_{m_3 m_3'} \tag{7.62}$$

3-j coefficient with one j zero:

$$\begin{pmatrix} j_1 & 0 & j_3 \\ m_1 & 0 & m_3 \end{pmatrix} = \delta_{j_1 j_3} \delta_{m_1,-m_3} \frac{(-1)^{j_1-m_1}}{\sqrt{2j_1+1}} \tag{7.63}$$

Symmetry under reversal of all m values:

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ -m_2 & -m_1 & -m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{7.64}$$

Even permutations of columns of a 3-j coefficient leave it unchanged. Odd permutations of columns of a 3-j coefficient produce a phase change of $(-1)^{j_1+j_2+j_3}$.	(7.68)

3-j coefficient with all m values zero:

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = 0 \qquad a+b+c \quad \text{odd} \tag{7.72}$$

$$\begin{aligned} \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} &= (-1)^g \frac{g!}{(g-a)!(g-b)!(g-c)!} \\ &\times \sqrt{\frac{(2g-2a)!(2g-2b)!(2g-2c)!}{(2g+1)!}} \qquad a+b+c \quad \text{even} \end{aligned} \tag{7.73}$$

Combined spin and orbital states:

$$\begin{aligned} &Y_{\ell,\ell+1/2,M} = \\ &\sqrt{\frac{\ell-M+1/2}{2\ell+1}}\, Y_{l,M+1/2}(\theta\phi)\chi_- + \sqrt{\frac{\ell+M+1/2}{2\ell+1}}\, Y_{l,M-1/2}(\theta\phi)\chi_+ \\ &Y_{\ell,\ell-1/2,M} = \\ &\sqrt{\frac{\ell+M+1/2}{2\ell+1}}\, Y_{l,M+1/2}(\theta\phi)\chi_- - \sqrt{\frac{\ell-M+1/2}{2\ell+1}}\, Y_{l,M-1/2}(\theta\phi)\chi_+ \end{aligned} \tag{7.75}$$

Inverse Clebsch-Gordan series:

$$\begin{aligned} D^{j_3}_{m_3'm_3}(\alpha\beta\gamma) &= (-1)^{m_3'-m_3}(2j_3+1) \\ &\times \sum_{\substack{m_1'm_2' \\ m_1 m_2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} D^{j_1}_{m_1'm_1}(\alpha\beta\gamma) D^{j_2}_{m_2'm_2}(\alpha\beta\gamma) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \end{aligned} \tag{7.87}$$

Clebsch-Gordan series for D-matrix elements:

$$\begin{aligned} D^{j_1}_{m_1'm_1}(\alpha\beta\gamma) D^{j_2}_{m_2'm_2}(\alpha\beta\gamma) &= (-1)^{m_3'-m_3}\, \delta_{m_3',-(m_1'+m_2')}\, \delta_{m_3,-(m_1+m_2)} \\ &\times \sum_{j_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} D^{j_3}_{m_3'm_3}(\alpha\beta\gamma) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \end{aligned} \tag{7.89}$$

Clebsch-Gordan series for reduced rotation matrix elements:

$$
\begin{aligned}
d^{j_1}_{m_1'm_1}(\beta)d^{j_2}_{m_2'm_2}(\beta) &= (-1)^{m_3'-m_3}\,\delta_{m_3',-(m_1'+m_2')}\,\delta_{m_3,-(m_1+m_2)} \\
&\times\sum_{j_3}(2j_3+1)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} d^{j_3}_{m_3'm_3}(\beta)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
\end{aligned}
\tag{7.90}
$$

Clebsch-Gordan series for Legendre polynomials:

$$
P_{\ell_1}(\cos\beta)P_{\ell_2}(\cos\beta) = \sum_{\ell_3}(2\ell_3+1)\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 P_{\ell_3}(\cos\beta) \tag{7.91}
$$

Integral of the product of three D-matrix elements:

$$
\begin{aligned}
&\int_R D^{j_1}_{m_1'm_1}(\alpha\beta\gamma)D^{j_2}_{m_2'm_2}(\alpha\beta\gamma)D^{j_3}_{m_3'm_3}(\alpha\beta\gamma)\,d\alpha\,\sin\beta\,d\beta\,d\gamma \\
&\qquad = 16\pi^2\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\
&\qquad R: \quad 0\le\alpha\le 4\pi \quad 0\le\beta\le\pi \quad 0\le\gamma\le 2\pi
\end{aligned}
\tag{7.98}
$$

Integral of the product of three reduced rotation matrix elements:

$$
\begin{aligned}
&\int_0^\pi d^{j_1}_{m_1'm_1}(\beta)\,d^{j_2}_{m_2'm_2}(\beta)\,d^{j_3}_{m_3'm_3}(\beta)\sin\beta\,d\beta\;\delta_{m_1'+m_2'+m_3',0}\,\delta_{m_1+m_2+m_3,0} \\
&\qquad = 2\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
\end{aligned}
\tag{7.99}
$$

Integral over the product of three Legendre polynomials:

$$
\int_0^\pi P_{\ell_1}(\cos\beta)P_{\ell_2}(\cos\beta)P_{\ell_3}(\cos\beta)\sin\beta\,d\beta = 2\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{7.100}
$$

Integral of the product of three spherical harmonics (Gaunt integral):

$$
\begin{aligned}
&\int_{R'} Y^*_{LM}(\theta\phi)Y_{\ell_1m_1}(\theta\phi)Y_{\ell_2m_2}(\theta\phi)\,d\phi\,\sin\theta\,d\theta \\
&= (-1)^M\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2L+1)}{4\pi}}\begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix}\begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix} \\
&\qquad R': \quad 0\le\phi\le 2\pi \quad 0\le\theta\le\pi
\end{aligned}
\tag{7.102}
$$

$$
\Delta(\ell_1\ell_2L) \qquad \ell_1+\ell_2+L \text{ even} \qquad m_1+m_2=M \tag{7.103}
$$

T4 IRREDUCIBLE SPHERICAL TENSOR OPERATORS

Irreducible Spherical Tensors

Basic definition of irreducible tensors:

$$T_{kq}(\alpha\beta\gamma) = \sum_{q'=-k}^{k} T_{kq'}(0) D^{k}_{q'q}(\alpha\beta\gamma) \tag{8.8}$$

Racah's requirements on commutators:

$$\left[J_0, T_{kq}\right] = \left[J_z, T_{kq}\right] = q T_{kq} \qquad q = -k, -k+1, \dots, k-1, k \tag{8.9}$$

$$\left[J_{\pm 1}, T_{kq}\right] = \sqrt{(k \mp q)(k \pm q + 1)}\, T_{kq\pm 1} \qquad q = -k, -k+1, \dots, k-1, k \tag{8.10}$$

Gradient operator commutators:

$$\begin{aligned} \left[L_\sigma, \nabla_q\right] &= (-1)^\sigma \sqrt{2} \langle 1 1 q + \sigma, -\sigma | 1 1 1 q \rangle \nabla_{\sigma+q} \\ &= (-1)^{\sigma+q} \sqrt{6} \begin{pmatrix} 1 & 1 & 1 \\ \sigma & q & -\sigma - q \end{pmatrix} \nabla_{\sigma+q} \end{aligned} \tag{8.17}$$

Hermitian adjoint of a tensor component:

$$T^{\dagger}_{kq} = (-1)^{p-q} T_{k,-q} \tag{8.19}$$

Matrix elements of tensor operators and adjoints:

$$\langle jm | T_{kq} | j'm' \rangle = (-1)^{p-q} \langle j'm' | T_{k,-q} | jm \rangle^{*} \tag{8.20}$$

Building-up formula for tensors:

$$\begin{aligned} T_{KQ}(A_1, A_2) &= \sum_{q_1, q_2} T_{k_1 q_1}(A_1) T_{k_2 q_2}(A_2) \\ &\quad \times (-1)^{k_1 - k_2 + Q} \sqrt{2K+1} \begin{pmatrix} k_1 & k_2 & K \\ q_1 & q_2 & -Q \end{pmatrix} \end{aligned} \tag{8.21}$$

Bipolar harmonics:

$$\begin{aligned} B_{LM}(\Omega, \Omega') &= \sum_{m_1, m_2} Y_{\ell_1 m_1}(\Omega) Y_{\ell_2 m_2}(\Omega') \\ &\quad \times (-1)^{\ell_1 - \ell_2 + M} \sqrt{2L+1} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \end{aligned} \tag{8.26}$$

Vector spherical harmonics:

$$\mathbf{Y}_{L\ell,M}(\theta\phi) = \sum_{\sigma} Y_{\ell,M-\sigma}(\theta\phi)\,\hat{\mathbf{e}}_{\sigma}\,(-1)^{\ell-1+M}\sqrt{2L+1}\begin{pmatrix} \ell & 1 & L \\ M-\sigma & \sigma & -M \end{pmatrix} \tag{8.27}$$

Scalar product of two irreducible tensors:

$$\begin{aligned} \mathbf{T}_k(A_1)\cdot\mathbf{T}_k(A_2) &\equiv (-1)^k\sqrt{2k+1}\,T_{00}(A_1,A_2) \\ &= \sum_q (-1)^q T_{kq}(A_1)T_{k-q}(A_2) \end{aligned} \tag{8.29}$$

Wigner-Eckart theorem:

$$\langle jm|T_{kq}|j'm'\rangle = (-1)^{2k}\langle j'km'q|j'kjm\rangle\langle j\|\mathbf{T}_k\|j'\rangle \tag{8.32}$$

Reduced Matrix Elements

Spherical harmonic tensor:

$$\begin{aligned} \langle \ell\|\mathbf{Y}_k\|\ell'\rangle &= \sqrt{\frac{2k+1}{4\pi}}\,(-1)^k\langle k\ell 00|k\ell\ell' 0\rangle \\ &= \sqrt{\frac{(2k+1)(2\ell+1)}{4\pi}}\,(-1)^{\ell}\begin{pmatrix} \ell & k & \ell' \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{8.35}$$

Angular momentum operator:

$$\langle j\|\mathbf{J}\|j'\rangle = \sqrt{j(j+1)}\,\delta_{jj'} \tag{8.48}$$

Formula for reduced matrix element (for nonzero Clebsch-Gordan coefficient):

$$\langle j\|\mathbf{T}_k\|j'\rangle = (-1)^{2k}\langle jm|T_{kq}|j'm'\rangle \,/\, \langle j'km'q|j'kjm\rangle \tag{8.49}$$

Reduced matrix element from sum rule:

$$\langle j\|\mathbf{T}_k\|j'\rangle = (-1)^{2k}\sum_{m'q}\langle j'km'q|j'kjm\rangle\langle jm|T_{kq}|j'm'\rangle \tag{8.50}$$

Absolute square of reduced matrix element:

$$\left|\langle j\|\mathbf{T}_k\|j'\rangle\right|^2 = \sum_{m'q}\left|\langle jm|T_{kq}|j'm'\rangle\right|^2 \tag{8.51}$$

For $\mathbf{T}_k$ Hermitian:

$$\sqrt{2j+1}\,\langle j\|\mathbf{T}_k\|j'\rangle = (-1)^{j-j'-p}\sqrt{2j'+1}\,\langle j'\|\mathbf{T}_k\|j\rangle^* \tag{8.52}$$

T5 THE 6-j COEFFICIENTS

Table of Formulas

6-j coefficients with smallest $j \le 1$ (algebraic), Table 9.1, p. 353.

Recoupling transformation in terms of Racah coefficients:

$$|(ed)c\gamma\rangle = \sum_f |(af)c\gamma\rangle \sqrt{(2e+1)(2f+1)}\, W(abcd;ef) \tag{9.10}$$

6-j coefficients in terms of Racah coefficients:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \equiv (-1)^{\Sigma} W(abcd;ef) \qquad \Sigma \equiv a+b+c+d \tag{9.11}$$

Recoupling transformation in terms of 6-j coefficients:

$$|(ed)c\gamma\rangle = \sum_f |(af)c\gamma\rangle \sqrt{(2e+1)(2f+1)}\, (-1)^{\Sigma} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \tag{9.13}$$

Triangle conditions for nonzero Racah or 6-j coefficient:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta & \Delta & \Delta \\ & & \end{Bmatrix} \quad \begin{Bmatrix} & & \Delta \\ \Delta & \Delta & \end{Bmatrix} \quad \begin{Bmatrix} & \Delta & \\ \Delta & & \Delta \end{Bmatrix} \quad \begin{Bmatrix} \Delta & & \\ & \Delta & \Delta \end{Bmatrix}$$

First expansion rule for 6-j coefficients:

$$\begin{aligned} &\sum_f (2f+1)(-1)^{\Sigma} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \begin{pmatrix} b & d & f \\ \beta & \delta & -\phi \end{pmatrix} \begin{pmatrix} a & f & c \\ \alpha & \phi & -\gamma \end{pmatrix} (-1)^{f-e-\alpha-\delta} \\ &= \begin{pmatrix} a & b & e \\ \alpha & \beta & -\varepsilon \end{pmatrix} \begin{pmatrix} e & d & c \\ \varepsilon & \delta & -\gamma \end{pmatrix} \end{aligned} \tag{9.14}$$

Second expansion rule for 6-j coefficients:

$$\begin{aligned} &(-1)^{\Sigma} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \begin{pmatrix} a & f & c \\ \alpha & \phi & -\gamma \end{pmatrix} \\ &= \sum_{\beta\delta} (-1)^{f-e-\alpha-\delta} \begin{pmatrix} a & b & e \\ \alpha & \beta & -\varepsilon \end{pmatrix} \begin{pmatrix} e & d & c \\ \varepsilon & \delta & -\gamma \end{pmatrix} \begin{pmatrix} b & d & f \\ \beta & \delta & -\phi \end{pmatrix} \end{aligned} \tag{9.15}$$

6-j coefficients in terms of 3-j coefficients:

$$
\begin{aligned}
&(-1)^{\Sigma}\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \\
&= \sum_{\alpha\beta\delta} (-1)^{f-e-\alpha-\delta}(2c+1)\begin{pmatrix} a & b & e \\ \alpha & \beta & -\varepsilon \end{pmatrix}\begin{pmatrix} e & d & c \\ \varepsilon & \delta & -\gamma \end{pmatrix} \\
&\times \begin{pmatrix} b & d & f \\ \beta & \delta & -\phi \end{pmatrix}\begin{pmatrix} a & f & c \\ \alpha & \phi & -\gamma \end{pmatrix} \qquad \Sigma = a+b+c+d
\end{aligned} \tag{9.16}
$$

General formula for 6-j coefficient:

$$
\begin{aligned}
&\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{a+b+c+d}\Delta(abe)\Delta(acf)\Delta(bdf)\Delta(cde) \\
&\times \sum_k (-1)^k (a+b+c+d+1-k)! \\
&\times \left[k!(e+f-a-d+k)!(e+f-b-c+k)!\right]^{-1} \\
&\times \left[(a+b-e-k)!(c+d-e-k)!(a+c-f-k)!(b+d-f-k)!\right]^{-1}
\end{aligned} \tag{9.20}
$$

$$
\Delta(abc) \equiv \sqrt{\frac{(a+b-c)!(a+c-b)!(b+c-a)!}{(a+b+c+1)!}}
$$

Orthogonality relations of 6-j coefficients:

$$
\sum_e (2e+1)(2f+1)\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}\begin{Bmatrix} a & b & e \\ d & c & g \end{Bmatrix} = \delta_{fg} \tag{9.21}
$$

Symmetry properties of 6-j coefficient:

A 6-j coefficient is invariant under interchange of any two columns and under interchange of the upper and lower arguments in each of any two columns.

(9.22)

Scalar Products of Irreducible Tensors

Scalar product of two irreducible tensors:

$$
\begin{aligned}
\mathbf{T}_k(A_1)\cdot\mathbf{T}_k(A_2) &\equiv (-1)^k\sqrt{2k+1}\,T_{00}(A_1,A_2) \\
&= \sum_q (-1)^q T_{kq}(A_1)T_{k-q}(A_2)
\end{aligned} \tag{9.26}
$$

Factorization theorem in coupled representation:

$$\begin{aligned}
&\left\langle (j_1 j_2) J M \middle| \mathbf{T}_k(A_1)\cdot\mathbf{T}_k(A_2) \middle| (j_1' j_2') J' M' \right\rangle \\
&= \delta_{JJ'}\delta_{MM'} \left\langle (j_1 j_2) J \middle\| \mathbf{T}_k(A_1)\cdot\mathbf{T}_k(A_2) \middle\| (j_1' j_2') J' \right\rangle \\
&= \delta_{JJ'}\delta_{MM'} (-1)^{j_1'+j_2+J} \sqrt{(2j_1+1)(2j_2+1)} \begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & k \end{Bmatrix} \\
&\quad \times \langle j_1 \| \mathbf{T}_k(A_1) \| j_1' \rangle \langle j_2 \| \mathbf{T}_k(A_2) \| j_2' \rangle
\end{aligned} \tag{9.27}$$

Projection theorem for rank-1 tensors:

$$\langle jm | \mathbf{T}_1 | j'm' \rangle = \frac{\langle jm | \mathbf{J}(\mathbf{J}\cdot\mathbf{T}_1) | jm' \rangle}{j(j+1)} \delta_{jj'} \tag{9.29}$$

Factorization theorem for rank-1 tensors in uncoupled representation:

$$\langle jm | \mathbf{J}(\mathbf{J}\cdot\mathbf{T}_1) | j'm' \rangle = \langle jm | \mathbf{J} | jm' \rangle \langle j \| \mathbf{J}\cdot\mathbf{T}_1 \| j \rangle \delta_{jj'} \tag{9.34}$$

Decomposition theorem for rank-1 tensor operator:

$$\langle jm | \mathbf{T}_1 | j'm' \rangle = \frac{\langle jm | \mathbf{J} | jm' \rangle \langle j \| \mathbf{J}\cdot\mathbf{T}_1 \| j \rangle}{j(j+1)} \delta_{jj'} \tag{9.35}$$

Reduced matrix element of rank-1 tensor operator:

$$\langle j \| \mathbf{T}_1 \| j' \rangle = \frac{\langle j \| \mathbf{J}\cdot\mathbf{T}_1 \| j \rangle}{\sqrt{j(j+1)}} \delta_{jj'} \tag{9.36}$$

Reduced matrix element of Legendre polynomial:

$$\begin{aligned}
&\left\langle (\ell_1 \ell_2) L \middle\| P_\mu \middle\| (\ell_1' \ell_2') L' \right\rangle = (-1)^{\mu+L} (2\ell_1+1)(2\ell_2+1) \\
&\quad \times \begin{pmatrix} \ell_1 & \mu & \ell_1' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_2 & \mu & \ell_2' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & L \\ \ell_2' & \ell_1' & \mu \end{Bmatrix} \delta_{LL'}
\end{aligned} \tag{9.46}$$

T6 THE 9-j COEFFICIENTS

9-j coefficients in terms of 3-j coefficients:

$$\begin{aligned}
\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} &= \sum \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} d & e & f \\ \delta & \varepsilon & \phi \end{pmatrix} \begin{pmatrix} g & h & i \\ \gamma & \eta & m \end{pmatrix} \\
&\quad \times \begin{pmatrix} a & d & g \\ \alpha & \delta & \gamma \end{pmatrix} \begin{pmatrix} b & e & h \\ \beta & \varepsilon & \eta \end{pmatrix} \begin{pmatrix} c & f & i \\ \gamma & \phi & m \end{pmatrix}
\end{aligned} \tag{9.58}$$

Symmetry properties of 9-j coefficients:

$$\boxed{\text{Under interchange of all rows with all columns a 9-}j\text{ coefficient is invariant.}} \tag{9.60}$$

$$\boxed{\text{Under interchange of two rows or two columns a 9-}j\text{ coefficient is multiplied by the phase }(-1)^P\text{ where }P=a+b+c+d+e+f+g+h+i.} \tag{9.61}$$

9-j coefficient with one argument zero:

$$\begin{Bmatrix} 0 & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = \frac{(-1)^{b+d+f+h}}{\sqrt{(2b+1)(2d+1)}} \begin{Bmatrix} d & e & f \\ b & i & h \end{Bmatrix} \delta_{bc}\,\delta_{dg} \tag{9.62}$$

9-j coefficient with one argument unity:

$$\begin{Bmatrix} 1 & b & b \\ d & e & f \\ d & h & i \end{Bmatrix} = \frac{\left[e(e+1)+i(i+1)-f(f+1)-h(h+1)\right]}{\sqrt{4b(b+1)(2b+1)d(d+1)(2d+1)}} \times (-1)^{b+d+f+h} \begin{Bmatrix} d & e & f \\ b & i & h \end{Bmatrix} \tag{9.64}$$

Orthogonality sum rule for 9-j coefficients:

$$\sum_{cf} (2c+1)(2f+1) \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} \begin{Bmatrix} a & b & c \\ d & e & f \\ j & k & i \end{Bmatrix} = \frac{\delta_{gj}\,\delta_{hk}}{(2g+1)(2h+1)} \tag{9.65}$$

Tensor matrix elements in the coupled scheme:

$$\begin{aligned} &\left\langle (j_1 j_2)J \right\| \mathbf{T}_K(A_1, A_2) \left\| (j_1' j_2')J' \right\rangle \\ &= \sqrt{(2j_1+1)(2j_2+1)(2J+1)(2J'+1)(2K+1)} \\ &\quad \times \begin{Bmatrix} j_1 & j_1' & k_1 \\ j_2 & j_2' & k_2 \\ J & J' & K \end{Bmatrix} \left\langle j_1 \right\| \mathbf{T}_{k_1}(A_1) \left\| j_1' \right\rangle \left\langle j_2 \right\| \mathbf{T}_{k_2}(A_2) \left\| j_2' \right\rangle \end{aligned} \tag{9.68}$$

States in j–j scheme in terms of L–S scheme:

$$\left|\left(\ell_1 s_1\right)j_1,\left(\ell_2 s_2\right)j_2;JM\right\rangle=\sum_{LS}\left|\left(\ell_1\ell_2\right)L,\left(s_1 s_2\right)S;JM\right\rangle \times\sqrt{(2L+1)(2S+1)(2j_1+1)(2j_2+1)}\begin{Bmatrix}\ell_1 & \ell_2 & L\\ s_1 & s_2 & S\\ j_1 & j_2 & J\end{Bmatrix} \tag{9.69}$$