Symmetry Reduced Exact Coherent Structures in Plane Couette Flow

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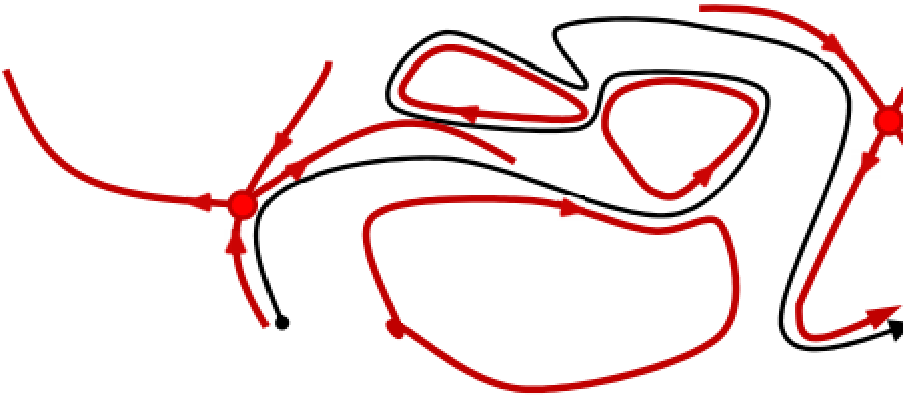
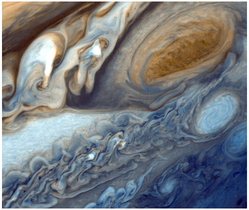


Turbulence

What is it, and why do we care?

- Turbulence is an irregular, chaotic, and dissipative flow phenomenon, with a vast number of (coupled) length scales.



- Shows up in many important systems - atmospheric and geophysical flows, turbine exhaust, stellar flows...

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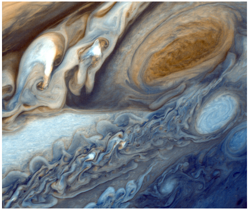

- Shows up in many important systems - atmospheric and geophysical flows, turbine exhaust, stellar flows...
- Direct numerical simulation (DNS) is very time consuming - need a simpler model!

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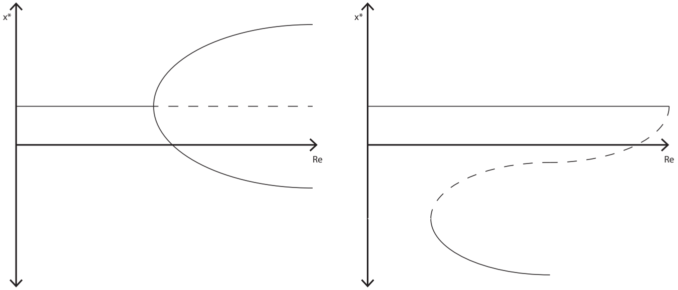
Transitions to Turbulence

- Of particular inteerest are transitions to turbulence from smooth (laminar) flow



Supercritical Transition   Subcritical Transition

Transitions to Turbulence

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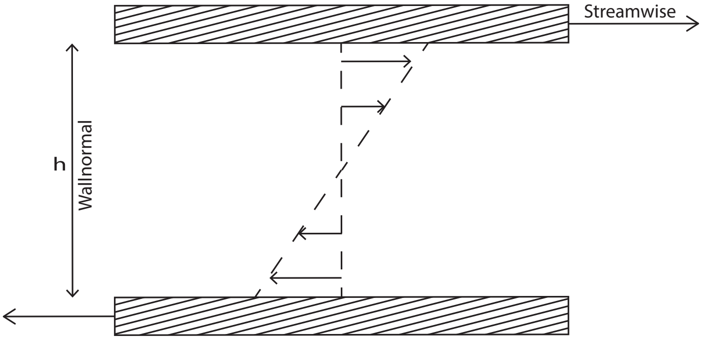
- Subcritical transitions are sudden and unpredictable (pipe flow, plane Couette flow)

The Navier-Stokes equation is the linear momentum conservation statement for Newtonian, incompressible fluids:

$$\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{v},$$

with the appropriate boundary conditions.

The System
Base Flow



If the Reynolds number is not small, the damping term is about the same size as the rest of the terms, and we have to solve the full Navier-Stokes equations (for Couette flow)

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- Mean field approaches, such as the Reynolds Averaged Navier Stokes have had a great deal of success
- RANS works by time averaging out the turbulent perturbations to some mean flow, resulting in a new equation of motion for the mean flow...
- ... but dynamical properties of turbulence are lost in the time-averaging process

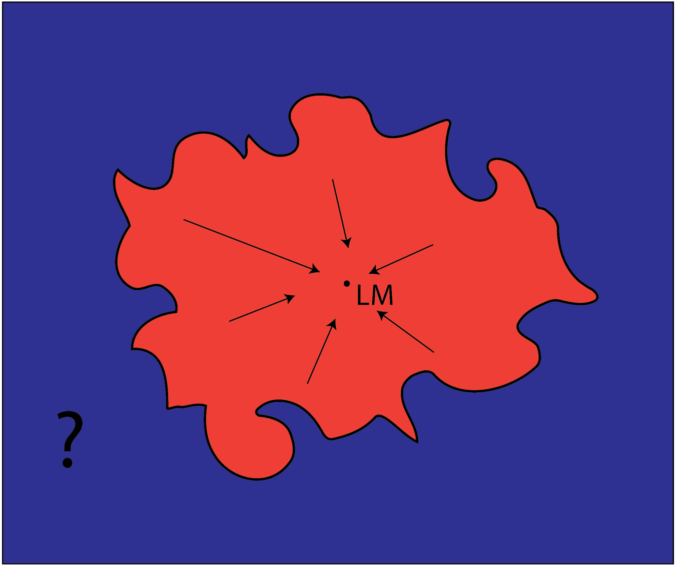
- Instead of time averaging the perturbation away, try to analyze its dynamics

Turbulence

Exact Coherent Structures

- Instead of time averaging the perturbation away, try to analyze its dynamics
- Imagine the perturbation as living in some infinite dimensional space of all possible velocity fields (the ‘state space’)

Laminar Basin of Attraction



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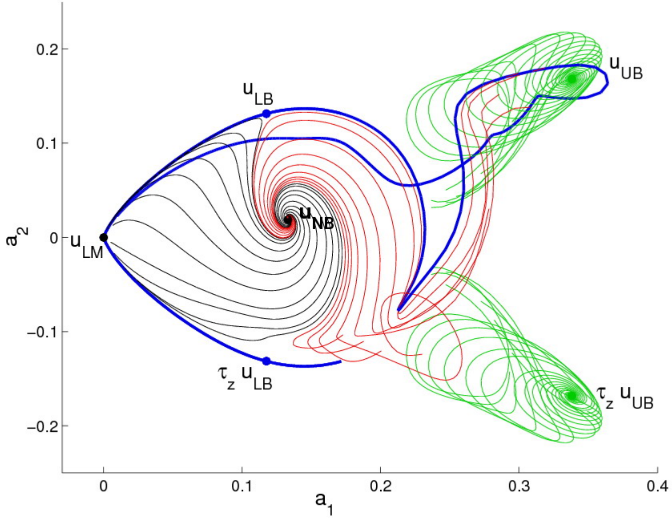

Halcrow et. al (2008)

Symmetry

- The full Navier-Stokes equations are invariant ($f_T(\sigma\mathbf{u}) = \sigma f_T(\mathbf{u})$) under any spatial rotation and translation

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  - $\sigma_{xy} \rightarrow$ Rotation by π about the z axis
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  - $\tau(l_x, l_z) \rightarrow$ Translation by (l_x, l_z) in the xz plane

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Relevance to Turbulence

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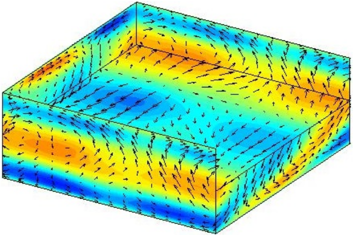
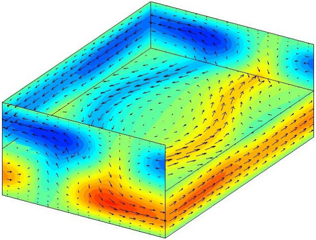
- Coherent structures can also exist in different inertial reference frames $\rightarrow$ travelling waves
- Symmetries can restrict the allowed directions of these travelling waves (Viswanath, 2007).
- Symmetric fields are also not necessarily representative of turbulence

Symmetric and Asymmetric Fields



A random, asymmetric field



A highly symmetric periodic orbit

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The Simulation

- I use Channelflow, a simulation library developed at UNH (Gibson, 2014) to analyze plane Couette flows.

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- In addition to Navier-Stokes integration, Channelflow can also find coherent structures (with a good initial guess)

The Simulation

Generating a good guess

- How do we hunt through the immense phase space? (> 60,000 dimensions)

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Borrero (2014)

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Borrero (2014)

- Look at recurrence graphs

The Simulation
Recurrence Graphs

- Can compare the flow at time t to time $t + T$ by calculating $d(\mathbf{u}(t), \mathbf{u}(t + T))$

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Recurrence Graphs

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- Use the physically motivated energy norm

$$|\mathbf{u}|^2 = \frac{1}{V}\int_V \mathbf{u}\cdot\mathbf{u}\,dV,$$

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- Minima of these plots should indicate the presence of a periodic orbit

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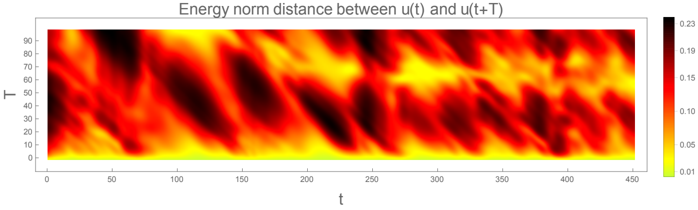
The Work

Recurrence Plots



The Simulation
Root Finding

- Once a promising candidate is found, it can be passed on to the root finder, to solve

$$\mathbf{u} - \sigma f_T(\mathbf{u}) = 0,$$

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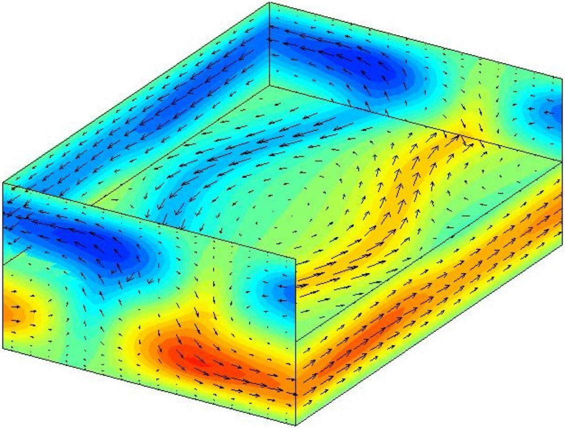
$$\mathbf{u} - \sigma f_T(\mathbf{u}) = 0,$$

- where σ encodes the travelling wave velocity.
- The root finder uses the Newton method, with the step in 60,000 dimensions calculated using the generalized minimum residual method.

P85.47

This orbit has period $T = 85.47$, and has symmetries generated by $\{e, \sigma_{xy}, \sigma_z\tau_{xz}\}$



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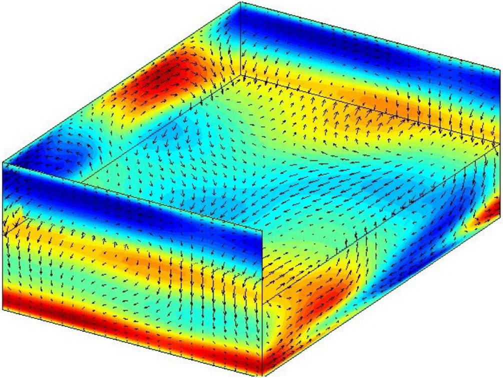
P8.32

This orbit has period $T = 8.32$, and has symmetries generated by $\{e, \sigma_{xy}\tau_{xz}\}$.



- Even though it's in a reduced symmetry subspace that allows travelling waves, it has no relative velocity.

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P8.32

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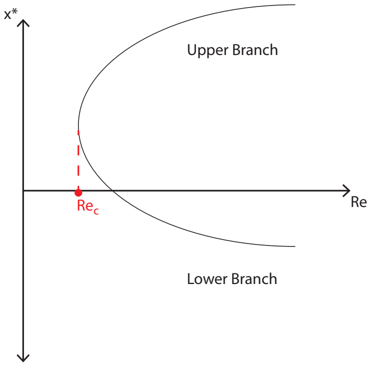
- Coherent structures cannot exist for all Reynolds numbers - $Re = 0$ means only the laminar state can exist
- Coherent structures will arise from some high-dimensional bifurcation



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- Can try to continue this solution for lower Reynolds numbers to locate the bifurcation
- After bifurcation, coherent structures tend to branch off in pairs - can this orbit's pair be located?

Future Work

- Try to find travelling coherent structures in the reduced symmetry space

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- Try to find coherent structures in spaces with even less symmetry than P8.32

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- Try to find travelling coherent structures in the reduced symmetry space
- Try to find coherent structures in spaces with even less symmetry than P8.32
- Find the bifurcation for P8.32 and continue the solution to find the other branch

Acknowledgments:

John Gibson at UNH for making the subject approachable

Daniel Borrero for being a great thesis advisor