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A Thesis
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Introduction

> I am an old man now, and when I die and go to heaven, there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former, I am optimistic
>
> Horace Lamb, 1932

Among the many successes of classical mechanics since Newton's *Principia*, the modern reader may notice one conspicuous absence - turbulence. Like a certain small, unnamed costal village in Armorica holding out against the Romans, turbulence has confounded the best efforts of physicists and engineers for over a century, with no real end in sight.

Historical approaches towards understanding turbulence began via a statistical approach, describing it as a random perturbation about some mean flow, resulting in Kolmogorov's famous scaling laws in 1941 **?**, and von Karman's so-called 'law of the wall' in 1930 **?**. These approaches, while perfectly suited to model the average behavior of turbulent flow, nevertheless fails to capture the dynamic behavior that would be the holy grail of fluid dynamics. The hope of the dynamic systems theory approach, lead by Hopf, Poincare and many other physicists is that this behavior can, to some extent, be captured in a meaningful way.

0.1 Dynamical Systems and Hopf's Dream

The prequel to Hopf and turbulence begins, somewhat unsurprisingly, with Newton. Newton showed in the *Principia* that his law of gravitation was consistent with his observations by solving the two-body problem, a procedure that is now routine in undergraduate classical mechanics courses. However, he was unable to solve the three-body problem - and neither was Gauss, Euler, d'Alembert or any other mathematical titan of the 18th and 19th centuries. In 1885, perhaps slightly frustrated with the unwillingness of nature to play ball with humans, King Oskar of Sweden announced a prize to the first person to the first person to find an exact analytic solution to the problem. Unfortunately for him, and for anyone else hoping for a tidy solution to the problem, Henri Poincare showed in 1887 that no general analytic solution existed for the problem (or, by extension for the n-body problem where $n > 2$), but fortunately for students of dynamical system's theory, setting the foundations for the geometric approach that is still used today. This is where Hopf comes in. Hopf had a vision of fluid flow as a vector in a infinite dimensional phase space, with viscosity forcing the phase space trajectory of the flow to lie in some finite manifold in the long term **?**. Hopf provides, as an example. the case where the viscosity μ is very large, in which case the manifold shrinks to a point corresponding to laminar flow, and speculates that as the viscosity decreases, new manifolds should arise from bifurcations, envisioning a maze of recurrent manifolds springing forth from the aether. Sadly for Hopf, the first electronic computer, ENIAC, had been built just two years earlier, the first silicon transistor was six years in the future, and high performance computing still decades away. With the resources available to him at the time, a numerical simulation of this phase space was impossible (to say nothing of analytic solutions), and Hopf had to remain satisfied with applying his ideas to approximations of the Navier-Stokes equations.

0.2 Computers and the Future

With the advent of modern computing however, numerical simulation of the phase space topology is within reach. In plane Couette flow (described in more detail in Section), cartographic efforts began with Nagata's demonstration of the existence of finite-amplitude turbulent perturbations from mean flow that were nevertheless equilibrium solutions in 1990 **?** and Kawahara and Kida's determination of *periodic* turbulent perturbations from mean flow **?** in 2001, and Viswanath's calculation of *relative* periodic orbits in 2007 **?**, which also introduced the numerical scheme that constitutes one of the core solvers used in this thesis. The development of the Channelflow software library by Gibson **? ?** is of particular note, as it has enabled the wider investigation of the phase space topology, and features heavily in this thesis. Indeed, the numerical schemes used within are formidable, and certainly beyond my ability to recreate within the thesis timescale, though I shall outline them in Section **??**. Given these tools, then, we can imagine that it may be possible to construct a web of periodic orbits, equilibria and their heteroclinc connections, and then predict to some accuracy the long-term dynamical behavior, based on transitions between these different states.

Chapter 1

Equations of Flow

> For every action, there is an equal
> and opposite regulatory body
>
> Anonymous

1.1 Formalisms

At the heart of fluid dynamics lie the Navier-Stokes equation, first derived by George Stokes in 1845, after a series of refinements leading back to Newton. Now if we were considering a point particle, we would begin with Newton's Second Law -

$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{F}, \tag{1.1}$$

write down the body force as a function of position, time, etc., and have our differential equation. With fluids, however, the treatment becomes somewhat more subtle - we are more concerned with the time-rate of deformation of the fluid particle than we are with the actual deformation - this being the difference between the Eulerian and Lagrangian descriptions of motion. Classical mechanics is typically framed in the Lagrangian context, so we will take a step back to develop the Eulerian context further.

1.1.1 The Control Volume

When asked to consider the mechanical evolution of some collection of bodies, two obvious methods would be readily apparent - we could either follow a particle (or collection of particles) on their merry way through space and time (the systems approach), or we could situate ourselves at some point in space, extend a 'bubble' around ourselves, and observe the properties of particles that enter and exit our bubble over time (the control volume approach). The systems approach will be familiar to anyone with a basic physics education, since it lends itself readily towards analysis of rigid-body motion. When considering fluids, however, the question of which fluid particle we should follow becomes non-trivial (Should we follow all? That be computationally difficult. Just one? Which one?) if we are using a systems approach, while the control volume approach remains as easy (or hard) as it was for rigid body motion. Historically, then, the control volume approach, also referred to as an **Eulerian** approach has been used to describe fluid dynamics (though work has been done on Lagrangian fluid mechanics), and this thesis will follow historical precedent.

1.1.2 The Fluid Particle

"Well", you might say, "All this talk of control volumes is all fine and dandy, but how do you plan to describe the motion of each molecule? Isn't that what you mean when you referred to fluid particles?" The answer, dear reader, is given by the continuum hypothesis. As you may have guessed, describing the motion of each and every molecule of fluid would be absurd - there are approximately 10^{21} atoms per milliliter of water, with six degrees of freedom each - solving 10^{21} coupled equations doesn't sound pleasant, or feasible. Furthermore, even if we ere able to write down this set of equations, finding the initial conditions of the fluid would be impossible (how was molecule # 19364829008283716 moving at time $t = 0$?).

Instead, we consider a fictitious "particle" of fluid, large enough so that we can take an average of the externally measurable quantities within (pressure, temperature, velocity, energy, etc.), but small enough so that we can approximately consider all these (averaged) variables as continuous. Perhaps this vagueness bothers you - does such a fluid particle even exist? As an example, let us consider water, with 10^{28} atoms per cubic meter. Imagine our fluid particles as cubes filling up space, with sides of length dl, giving a total volume of dl^3. First, let us make dl small enough that the external variables appear continuous - how about one micron? That gives the volume of a fluid particle as one cubic micrometer. For scale, consider that the reference volume of the human red blood cell ranges from 80-100 cubic micrometers**?** - this seems acceptably small. The number of water molecules within each fluid particle is then

$$10^{28} dl^3 = 10^{28} \times 10^{-15} = 10^{13}, \tag{1.2}$$

or about 10 trillion water molecules, which is certainly sufficient to achieve a meaningful average. Having defined a fluid particle in this way allows us to behave as if these external variables have well defined values at every point in space, which greatly simplifies the following analysis.

1.2 Mass Conservation

While not a part of the Navier-Stokes equations (which are a statement about conservation of linear momentum), conservation of mass is essential in solving fluid problems, and will serve as an easy demonstration of the control volume principle. Consider a volume Ω which is fixed in space, and has some mass density $\rho = \rho(\mathbf{x}, t)$ and some fluid velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ that are generically functions of time and space, allowing us to define the **mass current density** $\mathbf{m} = \mathbf{v}\rho$. We would prefer that our equations do not allow mass to disappear (excluding high-energy physics, naturally), and would additionally prefer a mathematical form of this statement.

The mass contained within the volume is given by

$$M = \int_\Omega \rho \ dV, \tag{1.3}$$

the flow of mass out of the volume through the surface $d\Omega$ of Ω is given by

$$M_{flow} = \int_{d\Omega} \mathbf{m} \cdot \mathbf{n} \ dA = \int_\Omega \nabla\cdot(\rho\mathbf{v}) \ \ dV, \tag{1.4}$$

by the divergence theorem . Now if mass is conserved, the sum of the rate of mass flow into (or out of) the volume and the rate of change of mass inside the volume must be zero, giving

$$\frac{\partial M}{\partial t} + M_{flow} = 0, \tag{1.5}$$

$$\frac{\partial}{\partial t}\left(\int_\Omega \rho \ dV\right) + \int_\Omega \nabla\cdot(\rho\mathbf{v}) \ \ dV = 0, \tag{1.6}$$

but since V is time independent, the time derivative commutes with the integral, giving

$$\int_\Omega \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) \ \ dV = 0, \tag{1.7}$$

but since Ω is arbitrary, the integrand must be zero everywhere, giving the statement of conservation of mass in differential form:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{1.8}$$

Other conservation laws can be written in a similar way; the generic form for a conserved quantity Φ with density ϕ and current ψ is

$$\frac{\partial \phi}{\partial t} + \nabla\cdot(\phi\psi) = 0. \tag{1.9}$$

Now, Equation 1.8 can be expanded further by using the chain rule for divergence, giving

$$\frac{\partial \rho}{\partial t} + \rho\nabla\cdot\mathbf{v} + \mathbf{v}\nabla\rho = 0. \tag{1.10}$$

If the flow is (approximately) incompressible, which will be true for small Mach numbers[1], then ρ must be constant, and Equation 1.10 becomes

$$\nabla\cdot\mathbf{v} = 0, \tag{1.11}$$

for both steady and unsteady flows (Steady, compressible flows would only drop the first term from Equation 1.8).

[1] The Mach number is the ratio of the fluid velocity to the speed of sound in the fluid. v_{sound} for water is 1497 ms^{-1} at room temperature and pressure.

1.3 Conservation of Linear Momentum

As mentioned earlier, the Navier-Stokes equations are simply a statement of conservation of linear momentum, along with certain assumptions about stress (an object that contains information about forces) and strain (an object that contains information about deformation), which are presented below.

1.3.1 Stress

For a control volume Ω with boundary $d\Omega$, there are in general three ways in which momentum can be change over time in Ω by transport through $d\Omega$ (i.e., forces on $d\Omega$):

1. Bulk, 'convective' flow across $d\Omega$
2. Surface-normal transfer through elastic collisions between molecules. This is the microscopic origin of pressure.
3. Transfer through stochastic motion of molecules through $d\Omega$, as in Figure ??. Since it is stochastic, time-averaged mass does not change, but momentum can still be transferred. THis leads to viscous stresses, and can be both normal and tangential (shear).

We define positive stress as stress that acts towards the control volume, and negative if they act away. Now, a stress on a fluid volume is not quite a vector, like force. Not only does it have a magnitude and direction, but it also has a plane that it acts from. Since there are three directions and three planes of action, stress objects generally have nine elements, and is a **second rank tensor**. That is, the viscous stress tensor $\mathcal{T}$ is identified by two subscripts, where the first subscript indicates the plane of action, and the second the direction of action. So $\mathcal{T}_{xy}$ would represent the viscous force on the (y, z) plane acting in the y direction. Note than in a Cartesian coordinate system, a second rank tensor can be written as a matrix.

1.3.2 Strain

Now that we can consider the forces on a fluid particle, we need to link these forces back to our external variables. In solids, this is easy - Hooke's Law, for instance, sets the strain proportional to the stress:

$$\sigma = \mathcal{C}\epsilon, \tag{1.12}$$

where σ is the Cauchy stress tensor, $\mathcal{C}$ is the (fourth order) stiffness tensor and ϵ is the infinitesimal strain tensor. However, for fluids, this is not the case - you can imagine that if you applied a constant force to a cube of water, it would deform continuously, without offering any resistance. Newton theorized that for continuously deformable fluids, the 1-D relationship between stress $\mathcal{T}$ and strain $\mathcal{S}$ should have the following form:

$$\mu\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} = \mu\frac{\mathrm{d}u}{\mathrm{d}x} = \mathcal{T}, \tag{1.13}$$

where μ is the viscosity and u is the velocity. Stokes extended this to three dimensions, giving the Newtonian constitutive relationship between stress and strain:

$$\mathcal{T}_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{1.14}$$

where δ_{ij} is the Kronecker delta function. Water, and most gases under normal conditions are Newtonian, but fluids like blood, quicksand and corn starch (to name a few) are not.

1.3.3 Surface Forces

Having written down the stress tensor $\mathcal{T}$ as a function of the velocity field, we now link it to the surface forces on a fluid particle. Recalling that stresses act over $d\Omega$ of the fluid particle, the total force is then simply

$$\mathbf{F} = \int_{d\Omega} \mathcal{T} \cdot \mathbf{n} \; dA, \tag{1.15}$$

where $\mathbf{n}$ is the surface normal.

1.3.4 Newton's Second Law

Newton's second law can be restated in a more useful form - assuming that mass is conserved,

$$\sum \mathbf{F} = M\mathbf{a}, \tag{1.16}$$

where the sum is over all possible external forces. For a fluid particle, we have two kinds of forces - body forces, like gravity or electromagnetism, and surface forces due to stress. We group the body forces $\mathbf{F}_b$ as

$$\mathbf{F}_b = \int_{\Omega} \rho \mathbf{f} \; dV, \tag{1.17}$$

where $\mathbf{f}$ is the **body force density**. Using Equation 1.15 to express the surface forces, Newton's Second Law becomes

$$\int_{\Omega} \rho \left(\mathbf{f} - \mathbf{a} \right) \; dV + \int_{d\Omega} \mathcal{T} \cdot \mathbf{n} \; dA = 0, \tag{1.18}$$

which can be written in differential form by the same trick used to generate Equation **??**, giving Cauchy's Equation of Motion

$$\rho \left(\mathbf{f} - \mathbf{a} \right) + \nabla \cdot \mathcal{T} = 0. \tag{1.19}$$

From this, the Navier-Stokes equation arise by a substitution of Equation 1.14 into Equation 1.19, giving (after tedious rearrangement by components),

$$\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{f} - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \mathbf{V} \tag{1.20}$$

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