Symmetry Breaking in Coherent Structures of Plane Couette Flow

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Agenda

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Turbulence

Why does it matter?

- Appears in physical flows - mixing, combustion, flow past bodies, etc.

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Turbulence

Why does it matter?

- Appears in physical flows - mixing, combustion, flow past bodies, etc.
- Difficult to predict and highly chaotic - direct numerical simulation (DNS) highly infeasible.

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- The approach taken towards characterizing turbulence was by considering statistical, mean field theories such as Kolmogorov's scaling law, or the law of the wall

Turbulence

Mean Flow

- The approach taken towards characterizing turbulence was by considering statistical, mean field theories such as Kolmogorov's scaling law, or the law of the wall
- Mean field theories cannot capture turbulent dynamics - but DNS takes too long

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The System
Navier-Stokes

The Navier-Stokes equation is the linear conservation statement for Newtonian, incompressible fluids:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{f}, \tag{1}$$

though in order to fully specify the flow, other conservation statements and boundary conditions are also necessary.

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though in order to fully specify the flow, other conservation statements and boundary conditions are also necessary.

- Isothermal system means that we need not consider the equation of state
- Incompressible system implies that $\nabla\cdot\mathbf{v} = 0$.

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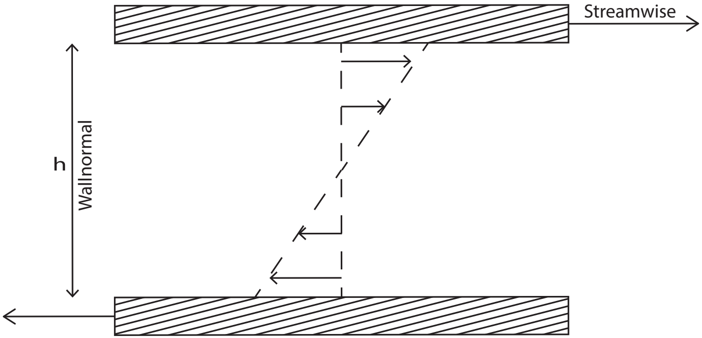
The System

Plane Couette



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The System
Plane Couette

The following geometry and assumptions are made

1. Fluid is constrained between two infinite parallel plates, each with some constant velocity
2. No-slip boundary conditions at the plates
3. Flow is unidirectional: $[u, v, w](x, y, z) = u(y)$
4. $\nabla p = 0$
5. Gravitational effects are negligible

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The System

Plane Couette

Under these assumptions, the Navier-Stokes equations reduce to a trivial ODE:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0, \tag{2}$$

with boundary conditions $u(0) = 0$, $u(h) = V$ (for example), which gives the bulk flow solution

$$u(y) = u_0 \frac{y}{h}. \tag{3}$$

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The System

Perturbations

If we relax the unidirectional requirement on the flow, we can use the Reynolds Decomposition to decompose the new flow field into the sum of the mean flow and a perturbation from it. In this case, we can get $\mathbf{u}' = \mathbf{u}_{bulk} + \mathbf{u}$, where $\mathbf{u}$ is a perturbation from the bulk flow. Then we get the following equation for $\mathbf{u}$:

$$\frac{\partial \mathbf{u}}{\partial t} + y\frac{\partial \mathbf{u}}{\partial x} + v\hat{\mathbf{x}} + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}, \nabla\cdot\mathbf{u} = 0 \qquad (4)$$

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State Space
Hopf's Vision

- Fluid flow as trajectories in an infinite dimensional phase space

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State Space
Hopf's Vision

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- However, viscosity forces trajectories to lie on some finite-dimensional manifold

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State Space
Hopf's Vision

- Fluid flow as trajectories in an infinite dimensional phase space
- However, viscosity forces trajectories to lie on some finite-dimensional manifold
- Can we map out this manifold? How does it depend on velocity?

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State Space
65,000 dimensions

- Divide computational domain into 3D grid

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- The state space vector can then be written as $\mathbf{v} = (u_{11}, v_{11}, w_{11}, u_{12} \ldots .)^T$

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State Space
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- Track components of vector field at each grid point
- The state space vector can then be written as $\mathbf{v} = (u_{11}, v_{11}, w_{11}, u_{12} \ldots)^T$
- The dimension of this space is then given by $3N_xN_yN_z$

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- Apart from the trivial laminar state, there exist other equilibrium states for **u**.

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- These appear to wall off the turbulent regions of the phase space

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Coherent Structures

Equilibria

- Apart from the trivial laminar state, there exist other equilibrium states for **u**.
- These appear to wall off the turbulent regions of the phase space
- The following solutions were computed by John Gibson and Predrag Civtanović at Georgia Tech.

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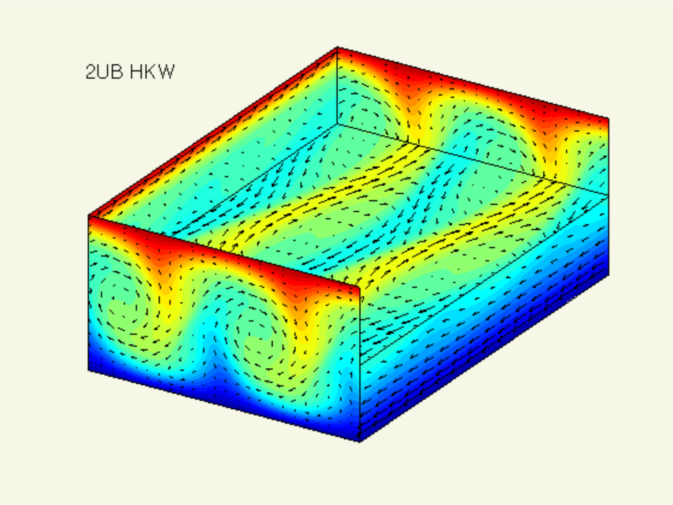
Coherent Structures

Equilibria



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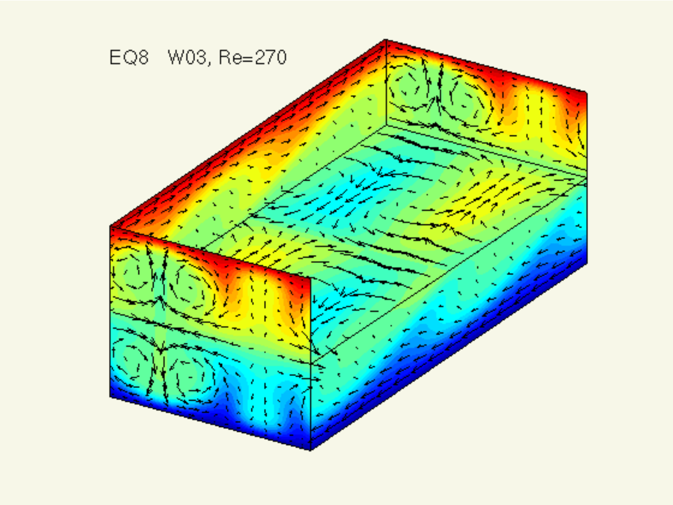
Equilibria



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Coherent Structures
Periodic Orbits

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Coherent Structures
Periodic Orbits

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- These exist in the middle of the turbulent region of phase space
- Do they have some influence on the turbulent dynamics?
- Can they help predict tubulent behavior?

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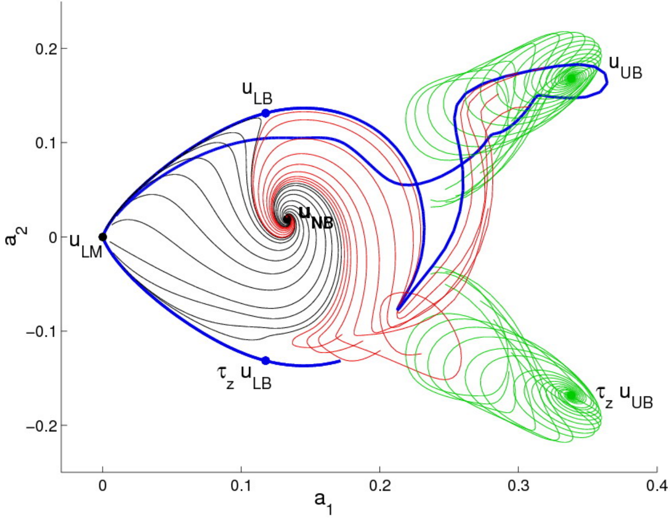
Coherent Structures
State Space



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Coherent Structures
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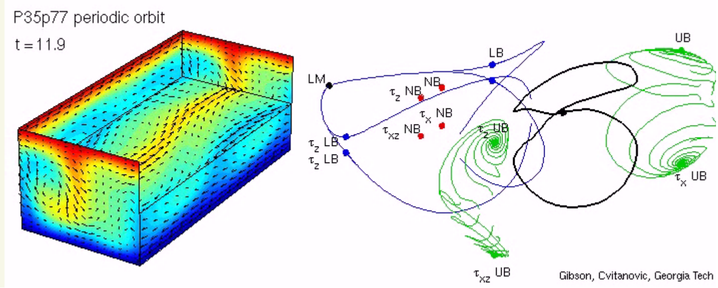

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Symmetry

The generators of symmetry for **u** are as follows

- $[u, v, w](x, y, z) \to [-u, v, w](-x, y, z)$

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- $[u, v, w](x, y, z) \rightarrow [-u, -v, -w](x, y, z)$
- $[u, v, w](x, y, z) \rightarrow [-u, v, w](x + l_x, y, z + l_z)$

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We will limit the explored symmetries to the S group, generated by the following operations:

- $s_1[u, v, w](x, y, z) = [u, v, -w](x + 0.5L_x, y, -z)$

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Symmetry
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- $s_2[u, v, w](x, y, z) = [-u, -v, w](-x + 0.5L_x, -y, z + 0.5L_z)$
- $s_3[u, v, w](x, y, z) = [-u, -v, -w](-x, -y, -z + 0.5L_z)$

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The Simulation

- I use the software system Channelflow, developed by John Gibson, to simulate the fluid flow and to solve for periodic orbits and equilibria.
- The minimum viable cell is used to speed up computation, with nondimensionalized lengths of $L_x = 1.75\pi$, $L_z = 1.2\pi$ and parallel plates at $y = \pm 1$

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- Periodic boundary conditions are imposed in the streamwise and spanwise directions, Dirichlet boundary conditions at the plates.
- This restricts the size of observable features in the physical flow

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The Simulation

The Hunt for the Red October

- How do we hunt through the immense phase space? (> 60,000 dimensions)

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The Simulation

The Hunt for the Red October

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- Start from a random initial condition

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The Simulation
The Hunt for the Red October

- How do we hunt through the immense phase space? (> 60,000 dimensions)
- Start from a random initial condition
- Integrate flow forward in time

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The Simulation
The Hunt for the Red October

- How do we hunt through the immense phase space? (> 60,000 dimensions)
- Start from a random initial condition
- Integrate flow forward in time
- Look at recurrence diagrams

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- We first need to define a distance function between two flow states

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- We first need to define a distance function between two flow states
- The physically motivated energy norm is used

$$d(\mathbf{u}, \mathbf{v}) = \frac{1}{V} \int_V \mathbf{u} \cdot \mathbf{v} dV \tag{5}$$

- We can now compare the flow at time t to time $t + T$ by calculating $d(\mathbf{u}(t), \mathbf{u}(t + T))$
- Minima of these plots should indicate the presence of a periodic orbit

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The Simulation
Root Finding

- Once a promising candidate is found, it can be passed on to the root finder, to solve

$$\mathbf{u} - f^T(\mathbf{u}) = 0 \tag{6}$$

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- The root finder primarily uses the Newton method, using the generalized minimum residual method to determine the Newton step.
- If the Newton step isn't good enough, a different step x_H is used, which minimizes $||\mathbb{A}x_h - b||^2$.
- As opposed to the Newton step x_N which solves $\mathbb{A}x_N = b$

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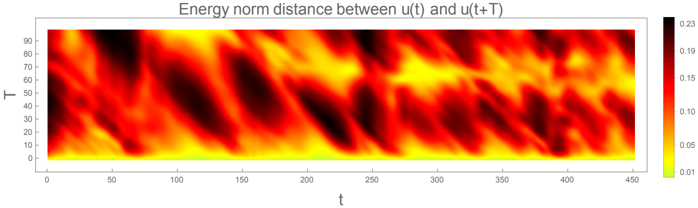
The Work
Recurrence Plots



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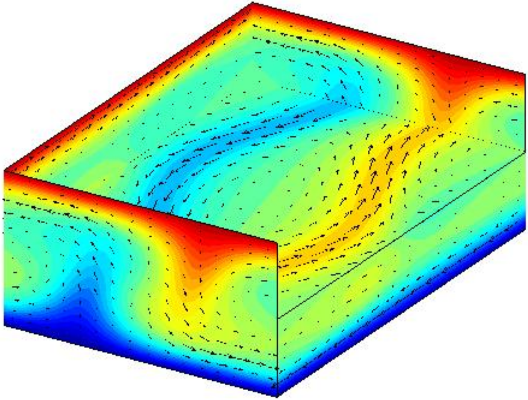
The Work

This orbit has period $T = 85.47$, and has symmetries that are not part of S.



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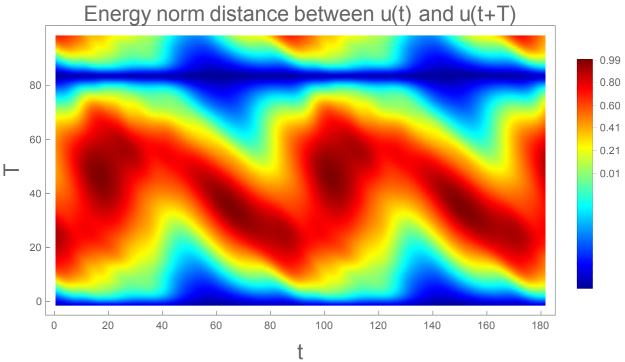
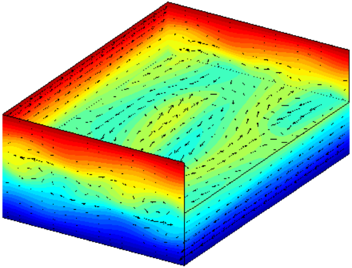

Energy norm distance between u(t) and u(t+T)
T
t
0
20
40
60
80
100
120
140
160
180
0.99
0.80
0.60
0.41
0.21
0.01

Out[73]=



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Future Work

- Locate more periodic orbits in the symmetric subspace

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Future Work

- Locate more periodic orbits in the symmetric subspace
- Remove symmetry constraints and attempt to locate asymmetric orbits

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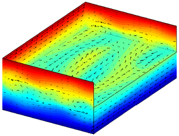
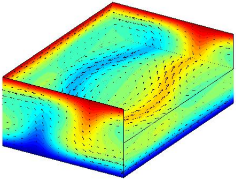


A random field



A highly symmetric field

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Daniel Borrerro for putting up with my lack of progress