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Introduction

> I am an old man now, and when I die and go to heaven, there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former, I am optimistic
>
> Horace Lamb, 1932

Among the many successes of classical mechanics since Newton's *Principia*, the modern reader may notice one conspicuous absence - turbulence. Like a certain small, unnamed costal village in Armorica holding out against the Romans, turbulence has confounded the best efforts of physicists and engineers for over a century, with no real end in sight.

Historical approaches towards understanding turbulence began via a statistical approach, describing it as a random perturbation about some mean flow, resulting in Kolmogorov's famous scaling laws in 1941 Kolmogorov (1991), and von Karman's so-called 'law of the wall' in 1930 Karman (1930). These approaches, while perfectly suited to model the average behavior of turbulent flow, nevertheless fails to capture the dynamic behavior that would be the holy grail of fluid dynamics. The hope of the dynamic systems theory approach, lead by Hopf, Poincare and many other physicists is that this behavior can, to some extent, be captured in a meaningful way.

0.1 Dynamical Systems and Hopf's Dream

The prequel to Hopf and turbulence begins, somewhat unsurprisingly, with Newton. Newton showed in the *Principia* that his law of gravitation was consistent with his observations by solving the two-body problem, a procedure that is now routine in undergraduate classical mechanics courses. However, he was unable to solve the three-body problem - and neither was Gauss, Euler, d'Alembert or any other mathematical titan of the 18th and 19th centuries. In 1885, perhaps slightly frustrated with the unwillingness of nature to play ball with humans, King Oskar of Sweden announced a prize to the first person to the first person to find an exact analytic solution to the problem. Unfortunately for him, and for anyone else hoping for a tidy solution to the problem, Henri Poincare showed in 1887 that no general analytic solution existed for the problem (or, by extension for the n-body problem where $n > 2$), but fortunately for students of dynamical system's theory, setting the foundations for the geometric approach that is still used today. This is where Hopf comes in. Hopf had a vision of fluid flow as a vector in a infinite dimensional phase space, with viscosity forcing the phase space trajectory of the flow to lie in some finite manifold in the long term Hopf (1948). Hopf provides, as an example. the case where the viscosity μ is very large, in which case the manifold shrinks to a point corresponding to laminar flow, and speculates that as the viscosity decreases, new manifolds should arise from bifurcations, envisioning a maze of recurrent manifolds springing forth from the aether. Sadly for Hopf, the first electronic computer, ENIAC, had been built just two years earlier, the first silicon transistor was six years in the future, and high performance computing still decades away. With the resources available to him at the time, a numerical simulation of this phase space was impossible (to say nothing of analytic solutions), and Hopf had to remain satisfied with applying his ideas to approximations of the Navier-Stokes equations.

0.2 Computers and the Future

With the advent of modern computing however, numerical simulation of the phase space topology is within reach. In plane Couette flow (described in more detail in Section), cartographic efforts began with Nagata's demonstration of the existence of finite-amplitude turbulent perturbations from mean flow that were nevertheless equilibrium solutions in 1990 Nagata (1990) and Kawahara and Kida's determination of *periodic* turbulent perturbations from mean flow Kawahara & Kida (2001) in 2001, and Viswanath's calculation of *relative* periodic orbits in 2007 Viswanath (2007), which also introduced the numerical scheme that constitutes one of the core solvers used in this thesis. The development of the Channelflow software library by Gibson Gibson et al. (2008) Gibson (2014) is of particular note, as it has enabled the wider investigation of the phase space topology, and features heavily in this thesis. Indeed, the numerical schemes used within are formidable, and certainly beyond my ability to recreate within the thesis timescale, though I shall outline them in Section **??**. Given these tools, then, we can imagine that it may be possible to construct a web of periodic orbits, equilibria and their heteroclinc connections, and then predict to some accuracy the long-term dynamical behavior, based on transitions between these different states.

Chapter 1

Equations of Flow

For every action, there is an equal
and opposite regulatory body

Anonymous

1.1 Formalisms

At the heart of fluid dynamics lie the Navier-Stokes equation, first derived by George Stokes in 1845, after a series of refinements leading back to Newton. Now if we were considering a point particle, we would begin with Newton's Second Law -

$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{F}, \tag{1.1}$$

write down the body force as a function of position, time, etc., and have our differential equation. With fluids, however, the treatment becomes somewhat more subtle - we are more concerned with the time-rate of deformation of the fluid particle than we are with the actual deformation - this being the difference between the Eulerian and Lagrangian descriptions of motion. Classical mechanics is typically framed in the Lagrangian context, so we will take a step back to develop the Eulerian context further.

1.1.1 The Control Volume

When asked to consider the mechanical evolution of some collection of bodies, two obvious methods would be readily apparent - we could either follow a particle (or collection of particles) on their merry way through space and time (the systems approach), or we could situate ourselves at some point in space, extend a 'bubble' around ourselves, and observe the properties of particles that enter and exit our bubble over time (the control volume approach). The systems approach will be familiar to anyone with a basic physics education, since it lends itself readily towards analysis of rigid-body motion. When considering fluids, however, the question of which fluid particle we should follow becomes non-trivial (Should we follow all? That be computationally difficult. Just one? Which one?) if we are using a systems approach, while the control volume approach remains as easy (or hard) as it was for rigid body motion. Historically, then, the control volume approach, also referred to as an **Eulerian** approach has been used to describe fluid dynamics (though work has been done on Lagrangian fluid mechanics), and this thesis will follow historical precedent.

1.1.2 The Fluid Particle

"Well", you might say, "All this talk of control volumes is all fine and dandy, but how do you plan to describe the motion of each molecule? Isn't that what you mean when you referred to fluid particles?" The answer, dear reader, is given by the continuum hypothesis. As you may have guessed, describing the motion of each and every molecule of fluid would be absurd - there are approximately 10^{21} atoms per milliliter of water, with six degrees of freedom each - solving 10^{21} coupled equations doesn't sound pleasant, or feasible. Furthermore, even if we ere able to write down this set of equations, finding the initial conditions of the fluid would be impossible (how was molecule # 19364829008283716 moving at time $t = 0$?).

Instead, we consider a fictitious "particle" of fluid, large enough so that we can take an average of the externally measurable quantities within (pressure, temperature, velocity, energy, etc.), but small enough so that we can approximately consider all these (averaged) variables as continuous. Perhaps this vagueness bothers you - does such a fluid particle even exist? As an example, let us consider water, with 10^{28} atoms per cubic meter. Imagine our fluid particles as cubes filling up space, with sides of length dl, giving a total volume of dl^3. First, let us make dl small enough that the external variables appear continuous - how about one micron? That gives the volume of a fluid particle as one cubic micrometer. For scale, consider that the reference volume of the human red blood cell ranges from 80-100 cubic micrometersFischer & Fischer (1983) - this seems acceptably small. The number of water molecules within each fluid particle is then

$$10^{28} dl^3 = 10^{28} \times 10^{-15} = 10^{13}, \tag{1.2}$$

or about 10 trillion water molecules, which is certainly sufficient to achieve a meaningful average. Having defined a fluid particle in this way allows us to behave as if these external variables have well defined values at every point in space, which greatly simplifies the following analysis.

1.2 Mass Conservation

While not technically a part of the Navier-Stokes equation (which is a statement about conservation of linear momentum), conservation of mass is nevertheless essential in solving fluid problems, and will serve as an easy demonstration of the control volume principle. Consider a volume Ω which is fixed in space, and has some mass density

$\rho = \rho(\mathbf{x}, t)$ and some fluid velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ that are generically functions of time and space, allowing us to define the **mass current density** $\mathbf{m} = \mathbf{v}\rho$. We would prefer that our equations do not allow mass to disappear (excluding high-energy physics, naturally), and would additionally prefer a mathematical form of this statement.

The mass contained within the volume is given by

$$M = \int_{\Omega} \rho \ dV, \tag{1.3}$$

the flow of mass out of the volume through the surface $d\Omega$ of Ω is given by

$$M_{flow} = \int_{d\Omega} \mathbf{m} \cdot \mathbf{n} \ dA = \int_{\Omega} \nabla \cdot (\rho \mathbf{v}) \ \ dV, \tag{1.4}$$

by the divergence theorem . Now if mass is conserved, the sum of the rate of mass flow into (or out of) the volume and the rate of change of mass inside the volume must be zero, giving

$$\frac{\partial M}{\partial t} + M_{flow} = 0, \tag{1.5}$$

$$\frac{\partial}{\partial t}\left(\int_{\Omega} \rho \ dV\right) + \int_{\Omega} \nabla \cdot (\rho \mathbf{v}) \ \ dV = 0, \tag{1.6}$$

but since V is time independent, the time derivative commutes with the integral, giving

$$\int_{\Omega} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \ \ dV = 0, \tag{1.7}$$

but since Ω is arbitrary, the integrand must be zero everywhere, giving the statement of conservation of mass in differential form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1.8}$$

Other conservation laws can be written in a similar way; the generic form for a conserved quantity Φ with density ϕ and current ψ is

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \psi) = 0. \tag{1.9}$$

Now, Equation 1.8 can be expanded further by using the chain rule for divergence, giving

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \nabla \rho = 0. \tag{1.10}$$

If the flow is (approximately) incompressible, which will be true for small Mach numbers[1], then ρ must be constant, and Equation 1.10 becomes

$$\nabla \cdot \mathbf{v} = 0, \tag{1.11}$$

for both steady and unsteady flows (Steady, compressible flows would only drop the first term from Equation 1.8).

[1]The Mach number is the ratio of the fluid velocity to the speed of sound in the fluid. v_{sound} for water is 1497 ms^{-1} at room temperature and pressure.

1.3 Conservation of Linear Momentum

As mentioned earlier, the Navier-Stokes equations are simply a statement of conservation of linear momentum, along with certain assumptions about stress (an object that contains information about forces) and strain (an object that contains information about deformation), which are presented below.

1.3.1 Stress

For a control volume Ω with boundary $d\Omega$, there are in general three ways in which momentum can be change over time in Ω by transport through $d\Omega$ (i.e., forces on $d\Omega$):

1. Bulk, 'convective' flow across $d\Omega$
2. Surface-normal transfer through elastic collisions between molecules. This is the microscopic origin of pressure.
3. Transfer through stochastic motion of molecules through $d\Omega$, as in Figure **??**. Since it is stochastic, time-averaged mass does not change, but momentum can still be transferred. THis leads to viscous stresses, and can be both normal and tangential (shear).

We define positive stress as stress that acts towards the control volume, and negative if they act away. Now, a stress on a fluid volume is not quite a vector, like force. Not only does it have a magnitude and direction, but it also has a plane that it acts from. Since there are three directions and three planes of action, stress objects generally have nine elements, and is a **second rank tensor**. That is, the viscous stress tensor $\mathcal{T}$ is identified by two subscripts, where the first subscript indicates the plane of action, and the second the direction of action. So $\mathcal{T}_{xy}$ would represent the viscous force on the (y, z) plane acting in the y direction. Note than in a Cartesian coordinate system, a second rank tensor can be written as a matrix.

1.3.2 Strain

Now that we can consider the forces on a fluid particle, we need to link these forces back to our external variables. In solids, this is easy - Hooke's Law, for instance, sets the strain proportional to the stress:

$$\sigma = \mathcal{C}\epsilon, \tag{1.12}$$

where σ is the Cauchy stress tensor, $\mathcal{C}$ is the (fourth order) stiffness tensor and ϵ is the infinitesimal strain tensor. However, for fluids, this is not the case - you can imagine that if you applied a constant force to a cube of water, it would deform continuously, without offering any resistance. Newton theorized that for continuously deformable fluids, the 1-D relationship between stress $\mathcal{T}$ and strain $\mathcal{S}$ should have the following form:

$$\mu \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} = \mu \frac{\mathrm{d}u}{\mathrm{d}x} = \mathcal{T}, \tag{1.13}$$

where μ is the viscosity and u is the velocity. Stokes extended this to three dimensions, giving the Newtonian constitutive relationship between stress and strain (for an incompressible fluid):

$$\mathcal{T}_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.14}$$

where δ_{ij} is the Kronecker delta function. Water, and most gases under normal conditions are Newtonian, but fluids like blood, quicksand and corn starch (to name a few) are not.

1.3.3 Surface Forces

Having written down the stress tensor $\mathcal{T}$ as a function of the velocity field, we now link it to the surface forces on a fluid particle. Recalling that stresses act over $d\Omega$ of the fluid particle, the total force is then simply

$$\mathbf{F} = \int_{d\Omega} \mathcal{T} \cdot \mathbf{n} \ dA, \tag{1.15}$$

where $\mathbf{n}$ is the surface normal.

1.3.4 Newton's Second Law

Newton's second law can be restated in a more useful form - assuming that mass is conserved,

$$\sum \mathbf{F} = M\mathbf{a}, \tag{1.16}$$

where the sum is over all possible external forces. For a fluid particle, we have two kinds of forces - body forces, like gravity or electromagnetism, and surface forces due to stress. We group the body forces $\mathbf{F}_b$ as

$$\mathbf{F}_b = \int_{\Omega} \rho\mathbf{f} \ dV, \tag{1.17}$$

where $\mathbf{f}$ is the **body force density**. Using Equation 1.15 to express the surface forces, Newton's Second Law becomes

$$\int_{\Omega} \rho\left(\mathbf{f} - \mathbf{a}\right) \ dV + \int_{d\Omega} \mathcal{T} \cdot \mathbf{n} \ dA = 0, \tag{1.18}$$

which can be written in differential form by the same trick used to generate Equation **??**, giving Cauchy's Equation of Motion

$$\rho\left(\mathbf{f} - \mathbf{a}\right) + \nabla \cdot \mathcal{T} = 0. \tag{1.19}$$

From this, the Navier-Stokes equation arise by a substitution of Equation 1.14 into Equation 1.19, giving (after tedious rearrangement by components),

$$\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{V} \cdot \nabla\right)\mathbf{V} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{V} \tag{1.20}$$

1.4 Plane Couette Flow

1.4.1 The Laminar Version

While Equation 1.20 is always correct (for Newtonian fluids), it is not always useful. Equation 1.20 is highly nonlinear, and difficult to solve analytically, outside of highly idealized scenarios[2]. These idealized scenarios, however, can be useful as an approximate model to real phenomena. We have already made such an approximation in deriving Equation 1.20, when we assumed that the fluid was incompressible.

One such idealized flow is called **plane Couette flow**, and is the focus of this thesis. In plane Couette flow, we model the (steady) incompressible flow between two infinite parallel plates, situated at $y = \pm h$, which can move in their own plane with some constant velocity. We can simplify the analysis by moving into the inertial reference frame of one of the plates, and re-orienting the axis so that one of the plates (say, the lower) is at rest, and the upper plate moves with a constant velocity along one coordinate axis (the 'streamwise' x-axis), with the y-axis as the 'wall-normal' coordinate, and the z-axis as the 'spanwise' coordinate. We then assume that the flow is purely streamwise (that is, $\mathbf{V} = u(x, y, z)\hat{\mathbf{x}}$), and that there are no pressure gradients in the fluid. Then, by symmetry, $u(x, y, z) = u(y)$, which reduces the Navier-Stokes equations to

$$\frac{\partial^2 u}{\partial y^2} = 0, \tag{1.21}$$

with no-slip boundary conditions

$$u(0) = 0, \tag{1.22}$$

$$u(h) = V_h, \tag{1.23}$$

which gives the laminar flow profile

$$u(y) = V_h \frac{y}{h}. \tag{1.24}$$

Consider then a small, potentially unsteady perturbation $\mathbf{v}(x, y, z, t)$ from this laminar state, so that the initial field is $\mathbf{u}(x, y, z, t) = \mathbf{v}(x, y, z, t) + y\hat{\mathbf{x}}$. Does this perturbation grow? Does it shrink? Does it oscillate periodically, or does it veer off chaotically? How can we characterize the state space of $\mathbf{v}$? This is the principle question of the work that this thesis is a part of.

1.4.2 Non-Dimensionalization

We are almost ready to model the system, but we must first address scale-invariance. The problem is as follows - there are currently 3 parameters in the model: the plate

[2] In fact, there is no proof that there are smooth solutions for all initial conditions to the Navier-Stokes equation

velocity V_h, the plate separation $2h$, and the viscosity $\nu = \dfrac{\mu}{\rho}$. Suppose we were to try to experimentally verify the results of a computational survey, but had to use a different plate velocity due to physical constraints - how could we be sure that the results of the experiment would correctly match the computational survey? The answer is that we non-dimensionalize all the dynamical variables using our parameters as a typical scale. If we do this, we find that our parameters have also shrunk, and we are left with just one - the Reynolds number

$$Re = \frac{V_h h}{\nu}. \tag{1.25}$$

1.4.3 Turbulent Perturbations

Using the perturbed, unsteady velocity field $\mathbf{u}(x, y, z, t) = \mathbf{v}(x, y, z, t) + y\hat{\mathbf{x}}$ in the non-dimensionalized Equation 1.20, we get

$$\frac{\partial \mathbf{v}}{\partial t} + y\frac{\partial \mathbf{v}}{\partial x} + v\hat{\mathbf{x}} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v}, \tag{1.26}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.27}$$

with boundary conditions

$$\mathbf{v}(x, \pm 1, z, t) = 0. \tag{1.28}$$

A note on nomenclature - I will refer to the unsteady perturbation as the 'velocity', and the full velocity as the 'total velocity' in the interests of brevity.

Chapter 2

Symmetry in plane Couette flow

Tyger! Tyger! burning bright,
In the forests of the night.
What immortal hand or eye,
Could frame thy fearful symmetry?

William Blake, *The Tyger*

Dynamical systems in physics often display symmetry. The electron wavefunction in hydrogen, or the gravitational motion of a planet around a star, for instance, display very high degrees of spatial symmetry. Understanding the symmetries of a system can be incredibly useful to an investigator, since they hint at conserved physical quantities (through Noether's Theorem), and can greatly reduce the complexity of the system in various ways[1]. Before I begin the discussion of the symmetries of plane Couette flow, I will first define what 'symmetry' means in this thesis. A system is said to be **equivariant** under a symmetric transformation of the dynamical system if a transformation that commutes with the time evolution of the system - that is, for a symmetric transformation S and a dynamical system $\dot{x} = f(x)$, $S\dot{x} = Sf(x) = f(Sx)$ implies that the system is equivariant under S.

The symmetry relations of plane Couette flowhave been derived extensively in GIBSON et al. (2009), which I will present here for the sake of flow[2].

2.1 Unbounded Navier-Stokes

If we do not impose boundary conditions on the Navier-Stokes equations on an infinite domain, the system will be equivariant under continuous rotational and translational symmetry, as well as the discrete **pointwise inversion** symmetryσ_{xz}, which has the following action on the system:

[1]In quantum mechanics, for instance, symmetries in the Hamiltonian imply the existence of operators that commute with it, which can allow one to greatly simplify the Hamiltonian matrix via simultaneous diagonalization for computational problems

[2]haha

$$\sigma_{xz}\mathbf{u}(\mathbf{x}) = -\mathbf{u}(-\mathbf{x}) \tag{2.1}$$

While the rotation or translation transformation can be easily conceptualized, the pointwise inversion can provide some difficulty. The easiest way of visualizing the transformation is to view it in a 2D domain instead of in the full 3D, as shown in Figure **??**. The proof of the equivariance of these transformations can be found in **?**.

2.2 Plane Couette Flow

If the domain is limited to $\mathbb{R}^2 \times [-1, 1]$ with the boundary conditions of plane Couette flow, we lose some of the equivariant transformations of the full, unrestricted problem, leaving us with two basic discrete symmetries: a rotation by pi about the z axis (denoted σ_x and a reflection about the z axis (denoted σ_z)[3], with together form a discrete symmetry group $D = D_1 \times D_1 = \{e, \sigma_x, \sigma_z, \sigma_{xz}\}$ of order 4, where

$$\sigma_x[u, v, w](x, y, z) = [-u, -v, w](-x, -y, z) \tag{2.2}$$
$$\sigma_z[u, v, w](x, y, z) = [u, v, -w](x, y, -z) \tag{2.3}$$
$$\sigma_{xz}[u, v, w](x, y, z) = [-u, -v, -w](-x, -y, -z) \tag{2.4}$$

The continuous symmetries are the two parameter steamwise-spanwise translations, which, when provided periodic boundary conditions, form a continuous $SO(2) \times SO(2)$ symmetry group

$$\tau(l_x, l_z)[u, v, w](x, y, z) = [u, v, w](x + l_x, y, z + l_z). \tag{2.5}$$

The group Γ of equivariant solutions is then any combination of these symmetry operations, given by $\Gamma = SO(2)_x \ltimes D_{1,x} \times SO(2)_z \ltimes D_{1,z}$[4]. For a solution $\mathbf{u}$ of plane Couette flow, the group s of symmetries that satisfies $s\mathbf{u} = \mathbf{u}$ is called the isotropy subgroup of $\mathbf{u}$ and is said to fix $\mathbf{u}$. Examples of such groups include the identity group $\{e\}$, which is typically the isotropy subgroup of turbulent solutions. Before I discuss the isotropy subgroup considered for this thesis, however, I will first highlight the useful properties of some particular symmetry subgroups to motivate the eventual choice of isotropy subgroup.

2.3 Properties of Γ

It should be evident that since plane Couette flowis equivariant under the continuous translations given in (2.5), trajectories can be traveling wave equilibria or relative periodic orbits: that is, if one moves into a different inertial frame, the trajectory is a regular equilibrium or periodic orbit. However, an initial condition that is fixed by σ_z

[3] The motivation for these subscripts will become apparent shortly

[4] $\ltimes$ is the semidirect product

cannot be translated in the spanwise direction without losing σ_z symmetry (except for the trivial case where $\frac{\partial \mathbf{u}_z}{\partial z} = 0$). Similarly, an initial condition that is fixed by σ_x cannot be translated in the streamwise direction without losing σ_x symmetry (and an initial condition that is fixed byσ_{xz} symmetry cannot be translated at all without losing σ_{xz} symmetry). Since these symmetries are also invariant[5], a trajectory with one of the discrete symmetries cannot have traveling waves in the direction corresponding to its subscript.

The presence of the periodic boundary conditions also implies that all solutions are fixed by the full-period translation $\tau(L_x, 0)$ and $\tau(0, L_z)$. However, solutions can also be fixed by any rational translation of the form $\tau(\frac{m}{n}L_x, \frac{m'}{n'}L_z$, or by the continuous translations. In the latter case, the velocity field is necessarily constant along the translation direction, but in the former, it implies that the periodic cell is subdivided into repeating subcells. In this case, we can simply reduce the domain to the subcell, which implies that we need not fix $\mathbf{u}$ under any translational symmetry other than the full period relation, which is required by the boundary conditions.

Finally, we can reduce the number of unique subgroups of Γ by considering its **conjugacy groups**. A group N and M are considered conjugate if for some $s \in \Gamma$, $N = s^{-1}Ms$ - that is, N and M are related by a coordinate transformation. This allows us to consider only one group out of a set of mutually conjugate groups (known as a conjugacy class), since any other group in the class is simply related by the application of a symmetry transformation. This becomes especially important when considering $O(2) = SO(2) \ltimes D_1$, since it is not an abelian group as reflections and translations about the same axis are noncomutative (Figure **??**). However, we can still recover a psuedo-commutative relation by considering Figure **??**. We can see that $\sigma_z\tau_z$ results in the object moving by l_z to the right, and the being mirrored across the z axis, at which point it is mirrored and l_z to the left of the origin. We can achieve the same effect by mirroring the object and moving it to the left - that is, applying the operation $\tau_z^{-1}\sigma_z$, leading us to conclude that $\sigma\tau = \tau^{-1}\sigma$. We can rewrite this as $\sigma\tau^2 = \tau^{-1}\sigma\tau$, so

$$\sigma_x\tau(l_x, 0) = \tau^{-1}(l_x/2, 0)\sigma_x\tau(l_x/2, 0), \tag{2.6}$$

which implies that σ_x and $\sigma_x\tau_x$ are part of the same conjugacy class - so if a isotropy group contains $\sigma_x\tau_x$, there is a simpler version of that group which contains σ_x instead. Note that if $l_x = L_x$, then we have $\sigma_x = \tau_x^{-1}\sigma_X\tau_x$, so reflection and translations commute for any half-integer cell shifts. In this thesis, I will work with either half-cell or or null shifts. The group of half cell shifts is denoted C_2, so the group

$$G = D_{1,x} \times C_{2,x} \times D_{1,z} \times C_{2,z} \subset \Gamma \tag{2.7}$$

is an abelian group of order 16, containing both the isotropy groups of half-cell and null shifts.

[5]That is, if the symmetry is satisfied at time $t = t_0$, it must be satisfied for all times.

2.4 Symmetry Groups of this Thesis

We can categorize the conjugacy classes of G by their order - in this thesis, we will work with order 2 and 4 classes. There are 15 subgrups of order-2 (since there are 15 non-identity elements in G), 35 subgroups of order 4 ($(15 \cdot 14)/(3 \cdot 2)$), 15 subgroups of order 8 and 1 subgroup of order 16, giving 67 subgroups of G. Luckily, the existence of conjugacy classes allow us to greatly simplify the number of distinct groups we need to consider. For order-2 subgroups, conjugacy between σ_x and $\sigma_x\tau_x$ and $\sigma_z\tau_z$ allows us to simplify down to just 8 distinct groups, which are generated by $\sigma_x, \sigma_z, \sigma_{xz}, \sigma_x\tau_z, \sigma_z\tau_x, \tau_x, \tau_z, \tau_{xz}$. Recalling the behavior of these symmetries, we can see that only σ_{xy} generates a group without travelling waves, σ_z and $\sigma_z\tau_x$ generate groups that allow travelling waves in the x direction, $\sigma_x\tau_z$ and $\sigma_x\tau_z$ generate groups that allow travelling waves in the z direction, and the pure translations allow travelling waves in any (in-plane) direction.

References

Fischer, S. L., & Fischer, S. P. (1983). Mean corpuscular volume. *Archives of Internal Medicine*, *143*(2), 282–283.

Gibson, J. F. (2014). Channelflow: A spectral Navier-Stokes simulator in C++. Tech. rep., U. New Hampshire. `Channelflow.org`.

GIBSON, J. F., HALCROW, J., & CVITANOVI, P. (2009). Equilibrium and travelling-wave solutions of plane couette flow. *Journal of Fluid Mechanics*, *638*, 243–266. `http://journals.cambridge.org/article_S0022112009990863`

Gibson, J. F., Halcrow, J., & Cvitanović, P. (2008). Visualizing the geometry of state space in plane couette flow. *Journal of Fluid Mechanics*, *611*, 107–130.

Hopf, E. (1948). A mathematical example displaying features of turbulence. *Commun. Pure Appl. Math.*, *1*(4), 303–322.

Karman, T. v. (1930). Mechanische aenlichkeit und turbulenz. *Nachrichten von der Gesellschaft der Wissenschaften zu Gttingen, Mathematisch-Physikalische Klasse*, *1930*, 58–76. `http://eudml.org/doc/59299`

Kawahara, G., & Kida, S. (2001). Periodic motion embedded in plane couette turbulence: regeneration cycle and burst. *J. Fluid Mech.*, *449*, 291–300.

Kolmogorov, A. N. (1991). The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, *434*(1890), 9–13.

Nagata, M. (1990). Three-dimensional finite-amplitude solutions in plane couette flow: bifurcation from infinity. *J. Fluid Mech.*, *217*, 519–527.

Viswanath, D. (2007). Recurrent motions within plane couette turbulence. *J. Fluid Mech.*, *580*, 339–358.