A model with the dependency of the quantum decoherence rate on the space metrics

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1 Introduction

A quantum scalar field in curved space coupled weakly to a simple two-state quantum system is considered. The total Hamiltonian is

$$\hat{H} = \frac{1}{2}\varepsilon\hat{\sigma}_z + \hat{H}_{\text{field}} + \hat{V}$$

$\hat{V}$ denotes an "interaction part":

$$\hat{V} = \int d\vec{x}\vartheta(\vec{x})\hat{\phi}(x)$$

and $\hat{H}_{\text{field}}$ is a quantized field defined by the Lagrangian

$$L = \int d^N x\sqrt{-g}\left(g^{ij}\partial_j\phi\partial_i\phi - (m^2 + \frac{1}{6}R^2)\phi^2\right)$$

$\vartheta(\vec{x})$ is a coupling constant. Operator $\hat{\phi}(x)$ mentioned corresponds to the operator of the similar notation defined in the quantization procedure described in [1].

We consider the evolution of the reduced density matrix of the system with the field modes traced out.

$$\hat{\rho}_R(t) = \mathrm{Tr}_\phi \hat{\rho}(t)$$

Only nondiagonal terms of the reduced density matrix evolve with time. We obtain the general formula for the evolution

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\vec{x}\sqrt{-g}\int d\vec{x'}\sqrt{-g}\int_0^T dt\int_0^T dt'\vartheta(\vec{x})\vartheta(\vec{x'})G(t,t',\vec{x},\vec{x'})\right] \tag{1}$$

where $G(t,t',\vec{x},\vec{x})$ is the Green's function of the field defined by the Lagrangian under discussion, eq. 2.1. That is

$$G(t,t',\vec{x},\vec{x'}) = \langle \text{vacuum} \mid \hat{\phi}(x,t), \hat{\phi}(x',t') \mid \text{vacuum}\rangle$$

Then we consider a pair of specific simple metrics. We observe that the decoherence rate depends on our definition of "vacuum state" of the field.

Let us assume we have quantization in one time-invariant coordinate frame ($g^{ij}_{,0} \equiv 0$) and defined the vacuum as $a \mid \text{vacuum}\rangle = 0$ where a is an annihilation operator in that frame in accordance with the notation in [1]. Let us assume we did quantization of the field in the qubit's rest frame. Then the decoherence rate is defined by the expression

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\omega\vartheta(\omega)|\frac{d\vec{k}}{d\omega}|\left(\frac{1}{2}|\upsilon'_{\vec{k}}(t)u_{\vec{k}}(t) - \upsilon_{\vec{k}}(t)u'_{\vec{k}}(t)|^2 + 1\right)\frac{\sin^2(\omega_{\vec{k}}T)}{\omega^2_{\vec{k}}}\right] \tag{2}$$

Where $\{\vec{k}\}$ is a set of the states that orthogonalize the Lagrangian in the space (again, just as in [1]) and $\upsilon_{\vec{k}}(t)$ is a normalized classical solution employed for quantizing the rest frame field and $u_{\vec{k}}(t)$ is the one for the vacuum definition frame.

2 The model

2.1 Statement of the problem considered

As from the layman's perception, one might speculate that the typical physics department as the institution consists of two groups of people with only two serious problems in life(besides the paperwork). West wing, say, full of people main hardship for whom is the extreme fragility of quantum systems to the collapse due to the unavoidable interaction with the environment. On the opposite side of the floor there are the "general relativity" people and their main daily woe seems to be the extreme weakness of the gravitational interaction. That weakness makes the study of the properties of the gravitational interaction by controllable experiments practically impossible and thus impedes advancement into the study of the interaction's nature. Thus it seems to be worth a try to attempt to discover the relationship between the two such "weaknesses" with the hope that they could cancel each other in some effects. One of such layman's attempts is presented to the reader's attention.

Let us formulate the problem for the purposes of this essay in the following manner. The goal is to consider the simplest possible case such that the interaction between the quantum system and the environment causes the decoherence of the system's quantum state and to study the dependence of the decoherence on the curvature of the metrics of the environment.

The appropriate "toy" system for such consideration to the best knowledge of the author is the generalized version of "independent boson system" [2].

$$\hat{H} = \frac{1}{2}\varepsilon\hat{\sigma}_z + \sum_{\vec{k}} \omega_{\vec{k}} b^{\dagger}_{\vec{k}} b_{\vec{k}} + \sum_{\vec{k}} \vartheta(\vec{k}) \frac{1}{2\sqrt{2\pi\omega}} \hat{\sigma}_z \left(b^{\dagger}_{\vec{k}} + b_{\vec{k}} \right)$$

Here the creation-annihilation operators $b^{\dagger}_{\vec{k}}, b_{\vec{k}}$ refer to the quantized mode $\vec{k}$ of the scalar field in the curved space. The Lagrangian is taken to be the one that is referred to as the conformal scalar field with the minimal coupling to the gravity

$$L = \int d^N x \sqrt{-g} \left(g^{ij} \partial_j \phi \partial_i \phi - \frac{1}{6} R^2 \phi \right)$$

We would as well discuss the one with the minimal coupling to the gravity

$$L = \int d^N x \sqrt{-g} \left(g^{ij} \partial_j \phi \partial_i \phi - \frac{1}{6} m^2 \phi \right)$$

I will refer to $\hat{V}$ as the interaction part of the system

$$\hat{V} = \sum_{\vec{k}} \vartheta(\vec{k}) \frac{1}{2\sqrt{2\pi\omega}} \hat{\sigma}_z \left(b^{\dagger}_{\vec{k}} + b_{\vec{k}} \right)$$

One may see that we generalized the interaction part to the space-time invariant case bearing in mind the generalized solution of the quantized scalar field

$$\hat{\phi(t)} = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{\omega}} \left[u(\vec{k}, \omega, t) b^{+} + \upsilon(\vec{k}, \omega, t) b \right]$$

One should emphasize here that $\vec{k}$ mode doesn't denote plain wave solutions, these are just orthogonal solutions for the Lagrangian with the given metric. The density matrix of the initial state is taken to be

$$\hat{\rho}(t=0) = \mid + \rangle \langle + \mid \otimes \mid \text{vacuum} \rangle \langle \text{vacuum} \mid$$

We consider the evolution of the reduced density matrix of the system with the field modes traced out.

$$\hat{\rho}_R(t) = \mathrm{Tr}_{\phi} \hat{\rho}(t)$$

The interaction part of the Hamiltonian commutes with the qubit part so we might expect that only the nondiagonal terms of the reduced density matrix are changing, that is only the decoherence occurs and no relaxation. That is, we are interested in the evolution of the term

$$\rho_{\uparrow\downarrow}(t) = \mathrm{Tr}_{k,N} \langle \uparrow, k, N \mid \hat{U}^{+}(t)\hat{\rho}(t=0)\hat{U}(t) \mid \downarrow, k, N\rangle \tag{3}$$

Here we introduced the notation of the quantum state $|\downarrow, k, N\rangle$ with N denoting the occupancy number of the $\vec{k}$ orthogonal Fock space. $U(t)$ is evolution operator.

2.2 Formal solution

Let us find a general formal expression for the evolution of the term 3 without specifying particular space metric. We use interaction representation so

$$\hat{U}(t) = T \exp\left[-\frac{i}{\hbar}\int_0^t V(\tau)d\tau\right]$$

We can show that the time-ordering operator can be dropped. All we need for that is to know that there is some covariant quantization of the field ([1] ,p.45) in some coordinate frame.

$$i\partial_t a^\dagger = [a^\dagger, H_0] = \omega_{\vec{k}}[a^\dagger, a^\dagger a] = -\omega_{\vec{k}} a^\dagger \tag{4}$$

$$a^\dagger(t) = e^{i\omega_{\vec{k}} t} a^\dagger(0) \tag{5}$$

$$a(t) = e^{-i\omega_{\vec{k}} t} a(0) \tag{6}$$

In interaction representation its evolution is

$$V(t) = \sum_{\vec{k}} \hat{\sigma}_z \left(\vartheta(\vec{k}) e^{i\omega_{\vec{k}} t} b^\dagger_{\vec{k}} + g^\star_{\vec{k}} e^{-i\omega_{\vec{k}} t} b_{\vec{k}}\right) \tag{7}$$

$$[V(t), V(0)] = 2i\sum_{\vec{k}} |\vartheta(\vec{k})|^2 \sin(\omega_{\vec{k}} t) \tag{8}$$

Now that we see that commutator of interaction part of hamiltonian with itself at different times is just a number, Time-ordering operation can be dropped due to the following general statement

$$T: \hat{H}_1\hat{H}_2 = \hat{H}_1\hat{H}_2 \tag{9}$$

$$T: \hat{H}_2\hat{H}_1 = \hat{H}_1\hat{H}_2 \tag{10}$$

$$T: \exp(\hat{H}_1 + \hat{H}_2) = \exp(\frac{1}{2}[\hat{H}_1, \hat{H}_2]) T: \exp(\hat{H}_1)\exp(\hat{H}_2) = \tag{11}$$

$$= \exp(\frac{1}{2}[\hat{H}_1, \hat{H}_2])\exp(\hat{H}_1)\exp(\hat{H}_2) = \exp(\hat{H}_1 + \hat{H}_2) \tag{12}$$

We just split the integral within the exponents as many times as we need to and see that the statement holds true. Now evolution operator is

$$\hat{U} = \exp\left[-\frac{i}{\hbar}\int_0^t V(\tau)d\tau\right] = \exp\left[-\frac{i}{\hbar}\int dk\int_0^t dt g_k\hat{\phi}(\vec{k},t)\right] = \tag{13}$$

$$= \exp\left(\sum_k \frac{1}{\hbar\omega_{\vec{k}}}\hat{\sigma}_z\left[\vartheta(\vec{k})(1-e^{i\omega_{\vec{k}}t})\hat{b}^\dagger_{\vec{k}} - g^\star_{\vec{k}}(1-e^{-i\omega_{\vec{k}}t})\hat{b}^\dagger_{\vec{k}}\right]\right) \tag{14}$$

We are going to employ several properties of the evolution operator of the form presented. In the quantum optics literature they refer to such operators as "Displacement operators". Let us introduce the notation of the following function to which we will refer as displacement operator

$$D(\xi) = \exp(\xi\hat{b}^\dagger - \xi^*\hat{b})$$

Let us enlist the following evident properties of the displacement operator that we are going to employ

$$D(\xi_1)D(\xi_2) = \exp\left(\frac{\xi_1\xi_2^\star - \xi_1^\star\xi_2}{2}\right)D(\xi_1+\xi_2) \tag{15}$$

$$\text{in particular, } D(\xi)D(\xi) = D(2\xi) \text{ and } D(\xi)D(-\xi) = 1 \tag{16}$$

$$D^\dagger(\xi) = D(-\xi) \tag{17}$$

And after some trivial transformations we arrive at the following formal expression for 3

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \langle\text{vacuum} \mid \exp\left[-\frac{i}{\hbar}\int dk\int_0^t dt 2\vartheta_k\hat{\phi}(\vec{k},t)\right] \mid \text{vacuum}\rangle$$

(We are going to neglect phase change for the final answer as it isn't what we are interested in)

This expression can be reformulated in the free-path integral terms

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \frac{\int D\phi(x,t)\exp\left[\frac{i}{\hbar}\int d^N x\sqrt{-g}\left(g^{ij}\partial_j\phi\partial_i\phi - m^2\phi\right) - \frac{i}{\hbar}\int dk\int_0^t dt 2\vartheta_k\hat{\phi}(\vec{k},t)\right]}{\int D\phi(x,t)\exp\left[\frac{i}{\hbar}\int d^N x\sqrt{-g}\left(g^{ij}\partial_j\phi\partial_i\phi - m^2\phi\right)\right]}$$

By taking the integral, we obtain

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\vec{x}\sqrt{-g}\int d\vec{x'}\sqrt{-g}\int_0^T dt\int_0^T dt'\vartheta(x)\vartheta(x')G(t,t',\vec{x},\vec{x'})\right] \tag{18}$$

where $G(t,t',\vec{x},\vec{x}^{)}$ is the Green's function of the field defined by the Lagrangian under discussion, eq. 2.1. That is

$$G(t,t',\vec{x},\vec{x'}) = \langle\text{vacuum} \mid \hat{\phi}(x,t),\hat{\phi}(x',t') \mid \text{vacuum}\rangle$$

or

$$\left\{\partial_\alpha\left[\sqrt{-g}g^{\alpha\beta}\right]\partial_\beta+m^2\right\}G(t,t',\vec{x},\vec{x'})=-i\delta(t-t',x-x')$$

This result can be as well be obtained by a number of more down-to-the earth methods, the most explicit one being writing down the Tailor series for an exponent, applying Wick's theorem, and comparing with the expansion of our expression.

3 Particular Metrics

3.1 Minkowski Space

For the Minkowski space, we could employ the traditional Second Quantization of the Fourier Transformation representation, that is we employ the solution

$$\hat{\phi}(\vec{k},t)=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{\sqrt{\omega}}\left[e^{ipx}b^+_{\vec{k}}+e^{-ipx}b_{\vec{k}}\right]$$

and for the free field we just have

$$G(t,t',\vec{k},\vec{k'})=G(t-t',\vec{k},\vec{k'})=\langle\text{vacuum}\mid\hat{\phi}(\vec{k},t-t'),\hat{\phi}(\vec{k}',0)\mid\text{vacuum}\rangle=\delta(\vec{k}-\vec{k'})e^{-i\omega_{\vec{k}}(t-t')}$$

So for the exponential factor we have only two integrals to take

$$\int_0^T dt\int_0^T dt'e^{-i\omega_{\vec{k}}(t-t')}=\frac{2i\sin^2(\omega T)}{\omega^2}$$

We see that in that case the expression 18 comes down to

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)}=\exp\left[-4\int d\vec{k}\vartheta^2(\vec{k})\frac{\sin^2(\omega_{\vec{k}}T)}{\omega^2_{\vec{k}}}\right]$$

The expression is similar to the one obtained for non-relativistic case [2] using slightly different approach.

3.2 All the time-homogenious metrics

Expression for the Minkowski space-time is an example of the general form of the solution that we always have when metric doesn't depend on time explicitly ($g^{ij}_{,0}\equiv 0$). In this case we would always have

$$G(t,t',\vec{k},\vec{k}')=G(t-t',\vec{x},\vec{x}')=\delta_{\vec{k},\vec{k'}}e^{-i\omega_{\vec{k}}t}\langle\text{vacuum}\mid a_{\vec{k}}a^\dagger_{\vec{k}}\mid\text{vacuum}\rangle+ \quad (19)$$

$$+\delta_{\vec{k},\vec{k'}}e^{+i\omega_{\vec{k}}t}\langle\text{vacuum}\mid a^\dagger_{\vec{k}}a_{\vec{k}}\mid\text{vacuum}\rangle \quad (20)$$

After the Fourier Tranformation by t we have

$$G(\omega,\vec{k},\vec{k}')=\int dte^{i\omega t}G(t-t',\vec{x},\vec{x}')=\delta_{\vec{k},\vec{k'}}\delta(\omega-\omega_{\vec{k}})\langle\text{vacuum}\mid a_{\vec{k}}a^\dagger_{\vec{k}}\mid\text{vacuum}\rangle+ \quad (21)$$

$$+\delta_{\vec{k},\vec{k'}}\delta(\omega+\omega_{\vec{k}})\langle\text{vacuum}\mid a^\dagger_{\vec{k}}a_{\vec{k}}\mid\text{vacuum}\rangle \quad (22)$$

And the formal solution 18 now looks neat and nice (we skipped several trivial transformations)

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\omega\vartheta(\omega)|\frac{d\vec{k}}{d\omega}|\langle\text{vacuum} \mid a^{\dagger}_{\vec{k}}a_{\vec{k}} + a_{\vec{k}}a^{\dagger}_{\vec{k}} \mid \text{vacuum}\rangle\frac{\sin^2(\omega_{\vec{k}}T)}{\omega^2_{\vec{k}}}\right] \tag{23}$$

Assuming it is a canonical quantization and employing commutation relation once again ([1] ,p.45), we obtain

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\omega\vartheta(\omega)|\frac{d\vec{k}}{d\omega}|\left(2\langle\text{vacuum} \mid a^{\dagger}_{\vec{k}}a_{\vec{k}} \mid \text{vacuum}\rangle + 1\right)\frac{\sin^2(\omega_{\vec{k}}T)}{\omega^2_{\vec{k}}}\right] \tag{24}$$

Here $|\frac{d\vec{k}}{d\omega}|$ is the density of states within the range of energy $\omega \pm d\omega$.

3.3 Accelerated frame (Unruh effect)

Following the deriviation presented in [3], p.102, we calculate the observed number of particles in accelarated (Rindler) spacetime for the vacuum state of inertial flat frame of space-time (in the sense of the one with the lowest energy). Let us just repeat the final result

$$\langle\text{vacuum} \mid a^{\dagger}_{\vec{k}}a_{\vec{k}} \mid \text{vacuum}\rangle = \frac{1}{\exp\left(\frac{2\pi\Omega c}{a}\right) - 1}$$

And for the diagonal term we have

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\omega\vartheta(\omega)|\frac{d\vec{k}}{d\omega}|\text{cth}\left[\frac{\pi\omega c}{a}\right]\frac{\sin^2(\omega_{\vec{k}}T)}{\omega^2_{\vec{k}}}\right] \tag{25}$$

One might interpret this expression as the decoherence of qubit state isolated on the accelerating ship in the universe of the ϕ-field in lowest energy state (vacuum) in the Minkowski(rest) frame.

3.4 Bogolubov's transformations

In the [3] the calculation for the accelarated field is described in the Bogolubov's transformation terms. We might generalize our answer a little bit using its explicit form.

Let us assume we have second quantization operators for quantized field in the qubit's rest frame and we have defined vacuum, say, to be the state with the minimal energy in some other coordinate frame. Let us assume we have Bogolubov's transformation between the two kinds of operators of the form

$$b^{\dagger} = \alpha_k\hat{a}^{\dagger}_k + \beta_{\vec{k}}\hat{a}_{-\vec{k}}$$

The coefficients are defined by the equations set

$$\upsilon_{\vec{k}}(t) = \alpha_{\vec{k}}u^{\dagger}_{\vec{k}}(t) + \beta_{\vec{k}}u^{\dagger}_{\vec{k}}(t) \tag{26}$$

Then using trivial transformations, we see that the new vacuum can be defined in the frame of interest's operators as

$$|\text{vacuum}\rangle = \prod_{\vec{k}} \alpha_{\vec{k}} \exp\left(\frac{\beta_{\vec{k}}}{\alpha_{\vec{k}}} b^\dagger_{\vec{k}} b^\dagger_{-\vec{k}}\right)|0\rangle$$

And the expression for the decoherence rate now have a form

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\omega\vartheta(\omega)|\frac{d\vec{k}}{d\omega}|\left(2|\beta_{\vec{k}}|^2+1\right)\frac{\sin^2(\omega_{\vec{k}}T)}{\omega^2_{\vec{k}}}\right] \tag{27}$$

Or, solving equation 26 directly,

$$\frac{\rho_{\uparrow\downarrow}(t)}{\rho_{\uparrow\downarrow}(0)} = \exp\left[-\int d\omega\vartheta(\omega)|\frac{d\vec{k}}{d\omega}|\left(\frac{1}{2}|v'_{\vec{k}}(t)u_{\vec{k}}(t) - v_{\vec{k}}(t)u'_{\vec{k}}(t)|^2+1\right)\frac{\sin^2(\omega_{\vec{k}}T)}{\omega^2_{\vec{k}}}\right] \tag{28}$$

3.5 FRW Universe

Following the derivation presented in [3], p.64 we consider space-homogeneous and isotropic Friedmann-Robertson-Walker(FRW) space-time. The metric is

$$ds^2 = dt^2 - a^2(t)d\vec{r}^2$$

We introduce new variable

$$\eta(t) = \int_{t_0}^{t} \frac{dt}{a(t)}$$

And after the transformations we have the effective Lagrangian

$$L = \left(\frac{d\chi}{d\eta}\right)^2 - (\vec{\nabla}\chi)^2 - \left[m^2 a(\eta) - \frac{a''(\eta)}{a(\eta)}\right]\chi^2$$

We employ space homogeneity and consider the evolution of Fourier modes $\chi_{\vec{k}}$. Classical solutions for that Lagrangian are determined by the equation of the following form

$$\chi''_{\vec{k}} + \left[\vec{k}^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right]\chi_{\vec{k}} = 0$$

Let us denote the two linearly independent solutions as $x_1(\eta), x_2(\eta)$ and define $u(\eta)$ for 26 as $u(\eta) = x_1(\eta)+ix_2(\eta)$ Now we can employ our technique. The only thing is to decide what do we mean by vacuum here as it isn't a time-invariant metric and the energy doesn't conserve. We can make assumptions, define the socond quantization technique and employ $v(\eta)$ obtained.

4 Discussion and conclusion

For a qubit weakly coupled to the conformal quantum scalar field in the space-time with arbitrary conformally representable metrics we obtained explicit expression for the decoherence rate, and linked it with the field's correlation functions. Then we considered few simplest examples of metrics.

We have seen that for our problem the decoherence rate dependance on the metrics is defined by the quantum average of the total number of particles operator within the frame of qubit's rest.

vacuum definition. Once we chose the vacuum state we are interested in we and it was clearly shown that are defined by the

We could speculate even futher. Let us assume we could in some away tune coupling constant function $\vartheta(\vec{k})$. We want "the coupling area" of the qubit to be coarse-grained enough compared to the Compton Length. Or equivalently

$$\vartheta(\vec{k}) \neq 0 : \frac{1}{k} \gg \frac{\hbar}{mc}$$

In this case we can approximate $\vartheta(\vec{k}) = \delta(k - k_0)$. Let us assume we have an opportunity to measure the decoherence rate of the same prepared state in the two different frames of qubit rest with metrics g_A^{ij} and g_B^{ij}. In this case we have the following identity

$$\frac{\log \rho_{\uparrow\downarrow}^A(t)}{\log \rho_{\uparrow\downarrow}^B(t)} = 1 + |\beta_{\vec{k}}|^2$$

References

[1] P.C.W. Davies N.D. Birrell. Quantum fields in curved space. 1982.

[2] G. Massimo Palma, Kalle A. Suominen, and Artur K. Ekert. Quantum computers and dissipation. *Proceedings: Mathematical, Physical and Engineering Sciences*, 452(1946):567–584, 1996.

[3] S.Winitzki V.F. Mukhanov. Introduction to quantum fields in classical backgrounds. *Openly accessible draft.*