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An unbreakable limit

The Fisher information imposes a fundamental limit on the precision with which an unknown parameter can be estimated from noisy data, as Dorian Bouchet explains.

In a game of darts, the most basic strategy is hitting the bullseye as often as possible. The average precision, reflected by the typical area covered by the darts, is then intrinsically limited by the player's skill.

In statistical estimation theory, throwing a dart at the bullseye translates into precisely estimating the value of an unknown parameter from noisy data. In such a problem, the average estimation precision is intrinsically limited by the random fluctuations of the noise. For example, we can consider the task of estimating the unknown height of a coral reef using noisy sonar data. As the measured echoes randomly fluctuate owing to both electronic and environmental noise, the estimated values of the coral reef height also necessarily fluctuate.

The ultimate limit to the precision of such estimations is calculated by introducing a key quantity called the Fisher information, which describes, on average, how strongly the probability density function of the noisy data depends on the unknown parameter. The Fisher information is large when small parameter variations have a significant impact on the measured data.

What does the Fisher information tell us about our ability to precisely estimate the unknown parameter? The answer lies in the Cramér–Rao inequality, which applies to any unbiased estimation. The inequality is simple: the inverse of the Fisher information sets a lower bound on the variance of the estimated values of the unknown parameter. Therefore, the Fisher information can be interpreted as the amount of information that noisy data carry about an unknown parameter; the larger the Fisher information, the more precise the estimations can be.

The concept of Fisher information dates back to the work of Ronald Fisher. In 1925, he

introduced a formula to calculate the intrinsic accuracy of a curve as a means of estimating a parameter[1], which was exactly the quantity that we now call Fisher information. Fisher's derivation, however, was valid only for large statistical samples. In 1943, Calyampudi Radhakrishna Rao presented Fisher's formula during a lecture at Calcutta University. One of the students asked whether such a result could be established for finite samples. Rao worked all night, and the next day he proved an inequality valid for any sample size[2]. At the same time, Harald Cramér independently discovered the same inequality[3]. Thus, the Cramér–Rao inequality was born.

The Fisher information is unquestionably a cornerstone in statistical signal processing, with many engineering applications, including radar technology and image analysis. Moreover, the Fisher information has also emerged as a central quantity in physics. Indeed, when all classical sources of noise have been removed, only quantum noise fluctuations remain. The Cramér–Rao inequality then establishes a fundamental limit of nature — one cannot estimate the value of a parameter with absolute certainty.

The Fisher information notably plays an important role in biophysics for the development of super-resolution microscopes based on single fluorescent molecules. These microscopes are nowadays widely used in biological applications, in particular for their ability to reveal the inner structure of cells. In this context, the Fisher information is used to assess the ultimate precision with which single fluorescent molecules can be localized. This drives experimental progress for the optimal exploitation of each fluorescence photon, in order to push the image resolution down to the nanometre scale[4].

We can look at another example from quantum metrology, which aims at precisely estimating unknown parameters by exploiting quantum effects. By calculating the quantum Fisher information, which is a quantum analogue of the Fisher information, it is possible to identify optimal quantum protocols that yield the ultimate precision. As an illustration, a canonical problem in quantum metrology consists in estimating a phase shift from interferometric measurements; the quantum Fisher information then enables us to compare the precision achievable with different input states, such as coherent states and highly entangled states. Such results lead to enhanced performances in many research applications, including gravitational wave detection and atomic clock technology[5].

Many years have passed since the pioneering works of Fisher, Cramér and Rao. Undoubtedly, their contributions to the theory of statistics have led to many important insights, specifically regarding the identification of optimal designs for precision measurements. A captivating question remains: to what extent can these insights — and the broader concept of information — further enhance our fundamental understanding of the laws of physics?

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