LECTURE 7: ROBUST LINEAR OPTIMIZATION

1. Motivation
2. Robust model
3. Solution methods

Motivation

- Example: Pharmaceutic decision making
  (from Prof. Tom Luo)

- One active agent A for treating a disease.
- Two possible drugs: D_1, D_2
- Two possible raw material: R_1, R_2

Data sheet

	D_1	D_2
Selling Price ($/1K packs)	6,200	6,900
Agent A (grams/1K packs)	0.5	0.6
Man Power (hr/1K packs)	90	100
Equipment Usage (hr/1K packs)	40	50
Operating Cost ($/1K packs)	700	800

	Buying Price ($/kg)	Content of A (g/kg)
R_1	100	0.01
R_2	199	0.02

Budget ($)	Man Power (hr)	Equipment (hr)	Storage (kg)
100,000	2,000	800	1,000

LP formulation

$$\text{Max } 6200D_1 + 6900D_2 - 100R_1 - 199R_2 - 700D_1 - 800D_2$$

s. t.

$$\begin{aligned}
0.01R_1 + 0.02R_2 - 0.5D_1 - 0.6D_2 &\geq 0 \quad \text{(Balance of A)} \\
R_1 + R_2 &\leq 1000 \text{ (Storage)} \\
90D_1 + 100D_2 &\leq 2000 \text{ (ManPower)} \\
40D_1 + 50D_2 &\leq 800 \quad \text{(Equipment)} \\
100R_1 + 199R_2 + 700D_1 + 800D_2 &\leq 100000 \text{ (Budget)} \\
R_1,\ R_2,\ D_1,\ D_2 &\geq 0
\end{aligned}$$

<u>Optimal Solution:</u>

$$z^* = 9205.79$$

$$x^* \begin{cases} R_1^* = 0, & R_2^* = 438.79 \\ D_1^* = 17.55, & D_2^* = 0 \end{cases}$$

<u>Another Feasible Solution:</u>

$$\bar{z} = 8294.5$$

$$\bar{x} \begin{cases} \bar{R}_1 = 877.73, & \bar{R}_2 = 0 \\ \bar{D}_1 = 17.467, & \bar{D}_2 = 0 \end{cases}$$

Situation analysis

Error / Uncertainty in Data

	Content of Agent A (g/Kg)	
R_1	$0.01 \rightarrow \pm 0.5\%$	$[0.00995, 0.00105]$
R_2	$0.02 \rightarrow \pm 2\%$	$[0.00196, 0.00204]$

① x^* becomes infeasible.

② To keep the same plan

$$\left\{ \begin{array}{ll} R_1^* = 0, & R_2^* = 438.79 \\ & D_2^* = 0 \end{array} \right\} \text{ remains the same}$$

D_1^* has to be reduced to $D_1^* \times 0.98 = 17.201$

③ z^* is reduced from \$ 9205.79 to \$7286.29.

④ This means the profit is reduced by 21%.

⑤ $\bar{x}$ remains feasible with a profit of \$8294.5

This is a more "robust" solution!

Robust LP model

$$\begin{aligned} \min \quad & z = c^T x + d \\ s.t. \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Data:

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix}_{(m+1)\times(n+1)}$$

Dimension: (m, n)

Common Practice:

(m, n) are certain

data set $\begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$ typically uncertain

Source of Uncertainty:

- some entries may be missing
- measurement error
- prediction error
- quality statistics

$\vdots$

Simple form robust LP

- Define

$$
\begin{aligned}
[A_0 - \Delta A,\ A_0 + \Delta A] &= \mathfrak{A} \\
[b_0 - \Delta b,\ b_0 + \Delta b] &= \mathfrak{B} \\
[c_0 - \Delta c,\ c_0 + \Delta c] &= \mathfrak{C} \\
[d_0 - \Delta d,\ d_0 + \Delta d] &= \mathfrak{D}
\end{aligned}
$$

- Consider

$$
\text{(RLP)} \quad \min \{c^T x + d \mid Ax \le b,\ x \ge 0\} \quad
\begin{aligned}
A &\in \mathfrak{A} \\
b &\in \mathfrak{B} \\
c &\in \mathfrak{C} \\
d &\in \mathfrak{D}
\end{aligned}
$$

<u>Question</u>: What does this (RLP) mean mathematically?

Assumptions in decision making

(A1): x must be determined "here and now".

(A2): Decision maker is fully responsible for all consequences of data uncertainty.

$\Rightarrow$ x must be "robust feasible". i.e.,

$$\begin{array}{rcl} Ax & \leq & b \\ x & \geq & 0 \end{array} \quad \forall\, A \in \mathfrak{A} \;\; and\; b \in \mathfrak{B}$$

(A3): Conservative decision is adopted.

$$\Rightarrow \quad \underset{\substack{c \,\in\, \mathfrak{C} \\ d \,\in\, \mathfrak{D}}}{\text{Min}} \;\; \text{Max}\{c^T x + d \mid x \text{ is robust feasible}\}$$

Mathematics involved

① $A \in \mathfrak{A} \Leftrightarrow A = A_0 + t_{\mathfrak{a}} \Delta A,\ t_{\mathfrak{a}} \in [-1,\ 1]$

$b \in \mathfrak{B} \Leftrightarrow b = b_0 + t_{\mathfrak{b}} \Delta b,\ t_{\mathfrak{b}} \in [-1,\ 1]$

$c \in \mathfrak{C} \Leftrightarrow c = c_0 + t_{\mathfrak{c}} \Delta c,\ t_{\mathfrak{c}} \in [-1,\ 1]$

$d \in \mathfrak{D} \Leftrightarrow d = d_0 + t_{\mathfrak{d}} \Delta d,\ t_{\mathfrak{d}} \in [-1,\ 1]$

② x is robust feasible

$$\Leftrightarrow \begin{cases} Ax & \leq b \quad \forall A \in \mathfrak{A} \text{ and } b \in \mathfrak{B} \\ x & \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (A_0 + t_{\mathfrak{a}} \Delta A)x \leq b_0 + t_{\mathfrak{b}} \Delta b, & \forall t_{\mathfrak{a}} \in [-1,\ 1] \\ x \geq 0 & \forall t_{\mathfrak{b}} \in [-1,\ 1] \end{cases}$$

③ $$\underset{\substack{c \in \mathfrak{C} \\ d \in \mathfrak{D}}}{\text{Min}}\ \text{Max}\{c^T x + d \mid x \text{ is robust feasible}\}$$

$$\Leftrightarrow \quad \begin{aligned} &\min \quad y \\ &\text{s. t.} \quad (c_0 + t_{\mathfrak{c}} \Delta c)^T x + (d_0 + t_{\mathfrak{d}} \Delta d) \leq y, \\ &\qquad\qquad \forall\, t_{\mathfrak{c}} \in [-1,\ 1] \\ &\qquad\qquad t_{\mathfrak{d}} \in [-1,\ 1] \\ &\qquad x \text{ is robust feasible} \end{aligned}$$

Semi-infinite LP

(RLP) becomes

$$
\begin{array}{ll}
& \text{Min} \quad y \\
\text{s. t.} & \\
(RLP)_{SI} & \left\{
\begin{array}{r}
(c_0 + t_{\mathfrak{c}} \Delta c)^T x + (d_0 + t_{\mathfrak{d}} \Delta d) \;\leq\; y, \\
\forall\, t_{\mathfrak{c}} \in [-1,\ 1] \\
t_{\mathfrak{d}} \in [-1,\ 1] \\
(A_0 + t_{\mathfrak{a}} \Delta A)x \;\leq b_0 + t_{\mathfrak{b}} \Delta b, \qquad\qquad \\
\forall t_{\mathfrak{a}} \in [-1,\ 1] \\
t_{\mathfrak{b}} \in [-1,\ 1] \\
x \;\geq\; 0
\end{array}
\right.
\end{array}
$$

Properties of (RLP)SI

- It is a linear programming problem with $n+1$ variables and infinitely many constraints

- It is a <u>semi-infinite linear programming problem.</u>

<u>Question</u>: How to solve $(RLP)_{SI}$?

Solution method 1

① <u>Discretization method</u>

- Pick k points from $[-1,\ 1]^4$
- Use these points to construct a regular linear program
- As $k \to\ +\infty$, LP solutions approach a solution of $(RLP)_{SI}$

- Pros: Solved by LP
- Cons: An approximation solution

Solution method 2

② <u>Cutting Plane method</u>

- Consider

$$\text{Min } \sum_{j=1}^{n} c_j x_j$$

$$(LSIP) \quad s.t. \quad \sum_{j=1}^{n} f_j(t) x_j \leq g(t), \ \forall \, t \in T$$

$$x_j \geq 0, \ j = 1, \ldots, n$$

Step 1: Let $\varepsilon > 0$ be sufficiently small, K and M be sufficiently large.

Choose any $t^1 \in T$ and set $k = 1$, $T_1 = \{t^1\}$, $z^0 = M$.

Solution method 2

Step 2: Find an optimal solution x^k to

$$\text{Min } \sum_{j=1}^{n} c_j x_j$$

$$(LP_k) \quad \text{s.t. } \sum_{j=1}^{n} f_j(t^i) x_j \le g(t^i), \quad i = 1, \dots, k$$

$$x_j \ge 0.$$

Set $z^k = c^T x^k$.

Let $\Phi_{k+1}(t) \triangleq g(t) - \sum_{j=1}^{n} f_j(t) x_j^k, \; \forall\, t \in T.$

Find a minimizer t^{k+1} of $\Phi_{k+1}(t)$ over T and

calculate $\Phi_{k+1}(t^{k+1})$.

Step 3: If $\Phi_{k+1}(t^{k+1}) \ge 0$ or $|z^k - z^{k-1}| < \epsilon$ for $k > K$, then stop and output x^k as an optimal solution.

Otherwise, set

$T_{k+1} = T_k \cup \{t^{k+1}\}$ and $k \leftarrow k + 1$.

Go to Step 2.

Computational bottleneck

Find a minimizer t^{k+1} of $\Phi_{k+1}(t)$ over T

$$\Phi_{k+1}(t) \triangleq g(t) - \sum_{j=1}^{n} f_j(t) x_j^k, \ \forall\, t \in T.$$