LECTURE 5: DUALITY AND SENSITIVITY ANALYSIS

1. Dual linear program
2. Duality theory
3. Sensitivity analysis
4. Dual simplex method

Introduction to dual linear program

- Given a constraint matrix **A**, right hand side vector **b**, and cost vector **c**, we have a corresponding linear programming problem:

$$
\begin{array}{llll}
 & \text{Min} & \mathbf{c}^T\mathbf{x} & \\
(\text{LP}) & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & \text{Primal Problem} \\
 & & \mathbf{x} \geq 0 &
\end{array}
$$

- Questions:

1. Can we use the same data set of (**A**, **b**, **c**) to construct another linear programming problem?
2. If so, how is this new linear program related to the original primal problem?

Guessing

- **A** is an *m* by *n* matrix, **b** is an *m*-vector, **c** is an *n*-vector.

- Can the roles of **b** and **c** can be switched?

  \- If so, we'll have *m* variables and *n* constraints.

  Correspondingly,

  (1) the transpose of matrix A should be considered to accommodate rows for columns, and vise versa?

  (2) nonnegative variables for free variables?

  (3) equality constraints for inequality constraints?

  (4) minimizing objective for maximizing objective?

Dual linear program

- Consider the following linear program

$$
\text{(D)} \quad \begin{aligned} \text{Max} \quad & \mathbf{b}^T\mathbf{w} \\ \text{s. t.} \quad & \mathbf{A}^T\mathbf{w} \le \mathbf{c} \quad \text{Dual Problem} \\ & \mathbf{w} \in R^m \end{aligned}
$$

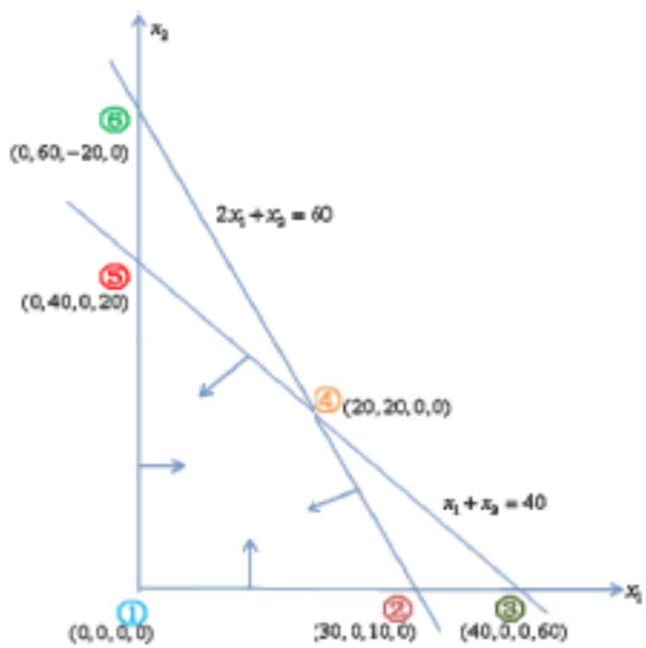
- Remember the original linear program

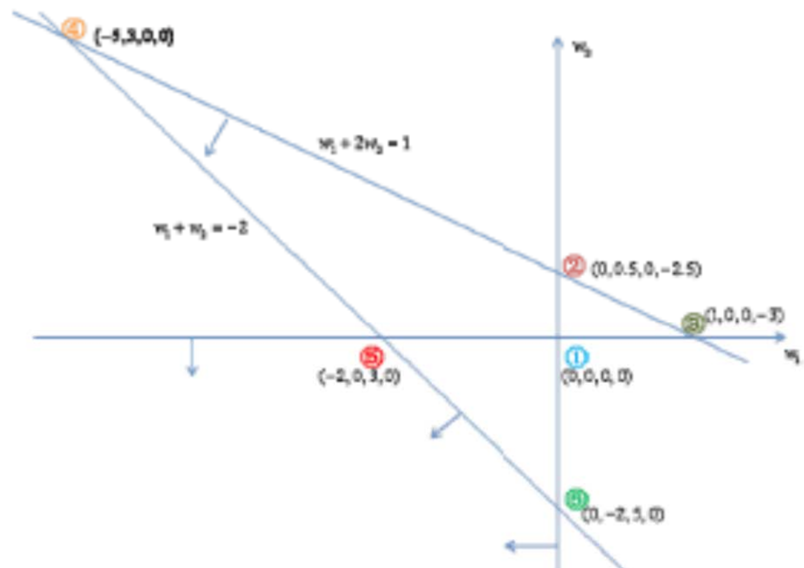
$$
\text{(LP)} \quad \begin{aligned} \text{Min} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{Primal Problem} \\ & \mathbf{x} \ge 0 \end{aligned}
$$

Example

$$
(P)\quad \begin{aligned} &\textit{Minimize} && x_1 - 2x_2 \\ &\textit{subject to} && x_1 + x_2 + x_3 = 40 \\ & && 2x_1 + x_2 + x_4 = 60 \\ & && x_1, x_2, x_3, x_4 \geq 0 \end{aligned}
$$

$$
(D)\quad \begin{aligned} &\textit{Maximize} && 40w_1 + 60w_2 \\ &\textit{subject to} && w_1 + 2w_2 + w_3 = 1 \\ & && w_1 + w_2 + w_4 = -2 \\ & && w_1 \leq 0, w_2 \leq 0 \\ & && w_3 \geq 0, w_4 \geq 0 \end{aligned}
$$





Dual of LP in other form

- Symmetric pair

$$
\begin{aligned}
\text{(P)} \quad \min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s. t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{(D)} \quad \max \quad & \mathbf{b}^T\mathbf{w} \\
\text{s. t.} \quad & \mathbf{A}^T\mathbf{w} \leq \mathbf{c} \\
& \mathbf{w} \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} + 0^T\mathbf{s} \\
\text{s. t.} \quad & \mathbf{A}\mathbf{x} - I\mathbf{s} = \mathbf{b} \\
& \mathbf{x}, \mathbf{s} \geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\max \quad & \mathbf{b}^T\mathbf{w} \\
\text{s. t.} \quad & \begin{bmatrix} \mathbf{A}^T \\ -I \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}
\end{aligned}
$$

Karmarkar's form LP

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s. t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \\
& x_1 + x_2 + \cdots + x_n = 1 \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & w_{m+1} \\
\text{s. t.} \quad & a_{11}w_1 + \cdots + a_{m1}w_m + w_{m+1} \leq c_1 \\
(DKLP) \quad & a_{12}w_1 + \cdots + a_{m2}w_m + w_{m+1} \leq c_2 \\
& \vdots \\
& a_{1n}w_1 + \cdots + a_{mn}w_m + w_{m+1} \leq c_n
\end{aligned}
$$

Dual (KLP) is always feasible. For example:

$w_1 = w_2 = \cdots = w_m = 0$

$w_{m+1} = \min\{c_1, c_2, \ldots, c_n\}$ then

$\mathbf{w} = (w_1, \ldots, w_m, w_{m+1})^T$ is feasible to (DKLP).

How are they related ?

We denote the original LP by (P) as the primal problem and the new LP by (D) as dual problem.

- (P) and (D) are defined by the same data set (**A**, **b**, **c**).
- Problem (D) is a linear program with *m* variables and *n* constraints. The right-hand-side vector and the cost vector change roles in (P) and (D).
- What else?

Many interesting questions

- Feasibility:
  - Can problems (P) and (D) be both feasible?
  - One is feasible, while the other is infeasible?
  - Both are infeasible?
- Basic solutions:
  - Is there any relation between the basic solutions of (P) and that of (D)? bfs ? optimal solutions?
- Optimality:
  - Can problems (P) and (D) both have a unique optimal solution?
  - Both have infinitely many?
  - One unique, the other infinitely many?

Examples

- Both (P) and (D) are infeasible:

$$(P)\quad \begin{array}{rrcrcl} \min & x_1 & - & 2x_2 & & \\ \text{s.t.} & -x_1 & + & x_2 & \geq & 1 \\ & x_1 & - & x_2 & \geq & 2 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

$$(D)\quad \begin{array}{rrcrcl} \max & w_1 & + & 2w_2 & & \\ \text{s.t.} & -w_1 & + & w_2 & \leq & 1 \\ & w_1 & - & w_2 & \leq & -2 \\ & w_1, & & w_2 & \geq & 0 \end{array}$$

- Both (P) and (D) have infinitely many optimal solutions:

$$(P)\quad \begin{array}{rrcrcl} \min & x_1 & - & x_2 & & \\ \text{s.t.} & -x_1 & + & x_2 & \geq & 1 \\ & x_1 & - & x_2 & \geq & -1 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

$$(D)\quad \begin{array}{rrcrcl} \max & w_1 & - & w_2 & & \\ \text{s.t.} & -w_1 & + & w_2 & \leq & 1 \\ & w_1 & - & w_2 & \leq & -1 \\ & w_1, & & w_2 & \geq & 0 \end{array}$$

The primal optimal solution set is $\{(x_1, x_2) | -x_1 + x_2 = 1,\ x_1, x_2 \geq 0\}$. The dual optimal solution set is $\{(w_1, w_2) | -w_1 + w_2 = 1,\ w_1, w_2 \geq 0\}$. Both of them have infinitely many optimal solutions.

Dual relationship

- If we take (D) as the primal problem, what'll happen?
- Convert (D) into its standard form by

$$\begin{aligned} \text{Max} \quad & \mathbf{b}^T \mathbf{w} \\ \text{s. t.} \quad & \mathbf{A}^T \mathbf{w} \le \mathbf{c} \\ & \mathbf{w} \in R^m \end{aligned}$$

1. taking **w** = **u** – **v**
2. adding slacks s
3. minimizing objective

$$(P) \quad \begin{aligned} -\text{Min} \quad & [-\mathbf{b}^T \mid \mathbf{b}^T \mid 0\,] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{s} \end{bmatrix} \\ \text{s. t.} \quad & [\,\mathbf{A}^T \mid -\mathbf{A}^T \mid I\,] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{s} \end{bmatrix} = \mathbf{c} \\ & \mathbf{u}, \mathbf{v}, \mathbf{s} \ge 0 \end{aligned}$$

New dual

$$(D) \quad \begin{aligned} -\text{Max} \quad & \mathbf{c}^T \mathbf{w} \\ \text{s. t.} \quad & \begin{bmatrix} A \\ -A \\ I \end{bmatrix} \mathbf{w} \le \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} \\ & \mathbf{w} \text{ unrestricted} \in R^n \end{aligned}$$

Dual of the dual = primal

- Rearrange the new dual

$$
\begin{array}{lll}
 & -\text{Max} & \mathbf{c}^T\mathbf{w} \\
(\text{D}) & \text{s. t.} & \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ I \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} -\mathbf{b} \\ \mathbf{b} \\ 0 \end{bmatrix} \\
 & & \mathbf{w} \text{ unrestricted} \in R^n
\end{array}
$$

- We have

$$
\begin{array}{ll}
-\text{Max} & \mathbf{c}^T\mathbf{w} \\
\text{s. t.} & \mathbf{A}\mathbf{w} \leq -\mathbf{b} \\
 & -\mathbf{A}\mathbf{w} \leq \mathbf{b} \\
 & \mathbf{w} \leq 0
\end{array}
$$

$$
\begin{array}{ll}
\text{Min} & -\mathbf{c}^T\mathbf{w} \\
\text{s. t.} & -\mathbf{A}\mathbf{w} = \mathbf{b} \\
 & \mathbf{w} \leq 0
\end{array}
$$

For $\mathbf{x} := -\mathbf{w}$, we have

$$
\begin{array}{ll}
\text{Min} & \mathbf{c}^T\mathbf{x} \\
\text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & \mathbf{x} \geq 0
\end{array}
$$

<u>Lemma 4.1</u> Dual of the Dual=Primal.

Example

$$
(P)\quad \begin{aligned}
&\textit{Minimize}\ x_1 - 2x_2 \\
&\textit{subject to}\ x_1 + x_2 + x_3 = 40 \\
&\qquad 2x_1 + x_2 + x_4 = 60 \\
&\qquad x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

$$
(D)\quad \begin{aligned}
&\textit{Maximize}\ 40w_1 + 60w_2 \\
&\textit{subject to}\ w_1 + 2w_2 + w_3 = 1 \\
&\qquad w_1 + w_2 + w_4 = -2 \\
&\qquad w_1 \leq 0, w_2 \leq 0 \\
&\qquad w_3 \geq 0, w_4 \geq 0
\end{aligned}
$$

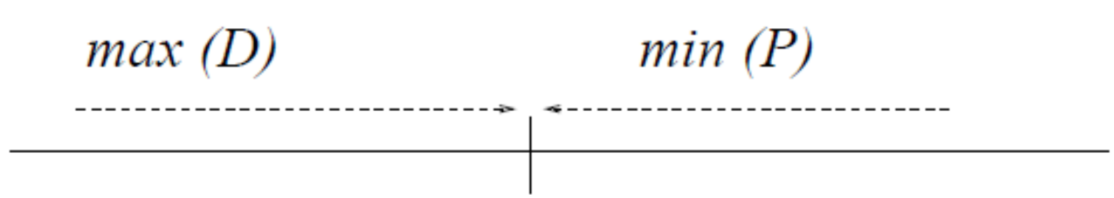
Weak duality theorem

If **x** is a primal feasible solution to (P) and **w** is a dual feasible solution to (D), then

$$\begin{aligned} \mathbf{c}^T\mathbf{x} &= \mathbf{x}^T\mathbf{c} \\ &\geq \mathbf{x}^T\mathbf{A}^T\mathbf{w} \\ &= \mathbf{b}^T\mathbf{w} \end{aligned}$$

(Weak Duality Theorem):

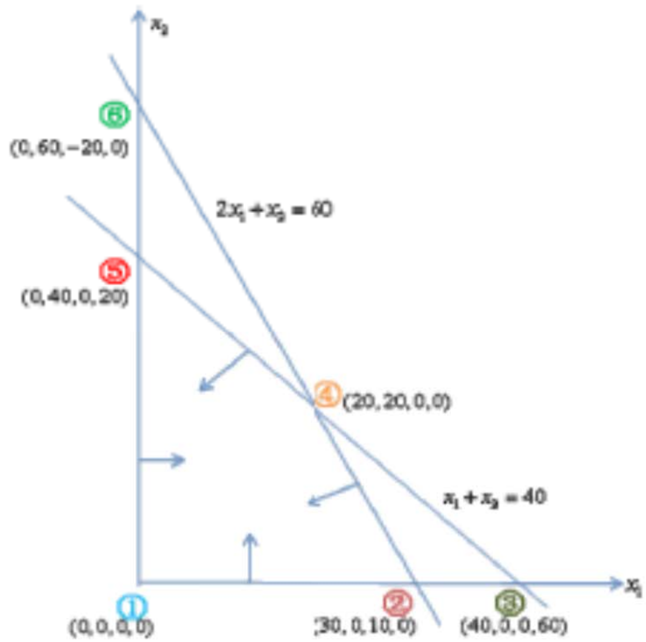
If **x** is a primal feasible solution to (P) and **w** is a dual feasible solution to (D), then $\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{w}$.

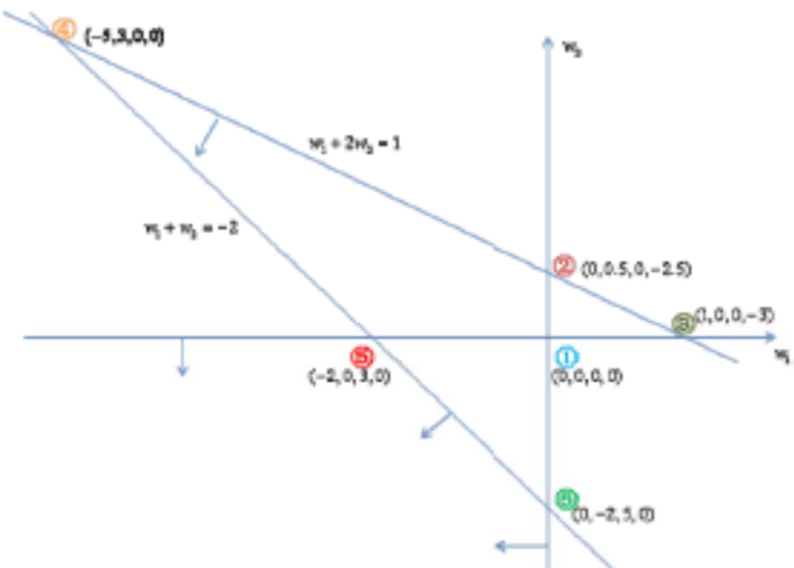


Example

$$(P)\quad \begin{aligned} \textit{Minimize}\quad & x_1 - 2x_2 \\ \textit{subject to}\quad & x_1 + x_2 + x_3 = 40 \\ & 2x_1 + x_2 + x_4 = 60 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$(D)\quad \begin{aligned} \textit{Maximize}\quad & 40w_1 + 60w_2 \\ \textit{subject to}\quad & w_1 + 2w_2 + w_3 = 1 \\ & w_1 + w_2 + w_4 = -2 \\ & w_1 \leq 0, w_2 \leq 0 \\ & w_3 \geq 0, w_4 \geq 0 \end{aligned}$$





Corollaries

1. If **x** is primal feasible, **w** is dual feasible, and

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w},$$

then **x** is primal optimal, and **w** is dual optimal.

2. If the primal is unbounded below, then the dual is infeasible.
(Is the converse statement true? -- watch for infeasibility)

3. If the dual is unbounded above, then the primal is infeasible.
(Is the converse statement true?)

Strong duality theorem

- Questions:
  - Can the results of weak duality be stronger?
  - Is there any gap between the primal optimal value and the dual optimal value?
- Strong Duality Theorem:

(a) If either the primal or the dual has a finite optimum, then so does the other and $\min \mathbf{c}^T\mathbf{x} = \max \mathbf{b}^T\mathbf{w}$ (No duality gap!)

(b) If either problem has an unbounded objective, then the other has no feasible solution.

Proof of strong duality theorem

- Proof:

Note that the dual of the dual is the primal and the fact that

"If **x** is primal feasible, **w** is dual feasible and

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w},$$

then **x** is primal optimal and **w** is dual optimal."

We only need to show that

"*if the primal has a finite optimal bfs* **x**, *then there exists a dual feasible solution* **w** *such that*

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}.\text{'}$$

Proof - continue

- Applying the simplex method at the optimal bsf **x** with basis B, we define $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$.

Then

$$
\begin{aligned}
\mathbf{c} - \mathbf{A}^T\mathbf{w} &= \begin{bmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{bmatrix} - \begin{bmatrix} \mathbf{B}^T \\ \mathbf{N}^T \end{bmatrix} \mathbf{w} \\
&= \begin{bmatrix} \mathbf{c}_B - \mathbf{B}^T(\mathbf{B}^T)^{-1}\mathbf{c}_B \\ \mathbf{c}_N - \mathbf{N}^T(\mathbf{B}^T)^{-1}\mathbf{c}_B \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ r_N \end{bmatrix} \geq 0
\end{aligned}
$$

Thus w is dual feasible and

$$
\begin{aligned}
\mathbf{c}^T\mathbf{x} &= \mathbf{c}_B^T\mathbf{x}_B \\
&= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} \\
&= \mathbf{w}^T\mathbf{b} \\
&= \mathbf{b}^T\mathbf{w}.
\end{aligned}
$$

(b) It is a direct consequence of the Weak Duality Theorem.

Implications

(1) The simplex multiplier $\mathbf{w}$ corresponding to a primal optimal solution $\mathbf{x}$ is a dual optimal solution.

(2) At each iteration of the simplex method, the simplex multiplier $\mathbf{w}$ always satisfies that
$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$.
However, $\mathbf{w}$ is not dual feasible unless $r_N \geq 0$.

(3) Revised Simplex Method

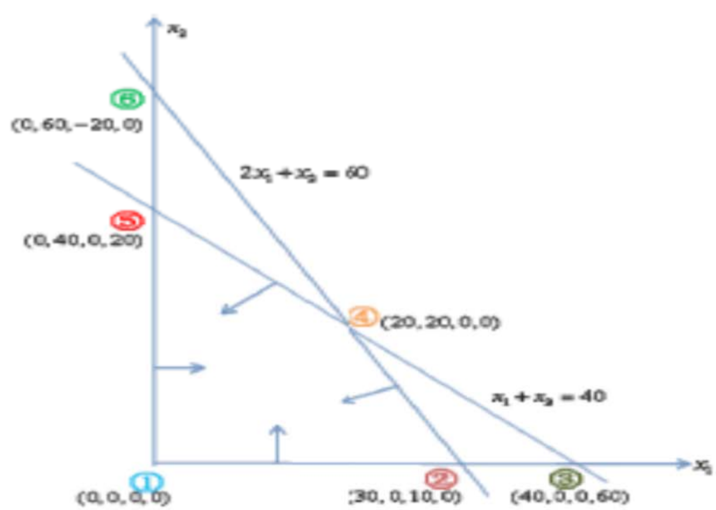
Keep primal feasibility
and $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$ (no duality gap)
but seeks for dual feasibility.

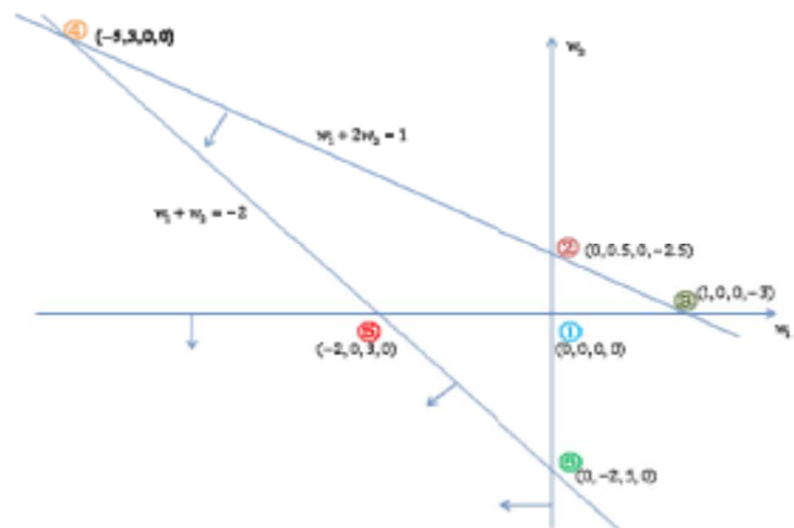
Corresponding basic solutions

- At a primal basic solution **x** with basis B, we defined a dual basic solution $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$

$$
(P)\quad
\begin{array}{ll}
\text{Minimize} & x_1 - 2x_2 \\
\text{subject to} & x_1 + x_2 + x_3 = 40 \\
& 2x_1 + x_2 + x_4 = 60 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

$$
(D)\quad
\begin{array}{ll}
\text{Maximize} & 40w_1 + 60w_2 \\
\text{subject to} & w_1 + 2w_2 + w_3 = 1 \\
& w_1 + w_2 + w_4 = -2 \\
& w_1 \leq 0, w_2 \leq 0 \\
& w_3 \geq 0, w_4 \geq 0
\end{array}
$$





Further implications of strong duality theorem

- Theorem of Alternatives

  "Existence of solutions of systems of equalities and inequalities"

- Famous Farkas Lemma (another form)

The system

$$\text{(I)} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

has no solution if and only if the system

$$\text{(II)} \quad \mathbf{A}^T\mathbf{w} \leq 0, \quad \mathbf{b}^T\mathbf{w} > 0$$

has solution.

Two systems

$$\text{(I)} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

$$\text{(II)} \quad \mathbf{A}^T\mathbf{w} \leq 0, \quad \mathbf{b}^T\mathbf{w} > 0$$

Either (I) or (II) has a solution but NOT both.

Proof of Farkas Lemma

Consider LP and its dual

$$(P)\quad \text{Min} \quad 0^T\mathbf{x} \qquad \text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

$$(D)\quad \text{Max} \quad \mathbf{b}^T\mathbf{w} \qquad \text{s. t.} \quad \mathbf{A}^T\mathbf{w} \leq 0$$

- Since w = 0 is dual feasible, we know
  (P) is infeasible if and only if (D) is unbounded above.
  Note that
- (P) is infeasible if and only if (I) has no solution.
- (D) is unbounded above if and only if (II) has a solution.
  Hence,
  (I) has no solution if and only if (II) has a solution.

Complementary slackness

- Consider the symmetric pair LP

$$
\begin{array}{lll}
\text{(P)} & \min & \mathbf{c}^T\mathbf{x} \\
& \text{s. t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
& & \mathbf{x} \geq 0
\end{array}
\qquad\qquad
\begin{array}{lll}
\text{(D)} & \max & \mathbf{b}^T\mathbf{w} \\
& \text{s. t.} & \mathbf{A}^T\mathbf{w} \leq \mathbf{c} \\
& & \mathbf{w} \geq 0
\end{array}
$$

Let $\mathbf{x}$ be primal feasible, $\mathbf{w}$ be dual feasible.

Define

$$
\begin{aligned}
\mathbf{s} &= \mathbf{A}\mathbf{x} - \mathbf{b} && \geq 0 \\
\mathbf{r} &= \mathbf{c} - \mathbf{A}^T\mathbf{w} && \geq 0
\end{aligned}
$$

$\mathbf{s} \in R^m$: <u>primal slackness</u>

$\mathbf{r} \in R^n$: <u>dual slackness</u>

Observations

1. If $\left.\begin{array}{l} r^T x = 0 \\ \text{and } \; w^T s = 0 \end{array}\right\}$ complementary slackness

then

$$\underline{(c^T - w^T A)}x = 0$$

and

$$w^T \underline{(Ax - b)} = 0$$

Hence

$c^T x = w^T Ax = w^T b = b^T w$

Thus $\mathbf{x}$ is primal optimal and $\mathbf{w}$ is dual optimal.

2. On the contrary side, for a feasible pair $(\mathbf{x}, \mathbf{w})$,

$$c^T x \geq w^T Ax \geq w^T b$$

If $\mathbf{x}$ is primal optimal and $\mathbf{w}$ is dual optimal, then $c^T x = w^T Ax = w^T b$

Hence

$$(c^T - w^T A)x = 0$$

and

$$w^T (Ax - b) = 0$$

Complementary slackness theorem

- Theorem:

  Let (P) and (D) be a "symmetric pair",

  $\mathbf{x}$ is primal feasible, $\mathbf{w}$ is dual feasible.

  Then $\mathbf{x}$, $\mathbf{w}$ are optimal solution pair if and only if

$$\begin{cases} r_j = 0 \text{ or } x_j = 0 \quad \forall\, j = 1, 2, \ldots, n. \\ s_i = 0 \text{ or } w_i = 0 \quad \forall\, i = 1, 2, \ldots, m. \end{cases}$$

Example

- Consider a relaxed knapsack problem:

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 4x_2 + 9x_3 + 2x_4 + 5x_5 \\
\text{s.t.} \quad & \\
& 4x_1 + 7x_2 + 10x_3 + 3x_4 + 7x_5 \leq 20 \\
& x_j \geq 0, \quad \text{for all } j
\end{aligned}
$$

- Its dual becomes

$$
\begin{aligned}
\min \quad & 20y \\
\text{s.t.} \quad & 4y \geq 3 \\
& 7y \geq 4 \\
& 10y \geq 9 \\
& 3y \geq 2 \\
& 7y \geq 5 \\
& y \geq 0
\end{aligned}
$$

4y* > 3   x*1 = 0

7y* > 4   x*2 = 0

10y* = 9   x*3 = ?   x*3 = 2   z* = 18

3y* > 2   x*4 = 0

7y* > 5   x*5 = 0

dual optimal solution is $y^* = 0.9$ $20 - (4x_1 + x_2 + 10x_3 + 3x_4 + 7x_5) = 0$

Complementary slackness for standard form LP

$$\text{(P)} \quad \min \quad \mathbf{c}^T\mathbf{x} \quad \text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

$$\text{(D)} \quad \max \quad \mathbf{b}^T\mathbf{w} \quad \text{s. t.} \quad \mathbf{A}^T\mathbf{w} \leq \mathbf{c}$$

the condition $\mathbf{w}^T\mathbf{s} = \mathbf{0}$ is always true.

The complementary slackness condition reduces to $\mathbf{r}^T\mathbf{x} = \mathbf{0}$.

Kuhn-Tucker condition

- Theorem:

$\mathbf{x}$ is optimal for the problem

$$
\begin{aligned}
(\mathrm{P}) \quad \min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

if and only if there exist $\mathbf{w}$ and $\mathbf{r}$ such that

$$(1)\mathbf{A}\mathbf{x} = \mathbf{b},\ \ \mathbf{x} \geq 0 \qquad \text{(Primal feasibilty)}$$

$$(2)\mathbf{A}^T\mathbf{w} + \mathbf{r} = \mathbf{c},\ \ \mathbf{r} \geq 0 \qquad \text{(Dual feasibilty)}$$

$$(3)\mathbf{r}^T\mathbf{x} = \mathbf{0} \qquad \text{(Complementary Slackness)}$$

Implication

- Solving a linear programming problem is equivalent to solving a system of linear inequalities.

Economic interpretation of duality

- Is there any special meaning of the dual variables?
- What is a dual problem trying to do?
- What's the role of the complementary slackness in decision making?

Dual variables

- Consider a nondegenerate linear program:

$$\begin{array}{llll} (\mathrm{P}) & \min & \mathbf{c}^T\mathbf{x} & \text{(minimize total cost)} \\ & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & \text{(satisfy demands)} \\ & & \mathbf{x} \geq 0 & \text{(different services)} \end{array}$$

Assume that $\mathbf{x}^*$ nondegenerate optimal bfs

$$\mathbf{x}^* = \begin{pmatrix} \mathbf{x}_B^* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \end{pmatrix}$$

$$z^* = \mathbf{c}^T\mathbf{x}^* = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$$

Since $\mathbf{x}_B^* = \mathbf{B}^{-1}\mathbf{b} > 0$,
we have $\mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b}) > 0$, when $\triangle\mathbf{b}$ is small enough!

Then

$$\bar{\mathbf{x}}^* = \begin{pmatrix} \bar{\mathbf{x}}_B^* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b}) \\ 0 \end{pmatrix}$$

is an optimal bfs to

$$\begin{array}{lll} (\bar{\mathrm{P}}) & \min & \mathbf{c}^T\mathbf{x} \\ & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} + \triangle\mathbf{b} \\ & & \mathbf{x} \geq 0 \end{array}$$

with $\bar{z}^* = \mathbf{c}_B^T\mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b})$.
(Why? No change in r_q!)

Dual variable for shadow price

- Moreover, we have

$$
\begin{aligned}
\triangle z &= \bar{z}^* - z^* \\
&= \mathbf{c}_B^T \mathbf{B}^{-1}(\mathbf{b} + \triangle \mathbf{b}) - c_B^T B^{-1}\mathbf{b} \\
&= \underbrace{\mathbf{c}_B^T \mathbf{B}^{-1}}_{\mathbf{w}^T} \triangle \mathbf{b}
\end{aligned}
$$

(simplex multiplier for (P) at optimum!)

Hence w_i is the " marginal price" of the ith demand.

- Note: dual variable w_i indicates the minimum unit price that one has to charge for additional demand *i*. It is also called shadow price or equilibrium price.

Dual LP problem

Consider the following production scenario:

- *n* products to be produced

$$x_j = \text{amount of product } j, \quad j = 1, 2, \ldots, n$$

- *m* resources in hand

$$b_i = \text{amount of resource } i, \quad i = 1, 2, \ldots, m$$

- market selling price for each product is known

$$c_1, c_2, \ldots, c_n$$

- technology matrix is given by

$[\, a_{ij} \,]$: each product j consumes a_{ij} units of resource i, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

A manufacturer's view

- Maximize total sales (hence profit)

$$\sum_{j=1}^{n} c_j x_j$$

- Resource limitation

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i,\ i = 1, 2, \ldots, m$$

- Production Requirement

$x_j \ge 0,\ j = 1, 2, \ldots, n$

$$\begin{array}{lll} \text{(P)} & \max & \mathbf{c}^T\mathbf{x} \\ & \text{s. t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & & \mathbf{x} \ge 0 \end{array}$$

<u>Getting Resources</u>

- m resources to purchase from a supplier
  w_i= unit price to purchase resource i,
  $i = 1, 2, \ldots, m$
- Free information market:
  Supplier knows your selling price c_j for product x_j and he/she wants to get the most out of you. *i.e.*

$$a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m \ge c_j,\ j = 1, 2, \ldots, n$$

Therefore, we have the dual

$$\begin{array}{lll} \min & \mathbf{b}^T\mathbf{w} & \text{(minimize total spending)} \\ \text{s. t.} & \mathbf{A}^T\mathbf{w} \ge \mathbf{c} & \text{(price accepted by the supplier)} \\ & \mathbf{w} \ge 0 & \end{array}$$

Complementary slackness condition

- Observation:

1. w_i^* is the maximum marginal price the manufacturer is willing to pay the supplier for resource *i*.
2. When resource *i* is not fully utilized, *i.e.*,
$$a_{i1}x_1^* + a_{i2}x_2^* + \ldots + a_{in}x_n^* < b_i$$
the complementary slackness condition implies $w_i^* = 0.$ This means the manufacturer is not willing to pay a penny for buying any additional amount !
3. When the supplier asks too much, *i.e.*,
$$a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_n > c_j$$
then $x_j = 0.$ This means the manufacturer is not going to produce any product *j* !

Sensitivity analysis

- Sensitivity is a post-optimality analysis of a linear program

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

in which, some components of (**A, b, c**) may change after obtaining an optimal solution $\mathbf{x}^*$ with an optimal basis $\mathbf{B}^*$ and an optimal objective value z^* .

Questions of interests:

Will $\mathbf{x}^*$ remain optimal ?

$\mathbf{B}^*$ remain optimal ?

or How will they change accordingly?

Fundamental concepts

- No matter how the data (A, b, c) change, we need to make sure that

1. Feasibility (**c** is not involved):

bfs **x** is feasible if and only $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$

2. Optimality (**b** is not involved):

$$\mathbf{x} \text{ is optimal} \iff r_q \geq 0 \;\; \forall\, q \in \mathbf{N}.$$

$$r_q \quad = c_q - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_q$$

Change in the cost vector

- Scenario: $\mathbf{x}^*$ optimal solution, $\mathbf{A} = [\mathbf{B}|\mathbf{N}]$

Given $\mathbf{c}' = \begin{bmatrix} \mathbf{c}'_B \\ \mathbf{c}'_N \end{bmatrix} \in R^n$ be a perturbation,

$$\mathbf{c} \longrightarrow \mathbf{c} + \alpha \mathbf{c}' = \begin{bmatrix} \mathbf{c}_B + \alpha \mathbf{c}'_B \\ \mathbf{c}_N + \alpha \mathbf{c}'_N \end{bmatrix} \triangleq \bar{\mathbf{c}},$$

$$\alpha \in R$$

Consider

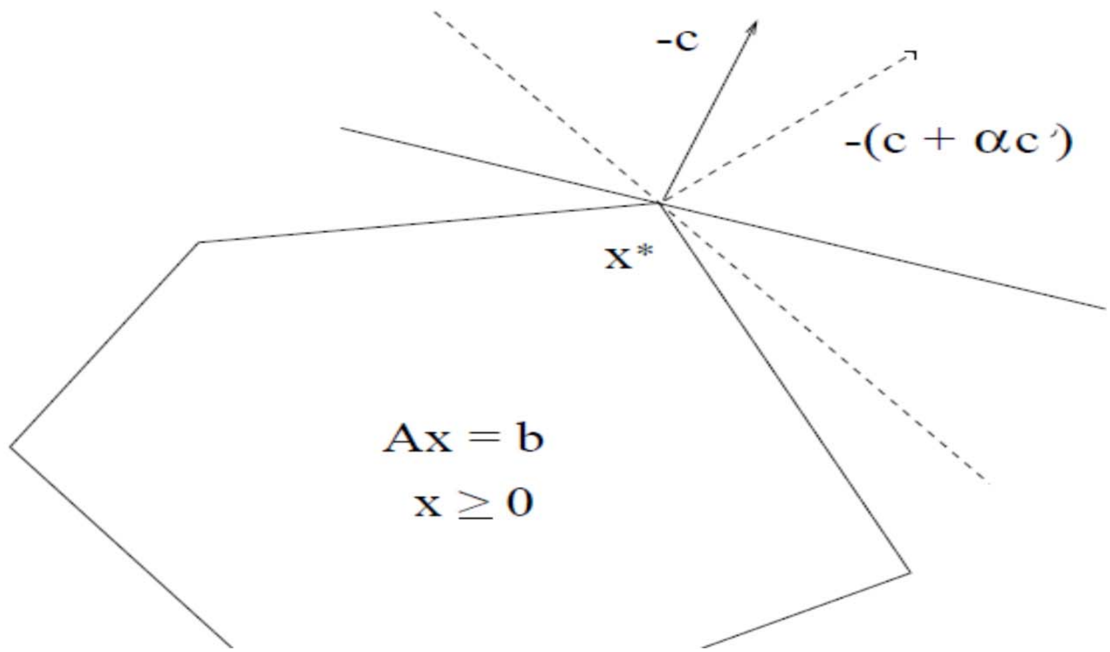
$$(\mathrm{P}') \quad \begin{aligned} \min \quad & z(\alpha) = \underline{(\mathbf{c} + \alpha \mathbf{c}')^T} \mathbf{x} = \bar{\mathbf{c}}^T \mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- **Question**:

  Within which range of $[\,\underline{\alpha},\ \bar{\alpha}\,]$, the current optimal solution $\mathbf{x}^*$ remains to be optimal?

Note: when c' = (0,,0,1,0, 0), we have the regular sensitivity analysis on each cost coefficient.

Geometric view



- Fact:

As α changes, $\quad \mathbf{x}^*(\alpha)$ changes,

$\mathbf{B}(\alpha)$ changes,

$z^*(\alpha)$ changes

Analysis

- Compare

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$(\mathrm{P}') \quad \begin{aligned} \min \quad & z(\alpha) = \underline{(\mathbf{c} + \alpha\mathbf{c}')^T}\mathbf{x} = \bar{\mathbf{c}}^T\mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

(1) (P) and (P$'$) have the same feasible domain, hence $\mathbf{x}^*$ is feasible to (P$'$) for any α.

(2) $\mathbf{x}^*$ remains optimal to (P$'$) if $\bar{r}_N^T = \bar{\mathbf{c}}_N^T - \bar{\mathbf{c}}_B^T\mathbf{B}^{-1}\mathbf{N} \geq 0$

i.e., $(\mathbf{c}_N + \alpha\mathbf{c}'_N)^T - (\mathbf{c}_B + \alpha\mathbf{c}'_B)^T\mathbf{B}^{-1}\mathbf{N} \geq 0$

$$\underbrace{(\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N})}_{r_N^T} + \alpha\underbrace{(\mathbf{c'}_N^T - \mathbf{c'}_B^T\mathbf{B}^{-1}\mathbf{N})}_{r'^T_N} \geq 0$$

$$\alpha r'^T_N \geq -r_N^T$$

Analysis - continue

(3) Case 1. For $r_q' > 0,\ q \in N$

$\alpha \geq \frac{-r_q}{r_q'}$ is required, thus

$$\underline{\alpha} = \max\{\frac{-r_q}{r_q'} \mid r_q' > 0,\ q \in \mathbf{N}\}$$

otherwise

$$\underline{\alpha} = -\infty,\ \text{if } r_q' \leq 0,\ \forall\, q \in \mathbf{N}$$

Case 2. For $r_q' < 0, q \in \mathbf{N}$

$\alpha \leq \frac{-r_q}{r_q'}$ is required, thus

$$\bar{\alpha} = \min\{\frac{-r_q}{r_q'} \mid r_q' < 0,\ q \in \mathbf{N}\}$$

otherwise

$$\bar{\alpha} = \infty,\ \text{if } r_q' \geq 0,\ \forall\, q \in \mathbf{N}$$

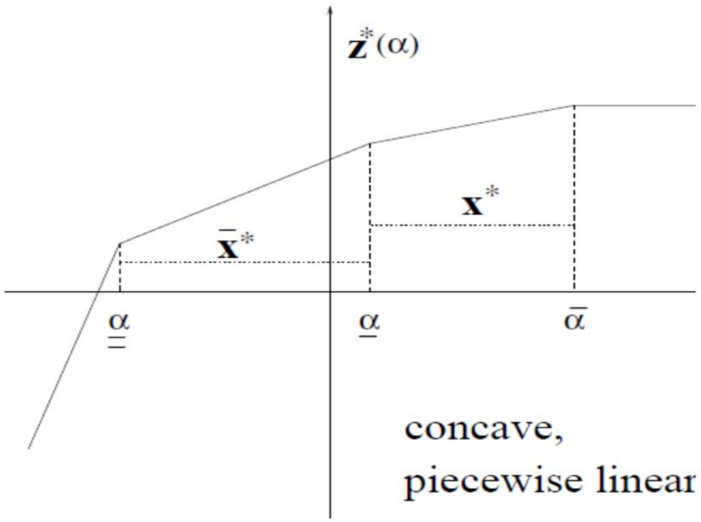
Analysis - continue

(5) For $\alpha \in [\ \underline{\alpha},\ \bar{\alpha}\ ]$, $\mathbf{x}^*$ remains optimal.

$$\begin{aligned} z^*(\alpha) &= (\mathbf{c}_B^T + \alpha \mathbf{c'}_B^T)\mathbf{B}^{-1}\mathbf{b} \\ &= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \alpha \underbrace{\mathbf{c'}_B^T\mathbf{B}^{-1}\mathbf{b}}_{k\ :\ \text{constant}} \\ &= z^* + k\ \alpha \end{aligned}$$

Thus $z^*(\alpha)$ is linear in α.

(6) .



(7) Take

$$\mathbf{c}' = e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow jth\ place$$

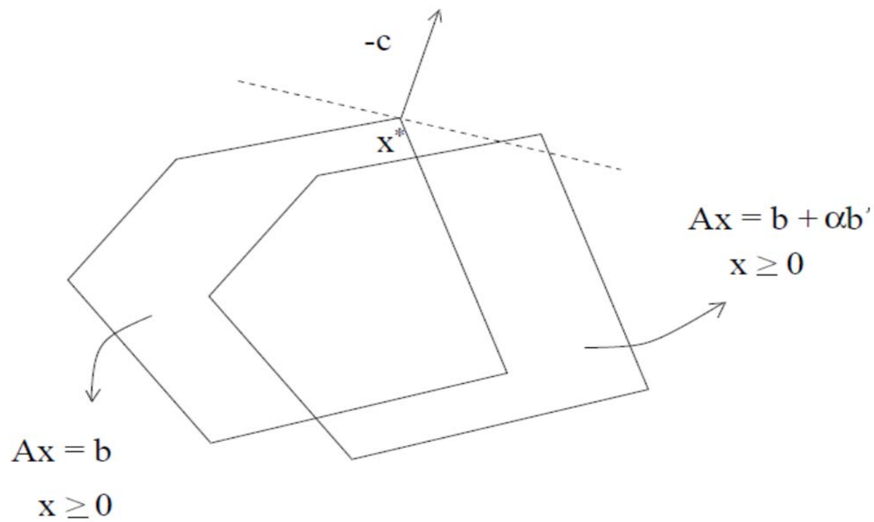
then $[\ c_j - \underline{\alpha},\ c_j + \bar{\alpha}\ ]$ gives stable range of the jth cost coefficient, or how sensitive the cost component is.

Change in the r-h-s vector

- Scenario:

Let $\mathbf{b}' \in R^m$ be a perturbation.

$$(P') \quad \begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} + \alpha\mathbf{b}' \\ & \mathbf{x} \geq 0 \end{aligned}$$



- Fact: $\mathbf{x}^*$ may become infeasible !
- Question:

Within which range $[\ \underline{\alpha},\ \bar{\alpha}\ ]$,

will $\mathbf{B}$ remain as an optimal basis?

Analysis

$$(P') \quad \begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} + \alpha\mathbf{b}' \\ & \mathbf{x} \geq 0 \end{array}$$

(1) $\mathbf{B}$ is an optimal basis if

(i) ${r_N}^T = \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} \geq 0$, and

(ii) $x(\alpha) = \left[\frac{\mathbf{B}^{-1}(\mathbf{b}+\alpha\mathbf{b}')}{\mathbf{0}}\right] \geq 0$

(2) (i) always holds, since $\mathbf{c}, \mathbf{B}, \mathbf{N}$ no change!

but (ii) is not always true.

(3) We need $\mathbf{B}^{-1}(\mathbf{b} + \alpha\mathbf{b}') \geq 0$

$i.e.,\ \underbrace{\mathbf{B}^{-1}\mathbf{b}}_{\bar{\mathbf{b}}} + \alpha\underbrace{\mathbf{B}^{-1}\mathbf{b}'}_{\bar{\mathbf{b}}'} \geq 0 \qquad i.e.,\ \alpha\bar{\mathbf{b}}' \geq -\bar{\mathbf{b}}$

(i) For $\bar{\mathbf{b}}'_p > 0,\ p \in \mathbf{B}$, we need

$$\alpha \geq \frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}}'_p}$$

(ii) For $\bar{\mathbf{b}}'_p < 0,\ p \in \mathbf{B}$, we need

$$\alpha \leq \frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}}'_p}$$

Thus $\underline{\alpha} = \begin{cases} \max\{\frac{-\bar{b}_p}{\bar{b}'_p} \mid \bar{b}'_p > 0, p \in \mathbf{B}\}, \\ -\infty \end{cases}$

Thus $\bar{\alpha} = \begin{cases} \min\{\frac{-\bar{b}_p}{\bar{b}'_p} \mid \bar{b}'_p < 0,\ p \in \mathbf{B}\}, \\ +\infty \end{cases}$

Analysis

(4) When $\alpha \in [\,\underline{\alpha},\ \bar{\alpha}\,]$,

$$\mathbf{x}^*(\alpha) = \left(\begin{array}{c} \mathbf{B}^{-1}(\mathbf{b}+\alpha\mathbf{b}') \\ \hline \mathbf{0} \end{array}\right)$$

$$= \left(\begin{array}{c} \mathbf{B}^{-1}\mathbf{b}+\alpha\mathbf{B}^{-1}\mathbf{b}' \\ \hline \mathbf{0} \end{array}\right)$$

$$= \left(\begin{array}{c} \mathbf{B}^{-1}\mathbf{b} \\ \hline \mathbf{0} \end{array}\right) + \alpha \left(\begin{array}{c} \mathbf{B}^{-1}\mathbf{b}' \\ \hline \mathbf{0} \end{array}\right)$$

$$= \mathbf{x}^* + \alpha \left(\begin{array}{c} \mathbf{B}^{-1}\mathbf{b}' \\ \hline \mathbf{0} \end{array}\right)$$

linear in α!

(5)

$$\begin{aligned} z^*(\alpha) &= \mathbf{c}_B^T \mathbf{x}^*(\alpha)_B \\ &= \mathbf{c}_B^T(\mathbf{x}_B^* + \alpha\mathbf{B}^{-1}\mathbf{b}') \\ &= \mathbf{c}_B^T\mathbf{x}_B^* + \alpha\underbrace{\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}'}_{k} \\ &= z^* + k\ \alpha \end{aligned}$$

again, linear in α!

Changes in the constraint matrix

- Since both feasibility and optimality are involved, a general analysis is difficult.
- We only consider simple cases such as adding a new variable, removing a variable, adding a new constraint.

Adding a new variable

- Why? A new product, service or activity is introduced.

$$
(P') \quad \begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} + c_{n+1}x_{n+1} \\ \text{s. t.} & \mathbf{A}\mathbf{x} + A_{n+1}x_{n+1} = \mathbf{b} \\ & \mathbf{x}, x_{n+1} \geq 0 \end{array}
$$

- Analysis:

(1) $\begin{bmatrix} \mathbf{x}^* \\ \hline 0 \end{bmatrix}$ is a bfs of (P′) with $[\,\mathbf{B} \mid \mathbf{N}, A_{n+1}]$.

(2) $\begin{bmatrix} \mathbf{x}^* \\ \hline 0 \end{bmatrix}$ is an optimal solution of (P′) if

$$r_{n+1} = c_{n+1} - \mathbf{c}_B^T\mathbf{B}^{-1}A_{n+1} \geq 0.$$

(3) If $r_{n+1} < 0$, then x_{n+1} enters the basis and continue the revised simplex method to find an optimal solution of (P′).

Removing a variable

- Why? An activity is no longer available.

(a) if $x_k^* = 0$, then $\mathbf{x}^*$ remains optimal by deleting x_k^*.

(b) if $x_k^* > 0$, then $\mathbf{x}_k$ has to leave the basis. Can this be done ? Consider

$$
\begin{array}{lll}
 & \min & x_k \\
\text{(Phase I)} & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & & \mathbf{x} \geq 0
\end{array}
$$

(1) $\mathbf{x}^*$ is a current bfs to start the revised simplex solution.

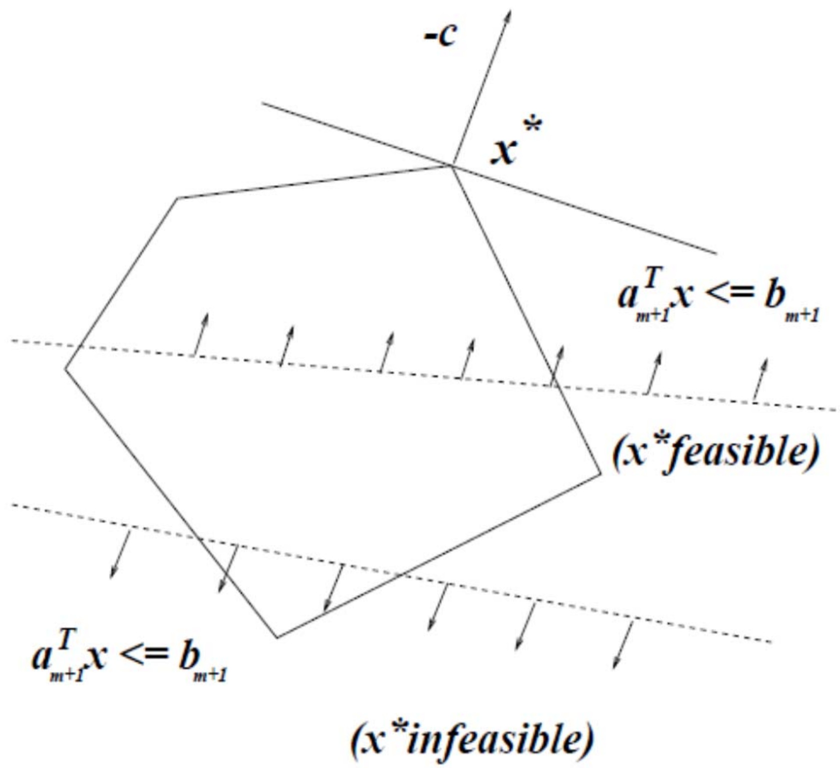
(2) If $z_{PhI}^* > 0$, then removing x_k will cause infeasibility.
If $z_{PhI}^* = 0$, then we can start from there to solve the new problem.

Adding a new constraint

- Why? A new restriction is enforces.

$$(P') \quad \begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{a}_{m+1}^T\mathbf{x} \leq b_{m+1} \\ & \mathbf{x} \geq 0 \end{array}$$

$$\mathbf{a}_{m+1}^T = (a_{m+1,1}, a_{m+1,2}, \ldots, a_{m+1,n})$$



Analysis

(1) If $\mathbf{a}_{m+1}^T \mathbf{x}^* \leq b_{m+1}$ then $\mathbf{x}^*$ remains optimal!

(2) If not, $\mathbf{x}^*$ is not feasible and we have to find a new basis of dimensionality $m+1$.

(3) Consider

$$\begin{array}{lll} & \min & \mathrm{c}_B^T \mathrm{x}_B + \mathrm{c}_N^T \mathrm{x}_N + 0 x_{n+1} \\ (P') & \text{s. t.} & \mathrm{B}\mathrm{x}_B + \mathrm{N}\mathrm{x}_N = \mathrm{b} \\ & & (\mathrm{a}_{m+1})_B^T \mathrm{x}_B + (\mathrm{a}_{m+1})_N^T \mathrm{x}_N + x_{n+1} = b_{m+1} \\ & & \mathrm{x}_B, \mathrm{x}_N, x_{n+1} \geq 0 \end{array}$$

$$\bar{\mathbf{B}} := \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ (\mathbf{a}_{m+1})_B^T & 1 \end{pmatrix}$$

then $\bar{\mathbf{B}}$ is a nonsingular $(m+1) \times (m+1)$ matrix, and

$$\bar{\mathbf{B}}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -(\mathbf{a}_{m+1})_B^T \mathbf{B}^{-1} & 1 \end{pmatrix}$$

i.e. $\bar{\mathbf{B}}$ is a basis for (P′)!

Analysis

(4) The reduced cost

$$
\begin{aligned}
r_q' &= c_q - \left[\begin{array}{c} \mathbf{c}_B \\ 0 \end{array}\right]^T \bar{\mathrm{B}}^{-1} \left[\frac{A_q}{a_{m+1,q}}\right] \\
&= c_q - \left[\, \mathbf{c}_B^T \mathrm{B}^{-1} \mid 0 \,\right] \left[\frac{A_q}{a_{m+1,q}}\right] \\
&= c_q - \mathbf{c}_B^T \mathrm{B}^{-1} A_q \\
&= r_q,\ \forall q \in \mathrm{N}
\end{aligned}
$$

Since B is an optimal basis to (P), we know

$r_q' = r_q \geq 0,\ \ \forall q \in \mathrm{N}$

i.e, $\bar{\mathrm{B}}$ provides a dual feasible solution

$\mathrm{w}^T = \mathbf{c}_{\bar{B}}^T \bar{\mathrm{B}}^{-1}$ for (P′).

(5) Define

$$
\begin{aligned}
\bar{\mathbf{x}}_{\bar{B}} &= \bar{\mathbf{B}}^{-1}\left(\frac{\mathbf{b}}{b_{m+1}}\right) \\
\bar{\mathbf{x}}_N &= \mathbf{0}
\end{aligned}
$$

then $\bar{\mathbf{x}} = \left(\frac{\bar{\mathbf{x}}_{\bar{B}}}{\bar{\mathbf{x}}_N}\right)$ is an optimal solution of (P′) if $\bar{\mathbf{x}}_{\bar{B}} \geq 0$.

(6) If

$$
\bar{\mathbf{x}}_{\bar{B}} = \bar{\mathrm{B}}^{-1}\left(\frac{\mathrm{b}}{b_{m+1}}\right) \ngeq 0
$$

then we can apply the dual simplex method with

$$
\begin{aligned}
\mathbf{w}^T &= \mathbf{c}_{\bar{B}}^T \bar{\mathbf{B}}^{-1} \\
&= (\mathbf{c}_B^T,\ 0) \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -(\mathbf{a}_{m+1})_B^T \mathbf{B}^{-1} & 1 \end{pmatrix} \\
&= (\mathbf{c}_B^T \mathbf{B}^{-1} \mid 0)
\end{aligned}
$$

to solve (P′).

Dual simplex method

- What's the dual simplex method?
  - It is a simplex based algorithm that works on the dual problem directly. In other words, it hops from one vertex to another vertex along some edge directions in the dual space.

- It keeps dual feasibility and complementary slackness, but seeks primal feasibility.

Motivation

- Applying the (revised) simplex method to solve the dual problem:

$$
\begin{array}{llll}
(\mathrm{D}) & \max & \mathbf{b}^T\mathbf{w} & m \text{ variables} \\
& \text{s. t.} & \mathbf{A}^T\mathbf{w} \le \mathbf{c} & n \text{ constraints}
\end{array}
$$

$$
\begin{array}{llll}
(\mathrm{D}') & (-)\min & -\mathbf{b}^T\mathbf{u} + \mathbf{b}^T\mathbf{v} + 0^T\mathbf{s} & 2m+n \text{ variables} \\
& \text{s. t.} & \mathbf{A}^T\mathbf{u} - \mathbf{A}^T\mathbf{v} + \mathbf{s} = \mathbf{c} & n \text{ constraints} \\
& & \mathbf{u}, \mathbf{v}, \mathbf{s} \ge 0 &
\end{array}
$$

(i) Dimensionality becomes larger.

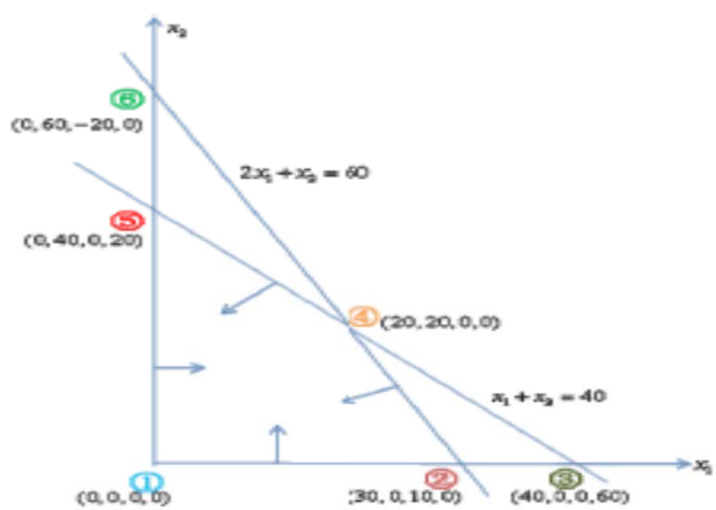
(ii) Keep (*dual*) feasibility
Maintain complementary slackness
Seek (*primal*) feasibility

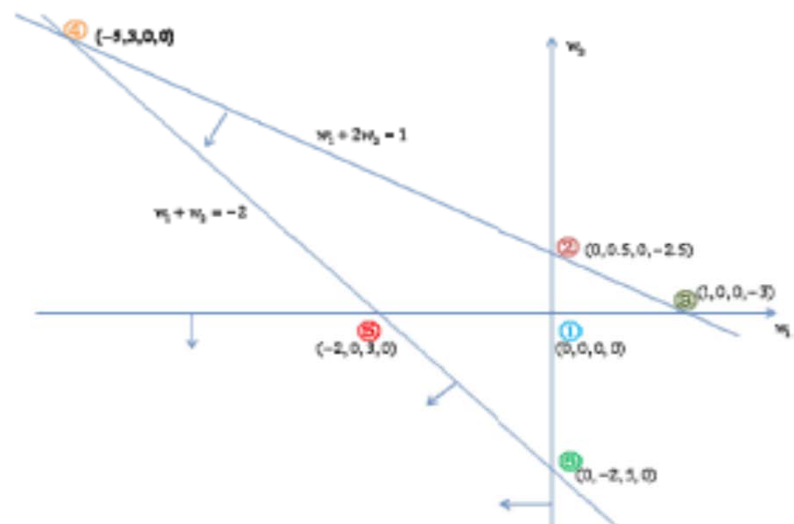
Corresponding basic solutions

- At a primal basic solution **x** with basis B, we defined a dual basic solution $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$

$$
(P)\quad
\begin{aligned}
\text{Minimize}\quad & x_1 - 2x_2 \\
\text{subject to}\quad & x_1 + x_2 + x_3 = 40 \\
& 2x_1 + x_2 + x_4 = 60 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

$$
(D)\quad
\begin{aligned}
\text{Maximize}\quad & 40w_1 + 60w_2 \\
\text{subject to}\quad & w_1 + 2w_2 + w_3 = 1 \\
& w_1 + w_2 + w_4 = -2 \\
& w_1 \le 0, w_2 \le 0 \\
& w_3 \ge 0, w_4 \ge 0
\end{aligned}
$$





Basic ideas of dual simplex method

- Starting with a dual basic feasible solution

(1) Start with a basis

$$\mathbf{A} = [\,\mathbf{B} \mid \mathbf{N}\,]$$

such that

$\mathbf{w}^T := \mathbf{c}_B^T \mathbf{B}^{-1}$ is dual feasible, *i.e.*,
$\mathbf{A}^T \mathbf{w} \leq \mathbf{c}$

(2) Further define

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \end{pmatrix}$$

then

$$\begin{aligned} \mathbf{r}^T \mathbf{x} &= (\mathbf{c}^T - \mathbf{w}^T \mathbf{A})\mathbf{x} \\ &= \mathbf{c}^T \mathbf{x} - \mathbf{w}^T \mathbf{A} \mathbf{x} \\ &= \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} \\ &= 0 \end{aligned}$$

i.e., complementary slackness condition holds.

Basic ideas of dual simplex method

- Checking optimality

(3) Since

$$\mathbf{A}\mathbf{x} = [\ \mathbf{B} \mid \mathbf{N}\ ][\ \frac{\mathbf{x}_B}{\mathbf{x}_N}\ ] = \mathbf{B}\mathbf{B}^{-1}\mathbf{b} = \mathbf{b}.$$

If $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq 0$, then
we have primal feasibility
and $\mathbf{x} = [\ \frac{\mathbf{x}_B}{\mathbf{x}_N}\ ]$ is primal optimal,
$\mathbf{w}^T := \mathbf{c}_B^T\mathbf{B}^{-1}$ is dual optimal.

Basic ideas of dual simplex method

- Pivoting move

If there exists $p \in \mathbf{B}$ such that $x_p < 0$

then A_p may leave the basis

and $x_p \longleftarrow 0$ becomes nbv.

And we have to pivot-in a nbv x_q for $q \in \mathbf{N}$.

Related issues

- Question 1: Which x_q is entering the basis?
- Analysis:

(1) We should keep dual feasibility and complementary slackness.

(2) Complementary slackness condition always holds according to the way we define $\mathbf{w}$ and $\mathbf{x}$.

(3) The real problem is to keep dual feasibility while x_p leaves and x_q enters.

Related issues

- Question 2:

  Where is the dual feasibility information?
- Guess: must be from the fundamental matrix and vector **c.**

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

$$\begin{aligned} \mathbf{c}^T\mathbf{M}^{-1} &= (\ \mathbf{c}_B^T \mid \mathbf{c}_N^T\ ) \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \\ &= (\ \mathbf{c}_B^T\mathbf{B}^{-1} \mid \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}\ ) \\ &= (\ \underset{\substack{\nearrow \\ \text{dual variable}}}{\mathbf{w}^T} \mid \underset{\substack{\nwarrow \\ \text{reduced cost}}}{\mathbf{r}_N^T}\ ) \end{aligned}$$

$$r_N \geq 0 \iff \text{dual feasibility}$$

<u>Answer:</u>

$$x_p \longrightarrow \text{nonbasic}$$

$$x_q \longleftarrow \text{basic}$$

$$\mathbf{M} \longleftarrow \text{update}$$

We check

$$\mathbf{c}^T\mathbf{M}^{-1} = (\ \mathbf{w}^T \mid \mathbf{r}_N^T\ )$$

for dual feasibility!

Related issues

- Question 3: When $x_p \longrightarrow \text{nonbasic}$

$$x_q \longleftarrow \text{basic}$$

$$\text{How will } \mathbf{M}^{-1} \text{ change?}$$

- Answer: Sherman-Morrison-Woodbury formula.

Sherman, J.; Morrison, W. J. (1949). "Adjustment of an Inverse Matrix Corresponding to Changes in the Elements of a Given Column or a Given Row of the Original Matrix (abstract)". Annals of Mathematical Statistics 20: 621. doi:10.1214/aoms/1177729959

Sherman-Morrison-Woodbury Formula

- Lemma: M : nonsingular $n \times n$ matrix.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in R^n$$

If $w := 1 + v^T M^{-1} u \neq 0$
then
$(M + uv^T)$ is nonsingular and

$$(M + uv^T)^{-1} = M^{-1} - (1/w)M^{-1}uv^T M^{-1}.$$

Proof: Check that

$$(M + uv^T)[M^{-1} - (1/w)M^{-1}uv^T M^{-1}] = I.$$

Proof from wikipedia.org/wiki

We verify the properties of the inverse. A matrix Y (in this case the right-hand side of the Sherman–Morrison formula) is the inverse of a matrix X (in this case $A + uv^T$) if and only if $XY = YX = \mathrm{I}$.

We first verify that the right hand side (Y) satisfies $XY = \mathrm{I}$. **Note that $v^T A^{-1} u$ is a scalar, and so $(1 + v^T A^{-1} u)$ can be factored out.**

$$\begin{aligned} XY &= (A + uv^T)\left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}\right) \\ &= AA^{-1} + uv^TA^{-1} - \frac{AA^{-1}uv^TA^{-1} + uv^TA^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u} \\ &= I + uv^TA^{-1} - \frac{uv^TA^{-1} + uv^TA^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u} \\ &= I + uv^TA^{-1} - \frac{(1 + v^TA^{-1}u)uv^TA^{-1}}{1 + v^TA^{-1}u} \\ &= I + uv^TA^{-1} - uv^TA^{-1} \\ &= I. \end{aligned}$$

In the same way, it is verified that

$$YX = \left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}\right)(A + uv^T) = I.$$

Observations

(1) It takes $o(n^3)$ elementary operations to invert the matrix $(M + uv^T)$ directly.

(2) The formula only takes $o(n^2)$ elementary operations.

(3) Sometimes, we call it *rank-one* updating method.

(4) For our case

$$\left.\begin{cases} x_p \longrightarrow \text{out} \\ x_q \longleftarrow \text{in} \end{cases}\right\} M \text{ becomes } \bar{M}$$

$$\bar{M} = M + e_q(e_p - e_q)^T.$$

Example

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \end{array}\right] \end{array}$$

$$M = \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \longrightarrow e_5^T \end{array}$$

$$x_1 \to \text{out},\ p = 1,\ \ x_5 \to \text{in},\ q = 5$$

$$\bar{M} = \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \longrightarrow e_1^T \end{array}$$

The qth row (e_q^T) of M is replaced by e_p^T in $\bar{M}$,

$$M + e_q(e_p - e_q)^T =$$

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] + \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right) \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & -1 \end{array}\right)$$

$$= \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] + \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \bar{M}.$$

Analysis

Let $u = e_q, \; v = (e_p - e_q)$

$$\bar{M}^{-1} = M^{-1} - \frac{M^{-1} + e_q(e_p - e_q)^T M^{-1}}{1 + (e_p - e_q)^T M^{-1} e_q}$$

Notice that

$$\begin{aligned} e_q^T M^{-1} &= q\text{th row of } M^{-1} \\ &= e_q^T \end{aligned}$$

Hence

$$\begin{aligned} \bar{M}^{-1} &= M^{-1} - \frac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{1 + e_p^T M^{-1} e_q - e_q^T e_q} \\ &= M^{-1} - \frac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{e_p^T M^{-1} e_q} \end{aligned}$$

Since $(\mathbf{w}, r_N^T) = c^T M^{-1}$, we have

$$\begin{aligned} (\bar{\mathbf{w}}, \bar{r}_N^T) &= c^T \bar{M}^{-1} \\ &= c^T \left(M^{-1} - \frac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{e_p^T M^{-1} e_q}\right) \\ &= (\mathbf{w}, r_N^T) - \frac{c^T M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{e_p^T M^{-1} e_q} \end{aligned}$$

We define

$$\begin{aligned} \boldsymbol{u}^T &= \boldsymbol{e}_p^T \mathbf{B}^{-1} \\ y_j &= \boldsymbol{u}^T \boldsymbol{A}_j \\ \gamma &= \frac{r_q}{y_q} \end{aligned}$$

then

$$\begin{aligned} \bar{\mathbf{w}} &= \mathbf{w} + \gamma u \\ \bar{r}_j &= r_j - \gamma y_j, \; \forall j \in \mathbf{N} - \{q\} \\ \bar{r}_p &= -\gamma \end{aligned}$$

Observations

1. $u^T = e_p^T \mathbf{B}^{-1} \Longrightarrow u^T$ is the pth row of $\mathbf{B}^{-1}$.
2. $y_q = u^T A_q = -d_p^q$ where $\mathbf{d}^q = -\mathbf{B}^{-1} A_q$.
3. To maintain dual feasibility, we need
   $\bar{r}_p = -\gamma = -\frac{r_q}{y_q} \geq 0$
   and
   $\bar{r}_j = r_j - \gamma y_j \geq 0, \ \forall j \in \mathbf{N} - \{q\}$.

<u>Case 1:</u>

If $\exists j \in \mathbf{N}$ such that $y_j < 0$
then $\frac{-r_j}{y_j} \geq -\gamma$ is required.
Therefore q is chosen by the min-ratio test, *i.e.*,

$$\frac{-r_q}{y_q} = \min\{\frac{-r_j}{y_j} \mid y_j < 0, j \in \mathbf{N}\}$$

<u>Case 2:</u>

If $y_j \geq 0, \ \forall j \in \mathbf{N}$

then $y_j = u^T A_j = e_p^T \mathbf{B}^{-1} A_j, \ j \in \mathbf{N}$.

and $e_p^T \mathbf{B}^{-1} \mathbf{A} = e_p^T \mathbf{B}^{-1} [\ \mathbf{B} \mid \mathbf{N}\ ] \geq 0$.

Thus $e_p^T \mathbf{B}^{-1} \mathbf{A}\mathbf{x} \geq 0, \ \forall \mathbf{x} \geq 0, \ \mathbf{x}$ feasible.

$\|$

$e_p^T \mathbf{B}^{-1} \mathbf{b} = e_p^T \mathbf{x}_B = x_p. \ \rightarrow\leftarrow \ x_p < 0$

Therefore there is no primal feasible solution!

Dual simplex method

Step 1 (starting with a feasible basic solution):

In the primal problem, given,

$\mathbf{B} = [\mathbf{A}_{j_1}, \mathbf{A}_{j_2}, \mathbf{A}_{j_3}, \ldots \mathbf{A}_{j_m}]$

$\widetilde{\mathbf{B}} = [j_1, j_2, j_3, \ldots j_m]$

A dual basic feasible solution $\mathbf{w}$ can be obtained by solving

$$\mathbf{B}^{\mathbf{T}}\mathbf{w} = \mathbf{c}_B$$

Compute the reduced cost $\mathbf{r}$ with

$$r_j = c_j - \mathbf{w}^{\mathbf{T}}\mathbf{A}_j, \quad \forall j \notin \widetilde{\mathbf{B}}$$

Step 2 (checking for optimality):

Compute $\mathbf{x}_B$ by solving

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}$$

If $\mathbf{x}_B \geq \mathbf{0}$, then STOP. The current solution

$$\left[\begin{array}{c} \mathbf{x}_B \\ \hline \mathbf{x}_N \end{array}\right] = \left[\begin{array}{c} \mathbf{B}^{-1}\mathbf{b} \\ \hline \mathbf{0} \end{array}\right]$$

is optimal.

Otherwise go to Step 3.

Step 3 (leaving the basis):

Choose a basic variable $x_{j_p} < 0$ with index $j_p \in \widetilde{\mathbf{B}}$.

Step 4 (checking for infeasibility):

Compute $\mathbf{u}$ by solving

$$\mathbf{B}^T\mathbf{u} = \mathbf{e}_p$$

Also compute

$$y_j = \mathbf{u}^T\mathbf{A}_j, \quad \forall j \notin \widetilde{\mathbf{B}}$$

If $y_j \geq 0$, $\forall j \notin \widetilde{\mathbf{B}}$; then STOP. The primal problem is infeasible.

Otherwise go to Step 5.

Dual simplex method

Step 5 (entering the basis):

Choose a nonbasic variable x_q by the minimum ratio test

$$\frac{-r_q}{y_q} = \min\left\{ \frac{-r_j}{y_j} \,\middle|\, y_j < 0, j \notin \widetilde{\mathbf{B}} \right\}.$$

Set

$$\frac{-r_q}{y_q} = -\gamma$$

Step 6 (updating the reduced costs):

$$r_j \leftarrow r_j - \gamma y_j \quad \forall j \notin \tilde{\mathrm{B}}, \quad j \neq q$$

$$r_{j_p} \leftarrow -\gamma$$

Step 7 (updating the current solution and basis):

Compute $\mathbf{d}$ by solving

$$\mathbf{B}\mathbf{d} = -\mathbf{A}_q$$

Set

$$\begin{aligned}
x_q &\leftarrow \alpha = \frac{x_{j_p}}{y_q} = \left(\frac{-x_{j_p}}{d_p}\right) \\
x_{j_i} &\leftarrow x_{j_i} + \alpha d_{j_i}, \qquad \forall j_i \in \widetilde{\mathbf{B}}, \quad i \neq p \\
\mathbf{B} &\leftarrow \mathbf{B} + [A_q - A_{j_p}]\mathbf{e}_p^T \\
\widetilde{\mathbf{B}} &\leftarrow \widetilde{\mathbf{B}} \cup \{q\} \setminus \{j_p\}
\end{aligned}$$

Go to Step 2.

Example

$$\begin{array}{lrrrrcl} \text{Minimize} & -2x_1 & -x_2 & & & & \\ \text{s.t} & x_1 & +x_2 & +x_3 & & = & 2 \\ & x_1 & +x_4 & & & = & 1 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

Step 1 (starting): Choose $\mathrm{B} = \{1, 4\}$

$$\mathrm{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Then the dual solution

$$\mathbf{w} = \mathbf{c}_B^T \mathbf{B}^{-1} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Computing r_j, $\forall j \notin \widetilde{\mathrm{B}}$, we have $r_2 = 1, r_3 = 2$, which implies that $\mathbf{w}$ is dual feasible.

Step 2 (checking for optimality):

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

the corresponding primal vector is infeasible.

Step 3 (leaving the basis):

$x_4 < 0$ (the second element in $\widetilde{\mathrm{B}}$), x_4 leaves the basis and let $p = 2$.

Step 4 (checking infeasibility):

$$\mathbf{u}^T = \mathbf{e}_2^T \mathbf{B}^{-1} = [-1 \quad 1]$$

and

$$y_2 = \mathbf{u}^T \mathbf{A}_2 = -1, \quad y_3 = \mathbf{u}^T \mathbf{A}_3 = -1$$

Step 5 (entering the basis):

Take the minimum ratio test

$$-\frac{r_2}{y_2} = \min\left\{ \frac{-1}{-1}, \frac{-2}{-1} \right\} = 1 = -\gamma$$

Therefore x_2 is entering the basis and $p = 2$.

Example - continue

Step 6 (updating the reduced cost):

$$r_4 = -\gamma = 1 \text{ and } r_3 = 2 - \gamma y_3 = 1$$

(note that r_2 has been changed from 1 to 0 as x_2 enters the basis.)

Step 7 (updating current solution and basis):

Solving for $\mathbf{d}$ in $\mathbf{Bd} = -\mathbf{A}_2$, we obtain

$$\mathbf{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Also

$$x_2 = \alpha = \frac{x_4}{y_2} = 1$$

$$x_1 = 2 - 1 \times 1 = 1$$

Thus the new primal vector has $x_1 = x_2 = 1$ (and nonbasic variables $x_3 = x_4 = 0$).

Since it's nonnegative, we know it's a optimal solution to the original linear program.

The corresponding optimal basis $\mathbf{B}$ becomes

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

How to start the dual simplex method?

$$(P)\quad \min\ \mathbf{c}^T\mathbf{x} \quad \text{s. t.}\ \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$$

$$(D)\quad \max\ \mathbf{b}^T\mathbf{w} \quad \text{s. t.}\ \mathbf{A}^T\mathbf{w} \leq \mathbf{c}$$

$\mathbf{A} = [\ \mathbf{B} \mid \mathbf{N}\ ]$, $\mathbf{B}_{m\times m}$ nonsingular matrix.

$\mathbf{w}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$

$\mathbf{A}^T\mathbf{w} \leq \mathbf{c}$? If not, consider:

$$(P')\quad \min\ \mathbf{c}^T\mathbf{x} \quad \text{s. t.}\ \mathbf{A}\mathbf{x} = \mathbf{B}e,\ \mathbf{x} \geq 0$$

$$(D')\quad \max\ e^T\mathbf{B}^T\mathbf{w} \quad \text{s. t.}\ \mathbf{A}^T\mathbf{w} \leq \mathbf{c}$$

$$e^T = (1, 1, \ldots, 1)$$

Observations

(1) $\mathbf{x} = \binom{e}{0}$ is a primal bfs of (P′).

(2) (D) and (D′) have the same feasible domain.

(3) Apply the revised simplex method to (P′), either it stops at an optimal solution, or find (P′) is unbounded.

(4) If it stops at an optimal solution, then $\mathbf{w}^{*T} = \mathbf{c}_{B^*}^T(\mathbf{B}^*)^{-1}$ is feasible to (D′). Hence $\mathbf{w}^*$ is feasible to (D).

(5) If (P′) is unbounded, then we find a feasible direction $\mathbf{d}$, such that $\mathbf{Ad} = 0$, $\mathbf{d} \underset{\neq}{\geq} 0$ and $\mathbf{c}^T\mathbf{d} < 0$.
Hence (P) is also unbounded
and (D) must be infeasible!

Remarks

(1) Solving a standard form LP by the dual simplex method is mathematically equivalent to solving its dual LP by the revised (primal) simplex method.

(2) The work of solving an LP by the dual simplex method is about the same as of by the revised (primal) simplex method.

(3) The dual simplex method is useful for the sensitivity analysis.

Complexity of the simplex method

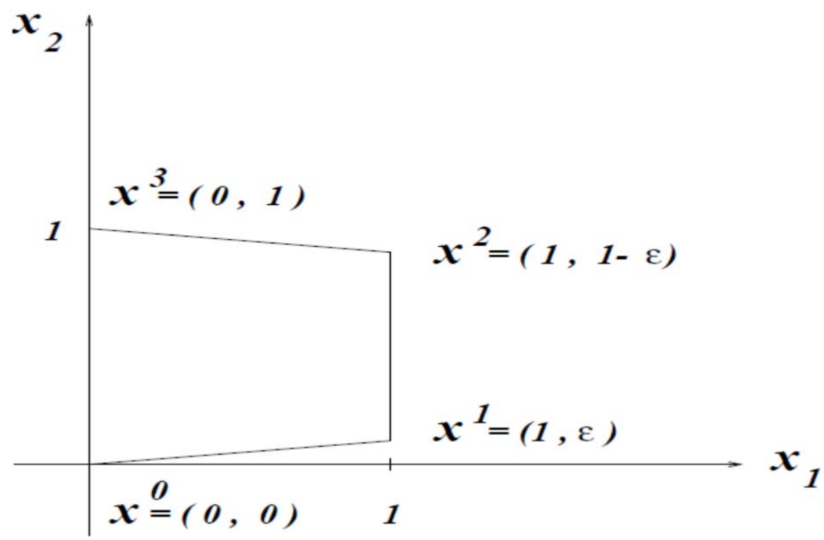
- Total # of elementary operations
$= $ (# of elementary operations at each iteration) $\times$ (# of iterations).
- # of elementary operations at each iteration of the revised simplex method $O(mn)$.
- From practical experience, the simplex method takes about (αm) iterations where $e^\alpha < \log_2(2 + n/m)$. Hence it is of $O(m^2 n)$.
- From the worst-case analysis, Klee and Minty [1972] showed a class of examples (in the d-dimensional space) which $2^d - 1$ iterations for the simplex method.

Worst case performance of the simplex method

Klee-Minty Example:

- Victor Klee, George J. Minty, “How good is the simplex algorithm?” in (O. Shisha edited) Inequalities, Vol. III (1972), pp. 159-175.

$$
\begin{array}{lll}
(2 \text{ dim}) & \min & -x_2 \\
& \text{s. t.} & x_1 \geq 0 \\
& & x_1 \leq 1 \\
& & x_2 \geq \epsilon x_1 \qquad (0 < \epsilon < \frac{1}{2}) \\
& & x_2 \leq 1 - \epsilon x_1 \\
& & x_1, x_2 \geq 0
\end{array}
$$

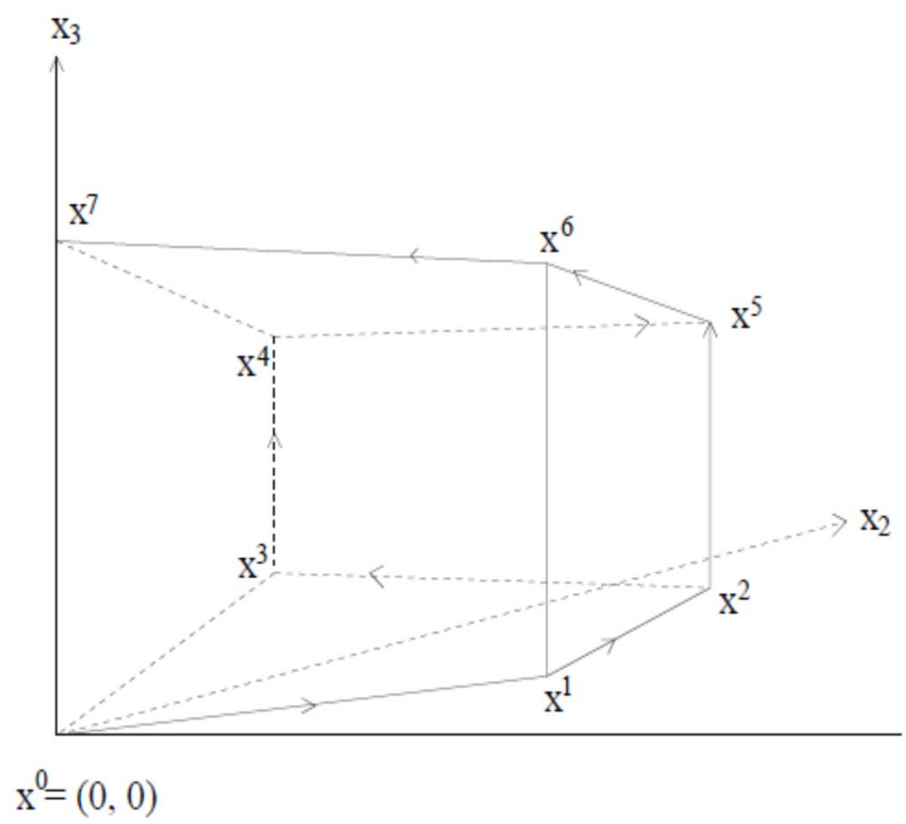


$$\mathbf{x}^0 \to \mathbf{x}^1 \to \mathbf{x}^1 \to \mathbf{x}^3 \text{ (optimal)}$$

$$2^2 - 1 = 3 \text{ iterations}$$

Klee-Minty Example

$$
\begin{aligned}
(3 \text{ dim}) \quad \min \quad & -x_3 \\
\text{s. t.} \quad & x_1 \geq 0 \\
& x_1 \leq 1 \\
& x_2 \geq \epsilon x_1 \\
& x_2 \leq 1 - \epsilon x_1 \\
& x_3 \geq \epsilon x_2 \\
& x_3 \leq 1 - \epsilon x_2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$



$2^3 - 1 = 7$ iterations

Klee-Minty Example

$$
\begin{aligned}
(d\ \text{dim}) \quad \min \quad & -x_d \\
\text{s. t.} \quad & x_1 \geq 0 \\
& x_1 \leq 1 \\
& x_2 \geq \epsilon x_1 \\
& x_2 \leq 1 - \epsilon x_1 \\
& \vdots \\
& x_d \geq \epsilon x_{d-1} \\
& x_d \leq 1 - \epsilon x_{d-1} \\
& x_i \geq 0
\end{aligned}
$$

$$2^d - 1 \text{ iterations}$$

Hence, in theory, the simplex method is not a polynomial-time algorithm. It is an *exponential time* algorithm!