Convex Invariants in Multipartite Tournaments

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Abstract

In the study of convexity spaces, the most common convex invariants are based on notions of independence with respect to taking convex hulls. In [15], H-independence, R-independence, and convex independence were studied to prove results about the Helly number, Radon number, and rank of a clone-free multipartite tournament under 2-path convexity. In this paper, we extend many of these results to general multipartite tournaments. In particular, we determine conditions under which a convexly independent set of vertices is R- and/or H-independent. We also investigate conditions under which an R-independent set is H-independent.

1 Introduction

Convexity in graphs and digraphs has been investigated in many different contexts. In each case, the convex sets are defined in terms of a particular type of path. Let T be a graph or digraph and let $\mathcal{P}$ be a set of paths in T. A subset $C \subseteq V(T)$ is $\mathcal{P}$-*convex* if, whenever $v, w \in C$, any path in $\mathcal{P}$ that originates at v and ends at w can involve only vertices in C. The most commonly studied type of convexity is *geodesic convexity*, where $\mathcal{P}$ is taken to be the set of geodesics in T (see [4], [3], [7] and [9]). Another type of convexity is *induced path convexity*, where $\mathcal{P}$ is the set of all chordless paths (see [5]). Other types of convexity include *path convexity* (see [12] and [17]), and *triangle path convexity* (see [2]). In this paper we consider *two-path convexity* where $\mathcal{P}$ is taken to be the set of all 2-paths in a digraph T. Two-path convexity was first studied in tournaments in [6], [20], and [11]. More recent results in multipartite tournaments are studied in [1], [13], [14], [16], and [15].

Several numerical invariants can be associated with a convex structure. Two of the most studied are the Helly and Radon numbers (see [10], [18], and [2]). These can each

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be defined using notions of independence (see [19, Chap. 3]). For a subset $S \subseteq V(T)$, the *convex hull* of S, denoted $C(S)$, is defined to be the smallest convex subset containing S. Let $F \subseteq V(T)$. We say F is *H-independent* if $\bigcap_{p \in F} C(F - \{p\}) = \emptyset$. The *Helly number* $h(T)$ is the size of a largest H-independent set. A partition $F = A \cup B$ with $C(A) \cap C(B) \neq \emptyset$ is called a *Radon partition* of F, and F is *R-independent* if F does not have a Radon partition. The *Radon number* $r(T)$ is the size of a largest R-independent set. Note that some authors define the Radon number to be the smallest number r such that every subset of size r has a Radon partition. This results in a Radon number one larger than the definition we use. It is well known that if F is H-independent then F is R-independent (see [19, p. 163]). This implies Levi's inequality, which is $h(T) \leq r(T)$.

One final type of independence we will consider is convex independence. We say F is *convexly independent* if, for each $p \in F$, we have $p \notin C(F - \{p\})$. The *rank* $d(T)$ is the size of a largest convexly independent set. Rank provides an upper bound on the number of elements of a convex set that are needed to generate the convex set using convex hulls. In [8], D. Haglin and M. Wolf used the fact that the collection of two-path convex subsets in a tournament has rank 2 to construct an algorithm for computing the convex subsets of a given tournament. Note that any set that is H- or R-independent must also be convexly independent, so rank is an upper bound for both the Helly and Radon numbers.

We call $v, w \in V(T)$ *clones* if they have the same inset and outset, and we say T is *clone-free* if $V(T)$ has no clones. Given $v \in V(T)$, we define $[v] = \{w \in V(T) : w$ and v are clones$\}$. In [15], we studied convex independence in clone-free multipartite tournaments and applied our results to Helly and Radon numbers in this context. In this paper, we seek to generalize these results to general multipartite tournaments.

For T a digraph, we denote an arc (v, w) by $v \to w$ and say that v dominates w. If $U, W \subseteq V(T)$, then we write $U \to W$ to indicate that every vertex in U dominates every vertex in W. We call T a *multipartite tournament* if it is possible to partition V into partite sets $P_1, P_2, \ldots, P_k$, $k \geq 2$ such that there is precisely one arc between each pair of vertices in different partite sets and no arcs between vertices in the same partite set. In the case when $k = 2$ we will also call T a *bipartite tournament*. If $A, B \subseteq V(T)$, we denote the convex hull of $A \cup B$ by $A \vee B$. If $v, w \in V$, we drop the set notation and write $\{v\} \vee \{w\}$ as $v \vee w$. Finally, we denote by T^* the digraph with the same vertex set as T, and where (v, w) is an arc of T^* if and only if (w, v) is an arc of T.

2 Convex Independence in Multipartite Tournaments

In [14] and [15], we studied properties of convexly independent sets under two-path convexity in multipartite tournaments. In this section, we present some notational conventions, definitions, and results that will be used throughout this paper.

The following is from [14]. Let T be a multipartite tournament, and let $U \subseteq V(T)$ be a convexly independent set. Then U can have a nonempty intersection with at most two partite sets. Thus, T has partite sets P_0 and P_1 such that $A = U \cap P_0$ and $B = U \cap P_1$ with $U = A \cup B$. It follows that $A \to B$ or $B \to A$. Note that T and T^* have the same convex subsets, so by relabelling P_0 and P_1 and reversing the arcs, if necessary, we can assume that $|A| \geq |B|$ and $A \to B$ if $B \neq \emptyset$.

Given $C \subseteq V(T)$, we define the following, which generalizes the analogous definition for clone-free multipartite tournaments in [14].

$$\begin{aligned} D_C^{\rightarrow} &= \{z \in V(T) : z \to x \text{ for some } x \in C, y \to z \text{ for all } y \in C - [x]\} \\ D_C^{\leftarrow} &= \{z \in V(T) : x \to z \text{ for some } x \in C, z \to y \text{ for all } y \in C - [x]\} \end{aligned}$$

The following appears in [13].

Theorem 2.1. Let T be a clone-free multipartite tournament. Let A and B form a convexly independent set, with $A \to B$ when both sets are nonempty.

1. If $A = \{x_1, \ldots, x_m\}$, $m \geq 2$, then one can order the vertices in A such that there exist $u_2, \ldots, u_m \in D_A^{\rightarrow}$ (resp., in $D_A^{\leftarrow}$ if $D_A^{\rightarrow} = \emptyset$) such that $u_i \to x_i$ (resp., $x_i \to u_i$).
2. If $|A| \geq 3$, then $D_A^{\rightarrow} \neq \emptyset$ if and only if $D_A^{\leftarrow} = \emptyset$, and $D_A^{\rightarrow}$ and $D_A^{\leftarrow}$ each lie in at most one partite set.
3. Suppose $A, B \neq \emptyset$. If $|A| \geq 2$, then $D_A^{\rightarrow}$ is in the same partite set as B, and if $|B| \geq 2$, then $D_B^{\leftarrow}$ is in the same partite set as A.
4. If $|A|, |B| \geq 2$, then $D_B^{\leftarrow} \to D_A^{\rightarrow}$.
5. Any vertex that distinguishes vertices in A must be in either $D_A^{\rightarrow}$ or $D_A^{\leftarrow}$ and any vertex that distinguishes vertices in B must be in $D_B^{\leftarrow}$ or $D_B^{\rightarrow}$. If $A, B \neq \emptyset$ or if $|A| \geq 3$, then any vertex that distinguishes vertices in A must be in $D_A^{\rightarrow}$ and any vertex that distinguishes vertices in B must be in $D_B^{\leftarrow}$.

When $|A| \geq 2$, Theorem 2.1(1) implies that one of $D_A^{\rightarrow}$ or $D_A^{\leftarrow}$ is nonempty, and when $|A| \geq 3$, Theorem 2.1(2) implies that the other is empty. In the case of $B = \emptyset$, we choose T or T^* so that $D_A^{\rightarrow} \neq \emptyset$ and let P_1 be the partite set containing $D_A^{\rightarrow}$. We will assume that this and the above notational conventions and choices have been made throughout the remainder of the paper.

Let $U = A \cup B$ be a convexly independent set. As noted in [15], $C(U)$ can be constructed in two ways. The most natural method is using the sets $C_k(U)$, defined as follows. Let $C_0(U) = U$ and for $k \geq 1$, let

$$C_k(U) = C_{k-1}(U) \cup \{w \in V(T) : x \to w \to y \text{ for some } x, y \in C_{k-1}(U)\}$$

Then $C(U) = \bigcup_{k=0}^{\infty} C_k(U)$. Another way to generate convex hulls is to define $\Delta_k(U)$ as follows. Let $\Delta_0(U) = A$, $\Delta_1(U) = B \cup C_1(A)$, and for $t \geq 2$, let $\Delta_t(U) = C_1(\Delta_{t-1}(U))$. As before, $C(U) = \bigcup_{i=0}^{\infty} \Delta_i(U)$.

The following generalizes the definition of $D_A^{\rightarrow}$ and $D_B^{\leftarrow}$.

Definition 2.2. Let $U = A \cup B$ be a convexly independent set with $A \to B$. For each $x \in U$, define $\overline{D}_t(x)$ for $t \geq 0$ as follows. If $x \in A$, then $\overline{D}_0(x) = \{x\}$, and if $x \in B$, then $\overline{D}_0(x) = \emptyset$ and $\overline{D}_1(x) = \{x\}$. Otherwise, we have

$$\begin{aligned}\overline{D}_{2k}(x) &= \{v \in \Delta_{2k}(U) : u \to v \text{ for some } u \in \overline{D}_{2\ell-1}(x), \ell \leq k\} \\ \overline{D}_{2k+1}(x) &= \{v \in \Delta_{2k+1}(U) : v \to u \text{ for some } u \in \overline{D}_{2\ell}(x), \ell \leq k\}\end{aligned}$$

We then define $D_t(x) = \bigcup_{k \leq t} \overline{D}_k(x)$ and $D(x) = \bigcup_{t=0}^{\infty} D_t(x)$.

Notice that $\overline{D}_k(x) \subseteq \overline{D}_{k+2}(x)$ for $k \geq 1$ if $x \in A$, and for $k \geq 2$ if $x \in B$. Our main result from [15] was the following.

Theorem 2.3. Let T be a clone-free multipartite tournament, and let $U = A \cup B$ be convexly independent. Suppose $|U| \geq 4$, and let $x, y, z \in U$.

1. For all $k, \ell \geq 0$, $\overline{D}_{2k}(x) \subseteq P_0$ and $\overline{D}_{2\ell+1}(x) \subseteq P_1$.
2. If $x \neq y$, then $\overline{D}_{2k}(x) \to \overline{D}_{2\ell+1}(y)$ for all $k, \ell \geq 0$.
3. Let $u \in \overline{D}_r(x)$, $v \in \overline{D}_s(y)$, where $x \neq y$, r and s have the same parity. If $x, y \in A$ and $\overline{D}_1(x), \overline{D}_1(y) \neq \emptyset$ or if $x, y \in B$ and $\overline{D}_2(x), \overline{D}_2(y) \neq \emptyset$, then $x \vee y = u \vee v$.
4. Let $u \in \overline{D}_m(x)$, $v \in \overline{D}_n(y)$, and $w \in \overline{D}_p(z)$, where x, y, and z are distinct. Then $x \vee y \vee z = u \vee v \vee w$.

We will also make use of the following result from [15].

Lemma 2.4. Let T be a clone-free multipartite tournament, let $U = A \cup B$ be a convexly independent set, and let $z \in V - (P_0 \cup P_1)$.

1. If $|U| \geq 4$ and z distinguishes two vertices in V_U, then $(V_U \cap P_0) \to z \to (V_U \cap P_1)$.
2. If $|U| \geq 3$ and z distinguishes two vertices in $U \cup D_A^{\to}$, then $(A \cup D_B^{\leftarrow}) \to z \to (B \cup D_A^{\to})$.

3 Induced Clone-Free Digraphs

From our work in [14] and [15], we have a great deal of knowledge about convexly independent sets in clone-free multipartite tournaments. The following helps us make connections between general multipartite tournaments and clone-free multipartite tournaments.

Definition 3.1. Let T be a directed graph.

1. We define the *induced clone-free digraph* of T, denoted by T_f, to be the multipartite tournament with vertex set $\{[v] : v \in V(T)\}$ and arcs given by $[v] \to [w]$ if and only if $v \to w$ in T.

2. For $U \subseteq V(T)$, define $[U] = \{[v] : v \in U\}$.

The following is immediate.

Lemma 3.2. For each $S \subseteq V(T_f)$, there exists a subset $U \subseteq V(T)$ with $[U] = S$. One such set is $\{v \in V(T) : [v] \in S\}$. Moreover, U can be chosen so that no two vertices in U are clones.

Given a convexly independent set $U \subseteq V(T)$, we write $U = A \cup B$ where A and B have the properties discussed in the previous section. We now relate the convex subsets in T and T_f.

Lemma 3.3. Let T be a multipartite tournament, and let $U \subseteq V(T)$ be a convexly independent set. For all $k \geq 0$, we have

1. $[C_k(U)] = C_k([U])$.
2. $[\Delta_k(U)] = \Delta_k([U])$.
3. $[\overline{D}_k(x)] = \overline{D}_k([x])$ for all $x \in U$.
4. $[D_k(x)] = D_k([x])$ for all $x \in U$.

Proof. For (1), we induct on k. For $k = 0$, we get $[C_0(U)] = [U] = C_0([U])$. For $k > 0$, let $[x] \in [C_k(U)]$. Then x or some clone of x is in $C_k(U)$, so $u_1 \to x \to u_2$ for some $u_1, u_2 \in C_{k-1}(U)$. Thus, $[u_1] \to [x] \to [u_2]$. By induction, $[u_1], [u_2] \in [C_{k-1}(U)] = C_{k-1}([U])$, so we have $[x] \in C_k([U])$. Thus, $[C_k(U)] \subseteq C_k([U])$. The other direction is similar, and (2) follows similarly.

For (3), the case $k = 0$ when $x \in A$ and the cases $k = 0, 1$ when $x \in B$ are trivial. For $k \geq 1$ when $x \in A$ or $k \geq 2$ when $x \in B$, let $[v] \in [\overline{D}_k(x)]$ with $v \in \overline{D}_k(x)$. If k is odd, there exist $u \in \overline{D}_{k-1}(x)$, $w \in \Delta_{k-1}(U)$ with $w \to v \to u$. By induction and (2), $[u] \in \overline{D}_{k-1}([x])$ and $[w] \in \Delta_{k-1}([U])$. Thus, $[v] \in \overline{D}_k([x])$, and so $[\overline{D}_k(x)] \subseteq \overline{D}_k([x])$. Similarly, $\overline{D}_k([x]) \subseteq [\overline{D}_k(x)]$, which gives us (3). The case when k is even is similar and part (4) follows immediately. $\square$

From this, we get the following.

Corollary 3.4. Let T be a multipartite tournament, and let $U \subseteq V(T)$ be a convexly independent set. Then

1. $[C(U)] = C([U])$.
2. $[D(x)] = D([x])$ for all $x \in U$.

We end this section with a lemma that describes when clones may be added to or taken away from a set without substantially affecting its convex hull.

Lemma 3.5. Let $U \subseteq V(T)$, $u \in U$, and $S \subseteq [u]$. Then

1. $C_k(U \cup S) = C_k(U) \cup S$ for all $k \geq 0$, and so $C(U \cup S) = C(U) \cup S$.

2. If $(U - S) \cap [u] \neq \emptyset$, then $C_k(U) - S \subseteq C_k(U - S)$ for all $k \geq 0$, and so $C(U) - S \subseteq C(U - S)$. Equality holds if U is convexly independent and $S \cap U \neq \emptyset$.

Proof. For (1), we induct on k. For $k = 0$, we have $C_0(U \cup S) = U \cup S = C_0(U) \cup S$. For $k \geq 1$, we clearly have $C_k(U) \cup S \subseteq C_k(U \cup S)$, so it suffices to show the other inclusion. Let $v \in C_k(U \cup S)$. If $v \in U \cup S$, then clearly $v \in C_k(U) \cup S$. Otherwise, there exist $w_1, w_2 \in C_{k-1}(U \cup S)$ with $w_1 \to v \to w_2$. By induction, $w_1, w_2 \in C_{k-1}(U) \cup S$.

If $w_1, w_2 \in C_{k-1}(U)$, then $v \in C_k(U)$. Since w_1 and w_2 are not clones, we cannot have $w_1, w_2 \in S$, so it suffices to prove the case where $w_1 \in C_{k-1}(U)$ and $w_2 \in S$. Since w_2 and u are clones, $w_1 \to v \to u$, and so $v \in C_k(U)$, proving the result.

For (2), let $v \in (U - S) \cap [u]$. We then have $[v] = [u]$, and so we can apply (1) with v in place of u, $U - S$ in place of U, and $U \cap S$ in place of S to get

$$C_k(U) = C_k((U - S) \cup (U \cap S)) = C_k(U - S) \cup (U \cap S)$$

and so $C_k(U) - S \subseteq C_k(U - S)$.

Now suppose U is convexly independent and $S \cap U \neq \emptyset$. Clearly, $C_k(U - S) \subseteq C_k(U)$, so it suffices to show $C_k(U - S) \cap S = \emptyset$ for all $k \geq 0$. If not, let k be minimal such that there exists $x \in C_k(U - S) \cap S$ for some k. If $k = 0$, then $x \in U - S$, which contradicts $x \in S$. If $k \geq 1$, then there exist $w_1, w_2 \in C_{k-1}(U - S)$ with $w_1 \to x \to w_2$. Since $x \in S \subseteq [u]$, we have $w_1 \to [u] \to w_2$, and so $[u] \subseteq C_k(U - S) \subseteq C(U - S)$. Since $S \cap U \neq \emptyset$, this contradicts the convex independence of U. $\square$

4 Convexly Independent Sets in General Multipartite Tournaments

In this section, we seek to use our understanding of convex independence in clone-free multipartite tournaments to help us understand convex independence in general multipartite tournaments. Throughout this section, let T be a (general) multipartite tournament. We begin by showing that convex independence of a set in T is preserved when it is passed down to T_f.

Lemma 4.1. Let $U \subseteq V(T)$ be a convexly independent set in T. Then $[U]$ is a convexly independent set of T_f.

Proof. For contradiction, assume that U is convexly independent and $[U]$ is convexly dependent. Then there exists $u \in U$ with $[u] \in C([U] - \{[u]\})$. Let $[u] \in C_k([U] - \{[u]\})$ with k minimal. Clearly, $k \geq 1$. Then there exist $[v], [w] \in C_{k-1}([U] - \{[u]\})$ with $[v] \to [u] \to [w]$. Since $[U] - \{[u]\} = [U - [u]]$ then $C_{k-1}([U] - \{[u]\}) = [C_{k-1}(U - [u])]$ by Lemma 3.3(1). Thus, we can assume that $v, w \in C_{k-1}(U - [u])$ so that $u \in C_k(U - [u]) \subseteq C(U - [u]) \subseteq C(U - \{u\})$, violating the convex independence of U. $\square$

This helps give us a better characterization of convex subsets in T.

Lemma 4.2. Let $U \subseteq V(T)$. Then U is convexly independent in T if and only if $[U]$ is convexly independent in T_f and for each $u \in U$ either

1. $[u] \cap U = \{u\}$ or
2. u does not distinguish any vertices in $C(U)$.

Proof. First assume that U is convexly independent. By Lemma 4.1, $[U]$ is also convexly independent. Now suppose for contradiction that $u' \in ([u] \cap U) - \{u\}$ and there exist $v, w \in C(U)$ with $v \to u \to w$. By Lemma 3.5(2), $v, w \in C(U) - \{u\} \subseteq C(U - \{u\})$. We get $u \in C(U - \{u\})$, which contradicts convex independence. Thus, u cannot distinguish vertices in $C(U)$.

For the converse, suppose that U is convexly dependent. Then there exists $u \in U$ such that $u \in C(U - \{u\})$. We have $u \in C_k(U - \{u\})$ for some $k \geq 1$, and so there exist $v, w \in C_{k-1}(U - \{u\}) \subseteq C(U)$ with $v \to u \to w$. Thus, u distinguishes vertices in $C(U)$, and so $[u] \cap U = \{u\}$. We then have $[U - \{u\}] = [U] - \{[u]\}$, and so $[u] \in [C(U - \{u\})] = C([U] - \{[u]\})$. This contradicts the convex independence of $[U]$, and the result follows. □

For the case $|[U]| \geq 3$, our next lemma helps us understand the convex independence of U without having to know $C(U)$.

Lemma 4.3. Let $U \subseteq V(T)$ with $|[U]| \geq 3$, and assume that $[U]$ is a convexly independent set in T_f. Then for all $u \in U$, u distinguishes two vertices in $C(U)$ if and only if $D(u) \neq \{u\}$.

Proof. Let $u \in U$ and assume $v \to u \to w$ for some $v, w \in C(U)$. When $u \in A$, $v \in \Delta_k(U)$ for some $k \geq 1$ and $v \in \overline{D}_l(u)$ for some odd $l \geq k$. Similarly, if $u \in B$ then $w \in \Delta_k(U)$ for some $k \geq 2$ and $w \in \overline{D}_l(u)$ for some even $l \geq k$. Either way, $D(u) \neq \{u\}$.

Conversely, assume $D(u) \neq \{u\}$. Suppose $u \in A$ and let k be the smallest positive odd integer such that $\overline{D}_k(u) \neq \{u\}$, say $v \in \overline{D}_k(u) - \{u\}$. Then $v \to [u]$. If $B \neq \emptyset$ then there is a $w \in B$ such that $u \to w$. Thus $v \to u \to w$ so u distinguishes vertices in $C(U)$. Now assume $B = \emptyset$, and note that $|[U]| \geq 3$, $[U]$ is convexly independent in T_f, and T_f is clone-free. By Theorem 2.1(1) there is a $[w] \in D_A^{\to}$ such that $[u] \to [w]$. It follows that $v \to u \to w$, so again u distinguishes vertices in $C(U)$. The argument when $u \in B$ is similar. □

Putting together Lemmas 4.2 and 4.3, we get

Theorem 4.4. Let T be a multipartite tournament, with $U \subseteq V(T)$. If $|[U]| \geq 3$ then U is convexly independent in T if and only if

1. $[U]$ is convexly independent in T_f and
2. For all $u \in U$, either $D(u) = \{u\}$ or $[u] \cap U = \{u\}$.

This result limits the vertices in a convexly independent set that can be clones.

Corollary 4.5. Let T be a multipartite tournament, and let $U \subseteq V(T)$ be a convexly independent set with $|[U]| \geq 3$. If P is a partite set with a nonempty intersection with U, then there exists at most one $[v] \in [P]$ with $|[v] \cap U| \geq 2$.

Proof. Let $[v], [w] \in [P]$ such that $[v] \cap U$ and $[w] \cap U$ each have at least two vertices. Since v and w are not clones, there exists $x \in V$ that distinguishes v and w (say $v \to x \to w$). But then either $w, x \in D(w)$ or $v, x \in D(v)$, which violates Theorem 4.4. □

It also turns out that, in most cases, any $[u] \in V(T_f)$ with $|[u] \cap U| \geq 2$ does not contribute significantly to $C(U)$.

Corollary 4.6. Let U be a convexly independent set with $|[A]| \geq 3$ or $|[U]| \geq 4$, and let $u \in U$ with $|[u] \cap U| \geq 2$. Then $C(U) - C(U - [u]) = [u] \cap U$.

Proof. The fact $[u] \cap U \subseteq C(U) - C(U - [u])$ follows directly from the convex independence of U, so we need only show that $C(U) - C(U - [u]) \subseteq [u] \cap U$. Suppose this is not the case, and let k be minimal such that there exists $y \in C_k(U)$ with $y \in C(U) - C(U - [u])$ and $y \notin [u] \cap U$. If $k = 0$, then $y \in C_0(U) = U$. Moreover, $y \notin C(U - [u])$ so we must have $y \in [u]$ and thus $y \in [u] \cap U$, a contradiction. Therefore, $k \geq 1$.

Since $k \geq 1$, there exist $z_1, z_2 \in C_{k-1}(U)$ with $z_1 \to y \to z_2$. Since $y \notin C(U - [u])$, we must have $z_1 \notin C(U - [u])$ or $z_2 \notin C(U - [u])$. By the minimality of k, this implies that $z_1 \in [u] \cap U$ or $z_2 \in [u] \cap U$. Without loss of generality, we can assume that $u = z_1$ or $u = z_2$. Suppose that $u = z_1$. Since $U = A \cup B$, we have $u \in A$ or $u \in B$. If $u \in B$, then $A \to B$ implies that $A \to u \to y$, and so u distinguishes two vertices in $C(U)$. By Lemma 4.2, this implies that U is not convexly independent, a contradiction. Thus, $u \in A$. If $u = z_2$ and $u \in A$ a similar argument using $y \to U \to B \cup D_{\overrightarrow{A}}$ gives a contradiction. Thus if $u = z_2$ then $u \in B$. Furthermore, since we cannot have both $z_1, z_2 \in [u]$, then the minimality of k implies that one of z_1 and z_2 is in $C(U - [u])$.

We first consider the case where $|[A]| \geq 3$. Let $a_1, a_2, a_3 \in A$ be distinct and not clones of one another. By Lemma 4.2 and Theorem 2.1(1), without loss of generality, there exist $v_2, v_3 \in D_{\overrightarrow{A}}$ such that $v_2 \to a_2$ and $v_3 \to a_3$.

If $u = z_1$, then as discussed above, $u \in A$ and $z_2 \in C(U - [u])$. If there exists some $a \in A - [u]$ with $a \to y$, then $a \to y \to z_2$, and so $y \in C(U - [u])$, a contradiction. Thus, $y \to A - [u]$. Since $|[A]| \geq 3$, y dominates at least two vertices in A that are not clones, and so $y \notin D_{\overrightarrow{A}}$. By Theorem 2.1(5), we must have $y \to A$, and so $y \to u$, a contradiction since $u = z_1$.

If $u = z_2$, then we have $u \in B$ and $z_1 \in C(U - [u])$. If we have $y \to v_2$, we get $z_1 \to y \to v_2$. Since $z_1, v_2 \in C(U - [u])$, this would imply $y \in C(U - [u])$, a contradiction. Thus, $v_2 \to y$ and, similarly, $v_3 \to y$. Since $a_1 \to v_2 \to a_2$, we have $v_2 \in a_1 \vee a_2 \vee u$. Then $v_2 \to y \to u$, $a_2 \to v_3 \to y$, and $v_3 \to a_3 \to v_2$ imply that $a_3 \in a_1 \vee a_2 \vee u$. This contradicts the convex independence of U.

We now move on to the case where $|[A]| \leq 2$. In this case, we must have $|[U]| \geq 4$. Since also $|[A]| \geq |[B]|$, we must have $|[A]| = |[B]| = 2$. So let $a_1, a_2 \in A$ and $b_1, b_2 \in B$, no two of which are clones, and let $v \in D_{\overrightarrow{A}}$, $w \in D_{\overleftarrow{B}}$ such that $v \to a_2$ and $b_2 \to w$. Since $|[u] \cap U| \geq 2$, Theorem 4.4 implies that $D(u) = \{u\}$. Thus, $a_1 \in [u]$ or $b_1 \in [u]$.

If $u = z_1$, then $u \in A$ and $z_2 \in C(U - [u])$ as before. Thus, without loss of generality, $u = a_1$. If $a_2 \to y$, then $a_2 \to y \to z_2$, and so $y \in C(U - [u])$, a contradiction. Thus, $y \to a_2$. If $w \to y$, then we have $b_2 \to w \to b_1$ and $w \to y \to z_2$. Again, this implies $y \in C(U - [u])$, a contradiction. Thus, $y \to w$. But now $u \to y \to a_2$, $y \to w \to b_1$, and $a_2 \to b_2 \to w$. This implies $b_2 \in u \vee a_2 \vee b_1$, which contradicts the convex independence of U.

Finally, if $u = z_2$, then $u \in B$ and $z_1 \in C(U - [u])$, and so we can assume that $u = b_1$. If $y \to b_2$, then $z_1 \to y \to b_2$, and so $y \in C(U - [u])$, a contradiction. Similarly, if $y \to v$, then $a_1 \to v \to a_2$ and $z_1 \to y \to v$ imply that $y \in C(U - [u])$, a contradiction. Thus, $\{b_2, v\} \to y$. But now $b_2 \to y \to u$, $a_1 \to v \to y$, and $v \to a_2 \to b_2$, which implies that $a_2 \in a_1 \vee u \vee b_2$. This contradicts the convex independence of U and completes the proof. □

For our main result, we generalize Theorem 2.3 to general multipartite tournaments.

Theorem 4.7. Let T be a multipartite tournament, and let $U = A \cup B$ be convexly independent. Suppose $|[U]| \geq 4$, and let $x, y, z \in U$.

1. For all $k, \ell \geq 0$, $\overline{D}_{2k}(x) \subseteq P_0$ and $\overline{D}_{2\ell+1}(x) \subseteq P_1$.

2. If $x \neq y$, then $\overline{D}_{2k}(x) \to \overline{D}_{2\ell+1}(y)$ for all $k, \ell \geq 0$.

3. Let $u \in \overline{D}_r(x)$, $v \in \overline{D}_s(y)$, where $[x] \neq [y]$, r and s have the same parity. If $x, y \in A$ and $\overline{D}_1(x), \overline{D}_1(y) \neq \emptyset$ or if $x, y \in B$ and $\overline{D}_2(x), \overline{D}_2(y) \neq \emptyset$, then $x \vee y = u \vee v$.

4. Let $u \in \overline{D}_m(x)$, $v \in \overline{D}_n(y)$, and $w \in \overline{D}_p(z)$, where $[x]$, $[y]$, and $[z]$ are distinct. Then $x \vee y \vee z = u \vee v \vee w$.

Proof. By Theorem 2.3, the theorem holds for clone-free multipartite tournaments. In particular, it holds for T_f. Part (1) follows directly, and (2) follows in the case $[x] \neq [y]$. Moreover, (2) is vacuously true if $[x] = [y]$, since Theorem 4.4(2) implies $D(x) = \{x\}$ and $D(y) = \{y\}$.

For (3), we assume $x, y \in A$, the case $x, y \in B$ being similar. Since (3) holds for T_f, we have $[x \vee y] = [u \vee v]$. We begin by showing that all clones of x and y are in $x \vee y$ and all clones of u and v are in $u \vee v$. Since $\overline{D}_1(x), \overline{D}_1(y) \neq \emptyset$, let $w_1 \in \overline{D}_1(x)$ and $w_2 \in \overline{D}_1(y)$. By (2) and the definitions, we have $x \to w_2 \to y \to w_1 \to x$. Clearly, $x \vee y = w_1 \vee w_2$. Since x' and x are clones, we have $w_1 \to x' \to w_2$, and so $x' \in w_1 \vee w_2 = x \vee y$. Similarly, any clone of y is in $x \vee y$.

For $u \vee v$, let r and s be even, the odd case being similar. If $r, s \geq 2$, we have $u' \in \overline{D}_{r-1}(x)$, $v' \in \overline{D}_{s-1}(y)$ with $u' \to u$ and $v' \to v$. Since also $v \to u'$ and $u \to v'$, we have $u' \vee v' = u \vee v$, and the result follows as with $x \vee y$. The case $u = x$ and $v = y$ was proven in the previous paragraph. If $u = x$ and $v \neq y$, let $q \in \overline{D}_1(x)$. We then have $x \to v' \to v \to q \to x$. As before, $x \vee v = q \vee v'$ and any clone of x or v is in $q \vee v'$. Thus all clones of x and y are in $x \vee y$ and all clones of u and v are in $u \vee v$. Since clones of elements of $x \vee y$ not in $[x] \cup [y]$ would be pulled into $x \vee y$ and similarly for $u \vee v$, this completes the proof. Part (4) follows similarly. □

This leads to the following analogue of Corollary 3.8 in [15].

Corollary 4.8. Let T be a multipartite tournament and let $U = A \cup B$ be a convexly independent set with $|U| \geq 4$. Then for $x \in U$ the $D(x)$ are pairwise disjoint.

Proof. It suffices to show that the $\overline{D}_t(x)$'s are pairwise disjoint for all $t \geq 0$. Suppose that $v \in \overline{D}_t(x) \cap \overline{D}_t(y)$, where $x, y \in U$ are distinct. We do the case of $v \in P_1$. The case $v \in P_0$ is similar. Clearly, we must have $t \geq 2$. Since $v \in \overline{D}_t(x)$, there exists $v' \in \overline{D}_{t-1}(x)$ with $v \to v'$. But since $v \in \overline{D}_t(y)$, Theorem 4.7(2) implies that $v' \to v$, a contradiction. $\square$

5 Helly, Radon, & Convex Independence

Our results in this section explore the relationship between H-, R-, and convex independence. This will help us determine when the Helly number, Radon number, and rank are equal.

The following helps us to determine how the addition of clones affects H-, R-, and convex independence.

Theorem 5.1. Let T be a multipartite tournament. Let $U \subseteq V(T)$, $u \in U$, and $S \subseteq [u]$. Suppose $|[u] \cap U| \geq 2$. If U is convexly independent (respectively R- or H-independent), then $U \cup S$ is also convexly independent (respectively R- or H-independendent).

Proof. Without loss of generality, assume $[u] \cap U \subseteq S$. Assume U is convexly independent and let $x \in U \cup S$. By Lemma 3.5(1),

$$C((U \cup S) - \{x\}) = C([U - \{x\}] \cup (S - \{x\})) = C(U - \{x\}) \cup (S - \{x\})$$

Clearly, $x \notin S - \{x\}$. Moreover, $x \notin C(U - \{x\})$ by convex independence, and so $x \notin C((U \cup S) - \{x\})$, which makes $U \cup S$ convexly independent.

If U is R-independent, suppose we have a nontrivial partition $U \cup S = R_1 \cup R_2$. Let $U_i = U \cap R_i$ and $S_i = S \cap R_i$ for $i = 1, 2$. By Lemma 3.5(1), we have

$$\begin{aligned} C(R_1) \cap C(R_2) &= (C(U_1) \cup S_1) \cap (C(U_2) \cup S_2) \\ &= (C(U_1) \cap C(U_2)) \cup (C(U_1) \cap S_2) \cup (C(U_2) \cap S_1) \cup (S_1 \cap S_2) \end{aligned}$$

Now $C(U_1) \cap C(U_2) = \emptyset$ since U is R-independent. Note that U is convexly independent, and so $U \cup S$ is also convexly independent. Therefore, $C(U_1) \cap S_2 = C(U_2) \cap S_1 = \emptyset$. Finally, $S_1 \cap S_2 \subseteq R_1 \cap R_2 = \emptyset$. Thus, $C(R_1) \cap C(R_2) = \emptyset$, and so there are no Radon partitions of $U \cup S$. This makes $U \cup S$ R-independent.

If U is H-independent, we seek to prove that $\bigcap_{x \in U \cup S} C((U \cup S) - \{x\}) = \emptyset$. As above

$$C((U \cup S) - \{x\}) = C((U - \{x\}) \cup (S - \{x\})) = C(U - \{x\}) \cup (S - \{x\})$$

for any $x \in U \cup S$. Since $\bigcap_{x \in U \cup S} C(U - \{x\}) \subseteq \bigcap_{x \in U} C(U - \{x\}) = \emptyset$ by the H-independence of U then $\bigcap_{x \in U \cup S} C((U \cup S) - \{x\}) = \bigcap_{x \in U \cup S} (C(U - \{x\}) \cup (S - \{x\})) \subseteq S$. Since U is H-independent, it is also convexly independent so $U \cup S$ is convexly independent. Thus if $x \in S$ then $x \notin C((U \cup S) - \{x\})$. Thus $\bigcap_{x \in U \cup S} C((U \cup S) - \{x\}) = \emptyset$ so $U \cup S$ is H-independent. $\square$

As in the previous section, let P_0 and P_1 be partite sets of T such that $U = A \cup B$ where $A = U \cap P_0$ and $B = U \cap P_1$. We also assume $|[A]| \geq |[B]|$, $A \to B$ and $D_A^{\to} \neq \emptyset$ when $B = \emptyset$. Our next result helps describe H-, R-, and convexly independent sets U when $|[U]|$ is small.

Theorem 5.2. Let $U \subseteq V(T)$, $U = A \cup B$ with $|[A]| \geq |[B]|$, $A \to B$ when $B \neq \emptyset$.

1. If $|[U]| = 1$, then U is H-, R-, and convexly independent.
2. If $|[U]| = 2$, then U is H-independent if and only if U is R-independent if and only if either
   (a) U is convexly independent, and $|[v] \cap U| \geq 2$ for at most one $[v] \in V(T_f)$ or
   (b) No vertices distinguish any two vertices in U.
3. If $|[U]| = 3$, then U is H-independent if and only if U is convexly independent and $[U]$ is H-independent.
4. If $|[U]| = 3$, then U is R-independent if and only if either
   (a) U is convexly independent and $|U| = 3$,
   (b) U is H-independent.

Proof. If $|[U]| = 1$, then $C(S) = S$ for all $S \subseteq U$, and so (1) follows.

For (2), let $u, v \in U$ with $[U] = \{[u], [v]\}$. If U is R-independent, then U is convexly independent. Further, suppose $|[u] \cap U|, |[v] \cap U| \geq 2$. Let $R_1 = \{u, v\}$, $R_2 = U - \{u, v\}$. Then there exist $u' \in [u] \cap R_2$ and $v' \in [v] \cap R_2$. Any vertex y that distinguishes u and v will also distinguish u' and v', and so $y \in (u \vee v) \cap (u' \vee v') \subseteq C(R_1) \cap C(R_2)$, making $R_1 \cup R_2$ a Radon partition. Thus, R-independence implies that either $|[v] \cap U| \geq 2$ for at most one $[v] \in V(T_f)$ or no vertices distinguish any two vertices in U.

Now suppose U is convexly independent and $|[v] \cap U| \geq 2$ for at most one $[v] \in V(T_f)$. If $U = \{u, v\}$, then clearly U is H-independent. If $|[v] \cap U| \geq 2$, and $U - [v] = \{u\}$ then $C(U - \{u\}) = U \cap [v]$. By Lemma 3.5(2), for each $w \in U \cap [v]$ we have $C(U - \{w\}) = C(U) - \{w\}$, giving us

$$\begin{aligned} \bigcap_{w \in U} C(U - \{w\}) &= (U \cap [v]) \cap \left(\bigcap_{w \in U \cap [v]} C(U) - \{w\} \right) \\ &= (U \cap [v]) \cap (C(U) - (U \cap [v])) = \emptyset \end{aligned}$$

This makes U H-independent. If no vertices distinguish u and v, then $C(S) = S$ for each $S \subseteq U$, and so U is H-independent. Since H-independence implies R-independence, this completes the proof of (2).

For (3), Let $u, v, w \in U$ with $[U] = \{[u], [v], [w]\}$. Suppose U is H-independent. By Lemma 4.1 and the fact that H-independence implies convex independence, both

U and $[U]$ are convexly independent. For contradiction, assume $[U]$ is H-dependent, and let $[y] \in C(\{[u],[v]\}) \cap C(\{[u],[w]\}) \cap C(\{[v],[w]\})$. Let $[y] \in C_k(\{[u],[v]\})$, with k minimal. By convex independence of $[U]$, we may assume $k \geq 1$. Then there exists $[z_1],[z_2] \in C_{k-1}(\{[u],[v]\}) = [C_{k-1}(\{u,v\})]$ with $[z_1] \to [y] \to [z_2]$. Without loss of generality, $z_1, z_2 \in C_{k-1}(\{u,v\})$, and so $[y] \subseteq C_k(\{u,v\}) \subseteq C(\{u,v\})$. Similarly, $[y] \subseteq C(\{u,w\}) \cap C(\{v,w\})$, violating the H-independence of U.

For the converse, suppose that U is convexly independent and $[U]$ is H-independent. Consider first the case $[A] = \{[u],[v]\}$ with $[v] \cap A = \{v\}$ and $[B] = \{[w]\}$. Applying Lemma 2.4(2) to $[z] \in T_f$ we see that any vertex $z \notin P_0 \cup P_1$ that distinguishes vertices in U must satisfy $A \to z \to B \cup D_{\vec{A}}$. But this would imply $[z] \in C(\{[u],[v]\}) \cap C(\{[u],[w]\}) \cap C(\{[v],[w]\})$, making $[U]$ H-dependent, a contradiction. Thus, no such z exists, and so $C([u] \cup [w]) = [u] \cup [w]$ and $C(U - \{v\}) = U - \{v\}$. By convex independence of U for each $x \in U - \{v\}$ we have $C(U - \{x\}) \subseteq C(U) - \{x\}$. Thus $\bigcap_{x \in U} C(U - \{x\}) = \emptyset$ making U H-indendent.

Finally, suppose $U = A$. If $|U| = 3$, then $U = \{u,v,w\}$. Since $[U]$ is H-independent, we have $[C(\{u,v\})] \cap [C(\{u,w\})] \cap [C(\{v,w\})] = \emptyset$, and so $C(\{u,v\}) \cap C(\{u,w\}) \cap C(\{v,w\}) = \emptyset$, making U H-independent. If $|U| \geq 4$, let $|[u] \cap U| \geq 2$, $[v] \cap U = \{v\}$ and $[w] \cap U = \{w\}$. Assume U is H-dependent, say $y \in \bigcap_{x \in U} C(U - \{x\})$. Applying Lemma 3.5(2)

$$\begin{aligned}
\bigcap_{x \in U} C(U - \{x\}) &= \left(\bigcap_{x \in [u] \cap U} C(U - \{x\}) \right) \cap C(([u] \cap U) \cup \{v\}) \cap C(([u] \cap U) \cup \{w\}) \\
&= (C(U) - ([u] \cap U)) \cap C(([u] \cap U) \cup \{v\}) \cap C(([u] \cap U) \cup \{w\}) \\
&= (C(([u] \cap U) \cup \{v\}) \cap C(([u] \cap U) \cup \{w\})) - ([u] \cap U)
\end{aligned}$$

Then $[y] \in C(\{[u],[v]\}) \cap C(\{[u],[w]\})$. Since $[U]$ is H-independent $[y] \notin C(\{[v],[w]\}) = [C(\{v,w\})]$ so $y \notin C(\{v,w\})$. By Corollary 4.6, $y \in C(U) - C(\{v,w\}) = [u] \cap U$ which is a contradiction. Thus U is H-independent.

For (4), let $u,v,w \in U$ with $[U] = \{[u],[v],[w]\}$. Assume U is R-independent. Suppose $|U| \geq 4$. Since U is clearly convexly independent, by (3), we need only show that $[U]$ is H-independent in T_f. For contradiction, assume that $[U]$ is H-dependent. Since T_f is clone-free, Theorem 4.8 in [15] implies that there exists $z \notin P_0 \cup P_1$ with $A \to z \to B \cup D_{\vec{A}}$. It suffices to produce a Radon partition of U.

By hypothesis, $|[A]| \geq 2$ so let $[u],[v] \subseteq A$. We can also assume that $D_{\vec{A}} \neq \emptyset$, and that there exists $q \in D_{\vec{A}}$ with $q \to v$. Since $|U| \geq 4$, it follows from Lemma 4.2 and Theorem 2.1 that either have $|[u] \cap U| \geq 2$ or $|[w] \cap U| \geq 2$ and $w \in B$.

In the case $|[u] \cap U| \geq 2$, let $u' \in U \cap ([u] - \{u\})$, let $R_1 = \{u,v\}$, and let $R_2 = U - R_1$. Since $u \to q \to v$ and $u \to z \to q$ we have $z \in C(\{u,v\}) = C(R_1)$. If $B \neq \emptyset$, we have $w \in B$. Then $u' \to z \to w$, and so $z \in C(\{u',w\}) \subseteq C(R_2)$. Similarly, if $B = \emptyset$, we have $w \in A$. The there is a $q' \in D_{\vec{A}}$ with $q' \to w$. Then $u' \to q' \to w$ and $u' \to z \to q'$, and so $z \in C(\{u',w\}) \subseteq C(R_2)$. Since $z \in C(R_1) \cap C(R_2)$, $R_1 \cup R_2$ is a Radon partition.

In the case $|[w] \cap U| \geq 2$ and $w \in B$, let $w' \in U \cap ([w] - \{w\})$, let $R_1 = \{u,w\}$, and let $R_2 = U - R_1$. We have $u \to z \to w$ and $v \to z \to w'$, and so again we have $z \in C(R_1) \cap C(R_2)$.

For the converse, H-independence implies R-independence, so we are left with the case $|U| = 3$. Let $U = R_1 \cup R_2$ be a Radon partition. Without loss of generality, $|R_1| = 1$, and so $C(R_1) = R_1$. But then $C(R_1) \cap C(R_2) \neq \emptyset$ implies $R_1 \cap C(R_2) \neq \emptyset$, and so U is convexly dependent. This completes the proof. □

We now consider bipartite tournaments. The following lemma follows similarly as Lemma 4.1 in [15].

Lemma 5.3. Let T be a bipartite tournament and let U be a convexly independent set with $|[U]| \geq 4$.

1. For each $t \geq 0$, $\Delta_t(U) = \bigcup_{x \in U} D_t(x)$.

2. $C(U) = \bigcup_{x \in U} D(x)$.

This gives us the following.

Theorem 5.4. Let T be a bipartite tournament and let U be a convexly independent set of T.

1. If $|[U]| \neq 2$, then U is H-independent.

2. If U has a nonempty intersection with two partite sets, then U is H-independent.

3. If there exists $u \in U$ such that $[u] \cap U = \{u\}$, then U is H-independent.

4. If $|[U]| = 2$, U is contained in a single partite set, and every vertex of U has a clone in U, then U is H-dependent.

Proof. We begin by proving (1). If $|[U]| = 1$, then U is H-independent by Theorem 5.2(1). If $|[U]| = 3$, Theorem 4.2 in [15] implies that $[U]$ is H-independent in T_f, and so U is H-independent by Theorem 5.2(3). If $|[U]| \geq 4$, we have, by Lemma 5.3(2), that $C(U - \{x\}) \subseteq \bigcup_{y \in (U - \{x\})} D(y)$, and so $\bigcap_{x \in U} C(U - \{x\}) \subseteq \bigcap_{x \in U} (\bigcup_{y \neq x} D(y)) = \emptyset$ since the $D(y)$'s are pairwise disjoint by Corollary 4.8. Thus U is H-independent.

We now prove (2). By (1), we need only consider the case $|[U]| = 2$, so let $[U] = \{[u], [v]\}$ with u and v in distinct partite sets. Since T is bipartite, no vertex can distinguish u and v. But then Theorem 5.2(2b) implies U is H-independent.

Part (3) follows from Theorem 5.2(2a), and so it suffices to prove (4). Let U be a convexly independent set contained in a single partite set of T such that $|[U]| = 2$ and each vertex in U has a clone in U. This violates Theorem 5.2(2a) and (2b), and so U is H-dependent. □

The next result follows directly.

Corollary 5.5. Let T be a bipartite tournament. Then $h(T) \neq d(T)$ if and only if for every maximum convexly independent set U we have $|[U]| = 2$, U is contained in a single partite set, and every vertex in U has a clone in U. Otherwise, we have $h(T) = r(T) = d(T)$.

As in [15], let $V_U = \bigcup_{u \in U} D(u)$. By Theroem 4.7, if $|[U]| \geq 4$ the $V_U \subseteq P_0 \cup P_1$. Let T_U denote the bipartite tournament induced by V_U. The following result shows what happens when partite sets other than P_0 and P_1 are taken into account.

Theorem 5.6. Let T be a multipartite tournament and let U be a convexly independent subset of V with $|[U]| \geq 4$. The following are equivalent.

1. U is H-independent
2. No vertex in $V - (P_0 \cup P_1)$ distinguishes two vertices in $U \cup D_A^{\rightarrow}$.
3. No vertex in $V - (P_0 \cup P_1)$ distinguishes two vertices in V_U.
4. $C(U) = V_U$.
5. There exist u, v, $w \in U$ with $[u]$, $[v]$ and $[w]$ distinct such that $\{u, v, w\}$ is H-independent.

Proof. Proving (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows as in Theorem 4.6 of [15], using Theorem 4.7. Theorem 5.5 and the fact that $|[U]| \geq 4$ imply that U is H-independent in T_U. If we assume $C(U) = V_U$, this implies that U is H-independent in T. This gives us both (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5)

We now prove (5) $\Rightarrow$ (2). Suppose that $z \in V - (P_0 \cup P_1)$ distinguishes two vertices in $U \cup D_A^{\rightarrow}$. Lemma 2.4(1) implies $(A \cup D_B^{\leftarrow}) \to z \to (B \cup D_A^{\rightarrow})$. Let u, v, $w \in U$ with $[u]$, $[v]$ and $[w]$ distinct . It follows that $z \in (u \vee v) \cap (u \vee w) \cap (v \vee w)$, and so $\{u, v, w\}$ is H-dependent, a contradiction. This proves the result. $\square$

Now we consider R-independence. It need not be equivalent to convex independence, but it is almost always equivalent to H-independence.

Theorem 5.7. Let T be a multipartite tournament and let $U = A \cup B$ be R-independent. Then U is H-dependent if and only if $|[U]| = |U| = 3$ and there exists a vertex z in a partite set disjoint from $U \cup D_A^{\rightarrow}$ with $A \to z \to (B \cup D_A^{\rightarrow})$.

Proof. Assume U is H-dependent. By Theorem 5.2(1) and (2), if $|[U]| \leq 2$, U is R-dependent, a contradiction. In the case $|[U]| \geq 4$, Theorem 5.6(3) and Theorem 2.4(1) imply that there exists $z \in V - (P_0 \cup P_1)$ such that $(V_U \cap P_0) \to z \to (V_U \cap P_1)$. Since $|A| \geq |B|$, we have $|[A]| \geq 2$, so let $u, v \in A$ be vertices that are not clones, and let $M = \{u, v\}$, $R = U - M$. Since $|[U]| \geq 4$, $|[R]| \geq 2$.

We show that $M \cup R$ is a Radon partition by proving that $z \in C(M) \cap C(R)$. In the case $B = \emptyset$, then $D_M^{\rightarrow}, D_R^{\rightarrow} \neq \emptyset$. Let $x \in D_M^{\rightarrow}$, $y \in D_R^{\rightarrow}$. Clearly, $x \in C(M)$ and $y \in C(R)$. We have $M \to z \to x$ and $R \to z \to y$, and so $z \in C(M) \cap C(R)$. In the case $|[B]| = 1$,

$B \subseteq R$ and $R \cap A \neq \emptyset$. If $w \in R \cap A$, $b \in B$, then $w \to z \to b$ implies $z \in C(R)$. As before, $z \in C(M)$, and so $z \in C(M) \cap C(R)$. Finally, if $|[B]| \geq 2$, then we get $z \in C(M)$ using the fact $A \to z \to D_A^{\to}$, and we get $z \in C(R)$ using the fact that $D_B^{\leftarrow} \to z \to B$. Again, this gives us $z \in C(M) \cap C(R)$. Thus, U is R-dependent, a contradiction.

Therefore, we have $|[U]| = 3$. By Theorem 5.2(4) and the fact that U is H-dependent, we get $|U| = 3$. The rest follows as in Theorem 4.7 of [15]. □

This gives a description of when the Helly number and Radon number are not equal.

Corollary 5.8. Let T be a multipartite tournament. Then $h(T) \neq r(T)$ if and only if $h(T) = 2$ and $r(T) = 3$. Furthermore, in this case for every convexly independent set $U = A \cup B$ of order 3 with $|[U]| = 3$, there exists a $z \in V(T)$ such that $A \to z \to (B \cup D_A^{\to})$.

Proof. Let U be a maximum R-independent set and assume $h(T) \neq r(T)$. Then U is H-dependent so by Theorem 5.7, $|U| = 3$. Thus $r(T) = 3$. Since any pair of vertices is H-independent, $h(T) \geq 2$, and so $h(T) = 2$. The converse is trivial.

Since a convexly independent set with 3 elements is automatically R-independent the last part also follows from Theorem 5.7. □

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