An Extremal Problem for the Neighborhood Lights Out Game

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Abstract

Neighborhood Lights Out is a game played on graphs. Begin with a graph and a vertex labeling of the graph from the set $\{0, 1, 2, \ldots, \ell - 1\}$ for $\ell \in \mathbb{N}$. The game is played by toggling vertices: when a vertex is toggled, that vertex and each of its neighbors has its label increased by 1 (modulo ℓ). The game is won when every vertex has label 0. For any $n \in \mathbb{N}$ it is clear that one cannot win the game on K_n unless the initial labeling assigns all vertices the same label. Given that the K_n has the maximum number of edges of any simple graph on n vertices it is natural to ask how many edges can be in a graph so that the Neighborhood Lights Out game is winnable regardless of the initial labeling. We find all such extremal graphs on n vertices that have $\binom{n}{2} - c$ edges for $c \leq \lceil \frac{n}{2} \rceil + 3$ and all those that have minimum degree $n - 3$. The proofs of our results require us to introduce a new version of the Lights Out game that can be played given any square matrix.

1 Introduction

The Lights Out game was originally created by Tiger Electronics. It has since been reimagined as a light-switching game on graphs. Several variations of the game have been developed (see, for example [CMP09] and [JPZ]), but all have some important elements in common. In each game, we begin with a graph G and a labeling of $V(G)$ with labels in $\mathbb{Z}_\ell$ for some $\ell \geq 2$. The vertices can be toggled so as to change the labels of some of the vertices. Finally, there is some desired labeling (usually the labeling with all labels being 0, called the *zero labeling* and denoted by $\mathbf{0}$) that marks the end of the game.

The most common variation of the Lights Out game is what we call the *neighborhood Lights Out game*. This is a generalization of Sutner's σ^+-game (see [Sut89]). Each time

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we toggle some $v \in V(G)$, the label of each vertex in the closed neighborhood of v, $N[v]$, is increased by 1 modulo ℓ. The game is won when the zero labeling is achieved. This game was developed independently in [GP13] and [Ara12] and has been studied in [AMW14], [AM14], [EEJ+10], [Par18], and [BBS19]. The original Lights Out game is the neighborhood Lights Out game on a grid graph with $\ell = 2$ and has been studied in [AS92], [GK07], and [Sut89].

It is possible for a Lights Out game to be impossible to win. Much of the work on Lights Out games has centered on the conditions under which winning the game is possible. Winnability depends on the version of the game that is played, the graph on which the game is played, and on ℓ. In our paper, we work with the neighborhood Lights Out game with labels in $\mathbb{Z}_\ell$ for arbitrary $\ell \geq 2$.

For each $n \geq 2$ there exist many labelings of K_n for which the neighborhood Lights Out game is impossible to win. In particular, any initial labeling in which not every vertex has the same label cannot be won. It is also true that K_n has the most edges of any simple graph on n vertices. It then makes sense to ask, given $n, \ell \geq 2$, what is the maximum size of a simple graph on n vertices with labels in $\mathbb{Z}_\ell$ for which the neighborhood Lights Out game can be won for every possible initial labeling? We call this maximum size $\max(n, \ell)$. In addition, we seek to classify the winnable graphs of maximum size among all graphs on n vertices with labels from $\mathbb{Z}_\ell$, which we call (n, ℓ)-*extremal graphs*.

For winnability we depend heavily on linear algebra methods similar to those in [AF98], [AMW14], [EEJ+10], and [GP13]. We discuss these methods in Section 2. These tools allow us to determine winnability in some dense graphs by considering winnability in a modified Lights Out game in their sparse complements, which we discuss further in Section 3. It turns out the complements of (n, ℓ)-extremal graphs often have the property that every non-pendant vertex is adjacent to a pendant vertex. We take the following definition from [Gra14].

Definition. A graph G is a *pendant graph*, if there exists a graph H such that $G = H \odot K_1$, which has vertex set $V(H) \cup \{w_v : v \in V(H)\}$ with $w_v \notin V(H)$ and edge set $E(H) \cup \{vw_v : v \in V(H)\}$. A *pendant tree* is a pendant graph that is also a tree.

The notation $H \odot K_1$ is for the *corona product* of H and K_1. The corona product was first introduced in [FH70] where it was used as a way of relating a product of graphs to the wreath product of groups. Note that $P_2 = K_1 \odot K_1$, $P_4 = P_2 \odot K_1$, and that no other paths are pendant graphs.

In Section 4, we apply our winnability results to determine partial results on the classification of (n, ℓ)-extremal graphs. In the case of n odd, we show that all (n, ℓ)-extremal graphs are complements of near perfect matchings. We also classify all (n, ℓ)-extremal graphs when n is even and ℓ is odd. In the remaining case we have the following conjecture.

Conjecture 1.1. For n, ℓ even then

$$\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + k\right)$$

if and only if k is the smallest nonnegative integer such that $\gcd(n-2k-1,\ell)=1$. In each case the (n,ℓ)-extremal graphs are precisely the complements of pendant graphs of order n that have size $\binom{n}{2}-\left(\frac{n}{2}+k\right)$.

We prove this conjecture for $0 \le k \le 3$ and in the family of all graphs that have minimum degree at least $n-3$.

2 Linear Algebra

Winnability in the Lights Out game on graphs can be studied by determining a strategy for toggling the vertices. But it can also be determined using linear algebra. We proceed as in [AF98] and [GP13].

Let G be a graph with $V(G)=\{v_1,v_2,\dots,v_n\}$. In the neighborhood game, if v_j is toggled x_j times, this increases the label of each vertex in $N[v_j]$ by x_j and has no effect on the labels of other vertices. Let $N(G)=[N_{ij}]$ be the neighborhood matrix of G (where $N_{ij}=1$ if and only if v_i is adjacent to v_j or $i=j$ and $N_{ij}=0$ otherwise). If $\pi : V(G)\to\mathbb{Z}_\ell$ is an initial labeling of G, it follows that the resulting label on each v_i is $\pi(v_i)+\sum_{j=1}^{n} N_{ij}x_j$. If we define the vector $\mathbf{b}$ with $\mathbf{b}[i]=\pi(v_i)$ and the vector $\mathbf{x}$ with $\mathbf{x}[i]=x_i$ then the resulting labeling is $\mathbf{b}+N(G)\mathbf{x}$. Thus, winning the neighborhood Lights Out game (that is, achieving the zero labeling) is equivalent to solving the matrix equation $N(G)\mathbf{x}=-\mathbf{b}$. In this linear algebra perspective we typically think of the initial labeling of the graph as a vector (which we called $\mathbf{b}$ above). When we determine winnability by playing the game we will typically think of the inital labeling as a function.

As described above, the neighborhood Lights Out game can be played by knowing the neighborhood matrix and an initial labeling. However, we can also play a Lights Out game using any matrix. Let $M=[m_{ij}]\in M_n(\mathbb{Z}_\ell)$ (the set of $n\times n$ matrices with entries in $\mathbb{Z}_\ell$), and define the *vertex set of M* as a set of n elements $V(M)=\{v_1,v_2,\dots,v_n\}$. We can then define the (M,ℓ)-Lights Out game (or simply the M-Lights Out game if the value of ℓ is clear) as follows:

- Label the elements of $V(M)$ with a vector $\mathbf{b}\in\mathbb{Z}_\ell^n$, where each v_i has label $\mathbf{b}[i]$.
- Play the game by toggling elements of $V(M)$. Each time v_j is toggled, add m_{ij} to the label of v_i for all $1\le i\le n$.
- As with the ordinary Lights Out game, we win the game when we achieve the labeling $\mathbf{0}$.

Example 2.1. Let G be a graph. For $M=N(G)$, we get the neighborhood Lights Out game. If we let M be the adjacency matrix $A(G)$, we get an analogue of the σ-game from Sutner (see [Sut89]), where toggling a vertex v increases the label of each vertex in the open neighborhood $N_G(v)$ of v by 1 and leaves the label of v unchanged. We call this the *adjacency Lights Out game.*

Throughout, we shorten the names of the neighborhood Lights Out game and the adajacency Lights Out game to the $(N(G), \ell)$-Lights Out game and the $(A(G), \ell)$-Lights Out game, respectively. We shorten even further to the (N, ℓ)-Lights Out game and the (A, ℓ)-Lights Out game when the graph is clear. Though the adjacency matrix and the neighborhood matrix are both symmetric there is no requirement that the matrix be symmetric to play the (M, ℓ)-Lights Out game. Now we introduce some terminology related to whether a given (M, ℓ)-Lights Out game can be won.

Definition. Let M, $V = V(M)$, and ℓ be as above.

1. We define $\mathcal{L}(V, \ell) = \{\pi \mid \pi : V \to \mathbb{Z}_\ell\}$. So $\mathcal{L}(V, \ell)$ is the set of labelings.
2. We call $\pi \in \mathcal{L}(V, \ell)$ (M, ℓ)-*winnable* if the (M, ℓ)-Lights Out game can be won with initial labeling π.
3. We say that V is (M, ℓ)-*always winnable*, or (M, ℓ)-*AW* for short, if all $\pi \in \mathcal{L}(V, \ell)$ are (M, ℓ)-winnable.

In the case that V is the vertex set of a graph G, we refer to G as being (M, ℓ)-AW, with the understanding that $V(M) = V(G)$. In these cases, M is often given by the neighborhood matrix or adjacency matrix. The following summarizes the connection between whether a given M-Lights Out game can be won and the linear algebraic properties of M. The proof follows from basic linear algebra.

Lemma 2.2. Let $M \in M_n(\mathbb{Z}_\ell)$ and $V(M) = \{v_1, v_2, \ldots, v_n\}$.

1. Let $\pi \in \mathcal{L}(V(M), \ell)$ and define $\mathbf{b}[i] = \pi(v_i)$. Then π is (M, ℓ)-winnable with toggling given by $\mathbf{x}$ if and only if $M\mathbf{x} = -\mathbf{b}$.
2. The vertex set $V(M)$ is (M, ℓ)-AW if and only if M is invertible over $\mathbb{Z}_\ell$.

In this paper, we focus on whether or not a graph G is $(N(G), \ell)$-AW, so we seek to determine whether or not a given neighborhood matrix is invertible. One straightforward way to apply linear algebra techniques is when two rows or columns of a matrix are identical.

Definition. Let $M \in M_n(\mathbb{Z}_\ell)$, and let $v, w \in V(M)$. We call v and w M-*twins* if the rows or columns of M represented by v and w are identical.

In graph theory, two vertices v and w are twins provided that have the same open neighborhood excluding v and w. Twin vertices that are adjacent in a graph result in identical rows and columns in the neighborhood matrix and thus are N-twins. Twin vertices that are not adjacent result in identical rows in the adjacency matrix and thus are A-twins. The following is immediate from considering the invertibility of the matrix.

Corollary 2.3. Let $M \in M_n(\mathbb{Z}_\ell)$, and suppose there exist M-twins in $V(M)$. Then the M-Lights Out game is not (M, ℓ)-AW for any ℓ.

Note that $\mathbb{Z}_\ell$ is generally not a field, but we can still use the determinant of a matrix to determine its invertibility. In particular, a matrix is invertible if and only if its determinant is a unit [Bro93, Corollary 2.21]. As in standard linear algebra, we can apply row operations to a matrix and leave the determinant unchanged or multiplied by a unit. In particular, the following *elementary row operations* have no effect on whether or not the determinant is a unit.

- Multiply a row of M by a unit in $\mathbb{Z}_\ell$.
- Add an integer multiple of one row of M to another row of M
- Switch two rows of M

We say that M is *row equivalent* to M' if and only if M can be turned into M' by applying a sequence of elementary row operations. Since elementary row operations do not change whether or not the determinant is a unit, if $M, M' \in M_n(\mathbb{Z}_\ell)$ such that M is row equivalent to M' then M is invertible if and only if M' is invertible. We can apply this to Lights Out games in which the matrices are row equivalent.

Corollary 2.4. Let $\ell \in \mathbb{N}$, $M, M' \in M_n(\mathbb{Z}_\ell)$, and let V be a vertex set for the M- and M'-Lights Out games. If M and M' are row equivalent, then V is (M, ℓ)-AW if and only if V is (M', ℓ)-AW.

Thus, we can determine whether or not a set V is (M, ℓ)-AW by applying some elementary row operations to M to obtain M', and then determining whether or not V is (M', ℓ)-AW.

We now apply this strategy to the neighborhood Lights Out game. Our general strategy is to use elementary row operations to transform $N(G)$ into a matrix whose Lights Out game is easy to play. Our first result using this technique will be for graphs that have a dominating vertex. Given graphs G and H we use $G \cup H$ to denote the disjoint union of the graphs.

Theorem 2.5. Let G be a graph. Then $\overline{G \cup K_1}$ is (N, ℓ)-AW if and only if G is (A, ℓ)-AW.

Proof. We have

$$N(\overline{G \cup K_1}) = \left[\begin{array}{c|c} N(\overline{G}) & 1 \\ \hline 1 & 1 \end{array}\right]$$

where the last row and column represent $V(K_1)$. We multiply each row except the last by the unit -1 and then add to each of those rows the last row. This turns every 1 of $N(\overline{G})$ into a 0 and vice versa, resulting in the adjacency matrix of G. Thus, we get that $N(\overline{G \cup K_1})$ is row equivalent to

$$M = \left[\begin{array}{c|c} A(G) & 0 \\ \hline 1 & 1 \end{array}\right].$$

By Corollary 2.4, it suffices to show that $V(\overline{G \cup K_1})$ is (M, ℓ)-AW if and only if G is (A, ℓ)-AW. Note that the (M, ℓ)-Lights Out game is played as the (A, ℓ)-Lights Out game on G, each vertex toggled in $V(G)$ adds 1 to the label of the vertex $v \in V(K_1)$, and toggling v increases its own label by 1 and has no other effect.

First suppose that G is (A, ℓ)-AW, and consider $\pi \in \mathcal{L}(V(\overline{G \cup K_1}), \ell)$. Since G is (A, ℓ)-AW, we can toggle the vertices of G in a way that wins the $A(G)$-Lights Out game for the labeling $\pi|_{V(G)}$. At this point, every vertex has label 0 except v. We then toggle v until it has label 0. In the M-Lights Out game toggling v has no effect on labels of other vertices, so this wins the M-Lights Out game. Thus $\overline{G \cup K_1}$ is (N, ℓ)-AW.

Conversely, suppose that G is not (A, ℓ)-AW. We then give $V(G)$ a labeling that is not $A(G)$-winnable. In the M-Lights Out game the only vertices that affect the labels of $V(G)$ are the vertices in $V(G)$, so this is not a winnable labeling for the M-Lights Out game. Thus, $\overline{G \cup K_1}$ is not (N, ℓ)-AW, which completes the proof. □

3 Winnability in Dense Graphs

In proving Theorem 2.5, we use elementary row operations to convert the neighborhood Lights Out game on a dense graph into something resembling the adjacency Lights Out game on a sparse graph. Since the extremal problem we are working on seeks dense, winnable graphs and playing the game on sparse graphs is typically easier, this technique works to our advantage. The next result allows us to make a graph denser by removing an edge from the complement graph when the complement graph is combined with P_4.

Theorem 3.1. Let G be a graph, $U \subseteq V(G)$ and v be an end vertex of P_4. Let H be the graph where $V(H) = V(G) \cup V(P_4)$ and $E(H) = E(G) \cup E(P_4) \cup \{uv : u \in U\}$. Then $\overline{H}$ is (N, ℓ)-AW if and only if $\overline{G \cup P_4}$ is (N, ℓ)-AW.

Proof. Let $V = V(\overline{G \cup P_4}) = V(\overline{H})$, and let P_4 in both $G \cup P_4$ and H be given by $vv_2v_3v_4$. Note that $\overline{P_4}$ is the path given by $v_2v_4vv_3$. By [GP13, Thm. 4.3], P_4 is (N, ℓ)-AW for all ℓ. It follows that in both $\overline{H}$ and $\overline{G \cup P_4}$, the subgraph induced by $\{v, v_2, v_3, v_4\}$ is (N, ℓ)-AW. Thus, we can toggle the vertices of P_4 in such a way that each vertex in P_4 has label zero.

We first assume $\overline{H}$ is (N, ℓ)-AW and show $\overline{G \cup P_4}$ is (N, ℓ)-AW. To that end, we let $\pi : V \to \mathbb{Z}_\ell$ and show that π is winnable on $\overline{G \cup P_4}$. As discussed above, we can assume that $\pi|_{V(P_4)} = 0$. Since $\overline{H}$ is (N, ℓ)-AW, π is winnable on $\overline{H}$. In this winning strategy, let each $w \in V(G)$ be toggled x_w times, and let v_2 be toggled x times. If we apply this strategy to $\overline{H}$ but refrain from toggling v, v_3, and v_4, this leaves v_2 and v_4 with label $x + \sum_{w \in V(G)} x_w$, v with label $\sum_{w \in V(G)-U} x_w$, and v_3 with label $\sum_{w \in V(G)} x_w$. Since v_4 is the only remaining vertex adjacent to v_2, v_4 must be toggled $-x - \sum_{w \in V(G)} x_w$ times. This will leave both v_2 and v_4 with label zero. Since v is the only remaining vertex adjacent to v_4, this means we do not toggle v at all. Thus, v_3 (the only remaining untoggled vertex) must make its own label zero by being toggled $-\sum_{w \in V(G)} x_w$ times. This completes winning the game on $\overline{H}$. An important observation is that the vertices of P_4 are collectively toggled $-2\sum_{w \in V(G)} x_w$ times, and none of those toggles come from v. Since each of v_2, v_3, and v_4 is adjacent to every vertex in $V(\overline{G})$, this implies that toggling the vertices of P_4 adds $-2\sum_{w\in V(G)} x_w$ to the labeling of each vertex in $V(G)$. Looked at another way, if we only toggle the vertices in $V(G)$, this leaves each such vertex with label $2\sum_{w\in V(G)} x_w$.

With the initial labeling π, we now apply the above toggling strategy to $V(G)$ in $\overline{G \cup P_4}$. By the above, each vertex in $V(G)$ has label $2\sum_{w\in V(G)} x_w$. Since v and each of the v_i are adjacent to all vertices in $V(G)$, it follows that toggling the vertices in $V(G)$ leaves v and each v_i with label $\sum_{w\in V(G)} x_w$. Each of v_2 and v_3 is now toggled $-\sum_{w\in V(G)} x_w$ times. This makes the label of v and each v_i zero. In addition, it adds $-2\sum_{w\in V(G)} x_i$ to the labels of $V(G)$, which gives each of them label zero as well.

We proceed similarly for the converse. Assume $\overline{G \cup P_4}$ is (N,ℓ)-AW, and let $\pi : V \to \mathbb{Z}_\ell$ be a labeling as above with $\pi\mid_{V(P_4)} = 0$. We need to prove that π is winnable on $\overline{H}$. As before, there is a winning toggling strategy for $\overline{G \cup P_4}$, where each $w \in V(G)$ is toggled x'_w times, and v_2 is toggled x' times. At this point, we determine the toggles for v and each remaining v_i as before, and it follows that the vertices are collectively toggled $-2\sum_{w\in V(G)} x'_w$ times. As before, this implies that toggling the vertices of $V(G)$ results in the label of each vertex in $V(G)$ being $2\sum_{w\in V(G)} x'_w$.

Again, we apply the above toggling strategy just to the vertices of $V(G)$ in $\overline{H}$. This leaves each of v_2, v_3, and v_4 with label $\sum_{w\in V(G)} x'_w$ and v with label $\sum_{w\in V(G)-U} x'_w$. We then win the game as follows: v_2 is toggled $-2\sum_{w\in U} x'_w - \sum_{w\in V(G)-U} x'_w$ times, v_3 is toggled $-\sum_{w\in V(G)} x'_w$ times, and v_4 is toggled $\sum_{w\in U} x'_w$ times. □

We can apply this result to complements of graphs that include components that are path graphs. For $k \in \mathbb{N}$ and G a graph we use kG to denote k disjoint copies of G.

Corollary 3.2. Let G be a graph of order n that is (N,ℓ)-AW.

1. No component of $\overline{G}$ can be P_k such that k is congruent to 3 mod 4.
2. At most one component of $\overline{G}$ can be P_k such that k is congruent to 1 modulo 4.
3. If $\overline{G}$ is an (n,ℓ)-extremal graph, then no component of G is a path of order more than 4.

Proof. For (1), let P be a component of $\overline{G}$ that is a path of order $4k+3$ with $k \in \mathbb{N}\cup\{0\}$. By Lemma 3.1, if we replace P in $\overline{G}$ with $kP_4 \cup P_3$, the complement of the resulting graph is (N,ℓ)-AW if and only if G is. Thus, we can assume $P = P_3$. However, the end vertices of the P_3 component in $\overline{G}$ are $N(G)$-twins in G, so G is not (N,ℓ)-AW by Corollary 2.3.

For (2), we apply Lemma 3.1 as above. If we have more than one component of $\overline{G}$ with order congruent to 1 modulo 4, we can assume that all such components are P_1. But the vertices of these components are all $N(G)$-twins, and so in order for G to be (N,ℓ)-AW, $\overline{G}$ can have at most one component be a path of order congruent to 1 modulo 4.

Finally, (3) follows from the fact that if we replace the component of $\overline{G}$ that is P_k with $k > 4$ with $P_{k-4} \cup P_4$, the complement of the resulting graph will be (N,ℓ)-AW with larger size, thus contradicting the assumption that $\overline{G}$ is (n,ℓ)-extremal. □

Our next main result is Theorem 3.7, where we use row operations to transform the neighborhood matrix into a matrix whose Lights Out game resembles the adjacency game. For this result we need to focus on some specific labelings, as defined below.

Definition. Let $\ell \in \mathbb{N}$, $r \in \mathbb{Z}_\ell$, $M \in M_n(\mathbb{Z}_\ell)$, $U \subseteq V(M)$ and $\pi \in \mathcal{L}(V(M), \ell)$. We say that π is (M, U, ℓ, r)-*winnable* if $\pi_{U,r} \in \mathcal{L}(V(M), \ell)$ is M-winnable, where

$$\pi_{U,r}(v) = \begin{cases} \pi(v) + r, & v \in U \\ \pi(v), & v \notin U \end{cases}$$

When $U = V(M)$, we write $\pi_{V(M),r} = \pi_r$ and say that π is (M, ℓ, r)-winnable.

In the Lights Out games we encounter in the proof of Theorem 3.7, we are concerned not only if certain labelings are winnable, but also how many toggles can be used to win the game for these labelings. Recall that $\mathbf{0}$ is the zero labeling, which assigns to every vertex a label of 0.

Definition. Let $\ell \in \mathbb{N}$, $M \in M_n(\mathbb{Z}_\ell)$, $r \in \mathbb{Z}_\ell$ and $U \subseteq V(M)$. We define the set of U-*toggling numbers* $T_U^M(r) \subseteq \mathbb{Z}_\ell$ as follows. We say $t \in T_U^M(r)$ if the elements of $V(M)$ can be toggled to win the M-Lights Out game with initial labeling $\mathbf{0}_{U,r}$ in such a way that the vertices in U are collectively toggled t times.

In both the neighborhood and adjacency Lights Out games, winning a particular game is equivalent to winning the game on each individual connected component. This simplifies the computation of toggling numbers in these cases. Let G be a graph with $U \subseteq V(G)$ and $M = N(G)$ or $A(G)$. If $G_1, G_2, \ldots, G_c$ are the connected components of G, and if $U_i = U \cap V(G_i)$, then $T_U^M(r) = \sum_{i=1}^c T_{U_i}^{M_i}(r)$, where $M_i = N(G_i)$ or $A(G_i)$, respectively.

Definition. Let $\ell \in \mathbb{N}$ and $M \in M_n(\mathbb{Z}_\ell)$. A *null toggle* is a toggling of $V(M)$ that leaves the labels of $V(M)$ unchanged. If $U \subseteq V(M)$, then the set of U-*null toggling numbers* $Nul_U^M \subseteq \mathbb{Z}_\ell$ are the set of possible numbers of toggles from U that are part of a null toggle of $V(M)$.

The following are immediate.

Lemma 3.3. Suppose we are playing the M-Lights Out game and that $\lambda, \pi \in \mathcal{L}(V(M), \ell)$.

1. Suppose that each $v \in V(M)$ is toggled t_v times to transform λ into π. Then toggling each $v \in V(M)$ t'_v transforms λ to π if and only if $t'_v = t_v + q_v$, where the q_v collectively form a null toggle.

2. Suppose that the vertices in $V(M)$ must collectively be toggled t times to transform λ into π. Then the vertices in $V(M)$ can be collectively toggled t' times to transform λ to π if and only if $t' = t + q$, where $q \in Nul_{V(M)}^M$.

The following is helpful when a certain toggling needs to be repeated to win a game.

Lemma 3.4. Suppose we are playing the M-Lights Out game, and let $r \in \mathbb{Z}$.

1. Suppose that, beginning with the labeling λ, each $v \in V(M)$ is toggled t_v times resulting in the labeling $\lambda + \pi$. Then toggling each $v \in V(M)$ rt_v times results in the labeling $\lambda + r\pi$.
2. Suppose that the M-Lights Out Game with initial labeling π can be won when each $v \in V(M)$ is toggled t_v times. Then the game can be won with initial labeling $r\pi$ when each $v \in V(M)$ is toggled rt_v times.

Our next result gives a useful characterization of nonempty U-toggling number sets.

Lemma 3.5. Let $n, \ell \in \mathbb{N}$ and $M \in M_n(\mathbb{Z}_\ell)$, let $U \subseteq V(M)$, and let $r \in \mathbb{N}$ be minimal such that $T_U^M(r) \neq \emptyset$. Then $r \mid \ell$, and $T_U^M(s) \neq \emptyset$ if and only if $r \mid s$.

Proof. It is easy to show that $\{s \in \mathbb{Z}_\ell : T_U^M(s) \neq \emptyset\}$ is an additive subgroup of $\mathbb{Z}_\ell$. The result follows easily. □

The next lemma helps us understand toggling numbers, null toggling numbers, and (A, ℓ, s)-winnability for graphs with a pendant vertex.

Lemma 3.6. Let G be a graph with a pendant vertex p. Let v be the neighbor of p in G, let G' be the graph induced by $V(G) - \{p, v\}$, and let $U = N_G(v) - \{p\}$. Then

1. If $s \in \mathbb{Z}_\ell$, then $T_{V(G)}^{A(G)}(s) = \{t - 2s : t \in T_{V(G')-U}^{A(G')}(s)\}$
2. $Nul_{V(G')-U}^{A(G')} = Nul_{V(G)}^{A(G)}$
3. Every $\pi \in \mathcal{L}(V(G), \ell)$ is $(A(G), \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$ if and only if every $\pi' \in \mathcal{L}(V(G'), \ell)$ is $(A(G'), V(G') - U, \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$.

Proof. For (1), let $t' \in T_{V(G)}^{A(G)}(s)$. With an initial labeling $\mathbf{0}_s$, we can then toggle the vertices of G a total of t' times to win the $(A(G), \ell)$-Lights Out game. Let t_U and t be the number of times that the vertices of U and $V(G') - U$ are toggled, respectively. The only neighbor of p is v, so v must be toggled $-s$ times for p to have label 0. This also leaves all vertices of U with label 0, and so G' has labeling $\mathbf{0}_{V(G')-U,s}$. Thus, $t \in T_{V(G')-U}^{A(G')}(s)$. The toggles t and t_U leave $V(G')$ with label 0 and v with label $t_U + s$. Then p must be toggled $-t_U - s$ times to win the game. Totaling the number of toggles, we get $t' = -s + t_U + t - s - t_U = t - 2s$, as desired.

Now let $t' = t - 2s$, where $t \in T_{V(G')-U}^{A(G')}(s)$. We begin with the labeling $\mathbf{0}_s$ on G. We toggle $V(G')$ as we would to win the $A(G')$-Lights Out game with initial labeling $\mathbf{0}_{V(G')-U,s}$ so that the vertices of $V(G') - U$ are toggled t times. Let t_U be the number of times the vertices of U are toggled. This leaves the vertices of $V(G') - U$ with label 0, the vertices of U with label s, v with label $s + t_U$, and p with label s. Then v is toggled $-s$ times and p is toggled $-t_U - s$ times to make all labels zero. We total these toggles to get $t + t_U - s - t_U - s = t - 2s = t'$. This implies $t' \in T_{V(G)}^{A(G)}(s)$, which completes (1).

For (2), let $q \in Nul_{V(G')-U}^{A(G')}$. Then the vertices of $V(G') - U$ can be toggled q times as part of a null toggle of G'. Let q_U be the number of toggles of vertices in U in this null toggle. If we perform these toggles on G, this leaves the labels of p and the vertices of G' unchanged, and increases the label of v by q_U. Then p is toggled $-q_U$ times to complete the null toggle. We get a total of $q + q_U - q_U = q$ toggles, which gives us $q \in Nul_{V(G)}^{A(G)}$. For the converse, let $q \in Nul_{V(G)}^{A(G)}$. Then $V(G)$ can be collectively toggled q times to leave the labels of $V(G)$ unchanged. Let q_U and q' be the number of toggles from U and $V(G') - U$, respectively. Note that we cannot toggle v at all in a null toggle, since it is the only vertex that can affect the label of p. Thus, v and p do not affect the labels of $V(G')$, which implies that the toggles of $V(G')$ listed above form a null toggle. Thus, $q' \in Nul_{V(G')-U}^{A(G')}$. Finally, we note that p must be toggled $-q_U$ times to complete the null toggle, which gives us $q = q_U + q' - q_U = q'$, and so $q \in Nul_{V(G')-U}^{A(G')}$.

For (3), assume each $\pi \in \mathcal{L}(V(G), \ell)$ is $(A(G), \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$. Let $\pi' \in \mathcal{L}(V(G'), \ell)$, and extend π' to a labeling $\pi \in \mathcal{L}(V(G), \ell)$ by defining $\pi(v) = \pi(p) = 0$. By assumption, π is $(A(G), \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$. Thus, we can win the (A, ℓ)-Lights Out game on G with initial labeling π_s. Since $\pi_s(p) = s$, v must be toggled $-s$ times to win this game. That leaves each vertex $u \in U$ with label $\pi'(u)$ and each $x \in V(G') - U$ with label $\pi'(x) + s$ (i.e. the labeling $\pi'_{V(G')-U,s}$ on $V(G')$). The remaining toggles on $V(G')$ must win this game. Thus, toggling $V(G')$ in the same way that we would win with initial labeling π_s on $V(G)$ also wins the (A, ℓ)-Lights Out game on G' with initial labeling ${\pi'}_{V(G')-U,s}$. This makes π' $(A(G'), V(G') - U, \ell, s)$-winnable, as desired.

For the converse, assume each labeling in $\mathcal{L}(V(G'), \ell)$ is $(A(G'), V(G') - U, \ell, s)$-winnable, and let $\pi \in \mathcal{L}(V(G), \ell)$. If we begin with the labeling π and v is toggled $-\pi(p)$ times, then the resulting labeling π' restricted to $V(G')$ is $(A(G'), V(G') - U, \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$ by assumption, making the $\pi'_{V(G')-U,s}$ labeling winnable. We claim that π is $(A(G), \ell, s)$-winnable. By construction, $\pi(u) = \pi'(u) + \pi(p)$ for each $u \in U$, and $\pi'(w) = \pi(w)$ for each $w \in V(G') - U$. We begin with the labeling π_s, and we have v toggled $-\pi(p) - s$ times. This leaves p with label 0, v with label $\pi(v) + s$, and $V(G')$ with the labeling $\pi'_{V(G')-U,s}$. By assumption, π' is $(A(G'), V(G') - U, \ell, s)$-winnable, so we can toggle the vertices in $V(G')$ to win the $\pi'_{V(G')-U,s}$ labeling. When this is done, every vertex in G' has label 0, v has some label a, and p has label 0. Now p can be toggled $-a$ times to win the game. The result follows. $\square$

The following theorem gives information about (A, ℓ)-winnability when a pendant vertex and its neighbor are removed from a graph, and (N, ℓ)-winnability when the complement of a graph has a pendant vertex.

Theorem 3.7. Let G be a graph with a pendant vertex p. Let v be the neighbor of p in G, and let G' be the graph induced by $V(G) - \{p, v\}$.

1. G is $(A(G), \ell)$-AW if and only if G' is $(A(G'), \ell)$-AW.
2. Let $r \in \mathbb{N}$ be minimum such that $T_{V(G)}^{A(G)}(r) \neq \emptyset$, and let $t \in T_{V(G)}^{A(G)}(r)$. Then $\overline{G}$ is $(N(\overline{G}), \ell)$-AW if and only if
    (a) All labelings in $\mathcal{L}(V(G), \ell)$ are $(A(G), \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$.

(b) For every $z \in \mathbb{Z}_\ell$, there exists $q \in Nul_{V(G)}^{A(G)}$ such that there is a solution to $(r+t)x \equiv z + q \pmod{\ell}$.

Proof. For (1), we first assume G is $(A(G),\ell)$-AW. Let G' have an arbitrary labeling. We extend this labeling to a labeling of G by giving each of p and v a label of 0. This labeling of G is winnable since G is $(A(G),\ell)$-AW, so we toggle the vertices of G' as we would in a winning toggling of G. If this does not make every label of G 0, then we have to toggle v to give G a zero labeling. However, this leaves p with a nonzero label. Since v is the only neighbor of p, this implies that toggling v makes the zero labeling impossible. Thus, the toggling we did for G' leaves all vertices in G' with label 0, and so G' is $(A(G'),\ell)$-AW.

If we assume G' is $(A(G'),\ell)$-AW and let G have an arbitrary labeling, we first toggle v so that p has label 0. The resulting labeling restricted to G' is winnable since G' is $(A(G'),\ell)$-AW. We can then toggle the vertices of G' so that all vertices of G' have label 0. This leaves all vertices with label 0, except perhaps v since v is the only vertex not in G' that is adjacent to a vertex in G'. We then toggle p until v has label 0, which wins that game. Thus, G is $(A(G),\ell)$-AW.

For (2), let $U = N_G(v) - \{p\}$. Then $N(\overline{G})$ looks like the following.

	$V(G')-U$	U	v	p
$V(G')-U$	$N(\overline{G'-U})$	$*$	1	1
U	$*$	$N(\overline{U})$	0	1
v	1	0	1	0
p	1	1	0	1

where $G'-U$ is the induced subgraph with vertex set $V(G')-U$ and the $*$ blocks are the entries that make the four top-left blocks $N(\overline{G'})$. We multiply each row except the last by the unit -1 and then add to each of those rows the last row to get

$M =$

	$V(G')-U$	U	v	p
$V(G')-U$	$A(G'-U)$	$\overline{*}$	-1	0
U	$\overline{*}$	$A(U)$	0	0
v	0	1	-1	1
p	1	1	0	1

where the $\overline{*}$ blocks are obtained from $*$ by changing the 1 entries to 0 and the 0 entries to 1. This makes the top-left four blocks $A(G')$. So the M-Lights Out game is played as the $A(G')$-Lights Out game on $V(G')$; toggling any vertex in $V(G')$ adds 1 to the label of p; toggling any vertex in U adds 1 to the label of v; toggling v adds -1 to every vertex in $(V(G)-U)\cup\{v\}$; and toggling p adds 1 to v and p.

We first assume $\overline{G}$ is $(N(\overline{G}),\ell)$-AW. Since M is row equivalent to $N(\overline{G})$, Corollary 2.4 implies $\overline{G}$ is (M,ℓ)-AW. In particular, given any vertex labeling, it is possible to make all labels of $V(G')$ zero in the (M,ℓ)-Lights Out game. However, v and the vertices of $V(G')$ are the only vertices that can change the labels of $V(G')$. Thus, after toggling v, the resulting labeling of $V(G')$ must be $(A(G'),\ell)$-winnable. But v merely changes every vertex of $V(G') - U$ by the same value s, where v is toggled $-s$ times. This implies that every labeling of $V(G')$ is $(A(G'), V(G') - U, \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$. By Lemma 3.6(3), this implies that every labeling of $V(G)$ is $(A(G), \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$. This gives us (2a).

For (2b), let $z \in \mathbb{Z}_\ell$, and consider the labeling where p has label $-z$ and all other labels are 0. This labeling is M-winnable by assumption, so let y_1 be the number of times v is toggled and y_2 be the number of times p is toggled in order to win the M-Lights Out game with this labeling. This results in each vertex of $V(G') - U$ having label $-y_1$, each vertex of U having label 0, v having label $y_2 - y_1$, and p having label $-z + y_2$.

At this point, we have only the vertices in $V(G')$ to toggle, which means the remaining toggles necessary to win the M-Lights Out game will also win the $(A(G'), V(G') - U, \ell, -y_1)$-Lights Out game. Thus, $T^{A(G')}_{V(G')-U}(-y_1) \neq \emptyset$. By Lemma 3.6(1), $T^{A(G)}_{V(G)}(-y_1) \neq \emptyset$, and so $-y_1 = rx$ for some $x \in \mathbb{Z}$ by Lemma 3.5. By assumption, $t \in T^{A(G)}_{V(G)}(r)$, and so $t = t' - 2r$ for some $t' \in T^{A(G')}_{V(G')-U}(r)$ by Lemma 3.6(1). Thus, there exists $t_U \in \mathbb{Z}$ such that we can collectively toggle the vertices of U t_U times and the vertices of $V(G') - U$ t' times to win the $(A(G'), V(G') - U, \ell, r)$-Lights Out Game. By Lemma 3.4(2), we can toggle the vertices of U and $V(G') - U$ xt_U and xt' times, respectively, to win the $(A(G'), V(G') - U, \ell, rx)$-Lights Out Game. By Lemma 3.3(2), in order to win the M-Lights Out game on G, we must toggle the vertices of G' $xt_U + xt' + k$ for some $k \in Nul^{A(G')}_{V(G')}$. If we let $k = q_1 + q_2$, where q_1 is the number of toggles from $V(G') - U$ and q_2 is the number of toggles from U in the null toggle, we have $q_1 \in Nul^{A(G')}_{V(G')-U}$. Note that by negating all toggles in this null toggle, we still get a null toggle, and so $-q_1 \in Nul^{A(G')}_{V(G')-U}$. This leaves all vertices in $V(G')$ with label 0, v with label $y_2 + xr + xt_U + q_2$ (by setting $-y_1 = xr$), and p with label $-z + y_2 + xt_U + xt' + q_1 + q_2$.

All of the toggles have been accounted for, and so the labels of v and p must be 0. We then eliminate y_2 in the resulting system of equations to get $(r - t')x = -z + q_1$. Recall that $t = t' - 2r$, and so $t' = t + 2r$. This gives us $(-r - t)x = -z + q_1$, and so $(r + t)x = z - q_1$. Now let $q = -q_1$. As noted above, $q \in Nul^{A(G')}_{V(G')-U}$. By Lemma 3.6(2), $Nul^{A(G')}_{V(G')-U} = Nul^{A(G)}_{V(G)}$, and so $q \in Nul^{A(G)}_{V(G)}$. Since $(r + t)x = z + q$, this proves (2b).

Now we assume that (2a) and (2b) hold, and we prove that $\overline{G}$ is $(N(\overline{G}), \ell)$-AW. Since M is row equivalent to $N(\overline{G})$, we need only prove that $\overline{G}$ is (M, ℓ)-AW. Let $\pi \in \mathcal{L}(V(\overline{G}), \ell)$ be an arbitrary labeling. By (2a), π is $(A(G), \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$, and so Lemma 3.6(3) implies that $\pi \mid_{V(G')}$ is $(A(G'), V(G') - U, \ell, s)$-winnable for some $s \in \mathbb{Z}_\ell$. Thus, v can be toggled $-s$ times to get an $A(G')$-winnable labeling on $V(G')$. We can then toggle the vertices of $V(G')$ to give every vertex in $V(G')$ a label of 0. This leaves v with label a and p with label b for some $a, b \in \mathbb{Z}_\ell$.

By Lemma 3.6(1), $t = t' - 2r$ for some $t' \in T^{A}_{V(G')-U}(r)$. Thus, there exists $t_U \in \mathbb{Z}_\ell$ such that we can toggle the vertices of $V(G') - U$ t' times and the vertices of U t_U times to win the $(A(G'), V(G') - U, \ell, r)$-Lights Out game. Lemma 3.6(2) implies that $N^{A(G)}_{V(G)} = N^{A(G')}_{V(G')-U}$, and so $q \in N^{A(G')}_{V(G')-U}$. As we reasoned above, $-q \in N^{A(G')}_{V(G')-U}$, and so there exists $q_U \in N_U^A$ such that $q_U - q \in N_{V(G')}^{A(G')}$. Now let x be a solution to (2b), where $z = a - b$. This gives us $(r - t')x = b - a - q$. If v is toggled $-xr$ times and p is toggled $-b - x(t_U + t') + (q - q_U)$ times, this leaves each vertex of $V(G') - U$ with label xr, each vertex of U with label 0, v with label $a - b + xr - x(t_U + t') + (q - q_U)$, and p with label $-x(t_U + t') + (q - q_U)$. By Lemma 3.4(2), we can then win the $(A(G'), V(G') - U, xr)$-Lights Out game (and thus make the labels of $V(G')$ to be 0) by toggling the vertices of U xt_U times and the vertices of $V(G') - U$ a total of xt' times. The vertices of $V(G')$ can be toggled in such a way that the vertices of U are toggled q_U times, the vertices of $V(G') - U$ are toggled $-q$ times, and these toggles collectively have no effect on the labels of $V(G')$. So we combine these to toggle the vertices of U collectively $xt_U + q_U$ times and the vertices of $V(G') - U$ collectively $xt' - q$ times. This leaves the vertices of $V(G')$ with label 0, p with label $(-x[t_U + t'] + [q - q_U]) + (xt_U + q_U + xt' - q) = 0$, and v with label

$$\begin{aligned} a - b + xr - x(t_U + t') + (q - q_U) + xt_U + q_U &= a - b + x(r - t') + q \\ &= a - b + (b - a - q) + q = 0 \end{aligned}$$

This wins the game and shows that $\overline{G}$ is $(N(\overline{G}), \ell)$-AW. $\square$

In Theorem 3.7(2), if G is (A, ℓ)-AW, then G is automatically (A, ℓ, s)-winnable for all $s \in \mathbb{Z}_\ell$. This automatically satisfies Theorem 3.7(2a) and makes $r = 1$. Furthermore, $A(G)$ is invertible, so the only null toggle is where no buttons are pushed, making $N_{V(G)}^{A(G)} = \{0\}$. This gives us the following.

Corollary 3.8. Let $\ell \in \mathbb{N}$, and let G be an (A, ℓ)-AW graph with a pendant vertex. Let $t \in T_{V(G)}^{A(G)}(1)$. Then $\overline{G}$ is (N, ℓ)-AW if and only if $\gcd(1 + t, \ell) = 1$.

Proof. Since G is (A, ℓ)-AW, part (2a) of Lemma 3.7 is automatically satisfied. Moreover, since G is (A, ℓ)-AW, A is invertible, which implies that $N_{V(G)}^{A(G)} = \{0\}$. The result then follows directly from Theorem 3.7. $\square$

Furthermore, for possible (n, ℓ)-extremal graphs with a dominating vertex, Theorem 3.7(1) gives us a way to eliminate most graphs with pendant vertices.

Corollary 3.9. Let G be a graph with a dominating vertex. If $\overline{G}$ has a pendant vertex that is not part of a component isomorphic to P_2, then G is not (n, ℓ)-extremal for any n and ℓ.

Proof. For contradiction, assume G is (n, ℓ)-extremal, that $\overline{G}$ has a pendant vertex p with neighbor v, and that p and v are not the only vertices in their connected component of $\overline{G}$. Thus, v has a neighbor other than p in $\overline{G}$. Let w be the dominating vertex in G, and let G' be the subgraph of $\overline{G}$ induced by $V(\overline{G}) - \{p, v, w\}$. If we remove the edges in $\overline{G}$ incident to v but not p, we get the graph $H = G' \cup P_2 \cup P_1$. Note that H has size smaller than $\overline{G}$, and so $\overline{H}$ has size greater than G. To contradict the assumption that G is (n, ℓ)-extremal, it then suffices to prove that $\overline{H}$ is (N, ℓ)-AW.

Since G is (N, ℓ)-AW, Theorem 2.5 implies that $\overline{G} - \{w\}$ is (A, ℓ)-AW. By Theorem 3.7(1), this implies that G' is (A, ℓ)-AW. Since P_2 is (A, ℓ)-AW for all ℓ, it follows that $G' \cup P_2 = H - \{w\}$ is (A, ℓ)-AW. By Theorem 2.5, $\overline{H}$ is (N, ℓ)-AW. This means G is not (n, ℓ)-extremal, a contradiction. □

One nice property of pendant graphs is that it is really easy to play the (A, ℓ)-Lights Out game on them. This is demonstrated in the following result.

Lemma 3.10. Let H be a graph, and $G = H \odot K_1$. Then G is (A, ℓ)-AW for all $\ell \in \mathbb{N}$. More specifically, if $\pi \in \mathcal{L}(V(G), \ell)$, then the following occur for any winning toggling in the A-Lights Out game with initial labeling π.

1. If $v \in V(H)$, then v is toggled $-\pi(w_v)$ times, where w_v is the pendant vertex in G adjacent to v.

2. If $w_v \in V(G) - V(H)$ is adjacent to $v \in V(H)$, then w_v is toggled $-\pi(v) + \sum_{x \in N_H(v)} \pi(w_x)$ times.

3. The vertices in $V(G) - V(H)$ are collectively toggled $\displaystyle\sum_{uv \in E(H)} (\pi(w_u) + \pi(w_v)) - \sum_{v \in V(H)} \pi(v)$ times.

Proof. Let $\pi \in \mathcal{L}(V(G), \ell)$. If each $v \in V(H)$ is toggled $-\pi(w_v)$ times, that will leave each w_v with label 0. Since each neighbor $x \in N_H(v)$ is toggled $-\pi(w_x)$ times, this leaves the label of v as $\pi(v) - \sum_{x \in N_H(v)} \pi(w_x)$. If each w_v is toggled $-\pi(v) + \sum_{x \in N_H(v)} \pi(w_v)$ times this makes all labels zero. Thus, G is (A, ℓ)-AW, and (1) and (2) follow.

It is clear from the above that each $v \in V(H)$ is toggled $-\pi(w_v)$ times. Each time a vertex $v \in V(H)$ is toggled, it adds $-\pi(w_v)$ to the label of each of its neighbors. Collectively, this adds $-\sum_{uv \in E(H)} (\pi(w_u) + \pi(w_v))$ to the labels of $V(H)$, and so the sum of the labels of H is $\sum_{v \in V(H)} \pi(v) - \sum_{uv \in E(H)} (\pi(w_u) + \pi(w_v))$. Thus, the vertices of $V(G) - V(H)$ are collectively toggled $\sum_{uv \in E(H)} (\pi(w_u) + \pi(w_v)) - \sum_{v \in V(H)} \pi(v)$ times. This gives us (3). □

This allows us to compute the $V(G)$-toggling numbers for pendant graphs.

Corollary 3.11. Let H be a graph, and let $G = H \odot K_1$.

1. If G has size m and order n, then $T^{A(G)}_{V(G)}(1) = \{2(m - n)\}$.

2. If G is a pendant forest with c components, then $T^{A(G)}_{V(G)}(1) = \{-2c\}$.

Proof. For (1), we determine $T^{A(G)}_{V(G)}(1)$ by making $\pi(v) = 1$ in Lemma 3.10 for all $v \in V(G)$. Thus, each $v \in V(H)$ is toggled -1 times, and so the vertices of $V(H)$ are collectively toggled $-|V(H)|$ times. Similarly, if for each $v \in V(H)$, we denote the pendant neighbor of v by w_v, then each w_v is toggled $-1 + |N_H(v)|$ times for all $v \in H$.

It follows that the total number of toggles in is $2|E(H)| - 2|V(H)|$. Using the facts that $|V(H)| = \frac{n}{2}$ and $|E(H)| = m - \frac{n}{2}$, the total number of toggles then comes to $2\left(m - \frac{n}{2}\right) - 2\left(\frac{n}{2}\right) = 2(m-n)$. Part (2) follows from (1) and the fact that $n = m + c$. $\square$

We can now determine the winnability of the complements of pendant graphs. Interestingly, the issue of whether or not a pendant graph is (N, ℓ)-AW depends entirely on the size and order of the pendant graph.

Lemma 3.12. Let G be a pendant graph of size m and order n, and let $\ell \in \mathbb{N}$. Then $\overline{G}$ is (N, ℓ)-AW if and only if $\gcd(2[n-m]-1, \ell) = 1$. Equivalently, if G is a graph of even order n and size $\binom{n}{2} - \left(\frac{n}{2} + k\right)$ such that $\overline{G}$ is a pendant graph then G is (N, ℓ)-AW if and only if $\gcd(n - 2k - 1, \ell) = 1$.

Proof. By Lemma 3.10, G is (A, ℓ)-AW, and so we can apply Corollary 3.8. By Corollary 3.11(1), $T_{V(G)}^{A(G)}(1) = \{2(m-n)\}$. By Corollary 3.8, $\overline{G}$ is (N, ℓ)-AW if and only if $\gcd(2(m-n)+1, \ell) = 1$. The second part follows from substituting $m = \frac{n}{2} + k$ to get $2(m-n)+1 = -(n-2k-1)$. $\square$

If $\overline{G}$ is a forest, then $n - m$ is the number of components of $\overline{G}$. This along with Lemma 3.12 gives us the following.

Corollary 3.13. Let G be a graph such that the components of $\overline{G}$ are all pendant trees. If c is the number of components of $\overline{G}$, then G is (N, ℓ)-AW if and only if $\gcd(2c-1, \ell) = 1$.

When we are classifying (n, ℓ)-extremal graphs in the next section, it will be helpful to replace connected components of the complement of one graph with another graph without affecting the (N, ℓ)-winnability of the original graph. The following guarantees that the conditions of Theorem 3.7(2) are unaffected by the replacement.

Corollary 3.14. Let G be a graph, let C be a connected component of G that is $(A(C), \ell)$-AW, and let C' be a graph such that

1. C' is $(A(C'), \ell)$-AW.
2. $T_{V(C')}^{A(C')}(1) = T_{V(C)}^{A(C)}(1)$

Let G' be the graph obtained by replacing C in G with C'. If G and G' both have a pendant vertex, then $\overline{G'}$ is (N, ℓ)-AW if and only if $\overline{G}$ is (N, ℓ)-AW.

Proof. Since the winnability of the adjacency game is determined by the winnability of the adjacency game on each connected component of a given graph, and since both C and C' are (A, ℓ)-AW, we have that a labeling is $(A(G), \ell, s)$-winnable (resp. $(A(G'), \ell, s)$-winnable) if and only if the labeling restricted to $G - C$ (resp. $G' - C'$) is $(A(G-C), s)$-winnable (resp. $(A(G'-C'), s)$-winnable). Since $G - C = G' - C'$, this condition is identical for both G and G'. Thus, either both of G and G' satisfy Theorem 3.7(2a) or neither does. A similar argument gives us that either both of G and G' satisfy Theorem 3.7(2b) or neither does. The result follows directly. $\square$

This leads us to a way of showing a given graph is not (n,ℓ)-extremal.

Corollary 3.15. Let G be a graph with a pendant vertex, and let C be a connected component of G that is (A,ℓ)-AW. If there exists a graph C' such that

1. C' is (A,ℓ)-AW.
2. $T^{A(C')}_{V(C')}(1) = T^{A(C)}_{V(C)}(1)$
3. C and C' have the same order.
4. C' has smaller size than C.

Then $\overline{G}$ is not (n,ℓ)-extremal.

Proof. Let G' be the graph identical to G except that the component C is replaced with C'. By Corollary 3.14, $\overline{G'}$ is (N,ℓ)-AW if and only if $\overline{G}$ is (N,ℓ)-AW. Furthermore, since C and C' have the same order, so do $\overline{G}$ and $\overline{G'}$. Finally, since C' has smaller size than C, $\overline{G'}$ has larger size than $\overline{G}$. Since $\overline{G'}$ and $\overline{G}$ have the same order and same winnability but $\overline{G'}$ has larger size, $\overline{G}$ cannot be (n,ℓ)-extremal. $\square$

4 Extremal Graphs

Recall that $\max(n,\ell)$ is the maximum number of edges in an (N,ℓ)-AW graph with n vertices. In this section we consider finding $\max(n,\ell)$ from two perspectives: finding all graphs with minimum degree $n-2$ or $n-3$ that are (n,ℓ)-extremal for any ℓ, and finding all combinations of n and ℓ such that the (n,ℓ)-extremal graph has $\binom{n}{2} - \left(\frac{n}{2}+k\right)$ edges for $0 \le k \le 3$. In both perspectives we are led to pendant graphs, which supports Conjecture 1.1.

4.1 Extremal Graphs With a Given Minimum Degree

In this section we find $\max(n,\ell)$ among all graphs with minimum degree $n-2$ and $n-3$. Note that the minimum degree of an (n,ℓ)-extremal graph can not be $n-1$ because K_n is not (N,ℓ)-AW. In fact, we know $\max(n,\ell) \le \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$ because of the following proposition.

Proposition 4.1. Let G be a graph on n vertices. If $|E(\overline{G})| < \left\lfloor \frac{n}{2} \right\rfloor$, then G is not (N,ℓ)-AW for any ℓ.

Proof. Since $|E(\overline{G})| < \left\lfloor \frac{n}{2} \right\rfloor$, at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ edges are removed from K_n to obtain G. Thus, at most $2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \le n-2$ vertices of K_n can have their degrees reduced by one or more to obtain G. Thus, at least two vertices in G are dominating vertices. So G has N-twins, and such a G is not (N,ℓ)-AW by Corollary 2.3. $\square$

On the flip side we know $\max(n,\ell) \geq \binom{n}{2} - (n-1)$ since the complement of any pendant tree is (N,ℓ)-AW for all ℓ by Corollary 3.13. We now consider when $\max(n,\ell) = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$. Let M_n be a perfect matching on n vertices when n is even and a near-perfect matching on n vertices when n is odd. The case where n is odd is simple.

Proposition 4.2. If n is odd, then

$$\max(n,\ell) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$$

for all $\ell \in \mathbb{N}$. Moreover, $\overline{M_n}$ is the unique (N,ℓ)-AW of maximum size on n vertices.

Proof. Note M_{n-1} is a pendant graph, specifically $\left(\frac{n-1}{2}K_1\right) \odot K_1$, so by Lemma 3.10, M_{n-1} is (A,ℓ)-AW for all ℓ. By Theorem 2.5, $\overline{M_n}$ is (N,ℓ)-AW. So, $\max(n,\ell) \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$ when n is odd. By Proposition 4.1, $\max(n,\ell) = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$. For any graph $G \neq \overline{M_n}$ with $\binom{n}{2} - \lfloor \frac{n}{2} \rfloor$ edges we must have two dominating vertices in G, which are N-twins. Thus, $\overline{M_n}$ is unique. □

In contrast to Proposition 4.2, when n is even, $\overline{M_n}$ is not (N,ℓ)-AW for every ℓ.

Proposition 4.3. If n is even, then

$$\max(n,\ell) = \tbinom{n}{2} - \tfrac{n}{2} \text{ if and only if } \gcd(n-1,\ell) = 1.$$

If n is even and $\gcd(n-1,\ell) = 1$, then $\overline{M_n}$ is the unique (N,ℓ)-AW graph of maximum size on n vertices.

Proof. Each component of M_n is a pendant tree. By Corollary 3.13 $\overline{M_n}$ is (N,ℓ)-AW if and only if $\gcd\left(2\left(\frac{n}{2}\right) - 1, \ell\right) = \gcd(n-1,\ell) = 1$. That $\overline{M_n}$ is the unique (N,ℓ)-AW graph of maximum size on n vertices follows from the proof of Proposition 4.1. □

By the proof of Proposition 4.1 the only (n,ℓ)-extremal graphs with minimum degree $n-2$ are M_n. Propositions 4.2 and 4.3 tell us that if G is an (n,ℓ)-extremal graph with minimum degree $n-2$, then $G = \overline{M_n}$.

In the case that G has minimum degree $n-3$ the complement has maximum degree 2. So the components of the complement graph are paths and cycles. We consider what happens with cycles in the following lemma. We denote the cycle graph C_n by $V(C_n) = \{v_i : 1 \leq i \leq n\}$, $E(C_n) = \{v_i v_{i+1}, v_n v_1 : 1 \leq i \leq n-1\}$. For $a, b \in \mathbb{Z}_\ell$, we define $\lambda_{a,b}$ to be the labeling where v_1 has label a, v_2 has label b, and the other vertices have label 0. Recall that for $s \in \mathbb{Z}_\ell$ we define $\mathbf{0}_s(v_i) = s$ for all $1 \leq i \leq n$.

Lemma 4.4. Let $\pi \in \mathcal{L}(V(C_n),\ell)$, where ℓ is even.

1. If π is the initial labeling of $V(C_n)$ in the (A,ℓ)-Lights Out game, then the vertices can be toggled so that the resulting labeling is $\lambda_{a,b}$ for some $a, b \in \mathbb{Z}_\ell$.

2. The (A,ℓ)-Lights Out game can be won with initial labeling $\lambda_{a,b}$, with v_1 toggled x times, and with v_2 toggled y times if and only if for each $1 \le i \le n$, v_i is toggled t_i times, where $t_{n-1}+x=0$, $t_n+a+y=0$, $t_1=x$ and for $2 \le i \le n$,

$$t_i = \begin{cases} -y & i \equiv 0 \pmod 4 \\ b+x & i \equiv 1 \pmod 4 \\ y & i \equiv 2 \pmod 4 \\ -b-x & i \equiv 3 \pmod 4 \end{cases} \tag{1}$$

3. The labeling $\lambda_{a,b}$ is (A,ℓ)-winnable precisely in the following circumstances.

   (a) When $n \equiv 0 \pmod 4$ and $a=b=0$.

   (b) When $n \equiv 1,3 \pmod 4$ and a and b have the same parity.

   (c) When $n \equiv 2 \pmod 4$ and a and b are both even.

4. The labeling $(\lambda_{a,b})_s$ can be toggled in the (A,ℓ)-Lights Out game to obtain the following labelings.

   - When $n \equiv 0 \pmod 4$, $\lambda_{a,b}$.
   - When $n \equiv 1 \pmod 4$, $\lambda_{a,b-s}$.
   - When $n \equiv 2 \pmod 4$, $\lambda_{a-s,b-s}$.
   - When $n \equiv 3 \pmod 4$, $\lambda_{a-s,b}$.

Proof. For (1), let k be maximum such that $\pi(v_k) \neq 0$. If v_{k-1} is toggled $-\pi(v_k)$ times, we get a labeling with $\pi(v_i)=0$ for all $i \ge k$. An easy induction argument shows that if we continue this process through toggling v_2, we end up with a labeling in which $\pi(v_i)=0$ for all $i \ge 3$. If we let a and b be the labels for v_1 and v_2, respectively, for this labeling, this results in the labeling $\lambda_{a,b}$.

For (2), it is clear that the game can be won if and only if $t_{n-1}+\lambda_{a,b}(v_n)+t_1=0$, $t_n+\lambda_{a,b}(v_1)+t_2=0$, and $t_i+\lambda_{a,b}(v_{i-1})+t_{i-2}=0$ for $3 \le i \le n$. Since $\lambda_{a,b}(v_n)=0$, $\lambda_{a,b}(v_1)=a$, $t_1=x$, and $t_2=y$, the first two equations are equivalent to $t_{n-1}+x=0$ and $t_n+a+y=0$. For $i=3$, we get $t_3=-\lambda_{a,b}(v_2)-t_1=-b-x$. For the remaining t_i's, we have $t_i=-0-t_{i-2}=-t_{i-2}$. The result follows from a straightforward induction argument.

For (3), it follows from (2) that the labeling $\lambda_{a,b}$ is winnable precisely when there exist $x,y \in \mathbb{Z}_\ell$ such that $t_{n-1}+x=0$, $t_n+a+y=0$, and when these values are compatible with Equation (1). When $n \equiv 0 \pmod 4$, this means $(-b-x)+x=0$ and $-y+a+y=0$. This gives us $a=b=0$. For $n \equiv 1 \pmod 4$, we get $-y+x=0$ and $(b+x)+a+y=0$, which simplifies to $x=y$ and $b+a+2x=0$. This system has a solution precisely when $b+a$ is even, which is true precisely when a and b have the same parity. The other cases follow similarly.

For (4), we begin with the labeling $(\lambda_{a,b})_s$, and then each v_i with $2 \le i < 4\left\lfloor \frac{n}{4} \right\rfloor$ and $i \equiv 2,3 \pmod 4$ is toggled $-s$ times. This results in the labeling where each v_i with

$1 \le i \le 4\left\lfloor \frac{n}{4} \right\rfloor$ has label $\lambda_{a,b}(v_i)$ and each v_i with $i > 4\left\lfloor \frac{n}{4} \right\rfloor$ has label $(\lambda_{a,b})_s(v_i)$. If $n \equiv 0 \pmod 4$, this gives us the labeling $\lambda_{a,b}$ over all vertices in C_n. For $n \equiv 1 \pmod 4$, v_1 is toggled $-s$ times to obtain $\lambda_{a,b-s}$. For $n \equiv 2 \pmod 4$, v_n and v_1 are each toggled $-s$ times to obtain $\lambda_{a-s,b-s}$. Finally, for $n \equiv 3 \pmod 4$, v_{n-1} and v_n are each toggled $-s$ times to obtain $\lambda_{a-s,b}$.

□

The next result helps us see how the presence of cycle components in a graph can affect its (A, ℓ, s)-winnability, which is crucial when we apply Theorem 3.7(2).

Lemma 4.5. Let G be a graph.

1. If G has a connected component that is a cycle of even order, then G has a labeling that is not (A, ℓ, s)-winnable for all $s \in \mathbb{Z}_\ell$.

2. If G has two connected components that are cycles, then G has a labeling that is not (A, ℓ, s)-winnable for all $s \in \mathbb{Z}_\ell$.

Proof. For (1), let C be a cycle component of G with even order, and define a labeling that is $\lambda_{1,0}$ on C and arbitrary on the remaining vertices of G. Since a labeling is winnable on a graph if and only if it is winnable on each connected component, it suffices to prove that $(\lambda_{1,0})_s$ is not winnable on C for all $s \in \mathbb{Z}_\ell$. If C has order divisible by 4, then Lemma 4.4(4) implies that the vertices of C can be toggled to achieve $\lambda_{1,0}$, which is not winnable by Lemma 4.4(3). If C has order not divisible by 4, then by Lemma 4.4(4), the vertices can be toggled to achieve the labeling $\lambda_{1-s,-s}$. Since $1-s$ and s can never both be even, Lemma 4.4(3) implies that $\lambda_{1-s,-s}$ is not winnable for all $s \in \mathbb{Z}_\ell$. In either case, $(\lambda_{1,0})_s$ is not winnable on C for all $s \in \mathbb{Z}_\ell$, and so $\lambda_{1,0}$ is not (A, ℓ, s)-winnable for all $s \in \mathbb{Z}_\ell$.

For (2), let C and C' be two cycle components of G. By (1), we can assume each of C and C' has odd order. We claim that any labeling π that restricts to $\lambda_{1,0}$ on C and $\lambda_{0,0}$ on C' is not (A, ℓ, s)-winnable for all $s \in \mathbb{Z}_\ell$. By Lemma 4.4(4), with initial labeling π_s, we can toggle the vertices of G to obtain a labeling that restricts either to $\lambda_{1-s,0}$ or $\lambda_{1,-s}$ on C and restricts either to $\lambda_{-s,0}$ or $\lambda_{0,-s}$ on C'. If s is even, then $1-s$ and 0 as well as 1 and $-s$ have opposite parity. If s is odd, then $-s$ and 0 have opposite parity. In any case, Lemma 4.4(3) implies that π_s is not winnable, and so π is not (A, ℓ, s)-winnable for all $s \in \mathbb{Z}_\ell$. □

Our next lemma helps us when we want to apply Theorem 2.5 to graphs with both a dominating vertex and a cycle component in its complement.

Lemma 4.6. If ℓ is even, then every cycle graph is not (A, ℓ)-AW.

Proof. By Lemma 4.4(3), if $a, b \in \mathbb{Z}_\ell$ have opposite parity, then $\lambda_{a,b}$ is not (A, ℓ)-winnable. The result follows. □

The following theorem gives us a connection between (n, ℓ)-extremal graphs and pendant graphs, in support of Conjecture 1.1.

Theorem 4.7. Let ℓ be even, and let G be an (n, ℓ)-extremal graph of even order with $\Delta(\overline{G}) \leq 2$. Then each connected component of $\overline{G}$ is either P_2 or P_4.

Proof. Suppose $\Delta(\overline{G}) = 1$. All dominating vertices in G are N-twins so by Corollary 2.3 G has at most 1 dominating vertex. However, G can not have only one dominating vertex since G has even order. Thus, G has no dominating vertices, and so $\overline{G}$ has no isolated vertices. It follows that each connected component of G is P_2.

In the case $\Delta(\overline{G}) = 2$, we first prove that $\overline{G}$ has at least one path component. If not, all connected components are cycles, and so $|E(\overline{G})| = |V(\overline{G})|$. However, note that any pendant tree of order $|V(G)|$ is (N, ℓ)-AW for all ℓ by Corollary 3.13. Since the pendant tree has size $|V(G)| - 1$, this implies that G is not (n, ℓ)-extremal. Thus, $\overline{G}$ has at least one path component (possibly P_1).

By Corollary 3.2, no component of $\overline{G}$ is P_k with $k \geq 5$ or $k = 3$. Furthermore two components of P_1 in $\overline{G}$ would be N-twins in G, which is prohibited by Corollary 2.3. If we have one component of P_1, Theorem 2.5 implies that all other connected components of $\overline{G}$ are (A, ℓ)-AW. This excludes cycles by Lemma 4.6. Since the remaining paths have even order, this would force G to have odd order, which is a contradiction.

So G is (N, ℓ)-AW and $\overline{G}$ has a pendant tree (P_2 or P_4) as a component. Thus, $\overline{G}$ has a pendant vertex, so we can use Theorem 3.7(2). This implies that $\overline{G}$ is (A, ℓ, s)-winnable for some $s \in \mathbb{Z}_\ell$. However, Lemma 4.5 implies that this can not happen if either $\overline{G}$ has more than one cycle component or if $\overline{G}$ has a cycle component of even order. Moreover, if $\overline{G}$ has precisely one cycle component, and if that connected component has odd order, this implies that G has odd order, which is a contradiction. Thus, $\overline{G}$ has no cycle components, and so each connected component is either P_2 or P_4, which completes the proof. $\square$

Note that the (n, ℓ)-extremal graphs given in Theorem 4.7 are pendant graphs. By Lemma 3.12 $kP_4 \cup \frac{n-4k}{2} P_2$ is (N, ℓ)-AW if and only if $\gcd(n - 2k - 1, \ell) = 1$. This implies Conjecture 1.1 for the family of graphs which have minimum degree at least $n - 3$.

4.2 Extremal Graphs with $\binom{n}{2} - (\frac{n}{2} + k)$ edges

In this section, we show which pairs (n, ℓ) have $\max(n, \ell) = \binom{n}{2} - (\lfloor \frac{n}{2} \rfloor + k)$ for $1 \leq k \leq 3$. Propositions 4.2 and 4.3 deal with the case of $k = 0$. Proposition 4.2 also allows us to consider only graphs of even order for the rest of the section.

The next proposition takes care of all cases where n is even and ℓ is odd. In contrast to Propositions 4.2 and 4.3 this proposition is not stated as a biconditional statement. There are, in fact, two situations in which $\max(n, \ell) = \binom{n}{2} - (\frac{n}{2} + 1)$. We address the following case separately since it is the only situation we have found in which an (n, ℓ)-extremal graph is not the complement of a pendant graph.

Proposition 4.8. Suppose that $n \geq 4$ is even. If ℓ is odd and $\gcd(n - 1, \ell) \neq 1$ then $\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + 1\right)$. In this case an (n, ℓ)-extremal graph is $\overline{C_3 \cup \left(\frac{n-4}{2}\right) P_2 \cup K_1}$.

Proof. We first show that $H = \overline{C_3 \cup \left(\frac{n-4}{2}\right) P_2 \cup K_1}$, which has $\binom{n}{2} - \left(\frac{n}{2} + 1\right)$ edges, is (N, ℓ)-AW. By Theorem 2.5, we need only prove that $C_3 \cup \left(\frac{n-4}{2}\right) P_2$ is (A, ℓ)-AW. Clearly,

P_2 is (A, ℓ)-AW for all $\ell \in \mathbb{N}$. We can see that C_3 is (A, ℓ)-AW if and only if ℓ is odd by row reducing the adjacency matrix of C_3. By playing on each component, $C_3 \cup \left(\frac{n-4}{2}\right) P_2$ is (A, ℓ)-AW, and so H is (N, ℓ)-AW.

We now show that H is (n, ℓ)-extremal. Since $\gcd(n-1, \ell) \neq 1$, Proposition 4.3 implies that M_n is not (n, ℓ)-extremal. By the uniqueness of M_n, $\max(n, \ell) \leq \binom{n}{2} - \left(\frac{n}{2} + 1\right)$. Since H is (N, ℓ)-AW and $|E(\overline{H})| = \binom{n}{2} - \left(\frac{n}{2} + 1\right)$ $\max(n, \ell) = \binom{n}{2} - \frac{n}{2} + 1$. $\square$

Since $\overline{M_n}$ is the (n, ℓ)-extremal graph in the case that n is odd and $\overline{C_3 \cup \left(\frac{n-4}{2}\right) P_2 \cup K_1}$ is an (n, ℓ)-extremal graph in the case that n is even and ℓ is odd, from here on we consider only cases where n and ℓ are both even. We now present the main theorem of this section using the language of Conjecture 1.1.

Theorem 4.9. For n, ℓ even and $0 \leq k \leq 3$

$$\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + k\right)$$

if and only if k is the smallest nonnegative integer such that $\gcd(n - 2k - 1, \ell) = 1$. In each case the (n, ℓ)-extremal graphs are precisely the complements of pendant graphs of order n that have size $\binom{n}{2} - \left(\frac{n}{2} + k\right)$.

We will prove this result using separate propositions for each k. When $k = 0$, Proposition 4.3 implies Theorem 4.9. The following lemma will help us for the cases $1 \leq k \leq 3$.

Lemma 4.10. Let $n \in \mathbb{N}$ be even and let $\ell \in \mathbb{N}$, and let G be a (N, ℓ)-AW graph with $|E(\overline{G})| = \frac{n}{2} + t$, where $t \geq 1$. Then the maximum degree of the complement is at most $t + 1$. That is, $\Delta(\overline{G}) \leq t + 1$.

Proof. We let $v \in V(\overline{G})$ and show $\deg(v) \leq t + 1$, where $\deg(v)$ is the degree of v in $\overline{G}$. Let $W = V(\overline{G}) - N_{\overline{G}}[v]$, and note that $\deg(v) = |N_{\overline{G}}(v)|$. Then $|W| = n - \deg(v) - 1$. In the graph $\overline{G}$, let k be the number of edges incident only to vertices in $N_{\overline{G}}(v)$, let r be the number of edges incident only to vertices in W, and let s be the number of edges between a vertex in $N_{\overline{G}}(v)$ and a vertex in W. Since $|E(\overline{G})| = \frac{n}{2} + t$, we have $\frac{n}{2} + t = \deg(v) + k + r + s$, and so $k + r + s = \frac{n}{2} + t - \deg(v)$.

Since G is (N, ℓ)-AW, it can not have any N-twins. Thus, no vertices in $N_{\overline{G}}(v)$ can be N-twins, so we can have at most one vertex in $N_{\overline{G}}(v)$ that is adjacent in G to every vertex except v. In other words, there are at least $\deg(v) - 1$ vertices in W that are adjacent in $\overline{G}$ to vertices other than v. There can be at most two such vertices for each of the k edges in $\overline{G}$ incident with two vertices in $N_{\overline{G}}(v)$, and at most one such vertex for each of the s edges between vertices in $N_{\overline{G}}(v)$ and W. This means that there are at most $2k + s$ such vertices in $N_{\overline{G}}(v)$. It follows that $\deg(v) - 1 \leq 2k + s$, and so $\deg(v) \leq 2k + s + 1$.

In order to prevent any vertices in W from becoming N-twins, we can have at most one vertex in W that is adjacent to every vertex in G. In other words, there are at least $|W| - 1 = n - \deg(v) - 2$ vertices in W with nonzero degree in $\overline{G}$. Similar reasoning as in the previous paragraph implies that there are at most $2r + s$ such vertices in W, and so $n - \deg(v) - 2 \leq 2r + s$. Thus, $\deg(v) \geq n - 2r - s - 2$.

Since we have $n-2r-s-2 \le \deg(v) \le 2k+s+1$, it follows that $n-2r-s-2 \le 2k+s+1$. This gives us $n-2k-2r-2s \le 3$. Since the left side of the equation is even, this actually gives us $n-2k-2r-2s \le 2$, and so $\frac{n}{2}-k-r-s \le 1$. Rearranging this a bit gives us $k+r+s \ge \frac{n}{2}-1$.

Now we use the fact $k+r+s = \frac{n}{2}+t-\deg(v)$ to get $\frac{n}{2}+t-\deg(v) \ge \frac{n}{2}-1$. Solving for $\deg(v)$ gives $\deg(v) \le t+1$. □

In the next proposition, we resolve the case of $k=1$.

Proposition 4.11. Suppose that n and ℓ are even and $n \ge 4$. Then

$$\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right) \text{ if and only if } \gcd(n-1,\ell) \ne 1 \text{ and } \gcd(n-3,\ell) = 1.$$

Moreover, the only (n,ℓ)-extremal graph in this case is the complement of the unique pendant graph of order n and size $\binom{n}{2} - \left(\frac{n}{2}+1\right)$, which is $\overline{P_4 \cup \left(\frac{n}{2}-2\right) P_2}$.

Proof. Suppose $\gcd(n-1,\ell) \ne 1$ and $\gcd(n-3,\ell) = 1$. Consider $H = P_4 \cup \left(\frac{n}{2}-2\right) P_2$. Note that H is a pendant graph with n vertices and $\frac{n}{2}+1$ edges. By Lemma 3.12, $\overline{H}$ is (N,ℓ)-AW if and only if $\gcd(n-3,\ell) = 1$. Since $\gcd(n-1,\ell) \ne 1$ it follows from Proposition 4.3 that $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right)$.

Suppose $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right)$. Then $\gcd(n-1,\ell) \ne 1$ (else $\max(n,\ell) = \binom{n}{2} - \frac{n}{2}$ by Proposition 4.3). By Lemma 4.10, if G is (N,ℓ)-AW with $|E(\overline{G})| = \frac{n}{2}+1$ edges then $\Delta(\overline{G}) \le 2$. So by Theorem 4.7 each connected component of $\overline{G}$ is either P_2 or P_4. The only such graph with $\frac{n}{2}+1$ edges is H. Thus $\gcd(n-3,\ell) = 1$. It is clear that H is the only pendant graph of order $\frac{n}{2}+1$. □

The following corollary combines Propositions 4.8 and 4.11 to state the situations in which $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right)$ exactly.

Corollary 4.12. For $n, \ell \in \mathbb{N}$,

$$\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right) \text{ if and only if } \gcd(n-1,\ell) \ne 1 \text{ and } \gcd(2,\ell) = 1 \text{ or } \gcd(n-3,\ell) = 1.$$

In this case the only (n,ℓ)-extremal graphs are $\overline{C_3 \cup \left(\frac{n-4}{2}\right) P_2 \cup K_1}$ and $\overline{P_4 \cup \left(\frac{n}{2}-2\right) P_2}$.

Proposition 4.11 implies Theorem 4.9 in the case that $k=1$. In the next proposition, we do the case $k=2$. The proof considers the possible degree sequences of the complements of (n,ℓ)-extremal graphs. To ease our explanation we introduce a notation. Let a *d-vertex* refer to a vertex of degree d. A *d^+-vertex* is a vertex of degree d or more.

Lemma 4.13. Suppose G is an (N,ℓ)-AW graph for some ℓ. Then any d-vertex in $\overline{G}$ with $d \ge 2$ must have at least $d-1$ neighbors that are 2^+-vertices.

Proof. Suppose that v is a d-vertex in $\overline{G}$ with $d \ge 2$ and that v has fewer than $d-1$ neighbors that are 2^+ vertices. Then v has two neighbors of degree 1 in $\overline{G}$, which results in G having N-twins. By Corollary 2.3, $\overline{G}$ is not (N,ℓ)-AW. □

Proposition 4.14. Let $n, \ell \in \mathbb{N}$ be even and $n \geq 6$. Then

$$\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+2\right) \text{ if and only if } \gcd(n-1,\ell) \neq 1,\ \gcd(n-3,\ell) \neq 1, \text{ and } \gcd(n-5,\ell) = 1.$$

Moreover, the (n,ℓ)-extremal graphs in this case are precisely the complements of pendant graphs of order n and size $\frac{n}{2}+2$: $\overline{(P_3 \odot K_1) \cup \frac{n-6}{2}P_2}$ and $\overline{2P_4 \cup \frac{n-8}{2}P_2}$ with the latter only possible when $n \geq 8$.

Proof. Suppose $\gcd(n-1,\ell) \neq 1$, $\gcd(n-3,\ell) \neq 1$ and $\gcd(n-5,\ell) = 1$. By Proposition 4.3 and Proposition 4.11 $\max(n,\ell) \leq \binom{n}{2} - \left(\frac{n}{2}+2\right)$. Consider $H = (P_3 \odot K_1) \cup \frac{n-6}{2}P_2$ which has size $\frac{n}{2}+2$. By Lemma 3.12, $\overline{H}$ is (N,ℓ)-AW if and only if $\gcd(n-5,\ell) = 1$. So, $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+2\right)$.

Now suppose that $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+2\right)$. Then $\gcd(n-1,\ell) \neq 1$ and $\gcd(n-3,\ell) \neq 1$ by Propositions 4.3 and 4.11. We will describe all G such that G is (N,ℓ)-AW for some ℓ and $E(\overline{G}) = \frac{n}{2}+2$ and show either that these graphs are not (n,ℓ)-extremal or that they are (N,ℓ)-AW if and only if $\gcd(n-5,\ell) = 1$.

Suppose that G is (N,ℓ)-AW with $E(\overline{G}) = \frac{n}{2}+2$. The degree sum of $\overline{G}$ is $n+4$. By Lemma 4.10, $\Delta(\overline{G}) \leq 3$. To avoid N-twins in G, $\overline{G}$ can have at most one 0-vertex. Thus the only possible degree sequences for $\overline{G}$ are $d_0 = (3,3,1,1,\ldots,1)$, $d_1 = (3,2,2,1,1,\ldots,1)$, $d_2 = (2,2,2,2,1,1,\ldots,1)$, $d_3 = (3,3,2,1,1,\ldots,1,0)$, $d_4 = (3,2,2,2,1,1,\ldots,1,0)$, and $d_5 = (2,2,2,2,2,1,1,\ldots,1,0)$. For degree sequence d_0, note that each of the 3-vertices has at least two pendant neighbors, which violates Lemma 4.13. If $\overline{G}$ has degree sequence d_1, Lemma 4.13 implies that all 2^+-vertices are in the same component. The other components of $\overline{G}$ must be a matching. So our options are $G_1 \cup \frac{n-4}{2}P_2$ where G_1 is shown in Appendix A or $H = (P_3 \odot K_1) \cup \frac{n-6}{2}P_2$. Note $\overline{H}$ is (N,ℓ)-AW if and only if $\gcd(n-5,\ell) = 1$ as shown above.

The graph G_1 has order 4 and size 4 and $T^A_{V(G_1)}(1) = -2$ (see Appendix A). By Lemma 2.2, G_1 is (A,ℓ)-AW because the adjacency matrix is invertible. The graph $P_4 = P_2 \odot K_1$ has order 4, size 3, is (A,ℓ)-AW by Corollary 3.10, and, by Corollary 3.11, $T^A_{V(P_4)}(1) = -2$. Thus, $\overline{G_1 \cup \frac{n-4}{2}P_2}$ is not (n,ℓ)-extremal by Corollary 3.15. If $\overline{G}$ has degree sequence d_2 then $\Delta(\overline{G}) = 2$ and so, by Theorem 4.7, each component is P_2 or P_4. This leaves just $2P_4 \cup \frac{n-8}{2}P_2$ which is a pendant graph and thus (N,ℓ)-AW if and only if $\gcd(n-5,\ell) = 1$ by Lemma 3.12.

Suppose $\overline{G}$ has degree sequence d_3, d_4 or d_5. In these cases $\overline{G}$ has an isolated vertex so by Corollary 3.9, any component with non-pendant vertices has no pendant vertices. Degree sequence d_3 is impossible because there are not enough 2^+ vertices to be in a component with a 3-vertex. If $\overline{G}$ has degree sequence d_4, this implies one of the components must have odd degree sum which is impossible. If $\overline{G}$ has degree sequence d_5 then $\Delta(\overline{G}) = 2$. By Theorem 4.7 if G is (n,ℓ)-extremal then each component of $\overline{G}$ is either P_2 or P_4. Since the number of 2-vertices is odd no such graph exists.

Thus if $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+2\right)$ then $\gcd(n-5,\ell) = 1$. Moreover the unique (n,ℓ)-extremal graphs are $\overline{(P_3 \odot K_1) \cup \frac{n-6}{2}P_2}$ and $\overline{2P_4 \cup \frac{n-8}{2}P_2}$ which are the complements of the only pendant graphs of order n and size $\frac{n}{2}+2$. $\square$

In the next proposition, we resolve the case $k = 3$ of Theorem 4.9. For the proof we again consider possible degree sequences of the complements of (n, ℓ)-extremal graphs.

Proposition 4.15. Let $n, \ell \in \mathbb{N}$ be even and $n \geq 8$ and $\ell \in \mathbb{N}$. Then

$$\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + 3\right) \text{ if and only if } \gcd(\ell, n - 2k - 1) \neq 1 \text{ for } 0 \leq k \leq 2, \text{ and } \gcd(\ell, n - 7) = 1.$$

In this case the unique (n, ℓ)-extremal examples are exactly those graphs whose complements are pendant graphs with $\frac{n}{2} + 3$ edges: $(C_3 \odot K_1) \cup \frac{n-6}{2} P_2$, $(P_4 \odot K_1) \cup \frac{n-8}{2} P_2$, $(K_{1,3} \odot K_1) \cup \frac{n-8}{2} P_2$, $(P_3 \odot K_1) \cup P_4 \cup \frac{n-10}{2} P_2$, and $3P_4 \cup \frac{n-12}{2} P_2$.

Proof. Suppose $\gcd(n-2k-1, \ell) \neq 1$ for $0 \leq k \leq 2$, and $\gcd(n-7, \ell) = 1$. By Propositions 4.3, 4.11 and 4.14 we know $\max(n, \ell) \leq \binom{n}{2} - \left(\frac{n}{2} + 3\right)$. Consider $G = (P_4 \odot K_1) \cup \frac{n-8}{2} P_2$ which has $\frac{n}{2} + 3$ edges. Since $\gcd(n - 2(3) - 1, \ell) = 1$, G is (N, ℓ)-AW by Lemma 3.12. So $\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + 3\right)$.

Suppose that $\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + 3\right)$. Then $\gcd(\ell, 2) \neq 1$ and $\gcd(n-2k-1, \ell) \neq 1$ for $0 \leq k \leq 2$ by Propositions 4.3, 4.11 and 4.14. We describe all G such that G is (N, ℓ)-AW and $E(\overline{G}) = \frac{n}{2} + 3$ and show that these graphs are either not (n, ℓ)-extremal or are (N, ℓ)-AW if and only if $\gcd(n - 7, \ell) = 1$.

Suppose that G is (N, ℓ)-AW with $E(\overline{G}) = \frac{n}{2} + 3$. The degree sum of $\overline{G}$ is $n + 6$. By Lemma 4.10, $\Delta(\overline{G}) \leq 4$. To avoid N-twins in G, $\overline{G}$ can have at most one 0-vertex.

We first consider the case when $\overline{G}$ has a 0-vertex. By Corollary 3.9 we know that $\overline{G}$ does not have a pendant vertex that is not part of a P_2 component. Thus the degree sequence of $\overline{G}$ has an even number of 1-vertices. Since the total number of vertices is even and we have a 0-vertex, we know the number of 2^+-vertices will be odd. By considering all integer partitions of 7 that when added to $(1, 1, \ldots, 1, 0)$ will satisfy having an even number of 2^+ vertices and no 5^+-vertex we get the following possible degree sequences:

- $d_0 = (4, 4, 2, 1, 1, \ldots, 1, 0)$
- $d_1 = (4, 3, 3, 1, 1, \ldots, 1, 0)$
- $d_2 = (4, 2, 2, 2, 2, 1, \ldots, 1, 0)$
- $d_3 = (3, 3, 2, 2, 2, 1, \ldots, 1, 0)$
- $d_4 = (2, 2, 2, 2, 2, 2, 2, 1, \ldots, 1, 0)$

By Lemma 4.13 degree sequences d_0 and d_1 can not have a realization that is (N, ℓ)-AW for any ℓ. In the case of d_2, by Corollary 3.9 and Lemma 4.13 all the 2^+-vertices form a component with no 1-vertices. The only realization of $(4, 2, 2, 2, 2)$ is the bowtie graph shown in Figure 1. Let H_{d_2} be the bowtie graph along with the required number of P_2 components and an isolated vertex. By Theorem 2.5, $\overline{H_{d_2}}$ is (N, ℓ)-AW if and only if $H_{d_2} - P_1$ is (A, ℓ)-AW. By row reducing the adjacency matrix, we find that the bowtie graph (and thus $H_{d_2} - P_1$) is (A, ℓ)-AW if and only if ℓ is odd. Thus, $\overline{H_{d_2}}$ is (N, ℓ)-AW

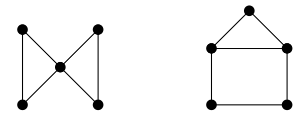

Figure 1: The bowtie graph with degree sequence $(4, 2, 2, 2, 2)$ on the left, and the house graph with degree sequence $(3, 3, 2, 2, 2)$ on the right. Both graphs appear in the proof of Proposition 4.15.

if and only if ℓ is odd, and so the graph corresponding to d_2 is not (n, ℓ)-extremal by Proposition 4.8.

Again, by Corollary 3.9 and Lemma 4.13 we find that for the case of d_3, the 2^+-vertices form a component with no 1-vertices. Considering the cases in which the two 3-vertices are adjacent and when they are not, we have that the only realizations of $(3, 3, 2, 2, 2)$ are the house graph (shown in Figure 1) and $K_{2,3}$. Let H_{d_3} be the house graph along with the required number of P_2 components and an isolated vertex. As in the previous paragraph, we use Theorem 2.5 to determine the N-winnability of $\overline{H_{d_3}}$ by row reducing the adjacency matrix of the house graph, and we find $\overline{H_{d_3}}$ is never (N, ℓ)-AW. Note that in $K_{2,3}$, the two 3-vertices are A-twins and thus, by Theorem 2.5, the complement of this realization is never (N, ℓ)-AW.

In the case of d_4 we note that $\Delta(\overline{G}) = 2$. By Theorem 4.7 if G is (n, ℓ)-extremal then each component of $\overline{G}$ is either P_2 or P_4. Since a realization of d_4 has a P_1 component, there is no such (n, ℓ)-extremal graph.

Therefore, given the hypotheses, there are no (n, ℓ)-extremal graphs with $\binom{n}{2} - \left(\frac{n}{2} + 3\right)$ edges and a dominating vertex.

Now suppose there is no 0-vertex in $\overline{G}$. To get a degree sum of $n + 6$ we need to add integer partitions of 6 to $(1, 1, \ldots, 1)$. Considering all integer partitions of 6 that have parts of size at most 3 and adding these to $(1, 1, \ldots, 1)$ we get the following possible degree sequences:

- $d_5 = (4, 4, 1, 1, \ldots, 1)$
- $d_6 = (4, 3, 2, 1, \ldots, 1)$
- $d_7 = (4, 2, 2, 2, 1, \ldots, 1)$
- $d_8 = (3, 3, 3, 1, \ldots, 1)$
- $d_9 = (3, 3, 2, 2, 1, \ldots, 1)$
- $d_{10} = (3, 2, 2, 2, 2, 1, \ldots, 1)$
- $d_{11} = (2, 2, 2, 2, 2, 2, 1, \ldots, 1)$

We eliminate d_5 and d_6 using Lemma 4.13. In the case of d_7 all of the 2-vertices must be adjacent to the 4-vertex. Considering the possible adjacencies among the 2-vertices

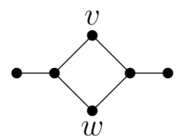

Figure 2: The graph G_{d_9} - the non-matching component of the only graph with degree sequence d_9 from Proposition 4.15 for which the two 3-vertices are not adjacent.

the possible graphs are $H_{d_7} = K_{1,3}\odot K_1 \cup \frac{n-4}{2}P_2$ and the graph $G_2 \cup \frac{n-6}{2}P_2$ where G_2 is given in Appendix A. Since H_{d_7} is a pendant graph we know $\overline{H_{d_7}}$ is (N,ℓ)-AW if and only if $\gcd(n-7,\ell) = 1$ by Lemma 3.12. By Lemma 2.2 and the fact that the adjacency matrix of G_2 is invertible we know G_2 is (A,ℓ)-AW. From Appendix A G_2 has order 6, size 6, and $T^A_{G_2}(1) = -2$. However, $P_3\odot K_1$ is (A,ℓ)-AW by Lemma 3.10, has order 6, size 5, and has $T^A_{V(P_3\odot K_1)} = -2$ by Corollary 3.11(2). Thus by Corollary 3.15, $\overline{G_2 \cup \frac{n-6}{2}P_2}$ is not (n,ℓ)-extremal.

Consider degree sequence d_8. By Lemma 4.13 all 3-vertices must be adjacent to each other. Thus the only possible graph is $H_{d_8} = (C_3\odot K_1) \cup \frac{n-6}{2}P_2$. Since H_{d_8} is a pendant graph we know $\overline{H_{d_8}}$ is (N,ℓ)-AW if and only if $\gcd(n-7,\ell) = 1$ by Lemma 3.12.

For degree sequence d_9 again by Lemma 4.13 all 2^+-vertices must be in the same component. We generate all possible graphs with degree sequence d_9 by considering whether or not the two 3-vertices are adjacent. If the two 3-vertices are not adjacent (in the complement graph) then they each must be adjacent to both of the degree 2 vertices, resulting in G_{d_9} which is given Figure 2. Since $\overline{G_{d_9} \cup \frac{n-6}{2}P_2}$ has N-twins (v and w in Figure 2) this graph is not (N,ℓ)-AW for any ℓ by Corollary 2.3.

Suppose the two 3-vertices are adjacent. We consider cases based on their number of common neighbors. If there are no common neighbors then, to avoid twins, we get $(P_4\odot K_1) \cup \frac{n-8}{2}P_2$ or $G_3 \cup \frac{n-6}{2}P_2$ where G_3 is given in Appendix A. For the former, the complement is (N,ℓ)-AW if and only if $\gcd(n-7,\ell) = 1$ by Lemma 3.12. For the latter we apply Corollary 3.15. By Lemma 2.2, $G_3 \cup \frac{n-6}{2}P_2$ is (A,ℓ)-AW. By Appendix A graph G_3 has order 6, size 6, and $T^A_{G_3}(1) = -2$. However, $P_3\odot K_1$ is (A,ℓ)-AW by Lemma 3.10, has order 6, size 5, and has $T^A_{V(P_3\odot K_1)} = -2$ by Corollary 3.11(2). Thus by Corollary 3.15, $\overline{G_3 \cup \frac{n-6}{2}P_2}$ is not (n,ℓ)-extremal.

Now suppose the two degree 3 vertices have one neighbor in common. Then we get the graph G_4 in Appendix A. We see G_4 has order 6, size 6, and has $T^A_{V(G_4)}(1) = \{-4\}$. Also G_4 is (A,ℓ)-AW by Lemma 2.2. However, by Corollary 3.11, $(P_2\cup P_1)\odot K_1 = P_4 \cup P_2$ has $T^A_{V(P_2\cup P_1)\odot K_1)}(1) = \{-4\}$. So by Corollary 3.15 G_4 is not (n,ℓ)-extremal. Finally, suppose the two degree 3 vertices have two neighbors in common. This results in G'_{d_9} given in Figure 3. Since $\overline{G'_{d_9} \cup \frac{n-4}{2}}$ has N-twins (v and w), it is not (N,ℓ)-AW for any ℓ by Corollary 2.3.

Next consider degree sequence d_{10}. In this case it is not necessarily true that all 2^+-vertices need to be in the same component. However, if the 2^+ vertices form more than

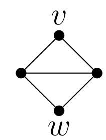

Figure 3: The graph G'_{d_9} - the non-matching component of only graph with degree sequence d_9 in which the two 3-vertices are adjacent and have two neighbors in common in the proof of Proposition 4.15.

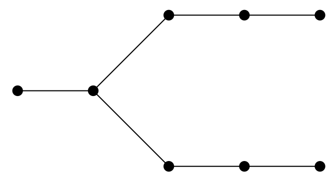

Figure 4: The second possibility for a high degree component for degree sequence d_{10} in which the degree 3 vertex is adjacent to two degree 2 vertices and one degree 1 component.

one component, it would have to be the case that one component had a 3-vertex and two 2-vertices by Lemma 4.13. In this case, the arguments in Proposition 4.14 for degree sequence d_1 apply.

Now suppose all the 2^+-vertices are in the same component. We consider cases based on the degrees of the vertices adjacent to the 3-vertex. This could either be two 2-vertices and one 1-vertex or three 2-vertices.

Suppose there are two 2-vertices and one 1-vertex adjacent to the 3-vertex. The two 2-vertices each have an additional neighbor (not the 3-vertex and not each other since then the 3-vertex would be forced to have three neighbors of degree 2). Call these additional neighbors v and w. If one of these vertices is degree 1 we end up with the graph G_5 in Appendix A. Using the standard method in Corollary 3.11 and that $P_4 \cup P_4$ has $T^A_{V(P_4 \cup P_4)} = -4$ we find $G_5 \cup \frac{n-8}{2}P_2$ is not (n, ℓ)-extremal by Corollary 3.15 . If v and w each have degree 2 we consider the possibility that they are adjacent to each other and if they are not. This yields the graphs in Figure 4 and graph G_6 in Appendix A. The graph in Figure 4 has a labeling that is not (A, ℓ, s)-winnable for all s, namely the labeling where one of the pendant vertices adjacent to a 2-vertex has label 1 and the remaining vertices have label 0. This would make G not (N, ℓ)-AW by Theorem 3.7(2). Since $P_3 \odot K_1$ has $T^A_{V(P_3 \odot K_1)} = -2$ by Corollary 3.11, $G_6 \cup \frac{n-6}{2}P_2$ is not (n, ℓ)-extremal by Corollary 3.15.

Now suppose there are three 2-vertices adjacent to the 3-vertex. Considering whether two of the 2-vertices are adjacent to each other or not we get graphs G_7 and G_8 in Appendix A. Since $T^A_{V(P_4 \cup 2P_2)}(1) = -6$ and $T^A_{V(P_4 \cup P_4)}(1) = -4$ by Corollary 3.11, we know neither $G_7 \cup \frac{n-6}{2}P_2$ nor $G_8 \cup \frac{n-8}{2}P_2$ is (n, ℓ)-extremal by Corollary 3.15.

If $\overline{G}$ has degree sequence d_{11} then $\Delta(\overline{G}) = 2$. By Theorem 4.7 if G is (n, ℓ)-extremal then each component of $\overline{G}$ is either P_2 or P_4. Thus, the graph must be $3P_4 \cup \frac{n-12}{2}P_2$, which is a pendant graph. This is (N, ℓ)-AW if and only if $\gcd(n-7, \ell) = 1$ by Lemma

3.12.

Therefore every possible graph that is (N, ℓ)-AW and has $E(\overline{G}) = \frac{n}{2} + 3$ is either not (n, ℓ)-extremal or has $\gcd(n-7, \ell) = 1$, as desired. □

Proof of Theorem 4.9. The cases $0 \le k \le 3$ are true by Propositions 4.3, 4.11, 4.14, 4.15, respectively. □

5 Open Problems

We close with three open problems related to our results.

(1) *Does Theorem 4.9 hold for $k \ge 4$?* We made much progress on this result by considering the possible degree sequences. However, when $k = 4$, there are 37 partitions of 7 and 8. Even with the additional restriction of Lemma 4.10 there are 23 different degree sequences to consider. Thus, we need an alternative method to solve the general problem.

(2) *What are the graphs of maximum size that are (N, ℓ)-AW for all ℓ?* The best candidates we have found are complements of pendant trees, which have size $\binom{n}{2} - (n-1)$. They are all (N, ℓ)-AW for all ℓ, but it is not clear that they are (n, ℓ)-extremal.

(3) *What are the (n, ℓ)-extremal graphs for other Lights Out games, such as the adjacency game?*

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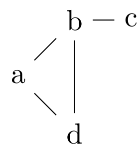
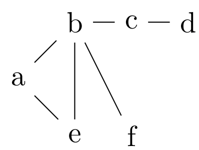
A Replacement Graphs

The following graphs are the connected components we replace when we apply Corollary 3.15. For each graph G, we show a picture of the graph; a table showing how many times each vertex is toggled to win the (A, ℓ)-Lights Out game with initial labeling $\mathbf{0}_1$; and $T^A_{V(G)}(1)$, which is obtained by adding the toggles from the table. Note that in each case $A(G)$ is invertible. Thus, there is precisely one way to win this Lights Out game, which is why there is only one toggling number for each graph.

$G_1 =$ 

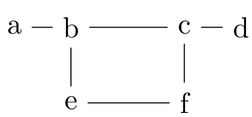
vertex	Number of Toggles
a	0
b	-1
c	-1
d	0

$T^A_{V(G_1)}(1) = -2$

$G_2 =$ 

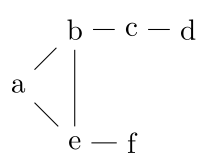
vertex	Number of Toggles
a	0
b	-1
c	-1
d	0
e	0
f	0

$T^A_{V(G_2)}(1) = -2$

$G_3 =$ 

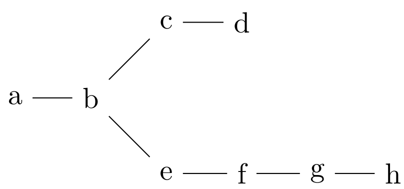
vertex	Number of Toggles
a	0
b	-1
c	-1
d	0
e	0
f	0

$T^A_{V(G_3)}(1) = -2$

$G_4 =$ 

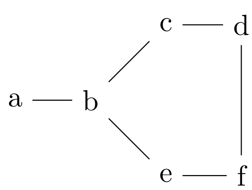
vertex	Number of Toggles
a	1
b	0
c	-1
d	-1
e	-1
f	-2

$T^A_{V(G_4)}(1) = -4$

$G_5 =$ 

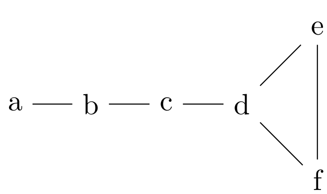
vertex	Number of Toggles
a	0
b	-1
c	-1
d	0
e	0
f	0
g	-1
h	-1

$T^A_{V(G_5)}(1) = -4$

$G_6 =$ 

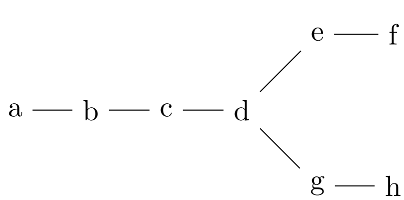
vertex	Number of Toggles
a	1
b	-1
c	-1
d	0
e	-1
f	0

$T^A_{V(G_6)}(1) = -2$

$G_7 =$ 

vertex	Number of Toggles
a	-2
b	-1
c	1
d	0
e	-1
f	-1

$T^A_{V(G_7)}(1) = -4$

$G_8 =$ 

vertex	Number of Toggles
a	-2
b	-1
c	1
d	0
e	-1
f	-1
g	-1
h	-1

$T^A_{V(G_8)}(1) = -6$