A Group Labeling Version of the Lights Out Game

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Abstract

The Lights Out game, originally an electronic game played on a 5×5 grid, is a solitaire game that can be played on any simple graph. It has been generalized to be played with multiple on-states, where each vertex state is represented by a label in $\mathbb{Z}_n$. In this paper, we define a new variant of the Lights Out game, where the vertex labels can come from any group H. This new game depends deeply on the group structure of H and gives us a significantly different game for cyclic groups. We investigate problems related to counting winnable labelings in the case $H = \mathbb{Z}_n$.

1 Introduction

A light-switching game is any game that has a collection of lights and a collection of switches such that flipping any switch changes the "states" of some subset of lights. Usually the possible states for the lights are "on" and "off". The game is usually considered to be won when all the lights are in the "off" state, or more generally, when the number of lights that are on is minimum. Examples of these games are the Berlekamp (or Gale-Berlekamp) light-switching game (see [BM00], [CS04], and [Sch11]), Merlin's Magic Square (see [Pel87] and [Sto89]), and others ([Ara00] and [CMP09]).

In this paper, we study a variation of a light-switching game called Lights Out. The original Lights Out game was an electronic handheld game by Tiger Electronics with a 5×5 grid of lighted buttons that begins with some lights on and some lights off. Here, the buttons play the role of both "switches" and "lights". Each time a button is toggled, the state of that button and the buttons adjacent to it above, below, and to either side, change from "on" to "off" or vice-versa. We can apply the notion of adjacency to any simple graph, and so the game generalizes easily to simple graphs. We can also think of

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the lighting pattern as a vertex labeling, where an "on" vertex has label 1 and an "off" vertex has a label 0. We can think of 0 and 1 as elements of $\mathbb{Z}_2$, which helps us interpret winning the game as solving a system of linear equations over the field $\mathbb{Z}_2$ (see [AF98] and [GKT97]). This enables us to use linear algebra to help us win Lights Out games when winning is possible.

It is possible that a particular Lights Out game may be impossible to win. Questions of when a game can be won have been central in the study of these games. These questions include which graphs and labeling sets make it possible to win all possible games (see [AS96], [EEJ$^+$10], [GP13], and [Par18]).

Sutner introduced the Lights Out game on graphs in [Sut89] and [Sut90] as an application of cellular automata to solve the "All-Ones Problem" (i.e. showing that the Lights Out game can be won on any graph when every light is on). Lights Out has also been interpreted in the context of Domination Theory (see [AS92], [AS96], [ACS98], and [GKW02]). Such interpretations were also approached from a linear algebraic perspective, as in [ASZ02].

Lights Out was generalized in [GP13] so that each vertex could have more than one "on" state. Each vertex state is represented by an element of $\mathbb{Z}_n$, where n is the number of states (including the "off" state, which is represented by 0). Collectively, these vertex states form a labeling of the vertices. In this game, each time a vertex is toggled, the label of the toggled vertex and each of its neighbors increases by one. As in the two-state Lights Out game, there have been a variety of ways the game is interpreted, including linear algebra ([GP13] and [EEJ$^+$10]), cellular automata ([AMW14], [AM14], and [AO16]), and Domination Theory ([GP13]). The linear algebra of this game is based on the *neighborhood matrix* of the graph G (denoted $N(G)$), which is the adjacency matrix with 1's down the diagonal. Thus, we call this game the $N(G)$-Lights Out game.

In this paper, we consider a variant of the Lights Out game that can be played with vertex labels from any group. Let G be a graph and let H be a group with binary operation $*$. We then define the *H-labeling Lights Out game* as follows. Whenever we have a vertex labeling $\pi : V(G) \to H$, if a vertex $v \in V(G)$ is toggled, this changes the labeling to π', where for each $w \in V(G)$, we have

$$\pi'(w) = \begin{cases} \pi(v) * \pi(w), & w = v \text{ or } vw \in E(G) \\ \pi(w), & \text{otherwise.} \end{cases}$$

The game is won when we achieve the *identity labeling* $\pi_0 : V(G) \to H$, where $\pi_0(v)$ is the identity element of H for all $v \in V(G)$. A labeling is *winnable* if vertices can be toggled to achieve the identity labeling. It is easy to see that a labeling is winnable on a simple graph G if and only if it is winnable on each of the connected components of the graph. As a result, all graphs in this paper will be simple and connected. In the case that H is commutative, we let the binary operation be $+$, and we call the identity labeling the *zero labeling*. In this paper, we focus on the game where $H = \mathbb{Z}_n$ for some $n \geq 2$.

Our main focus in this paper is proving theorems related to counting winnable labelings. In Section 2, we prove Theorem 2.4, which reduces the problem of counting winning labelings on G to the case that $H = \mathbb{Z}_{2^k}$ for some $k \geq 1$.

In Section 3, we put a natural digraph structure on the set of all vertex labelings of a graph over $\mathbb{Z}_{2^k}$. There is also a natural group structure on the set of labelings. We prove that the labelings in the weakly connected component containing the zero labeling (called the *zero component*) form a subgroup of the group of all labelings and that the cosets of this subgroup are precisely the sets of labelings that induce weakly connected components of the digraph. This implies that every weakly connected component has the same cardinality and the the order of the zero component divides evenly into the total number of labelings.

Finally, in Section 4, we prove that all labelings in the zero component are winnable in the case $H = \mathbb{Z}_2$ (Theorem 4.5).

2 Counting Winnable Labelings

The $N(G)$-Lights Out game has two important properties that make certain aspects of the game convenient. First, the result of toggling vertices depends only on how many times each vertex is toggled, not the order in which they are toggled. Second, if one starts at a given labeling and toggles some vertices, it is always possible to toggle the vertices in such a way as to restore the graph to the original labeling. A consequence of this is that it is never possible to begin with a winnable labeling, make some unwise toggles, and end up with a labeling that is not winnable.

Neither of the above properties holds in the $\mathbb{Z}_n$-labeling Lights Out game, although as we see in Section 4, it is an open question as to whether it is possible to start with a winnable labeling and toggle the graph to a non-winnable labeling. Not having commutativity in toggling, and facing the possibility that one can toggle from one labeling to another but not being able to toggle back to the original labeling suggests that winning the $\mathbb{Z}_n$-labeling Lights Out game will be more difficult than winning the $N(G)$-Lights Out game. This will be borne out in Theorem 2.4.

Recall that the elements of $\mathbb{Z}_n$ for $n \in \mathbb{N}$ are congruence classes modulo n. Thus, if $S \in \mathbb{Z}_n$, we can also think of $S \subseteq \mathbb{Z}$. For $S \in \mathbb{Z}_n$ and $d \in \mathbb{N}$, we define $dS = \{ds : s \in S\}$. The following is straightforward to prove.

Lemma 2.1. Let $d, n \in \mathbb{N}$, and let $S, T \in \mathbb{Z}_n$.

1. If $d \mid n$ and $s \in S$, then $d \mid s$ if and only if $d \mid s'$ for all $s' \in S$.

2. We have $dS \in \mathbb{Z}_{dn}$. Specifically, if $s \in S$, then dS is the congruence class in $\mathbb{Z}_{dn}$ that contains ds.

3. We have $d(S + T) = dS + dT$.

In the spirit of Lemma 2.1(1), if $d \mid n$, we say that $d \mid S$ when $d \mid s$ for all $s \in S$ and $d \nmid S$ if $d \nmid s$ for all $s \in S$. We use this to establish relationships between odd integer factors of n and elements of $\mathbb{Z}_n$, as in the following lemma.

Lemma 2.2. Let $n = 2^k d$, where $d, k \in \mathbb{N}$ with d odd. Let G be a graph with labeling $\pi : V(G) \to \mathbb{Z}_n$. If $v \in V(G)$ and $d \nmid \pi(v)$, then toggling any vertex in G will result in at least one vertex having a label that is not a multiple of d.

Proof. The vertex we toggle can be v, a vertex adjacent to v, or a vertex not adjacent to v. We show in each case that at least one vertex ends up with at label that is not a multiple of d. If v is toggled, v ends up with the label $2\pi(v)$. Since $d \nmid \pi(v)$ and d is odd, it follows that $d \nmid 2\pi(v)$. Thus, v has a label that is not a multiple of d.

Suppose we toggle a vertex w that is adjacent to v. After w is toggled, w has the label $2\pi(w)$ and v has the label $\pi(w) + \pi(v)$. If $d \nmid 2\pi(w)$, then w has a label that is not a multiple of d. If $d \mid 2\pi(w)$, then d odd implies $d \mid \pi(w)$. But then $d \nmid \pi(v)$ implies $d \nmid \pi(w) + \pi(v)$, and so v has a label that is not a multiple of d.

If the toggled vertex is not adjacent to v, then the label for v is unchanged, and is thus not a multiple of d. $\square$

The next lemma helps us show connections between the Group Labeling Lights Out game over $\mathbb{Z}_m$ and $\mathbb{Z}_n$, where $m \mid n$.

Lemma 2.3. Let $n = dm$, where $d, m \in \mathbb{N}$, and let G be a graph. Let $\pi : V(G) \to \mathbb{Z}_m$ be a labeling, and let $\pi' : V(G) \to \mathbb{Z}_m$ be the labeling that results from toggling $w \in V(G)$. Define $\lambda : V(G) \to \mathbb{Z}_n$ by $\lambda(v) = d\pi(v)$. Then

1. λ is well defined, and toggling w with the labeling λ results in the labeling $\lambda' : V(G) \to \mathbb{Z}_n$, where $\lambda' = d\pi'$.

2. π is winnable if and only if λ is winnable.

Note that Lemma 2.1(2) implies that λ has an appropriate codomain.

Proof. For (1), to prove λ is well defined, we need to show that each $\lambda(v)$ does not depend on the representative of $\pi(v)$. This follows from the fact that if $a \equiv b \pmod{m}$, then $ad \equiv bd \pmod{dm}$. To prove that $\lambda' = d\pi'$, consider $v \in V(G)$. If v is not equal to or adjacent to w, then $\pi'(v) = \pi(v)$ and $\lambda'(v) = \lambda(v)$. We then have

$$\lambda'(v) = \lambda(v) = d\pi(v) = d\pi'(v).$$

If v is equal to or adjacent to w, we have $\pi'(v) = \pi(w) + \pi(v)$ and $\lambda'(v) = \lambda(w) + \lambda(v)$. We then have

$$\lambda'(v) = \lambda(w) + \lambda(v) = d\pi(w) + d\pi(v) = d(\pi(v) + \pi(w)) = d\pi'(v).$$

Note that the second to last equality follows from Lemma 2.1(3). In any case, we have $\lambda'(v) = d\pi'(v)$, and so $\lambda' = d\pi'$.

For (2), an easy induction argument yields that for any sequence of toggles that results in the labelings $\pi' : V(G) \to \mathbb{Z}_m$ and $\lambda' : V(G) \to \mathbb{Z}_n$, we have $\lambda' = d\pi'$. So if π is winnable, there exists a sequence of toggles over $\mathbb{Z}_m$ that results in π', where π' is the zero labeling. It follows that $\lambda' = d\pi' = d \cdot 0 = 0$, and so λ is winnable over $\mathbb{Z}_n$. For the converse, we assume that λ is winnable. Then there exists a sequence of toggles over $n = dm$ such that $\lambda' = 0$. Thus, for all $v \in V(G)$, $d\pi'(v) \equiv 0 \pmod{dm}$. Since $d \neq 0$, $\pi'(v) \equiv 0 \pmod{m}$ for all $v \in V(G)$. It follows that π is winnable over $\mathbb{Z}_m$. $\square$

This leads to the main theorem of the section, which reduces the counting of winnable labelings to the case $n = 2^k$.

Theorem 2.4. For the Group Labeling Lights Out Game on the graph G with labels in $\mathbb{Z}_n$,

1. If n is odd, then the only winnable labeling is the zero labeling.
2. If $n = 2^k d$, where d is odd, and if $\pi : V(G) \to \mathbb{Z}_n$ is a winnable labeling, then for all $v \in V(G)$, $d \mid \pi(v)$.
3. If $n = 2^k d$, where d is odd, then there is a one-to-one correspondence between winnable labelings of G with labels in $\mathbb{Z}_n$ and winnable labelings of G with labels in $\mathbb{Z}_{2^k}$.

Proof. For (1), suppose there is a nonzero winnable labeling of G. Suppose we toggle vertices from this labeling to achieve the zero labeling, and let $\pi : V(G) \to \mathbb{Z}_n$ be the labeling that occurs just before the last toggle that wins the game. Clearly, π is not the zero labeling. Thus, if v is the last vertex that is toggled, then $\pi(v) \neq 0$. Since toggling v wins the game, $2\pi(v) = 0$. But n is odd, which makes 2 a unit in $\mathbb{Z}_n$. Thus, $\pi(v) = 0$, which is a contradiction.

For (2), suppose $d \nmid \pi(v)$ for some $v \in V(G)$. Since $d \mid 0$, d divides every vertex label of the zero labeling. Thus, there is some point in the game where we have a labeling in which not every vertex label is a multiple of d, but after the next vertex is toggled, every vertex label is a multiple of d. This contradicts Lemma 2.2, and so the vertex labels in every winnable labeling are divisible by d.

For (3), let $\mathcal{W}_{2^k}$ and $\mathcal{W}_n$ be the set of winnable labelings of G over $\mathbb{Z}_{2^k}$ and $\mathbb{Z}_n$, respectively, and define the function $\phi : \mathcal{W}_{2^k} \to \mathcal{W}_n$ by $\phi(\pi) = d\pi$. By Lemma 2.3(2), since π is winnable, so is $\phi(\pi)$. It suffices to prove that ϕ is bijective. To prove it is injective, suppose $\phi(\pi_1) = \phi(\pi_2)$. We then have $d\pi_1 = d\pi_2$ in $\mathbb{Z}_n$. Thus, for all $v \in V(G)$, $d\pi_1(v) \equiv d\pi_2(v) \pmod{2^k d}$. We get $\pi_1(v) \equiv \pi_2(v) \pmod{2^k}$, and so $\pi_1 = \pi_2$.

To prove ϕ is surjective, let $\rho \in \mathcal{W}_n$. By part (2), $d \mid \rho(v)$ for all $v \in V(G)$. Thus, for each $z \in \rho(v)$, we have $z = dy_z$ for some $y_z \in \mathbb{Z}$. Note that $dy \equiv dy_z \pmod{2^k d}$ if and only if $y \equiv y_z \pmod{2^k}$, and so all y_z for $z \in \rho(v)$ are in the same congruence class modulo 2^k (call it s_v). So if we define $\pi(v) = s_v$ for all $v \in V(G)$, we have $\pi : V(G) \to \mathbb{Z}_{2^k}$. Also note that this definition of π gives us $\rho = d\pi$, since $\rho(v) = ds_v = d\pi(v)$ for all $v \in V(G)$. Since ρ is winnable, Lemma 2.3(2) implies that π is winnable also. Since $\phi(\pi) = d\pi = \rho$, this proves that ϕ is surjective, which completes the proof. $\square$

In the spirit of the above theorem, we assume for the rest of the paper that we are playing the $\mathbb{Z}_{2^k}$-labeling Lights Out game for some $k \in \mathbb{N}$.

3 The Labeling Digraph for the Group Labeling Lights Out Game

Now that we have reduced the problem of counting winnable labelings to the case where our labeling set is $\mathbb{Z}_{2^k}$, we look a little closer at the relationship between the labelings with respect to toggling vertices. We do this with a digraph whose vertices are the labelings and whose arcs are the toggles that transform one labeling into another.

Definition 3.1. Suppose that we are playing a light-switching game on a set S over a labeling set U. The *labeling digraph* for this game is a digraph $D = (\mathcal{L}, \mathcal{T})$. The vertex set $\mathcal{L}$ is the set of all labelings of S with labels in U. The arc set $\mathcal{T}$ consists of all $L_1 \to L_2$ with $L_1, L_2 \in \mathcal{L}$, $L_1 \neq L_2$, where one can transform the labeling L_1 to the labeling L_2 by toggling a single switch once. We may also consider each arc to be colored by the switch that is toggled to transform one labeling to another.

Note that while toggling a vertex with label 0 causes no change in the labeling, we ignore the loops in the labeling digraph such toggles would cause.

It turns out that the labeling digraph D can tell us much about winnable labelings in the $\mathbb{Z}_{2^k}$-labeling Lights Out game. For instance, if π is a winnable labeling, then the toggles used to get from π to the zero labeling π_0 is represented as a directed path in D from π to π_0. In particular, all winnable labelings must be a part of the weakly connected component of D that contains π_0. We call this weakly connected component the *zero component* of D. Two natural questions arise.

Question 3.2.

1. What is the order of the zero component of the labeling digraph for a $\mathbb{Z}_{2^k}$-labeling Lights Out game?
2. Are there any labelings in the zero component that are not winnable?

If we can answer the first question, and if we can verify that the answer to the second question is "no", then the number of winnable labelings is the order of the zero component, and we can use that to determine the number of winnable labelings..

In this section, we take a small step toward answering the first question. Specifically, we prove that all weakly connected components of the labeling digraph for the $\mathbb{Z}_{2^k}$-labeling Lights Out game have the same order, which implies that the order of the zero component divides the number of possible labelings.

We actually take this a step further. We can put a group structure on the set of labelings. Given labelings π and ρ, we define the labeling $\pi+\rho$ to be given by $(\pi+\rho)(v) = \pi(v) + \rho(v)$. Under this operation, the zero labeling is the identity element. In the course of determining the bijection between the vertex sets (i.e. sets of labelings) of the weakly connected components, we show that the labelings in the zero component form a subgroup of the group of all labelings and that the sets of labelings that induce the weakly connected components of D are the cosets of this subgroup.

Before we get to the main results, we consider what happens when we repeatedly toggle a single vertex v.

Lemma 3.3. For the $\mathbb{Z}_{2^k}$-labeling Lights Out game on a graph G, let $v \in V(G)$. Let π be a vertex labeling.

1. If v is repeatedly toggled so that its label is 0, the resulting labeling for every $w \in V(G)$ with w adjacent to v is $\pi(w) - \pi(v)$. Every other vertex label is unaffected.

2. If v is toggled k times, then the label of v becomes 0.

Proof. For (1), assume that we toggle v so that its label is 0. This toggling adds $-\pi(v)$ to the label of v. Each time v is toggled, the labels of each adjacent vertex w are increased in the same way as v, and so after k toggles, this adds $-\pi(v)$ to the label of w. The resulting label is $\pi(w) - \pi(v)$. Non-adjacent vertices are unaffected by all toggles of v, which completes the proof of (1).

For (2), each time v is toggled, its label doubles in value. Since $2^k = 0$ in $\mathbb{Z}_{2^k}$, after k toggles, the label of v becomes $2^k \cdot \pi(v) = 0$. □

This leads us to the main result of this section.

Theorem 3.4. Let G be a graph, and let $D = (\mathcal{L}, \mathfrak{T})$ be the Lights Out labeling digraph of all vertex labelings on G. If $\rho \in \mathcal{L}$ is a fixed labeling, define the function $\phi_\rho : \mathcal{L} \to \mathcal{L}$ by $\phi_\rho(\pi) = \pi + \rho$. Then ϕ_ρ defines a bijection on the weakly connected components of D.

Proof. We begin by proving that if C is a weakly connected component of D, then $\phi_\rho(V(C))$ induces a connected subgraph of D. By an easy induction argument, it suffices to show that if $\pi \to \lambda$, then $\phi_\rho(\pi)$ and $\phi_\rho(\lambda)$ are the end vertices of an undirected path in D. Note that since $\pi \to \lambda$, it must be true that we obtain λ from π by toggling some vertex v. Thus, for each $u \in V(G)$, we have

$$\lambda(u) = \begin{cases} \pi(u) + \pi(v), & u = v \text{ or } uv \in E(G) \\ \pi(u), & \text{otherwise.} \end{cases} \tag{1}$$

Now we look at $\phi_\rho(\pi) = \pi + \rho$ and $\phi_\rho(\lambda) = \lambda + \rho$. On each of these labelings, we toggle the vertex v k times to get the labelings π' and λ', respectively. By Lemma 3.3, we get

$$\begin{aligned} \pi'(u) &= \begin{cases} \phi_\rho(\pi)(u) - \phi_\rho(\pi)(v), & u = v \text{ or } uv \in E(G) \\ \phi_\rho(\pi)(u), & \text{otherwise} \end{cases} \\ &= \begin{cases} \pi(u) + \rho(u) - \pi(v) - \rho(v), & u = v \text{ or } uv \in E(G) \\ \pi(u) + \rho(u), & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \lambda'(u) &= \begin{cases} \phi_\rho(\lambda)(u) - \phi_\rho(\lambda)(v), & u = v \text{ or } uv \in E(G) \\ \phi_\rho(\lambda)(u), & \text{otherwise} \end{cases} \\ &= \begin{cases} \lambda(u) + \rho(u) - \lambda(v) - \rho(v), & u = v \text{ or } uv \in E(G) \\ \lambda(u) + \rho(u), & \text{otherwise.} \end{cases} \end{aligned}$$

We then use Equation (1) to get

$$\begin{aligned}\lambda'(u) &= \begin{cases}\pi(u)+\pi(v)+\rho(u)-\pi(v)-\pi(v)-\rho(v), & u=v \text{ or } uv\in E(G)\\ \pi(u)+\rho(u), & \text{otherwise}\end{cases}\\ &= \begin{cases}\pi(u)+\rho(u)-\pi(v)-\rho(v), & u=v \text{ or } uv\in E(G)\\ \pi(u)+\rho(u), & \text{otherwise.}\end{cases}\end{aligned}$$

It is clear from the formulas above that $\pi' = \lambda'$. Since we toggled v k times to get from $\phi_\rho(\pi)$ to π' and from $\phi_\rho(\lambda)$ to $\lambda' = \pi'$, these toggles form an undirected path from $\phi_\rho(\pi)$ to $\phi_\rho(\lambda)$.

Now let C' be the weakly connected component with $\phi_\rho(V(C)) \subseteq V(C')$. To show that $\phi_\rho : V(C) \to V(C')$ is a bijection, it suffices to prove that ϕ_ρ has an inverse $\phi_\rho^{-1} : V(C') \to V(C)$. Consider the function $\phi_{-\rho}$ with the domain restricted to $V(C')$. An easy computation shows that $\phi_{-\rho} \circ \phi_\rho$ and $\phi_\rho \circ \phi_{-\rho}$ are both identity maps. It suffices to show that $\phi_{-\rho}(V(C')) \subseteq V(C)$. From our work in the previous paragraph, $\phi_{-\rho}(V(C')) \subseteq V(C'')$, where C'' is a weakly connected component of the labeling digraph. Let $\pi \in V(C)$. By assumption $\pi + \rho = \phi_\rho(\pi) \in V(C')$. Thus, $\pi = \phi_{-\rho}(\pi+\rho) \in V(C'')$. Since the weakly connected components define a partition of $\mathcal{L}$ and $\pi \in V(C) \cap V(C'')$, it follows that $C = C''$, and so $\phi_{-\rho}(V(C')) \subseteq V(C)$. The theorem follows from this. □

We use this theorem to prove some results about the group structure of $\mathcal{L}$.

Corollary 3.5. Let $\mathcal{L}$ be the group of vertex labelings of the graph G over the labeling set $\mathbb{Z}_{2^k}$.

1. The set of labelings that induces the zero component of the $\mathbb{Z}_{2^k}$-labeling Lights Out game is a subgroup of $\mathcal{L}$.

2. The sets of labelings that induce the weakly connected components of the labeling digraph are precisely the cosets of the zero component.

Proof. Let Z be the zero component of $\mathcal{L}$. For (1), we use the subgroup test to show that $V(Z)$ is a subgroup of $\mathcal{L}$. We have $\pi_0 \in V(Z)$, so $V(Z)$ is nonempty. It suffices to show that for all $\pi, \lambda \in V(Z)$, we have $\pi - \lambda \in V(Z)$. Since $\pi - \lambda = \phi_{-\lambda}(\pi)$ and $\pi \in V(Z)$ it suffices to show that $\phi_{-\lambda}(V(Z)) = V(Z)$. By Theorem 3.4, there exists a weakly connected component C of the labeling digraph such that $\phi_{-\lambda}(V(Z)) = V(C)$. Note that $\lambda \in V(Z)$ and $\phi_{-\lambda}(\lambda) = 0 \in V(Z)$, and so $\lambda \in V(C) \cap V(Z)$. Since the weakly connected components define a partition of $\mathcal{L}$, this implies that $V(C) = V(Z)$. Thus, $\phi_{-\lambda}(V(Z)) = V(Z)$, and so $\pi - \lambda \in V(Z)$.

For (2), the cosets of $V(Z)$ are precisely the sets $\lambda + V(Z)$ with $\lambda \in \mathcal{L}$. Since $\lambda + V(Z) = \phi_\lambda(V(Z))$, then Theorem 3.4 implies that every coset is a weakly connected component of the labeling digraph. Conversely, let C be a weakly connected component of the labeling digraph, and let $\lambda \in V(C)$. Then $\lambda + V(Z) = \phi_\lambda(V(Z))$ induces a weakly connected component of the labeling digraph. Since $\lambda \in V(C) \cap (\lambda + V(Z))$, we have $V(C) \cap (\lambda + V(Z)) \neq \emptyset$, and so $V(C) = \phi + V(Z)$. Thus, every set of labelings that induces a weakly connected component of the labeling digraph is a coset of $V(Z)$, which completes the proof. □

Each coset of a group has the same cardinality, and the number of vertex labelings over $\mathbb{Z}_{2^k}$ is $2^{k|V(G)|}$. This and Corollary 3.5 gives us the following.

Corollary 3.6. *Let G be a graph, and let $k \in \mathbb{N}$. If Z is the zero component of the labeling digraph for the $\mathbb{Z}_{2^k}$-labeling Lights Out game, then $|V(Z)| \mid 2^{k|V(G)|}$.*

4 The Trap Problem

The results of the previous section give us some insight into the order of the zero component of the labeling digraph. However, our primary interest is winnable labelings. As mentioned before, all winnable labelings must be in the zero component. The question we address in this section is whether or not there are labelings in the zero component that are not winnable.

Any labeling connected to a winnable labeling by an arc must be in the zero component. Thus, one way to have a non-winnable labeling in the zero component is to have a winnable labeling that can be transformed into a non-winnable labeling by a single vertex toggle. We can think of this vertex (along with the winnable labeling) as a trap that is set to cause a player to go from a winnable situation to a situation in which it is impossible to win. This motivates the following definition.

Definition 4.1. Let G be a graph, $k \in \mathbb{N}$, and let $\pi : V(G) \to \mathbb{Z}_{2^k}$ be a winnable labeling in the $\mathbb{Z}_{2^k}$-labeling Lights Out game.

1. A vertex $v \in V(G)$ is π-*winnable* if toggling v once in the labeling π results in a winnable labeling.
2. A vertex $v \in V(G)$ is a π-*trap* if v is not π-winnable. We call v a *trap* if it is a π-trap for some winnable labeling π.

Note that any vertex with label 0 in a winnable labeling π is trivially π-winnable. As indicated before the definition, if G has a trap, then there is a non-winnable labeling in the zero component of the labeling digraph. It turns out that the converse of this is true as well.

Lemma 4.2. *Let G be a graph, and let $k \in \mathbb{N}$. The zero component of the $\mathbb{Z}_{2^k}$-labeling Lights Out game has a non-winnable labeling if and only if G has a trap.*

Proof. We have already shown that if G has a trap, then the zero component has a non-winnable labeling. It then suffices to show that if the zero component has a non-winnable labeling, then G has a trap. Let π be a non-winnable labeling in the zero component, and let π_0 be the zero labeling. Since the zero component is connected, there is an undirected path between π_0 and π. By an easy induction argument on the length of this path, we can assume that there exists some labeling λ in the zero component such that λ is winnable and there is an arc between λ and π. Suppose $\pi \to \lambda$. Since λ is winnable, this would make π winnable as well, a contradiction. Thus, $\lambda \to \pi$, and so the vertex v that is toggled to get from λ to π is a λ-trap. This makes v a trap, which completes the proof. $\square$

So the existence of non-winnable labelings in the zero component is equivalent to the existences of traps. Our conjecture is that traps do not exist for the $\mathbb{Z}_{2^k}$-labeling Lights Out game for all graphs. Another way of stating this is that for all $v \in V$ and for all winnable labelings $\pi \in \mathcal{L}$, v is π-winnable. For the remainder of this section, we make some progress in proving this result generally and proving the result completely for the case $k = 1$. The following lemma helps in this cause with vertices that are adjacent to vertices with a label of zero.

Lemma 4.3. Let G be a graph, and let $\pi : V(G) \to \mathbb{Z}_{2^k}$ be a vertex labeling of G. Let $v, w \in V(G)$, $\pi(w) = 0$, and $vw \in E(G)$. Let π' be the labeling obtained from π by toggling the vertex v once. Then one can toggle v and w to get from the labeling π' to the labeling π.

Proof. Our approach is to alternately toggle v until it has a label of 0 and toggle w until it has a label of 0. Since v has already been toggled once to get π', we need only toggle it $k-1$ more times to get a label of 0. By Lemma 3.3, this decreases the labels of v and each adjacent vertex by $\pi(v)$. In particular, w will have a label of $-\pi(v)$. When we then toggle w k times, Lemma 3.3 implies that w and every vertex adjacent to w have their labels increased by $\pi(v)$.

After each iteration of toggling v and then toggling w, we have that v, w, and every vertex adjacent to both v and w have their labels decreased by $\pi(v)$ and then increased by $\pi(v)$. The net result is that these vertices have the same label as they did under π. Any vertex adjacent neither to v nor w has its label unchanged as well since its label is unaffected by the toggling. That leaves vertices adjacent to v but not w (which have their labels decreased by $\pi(v)$) and vertices adjacent to w but not v (which have their labels increased by $\pi(v)$). If we iterate this alternating toggling of v and w 2^k times, we end up with the labels of vertices adjacent to v but not w decreased by $2^k\pi(v) = 0$ and the labels of vertices adjacent to w but not v increased by $2^k\pi(v) = 0$. At this point, all labels are unchanged, so this process gets us back to the labeling π, which proves the lemma. $\square$

If π is a winnable labeling, the above lemma implies that if we toggle any vertex with a nonzero labeling that is adjacent to a vertex with label 0, we can alternately toggle these two vertices to return to the original winnable labeling. This gives us the following.

Theorem 4.4. Suppose that π is a winnable labeling for the $\mathbb{Z}_{2^k}$-labeling Lights Out game, and let $v, w \in V(G)$ with $vw \in E(G)$ and $\pi(w) = 0$. Then v is π-winnable.

This leaves us with the case of vertices with a nonzero labeling whose adjacent vertices all have nonzero labelings. It would be nice to have a version of Lemma 4.3 to bring this theorem to us. Unfortunately, the conclusion of Lemma 4.3 is false in general for vertices adjacent only to vertices with nonzero labels. As an example, consider any graph under the $\mathbb{Z}_2$-labeling Lights Out game, and suppose we start with the labeling that gives each vertex a label of 1. Toggling any vertex results in a labeling where at least one vertex has a label of 0. Moreover, it is easy to see that for every subsequent toggle, the label of the vertex toggled also ends up as 0. Thus, we can never return to our original labeling as we did in Lemma 4.3. Other methods are then necessary to prove the general case.

We now turn to the trap problem for the $\mathbb{Z}_2$-labeling Lights Out game. This game is a bit easier to play than the general game for at least two reasons. First, any time a vertex is toggled once, it ends up with a label of 0. Second, this game is played identically to the original Lights Out game except that toggling a vertex with a label of 0 has no effect on the labeling. This leads us to the solution to the trap problem for this game.

Theorem 4.5. Let G be a graph, and let $\pi : V(G) \to \mathbb{Z}_2$ be a labeling in the zero component of the Lights Out labeling digraph of G. Then π is winnable.

Proof. By Lemma 4.2, it suffices to show that if π is winnable, then every vertex in G is π-winnable. For contradiction, we assume that π is winnable and $v \in V(G)$ is a π-trap. It follows that $\pi(v) = 1$. Moreover, since π is winnable, there exists at least one π-winnable vertex w. By an easy induction argument on the distance between v and w, we can assume that v and w are adjacent. If $\pi(w) = 0$, then v is π-winnable by Theorem 4.4, so we can assume that $\pi(w) = 1$ and, more generally, that $\pi(x) = 1$ for all vertices x adjacent to v.

Suppose first that w is adjacent to a vertex u with $\pi(u) = 1$, $u \neq v$, and $uv \notin E(G)$. We begin by toggling v and show that the resulting labeling is winnable. After toggling v, we toggle (in the given order) u, w, v, and u to get the labeling π'. In each case, we are toggling a vertex whose label is 1. Consider all toggles made, starting with our original toggling of v. We have toggled v twice, which has the effect of adding 2 to the labels of both v and each vertex adjacent to v. Thus, the toggles of v have no net effect on π. Similarly, the toggles of u have no net effect on π. This leaves only one toggle of w, and so π' is the labeling obtained by toggling w once. Since w is π-winnable, π' is winnable. Since π' can be obtained from π by first toggling v, this makes v π-winnable, which is a contradiction. Thus, every vertex adjacent to w that is neither v nor a neighbor of v has label 0.

We now prove by induction that every vertex z that is distance $\ell \geq 2$ away from v has label 0. We do $\ell = 2$ and $\ell = 3$ as our base cases. For $\ell = 2$, assume for contradiction that we have $u \in V(G)$ with $vu, uz \in E(G)$, $vz \notin E(G)$, and $\pi(z) = 1$. Recall that since u is adjacent to v, $\pi(u) = 1$. Since $vz \notin E(G)$, our work in the previous paragraph implies $u \neq w$ and $wu, wz \notin E(G)$. We are thus left with the path $wvuz$, where there are no edges among the path vertices other than the edges in the path and perhaps uw. In the case $uw \notin E(G)$, we toggle the vertices w, u, v, w, and u in the order listed. In each case, we are toggling a vertex with a label of 1 that is either π-winnable (i.e. the first toggle of w) or adjacent to a vertex with label 0. Theorem 4.4 then implies that the resulting labeling is winnable. Since w and u are each toggled twice, they have no net effect on the resulting labeling, which means the resulting winnable labeling can be obtained by toggling v once. In the case $uw \in E(G)$, we toggle the vertices w, z, u, v, u, w, u, z, and u in the order listed. As in the previous case, this results in a winnable labeling that has the same effect as toggling v once. In either case, v is π-winnable.

For $\ell = 3$, let $d(v, z) = 3$, and let vz_1z_2z be a path in G of minimum length. Thus, the only edges among the path vertices are the edges of the path. We have $\pi(z_1) = 1$, and by the $\ell = 2$ case $\pi(z_2) = 0$. Assume for contradiction that $\pi(z) = 1$. We then toggle, in order, z_1, z_2, z_1, v, and z_2. Only vertices with label 1 are toggled. Moreover, each time a vertex is toggled, it is adjacent to a vertex of label 0. By Theorem 4.4, the resulting labeling is winnable. However, the two toggles each of z_1 and z_2 have no net effect on the labeling, and so the resulting labeling is equivalent to merely toggling v. This contradicts our assumption that v is not π-winnable.

For $\ell \geq 4$, let $z \in V(G)$ with $\pi(z) = 1$, and let $vz_1 \cdots z_{\ell-1}z$ be a path of minimum length. By induction, $\pi(z_i) = 0$ for all $2 \leq i \leq \ell - 1$ Since z_1 is adjacent to a vertex with label 0 (namely z_2), Theorem 4.4 implies that z_1 is π-winnable. Furthermore, z_1 is adjacent to v, and so without loss of generality, we can assume $z_1 = w$. Since z and w are not adjacent, toggling both vertices in either order leads to the same labeling. Note that toggling w and z results in a winnable labeling, since w is π-winnable and z is adjacent to a vertex of label 0 (namely, $z_{\ell-1}$) when it is toggled. It follows that if π' is the labeling obtained from π by toggling z, then w is π'-winnable. However, $\pi'(z_{\ell-1}) = 1$, and $z_{\ell-1}$ is distance $\ell - 1 < \ell$ from v, which contradicts the induction hypothesis.

By Theorem 4.4, since v is not π-winnable, $\pi(u) = 1$ for all u adjacent to or equal to v. Since we proved above that $\pi(u) = 0$ for all u not equal to or adjacent to v, the set of all vertices that have label 1 in π is precisely $N[v]$. However, this implies that toggling v results in the zero labeling, which makes v π-winnable. This is a contradiction, which completes the proof. $\square$

5 Open Problems

We close this paper with some possible future lines of inquiry.

1. Theorems 2.4 and 3.4 give us some insight into how many winnable labelings a graph can have. It would be nice to have some explicit counting of such labelings.

2. While the weakly connected components of the labeling digraph all have the same order, they are certainly not isomorphic to one another as digraphs. Moreover, the correspondence given in Theorem 3.4 is not a graph homomorphism. One may ask if there are any structural similarities between the weakly connected components.

3. If $\mathcal{L}$ is the group of vertex labelings of a given graph and Z is the zero component of the labeling digraph, then $\mathcal{L}/V(Z)$ is a group whose elements induce the weakly connected components of the labeling digraph by Corollary 3.5. What can be said about the structure of this group?

4. While we solved the trap problem for $\mathbb{Z}_2$, it would be nice if we were able to resolve the trap problem for $\mathbb{Z}_{2^k}$.

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