The Lights Out Game on Subdivided Caterpillars

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Abstract

We investigate the general Lights Out game on subdivided caterpillars. In the Lights Out game, the vertices of a graph G are labeled with elements of $\mathbb{Z}_k$. When a vertex is toggled, the labels of that vertex and all adjacent vertices are increased by 1 modulo k. The goal is to make the labels of all vertices 0. A graph in which the game can always be won regardless of the initial labeling is called *always-winnable*. In this paper, we determine a large class of non-always-winnable subdivided caterpillars. For the remaining cases, we reduce the determination of the winnability of a subdivided caterpillar to the winnability of certain subgraphs that are ordinary caterpillars.

1 Introduction

The game Lights Out, originally a handheld game by Tiger Electronics, has been generalized to graphs. It is part of a larger class of "light-switching" games in which we have a collection of lights that can be on, off, or have multiple on-states (which can be interpreted as different colors or different intensities of the same color). In addition, there are certain "switches" which, when toggled, change the state of a given subset of the lights. Given an initial light pattern, the goal of the game is to toggle the switches in such a way that all the lights are off. Such games include the Berlekamp Light-Switching game (see [5] and [14]), Merlin's Magic Square (see [13] and [15]), and others (see [4] and [8]).

In our version of the Lights Out game, introduced in [10] and studied in [9], the lights are the vertices of a graph G, and each vertex is labeled

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with an element of $\mathbb{Z}_k$. The switches are the vertices of G, where toggling the vertex $v \in V(G)$ increases the labels of v and the vertices adjacent to v by 1 modulo k. The $k = 2$ version of the game has been studied in [3], [11], and [16]. As noted in [10] and [11], there is a connection between the Lights Out game and Domination Theory. The parity domination version of the Lights Out problem can be found in [1] [2], [7], and [12]. The connection between Lights Out and domination theory can also be found in non-z (mod k) dominating sets introduced in [6].

Let G be a graph. Our focus in this paper is to determine the circumstances under which the Lights Out game can be won on G. Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\pi : V(G) \to \mathbb{Z}_k$ be a labeling of the vertices in G. We call π *winnable over* $\mathbb{Z}_k$ if it is possible to toggle the vertices of G in such a way that all vertices have label 0. We call G *always-winnable (or AW) over* $\mathbb{Z}_k$ if every labeling of $V(G)$ is winnable over $\mathbb{Z}_k$.

More specifically, we study the winnability of the Lights Out game on subdivided caterpillars. A *caterpillar* is a graph in which the vertex set is $S \cup L$, where S induces a path (called the *spine*) and L consists of leaves that are adjacent to vertices in S. A *subdivided caterpillar* is similar to a caterpillar, except we replace L above with paths of arbitrary length (called *legs*) where one endpoint of each leg is a vertex in the spine. We prove three main results. The first is Theorem 3.3, which determines a large class of subdivided caterpillars that are not AW. The second is Theorem 3.4, which reduces the problem of determining the winnability of certain subdivided caterpillars to determining the winnability of certain subgraphs that are ordinary caterpillars. The last result is Theorem 3.7, which shows how the AW subdivided caterpillars of Theorem 3.4 can be used to construct all AW subdivided caterpillars. This reduces the problem of determining the AW subdivided caterpillars to determining the AW caterpillars.

2 Winnability in Path Graphs

Since the legs of subdivided caterpillars are paths, it will be helpful to have some results on the winnability of paths. So let P_n be the graph with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$.

Definition 2.1. Let P_n be defined as above, and let $k \in \mathbb{N}$. For each $z \in \mathbb{Z}_k$, we define the labeling $\pi_z : V(P_n) \to \mathbb{Z}_k$ by $\pi_z(v_1) = z$ and $\pi_z(v_i) = 0$ for all $2 \leq i \leq n$.

The following is from Lemma 4.2 in [10].

Theorem 2.2. For each labeling of $V(P_n)$ by $\mathbb{Z}_k$, it is possible to toggle the vertices of P_n so that the resulting labeling is π_z for some $z \in \mathbb{Z}_k$.

For a given vertex, its label is increased by one every time it or one of its neighbors is toggled. The order in which this is done does not matter. With this in mind, if $V(G)$ has the labeling π_z, we can then win the Lights Out game (if it is possible) by toggling the vertices beginning with v_1, toggling v_2 so that v_1 has label 0, toggling v_3 so that v_2 has label 0, etc, until we come to v_n. We let t_i be the number of times v_i is toggled, and let d_i be the label for v_i after v_i has been toggled. It is then clear that the labeling π_z is winnable if and only if $d_n = 0$. We also have $t_i = -d_{i-1}$ for all i to make sure that v_{i-1} has label 0 after v_i has been toggled. Moreover, we have $d_1 = z + t_1$ and $d_i = t_i + t_{i-1}$ for each $i \geq 2$. An easy induction argument gives us

$$d_i = \begin{cases} -t_1 & i \equiv 0 \pmod 3 \\ z + t_1 & i \equiv 1 \pmod 3 \\ -z & i \equiv 2 \pmod 3 \end{cases} \tag{1}$$

and

$$t_i = \begin{cases} z & i \equiv 0 \pmod 3 \\ t_1 & i \equiv 1 \pmod 3 \\ -z - t_1 & i \equiv 2 \pmod 3 \end{cases} \tag{2}$$

By setting $d_n = 0$ and using Equation (1), we get the following.

Lemma 2.3. Let P_n have an initial labeling of π_z with $z \neq 0$.

1. If $n \equiv 0 \pmod 3$, then one can win the Lights Out game without toggling v_1.

2. If $n \equiv 1 \pmod 3$, then one can win the Lights Out game with $t_1 = -z$.

3 Winnability in Subdivided Caterpillars

Theorem 2.2 and Equations (1) and (2) suggest that the lengths modulo 3 of the legs of a subdivided caterpillar are important in the Lights Out game, so we define the following.

Definition 3.1. Let G be a subdivided caterpillar with spine $S \subseteq V(G)$. For each $i \in \{0, 1, 2\}$ and $v \in S$, we define $L_i(v)$ to be the set of all legs in G with end vertex v and whose length is congruent to i modulo 3. We also define $L_i(G) = \bigcup_{v \in S} L_i(v)$.

The following lemma uses Theorem 2.2 and the sets $L_i(G)$ to achieve labelings of a convenient form. Note that for a set $T \subseteq V(G)$, we have that $N(T)$ is the set of all vertices of G that are neighbors to some vertex in S.

Lemma 3.2. Let G be a subdivided caterpillar with spine $S \subseteq V(G)$ and an initial labeling $\pi : V(G) \to \mathbb{Z}_k$.

1. The vertices of G can be toggled to give us a labeling in which all vertices in $V(G) - S - (N(S) \cap L_2(G))$ have label 0.
2. If $|L_2(v)| \leq 1$ for all $v \in S$, then the vertices of G can be toggled to give us a labeling in which all vertices in $V(G) - S$ have label 0.

Proof. For (1), we apply Theorem 2.2 to each leg in G not including the spine vertices to get a labeling of $V(G)$ in which all vertices that are on the legs but not adjacent to the spine have label 0. Suppose $v_1 \in L_0(v)$ for some $v \in S$ with v adjacent to v_1. Since $v_1 \in L_0(v)$, the leg containing v_1 is of the form $vv_1v_2 \cdots v_{3k}$ for some $k \in \mathbb{N}$. Since the path $v_1v_2 \cdots v_{3k}$ is isomorphic to P_{3k}, Lemma 2.3(1) implies that we can toggle the vertices of $v_1v_2 \cdots v_{3k}$ in such a way that v_1 is not toggled and each v_i has label 0. If $v_1 \in L_1(v)$ with v adjacent to v_1, the leg containing v_1 is of the form $vv_1v_2 \cdots v_{3k+1}$. By Lemma 2.3(2), we can toggle $v_1, v_2, \ldots,$ and v_{3k+1} until all v_i have label 0.

If we apply the above to all vertices in $L_0(G)$ and $L_1(G)$, we obtain a labeling of $V(G)$ in which all vertices have label 0 except perhaps the vertices in S and the vertices in $L_2(G)$ that are adjacent to vertices in S.

For (2), we first toggle the vertices so that the resulting labeling satisfies (1). For each $v \in S$ such that $|L_2(v)| = 1$, we then toggle v so that the vertex in $L_2(v)$ that is adjacent to v has label 0. Of course, now this has made the vertices in $L_0(v)$ and $L_1(v)$ that are adjacent to v have a non-zero label. However, we can toggle the vertices in these legs as we did in the proof of (1) to get a labeling in which all vertices not on the spine have label 0. $\square$

We now use $|L_2(G)|$ to determine a class of subdivided caterpillars that are not always-winnable.

Theorem 3.3. Let G be a subdivided caterpillar with spine $S \subseteq V(G)$, and let $v \in S$. If $|L_2(v)| \geq 2$, then for all $k \geq 2$, G is not AW over $\mathbb{Z}_k$.

Proof. Let one of the paths in $L_2(v)$ be given by $v_1v_2 \cdots v_{3k+2}v$, and let another path in $L_2(v)$ be given by $vw_1w_2 \cdots w_{3m+2}$. Let $\pi : V(G) \to \mathbb{Z}_k$ be the labeling given by $\pi(v_1) = 1$ and $\pi(w) = 0$ for all $w \neq v_1$. We claim that this labeling is not winnable.

First note that the only vertex in the path $v_1 \cdots v_{3k+2}vw_1 \cdots w_{3m+2}$ that may be adjacent to a vertex outside the path is v. Thus, the vertices $v_1, v_2, \ldots, v_{3k+2}$, and v must be toggled in the same way as we would toggle the path in isolation. After w_1 is toggled, the remaining w_i must also be toggled as in the case of an isolated path. So let t be the number of times

v_1 is toggled. The vertex v behaves as a v_i with $i \equiv 0 \pmod 3$. Thus, v is toggled once by Equation (2). It follows that at this point, w_1 has label 1. Now we proceed with the path $w_1 w_2 \cdots w_{3m+2}$ as if we began with w_1 having label 1. By Equation (1), we have $d_{3m+2} = 1$. Since $d_{3m+2} \neq 0$, we cannot win the game. Thus, π is not winnable, and so G is not always-winnable over $\mathbb{Z}_k$. $\square$

In the case $L_2(G) = \emptyset$, the problem reduces to ordinary caterpillars.

Theorem 3.4. Let G be a subdivided caterpillar with spine $S \subseteq V(G)$ such that $L_2(G) = \emptyset$. Let G' be the caterpillar that is the subgraph of G induced by $S \cup (N(S) \cap L_1(G))$. For each $k \in \mathbb{N}$, G is always-winnable over $\mathbb{Z}_k$ if and only if G' is always-winnable over $\mathbb{Z}_k$.

Note that the G' above can also be described as the caterpillar with spine S such that each $v \in S$ has $|L_1(v)|$ leaves adjacent to it.

Proof. Assume that G is always-winnable over $\mathbb{Z}_k$, and let π be any labeling of $V(G')$. We can consider π a labeling of $V(G)$ by defining $\pi(v) = 0$ for all $v \in V(G) - V(G')$. Since G is always-winnable over $\mathbb{Z}_k$, the labeling π is winnable on G. We toggle the vertices in G' as we would as part of winning the game in G, and we claim that this toggling wins the game in G'. Let π_f be the labeling of $V(G')$ that results after we have toggled all vertices in G'. It suffices to prove that $\pi_f(v) = 0$ for all $v \in V(G')$. For contradiction, assume that $\pi_f(v) \neq 0$ for some $v \in V(G')$. Observe that G is always-winnable, and all vertices in $V(G')$ have already been toggled. Thus, we must be forced to toggle some vertex $w_1 \in V(G) - V(G')$ that is adjacent to v a nonzero number of times in the course of winning the game in G, and we must be able to win the game in G without toggling vertices in G'. Since $w_1 \notin V(G')$, we must have either $w_1 \in L_0(G) - S$ and $v \in S$ or $v, w_1 \in L_1(G) - S$.

In either case, we have w_1 in a leg in which all or part of the leg is the path $v w_1 w_2 \cdots w_{3k}$, where w_{3k} is the end vertex of the leg that is not on the spine. Note that no vertices adjacent to w_i for $i \geq 2$ have been toggled yet, and so the path $w_1 w_2 \cdots w_{3k}$ has labeling π_z for some $z \in \mathbb{Z}_k$. To win the game in G, we must toggle the vertices in $w_1 w_2 \cdots w_{3k}$ as we would in the graph P_{3k}. By Equation (2), we must toggle w_1 0 times. This contradicts the assumption that w_1 is toggled a nonzero number of times.

For the converse, assume that G' is always-winnable over $\mathbb{Z}_k$. We let π be a labeling of $V(G)$, and we show that π is winnable. By Lemma 3.2(2), we can assume that $\pi(v) = 0$ for all $v \notin S$. The remaining vertices that have nonzero labels are in $V(G')$. Since G' is always-winnable, we can toggle the vertices in $V(G')$ in such a way that all vertices in $V(G')$ have label 0. Thus, the only vertices in $V(G)$ that have nonzero labels are in

$V(G) - V(G')$ and are adjacent to a vertex in $V(G')$. Let $w_1 \in V(G) - V(G')$ have a nonzero label. Then either $w_1 \in L_0(v)$ or $w_1 \in L_1(v)$ for some $v \in S$. Thus, w_1 is part of a leg of the form $vw_1w_2 \cdots w_{3k}$ or $vww_1w_2 \cdots w_{3k}$, where $w \in V(G')$ (and thus w has label 0). Since $w_1w_2 \cdots w_{3k}$ is isomorphic to P_{3k}, Lemma 2.3(1) implies that we can toggle $w_2, \ldots, w_{3k}$ in such a way that w_i all have label 0 for $i \geq 1$. This does not change the label of v or w. If we apply this to all legs of G, we end up with all vertices having label 0. Thus, G is always-winnable. □

Finally, we look at the case where $L_2(G) \neq \emptyset$ and $|L_2(v)| \leq 1$ for each $v \in S$. In [2], the authors used a certain construction to "paste" together non-AW caterpillars to form another non-AW caterpillar in the case $k = 2$. The same construction was used in [10] for the case where k is a prime power. In this construction, caterpillars are arranged in a given order. For each pair of adjacent caterpillers, a new spinal vertex is introduced with arbitrarily many leaves. This new spinal vertex shares an edge with an end vertex from the spine of each tree in the pair.

We use a similar construction to arrange AW subdivided caterpillars in order and transform them into other AW caterpillars. The construction is almost identical to the one described above, except instead of connecting the end vertices of adjacent caterpillars to single vertices that have arbitrarily many leaves, we connect the end vertices of adjacent subdivided caterpillars to subdivided caterpillars in which each vertex v in the spine satisfies $|L_2(v)| = 1$.

Definition 3.5. For $n \geq 2$, let $G_1, \ldots, G_m$ be subdivided caterpillars. A *winnable attachment* of $(G_1, \ldots, G_m)$ is a subdivided caterpillar G created from $G_1, G_2, \ldots, G_m$, along with the caterpillars $T_0, T_1, \ldots, T_m$, where T_0 or T_m may be empty, such that

1. $|L_2(v)| = 1$ for each v in a spine of one of the nonempty T_i's.
2. $V(G)$ consists of all vertices from the G_i's and T_j's (taken over all i and j).
3. $E(G)$ consists of the following.
   (a) $E(G_i) \cup E(T_j)$ (taken over all i and j).
   (b) For each $0 \leq i \leq n-1$, an edge that connects one end vertex of the spine $T_i \neq \emptyset$ to an end vertex of the spine of G_{i+1}.
   (c) For each $1 \leq i \leq n$, an edge the connects the other end vertex in the spine of G_i to an end vertex in the spine of $T_i \neq \emptyset$ that does not share an edge with an end vertex in the spine of G_{i+1}.

Essentially, a winnable attachment consists of placing vertices with a single leg of length 2 modulo 3 between the given subdivided caterpillars and attaching the spines by edges between their end vertices. Note that there are no restrictions on $L_0(T_i)$ and $L_1(T_i)$. We get the following.

Lemma 3.6. Let G be a winnable attachment of $(G_1, \ldots, G_m)$, where each G_i is an always-winnable subdivided caterpillar. Then G is always-winnable.

Proof. Let $S \subseteq V(G)$ be the spine of G, and let π be a labeling of $V(G)$. By definition, we have $|L_2(v)| = 1$ if v is in the spine of T_i. Since each G_i is AW, Theorem 3.3 implies that $|L_2(v)| \leq 1$ if v is in the spine of G_i. Thus, $|L_2(v)| \leq 1$ for all vertices $v \in S$, so we can assume that $\pi(w) = 0$ for all vertices $w \notin S$ by Lemma 3.2(2). Since each G_i is always-winnable, we can toggle the vertices in each G_i so that each vertex in $V(G_i)$ has label 0. When this has been done, the only vertices that have a nonzero label are vertices $v \in S \cap V(T_i)$ for some i. For every such v, we must then have $|L_2(v)| = 1$. Let $vv_1 \cdots v_{3k+2}$ be the unique leg in $L_2(v)$. This path is isomorphic to P_{3k+3}. By Lemma 2.3(1), we can toggle the vertices v_i so that v and all v_i have label 0. If we do this to every vertex v with $|L_2(v)| = 1$, we end up with all vertices of G having label 0. Thus, G is always-winnable. $\square$

We are now ready to classify the always-winnable subdivided caterpillars in terms of always-winnable caterpillars with $L_2(G) = \emptyset$.

Theorem 3.7. Let G be a subdivided caterpillar with spine $S \subseteq V(G)$ with $L_2(G) \neq \emptyset$ and $|L_2(v)| \leq 1$ for all $v \in S$. Then G is always-winnable if and only if one of the following occur.

1. $|L_2(v)| = 1$ for all $v \in S$.
2. There exist always-winnable subdivided caterpillars G_i, $1 \leq i \leq m$ such that $L_2(G_i) = \emptyset$ for all i and G is a winnable attachment of $(G_1, G_2, \ldots, G_m)$.

Proof. If G is a winnable attachment of always-winnable subdivided caterpillars with $L_2 = \emptyset$, then G is always-winnable by Lemma 3.6. If G is a subdivided caterpillar with $|L_2(v)| = 1$ for all $v \in S$, we can use Lemma 2.3(1) as in the proof of Lemma 3.6 to prove that G is always-winnable. Thus, it suffices to prove that every always-winnable subdivided caterpillar satisfies either (1) or (2).

So assume G is always-winnable. By Theorem 3.3, we have $|L_2(v)| \leq 1$ for all $v \in S$. If we have $|L_2(v)| = 1$ for all $v \in S$, then we have (1). Otherwise, we have at least one $v \in S$ with $L_2(v) = \emptyset$. Let S be given by the path $v_1 v_2 \cdots v_n$. We construct the graphs T_i and G_j in the winnable attachment using the following algorithm. Note that we need only articulate what the spines of T_i and G_j are. The legs of each T_i and G_j are the legs of G that correspond to that portion of the spine.

1. If $L_2(v_1) = \emptyset$, let $T_0 = \emptyset$. If $L_2(v_1) \neq \emptyset$, let k_0 be minimal such that $L_2(v_{k_0}) = \emptyset$. We then let T_0 have spine $v_1 v_2 \cdots v_{k_0-1}$.

2. For each $i \geq 1$, suppose we have constructed T_{i-1} with spine $v_{\ell_{i-1}} v_{\ell_{i-1}+1} \cdots v_{k_{i-1}-1}$ such that $|L_2(v_i)| = 1$ for all $\ell_{i-1} \leq i \leq k_{i-1} - 1$ and $L_2(v_{k_{i-1}}) = \emptyset$. Let ℓ_i be minimal so that $\ell_i > k_{i-1}$ and $L_2(v_{\ell_i}) \neq \emptyset$. Then G_i has spine $v_{k_{i-1}} v_{k_{i-1}+1} \cdots v_{\ell_i - 1}$. If no ℓ_i exists as described above, let G_i be as above with $\ell_i = n+1$ and stop the algorithm.

3. If $i = 1$ and $T_0 = \emptyset$, let G_i be as above with $k_{i-1} = 1$.

4. For each $i \geq 1$, suppose we have constructed G_i with spine $v_{k_{i-1}} v_{k_{i-1}+1} \cdots v_{\ell_i - 1}$ with $L_2(v_i) = \emptyset$ for all $k_{i-1} \leq i \leq \ell_i - 1$ and $|L_2(v_{\ell_i})| = 1$. Let k_i be minimal with $k_i > \ell_i$ and $L_2(v_{k_i}) = \emptyset$. Then T_i has spine $v_{\ell_i} v_{\ell_i+1} \cdots v_{k_i - 1}$. If no k_i exists as described above, let T_i be as above with $k_i = n + 1$ and stop the algorithm.

Let $G_1, \cdots, G_m$ be the G_i generated by the above algorithm. It suffices to prove that G is a winnable attachment of $(G_1, \cdots, G_m)$ and that each G_i is an always-winnable subdivided caterpillar with $L_1(G_i) = \emptyset$. It is easy to show that every vertex v in the spine of each T_i satisfies $L_2(v) \neq \emptyset$ (and therefore $|L_2(v)| = 1$), and that every vertex v in the spine of each G_i satisfies $L_2(v) = \emptyset$. Also, it is clear that G is a winnable attachment of $(G_1, \cdots, G_m)$. Thus, it suffices to prove that each G_i is always-winnable.

Let π be a labeling of G_i. By Lemma 3.2(2), we can assume that $\pi(v) = 0$ for all v not on the spine of G_i. We extend π to a labeling on G by defining $\pi(w) = 0$ for all $w \in V(G) - V(G_i)$. Since G is always-winnable, we can toggle the vertices in $V(G)$ in such a way that all vertices have label 0. We toggle the vertices in $V(G_i)$ as we would as part of winning the game on G, and we claim that this wins the game on G_i. Note that at this point only vertices in $V(G) - V(G_i)$ need to be toggled in order to win the game on G. Also, the only vertices in $V(G_i)$ that are adjacent to vertices in $V(G) - V(G_i)$ are $v_{\ell_{i-1}}$ (if T_{i-1} is nonempty) and $v_{k_i - 1}$ (if T_i is nonempty). Thus, all vertices of G_i that are neither $v_{\ell_{i-1}}$ nor v_{k_i-1} have label 0. Let a be the label of $v_{\ell_{i-1}}$ and let b be the label of v_{k_i-1}. It suffices to show that $a = b = 0$.

We prove the result for $v_{\ell_{i-1}}$. The proof for v_{k_i-1} is similar. If $i = 1$ and $T_0 = \emptyset$, then there are no vertices in $V(G) - V(G_i)$ that are adjacent to $v_{\ell_{i-1}}$. Since our toggling was part of a winning strategy for G, it follows that

$a = 0$. Otherwise, T_{i-1} is nonempty, and the only vertex in $V(G) - V(G_i)$ that is adjacent to $v_{\ell_{i-1}}$ is $v_{\ell_{i-1}-1}$. Note that $v_{\ell_{i-1}-1}$ is in the spine of T_{i-1},and so $|L_2(v_{\ell_{i-1}-1})| = 1$. Let $v_{\ell_{i-1}-1}w_1 \cdots w_{3k+2}$ be the unique leg in $L_2(v_{\ell_{i-1}-1})$. Since $v_{\ell_{i-1}-1}$ is the only vertex adjacent to $v_{\ell_{i-1}}$ that has yet to be toggled, we must toggle it $-a$ times. This leaves w_1 with label $-a$ and w_i with label 0 for all $i \geq 2$. However, we must be able to toggle the w_i's in such a way that they all have label 0. Since the labeling of $w_1 \cdots w_{3k+2}$ is π_z, Equation (1) implies that w_{3k+2} ends up with label $-a$. Since w_{3k+2} must have label 0, we must have $a = 0$. This completes the proof. □

4 Conclusion

Theorem 3.3 characterizes the winnability of subdivided caterpillars G with $|L_2(v)| \geq 2$ for some $v \in V(G)$. Furthermore, Theorems 3.4 and 3.7 reduce the problem of characterizing the winnability of subdivided caterpillars G with $|L_2(v)| \leq 1$ for all $v \in V(G)$ to the problem of characterizing the winnability of ordinary caterpillars. Thus, the problem of determining winnability for all subdivided caterpillars has been reduced to determining winnability for ordinary caterpillars.

In [2], non-AW caterpillars (and thus AW caterpillars) were characterized for $k = 2$. Some progress was made in [10] in the case that k is a prime power, but the non-AW caterpillars have not been explicitly characterized in this case. When k is not a prime power, it is even possible that the construction used in [2] and [10] to create non-AW caterpillars from other non-AW caterpillars may actually produce an AW graph ([10, Example 5.7]). This complicates the task of classifying non-AW caterpillars in this case.

References

[1] A.T. Amin and P.J. Slater. Neighborhood domination with parity restrictions in graphs. *Congr. Numer.*, 91:19–30, 1992.

[2] A.T. Amin and P.J. Slater. All Parity Realizable Trees. *J. Combin. Math. Combin. Comput.*, 20:53–63, 1996.

[3] M. Anderson and T. Feil. Turning Lights Out with Linear Algebra. *Math. Mag.*, 71:300–303, 1998.

[4] P.V. Araujo. How to turn all the lights out. *Elem. Math.*, 55:135–141, 2000.

[5] J. Carlson and D. Stolarski. The correct solution to Berlekamp's switching game. *Discrete Math.*, 287:145–150, 2004.

[6] Y. Caro and M.S. Jacobson. On non-z (mod k) dominating sets. *Discuss. Math. Graph Theory*, 23:189–199, 2003.

[7] Y. Caro and W.F. Klostermeyer. The odd domination number of a graph. *J. Combin. Math. Combin. Comput.*, 44:65–83, 2003.

[8] D. Craft, Z. Miller, and D. Pritikin. A solitaire game played on 2-colored graphs. *Discrete Math.*, 309:188–201, 2009.

[9] S. Edwards, V. Elandt, N. James, K. Johnson, Z. Mitchell, and D. Stephenson. Lights Out on finite graphs. *Involve*, 3:17–32, 2010.

[10] A. Giffen and D.B. Parker. On Generalizing the "Lights Out" Game and a Generalization of Parity Domination. *Ars Combin.*, 111:273–288, 2013.

[11] J. Goldwasser and W. Klostermeyer. Maximization versions of "lights out" games in grids and graphs. *Congr. Numer.*, 126:99–111, 2007.

[12] J. Goldwasser, W. Klostermeyer, and H. Ware. Fibonnacci polynomials and parity domination in grid graphs. *Graphs Combin.*, 18:271–283, 2002.

[13] D. Pelletier. Merlin's Magic Square. *Amer. Math. Monthly*, 94:143–150, 1987.

[14] U. Schauz. Colorings and Nowhere-Zero Flows of Graphs in Terms of BerlekampÕs Switching Game. *Electronic J. Combin.*, 18:#P65, 2011.

[15] D.L. Stock. Merlin's Magic Square Revisited. *Amer. Math. Monthly*, 96:608–610, 1989.

[16] K. Sutner. Linear cellular automata and the Garden-of-Eden. *Math. Intelligencer*, 11:49–53, 1989.