Feedback Matrices and the Lights Out Game on Directed Graphs

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Abstract

For a directed graph D, the neighborhood lights out game begins with a labeling of $V(D)$ with elements of $\mathbb{Z}_r$ for $r \geq 2$. When a vertex v is toggled, the labels of v and any vertex that v dominates are increased by 1 mod r. The game is won when each vertex has label 0. We say that D is r-Always Winnable (also written r-AW) if the game can be won for every initial labeling with elements of $\mathbb{Z}_r$. We introduce the *feedback matrix* of D, which is a matrix derived from ordering $V(D)$ and using the feedback arcs from this ordering. Feedback matrices are useful in determining whether or not a directed graph is r-AW. We use this to determine whether various directed graphs, including some upset tournaments and tournaments with feedback arc sets that are certain unions of disjoint directed paths, are r-AW.

1 Introduction

Lights out is a one-player game that can be played on graphs and digraphs. It began as an electronic game created by Tiger Electronics in 1995 that was played on a 5×5 grid. There are now several variations of the game. Some are direct generalizations of the original game, like the σ^+-game in [Sut90] and the neighborhood lights out game developed independently in [GP13] and [Ara12]. This was generalized further to a matrix-generated version in [KP24]. Other versions are explored in [Pel87], [CMP09], and [PZ21].

The basic elements are the same in each version of the game. We begin with a labeling of the vertices, usually by elements of $\mathbb{Z}_r$ for some $r \geq 2$. We play the game by toggling the vertices, which changes the labels of some vertices in a way that usually depends on the adjacency of vertices with the toggled vertex. The game is won when we achieve some desired labeling, usually where each vertex has label 0.

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A directed graph version of the neighborhood lights out game was introduced in [DP]. For a digraph D, we denote the vertex set by $V(D)$ and the arc set by $A(D)$. In the neighborhood lights out game, we begin with a *labeling* $\lambda : V(D) \to \mathbb{Z}_r$ of the vertices with elements of $\mathbb{Z}_r$. Each time a vertex v is toggled, the label of v and each vertex it dominates (i.e. each w such that $vw \in A(D)$) is increased by 1 modulo r. The object of the game is to toggle the vertices so that all vertices have label 0. If N is the neighborhood matrix of D (i.e. $N = A+I$, where A is the adjacency matrix and I is the identity matrix), we call this game the (N, r)-lights out game, as defined in [KP24]. We drop the N in our notation for this paper and simply call it the r-lights out game, since we only study the neighborhood version of the game here. A labeling λ with labels in $\mathbb{Z}_r$ is called r-winnable if it is possible to win the game when we begin with the labeling λ. We say that D is r-always winnable (or r-AW) if every labeling of $V(D)$ with labels in $\mathbb{Z}_r$ is r-winnable.

The general problem we address is which digraphs are r-AW for $r \geq 2$. Our strategy for this problem has two stages. In the first stage, we order the vertices of D and look at the resulting *feedback arcs* and some related concepts as defined below.

Definition. Let D be a digraph with vertex ordering $\sigma = v_1, v_2, \ldots, v_n$.

- The *feedback arc set with respect to* σ (denoted $A_\sigma(D)$) is the set of all arcs $v_j v_i \in A(D)$ such that $i < j$.
- A *minimum feedback arc set* is a feedback arc set of minimum cardinality among all possible vertex orderings of D.
- If $vw \in A_\sigma(D)$, we call v a *tail feedback vertex* and w a *head feedback vertex.*
- If $v \in V(D)$ such that $vw, wv \notin A_\sigma(D)$ for all $w \in V(D)$, we call v a *non-feedback vertex.*
- We define $F_\sigma(D)$ to be the set of all feedback vertices, $H_\sigma(D)$ to be the set of all head feedback vertices, and $T_\sigma(D)$ to be the set of all tail feedback vertices.

We should note that there are two different definitions of feedback arc sets in the literature. Our definition is equivalent to the one given in [IN04], where a feedback arc set is defined to be a set of arcs that when reversed makes the resulting graph acyclic. The standard definition (see [BJG09] and [Kud22]) replaces *reversed* with *removed.* The definitions are not equivalent in general but *are* equivalent for *minimum* feedback arc sets.

For the second stage, we use the vertex ordering and feedback arc set to reduce winning the game to solving a manageable system of linear equations. The coefficients of these equations are elements of $\mathbb{Z}_r$. Since $\mathbb{Z}_r$ is not a field when r is composite, we require some methods from linear algebra over commutative rings to study the system's solutions. We use the following facts.

Proposition 1.1. [Bro93] Let A be an $n \times n$ matrix over a commutative ring R with identity.

1. A is invertible if and only if $\det(A)$ is a unit in R.

2. If we take a multiple of one row of A and add it to another row, then the determinant of the resulting matrix is $\det(A)$.

3. If we switch two rows of A, the resulting matrix has determinant $-\det(A)$.

4. If we multiply one row of A by $r \in R$, the resulting matrix has determinant $r\det(A)$.

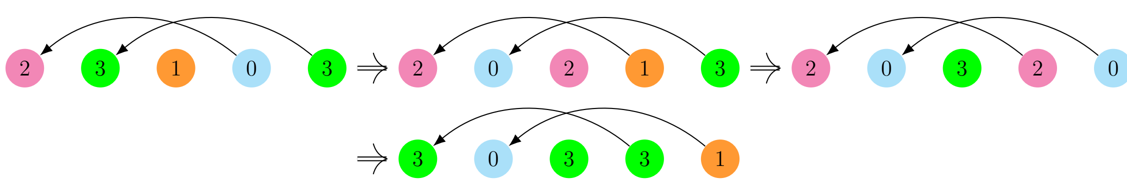
We also use the following well known determinant result on block matrices.

Proposition 1.2. Let R be a commutative ring, let A be an $m \times m$ matrix over R, and let B be an $n \times n$ matrix over R for $m, n \in \mathbb{N}$. We then have

$$\det\left(\begin{bmatrix} A & \mathbf{0} \\ * & B \end{bmatrix}\right) = \det\left(\begin{bmatrix} A & * \\ \mathbf{0} & B \end{bmatrix}\right) = \det(A)\det(B)$$

In [IN03], Isaak and Narayan classify minimum feedback arc sets of tournaments that are disjoint unions of directed paths. Note that a *tournament* is a digraph in which every pair of vertices has precisely one arc between them. While the results of this paper do not explicitly assume our feedback arc sets to be minimum, our main theorems are oriented toward tournaments with feedback arc sets that are disjoint unions of directed paths. Some of these results are on *upset tournaments*, which are studied in [BL83] and formally defined in [PS98]. These are tournaments having feedback arc sets that arc-induce a directed path digraph.

Since our main results are on tournaments, for convenience we denote a given tournament by indicating the ordering of the vertices and its feedback arcs. All arcs not shown are assumed to go from left to right. Here is what that looks like on a game with labels in $\mathbb{Z}_4$ where we toggle the second, third, and fourth vertices.



In Section 2, we define *feedback matrices* and prove that a digraph is r-AW if and only if its feedback matrix is invertible over $\mathbb{Z}_r$. In Section 3, we use our results on feedback matrices to determine when certain digraphs are r-AW based on whether or not some of their subdigraphs are r-AW. We then turn our attention to upset tournaments in Sections 4 and 5, where we determine the feedback matrices and their determinants for upset tournaments whose feedback arcs are all between consecutive vertices in the vertex ordering. Finally, in section 6, we prove a winnability result on tournaments whose feedback arc sets are certain disjoint unions of directed paths.

Our digraph notation and terminology are primarily the same as in [BJG09]. A digraph D consists of a set of *vertices* $V(D)$ along with a set $A(D)$ of ordered pairs of vertices called *arcs*. For $v, w \in V(D)$, we usually express the arc as vw or $v \to w$. In this arc, we call v the *tail* and w the *head* of the arc and say that v dominates w. The *outdegree* of a vertex is the number of vertices it dominates, and the *score-list* of a digraph is a multiset of the outdegrees of its vertices. If $U \subseteq V(D)$, we define the subdigraph *induced* by U to be the subdigraph that includes all vertices in U and all arcs between vertices in U. If $B \subseteq A(D)$, we define the subdigraph *arc-induced* by B to have arc set B and vertex set consisting of all vertices incident with an arc in B.

2 Feedback Matrices

Systems of linear equations play a central role in determining winning togglings in the neighborhood lights out game on both graphs and digraphs. The most common way this is done is as in [AF98], [AMW14], [GP13], [KP24], and [EEJ+10]. For a graph or digraph D, we create a vertex ordering $\sigma = v_1, v_2, \ldots, v_n$ of the vertices and express the initial labeling $\lambda : V(D) \to \mathbb{Z}_r$ as a column vector $\mathbf{b}$, where $\mathbf{b}[j] = \lambda(v_j)$. If each vertex v_i is toggled x_i times mod r, we can also express this as a column vector $\mathbf{x}$, where $\mathbf{x}[i] = x_i$. It follows that the final label for each v_j is $\lambda(v_j) + \sum_{i=1}^{n} a_{ij}x_i$, where $a_{ij} = 1$ if either $i = j$ or $v_iv_j \in A(D)$, and $a_{ij} = 0$ otherwise. Setting each label equal to 0, we get the matrix equation $N\mathbf{x} + \mathbf{b} = \mathbf{0}$, where $N = [a_{ij}]$ is the neighborhood matrix of D.

While this matrix equation can be applied to all neighborhood lights out games, determining the existence of solutions of these systems is often difficult. In [GP13] and [DP], we found smaller and simpler matrix equations that determine when the game can be won by first toggling the vertices to obtain a helpful labeling. For digraphs, we do this by toggling the vertices in the order given by σ. At each step, we toggle the given vertex until is has label 0. After the last vertex has been toggled, the only vertices (if any) that have a nonzero label are among the head feedback vertices, giving us the following.

Lemma 2.1. ([DP, Lem. 4.4]) Let D be a digraph with vertex ordering σ, and let $\lambda : V(D) \to \mathbb{Z}_r$ be a labeling of $V(D)$. Then the vertices of D can be toggled such that each vertex not in $H_\sigma(D)$ has label 0.

This enables us to assume that all initial labelings are zero on $V(D) - H_\sigma(D)$. Toggling from this point is much more straightforward. The following terminology will be helpful.

Definition. Let D be a digraph with vertex ordering σ.

- Let $v, w \in V(D)$ such that $v = w$ or v precedes w in σ. The *interval between v and w* is the set of all vertices between v and w in σ, inclusive. We can also simply say *interval* without reference to v and w.

- A *non-feedback interval* is an interval consisting of non-feedback vertices.

The next lemma describes how vertices in $V(D) - H_\sigma(D)$ must be toggled when $\lambda(V(D) - H_\sigma(D)) = \{0\}$ to have a possibility of winning the game.

Lemma 2.2. Let D be a digraph with vertex ordering $\sigma = v_1, v_2, \ldots, v_n$. Let $\lambda : V(D) \to \mathbb{Z}_r$ be a labeling such that $\lambda(V(D) - H_\sigma(D)) = \{0\}$. Suppose we perform a toggling of $V(D)$ that leaves all vertices in $V(D) - H_\sigma(D)$ with label 0 and such that each $w \in H_\sigma(D)$ is toggled x_w times.

1. Let $v \in V(D) - H_\sigma(D)$, and suppose we toggle the vertices in the order given by σ. If ℓ is the label of v just before we toggle v, then v must be toggled $-\ell$ times.

2. If D is a tournament, let v be a non-feedback vertex whose predecessor in σ is also a non-feedback vertex. Then v is toggled zero times.

3. If D is a tournament, let v be the first vertex in a non-feedback interval, let $u \in T_\sigma(D)$ such that $v < u$ in σ, and let $H_u = \{w \in H_\sigma(D) : w < v \text{ and } uw \in A_\sigma(D)\}$. After v is toggled, the label of u is $\lambda(u) - \sum_{w \in H_u} x_w$.

Proof. For (1), assume $v \in V(D) - H_\sigma(D)$. Thus, v dominates every vertex that follows it in σ. This implies that once it is time to toggle v, there are no untoggled vertices that dominate v. It follows that after toggling v, it must be left with label 0. This requires v to be toggled $-\ell$ times.

For (2), since the order we toggle the vertices does not affect the number of times we toggle each vertex for a winning toggling, we can assume that we toggle the vertices in the order given by σ. Let $v \in V(D) - F_\sigma(D)$. Then $v = v_k$ for some $1 \le k \le n$. Let m be maximum such that each $v_i \notin F_\sigma(D)$ for all $k - m \le i \le k$. By assumption, v_{k-1} is not a feedback vertex, so $m \ge 1$. Consider v_i with $k - m \le i \le k$. Since v_i is not a feedback vertex, every time a vertex before v_{k-m} is toggled, it increases the label of v_i by 1 mod r. Since all non-feedback vertices start with label 0, when it is time to toggle v_{k-m}, v_i has the same label ℓ as v_{k-m}. By (1), v_{k-m} is toggled $-\ell$ times, which leaves v_i with label 0. We now prove our result by induction on m. For $m = 1$, we have just toggled v_{k-1}, so it is time to toggle v_k. Since the label of v_k is 0, (1) implies that v_k is toggled 0 times. For $m > 1$, by induction, v_i gets toggled 0 times for $k - m + 1 \le i \le k - 1$. Thus, when it is time to toggle v_k, its label is still 0. By (1), v_k is toggled 0 times, which proves the result.

For (3), let x be the sum of all toggles of vertices that come before v. Since v is a non-feedback vertex and $\lambda(v) = 0$, right before v is toggled its label is x. Since u dominates the vertices in H_u, toggling vertices in H_u has no effect on the label of u. Thus, the label of u right before v is toggled is $\lambda(u) + x - \sum_{w \in H_u} x_w$. By (1), v is toggled $-x$ times, leaving u with label $(\lambda(u) + x - \sum_{w \in H_u} x_w) - x = \lambda(u) - \sum_{w \in H_u} x_w$. □

If D is a tournament, Lemma 2.2(2) implies that the first vertex in a non-feedback interval is the only vertex in the non-feedback interval that has any influence on whether or not D is r-AW. Thus, the length of a non-feedback interval has no effect on whether or not a digraph is r-AW.

The next lemma shows how we can determine when a lights out game can be won using a system of linear equations based on the head feedback vertices of D.

Lemma 2.3. Let D be a digraph with vertex ordering $\sigma = v_1, v_2, \dots, v_n$. Let $\lambda : V(D) \to \mathbb{Z}_r$ be a labeling such that $\lambda(V(D) - H_\sigma(D)) = 0$. Suppose that each $w \in H_\sigma(D)$ is toggled x_w times with initial labeling λ.

1. For all $v \in V(D) - H_\sigma(D)$, there exist unique $y_v \in \mathbb{Z}_r$ such that if each $v \in V(D) - H_\sigma(D)$ is toggled y_v times, then each $v \in V(D) - H_\sigma(D)$ has label 0.

2. For all $v \in V(D) - H_\sigma(D)$, $w \in H_\sigma(D)$, there exist unique $a_{v,w} \in \mathbb{Z}_r$ that do not depend on the x_w such that $y_v = \sum_{w \in H_\sigma(D)} a_{v,w} x_w$.

3. If each $v \in V(D) - H_\sigma(D)$ is toggled y_v times, then for each $u, w \in H_\sigma(D)$, there exist unique $h_{u,w} \in \mathbb{Z}_r$ that do not depend on the x_w such that the final labeling of u is $\lambda(u) + \sum_{w \in H_\sigma(D)} h_{u,w} x_w$.

4. For each $u, w \in H_\sigma(D)$, let $h_{u,w}$ be as in (3). Then λ is r-winnable if and only if there exists $x_w \in \mathbb{Z}_r$ for each $w \in H_\sigma(D)$ such that $\lambda(u) + \sum_{w \in W} h_{u,w} x_w = 0$ for all $u \in H_\sigma(D)$.

Proof. For (1), let $v \in V(D) - H_\sigma(D)$. Then $v = v_k$ for some $1 \leq k \leq n$. We prove our result by induction on k. If $k = 1$, since $v_1 = v$ is not a head feedback vertex, it begins the game with label 0. Since it is the first vertex in the ordering σ, it is the first to be toggled. By Lemma 2.2(1), v_1 is toggled 0 times, which is unique. Furthermore, since no vertex dominates v, its label remains 0 for the entire game, which proves the $k = 1$ case. For $k > 1$, the label ℓ of v just before it is toggled is the sum of the numbers of toggles on the vertices that are toggled before v and dominate v. This includes some or all of the head feedback vertices (whose numbers of toggles are unique by assumption) and all of the preceding vertices not in $H_\sigma(D)$ (whose numbers of toggles are unique by induction). Thus, ℓ is uniquely determined by how many times each preceding vertex is toggled and its arc relationship with v. By Lemma 2.3(1), v must be toggled $-\ell$ times, which is also unique. After v is toggled, there are no remaining vertices to be toggled that dominate v. Thus, the label of v remains 0 throughout the game, which proves the induction step.

We proceed similarly for (2). If $v \in V(D) - H_\sigma(D)$, then $v = v_k$ for some $1 \leq k \leq n$, and we induct on k. For $k = 1$, v_1 is toggled 0 times as noted above. Since the x_w can have any value in $\mathbb{Z}_r$, the only way for 0 to be linear in the x_w's is for $a_{v,w} = 0$ for all $w \in H_\sigma(D)$, which proves the $k = 1$ case. For $k > 1$, as noted above, the label of v right before it is toggled is the sum of all numbers of toggles on the vertices that are toggled before v and dominate v. This may include some or all of the head feedback vertices w, which are toggled x_w times. It may also include preceding vertices u not in $H_\sigma(D)$, which are toggled $\sum_{w \in H_\sigma(D)} a_{u,w} x_w$ for some constants $a_{u,w}$ by induction. Each of these numbers of toggles are linear in the x_w's, and so adding them will also be a linear expression in the x_w's. Thus, the label ℓ of v right before it is toggled is a linear expression in the x_w's. By Lemma 2.2(1), v is toggled $-\ell$ times, which is also a linear expression in the x_w's. Also, since the coefficients $a_{u,w}$ are unique for all preceding vertices u and similarly for the x_w toggles for head feedback vertices w that dominate v, the coefficients $a_{v,w}$ are also unique. This completes the proof.

For (3), let $v \in H_\sigma(D)$. The final label of u is the sum of $\lambda(u)$, x_u (since toggling u affects its label), and the numbers of toggles of the vertices that dominate u. Each of these vertices w is either in $H_\sigma(D)$ (which increases the label of u by $x_w \bmod r$); or not in $H_\sigma(D)$ (which, by (2), adds a unique linear expression in the x_w's to the label of u). Adding them together results in a unique linear expression in the x_w's, which proves the result.

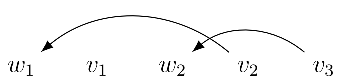
For (4), λ is r-winnable if and only if we can choose an appropriate number of toggles for each vertex so that the final label of each vertex is 0. By (1), these numbers of toggles are completely determined by our choice of x_w for each $w \in H_\sigma(D)$. Once we choose each x_w and each $v \in V(D) - H_\sigma(D)$ is toggled y_v times, the final label of each vertex in $V(D) - H_\sigma(D)$ is 0 by (1), and the final label of each $u \in H_\sigma(D)$ is $\lambda(u) + \sum_{w \in W} h_{u,w} x_w$ by (3). Since λ is r-winnable if and only if each of these labels is 0, this completes the proof. □

We can express the system of linear equations given in Lemma 2.3(4) as a matrix equation. For a vertex ordering σ of D, let $H_\sigma(D) = \{w_1, w_2, \ldots, w_m\}$. For convenience, we can write h_{w_i,w_j} as h_{ij}, which gives us the matrix $M_\sigma(D) = [h_{ij}]$. We define the column vectors $\mathbf{b}$ and $\mathbf{x}$ by $\mathbf{b}[i] = \lambda(w_i)$ and $\mathbf{x}[i] = x_i$. Then the system of equations becomes $M_\sigma(D)\mathbf{x} + \mathbf{b} = \mathbf{0}$. The next result follows directly from this equation.

Theorem 2.4. Let $r \in \mathbb{N}$. A digraph D is r-AW if and only if $\det(M_\sigma(D))$ is a unit in $\mathbb{Z}_r$.

We call $M_\sigma(D)$ the *feedback matrix of D for σ*.

Example 2.5. Consider the following tournament D.



To compute $M_\sigma(D)$, we first note that w_1 and w_2 are the head feedback vertices, so the initial labeling gives w_1 and w_2 arbitrary labels $\lambda(w_1)$ and $\lambda(w_2)$, respectively, and the remaining vertices label 0. Then w_1 and w_2 are toggled x_1 and x_2 times, respectively. We toggle from left to right. After toggling w_1, v_1 has label x_1, so we toggle it $-x_1$ times. That leaves w_1 with label $\lambda(w_1) + x_1$, v_2 with label $-x_1$, and the other vertices with their original labels. Then w_2 is toggled x_2 times. At this point, w_2 has label $\lambda(w_2) + x_2$, v_2 has label $-x_1 + x_2$, and v_3 has label 0, so v_2 is toggled $x_1 - x_2$ times. That leaves w_1 with label $\lambda(w_1) + 2x_1 - x_2$ and v_3 with label $x_1 - x_2$, so v_3 is toggled $-x_1 + x_2$ times. This leaves w_2 with label $\lambda(w_2) + 2x_2 - x_1$. At this point, all vertices have label 0 except w_1 with label $\lambda(w_1) + 2x_1 - x_2$ and w_2 with label $\lambda(w_2) - x_1 + 2x_2$. We win the game by solving $\lambda(w_1) + 2x_1 - x_2 = 0$ and $\lambda(w_2) - x_1 + 2x_2 = 0$. The feedback matrix we get from this is $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, which has determinant $2 \cdot 2 - (-1) \cdot (-1) = 3$. Thus, D is r-AW if and only if $\gcd(r, 3) = 1$.

3 Determining Winnability from Subdigraphs

In determining when a digraph is r-AW, it can be helpful to reduce the problem to determining when some of its subdigraphs are r-AW. We ask the following:

Question 3.1. Let D_1 and D_2 be digraphs with vertex orderings σ_1 and σ_2, respectively. How can we combine D_1 and D_2 into a larger digraph D whose vertices are ordered in a way consistent with σ_1 and σ_2 such that $\det(M_\sigma(D)) = \det(M_{\sigma_1}(D_1)) \det(M_{\sigma_2}(D_2))$?

By Theorem 2.4 and the fact that $\det(M_\sigma(D))$ is a unit in $\mathbb{Z}_r$ if and only if $\det(M_{\sigma_1}(D_1))$ and $\det(M_{\sigma_2}(D_2))$ are units in $\mathbb{Z}_r$, this would imply that D is r-AW if and only if D_1 and D_2 are r-AW. Before getting to the main results, we prove a lemma.

Lemma 3.2. Let D be a digraph with vertex ordering σ and let $\lambda : V(D) \to \mathbb{Z}_r$ be a labeling. For each assignment A of values in $\mathbb{Z}_r$ to the variables $s_1, s_2, \ldots, s_n$, define $\lambda_A : V(D) \to \mathbb{Z}_r$ by $\lambda_A(v) = \lambda(v) + \sum_{i=1}^{n} a_{v,i} s_i$, where each $a_{v,i}$ is constant with respect to the s_i. Suppose each $w \in H_\sigma(D)$ is toggled y_w times with initial labeling λ and x_w times with initial labeling λ_A, and that we toggle the remaining vertices for both labelings so that all vertices that are not head feedback vertices have label 0. If the final label for each $v \in H_\sigma(D)$ is $\lambda(v) + \sum_{w \in H_\sigma(D)} b_{v,w} y_w$ when the initial labeling is λ, where each $b_{v,w}$ is a constant with respect to the y_w, then the final label of v is $\lambda(v) + z_v + \sum_{w \in H_\sigma(D)} b_{v,w} x_w$ when the initial labeling is λ_A, where z_v is a linear combination of the s_i.

Proof. We first prove, for each vertex $v \in V(D)$, that if v is toggled $\sum_{w \in H_\sigma(D)} c_{v,w} y_w$ times under the labeling λ, where each $c_{v,w}$ is constant, then v is toggled $t_v + \sum_{w \in H_\sigma(D)} c_{v,w} x_w$ times under the labeling λ_A, where t_v is a linear combination of the s_i. We prove this by induction on the position of v in σ. If v is the first vertex in D, it cannot be a tail feedback vertex, since the corresponding head feedback vertex would have to come before v, which is impossible. If v is a non-feedback vertex, then v is toggled zero times under λ by Lemma 2.2(1). We also have $\lambda_A(v) = \lambda(v) + \sum_{i=1}^{n} a_{v,i} s_i = \sum_{i=1}^{n} a_{v,i} s_i$ since $\lambda(v) = 0$. By Lemma 2.2(1), v is toggled $-\sum_{i=1}^{n} a_{v,i} s_i = 0 + \sum_{i=1}^{n} (-a_{v,i}) s_i$ times under λ_A, so the claim follows with $t_v = \sum_{i=1}^{n} (-a_{v,i}) s_i$. The remaining case is when v is a head feedback vertex. When λ_A is the initial labeling, v is toggled x_v times, and when λ is the initial labeling, v is toggled y_v times. So the claim follows with $t_v = 0$, which proves the base case.

Now assume v is not the first vertex. If v is a head feedback vertex, then the claim follows as above with $t_v = 0$. If v is not a head feedback vertex, then Lemma 2.2(1) implies that whatever label v has, say ℓ for initial labeling λ and ℓ_A for initial labeling λ_A, v is toggled $-\ell$ times or $-\ell_A$ times, respectively. Note that ℓ is the sum of the toggles of the vertices that precede and dominate v in $V(D)$ under λ, and ℓ_A is $\lambda_A(v)$ plus the sum of the toggles of the vertices that precede and dominate v under λ_A. If $v_1, v_2, \ldots, v_k$ are the vertices in D that precede and dominate v, then by assumption v_i is toggled $\sum_{w \in H_\sigma(D)} c_{v_i,w} y_w$ times under λ. It follows that $\ell = \sum_{i=1}^{k} \left(\sum_{w \in H_\sigma(D)} c_{v_i,w} y_w \right)$. By induction, under λ_A each v_i is toggled $t_{v_i} + \sum_{w \in H_{\sigma_2}(D_2)} c_{v_i,w} x_w$ times, where t_{v_i} is a linear combination of the s_j. We use this and the fact that $\lambda_A(v) = \lambda(v) + \sum_{i=1}^{n} a_{v,i}s_i = \sum_{i=1}^{n} a_{v,i}s_i$ to get

$$\begin{aligned}\ell_A &= \sum_{i=1}^{n} a_{v,i}s_i + \sum_{i=1}^{k}\left(t_{v_i} + \sum_{w\in H_\sigma(D)} c_{v_i,w}x_w\right)\\ &= \left(\sum_{i=1}^{n} a_{v,i}s_i + \sum_{i=1}^{k} t_{v_i}\right) + \sum_{i=1}^{k}\left(\sum_{w\in H_\sigma(D)} c_{v_i,w}x_w\right)\end{aligned}$$

Since $t_v = \sum_{i=1}^{n} a_{v,i}s_i + \sum_{i=1}^{k} t_{v_i}$ is a linear combination of the s_j, the claim follows.

We can now look at the final labels of each $v \in V(D)$ under λ and λ_A, respectively. These labels are $\lambda(v) + f_v$ and $\lambda_A(v) + f'_v = \lambda(v) + \sum_{i=1}^{n} a_{v,i}s_i + f'_v$, where f_v is the sum of toggles of all vertices in $V(D)$ that dominate or equal v under λ, and f'_v is the sum of all toggles of vertices in $V(D)$ that dominate or equal v under λ_A. By our claim above, if $f_v = \sum_{w\in H_\sigma(D)} c'_{v,w}y_w$, then $f'_v = t'_v + \sum_{w\in H_\sigma(D)} c'_{v,w}x_w$ (here, $c'_{v,w}$ and t'_w are the sums of all $c_{v,w}$ and t_w from vertices that dominate/equal v). This makes the final labels

$$\lambda(v) + \sum_{w\in H_\sigma(D)} a'_{v,w}y_w \quad\text{and}\quad \lambda(v) + (t'_v + \sum_{i=1}^{n} a_{v,i}s_i) + \sum_{w\in H_\sigma(D)} a'_{v,w}x_w$$

Since $z_v = t'_v + \sum_{i=1}^{n} a_{v,i}s_i$ is a linear combination of the s_j, the lemma follows. □

We present three ways of combining digraphs that satisfy Question 3.1. In the first construction, D_1 and D_2 can be any digraphs, and any arcs in D between D_1 and D_2 originate in D_1 and terminate in D_2.

Theorem 3.3. Let D be a digraph with vertex ordering σ, and let D_1 and D_2 be induced subdigraphs of D that satisfy the following.

1. $V(D_1)$ and $V(D_2)$ partition $V(D)$.
2. All vertices of D_1 precede all vertices of D_2.
3. If $v \in V(D_1)$ and $w \in V(D_2)$, then $wv \notin A(D)$.

If σ_1 and σ_2 are the restriction of σ to $V(D_1)$ and $V(D_2)$, respectively, then $\det(M_\sigma(D)) = \det(M_{\sigma_1}(D_1))\det(M_{\sigma_2}(D_2))$.

Proof. To construct $M_\sigma(D)$, we begin with an arbitrary labeling $\lambda : V(D) \to \mathbb{Z}_r$ that is zero on all but the head feedback vertices. Let the restriction of λ to $V(D_1)$ and $V(D_2)$ be λ_1 and λ_2, respectively. For each $w \in H_\sigma(D)$, let x_w be the number of times v is toggled when λ is the initial labeling.

We toggle the vertices in the order given by σ. Since the vertices of D_1 are toggled first, that implies they are toggled in exactly the same way as they would without D_2. Moreover, by (2), toggling vertices in $V(D_2)$ has no effect the labels of $V(D_1)$. Thus, once all vertices in $V(D_1)$ are toggled, they have their final label, and so the upper left $|H_{\sigma_1}(D_1)| \times |H_{\sigma_1}(D_1)|$ matrix block of $M_\sigma(D)$ is $M_{\sigma_1}(D_1)$, and the upper right block is the zero matrix. This gives $M_\sigma(D)$ the form $\begin{bmatrix} M_{\sigma_1}(D_1) & \mathbf{0} \\ * & * \end{bmatrix}$. Also by (2), after toggling all vertices in $V(D_1)$, $V(D_2)$ has the labeling λ_t, where for each $v \in V(D_2)$, $\lambda_t(v) = \lambda(v)+t_v$, where t_v is the sum of the number of toggles of all vertices in $V(D_1)$ that dominate v. Note that by Lemma 2.3(2), t_v is a linear combination of all x_w with $w \in H_{\sigma_1}(D_1)$. We now have satisfied the hypotheses of Lemma 3.2, with x_w for $w \in V(D_1)$ playing the role of the s_i, λ_t playing the role of λ_A, and t_v playing the role of $\sum_{i=1}^n a_{v,i}s_i$. We can conclude that if the final label for each $v \in V(D_2)$ is $\lambda(v) + \sum_{w \in H_{\sigma_2}(D_2)} b_{v,w}y_w$ with initial labeling λ_2, then the final label for v is $\lambda(v) + z_v + \sum_{w \in H_{\sigma_2}(D_2)} b_{v,w}x_w$ with initial label λ_t, where z_v is a linear combination of all x_w, $w \in H_{\sigma_1}(D_1)$. Since the final label for v under the labeling λ_t is $\lambda(v) + z_v + \sum_{w \in H_{\sigma_2}(D_2)} b_{v,w}x_w$ and the coefficients from z_v are only of x_w, $w \in H_{\sigma_1}(D_1)$, the coefficients of z_v become the entries of $M_\sigma(D)$ below the block $M_{\sigma_1}(D_1)$. The remaining lower right block of the matrix is filled with the $b_{v,w}$. Since the final label for v under the labeling λ_2 is $\lambda(v)+\sum_{w \in H_{\sigma_2}(D_2)} b_{v,w}y_w$, the $b_{v,w}$ are the entries for $M_{\sigma_2}(D_2)$ as well. Thus, $M_\sigma(D)$ has the form $\begin{bmatrix} M_{\sigma(D_1)} & \mathbf{0} \\ * & M_{\sigma_2}(D_2) \end{bmatrix}$, and so Proposition 1.2 implies its determinant is $\det(M_{\sigma_1}(D_1)) \det(M_{\sigma_2}(D_2))$, which completes the proof. □

This result gives us a much simpler proof for a theorem in [DP] about strongly connected components of digraphs. A digraph D is strongly connected if for every $v, w \in A(D)$, there is a walk both from v to w and from w to v. A strong component of a digraph D is a maximal subdigraph that is strongly connected. It is well-known that the vertex sets of the strong components partition $V(D)$ and have an *acyclic ordering* (i.e. if $D_1, D_2, \ldots, D_t$ are the strong components, then if $vw \in A(D)$ with $v \in V(D_i)$, $w \in V(D_j)$, then $i \leq j$).

Corollary 3.4. Let D be a digraph with strong components $D_1, D_2, \ldots, D_t$ written in an acyclic ordering. Let σ be any ordering of $V(D)$ consistent with the acyclic ordering of the strong components, and for each $1 \leq i \leq t$, let σ_i be the restriction of σ to $V(D_i)$.

1. $\det(M_\sigma(D)) = \prod_{i=1}^t \det(M_{\sigma_i}(D_i))$
2. [DP, Thm. 3.3] D is r-AW if and only if each D_i is r-AW.

Proof. Part (1) follows from an easy induction using Theorem 3.3, and (2) follows from (1) and Theorem 2.4. □

If we order the vertices of D so that we begin with a set of vertices V_1, followed by a non-feedback interval I, followed by the remaining vertices V_2, then each vertex in I induces a strong component. Furthermore, the feedback matrix of the strong components in I are [1], so their determinants are 1. Corollary 3.4 immediately gives us the following.

Corollary 3.5. Let D be a digraph with vertex ordering σ such that we have the partition $V(D) = V_1 \cup I \cup V_2$ such that V_1 precedes I, which precedes V_2 under σ. Let σ_1 and σ_2 be the restriction of σ to V_1 and V_2, respectively, and let D_1 and D_2 be the subdigraphs induced by V_1 and V_2, respectively. If all vertices in I are non-feedback vertices and there are no feedback arcs originating in V_2 and terminating in V_1, then $\det(M_\sigma(D)) = \det(M_{\sigma_1}(D_1))\det(M_{\sigma_2}(D_2))$.

Our final two constructions require D to be a tournament with D_1 and D_2 either subtournaments of D or closely related to subtournaments of D. The constructions are similar to that of Theorem 3.3, except for some overlap in the order between the vertices of D_1 and D_2.

Theorem 3.6. Let D_1 and D_2 be tournaments with vertex orderings σ_1 and σ_2, respectively, where the vertex sets are given, in the order from σ_1 and σ_2, by the partitions $V(D_1) = V_1 \cup I_1 \cup \{u\}$ and $V(D_2) = \{v\} \cup I_2 \cup V_2$, where I_1 and I_2 are non-feedback intervals. Let D and vertex ordering σ be one of the following.

1. The vertices in order are $V(D) = V_1 \cup \{v\} \cup I \cup \{u\} \cup V_2$, where I is a non-feedback interval. The feedback arcs are given by $A_\sigma(D) = A_{\sigma_1}(D_1) \cup A_{\sigma_2}(D_2)$.
2. The vertices in order are $V(D) = V_1 \cup I_1 \cup \{v\} \cup \{u\} \cup I_2 \cup V_2$. The feedback arcs are given by $A_\sigma(D) = A_{\sigma_1}(D_1) \cup A_{\sigma_2}(D_2)$.

Then $\det(M_\sigma(D)) = \det(M_{\sigma_1}(D_1))\det(M_{\sigma_2}(D_2))$

If we use "__" to represent the non-feedback intervals, we can visualize the ordering on $V(D_1)$ as V_1 __ u, on $V(D_2)$ as v __ V_2, and on the two versions of $V(D)$ as V_1 v __ u V_2 and V_1 __ v u __ V_2. For $j \in \{1, 2\}$, let T_j be the tournament induced by V_j.

Proof. For (1), first recall that the length of any non-feedback interval does not affect the feedback matrix. Thus, when we refer to $V(D_1)$ and $V(D_2)$ as "subtournaments" of D, we abuse notation by allowing $V(D_1) = V_1 \cup I \cup \{u\}$ and $V(D_2) = \{v\} \cup I \cup V_2$. We observe that u cannot be a head feedback vertex in D, since the corresponding tail feedback vertex would come after u in $V(D_1)$ under σ_1, which is impossible. Thus, u is either a non-feedback vertex or a tail feedback vertex. Similarly v is either a non-feedback vertex or a head feedback vertex.

In the case that u is a non-feedback vertex in D_1 (and thus non-feedback in D), let D_2' be the subtournament induced by $V(D) - V_1$ (which we can visualize by v __ u V_2), with the restriction of σ to V_1 and $V(D_2')$ be given by σ_1' and σ_2', respectively. Since u is a non-feedback vertex, D_2 differs from D_2' only by the length of its non-feedback interval, so $M_{\sigma_2}(D_2) = M_{\sigma_2'}(D_2')$. Since every vertex in V_1 dominates every vertex in $V(D_2')$, Theorem 3.3 implies $\det(M_\sigma(D)) = \det(M_{\sigma_1'}(T_1))\det(M_{\sigma_2'}(D_2')$. We apply Corollary 3.5 to get $\det(M_{\sigma_1}(D_1)) = \det(M_{\sigma_1'}(T_1))$. We get

$$\det(M_\sigma(D)) = \det(M_{\sigma_1'}(T_1))\det(M_{\sigma_2'}(D_2')) = \det(M_{\sigma_1}(D_1))\det(M_{\sigma_2}(D_2))$$

which proves the case. The proof is similar when v is a non-feedback vertex.

This leaves us with the case that u is a tail feedback vertex and v is a head feedback vertex. We begin the construction of $M_\sigma(D)$ by toggling the vertices in V_1, which proceeds exactly as if we were to toggle $V_1 \subseteq V(D_1)$ separately from D. If x is the sum of the number of times we toggle the vertices in V_1, the next vertex v begins with label x and is toggled x_v times. By Lemma 2.2(3), after toggling the non-feedback vertex that follows v, the label of u is $\lambda(u) - \sum_{w\in H_u} x_w = -\sum_{w\in H_u} x_w$, where H_u is the set of all vertices in $V(D_1)$ that u dominates. This is identical to the effect of toggling the first non-feedback vertex of I for D_1 separate from D. Thus, the labels of $V(D_1)$ as linear combinations of x_w, $w \in H_{\sigma_1}(D_1)$ is the same for the lights out game on D and the lights out game one D_1, and the toggles from V_2 have no effect on $V(D_1)$. This gives $M_\sigma(D)$ the form $\begin{bmatrix} M_{\sigma_1}(D_1) & \mathbf{0} \\ * & * \end{bmatrix}$. Furthermore, after all vertices of I have been toggled, all vertices of $p \in V_2$ have label $\lambda(p) - x_v$ if p dominates v or $\lambda(p)$ otherwise. These labels are identical to the result of toggling all vertices in $\{v\} \cup I$ in $V(D_2)$ separately from D.

Next, u is toggled $t_u = \sum_{w\in H_u} x_w$ times, leaving the vertices $p \in V_2$ with label $\lambda(p) - x_v + t_u$ if the vertex dominates v or $\lambda(p) + t_u$ otherwise. This is equivalent to what we would get toggling the vertices in D_2 with initial labeling $\lambda' : V(D_2) \to \mathbb{Z}_r$ given by

$$\lambda'(p) = \begin{cases} \lambda(p), & p \in \{v\} \cup I \\ \lambda(p) + t_u, & p \in V_2 \end{cases}$$

Since both 0 and t_u are linear combinations of x_w for $w \in V(D_1)$, we can apply Lemma 3.2 and argue as in Theorem 3.3 to show that $M_\sigma(D)$ has the form $\begin{bmatrix} M_{\sigma_1}(D_1) & \mathbf{0} \\ * & M_{\sigma_2}(D_2) \end{bmatrix}$. Thus, Proposition 1.2 implies $\det(M_\sigma(D)) = \det(M_{\sigma_1}(D_1))\det(M_{\sigma_2}(D_2))$, which proves (1).

For (2), we proceed similarly as in (1). The cases where one or more of u and v is non-feedback follows similarly as in (1), which leaves the case where u is a tail feedback vertex and v is a head feedback vertex. Recall that we can visualize $V(D)$, in order, by $V_1 \_\_ \, v \; u \; \_\_ \, V_2$, where each "$\_\_$" represents a non-feedback interval. For each $w \in H_\sigma(D)$, let x_w be the number of times w is toggled. Also, let x be the sum of the numbers of toggles of all vertices in V_1, and again let H_u be the set of vertices in $V(D_1)$ that u dominates. As we toggle the vertices in order, the vertices in V_1 and the non-feedback interval preceding v are toggled identically to $V(D_1)$ separate from D. Furthermore, right before v is toggled, all vertices in $V(D_2)$ have their original labels since the first vertex in the non-feedback interval before v is toggled $-x$ times, cancelling the x toggles from V_1. Also, u has label $\lambda(u) - \sum_{w\in H_u} x_w = -\sum_{w\in H_u} x_w$ by Lemma 2.2(3). After v is toggled x_v times, u has label $x_v - \sum_{w\in H_u} x_w$. Note that $-\sum_{w\in H_u} x_w$ is precisely the label u would have from toggling $V(D_1)$ separately from D. So when u is toggled $-x_v + \sum_{w\in H_u} x_w$ times, the resulting labels on the vertices dominated by u are the same as they would be for D_1 by itself plus $-x_v$. Furthermore, none of the remaining vertices in D affect the labels of $V(D_1)$. We conclude that $M_\sigma(D)$ has the form $\begin{bmatrix} M_{\sigma_1}(D_1) & A \\ * & * \end{bmatrix}$, where the entries of A are -1 in the first column in rows that correspond to the vertices in $V(D_1)$ dominated by v and zero otherwise.

After u is toggled, we get the label of the non-feedback vertex following u by adding all toggles of vertices that precede it. We get $x + (-x) + x_v + (-x_v + \sum_{w \in H_u} x_w) = \sum_{w \in H_u} x_w$. Thus, we toggle it $-\sum_{w \in H_u} x_w$ times. For $p \in I_2 \cup V_2$, this sets the label of p to $\lambda(p) - x_v$ if p dominates v and $\lambda(p)$ otherwise. These are precisely the labels we would get playing the game on D_2 after toggling v and the following non-feedback vertex. We then toggle the remaining vertices exactly as they would be toggled if we considered D_2 separately from D (i.e. the vertices of D_1 contribute no matrix entries to the lower left block). This gives $M_\sigma(D)$ the form $\begin{bmatrix} M_{\sigma_1}(D_1) & A \\ 0 & M_{\sigma_2}(D_2) \end{bmatrix}$. It follows that $\det(M_\sigma(D)) = \det(M_{\sigma_1}(D_1)) \det(M_{\sigma_2}(D_2))$, completing the proof. $\square$

4 Feedback Matrices for Some Upset Tournaments

An *upset tournament* on n vertices is generally defined as any tournament with $n \geq 4$ whose score-list is $\{1, 1, 2, 3, \ldots, n-4, n-3, n-2, n-2\}$. It was shown in [PS98] that a tournament is an upset tournament if and only if it is strong and has a feedback arc set that arc-induces a directed path. Here we allow upset tournaments with $n = 3$ such that the resulting feedback arc set arc-induces a directed path between the last and first vertices. We exclude $n = 2$ since it gives us a transitive tournament. In this section, we determine feedback matrices for certain upset tournaments.

We analyze when upset tournaments are r-AW by computing the determinant of a suitable feedback matrix. Our approach to this computation depends on whether the feedback arcs are all between nonconsecutive vertices, all between consecutive vertices, or between a combination of consecutive and nonconsecutive vertices in the vertex ordering. For the first two cases, we have the following definitions.

Definition. Consider the following tournaments with m feedback vertices.

1. If $m \geq 2$, we define P_m^{nc} to be any tournament such that there exists a vertex ordering σ of $V(P_m^{nc})$ such that $A_\sigma(P_m^{nc})$ arc-induces a directed path on m vertices and each feedback arc is between nonconsecutive vertices with respect to σ.

2. If $m \geq 3$, we define P_m^c to be the tournament on m vertices such that there exists a vertex ordering $\sigma = v_1, v_2, \ldots, v_m$ of $V(P_m^c)$ with $A_\sigma(T) = \{v_{i+1}v_i : 1 \leq i \leq m-1\}$.

There are infinitely many tournaments that can be P_m^{nc} for each $m \geq 2$, but they differ only in the length of their non-feedback intervals. Thus, their feedback matrices are identical. In [DP], while we had not yet developed the language and supporting theorems of feedback matrices, our work essentially determined that $M_\sigma(P_m^{nc})$ is the $(m-1) \times (m-1)$

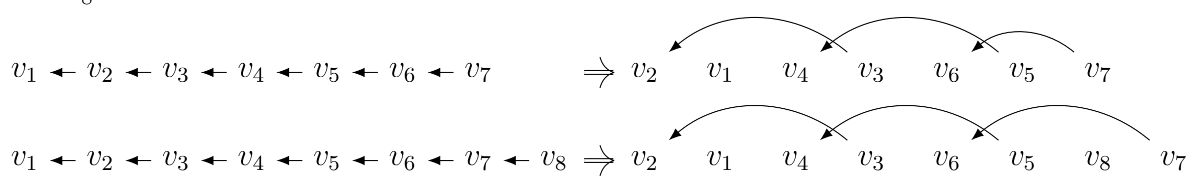
matrix given as follows: $M_\sigma(P_2^{nc}) = [2]$, $M_\sigma(P_3^{nc}) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, and for $m \geq 4$,

$$M_\sigma(P_m^{nc}) = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Recall the Fibonacci sequence, where $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. We get the following.

Theorem 4.1. [DP, Thm. 5.2] Let $m \geq 2$. Then $\det(M_\sigma(P_m^{nc})) = F_{m+1}$, and so P_m^{nc} is r-AW if and only if $\gcd(r, F_{m+1}) = 1$.

For each $m \geq 3$, P_m^c is unique up to isomorphism. The vertex ordering given in the definition does not yield a minimum feedback arc set. We get a minimum feedback arc set by swapping the order of v_{2i-1} and v_{2i} for each $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, as shown below with P_7^c and P_8^c.



In general, if we relabel the subscripts to reflect the new vertex ordering σ_{min}, we get $A_{\sigma_{min}}(P_{2k+2}^c) = \{v_{2i}v_{2i-3} : 2 \leq i \leq k+1\}$ and $A_{\sigma_{min}}(P_{2k+1}^c) = \{v_{2k+1}v_{2k-1}\} \cup \{v_{2i}v_{2i-3} : 2 \leq i \leq k\}$. We use σ_{min} when computing feedback matrices and their determinants in Theorems 4.4 and 5.5.

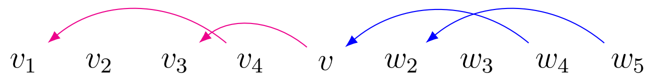
The remaining upset tournaments have a mix of consecutive and nonconsecutive feedback arcs. The following construction helps us to articulate this, combining the ideas of concatenation of graphs (see e.g. [KA90]) and the disjoint union of tournaments in [ZG23].

Definition. Let T and T' be vertex disjoint tournaments with vertex orderingss $\sigma = v_1, v_2, \ldots, v_m$ and $\sigma' = w_1, w_2, \ldots, w_n$, respectively. The *ordered vertex concatenation of T and T'* is the tournament $T * T'$ satisfying the following.

- $V(T * T')$ is the result of taking $V(T) \cup V(T')$ and identifying v_m and w_1 as one vertex v.
- The vertex ordering on $V(T * T')$, unless otherwise specified, is given by $\sigma_c = v_1, v_2, \ldots, v_{m-1}, v, w_2, \ldots, w_n$.
- $A_{\sigma_c}(T * T') = A_\sigma(T) \cup A_{\sigma'}(T')$.

Example 4.2. Let T and T' be the following tournaments.

To construct $T * T'$, we identify v_5 and w_1 into one vertex v, and keep all feedback arcs from T and T', as follows.



Note that the arc v_5v_3 becomes vv_3, and the arc w_4w_1 becomes w_4v. Also, each v_i for $i \leq 4$ dominates each w_j for $j \geq 2$.

For any upset tournament T, let σ be a vertex ordering such that $A_\sigma(T)$ is a directed path. If either the first two or last two vertices in σ form a feedback arc, we can switch their order to make the first and last feedback arcs be between nonconsecutive vertices. Thus, we can write $T = P^{nc}_{m_0} * P^c_{n_1} * P^{nc}_{m_1} * \cdots * P^c_{n_t} * P^{nc}_{m_t}$ for $m_i, n_i \geq 2$.

If we have some $n_i = 2$, let $D_1 = P^{nc}_{m_0} * P^c_{n_1} * \cdots * P^{nc}_{m_{i-1}}$ and $D_2 = P^{nc}_{m_i} * P^c_{n_{i+1}} * \cdots * P^{nc}_{m_t}$. If we switch the order of the two vertices in $P^c_{n_i}$, the resulting tournament satisfies the hypotheses of Theorem 3.6(2). The following is immediate.

Corollary 4.3. Let D_1 and D_2 be upset tournaments with vertex orderings σ_1 and σ_2, respectively, and let σ be the vertex ordering on $D_1 * P^c_2 * D_2$ that is identical to σ_c except the order of the vertices of P^c_2 are switched. If the first and last feedback arcs of each D_i are between nonconsecutive vertices, then $\det(M_\sigma(D_1 * P^c_2 * D_2)) = \det(M_{\sigma_1}(D_1)) \det(M_{\sigma_2}(D_2))$.

We now turn our attention back to $M_{\sigma_{min}}(P^c_m)$, where σ_{min} is the vertex ordering giving a minimum feedback arc set. As suggested by the examples following the definition of σ_{min}, we treat the cases of m odd and m even slightly differently. The matrices we eventually prove to be $M_{\sigma_{min}}(P^c_m)$ can be defined as follows. Let $k = \lfloor \frac{m-1}{2} \rfloor$, and define C_m to be the following $k \times k$ matrices. For $m \leq 8$ (i.e. $k \leq 3$),

$$C_3 = C_4 = [2], \quad C_5 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix},$$

$$C_7 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \quad C_8 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

For $k \geq 4$,

$$C_{2k+1} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 1 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 1 & -1 & 2 \end{bmatrix}, \quad C_{2k+2} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 1 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 1 & 0 & 2 \end{bmatrix}$$

Theorem 4.4. For all $m \geq 3$, $M_{\sigma_{min}}(P_m^c) = C_m$.

Proof. We begin our computation of $M_{\sigma_{min}}(P_m^c)$ with the case of m even, so $m = 2k+2$ for some $k \geq 1$. It is easy to check individually that the theorem holds for $k \leq 3$ (i.e. $m \leq 8$). For $k \geq 4$, the head feedback vertices are $\{v_{2i+1} : 0 \leq i \leq k-1\}$. Let λ be a labeling on $V(P_m^c)$ that is zero on $V(P_m^c) - H_{\sigma_{min}}(P_m^c)$, and let x_{2i+1} be the number of times v_{2i+1} is toggled for each $0 \leq i \leq k-1$. The tail feedback vertices are $\{v_{2i} : 2 \leq i \leq k+1\}$. That leaves v_2 and v_{2k+1} as the non-feedback vertices. Here is what the first 8 vertices look like.

$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8$

After v_1 is toggled x_1 times (giving it a label of $\lambda(v_1)+x_1$), Lemma 2.2(1) implies that v_2 must be toggled $-x_1$ times. This brings all vertices after v_2 except v_4 back to their original labels. After v_3 is toggled x_3 times, the label of v_4 is obtained by adding all of these toggles except x_1 (since v_1 does not dominate v_4), leaving v_4 with a label of $-x_1+x_3$. By Lemma 2.2(1), v_4 is toggled $x_1 - x_3$ times, leaving v_1 with label $\lambda(v_1) + 2x_1 - x_3$ and v_5 with label $\lambda(v_5) + x_1$. Similarly, after v_5 is toggled x_5 times, v_6 has label $x_1 - x_3 + x_5$, and so it is toggled $-x_1 + x_3 - x_5$ times. We now have v_3 with label $\lambda(v_3) - x_1 + 2x_3 - x_5$, v_5 with label $\lambda(v_5) + x_1 + x_5$, and v_7 with label $\lambda(v_7) + x_3$. After v_7 is toggled x_7 times to get a label of $\lambda(v_7) + x_3 + x_7$, v_8 has label $x_3 - x_5 + x_7$ and is thus toggled $-x_3 + x_5 - x_7$ times. This leaves v_5 with label $x_1 - x_3 + 2x_5 - x_7$ (notice the coefficients 1, -1, 2, -1, which are consistent with C_m). By an induction argument, for $4 \leq i \leq k$, we can assume that right before we toggle v_{2i}, v_{2i-3} has label $\lambda(v_{2i-3}) + x_{2i-7} + x_{2i-3}$, v_{2i-2} has label 0, v_{2i-1} has label $\lambda(v_{2i-1}) + x_{2i-5} + x_{2i-1}$, and v_{2i} has label $x_{2i-5} - x_{2i-3} + x_{2i-1}$. Also, for the label of every v_j with $j > 2i$ every toggle at this point has been canceled out except for x_{2i-5} (from v_{2i-2}) and x_{2i-1} (from v_{2i-1}). Then v_{2i} is toggled $-x_{2i-5} + x_{2i-3} - x_{2i-1}$ times, leaving v_{2i-3} with label $\lambda(v_{2i-3}) + x_{2i-7} - x_{2i-5} + 2x_{2i-3} - x_{2i-1}$. After v_{2i+1} is toggled x_{2i+1} times, we add the uncanceled toggles from v_{2i-2}, v_{2i}, and v_{2i+1} to give v_{2i+2} a label of $x_{2i-5} - x_{2i-5} + x_{2i-3} - x_{2i-1} + x_{2i+1} = x_{2i-3} - x_{2i-1} + x_{2i+1}$, completing the induction. The last four vertices look like the following.

$v_{2k-1} \quad v_{2k} \quad v_{2k+1} \quad v_{2k+2}$

By our induction argument, after v_{2k-1} is toggled, v_{2k} has label $x_{2k-5} - x_{2k-3} + x_{2k-1}$ and v_{2k-1} has label $\lambda(v_{2k-1}) + x_{2k-5} + x_{2k-1}$. After v_{2k} is toggled $-x_{2k-5} + x_{2k-3} - x_{2k-1}$ times, v_{2k-3} has label $\lambda(v_{2k-3}) + x_{2k-7} - x_{2k-5} + 2x_{2k-3} - x_{2k-1}$ and the non-feedback vertex v_{2k+1} has label x_{2k-3}. Then v_{2k+1} is toggled $-x_{2k-3}$ times. This leaves v_{2k+2} with label $-x_{2k-1}$. Then v_{2k+2} is toggled x_{2k-1} times to obtain label 0, which leaves v_{2k-1} with label $\lambda(v_{2k-1}) + x_{2k-5} + 2x_{2k-1}$. Setting all labels to 0, we get the following system of equations.

$$\lambda(v_1) + 2x_1 - x_3 = 0, \quad \lambda(v_3) - x_1 + 2x_3 - x_5 = 0, \quad \lambda(v_{2k-1}) + x_{2k-5} + 2x_{2k-1} = 0$$

$$\lambda(x_{2i+1}) + x_{2i-3} - x_{2i-1} + 2x_{2i+1} - x_{2i+3} = 0 \text{ for } (2 \leq i \leq k-2)$$

The matrix for this system of equations is precisely C_{2k+2} for $k \geq 4$. This proves the even case.

Only the case of $n = 2k + 1$, $k \geq 1$, remains. It is easy to check the cases for $k \leq 3$. For $k \geq 4$, the proof is identical to the even case up to the toggling of v_{2k}. After this, the only difference is that v_{2k+1} (not v_{2k+2}) dominates v_{2k-1}, and so there is no non-feedback vertex before v_{2k+1}. So right before v_{2k+1} is toggled, it has label $x_{2k-3} - x_{2k-1}$ and v_{2k-1} has label $x_{2k-5} + x_{2k-1}$. Then x_{2k+1} is toggled $-x_{2k-3} + x_{2k-1}$ times, giving x_{2k-1} a final label of $\lambda(v_{2k-1}) + x_{2k-5} - x_{2k-3} + 2x_{2k-1}$. So all rows of the feedback matrix are identical to those of $M_{\sigma'}(P^c_{2k+2})$ except the last row, which with its extra entry of -1 makes the feedback matrix C_{2k+1}. □

5 Winnability of the Lights Out Game on P^c_n

In this section, we find the determinants of C_m for $m \geq 3$, which then tell us when P^c_m is r-AW. Note that while we generally consider the entries of matrices to be in $\mathbb{Z}_r$ for $r \geq 2$, we are only interested in whether or not the determinant is a unit in $\mathbb{Z}_r$. Thus, we can consider the entries of the matrices to be integers.

Here are some sequences of integers that will be helpful to us as we work with the matrices.

Definition. Consider the sequences $\{\alpha_j\}$, $\{\beta_j\}$, $\{\gamma_j\}$, and $\{\gamma'_j\}$ defined as follows.

1. Let $\alpha_{-3} = 0$, $\alpha_{-2} = -1$, $\alpha_{-1} = -1$, and $\alpha_j = \alpha_{j-1} - 2\alpha_{j-2} + \alpha_{j-3}$ for all $j \geq 0$.

2. Let $\beta_0 = 0$, $\beta_1 = -1$, $\beta_2 = 0$, and $\beta_j = \beta_{j-1} - 2\beta_{j-2} + \beta_{j-3}$ for all $j \geq 0$.

3. Let $\gamma_0 = 1$, $\gamma_1 = 2$, $\gamma_2 = 4$, and $\gamma_j = 2\gamma_{j-1} - \gamma_{j-2} + \gamma_{j-3}$ for all $j \geq 3$.

4. Let $\gamma'_0 = 1$, $\gamma'_1 = 2$, $\gamma'_2 = 3$, and $\gamma'_j = 2\gamma'_{j-1} - \gamma'_{j-2} + \gamma'_{j-3}$ for all $j \geq 3$.

All of the above sequences appear in [Slo]. The sequence $\{\alpha_j\}$ is the negative of the sequence A077954; $\{\beta_j\}$ is the negative of the sequence A078019; $\{\gamma_j\}$ is the sequence A005251; and $\{\gamma'_j\}$ is the sequence A005314.

The following lemma lists some identities for $\{\alpha_j\}$ and $\{\beta_j\}$. All identities involved satisfy the recurrence relation $x_j = x_{j-1} - 2x_{j-2} + x_{j-3}$ (including linear combinations of α_j and β_j since the recurrence relation is linear and homogeneous). Therefore, it is only necessary to verify that the identities are true for the first three values of j (e.g. $j = -1, 0, 1$ for (1)). These are easily checked.

Lemma 5.1. Let $\{\alpha_j\}$ and $\{\beta_j\}$ be defined as above.

1. For $j \geq -1$, $\alpha_j = \beta_{j+2} - \beta_{j+1}$.

2. For $j \geq 1$, $\alpha_j = \beta_{j-1} - 2\beta_j$.

3. For $j \geq 1$, $\beta_j = \alpha_{j-3} - \alpha_{j-4}$.

4. For $j \geq 3$, $\beta_j = \alpha_{j-6} - 2\alpha_{j-5}$

We have the following identities for $\{\gamma_j\}$ and $\{\gamma'_j\}$.

Lemma 5.2. For all $j \geq 0$,

1. $\gamma_j = \alpha_{j-1}\beta_{j+1} - \alpha_j\beta_j = \alpha_j\alpha_{j-1} - \beta_{j+3}\beta_{j+1}$.
2. $\gamma'_j = \beta_{j+1}\alpha_{j-2} - \beta_{j+2}\alpha_{j-3} = \beta_{j+1}^2 - \alpha_j\alpha_{j-3}$.

Proof. We begin with (1). Using Lemma 5.1, we substitute $\beta_{j+3} = \alpha_j - \alpha_{j-1}$ and $\alpha_{j-1} = \beta_{j+1} - \beta_j$ into $\alpha_j\alpha_{j-1} - \beta_{j+3}\beta_{j+1}$ to get $\alpha_{j-1}\beta_{j+1} - \alpha_j\beta_j$. This shows the last two quantities in (1) are equal. One can easily check that (1) holds for $j = 0, 1, 2$. It suffices to show that $x_j = \beta_{j+1}\alpha_{j-2} - \beta_{j+2}\alpha_{j-3}$ satisfies $x_j = 2x_{j-1} - x_{j-2} + x_{j-3}$ (i.e. the same recurrence relation as γ_j). We can substitute the recursive part of the definition for β_{j+1} and α_j to get

$$\begin{aligned}
x_j &= \alpha_{j-1}\beta_{j+1} - \alpha_j\beta_j \\
&= \alpha_{j-1}[\beta_j - 2\beta_{j-1} + \beta_{j-2}] - [\alpha_{j-1} - 2\alpha_{j-2} + \alpha_{j-3}]\beta_j \\
&= 2[\alpha_{j-2}\beta_j - \alpha_{j-1}\beta_{j-1}] + \alpha_{j-1}\beta_{j-2} - \alpha_{j-3}\beta_j \\
&= 2x_{j-1} + \alpha_{j-1}\beta_{j-2} - \alpha_{j-3}\beta_j
\end{aligned}$$

We substitute the recursive definition for α_{j-1} and β_j to get

$$\begin{aligned}
x_j &= 2x_{j-1} + [\alpha_{j-2} - 2\alpha_{j-3} + \alpha_{j-4}]\beta_{j-2} \\
&\quad - \alpha_{j-3}[\beta_{j-1} - 2\beta_{j-2} + \beta_{j-3}] \\
&= 2x_{j-1} + \alpha_{j-2}\beta_{j-2} - 2\alpha_{j-3}\beta_{j-2} + \alpha_{j-4}\beta_{j-2} \\
&\quad - \alpha_{j-3}\beta_{j-1} + 2\alpha_{j-3}\beta_{j-2} - \alpha_{j-3}\beta_{j-3} \\
&= 2x_{j-1} + [\alpha_{j-2}\beta_{j-2} - \alpha_{j-3}\beta_{j-1}] + [\alpha_{j-4}\beta_{j-2} - \alpha_{j-3}\beta_{j-3}] \\
&= 2x_{j-1} - x_{j-2} + x_{j-3}
\end{aligned}$$

This gives us (1), and (2) follows similarly. □

We find the determinants for the C_m, for the most part, by row-reduction. To manage this, we will use sequences of matrix entries to keep track of how many rows in the echelon form have been established and what entries lie on the main diagonal at the end.

We begin with the case $m \geq 9$. Let $k = \lfloor \frac{n-1}{2} \rfloor$, so $k \geq 4$ and C_m is $k \times k$. For each $3 \leq j \leq k-1$, we move row j up two rows (which leaves the determinant unchanged). We then use elementary row operations to shift the leading entries of the two rows that are moved down one column to the right. Here is what this looks like in general for rows $j-2$, $j-1$, and j. Only columns with nonzero entries are shown.

$$\begin{bmatrix} a_{j-1} & b_{j-1} & c_{j-1} & 0 \\ a'_{j-1} & b'_{j-1} & c'_{j-1} & 0 \\ 1 & -1 & 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 \\ a_{j-1} & b_{j-1} & c_{j-1} & 0 \\ a'_{j-1} & b'_{j-1} & c'_{j-1} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & b_{j-1} + a_{j-1} & c_{j-1} - 2a_{j-1} & a_{j-1} \\ 0 & b'_{j-1} + a'_{j-1} & c'_{j-1} - 2a'_{j-1} & a'_{j-1} \end{bmatrix} \tag{1}$$

The third matrix is obtained by adding $-a_{j-1}$ times the first row to the second row and $-a'_{j-1}$ times the first row to the third row. Note that the indices of the sequences match the row with entries a'_{j-1}, b'_{j-1}, and c'_{j-1}. Moreover, the first two rows of C_m give us $a_2 = 2$, $b_2 = -1$, $c_2 = 0$, $a'_2 = -1$, $b'_2 = 2$, and $c'_2 = -1$. Since the nonzero entries in rows $j-1$ and j of the last matrix above correspond to the entries in the sequence with index j, we get the following.

$$\begin{array}{lll} a_j = b_{j-1} + a_{j-1}, & b_j = c_{j-1} - 2a_{j-1}, & c_j = a_{j-1} \\ a'_j = b'_{j-1} + a'_{j-1}, & b'_j = c'_{j-1} - 2a'_{j-1}, & c'_j = a'_{j-1} \end{array} \tag{2}$$

We use the above to prove the following identities.

Lemma 5.3. Let $\{a_j\}$, $\{b_j\}$, $\{c_j\}$, $\{a'_j\}$, $\{b'_j\}$, and $\{c'_j\}$ be defined as above. For all $j \geq 2$, $a_j = \beta_{j+1}$, $b_j = \alpha_j$, $c_j = \beta_j$, $a'_j = \alpha_{j-3}$, $b'_j = \beta_{j+1}$, and $c'_j = \alpha_{j-4}$.

Proof. By applying the row-reduction algorithm from Equation 1 above twice to C_m, it is easy to check that the identities are true for $j = 2, 3, 4$. It suffices to prove that each of $\{a_j\}$, $\{b_j\}$, $\{c_j\}$, $\{a'_j\}$, $\{b'_j\}$, and $\{c'_j\}$ satisfy the recurrence relation $x_j = x_{j-1} - 2x_{j-2} + x_{j-3}$ for all $j \geq 5$.

By Equations 2, we have $a_j = b_{j-1} + a_{j-1}$, $b_{j-1} = c_{j-2} - 2a_{j-2}$, and $c_{j-2} = a_{j-3}$. By substituting the third of these equations into the second equation, substituting the second equation into the first equation, and rearranging terms, we get $a_j = a_{j-1} - 2a_{j-2} + a_{j-3}$. Similarly, $a'_j = a'_{j-1} - 2a'_{j-2} + a'_{j-3}$. Since $c_j = a_{j-1}$ and $c'_j = a'_{j-1}$, it is clear that $\{c_j\}$ and $\{c'_j\}$ satisfy the recurrence relation. Finally, the second and fifth equations in Equations 2 show that $\{b_j\}$ and $\{b'_j\}$ are linear combinations of $\{a_{j-1}\}$, $\{c_{j-1}\}$, $\{a'_{j-1}\}$, and $\{c'_{j-1}\}$. Since the recurrence relation is linear homogeneous, that implies $\{b_j\}$ and $\{b'_j\}$ satisfy the recurrence relation as well. This completes the proof. $\square$

By applying the row operations through row $k-1$ for C_m, we get the following.

Lemma 5.4. Let $k \geq 4$. Then C_{2k+2} can be row reduced in a way that does not change the determinant to a matrix of the form

$$\left[\begin{array}{c|ccc} T'_{k-3} & * & * & * \\ \hline 0 & \beta_k & \alpha_{k-1} & \beta_{k-1} \\ 0 & \alpha_{k-4} & \beta_k & \alpha_{k-5} \\ 0 & 1 & 0 & 2 \end{array}\right]$$

and similarly with C_{2k+1} to a matrix of the form

$$\left[\begin{array}{c|ccc} T'_{k-3} & * & * & * \\ \hline 0 & \beta_k & \alpha_{k-1} & \beta_{k-1} \\ 0 & \alpha_{k-4} & \beta_k & \alpha_{k-5} \\ 0 & 1 & -1 & 2 \end{array}\right]$$

where T'_{k-3} is an $(k-3) \times (k-3)$ upper triangular matrix with 1's on the main diagonal.

We can use the above to get the following determinants.

Theorem 5.5. Let $k \geq 1$.

1. $\det(C_{2k+2}) = \gamma_k$.
2. $\det(C_{2k+1}) = \gamma'_k$.

Proof. It is easy to check the determinants in both cases for $k \in \{0, 1, 2, 3\}$. For $k \geq 4$, we use Lemma 5.4, the fact that $\det(T'_{k-3}) = 1$, and Proposition 1.2 to get

$$\det(C_{2k+2}) = \det(T'_{k-3}) \det\left(\begin{bmatrix} \beta_k & \alpha_{k-1} & \beta_{k-1} \\ \alpha_{k-4} & \beta_k & \alpha_{k-5} \\ 1 & 0 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \beta_k & \alpha_{k-1} & \beta_{k-1} \\ \alpha_{k-4} & \beta_k & \alpha_{k-5} \\ 1 & 0 & 2 \end{bmatrix}\right)$$

$$\det(C_{2k+1}) = \det(T'_{k-3}) \det\left(\begin{bmatrix} \beta_k & \alpha_{k-1} & \beta_{k-1} \\ \alpha_{k-4} & \beta_k & \alpha_{k-5} \\ 1 & -1 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \beta_k & \alpha_{k-1} & \beta_{k-1} \\ \alpha_{k-4} & \beta_k & \alpha_{k-5} \\ 1 & -1 & 2 \end{bmatrix}\right)$$

We use the above and Lemma 5.1 to get

$$\begin{aligned}
\det(C_{2k+2}) &= [2\beta_k^2 + \alpha_{k-1}\alpha_{k-5}] - [\beta_{k-1}\beta_k + 2\alpha_{k-1}\alpha_{k-4}] \\
&= [2\beta_k - \beta_{k-1}]\beta_k + \alpha_{k-1}[\alpha_{k-5} - 2\alpha_{k-4}] \\
&= -\alpha_k\beta_k + \alpha_{k-1}\beta_{k+1} = \gamma_k \\
\det(C_{2k+1}) &= [2\beta_k^2 + \alpha_{k-1}\alpha_{k-5} - \beta_{k-1}\alpha_{k-4}] \\
&\quad - [\beta_{k-1}\beta_k + 2\alpha_{k-1}\alpha_{k-4} - \beta_k\alpha_{k-5}] \\
&= [\alpha_{k-1}\alpha_{k-5} - 2\alpha_{k-1}\alpha_{k-4} + \beta_k\alpha_{k-5}] \\
&\quad + [-\beta_{k-1}\alpha_{k-4} - \beta_{k-1}\beta_k + 2\beta_k^2] \\
&= [\alpha_{k-1}\alpha_{k-5} - 2\alpha_{k-1}\alpha_{k-4} + \beta_k\alpha_{k-5} - \mathbf{2\beta_k\alpha_{k-4}}] \\
&\quad + [-\beta_{k-1}\alpha_{k-4} - \beta_{k-1}\beta_k + 2\beta_k^2 + \mathbf{2\beta_k\alpha_{k-4}}] \\
&= [\alpha_{k-1} + \beta_k][\alpha_{k-5} - 2\alpha_{k-4}] \\
&\quad - [\beta_{k-1} - 2\beta_k][\alpha_{k-4} + \beta_k] \\
&= \beta_{k+1}^2 - \alpha_k\alpha_{k-3} = \gamma'_k
\end{aligned}$$

□

By using Lemma 4.4, the following is immediate.

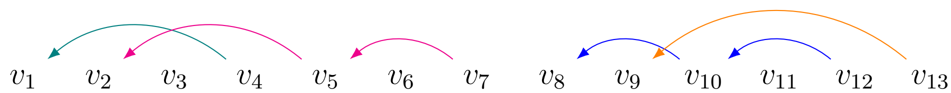
Corollary 5.6. Let $r \geq 2$, $k \geq 1$.

1. P^c_{2k+2} is r-AW if and only if $\gcd(r, \gamma_k) = 1$.
2. P^c_{2k+1} is r-AW if and only if $\gcd(r, \gamma'_k) = 1$.

6 When $A_\sigma(D)$ is a Union of Directed Paths

Our last result is a quick application of Corollary 3.4 and Theorem 3.6 to tournaments whose feedback arc sets are certain vertex-disjoint unions of directed paths. An example of such a tournament is given below.

Example 6.1.



As the above example shows, there are many ways the paths can interact with each other, from no interaction at all (e.g. the magenta and blue paths) to having arcs in one path surrounded by arcs in another path (e.g. the blue and orange paths). We focus on cases where the interaction is minimal. To describe more precisely what we mean by "interaction", we need to develop some terminology.

Definition. Let D be a digraph with vertex ordering σ such that $A_\sigma(D)$ is a disjoint union of directed paths.

- Let P be a directed path in $A_\sigma(D)$ whose first and last vertices with respect to σ are v and w. We say that P *encloses* the interval between v and w, inclusive, and we denote the interval between v and w by I_P.
- Two directed paths P and P' are *overlapping* if $I_P \cap I_{P'} \neq \emptyset$. Otherwise, P and P' are *non-overlapping*.
- Two directed paths in $A_\sigma(D)$ are *minimally overlapping* if the initial arc wv of one of the paths and the terminal arc $w'v'$ of the other path satisfy $v < v' < w < w'$ under σ.
- If P is a directed path in $A_\sigma(D)$, we use T_P to denote the tournament whose vertices are the vertices enclosed by P and whose feedback arc set is $A(P)$.
- Let P and P' be two directed paths in $A_\sigma(D)$ that do not have the same terminal vertex. We say $P < P'$ if the terminal vertex of P comes before the terminal vertex of P' under σ.

In Example 6.1, the green path encloses the interval between v_8 and v_{12}. The blue and magenta paths are minimally overlapping, but no other two directed paths are. The order of the paths is *blue* $<$ *magenta* $<$ *green* $<$ *orange*.

Suppose we have a tournament T with vertex ordering σ such that $A_\sigma(T)$ is a disjoint union of directed paths that are totally ordered under $<$, each consecutive pair of these directed paths is minimally overlapping or non-overlapping, and every pair of non-consecutive directed paths is non-overlapping. There are three ways that consecutive paths P and P' (with P preceding P') can have overlapping arcs.

1. P and P' can be non-overlapping.

2. P and P' have overlapping arcs, and there is a non-feedback interval between the terminal vertex of P' and the initial vertex of P.

3. P and P' have overlapping arcs, the terminal vertex v of P' and the initial vertex u of P are consecutive, and there are non-feedback intervals immediately before and after v and w.

4. P and P' have overlapping arcs, and one or the other of the initial vertex of P or the terminal vertex of P' both immediately precedes and is immediately preceded by feedback vertices.

For (1), P and P' are in different strong components, so we can apply Corollary 3.4. For (2) and (3), we have the following for the initial arc of P and the terminal arc of P', where the horizontal lines represent non-feedback intervals. For both of these, we can apply Theorem 3.6.

Finally, for (4), we get something that resembles the minimum feedback ordering for P_m^c, where m is odd (see the visual representation of P_7^c and P_8^c following Theorem 4.1 for an example). This is a more difficult case, which we avoid for our next theorem. If we assume that either (1), (2) or (3) hold and apply Corollary 3.4 and Theorem 3.6, we get the following.

Theorem 6.2. Let T be a tournament with vertex ordering σ such that $A_\sigma(T)$ is a disjoint union of directed paths $\bigcup_{i=1}^n P_i$, where $P_i < P_j$ if and only if $i < j$. Suppose also that P_i and P_{i+1} either minimally overlap or are non-overlapping for all $1 \leq i \leq n-1$, and P_i and P_j are non-overlapping for $|i-j| \geq 2$. If σ_i is the restriction of σ to T_i for each $1 \leq i \leq n$, then

1. $\det(M_\sigma(T)) = \prod_{i=1}^n \det(M_{\sigma_i}(T_{P_i}))$

2. T is r-AW if and only if each T_{P_i} is r-AW.

7 Future Work

There are at least two avenues of research that we can take from our work here.

- Upset Tournaments: Theorem 4.1, Corollary 4.3, and Corollary 5.6 determine r-AW upset tournaments when the feedback arcs are either all between nonconsecutive vertices; all between consecutive vertices; or when each feedback arc between consecutive vertices has no more than one arc. All that remains are the cases where the ordered vertex concatenation includes P_n^c with $n \geq 3$.

- Feedback Arc Sets that are Disjoint Unions of Directed Paths: Theorem 6.2 determines r-AW tournaments when the disjoint directed paths overlap in limited ways. It would be nice to at least complete the case of minimally overlapping paths. More ambitiously, we could explore more complex ways that the directed paths can interact.

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