Section 2.1 Exercises

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2.1 (a) The sentence: the integer 123 is prime, is a valid statement and it has a truth value of False.

(b) The sentence: the integer 0 is even, is a valid statement and it has a truth value of True.

(c) The sentence: Is $5 \times 2 = 10$? is not a valid statement since it is a question and is not declarative.

(d) The sentence: $x^2 - 4 = 0$ is an open sentence, however we are not given the domain for the variable x. If we assume that x can be anything in the real numbers then this statement has a truth value of True when $x = -2$ or when $x = 2$ and it has a truth value of False otherwise.

(e) The sentence: multiply $5x + 2$ by 3, is not a valid statement as it is an imperative command.

(f) The sentence: $5x+3$ is an odd integer is an open statement and if assume that the domain of x is the integers, as seems appropriate given that the sentence refers to the integers for the range of the function, then it is an open statement over the domain of $\mathbb{Z}$. This statement happens to have a truth value of True when x is even and a value of False when x is odd.

Proof. $5x + 3$ is an odd integer when x is an even integer. Since x is even it can be rewritten as $x = 2k$ for some $k \in \mathbb{Z}$. So we have $5(2k) + 3 = 10k + 3 = 2(5k) + 3 = 2(5k) + (2(1) + 1) = 2(5k) + 2 + 1 = 2(5k + 2) + 1$. If we let $l = 5k + 2$ we can rewrite $2(5k + 2) + 1$ as $2l + 1$. Since the definition of an odd integer is any integer that can be expressed as 2 times some arbitrary integer plus 1 we must conclude that the statement $5x + 3$ is an odd integer *is* true when x is an even integer. Q.E.D.

Proof. $5x + 3$ is not an odd integer when x is an odd integer. For the sake of simplicity and time we can simply prove this by finding a single counter example; it is a relatively simple matter to establish a more long-form proof, but doing so is unnecessary in this case. If we let $x = 1$ then we have that $5x + 3 = 8$ which is not an odd integer, since we cannot write 8 as 2 times some integer $k + 1$. In fact we can write 8 as 2 times 4 which is an even integer not an odd integer. Q.E.D.

(g) The sentence: "What an impossible question!" is not a valid statement as it is simply an exclamatory sentence.

2.3 (a) The statement: $\emptyset \in \emptyset$ is False as the empty set, by definition has no elements and thus cannot contain the empty set.

(b) The statement: $\emptyset \in \{ \emptyset \}$ is True, we can simply observe that the definition of being an element of a set means that that item is one of the members of the set. Reading off the list of items in the set $\{ \emptyset \}$ we can clearly see that yes $\emptyset$ is in fact in the set.

(c) The statement: $\{ 1, 3 \} = \{ 3, 1 \}$ is True, we can use the definition of set equality to see this. Specifically, sets are considered to be equal if they have the same members, since both of these sets contain the elements: 1 and 3 and no other explicit elements they are equal.

(d) The statement: $\emptyset = \{ \emptyset \}$ is False, one way to think about this is to look at the cardinality of each of these two sets. The first set namely $\emptyset$ is a set containing 0 elements, thus having a cardinality of 0. The second set namely $\{ \emptyset \}$ is a set with a single element $\emptyset$ having a cardinality of 1. So these two sets have different elements and even different numbers of elements, so given the definition of set quality which we have previously discussed; namely the requirement that the sets have the same elements these two sets cannot possibly be equal.

(e) The statement: $\emptyset \subset \{ \emptyset \}$ is True, we can observe that the definition of a subset is that a set meeting this criteria must have at least some of the elements of the superset and none that the superset doesn't have. In this case we have the $\emptyset$ or $\{\ \}$ which is a subset of *every* set by its definition.

(f) The statement: $1 \subseteq \{1\}$ is False, we can use the definition of subset to determine this. In order for something to be a subset of a set, that thing itself must be a *set* in this case the statement is asking if the integer 1 is a subset of the set containing the integer 1. This is a nonsensical question as the integer 1 is *not* a set so it cannot, by definition, be a subset of *any* sets.

2.5 Given the open sentence $P(x) : 3x - 2 > 4$ we can first observe that $3x - 2 = 4$ when $x = 2$, by some basic arithmetic namely $3x - 2 = 4$ $\rightarrow 3x = 6 \rightarrow x = 2$. This fact will allow us to determine the values of x for which this sentence is both true and false.

(a) Given the open sentence $P(x) : 3x - 2 > 4$ we can use the fact that $3x - 2 = 4$ when $x = 2$ to establish the lower-bound for the first value of x that satisfies this inequality namely $x = 3$. So the interval for values of x which satisfy this inequality is: $[3, \infty)$.

(b) Given the open sentence $P(x) : 3x - 2 > 4$ we can use the fact that $3x - 2 = 4$ when $x = 2$ to establish the upper-bound for the last value of x that does not satisfy this inequality namely $x = 2$. So the interval for values of x which do not satisfy this inequality is: $(-\infty, 2]$.

2.7 We are given $P(n) : n$ and $n+2$ are primes and are told that this is an open sentence over the domain $\mathbb{N}$. Six positive integers n that satisfy this open sentence in the domain are: $\{3, 5, 11, 17, 29, 41\}$.

2.9 We are given $S = \{3, 5, 7, 9\}$ and our task is to find some open sentence $P(n)$ over the domain S such that $P(n)$ is true for half of the integers in S and false for the other half. One such example for $P(n)$ is $P(n) : x > 6$. In this case exactly half of the elements of S satisfy this condition, namely $\{7, 9\}$ and the other half does not: $\{3, 5\}$.