Chapter 1 Exercises

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1.1 D and e are sets as they are in curly braces. The other examples have some items just listed by themselves.

1.2 (a) $\{ x \in S \mid x \in \mathbb{N} \}$
   (b) $\{ x \in S \mid x \geq 0 \}$
   (c) $\{ x \in S \mid x < 0 \}$
   (d) $\{ x \in S \mid |x| > 1 \}$

1.3 (a) $|A| = 5$
   (b) $|B| = 21$
   (c) $|C| = 50$
   (d) $|D| = 2$
   (e) $|E| = 1$
   (f) $|E| = 2$

1.4 (a) $A = \{ -3, -2, -1, 0, 1, 2, 3, 4 \}$
   (b) $B = \{ -2, -1, 0, 1, 2 \}$
   (c) $C = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$
   (d) $D = \{ 0, 1 \}$
   (e) $E = \emptyset$

1.5 (a) $A = \{ x \in \mathbb{Z} \mid x < 0 \}$
   (b) $B = \{ x \in \mathbb{Z} \mid x^2 < 10 \}$
   (c) $C = \{ x \in \mathbb{Z} \mid 0 < x^2 < 5 \}$

1.6 (a) $A = \{ 2x + 1 \mid x \in \mathbb{Z} \} = \{ \ldots, -3, -1, 1, 3, 4 \ldots \}$

(b) $B = \{ 4n \mid n \in \mathbb{Z} \} = \{ \ldots, -16, -8, -4, 0, 4, 8, 16, \ldots \}$

(c) $C = \{ 3q + 1 \mid q \in \mathbb{Z} \} = \{ \ldots, -7, -4, 1, 4, 7 \ldots \}$

1.7 (a) $A = \{ 3x + 2 \mid x \in \mathbb{Z} \}$

(b) $B = \{ 5x \mid x \in \mathbb{Z} \}$

(c) $C = \{ x^3 \mid x \in \mathbb{Z} \}$

1.8 (a) $A = \{ -4, -3, -2, 2, 3, 4 \}$

(b) $\frac{9}{4}, \frac{10}{4},$ and $\frac{21}{8}$

(c) $C = \{ 2, \sqrt{2} \}$

(d) $D = \{ 2 \}$

(e) $|A| = 6, |C| = 2,$ and $|D| = 1$

1.9 $B = \{ 5, 7, 8, 10, 13 \}$ so $C = \{ 5, 8 \}$

1.10 (a) $A = \{ 1 \}, B = \{ 1 \},$ and $C = \{ 1, 2 \}$

(b) $A = \{ 1 \}, B = \{ \{ 1 \}, 2 \},$ and $C = \{ \{ \{ 1 \}, 2 \} \}$

(c) $A = \{ 1 \}, B = \{ \{ 1 \}, 2 \},$ and $C = \{ 1, 2 \}$

1.11 The interval I can just be (a, b) since the two intervals would be equal and thus subsets. If we let $c = \dfrac{b + a}{2}$ then the interval I would be centered at c; as it would be the center of the original interval.

1.12 $A = B = C = D = E$

1.13 Skipping this because the problem is pretty straightforward and it's really complicated to draw Venn diagrams in LaTeX (aka the benefits are not worth the effort).

1.14 (a) $\mathcal{P}(A) = \{ \emptyset, \{ 1 \}, \{ 2 \}, \{ 1, 2 \} \}$ and $|\mathcal{P}(A)| = 4$

(b) $\mathcal{P}(A) = \{ \emptyset, \{ \emptyset \}, \{ 1 \}, \{ \emptyset, 1 \}, \{ \{ a \} \}, \{ \emptyset, \{ a \} \} \}, \{ 1, \{ a \} \}, \{ \emptyset, 1, \{ a \} \}$ and $|\mathcal{P}(A)| = 8$

1.15 $\mathcal{P}(A) = \{ \emptyset, \{ 0 \}, \{ \{ 0 \} \}, \{ \emptyset, \{ 0 \} \}, \{ 0, \{ 0 \} \}, \{ \emptyset, 0, \{ 0 \} \} \}$

1.16 Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality. $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$
$\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
$|\mathcal{P}(\mathcal{P}(\{1\}))| = 4$

1.17 $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{\emptyset, \{\emptyset\}\}, \{0, \{\emptyset\}\}, \{0, \emptyset, \{\emptyset\}\}\}$
$|\mathcal{P}(A)| = 8$

1.18 $A = \{\emptyset, 0, \{\emptyset\}, \{0\}\}$
$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{0\}, \{\{\emptyset\}\}, \{\{0\}\}, \{\emptyset, 0\},$
$\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{0\}\}, \{\{0\}, \{\emptyset, 0, \{\emptyset\}\}\},$
$\{\emptyset, 0, \{0\}\}, \{\emptyset, 0, \{0\}, \{\emptyset\}\}\}$
// I am stopping here this is really tedious, I know I am missing a few sets, but it isn't valuable enough to spend more time on. May come back with pen and paper.

1.19
- (a) $S = \emptyset$
- (b) $S = \{1\}$
- (c) $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$
- (d) $\{1, 2, 3, 4, 5\}$

1.20
- (a) This statement could be either true or false; we don't have enough information. Specifically, if the *only* element of A is 1, then it would be true. However if $A = \{1, \{1\}\}$ then it would be false.
- (b) *Proof.* Since we know that $A \subset \mathcal{P}(B) \subset C$ and that $|A| = 2$ then we know that $|\mathcal{P}(B)|$ must be less than $|C|$; since a proper subset must have less elements than its proper superset. So we have the following inequality $|A| < |\mathcal{P}(B)| < |C|$.

  Since the cardinality of a set must be a nonzero positive integer we know that the smallest possible cardinality for $\mathcal{P}(B)$ is 3. However, we also know that the cardinality of a power set $\mathcal{P}(K)$ must be $2^{|K|}$ for some set K. Thus, the cardinality of $\mathcal{P}(B)$ cannot be 3 (the next largest integer after 2; which is $|A|$) because $\log_2 3$ is not an integer. However, it *can* be 4. So the smallest possible cardinality of $\mathcal{P}(B) = 4$. Since $|\mathcal{P}(B)| < |C|$ the smallest possible cardinality of $C = 5$. Q.E.D.
- (c) *Proof.* We are given that $|B| = |A| + 1$. We also know that the cardinality of a power set of a set K must be $2^{|K|}$. So in this case,

$|\mathcal{P}(A)| = 2^{|A|}$ and $|\mathcal{P}(B)| = 2^{|B|} = 2^{|A+1|}$. From this equation we can observe that regardless of the cardinality of B it must always be at least 2^1 or simply 2 more than the cardinality of A. Q.E.D.

(d) *Proof.* We have four sets: $A, B, C,$ and D which are subsets of $\{1, 2, 3\}$. Furthermore, we know that $|A| = |B| = |C| = |D| = 2$. There are only 3 subsets of $\{1, 2, 3\}$ with a cardinality of 2, namely $\{1, 2\}, \{1, 3\}$, and $\{2, 3\}$. Since we have four sets all of which are subsets of $\{1, 2, 3\}$ with a cardinality of 2 we must conclude that at least one of them are the same set. Q.E.D.

1.21 One approach we can take to find a solution that satisfies all of the conditions is attempting to minimize the sums of the elements in each set. We know from (a) that A must contain 1, from (b) that A and C must contain 2, and that A must contain 3 from (c). So we know that the minimum sum of A is 6. If we choose 3 to be in C from (c) then the difference in the sums of A and C is exactly 1 which satisfies (f). So we'd like to maintain that if we can. Observing that (d) requires 4 to be in an even number of any of the sets we can choose to add it to both A and C, thus maintaining the difference of 1 for their sums. If we then choose to satisfy (e) by adding 5 to B we have satisfied all of the conditions leaving us with $B = \{1, 5\}$.

1.22 (a) $A \cup B = \{1, 5, 9, 13, 3, 15\}$

(b) $A \cap B = \{9\}$

(c) $A - B = \{1, 5, 13\}$

(d) $B - A = \{3, 15\}$

(e) $\overline{A} = \{3, 7, 11, 15\}$

(f) $A \cap \overline{B} = \{1, 5, 13\}$

1.23 To solve this problem, first observe that since $|A - B| = 3$, there must be 3 elements in A that are not in B. Similarly, since $|B - A| = 3$ there must be 3 elements in B that are not in A. Further, since $|A \cap B| = 3$, there must be 3 elements that they have in common.

We may choose any two sets that satisfy these conditions, so for the sake of simplicity let us choose $A = \{1, 2, 3, 7, 8, 9\}$ and $B = \{4, 5, 6, 7, 8, 9\}$. In this case we have that $A - B = \{1, 2, 3\}$ with a cardinality of 3 and we have $B - A = \{4, 5, 6\}$ with a cardinality of 3 and finally we have that $A \cap B = \{7, 8, 9\}$ with a cardinality of 3. (I did draw Venn diagram, but it is on paper).

1.24 *Proof.* First observe that since $B - A = C - A$ everything that is in B must be in C and vice versa except for whatever is in A. Further, since $B \neq C$ the elements that are shared between A and B must be different than those shared between A and C. With these facts in hand we may choose any three sets that satisfy these conditions. For the sake of simplicity let us choose $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4, 5\}$. In this case $B - A = \{5\}$ and $C - A = \{5\}$ thus $B - A = C - A$ and $B \neq C$. Q.E.D.