Math 221: Calculus III

Project 2 Topic and Questions

Applications of Lagrange Multipliers

With the introduction of Lagrange Multipliers, many previously discussed formulas, and formulas for new directions, could now be derived. The project will analyze two old formulas but framed within an optimization context.

Topic Questions

1. Recall that we could calculate the minimum distance between a point in space $\mathbf{x}_0$ and a line parameterized as $\mathbf{r}(s) = \mathbf{p}_0 + s\mathbf{v}$. We could approach this as a geometry problem, by projecting a vector from the origin to the point $\mathbf{x}_0$ onto the line $\mathbf{r}(s)$, then computing the length of that segment.

   Instead, we will write this as a minimization problem. We wish to minimize the distance, which will be our function, and the constraint is the point must be on the line. Thus, this can be written as

   $$\begin{aligned} &\text{Minimize } \ \frac{1}{2}||\mathbf{r}(s) - \mathbf{x}_0||^2 \\ &\text{subject to } \ \mathbf{r}(s) = \mathbf{p}_0 + s\mathbf{v} \end{aligned}$$

   Using the method of Lagrange multipliers, find the critical point. Argue this critical point will provide a minimum distance by arguing that the maximum distance will not exist. Finally, provide a formula for the minimum distance.

   Note: These formulas should be derived for any dimension. It may be helpful to start in two- or three dimensions as a method of checking your work. However, general vectors as presented above should be used.

2. This can also be extended to planes. Once again, suppose $\mathbf{x}_0$ is a point in space and a (hyper)plane is given by the vector equation $\big(\mathbf{r} - \mathbf{r}_0\big) \cdot \mathbf{n} = 0$, where $\mathbf{r}_0$ is a point on the plane and $\mathbf{n}$ is the normal vector to the plane. This may be written as

   $$\begin{aligned} &\text{Minimize } \ \frac{1}{2}||\mathbf{r} - \mathbf{x}_0||^2 \\ &\text{subject to } \ \big(\mathbf{r} - \mathbf{r}_0\big) \cdot \mathbf{n} = 0 \end{aligned}$$

   Similar to the previous question, find the critical point using Lagrange multipliers. Argue this also produces a minimum instead of a maximum. Then, provide a formula for the minimum distance.

3. Finally, Lagrange multipliers can be extended to topics outside of the course. As long as the problem can be phrased within the context of minimizing or maximizing a function over a (set of) constraint(s), this method can commonly be applied.

   For example, in thermodynamics, there is a principle that physical systems tend to the highest entropy state, which is referred to as thermodynamic equilibrium. Thus, for some particle, we can define a discrete distribution of states the particle can be in. If we define an appropriate entropy function, then we can maximize this function constrained to some assumptions about the distribution or physical system.

   Let $\mathbf{p} = \langle p_1, p_2, ..., p_N \rangle$ be a discrete distribution of state variables, i.e. there is a probability p_i of a particle being in state i. We will define our entropy function to be the *Shannon entropy function*. Altogether, this becomes:

   (a) Maximize $f(\mathbf{p}) = -\sum_{i=1}^{N} p_i \ln(p_i)$

   subject to $\sum_{i=1}^{N} p_i = 1$

   Our constraint comes from the probability of *any* event happening must be 1. Thus, the sum of probability states must sum to 1. Show the critical point will have the same value in each index. What distribution does this correspond to?

   (b) Maximize $f(\mathbf{p}) = -\sum_{i=1}^{N} p_i \ln(p_i)$

   subject to $\sum_{i=1}^{N} p_i = 1$ and $\sum_{i=1}^{N} x_i p_i = \bar{x}$

   where x_i and $\bar{x}$ are constant real numbers. Note this is similar to the previous question, but now we require the mean of the distribution to be fixed.

   Find a general form for the distribution of p_i by evaluating the critical point(s). This form *will* include the Lagrange multiplier. Solving explicitly is too difficult.

   (c) Maximize $f(\mathbf{p}) = -\sum_{i=1}^{N} p_i \ln(p_i)$

   subject to $\sum_{i=1}^{N} p_i = 1$ and $\sum_{i=1}^{N} x_i p_i = \bar{x}$ and $\sum_{i=1}^{N} x_i^2 p_i = \bar{x}^2$

   Similar to (b), this now requires the mean and variance of the distribution to be given fixed values. Again, evaluate the critical point(s) and find the distribution by solving for p_i. This form will include two Lagrange multipliers.

An application of this approach in biology can be seen here:

`https://www.sciencedirect.com/science/article/pii/S2405844018301695`

Additional Topics

The suggestions below are a few avenues that could be explored:

- One could argue that there may be a method to define a minimum distance between any point $\mathbf{x}_0$ and any implicitly defined object $F(\mathbf{x}) = 0$. This could be investigated.

- Another application of Lagrange multipliers could be discussed. These commonly show up in mechanics, physics, and other applications of motion. Similarly, certain questions in statistics can be phrased as Lagrange multipliers. You may wish to look for one that sounds interesting.

- There is a generalization of Lagrange multipliers called Karush-Kuhn-Tucker (KKT) multipliers. Investigating what these do, how they relate to Lagrange multipliers, and a potential application would suffice as an additional topic.