Name: ______________________________

Eigenvectors and Eigenvalues

Consider the matrix

$$A = \begin{bmatrix} 9 & -3 & -7 \\ 7 & -1 & -7 \\ 3 & -3 & -1 \end{bmatrix}$$

1. Show the eigenvalues are $\lambda = 6, 2$, and -1 from the characteristic polynomial. Then, find their corresponding eigenvectors.

2. In general, finding these eigenvalues can be difficult. Roots of polynomials are not easy to solve in general. One strategy to estimate eigenvalues is to use the basis of eigenvectors, which will eventually find the largest eigenvalue for a matrix given enough computation.

(a) If $\mathbf{x}_0$ is a vector (point), we can write it as a linear combination of our eigenvalues and eigenvectors. For the vector $\mathbf{x}_0 = [2, -3, 8]^T$, write it as a linear combination of the eigenvectors from matrix A, i.e.

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3,$$

where $\mathbf{v}_i$ is from Question 1 and the coefficients c_i are to be determined.

Hint: Set up an appropriate augmented matrix.

(b) Now, assume A is a 3×3 matrix with three independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Show that we can write

$$A\mathbf{x}_0 = \lambda_1 c_1\mathbf{v}_1 + \lambda_2 c_2\mathbf{v}_2 + \lambda_3 c_3\mathbf{v}_3$$

using the definition of eigenvalues and eigenvectors and matrix properties.

(c) Further, this can be repeated. Now show that by repeating (iterating) this process, we can write

$$A^k\mathbf{x}_0 = (\lambda_1)^k c_1\mathbf{v}_1 + (\lambda_2)^k c_2\mathbf{v}_2 + (\lambda_3)^k c_3\mathbf{v}_3$$

Notice if the eigenvalues are ordered such that $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then for a large enough k, $A^k\mathbf{x}_0 \approx (\lambda_1)^k c_1\mathbf{v}_1$.

(d) If $\mathbf{w}$ is any vector, use the concluding statement in part (c) to show that

$$\frac{\mathbf{w}\cdot A^k\mathbf{x}_0}{\mathbf{w}\cdot A^{k-1}\mathbf{x}_0} \approx \lambda_1$$

Thus, for large enough k, this will be a reasonable estimate of the largest (in absolute magnitude) eigenvalue.

The Power Method

3. The previous question provided the foundation for an algorithm for finding large eigenvalues and its associated eigenvector called the *power method*.

Choose an initial point $\mathbf{x}_0 \in \mathbb{R}^3$. Do not choose an eigenvector of matrix A from Question 1, but some other vector. Then, compute the following using matrix A from Question 1:

$$
\begin{aligned}
\mathbf{x}_0 &= \text{chosen by student} & & \\
\mathbf{x}_1 &= A\mathbf{x}_0 & \lambda_1^{(1)} &= \frac{\mathbf{x}_0 \cdot \mathbf{x}_1}{\mathbf{x}_0 \cdot \mathbf{x}_0} \\
\mathbf{x}_2 &= A\mathbf{x}_1 & \lambda_1^{(2)} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1} \\
&\vdots & &\vdots \\
\mathbf{x}_5 &= A\mathbf{x}_4 & \lambda_1^{(5)} &= \frac{\mathbf{x}_4 \cdot \mathbf{x}_5}{\mathbf{x}_4 \cdot \mathbf{x}_4}
\end{aligned}
$$

In this case, we use the previous eigenvector estimate for $\mathbf{w}$ from Question 2. After 5 iterations, you should see λ_1 beginning to look like the largest eigenvalue from Question 1. You may use software to do these computations or you may do them by-hand. You're also welcome to round your answers at each step to a few decimal places.