Lecture 2: Gaussian Elimination

Section 1.1-1.2 in Linear Algebra and Its Applications

Daniel Poll

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Definition: Matrix

Let m and n be natural numbers. An $m \times n$ matrix is a rectangular array (or table) consisting of m rows and n columns. We write the matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where a_{ij} is a real number on the i^{th} row and j^{th} column of A.

Note: Let $r = \min(m, n)$. We call $a_{11}, a_{22}, \dots, a_{rr}$ the *main diagonal entries* of A.

Definition: Coefficient and Augmented Matrix

A *coefficient matrix* is a matrix consisting entirely of the coefficients of the variables in a system of linear equations.

An *augmented matrix* is a matrix consisting of the coefficients of the variables and an additional column of the constant terms in a system of equations appended to the last column of the coefficient matrix.

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Example. The system of equations below has the following coefficient matrix and augmented matrix:

System:

$$\begin{aligned} 3x_1 + 2x_2 - 4x_3 &= 7 \\ 5x_1 - 9x_2 + 8x_3 &= 1 \\ 6x_1 - 2x_2 - 3x_3 &= 4 \end{aligned}$$

Coefficient Matrix:

$$\begin{bmatrix} 3 & 2 & -4 \\ 5 & -9 & 8 \\ 6 & -2 & -3 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc:c} 3 & 2 & -4 & 7 \\ 5 & -9 & 8 & 1 \\ 6 & -2 & -3 & 4 \end{array}\right]$$

Definition: Row Echelon Form of a Matrix

A matrix is said to be in *Row Echelon Form* (REF) if the following are true:

- Each leading entry of a row is strictly to the left of all other rows below it.
- All entries below a leading entry are zero.
- Rows of all zeros are below any rows containing a leading entry.

$$\begin{bmatrix} * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Definition: Reduced Row Echelon Form of a Matrix

A matrix is said to be in *Reduced Row Echelon Form* (RREF) if the matrix is in REF form and the additional properties hold:

- All leading entries are 1.
- All other entries in the leading entry's column are 0.

$$\begin{bmatrix} 1 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: Additional Definitions

- A *pivot* is a nonzero number in a leading entry of a matrix used to create zeros. A *pivot column* is a column that contains a pivot.
- A *basic variable* is a variable corresponding to a pivot column within an augmented matrix.
- A *free variable* is a variable that is not a basic variable.
- The *general solution* is found by solving each equation for the basic variables in terms of the free variables.

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Example. For the following system of equations and it's associated augmented matrix,

$$\begin{array}{rrrrrl} x_1 & +\,6x_2 & & +\,3x_4 & & = 0 \\ & & 4x_3 & -\,9x_4 & & = 5 \\ & & & & 2x_5 & = 7 \end{array} \quad \longrightarrow \quad \left[\begin{array}{ccccc:c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & -9 & 0 & 5 \\ 0 & 0 & 0 & 0 & 2 & 7 \end{array}\right]$$

find the number of each pivot column, basic variables, and free variables.

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Solution. Pivot columns: 1,3,5 Basic Variables: x_1, x_3, x_5 Free variables: x_2, x_4

Properties: Elementary Row Operations

Each of these operations produces on an augmented matrix produces an row equivalent matrix

- Interchange any rows within the matrix

$$R_i \longleftrightarrow R_j$$

- Multiply a row by a (nonzero) constant

$$cR_i \longrightarrow R_i$$

- Add a multiple of one row to another row

$$cR_i + R_j \longrightarrow R_j$$

Algorithm: Gaussian Elimination

Begin with augmented matrix A of size $m \times n$

For $i = 1, 2, ..., m$

  If leading coefficient is 0:

    Interchange with row $j > i$ such that $a_{ji} \neq 0$

  Divide equation i by leading coefficient[†]

  For $j = i, i + 1, .., m$

    Add multiple of equation i to equation j to eliminate a_{ji}

    If equation j has all zero coefficients, move to end of equations.

End

[†]Not required, but generally a recommended practice.

Example. Reduce the following augmented matrix to row echelon form.

$$\left[\begin{array}{ccc:c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \text{ and } R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right]$$

Example. Repeat the previous example, but put into row reduced echelon form.

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Solution. The strategy will be to do elimination, but backwards. So, we'll start with the right-most pivot column and eliminate all of the elements above it:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right] \xrightarrow{(1/5)R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-5R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{3R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This makes it easy to read the solution for the basic variables, i.e.
$x_1 = 5, x_2 = 3, x_3 = -1$.

The algorithm for this is generally referred to as *Gauss-Jordan elimination*.

Useful MATLAB Code:

We can import matrices as a two-dimensional array into MATLAB. To do this by-hand, we'll begin with a bracket, then separate entries by a comma or space. This fills out a row. To start a new row, use a semicolon (;). Then, close the bracket when done.

```
A = [1, 0, -3, 8; 2, 2, 9, 7; 0, 1, 5, -2]
```

This stores our matrix as the variable 'A'. To perform row reduced echolon form, we can use the function `rref()`:

```
rfA = rref(A)
```

This stores our row-reduced matrix as the variable 'rfA'.