Name: ______________________________

## Solving a Large Traffic Model

1. Recall the network problem from in-class and on the second homework. Directed networks have a variety of applications as we've discussed including modeling traffic behavior, electrical circuits, and force balance during mechanical equilibrium.

(a) Draw a network diagram such as the ones presented in-class. Do not collaborate with other students on your diagram. Your network diagram should contain at a minimum 5 nodes and 12 edges, such that at least 8 of the edges must be drawn between the nodes. The other edges are allowed to leave or enter the network. Make 6 of these edges unknown (variables labeled $x_i$) while the rest are numbers chosen by the student.

(b) Write the corresponding system of linear equations to the network diagram above. Then, write the corresponding augmented matrix. Finally, reduce the augmented matrix to reduced row-echelon form. You may use code or software (such as MATLAB) to do this. However, if you do, cite your source and/or code used to generate your reduced row-echelon matrix.

(d) Interpret and explain the reduced row-echelon matrix in the context of your diagram.

## Spline Interpolation

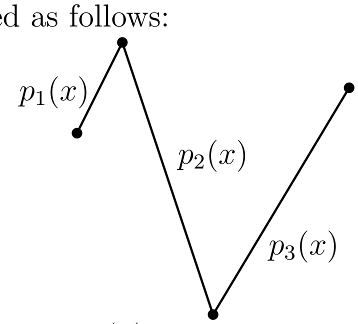
A common question in engineering is how to best model and estimate a proposed design, such as the wings of an aircraft. Then, to simulate their effectiveness against various physical processes (air resistance, drag, turbulence, etc.), an accurate and robust representation of the aircraft wing's geometry should be constructed. One method to accomplish this is to consider a sequence of polynomials between multiple points in space. These are called *splines*. More information about their history, in particular at Boeing, can be found here:

`https://sites.me.ucsb.edu/~moehlis/ME17/splines.pdf`

We will focus on a particular case of this. Our points will be in two-dimensions, i.e. of the form $(x_i, y_i)$. Further, we will look at two cases: *linear splines* and *cubic splines*.

2. Consider the points $(1,3),(2,5),(4,-1)$ and $(7,4)$. We will construct three linear polynomials (lines), which have unknown coefficients labeled as follows:

$$\begin{aligned} p_1(x) &= a_1 + b_1 x \\ p_2(x) &= a_2 + b_2 x \\ p_3(x) &= a_3 + b_3 x \end{aligned}$$




For these splines, we require certain conditions. Namely, $p_1(x)$ should pass through the first two points, then $p_2(x)$ passes through $(2,5)$ and $(4,-1)$, and finally $p_3(x)$ should pass through the last two points (see picture).

(a) Write down a linear system of equations to solve for the coefficients. This should be a six equations with six unknowns.

(b) Write the linear system of equations from (a) as an augmented matrix. Then, put the augmented matrix into reduced row-echelon form. You may again use software for this, but you should cite your source and/or provide the code used.

(c) Using the information from (b), state what the three lines $p_1(x), p_2(x)$ and $p_3(x)$ are. You may wish to plot or graph these to check your work.

3. Let's now consider the case of cubic splines. Consider three points $(0, 1), (2, 4)$, and $(6, -2)$. We will construct two cubic polynomials, the first between the first two points, called $s_1(x)$, and another polynomial between the last two, called $s_2(x)$. We define the cubic polynomials with coefficients as follows:

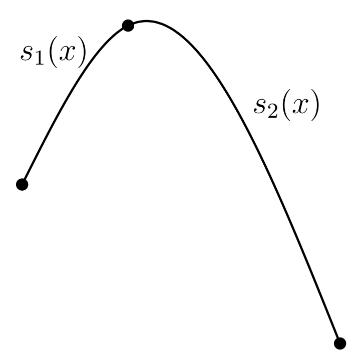
$$s_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$$
$$s_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3$$

Again, there are certain conditions. First, we require $s_1(x)$ to pass through the first two points. Then, we require $s_2(x)$ to pass through the last two points. Since we have 8 unknown coefficients, we require four more conditions.

These conditions are as follows:

$$s_1'(2) = s_2'(2), \qquad s_1''(2) = s_2''(2)$$
$$s_1''(0) = 0, \qquad s_2''(6) = 0$$

The last two conditions are typically called *free boundary conditions.*




Note these require derivatives from Math 120. If you need help with this, please drop-by office hours.

(a) Write the linear system of equations for the above conditions. You will have eight equations and eight unknowns.

(b) Write this linear system of equations as an augmented matrix. Then, put this augmented matrix into reduced row-echelon form. Again, you may use external resources, but cite the source and/or write your code.

(c) Using part (b), state the two cubic polynomials $s_1(x)$ and $s_2(x)$ are now that the coefficients have been determined. You may wish to plot these to check your work.

## Fitting an Exponential Function Using Linear Systems

4. (Bonus) A common question is how to fit an exponential function

$$Y(t) = Ae^{rt} \tag{1}$$

to some data set. There is a general approach for an arbitrary amount of data. However, we will need to progress further in the course to address the general case. For now, let's assume we have two data points, $(1, 200)$ and $(10, 500)$.

(a) By taking the natural logarithm of Eq. (1) above, explain how this can be written as $p(t) = a_0 + a_1 t$ by determining what $a_0$ and $a_1$ are above.

(b) By using the two data points given above, write down a system of equations that solve for $a_0$ and $a_1$. Note: Don't forget that you applied the logarithm, which also applies to the data.

(c) Write the corresponding augmented matrix, then put the augmented matrix into reduced row-echelon form. You may use software, such as MATLAB, to perform the row operations. If you choose to do this, write the snippet of code used below.

(d) Using your results from (c), find $A$ and $r$ in terms of $a_0$ and $a_1$. Then, plot the data points and the exponential function. You may use software, such as Desmos or MATLAB, or you may provide a written sketch.