Lecture 1.2: Limit Properties. Techniques of Limit Computation

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1 Basics

1. **Limit of a constant**
$$\lim_{x \to a} C = C$$
The limit of a constant is always the constant (think because the plot is just a horizontal line).

2. **Limit of x**
$$\lim_{x \to a} x = a$$
Think because $f(x) = x$ is just identity, so point a on x is also a on y.

3. **Zero limit**
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

2 Properties

Given two functions with limits that exist:

$$\lim x \to af(x) = L_1$$
$$\lim x \to ag(x) = L_2$$

1. **Function joining and separation**
$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
This operation works both ways (can also join).

2. **Same thing for multiplication**
$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

3. **For division**

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \lim_{x \to a} g(x) \neq 0$$

4. **Exponents**

$$\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \to \lim_{x \to a} \sqrt[n]{f(x)} \to \sqrt[n]{\lim_{x \to a} f(x)}$$

2.1 Example

$$\begin{aligned}
\lim_{x \to 2}(x^3 - 2x + 7) &\to \lim_{x \to 2} x^3 - \lim_{x \to 2} 2x + \lim_{x \to 2} 7 \\
&\to \left[\lim_{x \to 2} x\right]^3 - \left(\lim_{x \to 2} 2\right) \cdot \left(\lim_{x \to 2} x\right) + \lim_{x \to 2} 7 \\
&\to 2^3 - 2 \cdot 2 + 7 \\
&\to 11 \\
f(2) &= 11
\end{aligned}$$

For any polynomial, all you need to do is plug the value at the limit as the variable in order to determine the limit. The general case for any P polynomial:

$$\lim_{x \to a} P(x) = P(a)$$

2.2 Holes and asymptotes

2.2.1 Holes

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

With limits, because you're not actually reaching the point at the limit, it's okay to simplify out a domain problem. So:

$$\begin{aligned}
&\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\
&\frac{(x+2)(x-2)}{x-2} \\
&x + 2 \\
&\lim_{x \to 2} = 4
\end{aligned}$$

If the domain issue can simplify, it's a hole in the graph. If it can't, then it's some type of asymptote. If a fraction reduces to zero over zero, I think he's saying that means it can be factored.

2.2.2 Asymptotes

$$\begin{aligned} \lim_{x\to5}\frac{x^3-3x-10}{x^2-10x+25} \\ \lim_{x\to5}\frac{(x-5)(x+2)}{(x-5)(x-5)} \\ \lim_{x\to5}\frac{x+2}{x-5} \end{aligned}$$

We have ourselves a vertical asymptote.
Sign analysis test
To determine the direction of the asymptotes in both directions, we need to construct a number line between the two values that will make the numerator and denominator equal to zero (here -2 and 5). Sample on both sides of the limit, plug into the function, and determine the sign on both sides.
Example.

$$\lim_{x\to1}\frac{x-1}{\sqrt{x}-1} \tag{1}$$

$$\lim_{x\to1}\frac{x-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1} \tag{2}$$

$$\lim_{x\to1}\frac{(x-1)(\sqrt{x}+1)}{x-1} \tag{3}$$

$$\lim_{x\to1}\sqrt{x}+1 \tag{4}$$

$$\lim_{x\to1}\sqrt{1}+1=2 \tag{5}$$

The logic behind (2) is to multiply by the *conjugate*, since you can't necessarily factor anything.
The trick to get the denominator in step (3) is:

$$\begin{aligned} (\sqrt{x}-1)\cdot(\sqrt{x}+1) \\ \sqrt{x}\cdot\sqrt{x}+\sqrt{x}-\sqrt{x}-1 \\ x-1 \end{aligned}$$

Forget what the formal name of that is... but it's the same way you'd distribute one term into another to get a classic polynomial. Been a minute. Yea so conjugate is the term you multiply another term by to get that middle term to cancel. Eg, the conjugate of $p-q$ is $p+q$.
Another example.

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \\
\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \\
\lim_{x \to 0} \frac{x}{x(\sqrt{1+x}+1)} \\
\lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} \\
\lim_{x \to 0} \frac{1}{\sqrt{1}+1} \\
\lim_{x \to 0} \frac{1}{2}
\end{aligned}
$$

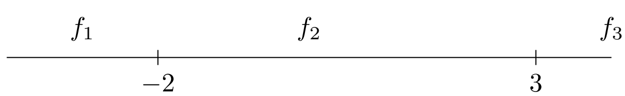
3 Piece-Wise Limits

We're going to find some one-sided limits and see if they are equal.

Example.

$$
f(x) = \begin{cases} \frac{1}{x+2} & \text{if } x < -2 \\ x^2 - 5 & \text{if } -2 > x \leq 3 \\ \sqrt{x+13} & \text{if } x > 3 \end{cases}
$$

A helpful approach is to plot a number line and draw the intervals of the piece-wise function:



If the functions have the same limit as they approach the interval between them, then the limit can be said to exist; otherwise, it doesn't exist.

Let's work through the first two functions with -2 as the limit:

$$
\begin{aligned}
f_1(x) &= \lim_{x \to -2} \frac{1}{x+2} \\
&= \lim_{x \to -2} \frac{1}{x+2} \cdot \frac{x-2}{x-2} \\
&= \lim_{x \to -2} \frac{x-2}{x^2-4}
\end{aligned}
$$

Hmm okay I'm stuck. Multiplying by the conjugate doesn't seem to free up the denominator from being zero when plugging in the limit.

Aha okay so if the function doesn't reduce to 0/0, that suggests there's an asymptote, not a hole. Then we need to do the sign analysis test to determine the direction of the asymptote.

$$
\begin{aligned}
f_1(x) &= \lim x \to -2 \frac{1}{-3+2} \\
&= -1 \\
\lim_{x \to -2^-} \frac{1}{x+2} &= -\infty
\end{aligned}
$$

Because the limit approaching -2 from the left does not equal the limit approaching -2 from the right (I didn't show computation for approaching from the right here), the limit is said to not exist.