Lecture 1.2: Limit Properties. Techniques of Limit Computation

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1 Basics

1. **Limit of a constant**
$$\lim_{x \to a} C = C$$
The limit of a constant is always the constant (think because the plot is just a horizontal line).

2. **Limit of x**
$$\lim_{x \to a} x = a$$
Think because $f(x) = x$ is just identity, so point a on x is also a on y.

3. **Zero limit**
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

2 Properties

Given two functions with limits that exist:

$$\lim x \to af(x) = L_1$$
$$\lim x \to ag(x) = L_2$$

1. **Function joining and separation**
$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
This operation works both ways (can also join).

2. **Same thing for multiplication**
$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

3. **For division**

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \lim_{x \to a} g(x) \neq 0$$

4. **Exponents**

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n \to \lim_{x \to a} \sqrt[n]{f(x)} \to \sqrt[n]{\lim_{x \to a} f(x)}$$

2.1 Example

$$\begin{aligned}
\lim_{x \to 2} (x^3 - 2x + 7) &\to \lim_{x \to 2} x^3 - \lim_{x \to 2} 2x + \lim_{x \to 2} 7 \\
&\to \left[\lim_{x \to 2} x \right]^3 - \left(\lim_{x \to 2} 2 \right) \cdot \left(\lim_{x \to 2} x \right) + \lim_{x \to 2} 7 \\
&\to 2^3 - 2 \cdot 2 + 7 \\
&\to 11 \\
f(2) &= 11
\end{aligned}$$

For any polynomial, all you need to do is plug the value at the limit as the variable in order to determine the limit. The general case for any P polynomial:

$$\lim_{x \to a} P(x) = P(a)$$

2.2 Holes and asymptotes

2.2.1 Holes

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

With limits, because you're not actually reaching the point at the limit, it's okay to simplify out a domain problem. So:

$$\begin{aligned}
&\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\
&\frac{(x+2)(x-2)}{x - 2} \\
&x + 2 \\
&\lim_{x \to 2} = 4
\end{aligned}$$

If the domain issue can simplify, it's a hole in the graph. If it can't, then it's some type of asymptote. If a fraction reduces to zero over zero, I think he's saying that means it can be factored.

2.2.2 Asymptotes

$$\begin{aligned}
\lim_{x \to 5} \frac{x^3 - 3x - 10}{x^2 - 10x + 25} \\
\lim_{x \to 5} \frac{(x-5)(x+2)}{(x-5)(x-5)} \\
\lim_{x \to 5} \frac{x+2}{x-5}
\end{aligned}$$

We have ourselves a vertical asymptote.

Sign analysis test

To determine the direction of the asymptotes in both directions, we need to construct a number line between the two values that will make the numerator and denominator equal to zero (here -2 and 5). Sample on both sides of the limit, plug into the function, and determine the sign on both sides.

Example.

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} \tag{1}$$

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \tag{2}$$

$$\lim_{x \to 1} \frac{(x-1)(\sqrt{x}+1)}{x-1} \tag{3}$$

$$\lim_{x \to 1} \sqrt{x}+1 \tag{4}$$

$$\lim x \to 1\sqrt{1}+1 = 2 \tag{5}$$

The logic behind (2) is to multiply by the *conjugate*, since you can't necessarily factor anything.

The trick to get the denominator in step (3) is:

$$\begin{aligned}
(\sqrt{x}-1) \cdot (\sqrt{x}+1) \\
\sqrt{x} \cdot \sqrt{x} + \sqrt{x} - \sqrt{x} - 1 \\
x - 1
\end{aligned}$$

Forget what the formal name of that is... but it's the same way you'd distribute one term into another to get a classic polynomial. Been a minute. Yea so conjugate is the term you multiply another term by to get that middle term to cancel. Eg, the conjugate of $p - q$ is $p + q$.

Another example.

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$$
$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$
$$\lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)}$$
$$\lim_{x\to 0} \frac{1}{\sqrt{1+x}+1}$$
$$\lim_{x\to 0} \frac{1}{\sqrt{1}+1}$$
$$\lim_{x\to 0} \frac{1}{2}$$

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