Lecture 1.3: Continuity of Functions

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1 Defining Continuity

Definition 1. *A function is* <u>*continuous*</u> *if it has no holes, breaks or asymptotes. Mathematically, a function is continuous at point c if:*

1. *$f(c)$ is defined.*
2. *$\lim_{x\to c} f(x)$ must exist.*
3. *$\lim_{x\to c} f(x) = f(c)$*

It's important to be specific about *where* a function is continuous or not. For example, a hole in a function is referred to as a *removable discontinuity.*[1]

Definition 2. *A* <u>*removable discontinuity*</u> *is a discontinuity that could be filled in with a single point.*

Definition 3. *A* <u>*jump discontinuity*</u> *is a visual 'jump' in the function as the function approaches a point (ie, the limit doesn't exist).*

1.1 Examples

Are the following functions continuous at $x = 2$?

$$f(x) = \frac{x^2-4}{x-2}$$
$$g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$
$$h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

[1] I got shook a bit initially thinking point c was in the range of function f but it's actually in the *domain*. Remember, $\lim_{x\to c} f(x)$ means x (the input) is approaching point c (in the domain). *Edit:* I guess actually a *point* (x, y) consists of both domain and range.

Case 2 is worth highlighting. We need to check whether $g(2) = \lim_{x \to 2} g(x)$.

$$\begin{aligned} \lim_{x \to 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x \to 2} x + 2 \\ g(2) = 3 &\neq 4 \end{aligned}$$

The function is not continuous at 2, because $g(2) \neq \lim_{x \to 2} g(x)$.

2 Endpoints

We need to establish a few facts in order to work with continuity of endpoints.

Theorem 1. *If a function f is continuous at every point between a and b, then f is continuous on the open interval (a, b).*

Also need to check whether a function is continuous at a point when approaching from either side.

Theorem 2. *Continuous from the left at point c:*

$$\lim_{x \to c^-} f(x) = f(c)$$

Continuous from the right at point c:

$$\lim_{x \to c^+} f(x) = f(c)$$

2.1 Examples

Prove the following function is continuous at $[-4, 4]$.

$$f(x) = \sqrt{16 - x^2}$$

I worked this out on the board, but the jist of it is to first check the open interval $(-4, 4)$ for domain problems.[2] Then, for the one sided limits at the endpoints, you actually are evaluating the limit of the function at that endpoint *from* the side (pos/neg) that the function is coming from. For example, to evaluate the -4 endpoint, you check

$$\lim_{x \to -4^+} f(x)$$

[2] This was kind of an unsatisfying proof. You just sort of intuitively check the interval and if it's okay then you declare it's all good.

I think unless your limit is at infinity or is some other type of domain problem, then you can just plug it in and it should be fine. Otherwise I guess you need to do a sign analysis test.
Left off at 35:45.