Lecture 0.3: Review of Trigonometry and Graphing Trigonometric Functions

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1 Angles

Plotting a vector and making an angle relative to the x-axis. The x-axis is called the "initial side", the vector line is called the "terminal side".

Counter clockwise rotation from the x-axis gives positive angles. Clockwise gives negative angles.

1.1 Degrees vs. radians

Radians involve π.

Coverting radians to degrees:

$$\begin{aligned} 2\pi &= 360^\circ \\ f(x^\circ) &= x^\circ \cdot \frac{\pi}{180^\circ} \\ g(r) &= r \cdot \frac{180^\circ}{\pi} \end{aligned}$$

$f(x^\circ)$ converts degrees to radians. $g(r)$ converts radians to degrees.

He's going through some plotting of radian angles... not sure how to take notes on that. A bit tricky, but if you have a radian (eg, $5\pi/2$), then you take the four cartesian quadrants, chop up the top and bottom halves equally by the denominator. The terminals of the four axes end points are as follows ($+x$: 0 or 2π, $+y$: $\pi/2$, -x: π, -y: $3\pi/2$). And you then take the numerator coefficient and count that many either counter clockwise (for positive numerator) or clockwise (for negative numerator).

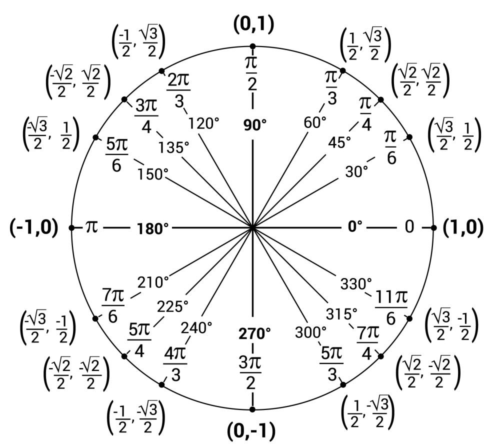
2 Trigonometric Functions

Sine, cosine, and tangent are *functions*. They need an input (usually an angle). Sine, cosine, and tangent can be used to find the angles of a right triangle (remember SohCahToa: sine, opposite over adjacent; cosine, adjacent over hypotenuse, tangent, opposite over adjacent).

There's also the reciprocals of the three trig functions.

$$\text{Secant: } sec(\theta) = \frac{1}{cos(\theta)} = \frac{Hyp}{Adj}$$
$$\text{Cosecant: } csc(\theta) = \frac{1}{sin(\theta)} = \frac{Hyp}{Opp}$$
$$\text{Cotangent : } cot(\theta) = \frac{1}{tan(\theta)} = \frac{Adj}{Opp}$$

For the unit circle (ie, circle with radius of 1), for any point (x, y) along the circumference of the circle, $cos(\theta) = x$, $sin(\theta) = y$, $tan(\theta) = y/x = sin(\theta)/cos(\theta)$. And the reciprocal functions are just the reciprocals of these outcomes.



He's stressing how important it is to know this unit circle. For example, one should know the sin, cos, and tan of $\{0, \pi/2, \pi, 3\pi/2, 2\pi\}$. And should also know $\{\pi/3, \pi/4, \pi/6\}$.

2.1 Trigonometric functions across the quadrants

Whether the trig functions are positive or negative depends on the quadrant. The mnemonic "**A**ll **S**tudents **T**ake **C**alculus" $\rightarrow$ each letter iterates through cartesian quadrants I:IV, A=All functions are positive in quadrant I, S=Sine (and cosecant) is positive in quadrant II, T=Tangent (and cotangent) is positive in quadrant III, C=Cosine (secant) is positive in quadrant IV.

2.2 Reference angles

The idea is to make an acute angle with the x-axis, find the trig function of that, then use the ASTC idea.

If the angle is already acute relative to x-axis, just take theta. Otherwise, find the angle with x that makes it acute. So for example, if $\pi > \theta > 0$, then you calculate the acute angle by $\pi - \theta$. Similar idea for quadrants III & IV.

2.3 Trig function for any angle

One can use the properties of the functions across the quadrants and the reference angles, along with knowledge of the unit circle, to determine for any input angle the output of any trig function.

Example: find sin, cos, tan of $5\pi/3$.

Steps:

1. Locate the quadrant.
   I think it's Quadrant IV

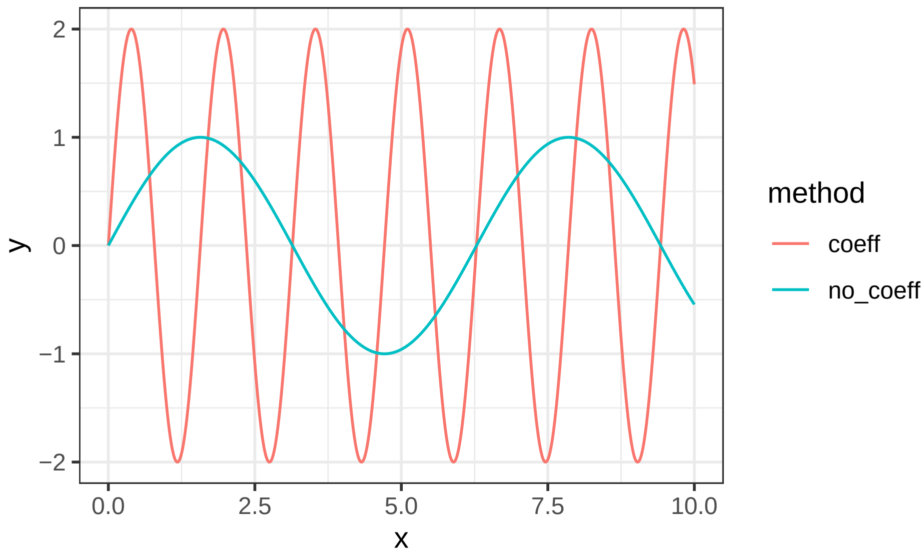
2. Find the reference angle.
   $2\pi - 5\pi/3 = -3\pi/3$
   Oh, fractions need a common denominator when adding or subtracting...
   $6\pi/3 - 5\pi/3 = \pi/3$

3. Find all trig functions of the reference angle.
   I guess this is where we use the unit circle.[1]

3 Graphing Trigonometric Functions

We're going to graph things like this: $y = A \cdot sin(Bx)$ or $y = A \cdot cos(Bx)$. To appreciate the impact of the coefficients, we'll compare $y = sin(x)$ against $y = 2 \cdot sin(4x)$.

[1] I had to take a little detour into the unit circle. Apparently the x y coordinates of the so-called "special angles/trianles" (30-60-90 and 45-45-90) are just givens and expressed as fractions of roots (so unsatisfying). So if want to find $sin(\pi/3)$ I have to go look up that the coordinates for the unit circle at $\theta = \pi/3$ are $(1/2, \sqrt{3}/2)$. Then obviously $sin(\pi/3) = y = \sqrt{3}/2$. And then you determine the sign of that by the quadrant.



The period of a wave function is the stretch over which it's unique (ie, before it repeats).

We get the amplitude and period by Amplitude $= |A|$ and Period $= 2\pi/B$.

You can break the period up into quarters to find the first positive and negative amplitudes and the first x axis crossover point.

3.1 Shifting the function

Shifting the sin or cos function by a constant (ie, $y = A \cdot cos(Bx - c)$). Which is rewritten as $y = A \cdot cos[B(x - c/b)]$. For reasons that aren't quite intuitive to me, you flip the sign in terms of the shift—$-c/b$ is a shift to the right, and vice versa for positive.

Example:

$$y = 3cos(2x + \frac{\pi}{2})$$
$$y = 3cos[2(x + \frac{\pi}{4})]$$
$$\text{Amplitude} = 3$$
$$\text{Period} = \frac{2\pi}{2}$$
$$\text{Period} = \pi$$
$$\text{Shift} = \frac{\pi}{4}$$

You'll shift to the left by $\pi/4$.

In closing, he's saying you'll need to know trig identites (eg, $sin^2 + cos^2 = 1$, $tan = sin/cos$). Half angle and double angle formulas ($sin(2x) = 2sin(x)cos(x)$).

4 Combining and Compoisition of Functions

This is a separate video but it's only 15 min so I'm tacking it on here.

We can add, subtract, multiply and divide functions.

Example:

$$f(x) = 1 + \sqrt{x-2}$$
$$g(x) = x - 3$$

$(f+g)(x) = f(x) + g(x) \rightarrow 1 + (\sqrt{x-2}) + (x-3)$.

Et cetera for the other types of operations. Beware of domain restrictions that arise because of division.

The domain of the resulting combined function is the intersection of the two previous functions. Meaning any restrictions that apply to original functions apply to the resulting function. You can't ever make a domain problem better by combining functions.

4.1 Compositions

Nesting functions inside functions.

$$f(x) = x^3 - 4$$
$$g(x) = \sqrt{x}$$
$$(fg)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^3 - 4$$
$$(gf)(x) = g(f(x)) = g(x^3 - 4) = \sqrt{x^3 - 4}$$

Can expand to more than two functions:

$$f(x) = \sqrt{x}$$
$$g(x) = \frac{1}{x}$$
$$h(x) = x^3$$
$$f(g(h(x))) = f(\frac{1}{x^3}) = \sqrt{\frac{1}{x^3}}$$