Lecture 1.1: An Introduction to Limits

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1 Limits

Limits are the basis of calculus. There are two goals in calculus:

1. Given any curve, find the slope of a curve at a point (find the tangent to a curve at a point).
2. Given any curve defined by a function, can you between two points find the area under the curve?

1.1 Tangent Problem

If you give me an arbitrary curve, and give me a point p, I need to find the slope of the curve at point p. If we have the slope and we have the point, we can make the tangent line.

Tangent line. A tangent line is a line that touches at exactly one point and is "parallel to the curve at that point.

One problem if we only have one point and no slope is we can't draw a line—we need at least one more point. We can find another point along the curve and connect the two to form a secant line.

Secant line. A secant line is a line that intersects a curve at two points (see Figure 1).

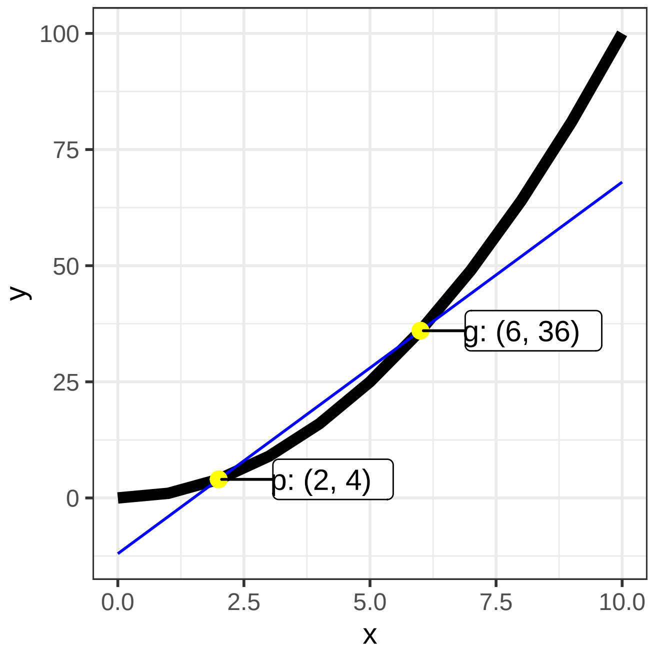

Figure 1: A secant line intersects two points on a curve.

So in Figure 1, how could we come up with a secant line that's a better approximation of the tangent line? We can incrementally move g closer and closer to p and observe that it becomes a better approximation.

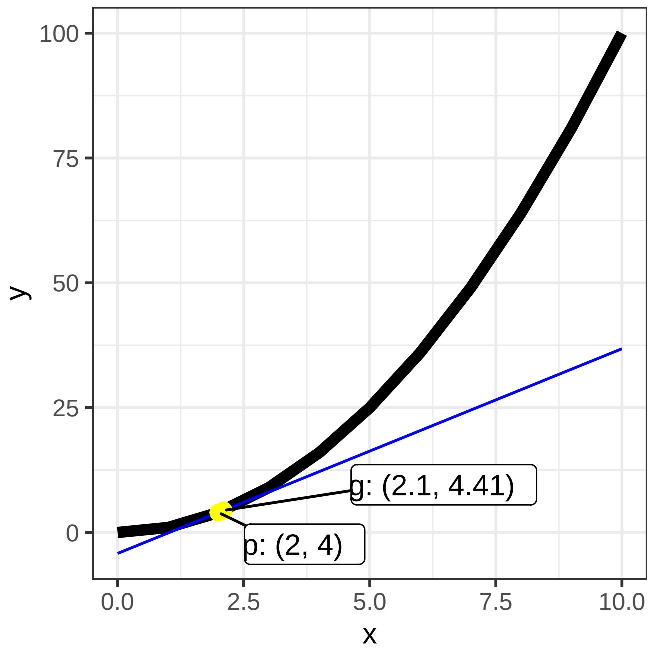

Figure 2: Moving points closer together results in a more accurate estimation of the tangent line

As points p and g come closer together in Figure 2, we see the line is a better approximation of the tangent line.

Note: p can't equal q because two points are needed to make a line.

Big idea for limits. How close can two points get to each other without being the same point? The solution is to let two points get infinitesimally close together without actually equalling each other. If this is the case, then the secant will be identical to the tangent. So a limit is letting something get infinitesimally close without touching it.

1.1.1 Example of finding tangent line

For function $f(x) = x^2$, can we find the tangent line at $(1, 1)$?

1. We can think of a general second point of the form $q : (x, x^2)$
2. Recall point slope formula: $y - y_1 = m(x - x_1)$
3. For a tangent line, the formula is similar but we'd substitute in the slope of the tangent line: $y - y_1 = M_{tan}(x - x_1)$

4. Substitute in the fixed point $(1,1)$.

5. Make the line into a secant, and move q closer to $(1,1)$ to find the tangent.

6. Derive a general slope for $M_{sec} = \frac{x^2-1}{x-1}$. This holds because we know the general form of $q : (x, x^2)$.

7. As $Q \to P$, then $M_{sec} \to M_{tan}$.

8. But notice that, in the general slope formula for M_{sec}, moving q all the way to p results in $0/0$, which doesn't work.

9. But also notice that the general slope formula for M_{sec} can be simplified:

$$\begin{aligned} M_{sec} &= \frac{x^2-1}{x-1} \\ &= \frac{(x+1)(x-1)}{x-1} \\ &= x+1 \end{aligned}$$

10. Remember, after a simplification, you don't get rid of domain restrictions. So, as $q \to p$, $M_{sec} = x + 1 \to 2$.

11. But you are allowed to make a jump to make a claim that $M_{tan} = 2$. And solving for slope intercept form gives you the equation of a tangent line at point $(1,1)$, which is $y = 2x - 1$.

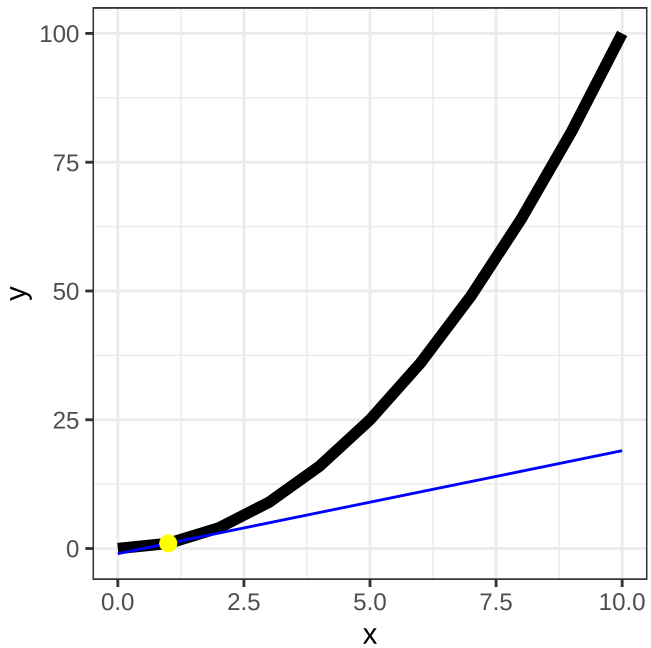

Figure 3: A tangent line at point $p : (1, 1)$

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