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Calculus

Contents

Part I

Fundamentals

1

Basic Properties of Numbers

> To be conscious that you are ignorant is a great step to knowledge. – Benjamin Disraeli

We're going to consider twelve properties of numbers, and the first nine are concerned with addition and multiplication.

1.1 Addition

Definition 1. *Regard addition as an operation which can be performed on a pair of numbers, $a + b$.*

He argues it might be intuitive to want to define addition over a series of numbers, but that series can just be broken down into summing of different pairs. This reveals that the order of the summation yields equivalent results, which gives our first property:

Property 1 (Associative law for addition)**.** *If a, b, and c are any numbers, then*

$$a + (b + c) = (a + b) + c$$

This generalizes straightforwardly to $a_1 + \ldots + a_n$. He notes that a reasonable approach to showing this extension is outlined in Problem 24.

1.1.1 Properties involving zero

Property 2 (Existence of an additive identity)**.** *If a is any number, then*

$$a + 0 = 0 + a = a.$$

Property 3 (Existence of additive inverses)**.** *For every number a, there is a number $-a$ such that*

$$a + (-a) = (-a) + a = 0$$

We can now use these three properties to prove a simple assertion, which is that if a number x satisfies $a + x = a$, $x = 0$.

$$
\begin{array}{ll}
\text{If} & a + x = a, \\
\text{then} & (-a) + (a + x) = (-a) + a = 0; \\
\text{hence} & ((-a) + a) + x = 0; \\
\text{hence} & 0 + x = 0 \\
\text{hence} & x = 0
\end{array}
$$

Ahhh that's pretty cool. He goes on to show that you can support basic algebraic maneuvers with these three properties.

Property 4 (Commutative law for addition)**.** *If a and b are any numbers, then*

$$a + b = b + a$$

He goes on to say that commutative law doesn't hold for other types of relations (eg, subtraction). But in order to have all algebra moves on the table, it's necessary to introduce multiplication.

1.2 Multiplication

Property 5 (Associative law of multiplication)**.** *If a, b and c are any numbers, then*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Property 6 (Existence of multiplicative identity)**.** *If a is any number, then*

$$a \cdot 1 = 1 \cdot a = a$$

$$\textit{Moreover, } 1 \neq 0$$

He's saying we have to list $1 \neq 0$ because there's no way to prove based only on the other properties—they would all hold if 0 was the only number.

Property 7 (Existence of multiplicative inverses)**.** *For every number $a \neq 0$, there is a number a^{-1} such that*

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

Property 8 (Commutative law of multiplication)**.** *If a and b are any numbers, then*

$$a \cdot b = b \cdot a$$

Points out how necessary $a \neq 0$ is in Property 7, as there is no number 0^{-1} satisfying $0 \cdot 0^{-1} = 1$. This is cool: just as how subtraction was defined in terms of addition, division is defined in terms of multiplication. Note that the usual form of an inverse (at least for an integer) is:

$$a^{-1} = \frac{1}{a}$$

So he says the symbol a/b can be thought of as $a \cdot b^{-1}$. We can show that, as long as $a \neq 0$, then for $a \cdot b = a \cdot c$, that $b = c$.

$$\begin{array}{ll} \text{If} & a \cdot b = 0 \text{ and } a \neq 0, \\ \text{then} & a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c); \\ \text{hence} & (a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c; \\ \text{hence} & 1 \cdot b = 1 \cdot c; \\ \text{hence} & b = c. \end{array}$$

It also follows from Property 7 that if $a \cdot b = 0$ then either $a = 0$ or $b = 0$ (implied that this includes the possibility that they're both zero). We can show that, if we know one of them *isn't* zero, that the other one must be:

$$\begin{array}{ll} \text{if} & a \cdot b = 0 \text{ and } a \neq 0, \\ \text{then} & a^{-1} \cdot (a \cdot b) = 0; \\ \text{hence} & (a^{-1} \cdot a) \cdot b = 0; \\ \text{hence} & 1 \cdot b = 0; \\ \text{hence} & b = 0. \end{array}$$

This concept of one or another variable must be equal to zero comes into play in the familiar factoring of binomial situations (eg, $(x - 1)(x - 2) = 0$), where we know $x = 1$ or $x = 2$.

1.3 Distribution

This next property gives us tremendous ability to prove lots of things:

Property 9 (Distributive law)**.** *If* a, b, *and* c *are any numbers, then*

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

With Property 9, we can now show the only case when $a - b = b - a$:

If	$a - b = b - a,$	
then	$(a-b) + b = (b-a) + b = b + (b-a);$	
then	$a = b + b - a;$	
then	$a + a = (b + b - a) + a = b + b;$	
Consequently	$a \cdot (1+1) = b \cdot (1+1),$	
and therefore	$a = b.$	

1.3.1 More zero stuff

With Property 9, we can do more reckoning around zero, such as showing that $a \cdot 0 = 0$. (Some of this upcoming logic gets pretty nifty). The proof:

$$\begin{aligned} a \cdot 0 &= a \cdot (0+0) && (1.1)\\ &= a \cdot 0 + a \cdot 0 && (1.2)\\ (a \cdot 0) - (a \cdot 0) &= a \cdot 0 + a \cdot 0 - a \cdot 0 && (1.3)\\ 0 &= a \cdot 0 && (1.4) \end{aligned}$$

Step 1.1 is by Proposition 2 (the additive identity).

He next talks about using Property 9 to show why multiplying two negative numbers equals a positive number. Starting with proving that $(-a) \cdot b = -(a \cdot b)$:

$$\begin{aligned} (-a) \cdot b + a \cdot b &= [(-a) + a] \cdot b && (1.5)\\ &= 0 \cdot b && (1.6)\\ &= 0 && (1.7) \end{aligned}$$

He says then by adding $-(a \cdot b)$ to both sides, we see that $(-a) \cdot b = -(a \cdot b)$. A few things I'm learning about proofs here:

- I think knowing where to get started in proofs is one of the hardest parts. Here he's implicitly starting with applying Proposition 3 (additive inverse), by relying on the logic that, if two quantities are the same, then adding the quantity to its inverse should reduce to zero. So if $(-a) \cdot b = -(a \cdot b)$, then we should be able to show that $(-a) \cdot b + a \cdot b = 0$.
- When we get to the end $(= 0)$, that's actually a shorthand for the initial expression equaling whatever we've arrived at (ie, $(-a) \cdot b + a \cdot b = 0$). So when he says something like "adding $-(a \cdot b)$ to both sides, that really means $(-a) \cdot b + a \cdot b - (a \cdot b) = 0 - (a \cdot b)$, which reduces to $(-a) \cdot b = -(a \cdot b)$.

We then go on to prove that $(-a)\cdot(-b) = a\cdot b$:

$$
\begin{aligned}
(-a)\cdot(-b) + [-(a\cdot b)] &= (-a)\cdot(-b) + (-a)\cdot b && (1.8)\\
&= -a\cdot(-b+b) && (1.9)\\
&= -a\cdot 0 && (1.10)\\
&= 0 && (1.11)\\
(-a)\cdot(-b) + [-(a\cdot b)] + (a\cdot b) &= 0 + (a\cdot b) && (1.12)\\
(-a)\cdot(-b) &= a\cdot b && (1.13)
\end{aligned}
$$

Step 1.8 applies Proposition 3 and applies the previous proof that $(-a)\cdot b = -(a\cdot b)$. Step 1.9 applies Proposition 9, and step 1.10 applies Proposition 3. He notes that this last proof is a *consequence* of the proposition—it simply follows from them once they're established. Goes on to hype up Proposition 9 a bit more, showing how essential it is for basic algebraic operations like factoring and multiplication.

1.4 Inequalities

Inequalities apparently feature prominently in calculus. Numbers satisfying $a > 0$ are called **positive**, while those satisfying $a < 0$ are called **negative**. If we denote the collection of all positive numbers with P, we can state the remaining three propositions in terms of P:

Property 10 (Trichotomy law)**.** *For every number a, one and only one of the following holds:*

(i) $a = 0$,

(ii) *a is in the collection P,*

(iii) *$-a$ is in the collection P.*

Property 11 (Closure under addition)**.** *If a and b are in P, then $a+b$ is in P*

Property 12 (Closure under multiplication)**.** *If a and b are in P, then $a\cdot b$ is in P.*

He says these properties should be paired with the following definitions:

$$
\begin{aligned}
a > b &\quad \text{if} \quad a - b \text{ is in } P;\\
a < b &\quad \text{if} \quad b > a;\\
a \geq b &\quad \text{if} \quad a > b \text{ or } a = b;\\
a \leq b &\quad \text{if} \quad a < b \text{ or } a = b.
\end{aligned}
$$

The first one here is the only one that's a bit tricky, which is essentially saying that as long as the difference is positive, then we can say $a > b$. He also points out that $a > 0$ if and only if a is in P.

Glossing over some details. He goes on to say that if $a < b$ and $b < c$, then $a < c$. He gets there with the following logic. First, suppose $a < b$ and $b < c$. Then

$$\begin{aligned} & b - a \text{ is in } P, \\ \text{and} \quad & c - b \text{ is in } P, \\ \text{so} \quad & c - a = (c - b) + (b - a) \text{ is in } P. \end{aligned}$$

Yea this logic of "something minus something is positive" is a little tricky. Let's zoom in on $b - a$ is in P provided that $a < b$. What we *don't* have is $b > a > 0$. If $b \in P$, then the only constraint on a is $a < b$—ie, a can be negative because, for example, $10-(-5) > 0$. If $b < 0$, then $a < b$ still satisfies $b - a \in P$.

He also shows, by Property 12, that $ab > 0$ if $a > 0,\ b > 0$ and also if $a < 0,\ b < 0$. Thus, $a^2 > 0$ if $a \neq 0$. The fact that $-a > 0$ if $a < 0$ sets us up to define the absolute value function:

Definition 2 (Absolute Value)**.**

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

He says the most straightforward way to deal with absolute values is to treating several cases separately.

Theorem 1. *For all real numbers a and b, we have*

$$|a + b| \leq |a| + |b|$$

Proof. Consider four cases:

$$\begin{array}{lll} (1) & a \geq 0, & b \geq 0; \\ (2) & a \geq 0, & b \leq 0; \\ (3) & a \leq 0, & b \geq 0; \\ (4) & a \leq 0, & b \leq 0; \end{array}$$

For case (1), we see $|a+b| = |a| + |b|$. For case (4), we have $a + b \leq 0$, and again equality holds:

$$|a + b| = -(a + b) = -a + (-b) = |a| + |b|$$

My note: Again with proofs, it's a little obtuse to me where the reasoning behind the initial claim $a + b \leq 0$ comes from. Like I can clearly see that it's true, but I don't know how I would have generated that as a starting point.

Case (2) starts to get tricky. He says we must prove that

$$|a + b| \leq a - b$$

It took me a minute to see where this comes from. I understand the right side of the inequality as being the result of passing the original expression $(|a| + |b|)$ through the absolute value function for $a \geq 0,\ b \leq 0$. He then proceeds to check whether this inequality holds true for cases where $a+b \geq 0$ and $a + b \leq 0$. For the first case $a + b \geq 0$,

$$\begin{aligned} a + b &\leq a - b \\ b &\leq -b \end{aligned}$$

Which is true when $b \leq 0$. Next, for $a + b \leq 0$ we show that

$$\begin{aligned} -a - b &\leq a - b \\ -a &\leq a \end{aligned}$$

It's hard for me to see where $-a - b$ comes from. Maybe it's something like

$$\begin{aligned} a + b &\leq 0 \\ |a + b| &= -(a + b) \\ &= -a + (-b) \end{aligned}$$

Yea best I can tell that's it. Just gotta think back to the functional form of the absolute value function and apply a negative sign to any input less than zero. And so this is an example of what he alluded to above, where one needs to treat absolute values with respect to several different cases that revolve around zero.

$\square$

He says that we can often find shorter methods for dealing with absolute values. For example, for the last theorem, we can take advantage of the fact that

$$|a| = \sqrt{a^2}$$

to show that

$$\begin{aligned} (|a + b|)^2 = (a + b)^2 &= a^2 + 2ab + b^2 \\ &\leq a^2 + 2|a| \cdot |b| + b^2 \\ &= |a|^2 + 2|a| \cdot |b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

Therefore, $|a + b| \leq |a| + |b|$ because $x^2 < y^2$ implies $x < y$. He finally points out that this theorem is an equality if a and b have the same sign or one is zero; the theorem is a less than inequality if a and b have opposite signs.