Lecture 1.2: Limit Properties. Techniques of Limit Computation

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1 Basics

1. **Limit of a constant**

$$\lim_{x \to a} C = C$$

The limit of a constant is always the constant (think because the plot is just a horizontal line).

2. **Limit of x**

$$\lim_{x \to a} x = a$$

Think because $f(x) = x$ is just identity, so point a on x is also a on y.

3. **Zero limit**

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

2 Properties

Given two functions with limits that exist:

$$\lim x \to af(x) = L_1$$
$$\lim x \to ag(x) = L_2$$

1. **Function joining and separation**

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

This operation works both ways (can also join).

2. **Same thing for multiplication**

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

3. **For division**

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \lim_{x\to a} g(x) \neq 0$$

4. **Exponents**

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n \to \lim_{x\to a} \sqrt[n]{f(x)} \to \sqrt[n]{\lim_{x\to a} f(x)}$$

2.1 Example

$$\begin{aligned}\lim_{x\to 2}(x^3 - 2x + 7) &\to \lim_{x\to 2} x^3 - \lim_{x\to 2} 2x + \lim_{x\to 2} 7\\ &\to \left[\lim_{x\to 2} x\right]^3 - \left(\lim_{x\to 2} 2\right)\cdot\left(\lim_{x\to 2} x\right) + \lim_{x\to 2} 7\\ &\to 2^3 - 2\cdot 2 + 7\\ &\to 11\\ f(2) &= 11\end{aligned}$$

For any polynomial, all you need to do is plug the value at the limit as the variable in order to determine the limit. The general case for any P polynomial:

$$\lim_{x\to a} P(x) = P(a)$$

2.2 Holes and asymptotes

2.2.1 Holes

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

With limits, because you're not actually reaching the point at the limit, it's okay to simplify out a domain problem. So:

$$\begin{aligned}&\lim_{x\to 2}\frac{x^2-4}{x-2}\\ &\frac{(x+2)(x-2)}{x-2}\\ &x+2\\ &\lim_{x\to 2} = 4\end{aligned}$$

If the domain issue can simplify, it's a hole in the graph. If it can't, then it's some type of asymptote. If a fraction reduces to zero over zero, I think he's saying that means it can be factored.

2.2.2 Asymptotes

$$\begin{aligned}\lim_{x\to 5}\frac{x^3-3x-10}{x^2-10x+25}\\ \lim_{x\to 5}\frac{(x-5)(x+2)}{(x-5)(x-5)}\\ \lim_{x\to 5}\frac{x+2}{x-5}\end{aligned}$$

We have ourselves a vertical asymptote.

Sign analysis test

To determine the direction of the asymptotes in both directions, we need to construct a number line between the two values that will make the numerator and denominator equal to zero (here -2 and 5). Sample on both sides of the limit, plug into the function, and determine the sign on both sides.

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