Lecture 1.2: Limit Properties. Techniques of Limit Computation

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1 Basics

1. **Limit of a constant**
$$\lim_{x \to a} C = C$$
The limit of a constant is always the constant (think because the plot is just a horizontal line).

2. **Limit of x**
$$\lim_{x \to a} x = a$$
Think because $f(x) = x$ is just identity, so point a on x is also a on y.

3. **Zero limit**
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

2 Properties

Given two functions with limits that exist:

$$\lim_{x \to a} f(x) = L_1$$
$$\lim_{x \to a} g(x) = L_2$$

1. **Function joining and separation**
$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
This operation works both ways (can also join).

2. **Same thing for multiplication**

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

3. **For division**

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \lim_{x \to a} g(x) \neq 0$$

4. **Exponents**

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n \to \lim_{x \to a} \sqrt[n]{f(x)} \to \sqrt[n]{\lim_{x \to a} f(x)}$$

2.1 Example

$$\begin{aligned}
\lim_{x \to 2} (x^3 - 2x + 7) &\to \lim_{x \to 2} x^3 - \lim_{x \to 2} 2x + \lim_{x \to 2} 7 \\
&\to \left[\lim_{x \to 2} x \right]^3 - \left(\lim_{x \to 2} 2 \right) \cdot \left(\lim_{x \to 2} x \right) + \lim_{x \to 2} 7 \\
&\to 2^3 - 2 \cdot 2 + 7 \\
&\to 11 \\
f(2) &= 11
\end{aligned}$$

For any polynomial, all you need to do is plug the value at the limit as the variable in order to determine the limit. The general case for any P polynomial:

$$\lim_{x \to a} P(x) = P(a)$$

2.2 Holes and asymptotes

2.2.1 Holes

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

With limits, because you're not actually reaching the point at the limit, it's okay to simplify out a domain problem. So:

$$\begin{aligned}
&\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\
&\quad \frac{(x+2)(x-2)}{x-2} \\
&\quad x + 2 \\
&\lim_{x \to 2} = 4
\end{aligned}$$

If the domain issue can simplify, it's a hole in the graph. If it can't, then it's some type of asymptote. If a fraction reduces to zero over zero, I think he's saying that means it can be factored.

2.2.2 Asymptotes

$$\begin{aligned}\lim_{x\to 5}\frac{x^3-3x-10}{x^2-10x+25}\\ \lim_{x\to 5}\frac{(x-5)(x+2)}{(x-5)(x-5)}\\ \lim_{x\to 5}\frac{x+2}{x-5}\end{aligned}$$

We have ourselves a vertical asymptote.
Sign analysis test
To determine the direction of the asymptotes in both directions, we need to construct a number line between the two values that will make the numerator and denominator equal to zero (here -2 and 5). Sample on both sides of the limit, plug into the function, and determine the sign on both sides.

(coming back after awhile; I think the below is an example of factoring a limit, not an example of the sign analysis test...)

Example.

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} \tag{1}$$

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1} \tag{2}$$

$$\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1} \tag{3}$$

$$\lim_{x\to 1}\sqrt{x}+1 \tag{4}$$

$$\lim_{x\to 1}\sqrt{1}+1=2 \tag{5}$$

The logic behind (2) is to multiply by the *conjugate*, since you can't necessarily factor anything.
The trick to get the denominator in step (3) is:

$$\begin{aligned}(\sqrt{x}-1)\cdot(\sqrt{x}+1)\\ \sqrt{x}\cdot\sqrt{x}+\sqrt{x}-\sqrt{x}-1\\ x-1\end{aligned}$$

Forget what the formal name of that is... but it's the same way you'd distribute one term into another to get a classic polynomial. Been a minute. Yea so conjugate is the term you multiply another term by to get that middle term to cancel. Eg, the conjugate of $p-q$ is $p+q$.

Another example.

$$
\begin{aligned}
&\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \\
&\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \\
&\lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)} \\
&\lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} \\
&\lim_{x\to 0} \frac{1}{\sqrt{1}+1} \\
&\lim_{x\to 0} \frac{1}{2}
\end{aligned}
$$

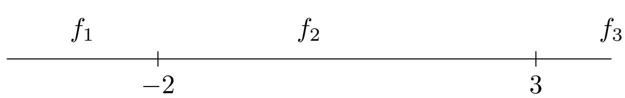
3 Piece-Wise Limits

We're going to find some one-sided limits and see if they are equal.

Example.

$$
f(x) = \begin{cases} \frac{1}{x+2} & \text{if } x < -2 \\ x^2 - 5 & \text{if } -2 > x \leq 3 \\ \sqrt{x+13} & \text{if } x > 3 \end{cases}
$$

A helpful approach is to plot a number line and draw the intervals of the piece-wise function:



If the functions have the same limit as they approach the interval between them, then the limit can be said to exist; otherwise, it doesn't exist.

Let's work through the first two functions with -2 as the limit:

$$
\begin{aligned}
f_1(x) &= \lim_{x\to -2} \frac{1}{x+2} \\
&= \lim_{x\to -2} \frac{1}{x+2} \cdot \frac{x-2}{x-2} \\
&= \lim_{x\to -2} \frac{x-2}{x^2-4}
\end{aligned}
$$

Hmm okay I'm stuck. Multiplying by the conjugate doesn't seem to free up the denominator from being zero when plugging in the limit.

Aha okay so if the function doesn't reduce to 0/0, that suggests there's an asymptote, not a hole. Then we need to do the sign analysis test to determine the direction of the asymptote.

$$\begin{aligned} f_1(x) &= \lim_{x \to -2} \frac{1}{-3+2} \\ &= -1 \end{aligned}$$

$$\lim_{x \to -2^-} \frac{1}{x+2} = -\infty$$

Because the limit approaching -2 from the left does not equal the limit approaching -2 from the right (I didn't show computation for approaching from the right here), the limit is said to not exist.

4 Limits of Trig Fucnctions

First thing to note is that $sin(x)$ and $cos(x)$ are continuous everywhere.

$$\begin{aligned} \lim_{x \to a} sin(x) &= sin(a) \\ \lim_{x \to a} cos(x) &= cos(a) \\ \lim_{x \to a} tan(x) &= \lim_{x \to a} \frac{sin(x)}{cos(x)} \\ &= \frac{\lim_{x \to a} sin(x)}{\lim_{x \to a} cos(x)} \\ &= \frac{sin(a)}{cos(a)} \\ \lim_{x \to a} tan(x) &= tan(a), cos(x) \neq 0, x \neq \pm\frac{\pi}{2} \end{aligned}$$

Example.

$$\lim_{x \to 1} cos\left(\frac{x^2-1}{x-1}\right) \tag{6}$$

$$cos\left[\lim_{x \to 1} \frac{x^2-1}{x-1}\right] \tag{7}$$

$$cos\left[\lim_{x \to 1} \frac{(x+1)(x-1)}{x-1}\right] \tag{8}$$

$$cos[\lim_{x \to 1} x+1] \tag{9}$$

$$cos(2) \tag{10}$$

I guess (7) is a legal move. His justification was "Cosine is continuous, so by composition...".

Example.

$$\lim_{x\to\pi/2}[3x^2 + cos\ x]$$
$$\lim_{x\to\pi/2} 3\cdot\left(\frac{\pi}{2}\right)^2 + 0$$
$$\lim_{x\to\pi/2} 3\cdot\left(\frac{\pi^2}{4}\right)$$
$$\lim_{x\to\pi/2} \frac{3\pi^2}{4}$$
$$\lim_{x\to\pi/2}[3x^2 + cos\ x] = \frac{3\pi^2}{4}$$

Checking against his solution. After substituting the limit in for x (step 2), you can drop the limit from the computations. Otherwise everything is good here.

4.1 Trig limit identities

Example.

$$\lim_{x\to 0}\frac{sin(x)}{x}$$

He goes through a crazy procedure of trig and the unit circle to eventually derive $1 \geq sin(x)/x \geq cos(x)$ and demonstrates the **squeeze theorem**, which says that when a function falls between two functions that share a limit, then the squeezed function also shares that limit.

Note: When you take the reciprocal when inequalities are involved you need to flip the inequality.

Alright so I guess there are a few trig limits (ie, identities) that are important to know. They are the following:

$$\lim_{x\to 0}\frac{sin(x)}{x} = 1$$
$$\lim_{x\to 0}\frac{1 - cos(x)}{x} = 0$$
$$\lim_{x\to 0}\frac{tan(x)}{x} = 1$$

These all have elaborate proofs that he walked through that I'm not reproducing here. These will all be important to use when solving for other limits.
Example.

$$\lim_{x\to 0} \frac{sin(2x)}{x}$$
$$\lim_{x\to 0} \frac{sin(2x)}{x} \cdot \frac{2}{2}$$
$$2 \cdot \lim_{x\to 0} \frac{sin(2x)}{2x}$$
$$u = 2x$$
$$2 \cdot \lim_{x\to 0} \frac{sin(u)}{u}$$
$$2 \cdot 1 = 2$$

Making a substitution with $u = 2x$ because as $2x$ approaches 0 so does u.

Example.

$$\lim_{x\to 0} \frac{sin(5x)}{sin(6x)}$$
$$\lim_{x\to 0} \frac{sin(5x)}{sin(6x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$
$$\lim_{x\to 0} \frac{\frac{sin(5x)}{x}}{\frac{sin(6x)}{x}}$$
$$\frac{5 \cdot \lim_{x\to 0} \frac{sin(5x)}{5x}}{6 \cdot \lim_{x\to 0} \frac{sin(6x)}{6x}}$$
$$\frac{5 \cdot 1}{6 \cdot 1}$$
$$\frac{5}{6}$$

Example.

$$\lim_{x\to 0} \frac{sin(x^2)}{x}$$

$$\lim_{x\to 0} \frac{sin(x^2)}{x} \cdot \frac{x}{x}$$

$$\lim_{x\to 0} \frac{x \cdot sin(x^2)}{x^2}$$

$$\lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{sin(x^2)}{x^2}$$

$$0 \cdot 1 = 0$$

I could really use a refesher on the algebraic rules on how you can move things around when multiplying fractions.

Ah so in the earlier examples when he pulled out a constant and dropped the limit notation, that's because the limit as x approaches any number when the function is a constant is just that constant (ie, $\lim_{x\to 0} 2 = 2$).

Example.

$$\lim_{x\to 0} \frac{sin^2(x)}{x}$$

$$\lim_{x\to 0} \frac{sin(x)}{x} \cdot \lim_{x\to 0} sin(x)$$

$$1 \cdot \lim_{x\to 0} sin(x)$$

$$0$$

Example.

$$\lim_{x\to 0} sin(\frac{1}{x})$$

$$\lim_{x\to 0} sin(\frac{1}{x}) \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{sin(\frac{1}{x})}{\frac{1}{x}} \cdot \lim_{x\to 0} \frac{1}{x}$$

$$\lim_{x\to 0} \frac{1}{x}$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

Hm okay got this one wrong. He's saying that the sine function will just oscillate faster and faster as x approaches zero, but it'll never reach zero, and since the two sides of the limit won't converge on the same number, the limit is said to not exist.

Example.

This one's funky, uses the squeeze theorem.

$$\lim_{x \to 0} x\ sin(\frac{1}{x})$$
$$-|x| \leq x\ sin(\frac{1}{x}) \geq |x|$$
$$\lim_{x \to 0} -|x| = 0$$
$$\lim_{x \to 0} |x| = 0$$
$$\lim_{x \to 0} x\ sin(\frac{1}{x}) = 0$$

Since we know that the range of sin is $[-1, 1]$, then scaling the output by x must fall between $[-x, x]$. We can define the limits for those absolute value functions and observe that they are equal to each other, so by the squeeze theorem the function of interest ($x\ sin(1/x)$) must equal that same limit.

Example.

$$\lim_{x \to 0} \frac{2 - cos(3x) - cos(4x)}{x}$$
$$\lim_{x \to 0} \frac{1 - cos(3x) + 1 - cos(4x)}{x}$$
$$\lim_{x \to 0} \frac{1 - cos(3x)}{x} + \lim_{x \to 0} \frac{1 - cos(4x)}{x}$$
$$\lim_{x \to 0} \frac{1 - cos(3x)}{x} \cdot \frac{3}{3} + \lim_{x \to 0} \frac{1 - cos(4x)}{x} \cdot \frac{4}{4}$$
$$3 \cdot \lim_{x \to 0} \frac{1 - cos(3x)}{3x} + 4 \cdot \lim_{x \to 0} \frac{1 - cos(4x)}{4x}$$
$$3 \cdot 0 + 4 \cdot 0$$
$$0$$

Some absolutely bananas fractions moves here. Step two is wild—like I can see it in reverse but I would've never thought that splitting the two would be legel. Then it was also news to me that you could multiply fractions by a 1/1 term separately with different factors for different terms separated by addition

/ subtraction (step 4). This is some serious fraction kung-fu.

Example.

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 - 3\ sin(x)}{x} \\
\lim_{x\to 0} \frac{x^2}{x} - \lim_{x\to 0} \frac{3\ sin(x)}{x} \\
0 - \lim_{x\to 0} \frac{3\ sin(x)}{x} \\
-3 \cdot \lim_{x\to 0} \frac{sin(x)}{x} \\
-3
\end{aligned}$$

Skipping a couple examples. He's stressing the importance of knowing trig identites for simplifying some of these expressions.

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\theta^2}{1 - cos(\theta)} \\
\lim_{\theta\to 0} \frac{\theta^2}{1 - cos(\theta)} \cdot \frac{1 + cos(\theta)}{1 + cos(\theta)} \\
\lim_{\theta\to 0} \frac{\theta^2 + \theta^2\ cos(\theta)}{1 - cos^2(\theta)} \\
\lim_{\theta\to 0} \frac{\theta(\theta + \theta\ cos(\theta))}{sin(\theta) \cdot sin(\theta)} \\
\lim_{\theta\to 0} \frac{\theta}{sin(\theta)} \cdot \lim_{\theta\to 0} \frac{\theta\ (1 + cos(\theta))}{sin(\theta)} \\
\lim_{\theta\to 0} \frac{\theta}{sin(\theta)} \cdot \lim_{\theta\to 0} \frac{1 + cos(\theta)}{sin(\theta)} \\
\lim_{\theta\to 0} \frac{1 + cos(\theta)}{sin(\theta)}
\end{aligned}$$

Nope. Don't distribute in step 3 and do the following:

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\theta^2\ (1 + cos(\theta))}{sin^2(\theta)} \\
\lim_{\theta\to 0} \frac{\theta^2}{sin^2(\theta)} \cdot \lim_{\theta\to 0} 1 + cos(\theta) \\
1 \cdot \lim_{\theta\to 0} 1 + cos(\theta) \\
2
\end{aligned}$$

In a previous example that I didn't go through here, he showed the follwing:

$$\lim_{x\to 0} \frac{x^2}{sin^2(x)}$$
$$\lim_{x\to 0} \left(\frac{x}{sin(x)}\right)^2$$
$$\lim_{x\to 0} \left(\frac{sin(x)}{x}\right)^{-2}$$
$$\left(\lim_{x\to 0} \frac{sin(x)}{x}\right)^{-2}$$
$$1^{-2} = 1$$