Lecture 1.1: An Introduction to Limits

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1 Limits

Limits are the basis of calculus. There are two goals in calculus:

1. Given any curve, find the slope of a curve at a point (find the tangent to a curve at a point).
2. Given any curve defined by a function, can you between two points find the area under the curve?

1.1 Tangent Problem

If you give me an arbitrary curve, and give me a point p, I need to find the slope of the curve at point p. If we have the slope and we have the point, we can make the tangent line.

Tangent line. A tangent line is a line that touches at exactly one point and is "parallel to the curve at that point.

One problem if we only have one point and no slope is we can't draw a line—we need at least one more point. We can find another point along the curve and connect the two to form a secant line.

Secant line. A secant line is a line that intersects a curve at two points (see Figure 1).

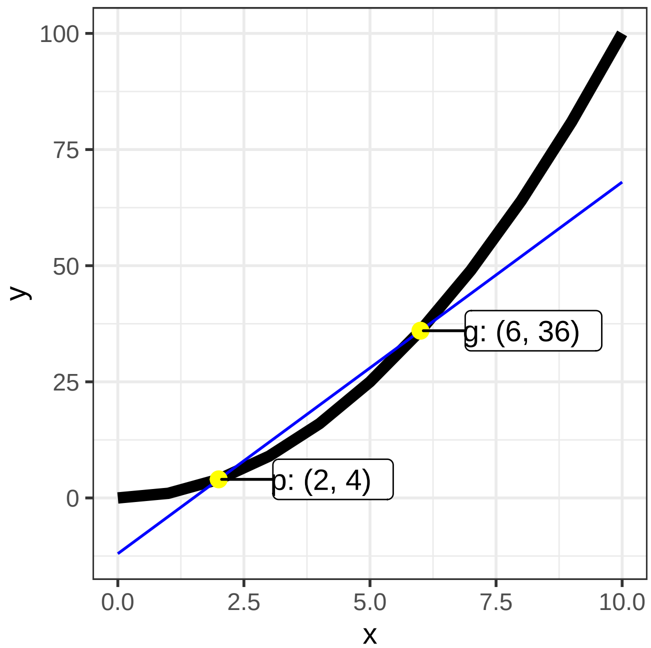

Figure 1: A secant line intersects two points on a curve.

So in Figure 1, how could we come up with a secant line that's a better approximation of the tangent line? We can incrementally move g closer and closer to p and observe that it becomes a better approximation.

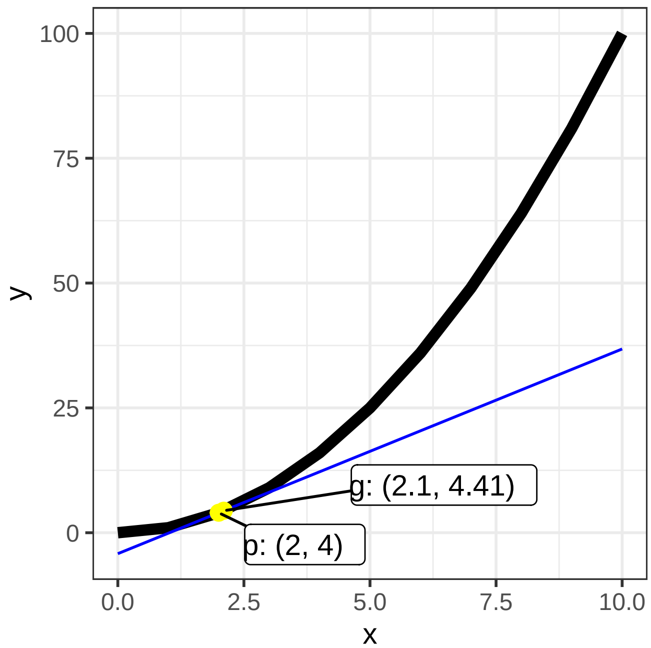

Figure 2: Moving points closer together results in a more accurate estimation of the tangent line

As points p and g come closer together in Figure 2, we see the line is a better approximation of the tangent line.

Note: p can't equal q because two points are needed to make a line.

Big idea for limits. How close can two points get to each other without being the same point? The solution is to let two points get infinitesimally close together without actually equalling each other. If this is the case, then the secant will be identical to the tangent. So a limit is letting something get infinitesimally close without touching it.

1.1.1 Example of finding tangent line

For function $f(x) = x^2$, can we find the tangent line at $(1, 1)$?

1. We can think of a general second point of the form $q : (x, x^2)$
2. Recall point slope formula: $y - y_1 = m(x - x_1)$
3. For a tangent line, the formula is similar but we'd substitute in the slope of the tangent line: $y - y_1 = M_{tan}(x - x_1)$

4. Substitute in the fixed point $(1, 1)$.
5. Make the line into a secant, and move q closer to $(1, 1)$ to find the tangent.
6. Derive a general slope for $M_{sec} = \frac{x^2-1}{x-1}$. This holds because we know the general form of $q : (x, x^2)$.
7. As $Q \to P$, then $M_{sec} \to M_{tan}$.
8. But notice that, in the general slope formula for M_{sec}, moving q all the way to p results in $0/0$, which doesn't work.
9. But also notice that the general slope formula for M_{sec} can be simplified:

$$\begin{aligned} M_{sec} &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x + 1)(x - 1)}{x - 1} \\ &= x + 1 \end{aligned}$$

10. Remember, after a simplification, you don't get rid of domain restrictions. So, as $q \to p$, $M_{sec} = x + 1 \to 2$.
11. But you are allowed to make a jump to make a claim that $M_{tan} = 2$. And solving for slope intercept form gives you the equation of a tangent line at point $(1, 1)$, which is $y = 2x - 1$.

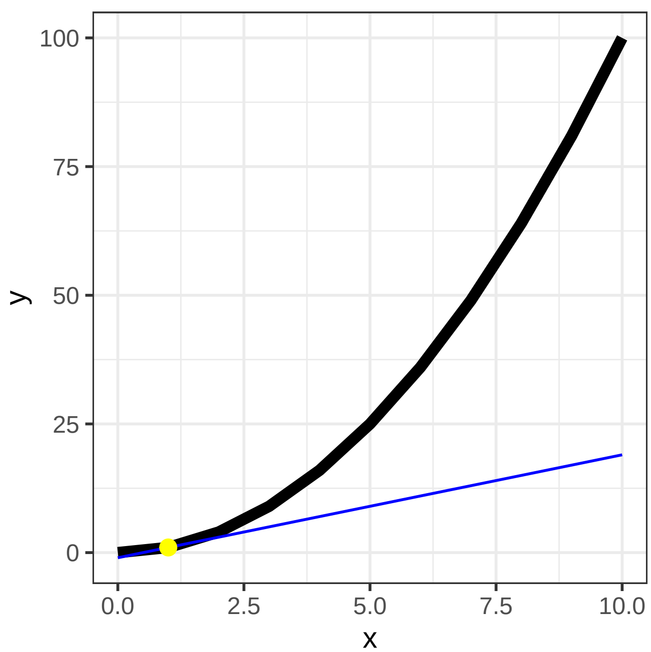

Figure 3: A tangent line at point $p : (1, 1)$

1.2 The area problem

The other big calculus idea is finding the area under a curve. The basic idea is that you're drawing many many tiny rectangles in that space under the curve, then taking the area of the rectangles. As the number of rectangles approaches infinity, the approximation of area under the curve approaches perfection.

1.3 Back to limits

A *limit* tells you what a function does as the variable approaches a specific value.

Example. For $f(x) = x^2$, what happens as $x \to 2$?

The way he's working through this is to put a table on the board mapping x to x^2 for values approaching 2 on both the high and low ends. The function has to approach the same value from both the left and the right for the limit to exist. Write a limit like: $\lim_{x \to 2} x^2 = 4$. We actually don't care about what happens when x *gets* to that number.

$$\lim_{x \to a} f(x) = L$$

To talk about a limit coming from the right hand side: $\lim_{x \to a^+} f(x) = L$. Add negative sign for from the left. These quantities can be pretty different depending if there's a big discontinuity in the function around the limit.

Note: In order from a limit to exist at a point, you must have $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$. This suggests that if you have a limit around a big discontinuity, that there is no limit "at a point". Ah, so this plus / minus notation is specific to finding the limit coming from a certain side. Without the sign, it's like the general limit and, if it converges at a point, it'll have a single solution.

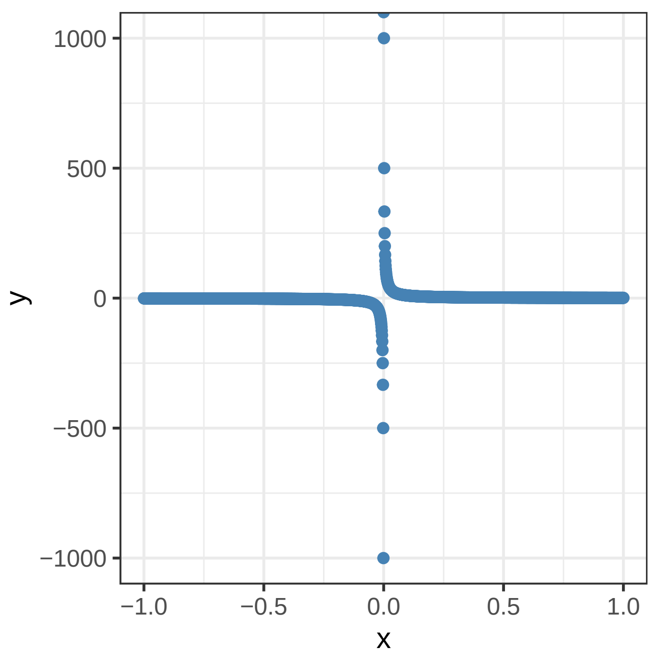

Figure 4: $\lim_{x \to 0} f(x)$

The limit is ∞ !!!

So actually, because the function goes to pos or neg infinity as it approaches from either side, it doesn't converge on a single limit, and so we'd say the limit doesn't exist.

He's talking a bit about the general case of thinking about an infinity limit as you approach some x: $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a^+} = f(x) - \infty$, $\lim_{x \to a^-} f(x)\infty$, $\lim_{x \to a^-} f(x) - \infty$. Should be able to understand what happens to the function at all four of these cases.