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THE WESTWARD INTENSIFICATION OF WIND-DRIVEN OCEAN CURRENTS

Henry Stommel

(Contribution No. 408, Woods Hole Oceanographic Instiution)

Abstract--A study is made of the wind-driven circulation in a homogeneous rectangular ocean under the influence of surface wind stress, linearised bottom friction, horizontal pressure gradients caused by a variable surface height, and Coriolis force.

An intense crowding of streamlines toward the western border of the ocean is discovered to be caused by variation of the Coriolis parameter with latitude. It is suggested that this process is the main reason for the formation of the intense currents (Gulf stream and others) observed in the actual oceans.

Introduction--Perhaps the most striking feature of the general oceanic wind-driven circulation is the intense crowding of streamlines near the western borders of the oceans. The Gulf Stream, the Kuroshio, and the Agulhas Current are examples of this phenomenon. The physical reason for the westward crowding of streamlines has always been obscure. The purpose of this paper is to study the dynamics of wind-driven oceanic circulation using analytically simple systems in an attempt to discover a physical parameter capable of producing the crowding of streamlines.

The phenomenon occurs along coastlines of such varied topography that it is clear that local topographic features do not significantly control the general streamline pattern. For the sake of simplicity the present study deals with flat rectangular oceans.

The formulation of the problem--A rectangular ocean is envisaged with the origin of a cartesian coordinate system at the southwest corner (see Fig. 1). The y axis points northward; the x axis eastward The shores of the ocean are at $x = 0, \lambda$ and $y = 0, b$. The ocean is considered as a homogeneous layer of constant depth D when at rest. When currents occur, as in the real oceans, the depth differs from D everywhere by a small variable amount h. The quantity h is much smaller than D. The total depth of the water column is therefore D + h, D being everywhere constant, and h a variable yet to be determined.

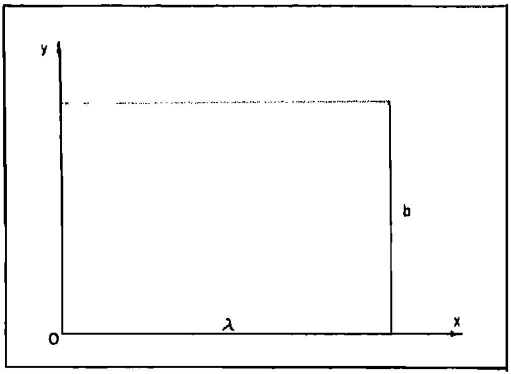

Fig. 1--Ocean basin dimensions and the coordinate system

The winds over the ocean are the Trades over the equatorial half of the rectangular basin, and Prevailing Westerlies over the poleward half. An expression for the wind stress acting upon a column of unit cross-section and depth D + h must include this dependence upon y. A simple functional form of the wind stress is taken as $-F\cos(\pi y/b)$.

In order to keep the ocean from accelerating, a frictional dissipative term is required. To keep the equations of motion as simple as possible the component frictional forces are taken as $-Ru$ and $-Rv$, where R is the coefficient of friction, and u and v are the x and y components of the velocity vector, respectively. The Coriolis parameter f is also introduced. In general it is a function of y.

The steady state equations of motion, with the inertial terms omitted because they are small, are written in the form

$$0 = f(D + h)v - F\cos(\pi y/b) - Ru - g(D + h)\partial h/\partial x \tag{1}$$

$$0 = -f(D+h)u - Rv - g(D+h)\partial h/\partial y \quad \ldots\ldots \quad (2)$$

To these, the equation of continuity must be added

$$\partial[(D+h)u]/\partial x + \partial[(D+h)v]/\partial y = 0 \quad \ldots\ldots \quad (3)$$

Cross-differentiation of (1) and (2) and use of (3) results in the following equation

$$v(D+h)(\partial f/\partial y) + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0 \quad \ldots\ldots \quad (4)$$

In the actual oceans h is so much smaller than D that to a first degree of approximation (4) may be rewritten as

$$\alpha v + \gamma \sin(\pi y/b) + \partial v/\partial x - \partial u/\partial y = 0 \quad \ldots\ldots \quad (5)$$

where the following definitions have been made

$$\alpha = (D/R)(\partial f/\partial y) \qquad \gamma = F\pi/Rb \quad \ldots\ldots \quad (6)$$

To the same degree of approximation the equation of continuity (3) may be replaced by

$$\partial u/\partial x + \partial v/\partial y = 0 \quad \ldots\ldots \quad (7)$$

A stream function ψ is introduced now by the following relations

$$u = \partial\psi/\partial y \qquad v = -\partial\psi/\partial x \quad \ldots\ldots \quad (8)$$

The equation (5) is now rewritten in terms of the stream function

$$\nabla^2\psi + \alpha\partial\psi/\partial x = \gamma\sin(\pi y/b) \quad \ldots\ldots \quad (9)$$

The remainder of this paper is simply a study of solutions of this equation. The boundary conditions upon (9) are that the shore of the ocean be a streamline

$$\psi(0,y) = \psi(\lambda,y) = \psi(x,0) = \psi(x,b) = 0 \quad \ldots\ldots \quad (10)$$

Formal solution of the problem--If f is a linear function of y then α is a constant. Equation (9) is nonhomogeneous and therefore cannot be solved directly by separation of variables. If the right hand member is dropped, the resulting homogeneous equation is soluble by separation. Adding the solution of the homogeneous equation to a particular integral of (9) gives the general solution of (9). A particular integral of (9) is, by inspection

$$-\gamma(b/\pi)^2\sin(\pi y/b) \quad \ldots\ldots \quad (11)$$

The homogeneous equation is

$$\nabla^2\psi + \alpha\partial\psi/\partial x = 0 \quad \ldots\ldots \quad (12)$$

To solve (9) by separation of variables let ψ be given in the following form

$$\psi = XY \quad \ldots\ldots \quad (13)$$

X is a function of x only, and Y is a function of y only. Equation (9) may therefore be written as the following system

$$Y'' + n^2Y = 0 \quad \ldots\ldots \quad (14)$$

$$X'' + \alpha X' - n^2X = 0 \quad \ldots\ldots \quad (15)$$

The primes represent total differentiation, and n^2 is determined by the conditions (10). The general solutions of these two equations are in series form

$$Y = \Sigma(c_j \sin n_j y + d_j \cos n_j y) \quad \ldots\ldots \quad (16)$$

$$X = \Sigma(p_j e^{A_j x} + q_j e^{B_j x}) \quad \ldots \quad (17)$$

The constants A_j and B_j have been defined thus

$$A_j = -\alpha/2 + \sqrt{\alpha^2/4 + n_j^2}, \text{ and } B_j = -\alpha/2 - \sqrt{\alpha^2/4 + n_j^2}$$

The quantities c_j, d_j, p_j, q_j, are undetermined constants. The general solution of (9) is therefore

$$\psi = XY - \gamma(b/\pi)^2 \sin(\pi y/b) \quad \ldots \quad (18)$$

This solution is very general but reduces to a simple closed form when the boundary conditions (10) are imposed. First of all, the d_j and c_j vanish except c_1 corresponding to $n_1 = \pi/b$. This constant c_1 may be absorbed into p_1 and q_1. Dropping subscripts the stream function now has the form

$$\psi = \gamma(b/\pi)^2 \sin(\pi y/b)[pe^{Ax} + qe^{Bx} - 1] \quad \ldots \quad (19)$$

where

$$p = (1 - e^{B\lambda})/(e^{A\lambda} - e^{B\lambda}) \text{ and } q = 1 - p \quad \ldots \quad (20)$$

The curves (ψ = const.) are the streamlines of the ocean currents.

The velocity components u and v may be obtained from (8) by simple differentiation of the stream function

$$u = \gamma(b/\pi)^2 \cos(\pi y/b)(pe^{Ax} + qe^{Bx} - 1) \quad \ldots \quad (21)$$

$$v = -\gamma(b/\pi)^2 \sin(\pi y/b)(pAe^{Ax} + qBe^{Bx}) \quad \ldots \quad (22)$$

The value of h at any point referred to the value of h at the origin may now be obtained by integration of (1) and (2).

$$h(x,y) = -(F/gD)(e^{Ax}p/A + e^{Bx}q/B) - (b/\pi)^2(F/gD)(pAe^{Ax} + qBe^{Bx})[(\cos \pi y/b) - 1]$$
$$- \left\{(f\gamma/g)(b/\pi)^2 \sin(\pi y/b) - (\partial f/\partial y)(\gamma/g)(b/\pi)^3[\cos(\pi y/b) - 1]\right\} \left\{pe^{Ax} + qe^{Bx} - 1\right\} \quad . \quad (23)$$

Discussion of the solution for certain ocean systems--In order to clarify the meaning of the foregoing section, it is advisable to see what role the various parameters play by working some numerical examples. Three cases are discussed. All involve the same effects of wind stress, bottom friction, and horizontal pressure gradients caused by variations of surface height. The role of the Coriolis force is different in each case. First it is assumed that the Coriolis parameter vanishes everywhere, the case of the non-rotating ocean. Secondly, it is assumed that the Coriolis parameter is constant everywhere, the case of the uniformly rotating ocean. In the third case it is assumed that the Coriolis parameter is a linear function of latitude. Of the three cases, the last one most nearly approximates the state of affairs in the real ocean.

For convenience of the numerical computations the dimensions of the ocean are taken as follows

$$\lambda = 10^9 \text{ cm} = 10{,}000 \text{ km}$$
$$b = 2\pi \times 10^8 \text{ cm} = 6249 \text{ km}$$
$$D = 2 \times 10^4 \text{ cm} = 200 \text{ m}$$

The maximum wind stress F is taken to be one dyne/cm^2.

The coefficient of friction R is the only quantity for which a value must be devised. If a value of R = 0.02 is assumed, the velocities in the resulting systems approach those observed in nature.

The case of the non-rotating ocean--In this case the constants p and q are particularly simple. Within one per cent, or as closely as graphs may be drawn, p and q are given by

$$p = e^{-\pi\lambda/b} \qquad q = 1$$

The equation for the stream function is therefore

$$\psi = \gamma(b/\pi)^2 \sin(\pi y/b)[e^{(x-\lambda)\pi/b} + e^{-x\pi/b} - 1].$$

The east-west and north-south symmetry of the streamlines is immediately evident from this equation, and the actual streamlines computed from it are exhibited in Figure 2.

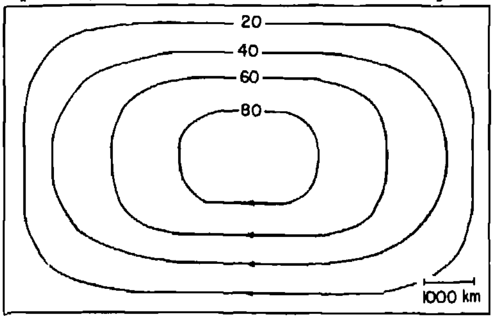

Fig. 2--Streamlines for the case of both the non-rotating and uniformly rotating oceans

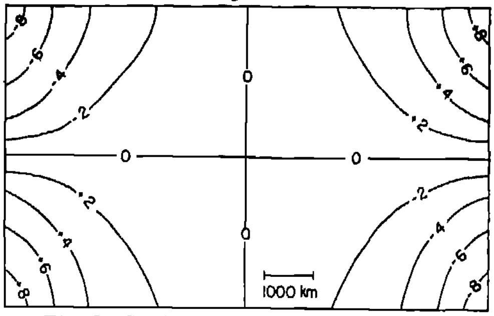

Fig. 3--Surface height contours for the non-rotating ocean in cm

The height contours are computed from (23). Since in this case there is no Coriolis force, the last two terms vanish. Height contours according to this equation are plotted in Figure 3. The general features of the non-rotating wind-driven system are a broad circulation exhibiting absolutely no tendency toward crowding of the streamlines.

The case of the uniformly rotating ocean--In the case where the Coriolis parameter is a constant 0.25×10^{-4}, the streamline diagram does not differ from that of the non-rotating basin. When the height contours are computed however from (23), a difference between the two cases becomes apparent. The third term which vanished in the first case does not vanish in this case. Height contours computed from this equation are exhibited in Figure 4. The large elevation in the central portion of the ocean provides horizontal pressure gradients that largely counterbalance the Coriolis forces. The height contours are not strictly parallel to the streamlines but nearly so.

The case where the Coriolis parameter is a linear function of latitude--In the real ocean the Coriolis force is a function of latitude. In low latitudes this function is nearly a linear one $f = y \times 10^{-13}$. The inequality in the absolute values of the quantities A and B that occurs in this case immediately makes clear the complete lack of east-west symmetry. The streamlines drawn from this formula are shown in Figure 5. The most striking feature of this figure is the intense crowding of stream-

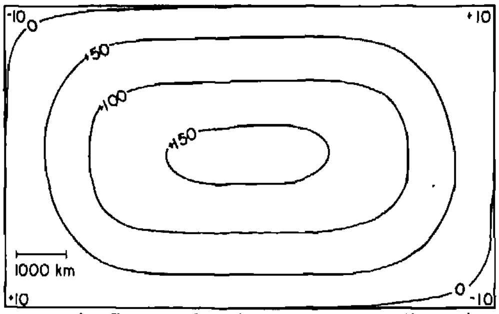

Fig. 4--Surface height contours for the uniformly rotating ocean in cm referred to an arbitrary level

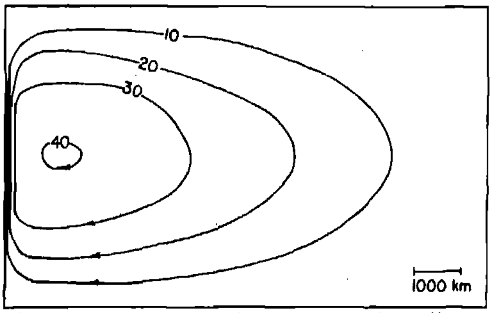

Fig. 5--Streamlines for the case where the Coriolis force is a linear function of latitude

lines toward the western border of the ocean. The rest of the streamline picture is broad and diffuse. The velocity of this system in the region corresponding to the location of the equatorial current (in the actual oceans) is about 20 cm/sec. At the western border the northward velocity amounts to as much as 240 cm/sec. The width of the region of strong northward currents is less than 100 km. The similarity that the velocity field of this simple case bears to that of the actual Gulf Stream suggests that the westward concentration of streamlines in the wind-driven oceanic circulation is a result of the variation of the Coriolis parameter with latitude.

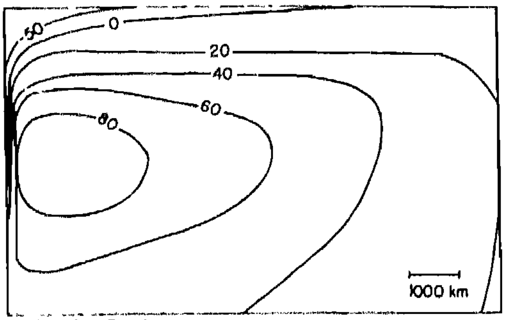

Fig. 6--Surface height contours in cm for the case where the Coriolis parameter is a linear function of latitude

The height contours computed from this case are shown in Figure 6. To extend the results of this study to the Southern Hemisphere the reader will notice that since α is unaffected by crossing the Equator and γ simply changes sign, all the diagrams may be transformed to below the Equator by simple reflection across the x axis. The crowding of streamlines is therefore toward the western border of each ocean, irrespective of hemisphere.

The artificial nature of this theoretical model should be emphasized, particularly the form of dissipative term. The writer thinks of this work as suggestive, certainly not conclusive. The many features of actual ocean structure omitted in the model should be evident.

Woods Hole Oceanographic Institution,
Woods Hole, Massachusetts

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