COMPUTING RESONANCES OF CHAOTIC SYSTEMS VIA CHEBYSHEV POLYNOMIALS AND HOFBAUER EXTENSIONS

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Abstract. The transfer operator is a fundamental operator in the study of dynamic systems, describing how a density of points evolves under some dynamic system. The resonances (eigenvalues) are of particular interest for understanding long term dynamics and trapping regions. While some analytic results exist, many simple maps do not have solutions. In this note we examine a novel computational approach to approximate the eigenvalues of the transfer operator using Chebyshev approximations in combination with Hofbauer extensions. This method is described generally, but with specific focus on the tent map. Computational results yield eigenvalues in agreement with other confirmed methods for computationally estimating the eigenvalues, and error estimates show that this approach is a fast and accurate technique for finding these eigenvalues.

1. Introduction. Dynamical systems are systems in which an iterative function describes the time dependence of a point in space. Essentially, the next state of the system is a function of the previous state or value:
$x_{n+1} = f(x_n)$ [2]. Chaotic systems are dynamical systems that are ultrasensitive to initial conditions, making it difficult to predict their long-term behaviour. These mathematical models describe many natural systems like the weather, population growth, the swing of pendulums, and financial markets [4].

The "**transfer operator**" L of a dynamic system is an operator that examines how a probability density Φ evolves under the dynamic of the system. That is, if we start with points initially distributed according to Φ and evolve each point by f, then the new distribution of points will be $L\Phi$. The formula for the transfer operator is simply given by the change of basis formula from probability [3]. That is:

$$L\Phi(x) = \sum_{y \in f^{-1}(x)} \Phi(y) \frac{1}{\mid f'(y) \mid}$$

We are often interested in the eigenvalues and eigenvectors of this operator. That is, probability densities ϕ such that $L\phi = \lambda\phi$. These are of interest because they form an orthogonal basis over the space of functions, and as such any density can be expressed as $\Phi = a_0\phi + a_1\phi_1 + ...$, and $L^n\Phi = a_0\lambda_0^n\phi_0 + a_1\lambda_1^n\phi_1$. That is, we can understand the dynamics of densities of particles over time by understanding the eigenvalues and eigenvectors of the transfer operator. Additionally, the eigenvectors tell us how quickly a "mode" of the system decays. That is if $\lambda_i << 1, \lambda_i^n \to 0$ quickly, and the mode is transient, whereas if $\lambda_i = 1$ the mode persists and the system eventually stabilises to that mode. That is, the eigenvalues allow us to predict trapping regions of the system and provide a means to predict the long-term behaviour of chaotic systems [3].

As the transfer operator acts on a function space, it is infinite dimensional. For systems described by Markov maps, we are able to accurately construct a discrete "Chebyshev-Lagrange" approximation of the transfer operator. However, for a map to be Markov, it must be 2:1 everywhere. Not many systems are described by such maps. For us to construct a finite dimensional approximation of the transfer operator we must be able to transform its map such that it can be treated as Markovian [5].

In this note, we examine using Hofbauer extension to make non-Markovian maps Markovian, and apply Chebyshev operator approximations to the edges of the Hofbauer extension to from our approximations.

Section 2 defines Hofbauer extensions, specifically in the context of the tent map. Section 3 defines Chebyshev operator approximations in general, explaining how they are computed and showing that their eigenvalues should approximate the true eigenvalues of the original operator. Section 4 examines how Hofbauer extensions can be used to construct "super adjacency matrices" and how incorporating Chebyshev operator approximations with the super adjacency matrix should lead to more accurate estimates of the eigenvalues. Section 5 compares the eigenvalues estimated from the super adjacency matrix with those estimated from the Chebyshev approximation of the full transfer operator, and with those estimated via Ulam's method. Section 6 looks at how the error of the eigenvalues depends on the order of the Chebyshev polynomials and the depth of the Hofbauer extension. Finlay, section **??** concludes the paper.

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2. Hofbauer Extension. The Hofbauer extension transforms non-Markovian systems into a "tower" of Markov systems, continuously breaking up the original map into overlapping segments that are 2:1 throughout. [1] We look at an example of this with the tent map, a non-Markovian map defined as:

$$f(x) = a \cdot \max(x, 1 - x),$$

See Fig. 1.

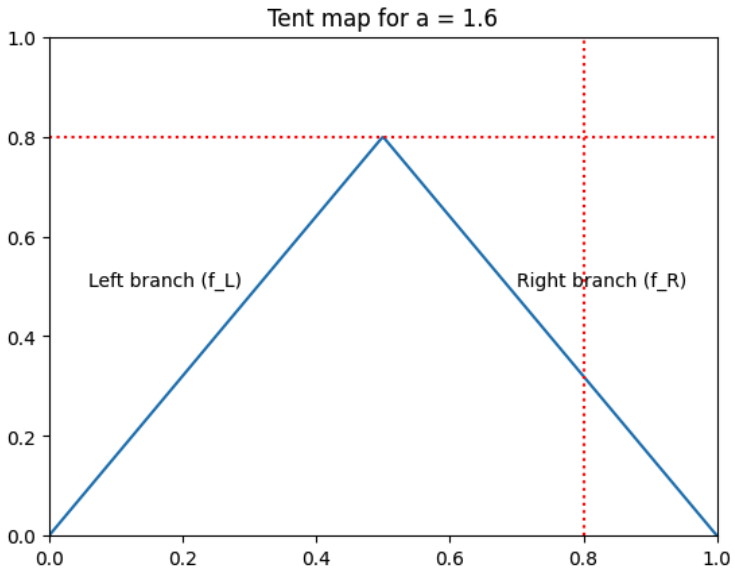

FIG. 1. *The tent map visualised for* $a = 1.6$

For $0 \leq f(x) < 0.8$ this map has two pre-images, at $f(x) = 0.8$ it has one and at $f(x) > 0.8$ it has zero. The goal of the Hofbauer extension is to split the map into segments that have exactly two pre-images. We begin with the full domain: $[0, 1]$ and calculate the range of the left and right branches: f_L, f_R. Both of these give us $[0, 0.8]$. Across $[0, 0.8]$, f_L produces the same range but f_R produces $[0.32, 0.8]$. We continue this process, computing the range of each branch for each new interval. This is represented in the directed graph below, where a red directed edge indicates the right branch mapping between the intervals of the nodes linked, a green edge indicating the left branch and blue for both. See Fig. 2.

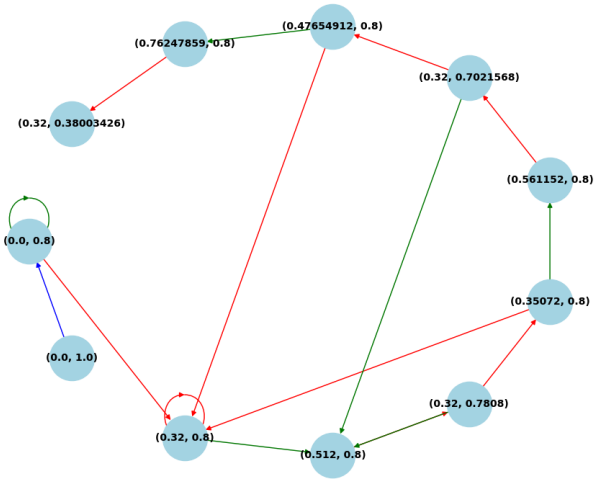

FIG. 2. *The digraph for the tent map with* $a = 1.6$

Performing this extension to a high depth, and storing the directed graph information in an adjacency matrix allows us to create an approximate Markov representation of a non-Markov system. Since we are only able to create a discrete approximation of the transfer operator for Markov systems, this is the first step in the process of computing the resonances of a non-Markov system.

3. Chebyshev Operator Approximation. The Chebyshev polynomials $T_n(x)$ of order n are defined such that $T_n(\cos(\theta)) = \cos(n\theta)$. Typically, Chebyshev polynomials are defined over the interval $[-1, 1]$. However, we will often be working on arbitrary ranges $[a, b] \subset [0, 1]$, so we define

$$T_n^{[a,b]} = T\left(\frac{2x - (a+b)}{b-a}\right) \tag{1}$$

as Chebyshev polynomials in the domain $[a, b]$ (we will often omit the superscript if it is obvious). Related to the Chebyshev polynomials are the Chebyshev nodes of order K defined as $x_k = \cos(\theta_k)$ with $\theta_k = \frac{2k+1}{2K}\pi$. Here we will similarly define $x_k^{[a,b]}$ as the Chebyshev nodes transformed from the interval $[-1, 1]$ to $[a, b]$ via the inverse of the transformation used in the argument of Eq. 1. Regardless of the choice of domain, $T_n^{[a,b]}\left(x_k^{[a,b]}\right) = \cos(n\theta_k)$.

The transfer operator is an operator, making it difficult to evaluate on a computer. As such we find a finite-dimensional matrix $\hat{L}$, and some basis of functions $\vec{\Phi} = [\Phi_1, \Phi_2, ..., \Phi_n]^T$ such that

$$(\hat{L}\vec{\Phi})(x) \approx [(L\Phi_0)(x), (L\Phi_1)(x), ..., (L\Phi_{n-1})(x)]^T \tag{2}$$

We choose the basis functions $\vec{\Phi}$ to be the Chebyshev Polynomials of order n, and require that

$$(\hat{L}\vec{T})(x_k) = [(LT_0)(x_k), (LT_1)(x_k), ..., (LT_{n-1})(x_k)]^T \tag{3}$$

For all Chebyshev nodes x_k of order K. The equality at these points ensures that Eq. 2 is satisfied (under certain conditions). Expanding Eq. 3 we get

$$L_{i,0}T_0(x_k) + L_{i,1}T_1(x_k) + ... + L_{i,n-1}T_{n-1}(x_l) = (LT_i)(x_k) \tag{4}$$

$$L_{i,0} + L_{i,1}\cos(\theta_k) + ... + L_{i,n-1}\cos((n-1)\theta_k) = (LT_i)(x_k) \tag{5}$$

for all $i < n$ and all $k < K$. Given that the values of θ_k are equally spaced, the coefficients $L_{i,j}$ can be efficiently estimated using a discrete cosine transformation. This gives us $\hat{L}$. Given some function Φ on the range $[a, b]$, we can estimate the action of L on Φ by first expanding Φ in terms of the Chebyshev polynomials. That is $\Phi \approx c_0T_0 + c_1T_1 + ... + c_{n-1}T_{n-1} = \vec{c} \cdot \vec{T}$. Operating on this with L and given its linear we get

$$Lf \approx c_0LT_0 + c_1LT_1 + ... + c_{n-1}LT_{n-1} \tag{6}$$

$$\approx \vec{c} \cdot (\hat{L}\vec{T}) \tag{7}$$

$$= (\hat{L}^T\vec{c}) \cdot \vec{T} \tag{8}$$

Given this approximation, we can approximate the eigenvalues and eigenvectors of L as follows. First take the true eigenvalues and eigenvectors as ϕ and λ such that $L\phi = \lambda\phi$, and also expres $\phi \approx \vec{c_\phi} \cdot \vec{T}$

$$\lambda\phi = L\phi \approx (\hat{L}^T\vec{c_\phi}) \cdot \vec{T} \tag{9}$$

$$\lambda\vec{c_\phi} \cdot \vec{T} \approx (\hat{L}^T\vec{c_\phi}) \cdot \vec{T} \tag{10}$$

$$\lambda\vec{c_\phi} \approx \hat{L}^T\vec{c_\phi} \tag{11}$$

Where the last line follows from the fact that the elements of $\vec{T}$ are orthogonal. So the eigenvalues of $\hat{L}^T$ are approximately the eigenvalues of L, and the eigenvectors allow us to approximately reconstruct the eigenvectors of L.

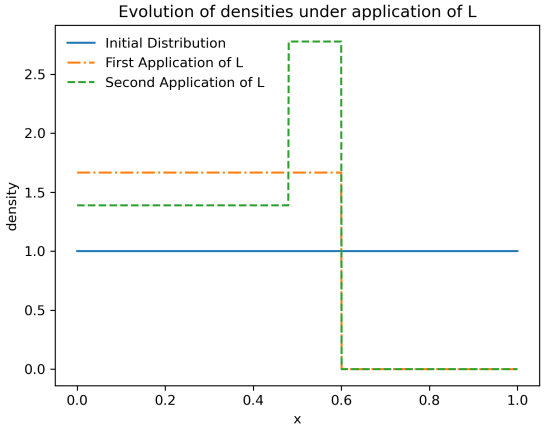

FIG. 3. *Probability density evolving under repeated application of L of the tent map with $a = 1.2$, note the sharp edges due to the domain being restricted.*

4. Super Adjacency Matrix. The primary limitation of the approximation outlined above is that for operations like the tent map, each time the function operates, the domain is changed (see section 2) causing large sections of the function to be mapped to zero (where the points are outside the domain). This causes $L\Phi$ to have "sharp" changes / "cliffs". These can be seen in Fig 3. As such, in order to get an accurate approximation for the eigenvalues and eigenvectors of L you need Chebyshev polynomials of exceedingly high degree. This limitation can be overcome by considering the restricted operations $L^{L,R}_{A\to B}$ that map functions over the domain A to the functions over the domain B with either the left (L) or right (R) branch of the map. We can take these and construct the "Super Adjacency Matrix" from the adjacency matrix of the Hofbauer extension as the operator that "keeps track" of where the content of the function Φ is mapped. For example, for the tent map with $a = 1.2$

$$L_S = \begin{pmatrix} L^L_{[0,1]\to[0,0.6]} + L^R_{[0,1]\to[0,0.6]} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & L^L_{[0,0.6]\to[0,0.6]} & L^R_{[0,0.6]\to[0.48,0.6]} & \vdots & \ddots \end{pmatrix} \tag{12}$$

Here we start with the vector $[f, 0, 0, ...]^T$ and each successive operation we look at where the "density" of f gets mapped by each branch and keep track of it. For example, after applying the tent map once, all the density of f has been mapped to the domain $[0, 0.6]$, the second entry of the vector is for functions on the domain $[0, 0.8]$ so we end up with $[0, Lf, 0, 0, ...]$. Applying again, the left branch maps density in $[0, 0.6]$ to density in $[0, 0.6]$, and the right branch maps density in $[0, 0.6]$ to $[0.48, 0.6]$ (functions in this domain are stored in the third element of the vector). We construct the final density by adding the entries of the vector together, assigning a value of 0 if the function is not defined at that point. This "bookkeeping" is such that when the map takes "density" and maps it to a restricted domain (which would typically introduce a hard edge that could not be approximated well by Chebyshev polynomials), we don't lose accuracy because we can simply use Chebyshev polynomials on the restricted domain. It is clear that the eigenvalues of this super adjacency matrix are the same as those of the operator itself, and because the Chebyshev operator approximations of the $L^{L,R}_{A\to B}$ are more accurate, using this in the computation should yield a higher accuracy for the eigenvalues of L, and if desired the eigenvectors of L_S could be used to reconstruct the eigenvectors of L.

5. Computational Results. Fig. 4 compares the approximation of the eigenvalues with the Hofbauer extension compared to simply approximating the transfer operator with Chebyshev polynomials. There is agreement for the first eigenvalue ($\lambda = 1$), and the flowing ≈ 5 eigenvalues are similar. Looking at Fig. 5 we see good agreement between Ulam's method and the Super adjacency method for the first few

eigenvalues, more so then compared to the direct estimate from the Chebyshev approximation, assuming that the eigenvalues from Ulam's method are closer to the true eigenvalues. This indicates that the Hofbauer extensions do improve the accuracy of estimation of the eigenvalues. The discrepancy of the eigenvalues at the essential spectrum is expected due to the density of the essential spectrum meaning that all points are a valid eigenvalues.

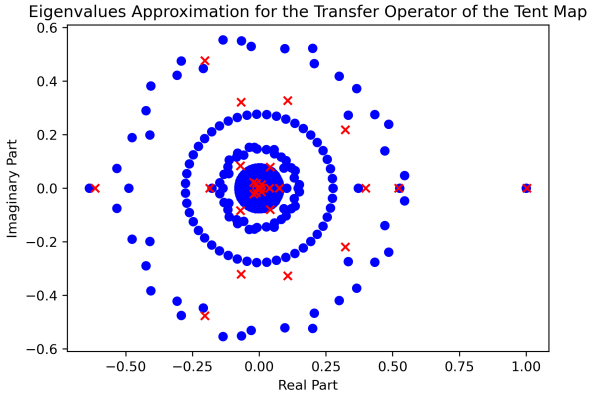

FIG. 4. *Comparison of eigenvalue approximation for the tent map ($a = 1.9$) using the super adjacency matrix (blue) vs a direct approximation of the transfer operator with Chebyshev operator approximations (red). Polynomials of order 100, and Hofbauer Extension of depth 50.*

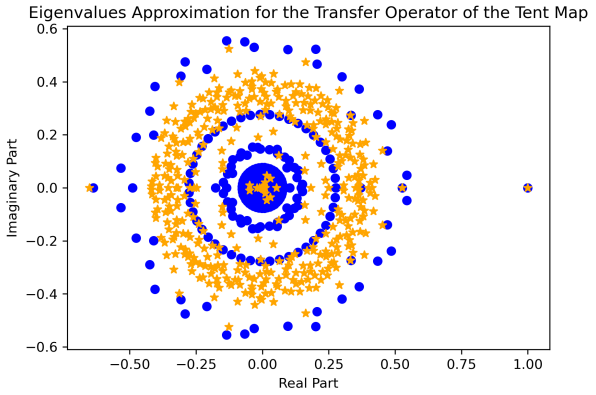

FIG. 5. *Comparison of eigenvalue approximation for the tent map ($a = 1.9$) using the super adjacency matrix (blue) vs Ulam's method (orange). Polynomials of order 100, Hofbauer Extension of depth 50, 1000 bins and 10,000 samples per bin.*

6. Eigenvalue Error. After obtaining the super adjacency matrix, a key measure of the accuracy of our Chebyshev operator approximation is the error in the largest eigenvalue. Theory predicts that the largest eigenvalue is typically 1, thus, we assess our approximation by comparing the computed largest eigenvalue to 1; the error is defined as

$$\text{Error} \;=\; \big|\lambda_{\max} - 1\big|.$$

This eigenvalue error is a crucial metric because it reflects how well our discretised operator (obtained via the Chebyshev-Lagrange approximation and the Hofbauer extension) captures the true long-term dynamics of the system.

We vary both the number of Chebyshev points K (from 10 to 200) and the "depth" of the Hofbauer extension (from 5 to 50), and record the resulting errors. Figure 6 plots these errors on a logarithmic scale.

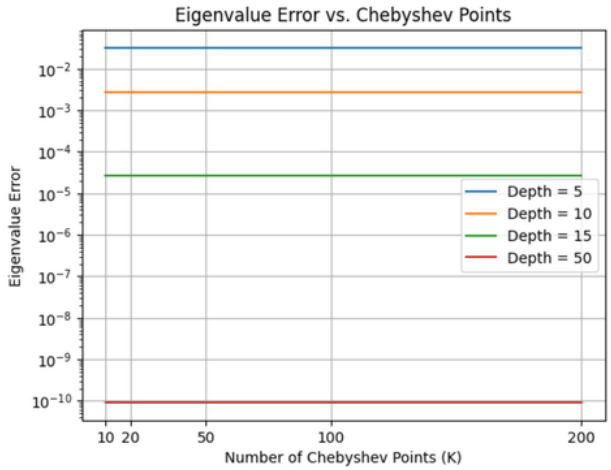

FIG. 6. *Eigenvalue error for varying Chebyshev points (K) and Hofbauer depth.*

For most depths, the error curves remain fairly flat (around 10^{-2} to 10^{-4}) as K increases, suggesting that once the main features of the transfer operator are captured, further increases in resolution provide limited additional benefit. Notably, one of the curves dips to about 10^{-10} for larger K at depth 50, indicating an extremely accurate approximation of the leading eigenvalue in that specific configuration. This highlights that, under certain conditions, a sufficiently refined Hofbauer extension and Chebyshev approximation can capture the operator with very high precision.

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