Contents

Chapter 1

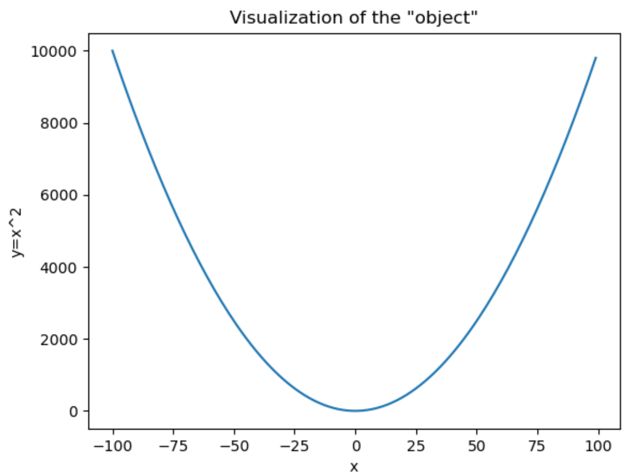
Coordinate Geometry

1.1 Introduction

Many of you have encountered some form of coordinate geometry in high school. For instance, the "standard" way to visualize a graph e.g. $f(x) = x^2$ is to visualize the points in 2-D space (x, y) where $y = x^2$. We give a demonstration in Python code.

```
import matplotlib.pyplot as plt
import numpy as np
X=np.arange(-100,100) #create list of numbers from -100 to 100
Y= X**2 #calculate the square of at each x
plt.plot(X,Y) #plot all the pairs of points in 2d plane
plt.xlabel('x')
plt.ylabel('y=x^2')
plt.title('Visualization of the "object"')
plt.show()
plt.show()
```

$$x \xrightarrow[\text{the “graph” object}]{} x^2$$



This is also known as the Cartesian plane, named after René Descartes who invented it in the 17th century.

1.2 Visualization of geometric objects

The Cartesian plane allows us to describe shapes with equations and perform calculations with them. We first define the playing field (the Cartesian plane and higher dimensional analogues) and the players.

Definition 1.1 (Real numbers)

The set of **real numbers**, denoted as $\mathbb{R}$, is (informally) the set of all the numbers that can be written out in decimal form.

Example 1.2

The following are real numbers:

1. The integers $0, \pm 1, \pm 2, \ldots$
2. Fractions in the form $\frac{a}{b}$, where a and $b \neq 0$ are integers.
3. Irrational numbers $\sqrt{2}$, π.

Remark. *The set of real numbers is known as a* ***complete field****. The definition of a complete field will be swept under the rug, but it guarantees a few things. The most important* *property: We will not "escape" the set by performing operations, possibly infinitely many.*

Definition 1.3 (N-dimensional space)

Let n be a positive integer. We denote the **n-dimensional real space** to be $\mathbb{R}^n$, consisting of all the n-tuples $(x_1, x_2, x_3, ..., x_n)$, where each x_j is a real number.
We call an n-tuple $(x_1, ..., x_n)$ a **point**, and two points $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are equal if $x_j = y_j$ for all j-th entries of the tuples.

Remark. *We sometimes use* $\mathbf{x}$ *to denote* $(x_1, ..., x_n)$ *to make notation cleaner.*

1.2.1 Lines

Now that we have introduced the playing field of n-dimensional space, we can start translating the axioms of euclidean geometry to this coordinate system.

Definition 1.4 (Lines)

In euclidean geometry, a line is defined by two points. We let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The **line** going from $\mathbf{x}$ to $\mathbf{y}$ is denoted $\overrightarrow{\mathbf{xy}}$.

What would this line look like? To get from $\mathbf{x}$ to $\mathbf{y}$, we have to traverse $y_1 - x_1$ units in the first coordinate, $y_2 - x_2$ units in the second, ..., $y_n - x_n$ in the last. We thus have a natural notation for the line $\overrightarrow{\mathbf{xy}}$.

$$\overrightarrow{\mathbf{xy}} = (y_1 - x_1, y_2 - x_2, ..., y_n - x_n).$$

This is very similar to a point as an n-tuple, but this is "spiritually" different to a point. This tuple represents the direction of line. One way to think of the correspondence between $(x_1, ..., x_n)$ point and $(x_1, ..., x_n)$ line is that $(x_1, ..., x_n)$ line is the line connecting $(0, 0, ...0)$ to $(x_1, ..., x_n)$ point. Because of this, we can identify a tuple as both the point and the line, and we call it a "vector" to abstract away from the actual geometric meaning.

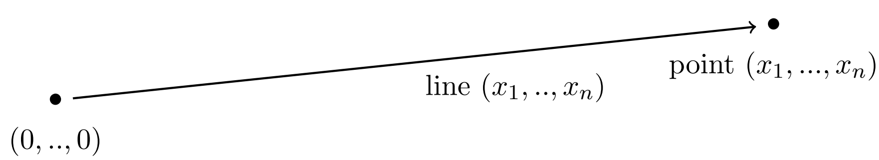


Figure 1.1: The correspondence between a line "vector" and a point "vector".

Notation. *Vectors are a very general notation of* $n-tuples$*. Depending on context, we use both of the following notations to denote the entries of* $\vec{v} \in \mathbb{R}^n$

- *"Ordered sets"* $(v_1, ..., v_n)$, *suitable dealing with points (and other geometric objects). It also looks cleaner when writing inline.*
- *"Column vectors"* $\begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_n \end{bmatrix}$ *when the ordered sets are complicated to read, and when working with matrix algebra.*

1.2.2 Operation with lines

We need to translate a few more things from euclidean geometry.

Proposition 1.5

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, then

$$\overrightarrow{\mathbf{xy}} + \overrightarrow{\mathbf{yz}} = \overrightarrow{\mathbf{xz}},$$

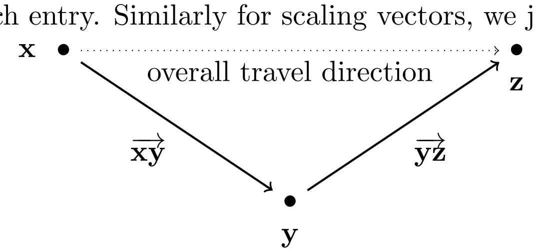
where $(a_1, ..., a_n) + (b_1, ..., b_n) \stackrel{\text{def}}{=} (a_1 + b_1, ..., a_n + b_n)$.

Proof.

$$\begin{aligned} \overrightarrow{\mathbf{xy}} + \overrightarrow{\mathbf{yz}} &= (y_1 - x_1, ..., y_1 - x_n) + (z_1 - y_1, ..., z_1 - y_n) \\ &= (y_1 - x_1 + z_1 - y_1, ..., y_n - x_n + z_n - y_n) \\ &= (z_1 - x_1, ..., z_n - x_n) \\ &= \overrightarrow{\mathbf{xz}} \end{aligned}$$

□

Geometrically, this means if you connect $\mathbf{x}$ to $\mathbf{y}$ to $\mathbf{z}$, the overall "direction of travel" you make is $\mathbf{x}$ to $\mathbf{z}$. This gives us a natural extension for addition of vectors by considering each entry. Similarly for scaling vectors, we just scale the entries along each dimension.



Definition 1.6 (Addition and scaling of vectors)

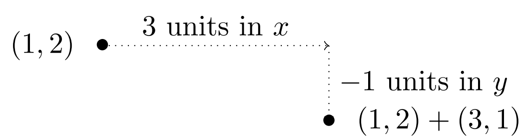
Let $\vec{a}, \vec{b}$ be two vectors in $\mathbb{R}^n$. We define the sum/difference of $\vec{a}$ and $\vec{b}$

$$\vec{a} + \vec{b} \overset{\text{def}}{=} (a_1 + b_1, ..., a_n + b_n), \quad \vec{a} - \vec{b} \overset{\text{def}}{=} (a_1 - b_1, ..., a_n - b_n)$$

and the scaling of $\vec{a}$ by a real number $c \in \mathbb{R}$

$$c\vec{a} \overset{\text{def}}{=} (ca_1, ca_2, ..., ca_n).$$

Remark. *Here we use the term "vectors", as we can in essence add points and lines together. How does one make sense of adding a line to a point? We can view this as translating the point along the path of the line, for instance, let us translate the point* $(1, 2)$ 3 *units in the first coordinate and* -1 *units in the second coordinate. This will give us* $(4, 1)$*.*



This way, we can write the line from $\vec{x}$ to $\vec{y}$ as $\vec{y} - \vec{x}$. The proof is a computational exercise.

Notation. *We now transferred for talking about points and the lines between points to addition. Therefore, we can overload the notation for points and lines as a vector* $\vec{v}$*, keeping in mind that they have the same arithmetic structure.*

In fact, most of our intuition for the real numbers translates to $\mathbb{R}$. For formality, we will list them here; in practice, we (almost always) take these properties for granted.

Proposition 1.7

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$. Then the following hold:

- *(Associativity)* $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
- *(Commutativity)* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- *(Identity)* The zero vector $\vec{0} \stackrel{\text{def}}{=} (0, 0, ..., 0) \in \mathbb{R}$ satisfies $\vec{v} + \vec{0} = \vec{v}$.
- *(Inverse)* The inverse of $\vec{v}$, $-\vec{v} \stackrel{\text{def}}{=} (-v_1, ..., -v_n)$ satisfies $\vec{v} + (-\vec{v}) = \vec{0}$.
- *(Scalar multiplication)* $a(b\vec{v}) = (ab)\vec{v}$.
- *(Scalar Identity)* $1\vec{v} = \vec{v}$.
- *(Distributivity 1)* $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.
- *(Distributivity 2)* $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.

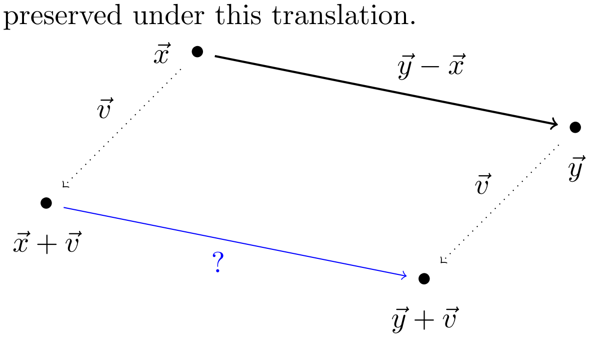
Remark. *All of these have good geometric intuition behind. For instance, the zero vector $\vec{0}$ is the "don't move" vector, corresponding to the point at the origin, or the "too short to be a line". The inverse of $\overrightarrow{\mathbf{xy}}$ is $\overrightarrow{\mathbf{yx}}$, where you go back from $\mathbf{x}$ to $\mathbf{y}$.*

Remark. *These 8 conditions are the axioms of a vector space. Later in the course, we will generalize the notion of vectors in $\mathbb{R}^n$ to other spaces (playing fields).*

Proposition 1.8

Lines are translation invariant. That is, for every $\vec{x}, \vec{y}, \vec{v} \in \mathbb{R}^n$, then the line from $\vec{x}$ to $\vec{y}$ is the same as the line from $\vec{x} + \vec{v}$ to $\vec{y} + \vec{v}$.

Let us illustrate what this statement is trying to convey. We have two points $\vec{x}$, $\vec{y}$; now we translate each of these points by $\vec{v}$, and we want the line between the points to be preserved under this translation.



The proof is one line: $(\vec{y} + \vec{v}) - (\vec{y} + \vec{v}) = \vec{y} - \vec{x} + \vec{v} - \vec{v} = \vec{y} - \vec{x}$. However, an immediate consequence of this is that we can "transport" vectors in space without distorting the vector. Colloquially, *5 miles South* to you describes the same direction and length as *5 miles South* to a person a few feet away. This justifies the way we visualize the correspondence between points and vectors - we "transport" the vectors to start from the origin $(0, ..., 0)$, and the end describes the point.

Remark. *Translation (and scaling) invariance is a property of Euclidean geometry. There are some exotic geometry systems that distort distance and direction through translation and scaling. One such example is the Poincaré metric.*

Another notion we can carry from Euclidean geometry is parallel lines. Here we not only define what it means for two vectors to be parallel (never touching), we also give a definition for two vectors to be parallel but point in opposite directions.

Definition 1.9 (Parallel and Antiparallel Vectors)

Let non-zero vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$. We say that $\vec{v}$ and $\vec{w}$ are **parallel** if there is some $c > 0$ such that $\vec{v} = c\vec{w}$. We say that $\vec{v}$ and $\vec{w}$ are **antiparallel** if there is some $c < 0$ such that $\vec{v} = c\vec{w}$.

Example 1.10

Find an expression for the points on the (infinite) line passing through $P(1, 1, 0)$ and $Q(0, 2, 2)$.

Let M be a point on the line $\overrightarrow{PQ}$, then $\overrightarrow{PM}$ is parallel to $\overrightarrow{PQ}$ (or $M = P$). Either way, we would have some $c \in \mathbb{R}$ such that

$$\overrightarrow{PM} = c\overrightarrow{PQ}.$$

Indeed, we can confirm that any point represented as $P + c\overrightarrow{PQ}$ will be on the line, as $c\overrightarrow{PQ}$ is parallel to $\overrightarrow{PQ}$. One exception is to check for when $c = 0$, but we already know P is on the line. We thus have an expression for the line to be

$$(1, 1, 0) + c(-1, 1, 2)$$

for all $c \in \mathbb{R}$. Another way to put it, the line is the **image** of the function $f(c) = (1, 1, 0) + c(-1, 1, 2)$ on $\mathbb{R}$.

Remark. *This form of representing a line is known as a **parametric representation** or a **parametrization** of the line. We will give a more concrete definition of parametrization when we move beyond straight lines to "curvy" curves and surfaces in* **??**.

Remark. *Using the parametric representation for a line $l(t) = (P_1, P_2, ..., P_n) + t(v_1, v_2, ..., v_n)$, we can take slices/cross sections across each of the j-th coordinates, and get*

$$x_j = P_j + tv_j \implies t = \frac{x_j - P_j}{v_j}$$

is the equation of a line in 2d. P_j, v_j are fixed, viewing in the variables $y = x_j, x = t$, this is the equation of a line with y-intercept x_j and slope v_j. Setting the values of t for all slices to be equal,

$$t = \frac{x_1 - P_1}{v_1} = \frac{x_2 - P_2}{v_2} = ... = \frac{x_j - P_j}{v_j} = ... = \frac{x_n - P_n}{v_n}$$

This form is known as the ***symmetric equations*** *of a line.*

Example 1.11

Find an expression for the points on the line connecting **between** $P(1, 1, 0)$ to $Q(0, 2, 2)$.

This will be a segment of the line from the previous example, so the answer would be the same expression, but we limit the domain of f to be an interval on $\mathbb{R}$. Let us examine what f does to a few values of c.

c	very negative	$c = 0$	$c = 1/2$	$c = 1$	very positive
f(c)	very off the segment	P	midpoint of $\overline{PQ}$	Q	very off the line

As we move from very negative c to very positive c, we start very away from the line segment, reach P at $c = 0$, Q at $c = 1$, then move away from the line segment. Indeed the values $(1, 1, 0) + c(-1, 1, 2)$ will be on the line segment for $c \in [0, 1]$.

Finally, the nice algebraic properties for adding and scaling vectors gives us a natural way to understand these vectors as a sum of n component vectors, one for each dimension. These *special* vectors deserve a name.

Definition 1.12 (Standard Basis Vectors and Vector Decomposition)

We denote in $\mathbb{R}^n$, the **standard basis vectors**

$$\begin{aligned}\vec{e}_1 &\stackrel{\text{def}}{=} (1,0,0,...,0,0,0),\\ \vec{e}_2 &\stackrel{\text{def}}{=} (0,1,0,...,0,0,0),\\ \vec{e}_{n-1} &\stackrel{\text{def}}{=} (0,0,0,...,0,1,0),\\ \vec{e}_n &\stackrel{\text{def}}{=} (0,0,0,...,0,0,1),\end{aligned}$$

so that $\vec{v} = \sum_{i=1}^n v_i\vec{e}_i$. In $\mathbb{R}^3$, we sometimes write

$$\vec{i} \stackrel{\text{def}}{=} \vec{e}_1, \quad \vec{j} \stackrel{\text{def}}{=} \vec{e}_2, \quad \vec{k} \stackrel{\text{def}}{=} \vec{e}_3$$

to accommodate the physicists.

Exercises

1. Compute $5\vec{v} - 2\vec{w}$ and $-3\vec{w}$ for the following pairs of vectors:
   (a) $\vec{v} = 2\vec{i} + 3\vec{j}$, $\vec{w} = 4\vec{i} - 9\vec{j}$
   (b) $\vec{v} = (1,2,-1)$, $\vec{w} = (2,-1,0)$
   (c) $\vec{v} = -2\vec{e}_3 + 4\vec{e}_5$, $\vec{w} = \vec{e}_1 - 4\vec{e}_5$
   (d) $\vec{v} = (\cos t, \sin t)$, $\vec{w} = (\cos t)\vec{e}_2 - (\sin t)\vec{e}_1$
2. Find the vector $\overrightarrow{PQ}$ connecting $P(7,2,9)$ to $Q(-2,1,4)$. Put your answer in the form $a\vec{i} + b\vec{j} + c\vec{k}$. Give a parametric representation for the line $\overline{QP}$ in the form '$f(t) = \mathbf{a} + t\mathbf{b}$ for $t \in [c,d]$' this time the image moves from Q to P as the variable t increases.

1.3 Length, angles and projections

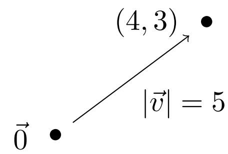
Definition 1.13 (Magnitude)

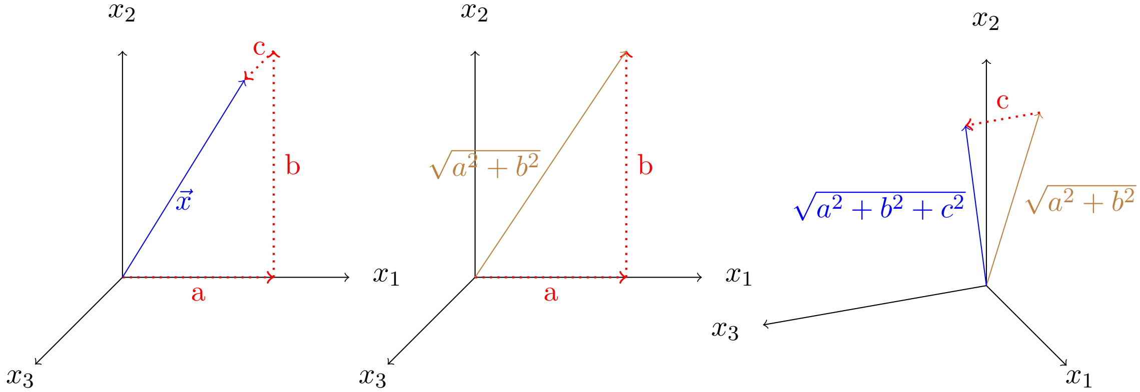
Let $\vec{v} \in \mathbb{R}^n$. The magnitude of $\vec{v}$ is denoted

$$|\vec{v}| \stackrel{\text{def}}{=} \sqrt{v_1^2 + v_2^2 + ...v_n^2}.$$

We build intuition through the lower dimensional cases. In $\mathbb{R}^2$, let us consider the point $(4, 3)$.

The magnitude of this vector is $\sqrt{3^2+4^2} = 5$. If this sounds very familiar, it is because this is indeed an application of Pythagorean theorem. In 3-dimensions, this still applies - take $\vec{x} = (a, b, c)$, we can traverse in each coordinate to apply Pythagorean theorem twice.





(a) $\vec{x} = (a, b, c)$. (b) Pyth. thm on a, b. (c) Reapply Pyth. thm.

Proposition 1.14

Let $\vec{v} \in \mathbb{R}^n, a \in \mathbb{R}$. Then

$$|a\vec{v}| = |a||\vec{v}|.$$

We have now transported the notions of length and scaling into the coordinate system, and this allows us to make "measurements" such as angles and area.

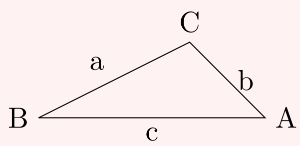
1.3.1 The Dot Product

Example 1.15

Let points A=$(5, 8)$, B=$(-2, 7)$, O=$(4, 5)$. Find the angle $\angle AOB$.

To solve this using only information about the lengths, recall the Law of Cosines:

Theorem 1.16 (Law of Cosines)

For any triangle 

,

$$c^2 = a^2 + b^2 - 2ab\cos(\angle BCA).$$

We now apply the Law of Cosines to $\triangle AOB$, so that

$$|\overrightarrow{AO}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{AO}||\overrightarrow{OB}\ |\cos(\angle AOB) = |\overrightarrow{AB}|^2.$$

Plugging in the values, we solve

$$((4-5)^2 + (5-8)^2) + ((-2-4)^2 + (7-5)^2) - 2\sqrt{(4-5)^2 + (5-8)^2} \times\sqrt{(-2-4)^2 + (7-5)^2}\cos(\angle AOB) = ((-2-5)^2 + (7-8)^2).$$

Simplifying, we get

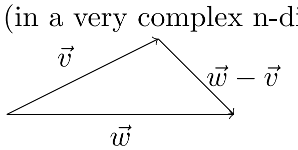
$$50 - 2\sqrt{10}\sqrt{40}\cos(\angle AOB) = 50 \implies \cos(\angle AOB) = 0.$$

So the angle is $\pi/2$.

Example 1.17

Find a closed form formula for the cosine of an angle formed by two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$.

Let the angle formed be θ. Using the intuition from 2-D space, we can form a triangle (in a very complex n-dimensional space). We write the Law of Cosine in terms of vectors



$$|\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| = |\vec{w} - \vec{v}|^2.$$

This expands to

$$\sum_{i=1}^n v_i^2 + \sum_{j=1}^n w_j^2 - 2\sqrt{\sum_{i=1}^n v_i^2}\sqrt{\sum_{j=1}^n w_j^2}\ \cos(\theta) = \sum_{i=1}^n (w_i - v_i)^2.$$

Rearranging $(a-b)^2 = a^2 + b^2 - 2ab$,

$$2\sqrt{\sum_{i=1}^n v_i^2}\sqrt{\sum_{j=1}^n w_j^2}\ \cos(\theta) = \sum_{i=1}^n 2w_iv_i$$

so

$$\cos(\theta) = \frac{\sum_{i=1}^{n} w_i v_i}{\sqrt{\sum_{i=1}^{n} v_i^2}\sqrt{\sum_{j=1}^{n} w_j^2}}.$$

Definition 1.18 (Dot Product)

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, the **dot product** between $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} \stackrel{\text{def}}{=} \sum_{i=1}^{n} v_i w_i.$$

Proposition 1.19

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $c \in \mathbb{R}$. The dot product satisfies the following properties:

- *(Symmetry)* $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- *(Linearity 1)* $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})(= \vec{v} \cdot (c\vec{w})$ by symmetry).
- *(Linearity 2)* $(\vec{v} + \vec{u}) \cdot \vec{w} = \vec{v} \cdot \vec{w} + \vec{u} \cdot \vec{w}$.
- *(Positive definiteness)* $\vec{v} \cdot \vec{v} \geq 0$, with equality if and only if $\vec{v} = \vec{0}$.

Proof. **TODO: write the proof, or put it as an exercise.** □

Corollary 1.20:

1. $|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$.
2. If $\vec{v}, \vec{w} \neq \vec{0}$, the angle between $\vec{v}$ and $\vec{w}$ is

$$\cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right).$$

   In particular, the angle is $\pi/2$ when $\vec{v} \cdot \vec{w} = 0$.

3. $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Since we can make sense of "angles" in higher dimensions, it is natural to generalize the notion of two vectors "perpendicular" to each other.

Definition 1.21 (Orthogonality)

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. We say $\vec{v}$ and $\vec{w}$ are **orthogonal** to each other if $\vec{v} \cdot \vec{w} = 0$.

Example 1.22

Find all vectors $\vec{v} \in \mathbb{R}^3$ such that $\vec{v} \cdot (-3\vec{i} + 2\vec{j} + 6\vec{k}) = 49$. Sketch the locus of corresponding points in $\mathbb{R}^3$.

The condition expands to

$$\begin{aligned} -3v_1 + 2v_2 + 6v_3 &= 49 \\ \implies v_3 &= \frac{49 + 3v_1 - 2v_2}{6} \end{aligned}$$

so that the set of all vectors is

$$\left\{ s\vec{i} + t\vec{j} + \left(\frac{49 + 3s - 2t}{6} \right) \vec{k} | s, t \in \mathbb{R} \right\}.$$

To plot this in 3D space, notice that this describes the equation of a plane $z = (49 + 3x - 2y)/6$. We pick any three points (of course we want those that are easy to calculate) $(0, 0, 49/6)$, $(0, 49/12, 0)$, $(-49/18, 0, 0)$.

```python
def function_to_plot(X,Y):
return (49+3*X-2*Y)/6
#create figure
fig=plt.figure(figsize=(10,8))
ax = fig.add_subplot(projection='3d')
ax.view_init(elev=10, azim=120) #change view 1
#ax.view_init(elev=10, azim=0) #change view 2
X,Y=np.meshgrid(np.linspace(-10,10,10), np.linspace(-10,10,10))
Z=function_to_plot(X,Y)
surface=ax.plot_surface(X, Y, Z, alpha=0.5,
label='graph surface')
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')
#plot the curve and the cross section to integrate
ax.scatter(0,0,0,color='red',label='origin')
ax.text(0,0,0,'origin')
ax.scatter(-3,2,6,color='black')
ax.text(-3,2,6,'(-3,2,6)')
plt.legend()
plt.show()
```

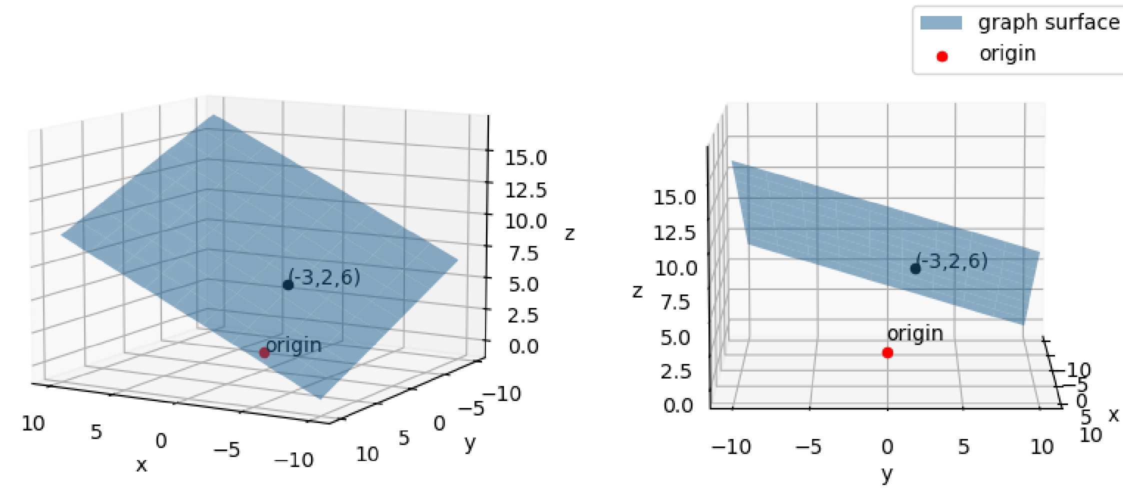


Figure 1.3: Two views of the plane

Interestingly, we see that $(-3, 2, 6)$ - the point corresponding to our vector - is a point on the plane!

Remark. *The form*

$$\left\{ s\vec{i} + t\vec{j} + \left(\frac{49 + 3s - 2t}{6}\right)\vec{k} | s, t \in \mathbb{R} \right\}$$

is a ***parametrization*** *of the plane described by* $-3x + 2y + 6z = 49$*, viewing this as a function* $\mathbb{R}^2 \to \mathbb{R}^3$.

1.3.2 Projection

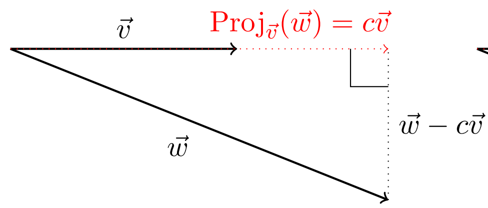
Definition 1.23 (Projection)

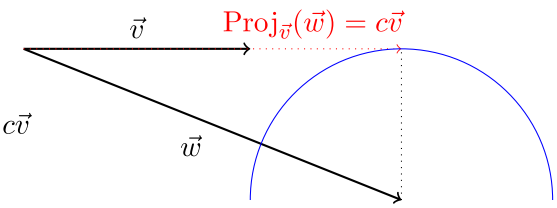
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, and $c \in \mathbb{R}$, such that $\vec{v} \cdot (\vec{w} - c\vec{v}) = 0$ (or equivalently, $\vec{w} - c\vec{v}$ orthogonal to $\vec{v}$).
We say that $c\vec{v} \overset{\text{def}}{=} \text{Proj}_{\vec{v}}(\vec{w})$ is the **vector projection of $\vec{w}$ on $\vec{v}$**.

To build intuition, it is always helpful to start with lower dimensions. We take a plane through $\vec{v}$ and $\vec{w}$. Looking at this slice, we can work in 2D.

We see that $c\vec{v}$ is the the point on the extension of $\vec{v}$ such that it is closest to the point $\vec{w}$. To geometrically show this idea, we draw a circle centered at $\vec{w}$ with radius $|\vec{w} - c\vec{v}|$.

The line generated by $\vec{v}$ is tangent to this circle, so the contact point at $c\vec{v}$ is indeed the closest point to $\vec{w}$.





We also see from this geometric construction that 1) the projection $c\vec{v}$ is unique thus well defined, 2) $c|\vec{v}| = |\vec{w}|\cos\theta$

Proposition 1.24

The projection is unique and is given by

$$\mathrm{Proj}_{\vec{v}}(\vec{w}) = \frac{\cos\theta|\vec{w}|}{|\vec{v}|} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|^2} = (\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}})\vec{v}.$$

This is the first time we show that something is unique. The standard argument goes as follows: Suppose another object a' satisfies all the properties you want for a. Then you can show $a = a'$, meaning every object that satisfies the properties is a, or equivalently, a is unique.

So what happens if we pick two points P, P' on $\vec{v}$ such that both make a right angle when connected to $\vec{w}$? We would have constructed a triangle $\triangle PP'\vec{w}$ with two right angles!

Proof. Let $c, c' \in \mathbb{R}$ such that $\vec{p}_1 = c\vec{v}$ and $\vec{p}_2 = c'\vec{v}$ are both satisfy the definition of $\mathrm{Proj}_{\vec{v}}(\vec{w})$. We want to show that $c = c'$, so we consider $\vec{u} = (c - c')\vec{v}$, the line between the two projections.
By a corollary in 1.3.1, we have two equations

$$|\vec{w} - \vec{p}_1|^2 = |\vec{w} - \vec{p}_2|^2 + |\vec{u}|^2$$
$$|\vec{w} - \vec{p}_2|^2 = |\vec{w} - \vec{p}_1|^2 + |\vec{u}|^2$$

We can solve for $|\vec{u}| = 0$, and by positive definiteness, $\vec{u} = \vec{0}$ and $c - c' = |\vec{u}|/|\vec{v}| = 0$.
To show our formula for projection works, we can just compute that

$$\vec{v}\cdot\left(\vec{w} - \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\right) = \vec{v}\cdot\vec{w} - \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)(\vec{v}\cdot\vec{v}) = 0.$$

□

Exercises

3. Determine if the following pairs of vectors are orthogonal:

   (a) $3\vec{e}_1 + 4\vec{e}_2$, $-4\vec{e}_1 + 3\vec{e}_2$

   (b) $(4, -1, 2)$, $(3, 0, -6)$

   (c) $3\vec{i} - 2\vec{j}$, $-2\vec{i} - 4\vec{k}$

   (d) $\vec{e}_1 + 3\vec{e}_3 + 5\vec{e}_5 + ... + (2k-1)\vec{e}_{2k-1}$, $2\vec{e}_2 + 4\vec{e}_4 + ... + (2k)\vec{e}_{2k}$

4. Refer to the plane in the example in the previous section 1.3.1. Show that $\mathrm{Proj}_{-3\vec{i}+2\vec{j}+6\vec{k}}(\vec{v})$ is the same for any vector $\vec{v}$ such that $\vec{v} \cdot (-3\vec{i} + 2\vec{j} + 6\vec{k}) = 49$, and compute this projection.

5. *(Geometry of a methane molecule)* Place four points $P(0,0,0)$, $Q(1,1,0)$, $R(1,0,1)$, $S(0,1,1)$ in $\mathbb{R}^3$.

   (a) Compute the distance between any two points of $PQRS$ and show that all 6 pairs are the same. This means that $PQRS$ forms a regular tetrahedron.

   (b) Verify that the geometric center of the tetrahedron $O(1/2, 1/2, 1/2)$ is equidistant to all of the vertices of the tetrahedron.

   (c) Compute the angle between two edges of the tetrahedron, rounded to 2 decimal places. By symmetry, you only have to compute one angle.

   (d) Compute the angle $\angle POQ$, rounded to 2 decimal places. Does this angle remind you of something from Chemistry?

1.4 Cross Product

Definition 1.25 (Cross Product)

Let $\vec{v}, \vec{w} \in \mathbb{R}^3$, we define the **cross product** of $\vec{v}$ and $\vec{w}$ to be

$$\vec{v} \times \vec{w} \stackrel{\text{def}}{=} \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}.$$

Remark. *Later on, we will introduce determinants of a matrix. The cross product can be* *understood as the determinant of the 'matrix'*

$$\begin{bmatrix} \vec{i} & v_1 & w_1 \\ \vec{j} & v_2 & w_2 \\ \vec{k} & v_3 & w_3 \end{bmatrix}.$$

This definition seems a bit unmotivating, so let us work through some examples.

Example 1.26

Compute the 9 cross products for each pair of the standard basis vectors in $\mathbb{R}^3$.

With some (heavy) computation, we find

$$\begin{array}{lll} \vec{i} \times \vec{i} = \vec{0} & \vec{i} \times \vec{j} = \vec{k} & \vec{i} \times \vec{k} = -\vec{j} \\ \vec{j} \times \vec{i} = -\vec{k} & \vec{j} \times \vec{j} = \vec{0} & \vec{j} \times \vec{k} = \vec{i} \\ \vec{k} \times \vec{i} = \vec{j} & \vec{k} \times \vec{j} = -\vec{i} & \vec{k} \times \vec{k} = \vec{0} \end{array}$$

Importantly, the cross product is **antisymmetric**, meaning $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$. You can see special cases with the standard basis from above and confirm the general case in an exercise.

Example 1.27

Compute $\vec{v} \cdot (\vec{v} \times \vec{w})$ and $\vec{w} \cdot (\vec{v} \times \vec{w})$.

Again with some heavy computation,

$$\begin{aligned}
\vec{v}\cdot(\vec{v}\times\vec{w}) &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\
&= v_1v_2w_3 - v_1v_3w_2 + v_2v_3w_1 - v_2v_1w_3 + v_3v_1w_2 - v_3v_2w_1 \\
&= 0, \\
\vec{w}\cdot(\vec{v}\times\vec{w}) &= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \cdot \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\
&= w_1v_2w_3 - w_1v_3w_2 + w_2v_3w_1 - w_2v_1w_3 + w_3v_1w_2 - w_3v_2w_1 \\
&= 0.
\end{aligned}$$

Which means $\vec{v}\times\vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$!

Example 1.28

Compute $|\vec{v}\times\vec{w}|^2 + (\vec{v}\cdot\vec{w})^2$.

We have

$$\begin{aligned}
|\vec{v}\times\vec{w}|^2 + (\vec{v}\cdot\vec{w})^2 =& (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 \\
& + (v_1w_1 + v_2w_2 + v_3w_3)^2 \\
=& (v_1w_1)^2 + (v_1w_2)^2 + (v_1w_3)^2 \\
& + (v_2w_1)^2 + (v_2w_2)^2 + (v_2w_3)^2 \\
& + (v_3w_1)^2 + (v_3w_2)^2 + (v_3w_3)^2 \\
=& (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^3) \\
=& |\vec{v}|^2|\vec{w}|^2.
\end{aligned}$$

Now we substitute $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$, we get

$$|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sqrt{1-\cos^2\theta} = |\vec{v}||\vec{w}|\sin\theta.$$

Where we know $\sin\theta \geq 0$ as θ is between 0 and π.

The value $|\vec{v}||\vec{w}|\sin\theta$ has a nice geometric meaning. It is the area spanned by the vectors $\vec{v}$ and $\vec{w}$.

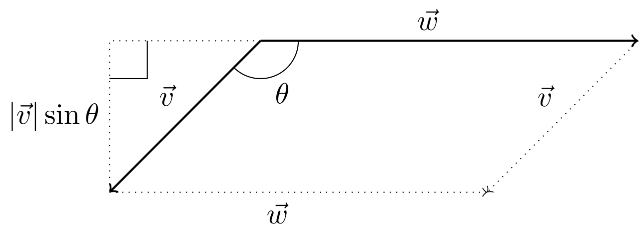


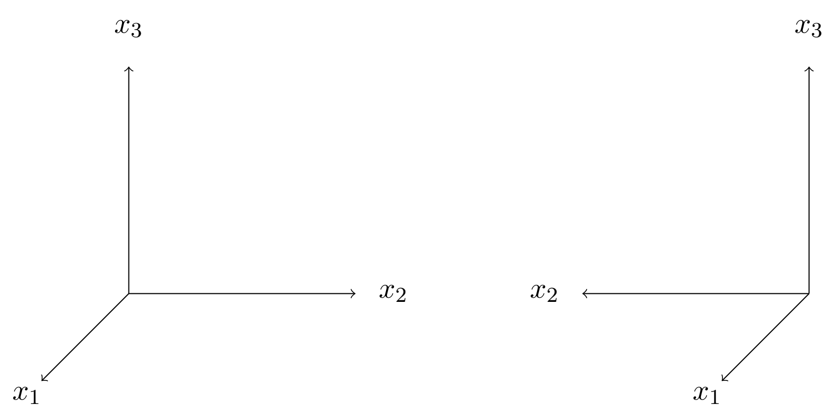
Figure 1.5: The area of a parallelogram formed by these vectors is the magnitude of the vector $\vec{v} \times |\vec{w}|$

Proposition 1.29

The direction of $\vec{v} \times \vec{w}$ is determined by the **right-hand rule** as follows:

> Using the right hand, align the index finger with the direction $\vec{v}$, and the middle finger with the direction of $\vec{w}$. Extend the thumb so that it is perpendicular to both the index finger and the middle finger. The thumb is pointing in the direction of $\vec{v} \times \vec{w}$.

This is a byproduct of the convention we use. In $\mathbb{R}^3$ we use what is known as a right-handed coordinate system - the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ align with the first three fingers of the right hand respectively. If we used a left-handed coordinate system, the rule would be left-handed instead. We now have a few properties about the cross product from our



(a) Right handed coordinate system (b) Left handed coordinate system

computation, the first you will verify on your own:

Proposition 1.30

Let $\vec{v}, \vec{w}, \vec{u} \in \mathbb{R}^n$, then

- latex sucks. I hate using tikz.
- *(distributivity)* $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$ and $(\vec{v} + \vec{u}) \times \vec{w} = \vec{v} \times \vec{w} + \vec{v} \times \vec{u}$.
- *(anti-symmetry)* $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
- $\vec{v} \times \vec{w}$ is orthogonal (perpendicular in 3D space) to $\vec{v}$ and $\vec{w}$.
- The magnitude $|\vec{v} \times \vec{w}|$ is given by $|\vec{v}||\vec{w}| \sin\theta$, with θ being the angle between $\vec{v}$ and $\vec{w}$, so
  (i) The magnitude $|\vec{v} \times \vec{w}|$ also corresponds to the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$.
  (ii) If $\vec{v}$ and $\vec{w}$ are parallel or antiparallel, $\vec{v} \times \vec{w} = \vec{0}$.

Remark. *Using distributivity of the cross product, you only need to memorize the cross product of the basis vectors, and write* $\vec{v} \times \vec{w} = \sum_{i=1}^{3} \sum_{j=1}^{3} v_i w_i (\vec{e}_i \times \vec{e}_j)$.

1.4.1 Triple products

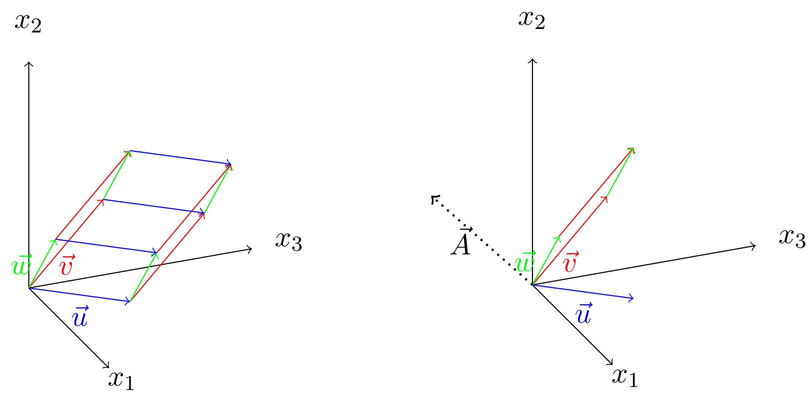
Example 1.31

Find the volume of the parallelepiped formed from $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The parallelepiped is a generalization of a parallelogram to higher dimensions. Using combinations of $\vec{v}, \vec{w}, \vec{u}$, you can make the frame of a 3d solid. As in the figure on the left, edges of the same color correspond to the same vector (and thus parallel). The volume of this solid is still *base* $\times$ *height*, where the base is a 2D parallelogram formed by two vectors and the height is determined by third vector. We make an arbitrary decision and set the base to be $\vec{v}$ and $\vec{w}$. (setting any two vectors would give the same result in the end!) The area of this base is given by the cross product

$$\vec{A} = \vec{v} \times \vec{w} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

Now we can determine the height of the paralellepiped from $\vec{u}$. We want to isolate the component of $\vec{v}$ that is orthogonal to the base. Equivalently, we want to find the component of $\vec{u}$ that is pointing in the direction of $\vec{A}$, a vector that is orthogonal to both $\vec{v}$ and $\vec{u}$!



(a) The parallelepiped, draw in an optical illusion fashion.

(b) We want to get the projection of $\vec{u}$ on $\vec{A}$.

The volume is thus

$$|\text{Proj}_{\vec{A}}(\vec{u})||\vec{v}| = |\frac{\vec{u}\cdot\vec{A}}{|\vec{A}|^2}| \times |\vec{A}| \times |\vec{A}| = |\vec{u}\cdot\vec{A}| = 2.$$

Proposition 1.32

The volume of the parallelpiped formed from $\vec{v}, \vec{w}, \vec{u}$ is

$$|(\vec{v}\times\vec{w})\cdot\vec{u}|$$

Remark. *This is also the expression of the (absolute value of) determinant of*

$$\begin{bmatrix} \vec{v} & \vec{w} & \vec{u} \end{bmatrix}$$

where $\vec{v}, \vec{w}, \vec{u}$ are written as column vectors. Using properties of the determinant (later chapters), you can show cycling the three vectors does not change the volume. (i.e. you can calculate using what order of the three vectors you want)

Remark. *You may notice that the expression*$(\vec{v} \times \vec{w}) \cdot \vec{u}$ *can take on negative volumes. In this case, the three vectors (taken in order) do not follow the right-hand rule. For instance, in the last example,* $\vec{u}$ *points in the 'opposite' direction as* $\vec{A}$*.*

Exercises

6. Find $\vec{v} \times \vec{w}$ for the following:
   (a) $\vec{v} = (4, -2, 0)$, $\vec{w} = (2, 1, -1)$
   (b) $\vec{v} = (3, 3, 3)$, $\vec{w} = (4, -3, 2)$
   (c) $\vec{v} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\vec{w} = \vec{i} - 3\vec{j} + 4\vec{k}$
7. Find the areas for the following shapes:
   (a) The parallelogram with vertices $P(0, 0, 0), Q(1, 1, 0), R(1, 2, 1), S(0, 1, 1)$.
   (b) The triangle with vertices $A(1, 9, 3), B(-2, 3, 0), C(3, -5, 3)$.
8. Find the volume of the parallelepiped formed from the vectors $\vec{v} = (1, 1, 0), \vec{w}(0, 2, -2), \vec{u} = (1, 0, 3)$.
9. A triangular kite has vertices $P(0, 0, 10), Q(2, 1, 10), S(0, 3, 12)$ and is displaced by the wind at a velocity of $(20\vec{i} + 6\vec{j} + 4\vec{k})/s$
   (a) Find the area of the kite.
   (b) After 1/2 seconds, find the volume of the space swept by the kite. (leave the answers in $[units]^3$)

1.5 Applications - Geometry of lines and planes

Definition 1.33 (Relations between lines)

For two (infinitely extending) lines in $\mathbb{R}^n$ parametrized in s and t respectively as $l_1 = P + t\vec{v}$, $l_2 = Q + s\vec{w}$, we say the lines are

- **Parallel**, if the $\vec{v}$ and $\vec{w}$ are parallel or antiparallel.
- **Intersecting**, if l_1 and l_2 exactly one point on both l_1 and l_2.
- **Skew**, if l_1 and l_2 are not parallel/antiparallel or intersecting.

Remark. *Lines do not have direction, so there usually is no need to distinguish between parallel and antiparallel lines. One may extend the definition of antiparallel to lines from* $\vec{v}$ *and* $\vec{w}$*.*

Proposition 1.34

Determination of parallel lines are independent of parametrization. Concretely,

> Let $P_1 + t_1\vec{v}_1$ and $P_2 + t_2\vec{v}_2$ be two parametrizations of l_1, $Q_1 + s_1\vec{w}_1$, $Q_2 + s_2\vec{w}_2$ be two parametrizations of l_2. If $\vec{v}_1 = c_1\vec{w}_1$ for some $c_1 \in \mathbb{R}$, then $\vec{v}_2 = c_2\vec{w}_2$ for some (possibly different)$c_2 \in \mathbb{R}$.

The proof is not very enlightening. However, the result of this guarantees that our definition of parallel lines is precise.

Proof. The idea is to show that $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$, and by the same logic $\vec{w}_1$ is a scalar multiple of $\vec{w}_2$. Since all vectors are non-zero in the parametrization, we will get the result of $\vec{v}_2$ a scalar multiple of $\vec{w}_2$.

To show $\vec{v}_1 = k\vec{v}_2$ for some k, we can pick two distinct points A, B on l_1. From the parametriation we can get $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $P_1 + \alpha_1\vec{v}_1 = A = P_2 + \alpha_2\vec{v}_2$ and $P_1 + \beta_1\vec{v}_1 = B = P_2 + \beta_2\vec{v}_2$. Therefore we get the vector

$$\overrightarrow{AB} = (\beta_1 - \alpha_1)\vec{v}_1,$$
$$\overrightarrow{AB} = (\beta_2 - \alpha_2)\vec{v}_2.$$

As we picked distinct points A and B, we can conclude $\overrightarrow{AB} \neq \vec{0}$ and thus $\beta_1 - \alpha_1 \neq 0, \beta_2 - \alpha_2 \neq 0$, so that

$$\vec{v}_1 = \frac{\beta_2 - \alpha_2}{\beta_1 - \alpha_1}\vec{v}_2.$$

□

Example 1.35

Determine whether the lines parametrized by $l_1(t) = (1, 2, 1) + t(1, 3, -2)$ and $l_2(t) = (3, 1, 0) + t(-2, -6, 4)$ are parallel, intersecting, or skew. Confirm that l_1 and l_2 describe two different lines.

We notice that $-2 \times (1, 3, -2) = (-2, -6, 4)$, so these lines are parallel. To confirm that these two lines are not the same, we notice that $(1, 2, 1)$ is a point on l_1, but if we attempt to solve

$$(1, 2, 1) = l_2(t) = (3, 1, 0) + t(-2, -6, -4) \implies (-2, 1, 1) = t(-2, -6, -4)$$

which no t can solve! Specifically, the first coordinate forces $t = 1$ and the second coordinate forces $t = -6$.

Example 1.36

Determine whether the lines parametrized by $l_1(t) = (1,2,1) + t(1,3,-2)$ and $l_2(t) = (0,3,9) + t(0,2,3)$ are parallel, intersecting, or skew.

$(1,3,2)$ is not a multiple of $(0,2,3)$, so the lines are not parallel. We might be tempted to solve for $l_1(t) = l_2(t)$ to check for intersection, but this misses a lot of cases! We need to compare all the points of l_1 with all the points of l_2, so we need two independent variables to describe where we are on each of the lines. That is, we solve for s, t in $l_1(t) = l_2(s)$,

$$
\begin{aligned}
(1,2,1) + t(1,3,-2) &= (0,3,9) + s(0,2,3) \\
\implies (t+1, 3t+2, -2t+1) &= (0, 2s+3, 3s+9) \\
\implies t = -1 \text{ and } 3t+2 &= 2s+3 \text{ and } -2t+1 = 3s+9
\end{aligned}
$$

$t = -1, s = -2$ solves this system of equations. We can plug in t and s in our original parametrization to find $(0,1,3)$ is indeed a point on both l_1 and l_2. We would have missed this if we set $l_1(t) = l_2(t)$!

Example 1.37

Determine whether the lines parametrized by $l_1(t) = (1,2,1) + t(1,3,-2)$ and $l_2(t) = (0,3,8) + t(0,2,3)$ are parallel, intersecting, or skew.

We repeat the same process as above to see that the lines are not parallel and solve for

$$
\begin{aligned}
(1,2,1) + t(1,3,-2) &= (0,3,8) + s(0,2,3) \\
\implies (t+1, 3t+2, -2t+1) &= (0, 2s+3, 3s+8) \\
\implies t = -1 \text{ and } 3t+2 &= 2s+3 \text{ and } -2t+1 = 3s+8
\end{aligned}
$$

This time, we do not have a solution - the first two equations forces $t = -1, s = -2$, and this does not solve the third. We therefore do not have a point of intersection, and the lines are skew.

Example 1.38

Determine if $P(5,6,9), Q(7,9,15), R(13,18,33)$ are **colinear** i.e. if they lie on the same line.

With the machinery we have built up, there are multiple ways to check if P, Q, R form a straight line. Here are a few ideas:

1. Check that $\overrightarrow{PQ}$ is parallel/antiparallel to $\overrightarrow{PR}$. Because the lines defined by $\overline{PQ}$ and $\overline{PR}$ are parallel and share a same point P, they are the same line.

2. Use the dot product to calculate the angle $\angle PQR = \pi$.

3. Use the cross product to calculate that $\overrightarrow{PQ} \times \overrightarrow{PR} = \vec{0}$. This means the triangle with vertices P, Q, R has no area and thus is a degenerate triangle.

Definition 1.39 (Characterization of Planes)

In euclidean geometry, planes can be characterized by any of the following ways:

- For **any three non-colinear points** P_1, P_2, P_3, there is a unique plane passing through P_1, P_2, P_3.
- For **any pair of intersecting lines**, there is a unique plane that contains both.
- For **a line** l **and a point** P, there is a unique plane that contains P and is perpendicular to l.
- For **a line** l **and a point** P **not on** l, there is a unique plane that contains both l and P.

Remark. *The third characterization is the hardest to visualize at first, but is also the easiest to describe with the analytical tools we have built towards. We can refer to example 1.22 in 1.3.1. The plane sketched is the unique plane that contains* $(-3, 2, 6)$*, such that each vector in the plane is orthogonal to* $(-3, 2, 6)$*, so the line parametrized by* $l(t) = t(-3, 2, 6)$ *is perpendicular to the plane at the point of intersection* $(-3, 2, 6)$*.*

Example 1.40

In $\mathbb{R}^3$, determine the equation of the plane that contains $P_0(x_0, y_0, z_0)$ and is perpendicular to the line parametrized by $l(t) = Q_0 + t\vec{N}$, $\vec{N} = (a, b, c)$.

First we can exploit **translation invariance** of $\mathbb{R}^3$ and move the line to $\tilde{l} = P_0 + t\vec{N}$. Because l and $\tilde{l}$ are parallel, any line perpendicular to l will also be perpendicular to $\tilde{l}$. Then by the characterization of a plane, any P on the plane satisfies the orthogonality relation $\overrightarrow{PP_0} \cdot \vec{N} = 0$. Expanding this, we get the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \implies ax + by + cz = ax_0 + by_0 + cz_0.$$

Definition 1.41 (Normal form of a plane)

Let $a, b, c, d \in \mathbb{R}$, with at least one of $a, b, c \neq 0$. The equation of a plane in ^{3} written as

$$ax + by + cz = d$$

is called a **normal form** of the equation of a plane.

Remark. *As all planes can be characterized this method, all equations of planes can be put in normal form.*

Remark. *The normal form of a plane is not unique. Pick your favorite non-zero number α, the equation $\alpha ax + \alpha by + \alpha c_z = \alpha d$ describes the same plane.*

Theorem 1.42 (Normal vectors of planes)

When written in normal form, the plane is perpendicular to the vector $\vec{N} = (a, b, c)$. We call this vector $\vec{N}$ the **normal vector**.

A problem in the exercise will guide you through the proof of this. The intuition behind the proof is the reverse direction of the equation $\overrightarrow{PP_0} \cdot \vec{N} = 0$ we derived from the last example.
We will now apply this theorem in a few examples.

Example 1.43

Find the equation of the plane that passes through the points $P(1, 0, 0), Q(0, 1, 0), R(0, 0, 1)$.

Method 1: One sees that (by coincidence) the sum of coordinates of each point are equal to 1, so immediately writes down $x + y + z = 1$. Despite being the fastest method, this is somewhat inconsistent.

Method 2: We set $ax + by + cz = d$ to be the equation, and plug in values for P, Q, R, giving the system of equations

$$\begin{aligned} a + 0b + 0c &= d \\ 0a + b + 0c &= d \\ 0a + 0b + c &= d \end{aligned}$$

This is an **underdetermined system**, meaning there are fewer equations than unknowns. The best we can say is that $a = b = c = d$. However, setting $a = b = c = d = k$ works for all $k \neq 0$, further confirming that the normal form is not unique. This method is reasonably fast when the system of equations are simple. When there are more non-zero coefficients, solving the system takes more time.

Method 3: We can compute the direction of the normal vector $\vec{N}$ of this plane. By the characterization of planes, $\vec{N}$ is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, two vectors in the plane. We can thus write $\vec{N}$ as a multiple of the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

. Using this normal vector (or any multiple of it) and applying the theorem, we get $x + y + z = d$, and substituting P into the equation will give $x + y + z = 1$. This method is more general and consistent, as the number of operations in a cross product is constant.

Example 1.44

On the plane given by $ax + by + cz = d$, find the point on the plane that is closest to the origin.

Let r be the distance, we draw a sphere centered at the origin with radius r. This sphere is thus tangent to the surface at one point, and the vector corresponding to this point will be perpendicular to the plane. By the theorem, we can denote the point $P(\alpha a, \alpha b, \alpha c)$, with α a constant to be determined. Substituting P into the equation,

$$\alpha(a^2 + b^2 + c^2) = d \implies \alpha = \frac{d}{a^2 + b^2 + c^2}$$

The closest distance from the origin is

$$|(\alpha a, \alpha b, \alpha c)| = |\alpha|\sqrt{a^2 + b^2 + c^2} = \left|\frac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

at the point

$$\left(\frac{da}{\sqrt{(a^2 + b^2 + c^2)}}, \frac{db}{\sqrt{(a^2 + b^2 + c^2)}}, \frac{dc}{\sqrt{(a^2 + b^2 + c^2)}}\right)$$

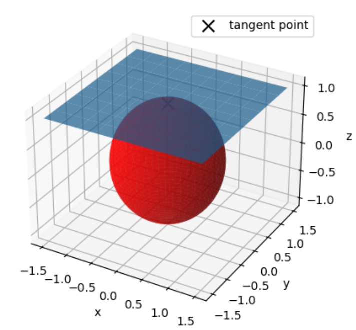


Figure 1.8: A toy example with the unit sphere of radius 1 and the plane described by $z = 1$.

1.5.1 Intersection of planes

Example 1.45

Find the intersection of the planes given by

$$6x + 2y - z = 2$$
$$\text{and } x - 2y + 3z = 5.$$

Method 1: We can solve the system of equations to find the line of intersection, and arrive at the set of symmetric equations. A similar method would be to solve for two points on the line of intersection, then get the parametric equation. **TODO, only if i feel like it**

Method 2: The line lies on the first plane, so is perpendicular to its normal vector $(6, -2, -1)$. The line also lies on the second plane, so is perpendicular to its normal vector $(1, -2, 3)$. Using the cross product, the line should point in the direction of

$$\begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -19 \\ -14 \end{bmatrix}$$

Now we just need to find one point P on the line of intersection, so that the parametric equation (in t) is $P + t(4, -19, -14)$. We can impose an additional restriction that $x = 0$ (or $y = 0$ or $z = 0$) and solve the simultaneous equations

$$2y - z = 2$$
$$-2y + 3z = 5.$$

and get $(0, 11/4, 7/2)$ is a point on the intersection. The final parametric representation is $l(t) = (0, 11/4, 7/2) + t(4, -19, -14)$.

Exercises

10. todo...

11. On the plane given by $ax + by + cz = d$, find the closest distance from $P(x_1, y_1, z_1)$ to the plane. (Hint: Refer to Example 1.44 in section 1.5)

1.6 End of Chapter Exercises

12. todo - proof based questions

Chapter 2

Multiple Integrals

2.1 Introduction

By this point, you should have noted that the geometry of $\mathbb{R}^n$ is intimately connected to calculus. Partial derivatives and tangent planes go hand in hand. However, if there's anything to take away from your first calculus course is that derivatives also go hand in hand with integrals. So you have every right to ask "What does it mean to do integrals in higher-dimensions?" This chapter will focus on developing the core concepts that surround this question. We will make sense of integrals in 2 and 3 dimensions by generalizing our intuition for the 1D case. We will see how — in certain situations — we can convert double and triple integrals into a simple iteration of singular integrals. Finally, we will derive the multidimensional version of the change of variables formula. Along the way, various applications to other disciplines will be presented, and you are encouraged to test your understanding via the exercises in each section.

To start our discussion, let's remind ourselves of what it means to integrate a real-valued function $f(x)$ along some interval $I = \{x \in \mathbb{R} : \ a \leq x \leq b\}$,

$$\int_a^b f(x)\ \mathrm{d}x \tag{2.1}$$

Intuitively, we interpret this as the signed area determined by $f(x)$ when $x \in I$. But how do we compute this area, in general? A good strategy is to partition the interval into very small chunks that are easier to deal with, and then add up their individual contributions to the integral. To that effect, let's partition the interval I into a sequence of intervals $I_1, I_2, \cdots, I_n$ according to a sequence of points $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. In other words, take $I_k = [x_{k-1}, x_k]$. This procedure is demonstrated in Figure 2.1. Now, take a representative point $\tilde{x}_k \in I_k$ from each interval. It doesn't matter what this point is exactly, as we are assuming the intervals are small enough such that the function doesn't change very much within each interval. The signed area of f on each of the I_k is approximately

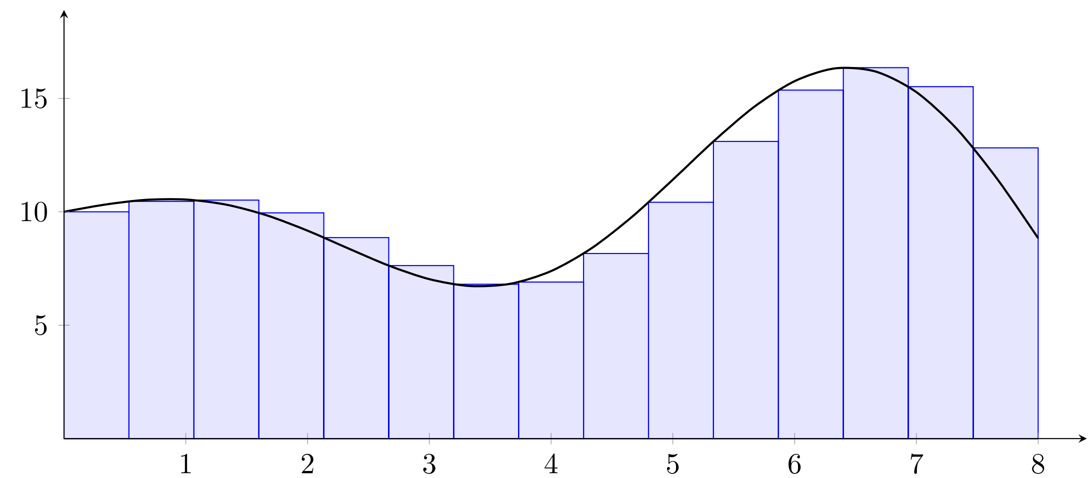


Figure 2.1: In this plot, we took $\tilde{x}_k = x_{k-1}$.

$f(\tilde{x}_k)(x_k - x_{k-1})$. Thus, since $I = I_1 \cup I_2 \cdots \cup I_n$, we can approximate the integral as follows:

$$\int_a^b f(x)\ \mathrm{d}x \approx \sum_{k=1}^{n} f(\tilde{x}_k)(x_k - x_{k-1}) \tag{2.2}$$

The right-hand side is a *Riemman sum*, as you learned in your calculus course. We can measure the accuracy of our approximation by the largest interval width $\sigma = \max_k\{x_k - x_{k-1}\}$. Then, we define[1] the *Riemman integral* to be:

$$\int_a^b f(x)\ \mathrm{d}x \overset{\text{def}}{=} \lim_{\sigma\to 0} \sum_{k=1}^{n} f(\tilde{x}_k)(x_k - x_{k-1}) \tag{2.3}$$

We can use this definition to calculate various quantities of relevance in science. For example, imagine someone gives you a rod of length L. The rod is made of an insulating material, and it is charged with a linear charge density $\rho(x)$, for $x \in [0, L]$. The total charge Q stored in the rod is given by the integral:

$$Q = \int_0^L \rho(x)\ \mathrm{d}x \tag{2.4}$$

In terms of the Riemman sum, we visualize the rod as a set of tiny point masses with charge $\rho(\tilde{x}_k)(x_k - x_{k-1})$ glued together, each contributing a small amount to the total charge of

[1] Note that the Riemman sum is not a function of σ, so there are some subtleties in how we define this limit. You don't need to worry about them for now. You just need to note that the Riemman sum is definitely constrained by σ.

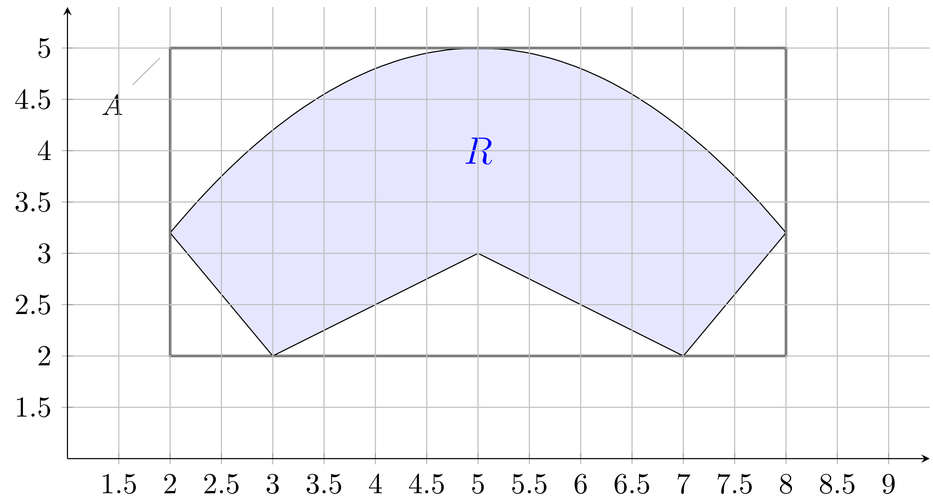


Figure 2.2: A set of small rectangles with a larger rectangle A bounding the region R.

the rod. But what if someone gave us a charged sheet instead? Suppose you are also given an area charge density $\rho(x, y)$, how would you calculate the total charge of this sheet? Well, there's no reason why the divide and conquer strategy shouldn't work here as well. Let's get to work!

Suppose the sheet is represented by a region $R \in \mathbb{R}^2$. We will focus on the rectangle $A = [x_0, x_m] \times [y_0, y_n]$ that "bounds" the region. It can be subdivided into a grid of smaller rectangles obtained from subdividing the intervals $[x_0, x_m]$ and $[y_0, y_n]$ as we did with the 1D case. For a visual aid, check Figure 2.2. We want to add up the contributions just from the rectangles that lie inside the region R, so we need to evaluate ρ at a select group of rectangles while avoiding the others. This is complicated in general, so as a computational trick, we will define an auxiliary function $\tilde{\rho}(x, y)$ that agrees with $\rho(x, y)$ inside R, but is zero outside R.[2] Then, choose points $(\tilde{x}_i, \tilde{y}_j)$ inside the (i, j)-rectangle. By analogy with our intuition for the single integral, the small amount of charge contributed from that rectangle is $\tilde{\rho}(\tilde{x}_i, \tilde{y}_j)\Delta x_i \Delta y_j$ (where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$). Thus, we can approximate the total charge Q of the charged sheet by

$$Q \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{\rho}(\tilde{x}_i, \tilde{y}_j)\Delta x_i \Delta y_j \tag{2.5}$$

At this point, you must be anticipating that we just need to take the limit as $\sigma_x = \max_i\{\Delta x_i\}$ and $\sigma_y = \max_j\{\Delta y_j\}$ go to zero, like we did for the 1D case. However, just

[2]We know that $\rho(x, y)$ is well-defined in R, but we have no idea what it does outside R. Maybe it describes the charge density of another object, or maybe it's not defined there. In the general case, we need to be careful.

as we needed to be careful when taking limits in $\mathbb{R}^2$ (recall that the limit can't depend on the path taken towards the limit point), we need to be careful here. If we take the limit as $\sigma_x \to 0$ first, then $\Delta x_i \to 0$ and we end up with a bunch of rectangles of finite height and zero width. That amounts to adding zeros and our strategy fails. Similarly if we take the limit as $\sigma_y \to 0$ first. We want both dimensions of the rectangles to become small "at the same time." A reasonable way of doing this is to declare that the largest dimension $\sigma = \max\{\sigma_x, \sigma_y\}$ of the rectangles must go to zero. Definition 2.2 is as precise as we can make this statement in this course. When the limit exists, f is said to be *integrable* over the region R. Note that being integrable depends on *both* f and R. Under different formalizations of how one actually takes this limit, the same function might be classified as integrable or not. Well-behaved functions might not be integrable if the region R is poorly-behaved, and vice versa. These pathological cases are not of interest right now, so from now on we will assume both R and f are "nice enough." In particular, if R is the union of finitely many elementary regions (as defined in the next section), then any continuous function f is integrable regardless of how you formally perform the limiting process.

2.2 Double Integrals

Based on the preceding discussion, we take Definition 2.2 to capture the basic notion of integrating over a region in $\mathbb{R}^2$. Note that the double integral is a fundamentally two-dimensional concept: the symbol $\iint$ is there just as an analogy to the one-dimensional case. As of right now, we have no idea how to compute double integrals for any reasonable function. A priori, you shouldn't look at them as two single integrals nested together. But it seems reasonable that they *should* be related to single integrals in some way. The results later in this section will indeed establish that this intuition is correct, providing a computational method to deal with multidimensional integrals. Before we engage with such powerful tools, we need first to look at these objects from a geometrical perspective.

Definition 2.1 (Double Integral)

If $R \subseteq \mathbb{R}^2$ is a finite region and $f : R \to \mathbb{R}$ is a function, then the double integral of f over the region R, is given by the following limit, when it exists:

$$\iint_R f(x, y)\, \mathrm{d}A \stackrel{\text{def}}{=} \lim_{\sigma \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{f}(\tilde{x}_i, \tilde{y}_j)\Delta x_i \Delta y_j$$

What general feature of integral is illustrated by Figure 2.1? For each x, $f(x)$ is the height of the graph of the f at that point, so when we multiply it by the length of a thin rectangle in the Riemman sum and add all pieces together, we get the signed area determined by $f(x)$ on the interval. How does this generalize? Assume now $f(x, y)$ is a function $R \to \mathbb{R}$. For each tiny rectangle at (x, y), we are multiplying the value of a function $f(x, y)$, which is the height of the graph of f at (x, y). Thus, when we look at the graph of a two-variable function, its double integral over some region R is just the net volume covered by $f(x, y)$ in that region. Let's take this opportunity to do some programming and visualize things. Assume $f(x, y) = \sqrt{1 - x^2 - y^2}$, defined on the unit disk centered at the origin. D. We will compute an approximation for $\iint_D f(x, y)\, \mathrm{d}A$ using Python.

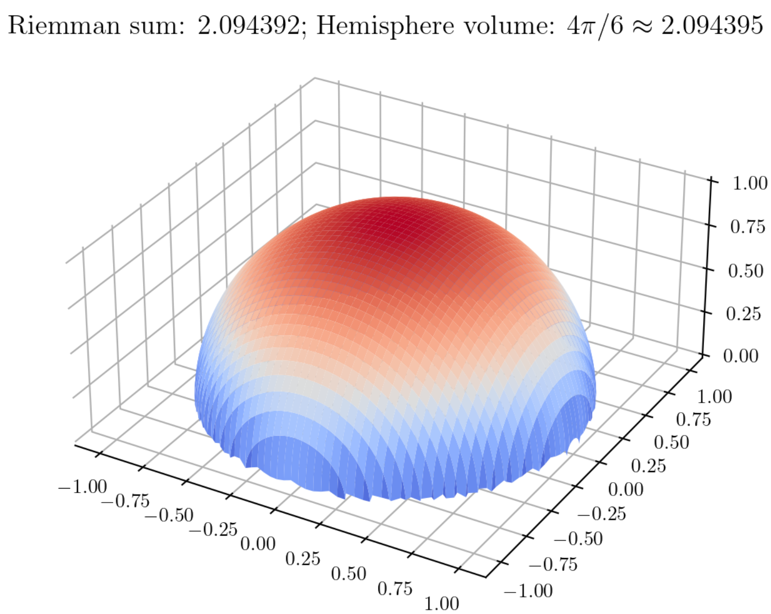


Figure 2.3: The hemisphere surface $z = \sqrt{1 - x^2 - y^2}$.

Using the code in the next page, we calculated the volume enclosed by the hemisphere surface and the xy-plane in Figure 2.3 to be 2.094392. From elementary geometry, we know the answer should be $4\pi/6 \approx 2.094395$. The error comes, of course, from the fact that the Riemman sum is just an approximation, not the exact answer. But as we increase the number of sampling points (i.e. make the rectangles smaller), these errors go away and the sum gets closer to the correct value, which is the integral. Our task now will be to learn how to get the exact value using some additional facts about double integrals.

```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import cm
# Use a 3d projection
fig, ax = plt.subplots( subplot_kw={"projection": "3d"},
                        figsize=(8,5), dpi=200)

# Increasing the number of sampling points will reduce
# the jagged edges around the equator and increasing accuracy
# of the integration.
sampling_points = np.linspace(-1, 1, 1000)

# X and Y are matrices containing the x and y coordinates of the sampling
    points
X, Y = np.meshgrid(sampling_points, sampling_points, indexing='ij')

# For each coordinate, we need not to sample at the last point as it
# belongs to the next interval [1, 1+delta]
Z = np.sqrt(1- X[:-1,:-1]*X[:-1,:-1] - Y[:-1,:-1]*Y[:-1,:-1])

surface = ax.plot_surface( X[:-1,:-1], Y[:-1,:-1], Z,
                           cmap=cm.coolwarm, linewidth=0,
                           antialiased=True)

ax.set_xticks([-1 + 0.25 *n for n in range(9)])
ax.set_yticks([-1 + 0.25 *n for n in range(9)])
ax.set_zticks([0.25 *n for n in range(5)])
ax.axes.set_aspect('equal')

# We took the sqrt of negative numbers many times and got NaN as a result,
# but we just want to ignore the function there, so we set it to 0
Z = np.nan_to_num(Z)

# Compute the sizes of each rectangle (these should be all the same
# except for issues with floating-point arithmetic, which is why
# we still have to compute them directly)

deltaX = np.diff(X, axis=0)
deltaY = np.diff(Y, axis=1)

# Compute Riemman sum
integral = np.sum(Z * deltaX[:,:-1] * deltaY[:-1,:])

fig.text(0.5, 0.9, s = fr'Riemman sum: ${integral:.6f}$; Hemisphere volume:
     $4\pi / 6 \approx {np.pi * 4/6:.6f}$', size=14, horizontalalignment='
    center')


fig.tight_layout()
fig.savefig('double_integral.png')
```

Definition 2.2 (Elementary Regions)

def

2.3 Iterated Integrals

2.4 Triple Integrals

2.5 Change of Variables