Chapter 1

Sample - vector calculus

Definition (Line Integral - scalar)

Let $U \subset \mathbb{R}^n$, $\gamma \subset U$ be a piecewise smooth curve, and $f : U \to \mathbb{R}$. The **line integral of f along** γ is defined as

$$\int_\gamma f \, ds \;=\; \int_a^b f(r(t)) \, |r'(t)| \, dt$$

where $r : [a, b] \to \gamma$ is any parameterization of γ.

Definition (Line Integral - vector)

Let $U \subset \mathbb{R}^n$, $\gamma \subset U$ be a piecewise smooth curve, and $f : U \to \mathbb{R}^n$. The **line integral of f along** γ is defined as

$$\int_\gamma f \, ds \;=\; \int_a^b f(r(t)) \cdot r'(t) \, dt$$

where $r : [a, b] \to \gamma$ is any parameterization of γ.

Geometric intuition

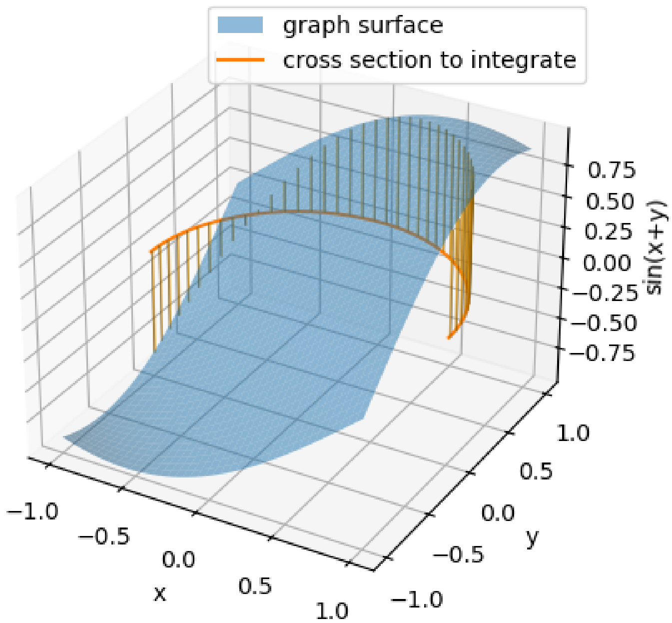
For a visualization, let's plot a 2D function (so visualizing the graph in 3D) in Python. We want to integrate the function $f(x, y) = \sin(x+y)$ along the semi-unit-circle in the top half plane, oriented counter-clockwise. This can be parametrized by $r(t) = (\cos(t), \sin(t))$ for $t \in [0, \pi/2]$.

```
import matplotlib.pyplot as plt
import numpy as np
#create figure
ax = plt.figure().add_subplot(projection='3d')

#plot sin(x+y)
X,Y=np.meshgrid(np.linspace(-1,1,40), np.linspace(-1,1,40))
Z=np.sin(X+Y)
surface=ax.plot_surface(X, Y, Z, alpha=0.5,
label='graph surface')
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('sin(x+y)')

#plot the curve and the cross section to integrate
curve=np.linspace(0,np.pi,40)
plot_curve=ax.plot(np.cos(curve), np.sin(curve), 0,
label='cross section to integrate')
curvevalues=np.sin(np.cos(curve)+np.sin(curve))
curveonsurface=ax.bar3d(np.cos(curve),np.sin(curve),0,
dx=0.01, dy=0.01, dz=curvevalues, alpha=0.5,
color='orange')

plt.legend()
plt.show()
```



graph surface
cross section to integrate
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
sin(x+y)
−1.0
−0.5
0.0
0.5
1.0
x
1.0
0.5
0.0
−0.5
−1.0
y