Large values of Dirichlet Polynomials and Zero Density Results of the Riemann Zeta Function

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Overview

The goal of this paper is to prove a new result on large values of Dirichlet polynomials. We first motivate this result through the study of primes in short intervals, using the Riemann Zeta function as a special case of a Dirichlet L-function. We then prove the Prime Number Theorem to demonstrate the connection of Zeta zeros and primes, with the Riemann Hypothesis leading to the tightest bound on the error term of the Prime Number Theorem. The Riemann Hypothesis has not be proved or disproved. Nevertheless, one can still obtain unconditional results on the error term through appealing to the density of Zeta zeros in the critical strip. We will provide the previous bound on the density of zeros and sketch its refinement by Guth and Maynard. Finally, extending from the ideas of Guth and Maynard's proof, we provide a generalization for the analogous Halász inequality lemma for Dirichlet L-functions.

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Statement Regarding Generative AI

No part of this thesis had been aided by generative AI.

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Chapter 0

Notation and Preliminaries

Notation

We denote summation over the natural numbers $\mathbb{N}$ to be over the positive integers. We always denote p as a prime, and by extension p_j, p_n etc. We denote $e(x) := \exp(2\pi i x)$. For asymptotic behaviors, we write $A \ll B$ if there is an absolute constant c such that $A < cB$, and $A \ll_\epsilon B$ if $A < cB$ with c possibly depending on ϵ. Similar to the notation in Guth-Maynard's paper, we write $A \asymp B$ if $A \ll B$ and $B \ll A$ and $A \sim B$ for $B < A \leq 2B$. We also write $A \lesssim B$ if $A \ll_\epsilon (qT)^\epsilon B$ for any $\epsilon > 0$ (or in the case of the Zeta function, $q = 1$). Words on top of a relation denote the theorem applied to derive the relation, for instance $\overset{\text{Poisson}}{=}$ and $\overset{\text{CS}}{\leq}$ means that Poisson summation and Cauchy-Schwarz is applied respectively.

Preliminaries

Here we provide some supplementary definitions and statements of theorems. These results are well-known.

Number Theory

Definition 0.1 (Dirichlet Characters). *Let $q \in \mathbb{N}$. A Dirichlet character $\chi : \mathbb{N} \to \mathbb{C}$ modulus q is an arithmetic function satisfying*

- *(Periodicity)* $\chi(n + q) = \chi(n) \forall n \in \mathbb{N}$.
- *(Complete multiplicativity)* $\chi(nm) = \chi(n)\chi(m) \forall n, m \in \mathbb{N}$.
- $|\chi(n)| = \begin{cases} 1, & \text{if } \gcd(\text{n}, \text{q}) = 1, \\ 0, & \text{otherwise.} \end{cases}$

Proposition 0.2. *There are $\phi(q)$ Dirichlet characters of modulus q.*

Proof. Taking residual classes mod q, we see that Dirchlet characters are in one-to-one correspondence with one-dimensional representations of the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$. Since this group is abelian, all of its irreducible representations are one-dimensional. Therefore, the number of Dirichlet characters equals the number of irreducible representations of the $(\mathbb{Z}/q\mathbb{Z})^\times$. It is known that the sum of squares of the dimensions irreducible representations equals the order of the group, so we have

$$\phi(q) = |(\mathbb{Z}/q\mathbb{Z})^\times| = \sum_{\text{irreducible representations } \varphi} (\dim \varphi)^2 = \sum_{\text{irreducible representations } \varphi} 1.$$

$\square$

Definition 0.3. *A Dirichlet character χ modulus q is induced by another character χ^* mod $m < q$ if they agree on all n such that $\gcd(q, n) = 1$. A Dirchlet character is primitive if it is not induced by another character. A Dirchlet character is principal if it is induced by the character $\chi_1(n) := 1(n) \equiv 1$, thus corresponds to the trivial representation.*

Theorem 0.4 (Möbius Inversion)**.** *The Möbius function μ is defined for $n \in \mathbb{N}$,*

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 p_2 ... p_k \text{ for distinct } p\text{'s} \\ 0, & \text{otherwise} \end{cases}$$

Suppose we have arithmetic functions f, g, and that

$$f(n) = \sum_{d|n} g(d)$$

Then the Möbius Inversion formula gives

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

Example 0.5. *On $\Re(s) > 1$, let $M_N(s) = \sum_{n \le N} \mu(n) n^{-s}$. Then setting $f(n) = 1$ for all n, $g(1) = 1$, $g(n) = 0$ for $n \ge 2$, we multiply M_N by ζ in Dirichlet series to get*

$$\zeta(s) M_N(s) = \sum_n \frac{a_n}{n^{-s}},$$

where $a_n = g(n)$ for all $n \le N$. Similarly, letting $M_N(s) = \sum_{n \le N} \chi(n)\mu(n) n^{-s}$ for some Dirichlet character χ, we get

$$L(s, \chi) M_N(s) = \sum_n \frac{a_n \chi(n)}{n^{-s}}$$

with the same a_n as in the previous equation.

Theorem 0.6 (Erdös-Kac)**.** *Let $\omega(n)$ be the number of prime divisors of n, ignoring multiplicity. Let $X_N \sim \text{unif}\{1, \text{N}\}$, the uniform distribution from 1 to N, and*

$$Y_N := \frac{\omega(X_N) - \log\log X_N}{\sqrt{\log\log X_N}}.$$

Then

$$\lim_{N \to \infty} P(a < Y_N < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} dt.$$

Harmonic Analysis

Theorem 0.7 (Fourier Inversion)**.** *In Schwartz space, the Fourier transform of $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ of $f \in \mathcal{S}(\mathbb{R}^d)$ is given by*

$$\hat{f}(\xi) := \mathcal{F} f(\xi) := \int_{\mathbb{R}^d} e(-\xi \cdot \mathbf{x}) f(\mathbf{x})\ d\mathbf{x}$$

has inverse given by

$$f(\mathbf{x}) = \mathcal{F}^{-1} \hat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e(\xi \cdot \mathbf{x}) \hat{f}(\xi)\ d\xi.$$

Theorem 0.8 (Discrete Fourier Inversion)**.** *Let g be an arithmetic function that is periodic mod q. We define the discrete Fourier transform of g to be*

$$\hat{g}(y) := \sum_{x \bmod q} g(x) e\Big(-\frac{xy}{q}\Big)$$

which is periodic mod q. This has inverse given by

$$g(x) = \frac{1}{q} \sum_{y \bmod q} \hat{g}(y) e\Big(\frac{xy}{q}\Big).$$

Theorem 0.9 (Plancherel/Parsavel). *We have*

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}$$

for Schwartz functions and

$$\sum_{x \ mod \ q} |g(x)|^2 = \frac{1}{q} \sum_{y \ mod \ q} |\hat{g}(y)|^2$$

for the discrete Fourier transform of functions periodic mod q.

Theorem 0.10 (Mellin Inversion). *The Mellin transform of a function $f : (0, \infty) \to \mathbb{C}$*

$$\tilde{f}(s) \coloneqq \mathcal{M}f(s) \coloneqq \int_0^\infty f(x) x^{s-1} \ dx$$

has inverse

$$\mathcal{M}^{-1}\tilde{f}(x) = \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) x^{-s} ds$$

on $a < c < b$ provided that the integral $\tilde{f}$ is absolute convergent on the strip $a < \Re(s) < b$.

Theorem 0.11 (Poisson Summation). *Let $f : \mathbb{R} \to \mathbb{C}$ be Schwartz. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{\xi \in \mathbb{Z}} \hat{f}(\xi).$$

Chapter 1

The Riemann Zeta Function and the Prime Number Theorem

1.1 Introduction to the Riemann Zeta Function

We give a quick introduction to the Zeta function in this section, including its product representation and analytic continuation.

Definition 1.1 (Zeta Function). *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$. The properties of the zeta function as they relate to the distribution primes. In particular, the Dirichlet series can be represented as a product of primes.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1.2}$$

Remark: This expresion also converges absolutely for $\Re(s) > 1$. Since

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{p^s}{p^s - 1} = 1 + \frac{1}{p^s - 1}$$

and $\sum (p^s - 1)^{-1}$ converges absolutely by comparison to the zeta function Dirichlet series.

Sketch of proof. Write $s = \sigma + it$. For each p,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p. We thus take for $m > N$

$$\begin{aligned}
\prod_{p \le N} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \le N} \left(\sum_{k=1}^{m} \frac{1}{p^{ks}} + O(2^{-m\sigma})\right) \\
&\overset{(*)}{=} \sum_{n=1}^{N} \frac{1}{n^s} + O(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}) + O(2^{-m\sigma}) \\
&= \zeta(s) + O(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}) + O(2^{-m\sigma})
\end{aligned}$$

Where we apply to Fundemental Theorem of Arithmetic in (*) to show that each term n^{-s} has coefficient 1 determined by the unique prime factorization. As $m \to \infty$, $2^{-m\sigma} \to 0$. Then we take $N \to \infty$, the tail of the infinite sum converges to zero too. □

Proposition 1.3. *ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too.*

We will prove Proposition 1.3 in two steps. First, we will extend ζ to $\sigma > 0$. Then, we will describe the continuation of the zeta function to the whole plane using by its functional equation: ζ has a line of symmetry across $\Re(s) = 1/2$.

Proposition 1.4. *Let $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then*

$$\xi(s) = \xi(1-s). \tag{1.3}$$

Extension of ζ to $\sigma > 0$. We apply integration by parts on Dirichlet series when $\sigma > 1$

$$\begin{aligned}
\zeta(s) &= \int_{1/2}^{\infty} x^{-s} d\lfloor x \rfloor \\
&= s\int_{1/2}^{\infty} \lfloor x \rfloor x^{-s-1} dx \\
&= s\int_{1}^{\infty} x^{-s} - \frac{\{x\}}{x^{-s-1}} dx \\
&= \frac{s}{s-1} - s\int_{1}^{\infty} \frac{\{x\}}{x^{-s-1}} dx
\end{aligned}$$

where in the last expression, the integral converges when $\sigma > 0$, and the pole at $s = 1$ arises from the first term. □

Proof of Proposition 1.4. Using

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx,$$

we make the substitution $x = \pi n^2 y$ to get

$$\begin{aligned}
\Gamma(s) &= \int_0^{\infty} e^{-\pi n^2 y} (\pi n^2 y)^{s-1} \pi n^2 dy \\
\implies \frac{\Gamma(s)}{\pi^s n^{2s}} &= \int_0^{\infty} e^{-\pi n^2 y} y^{s-1} dy
\end{aligned}$$

So that by the Monotone Convergence Theorem,

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n=1}^{\infty} \frac{\Gamma(s/2)}{\pi^{s/2} n^s} \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 x} x^{s/2-1} dx \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} \left(e^{-\pi n^2 x}\right) x^{s/2-1} dx.
\end{aligned}$$

We now let

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}, \quad \theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 2\omega(x) + 1,$$

and apply Poisson Summation to

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i k u/\sqrt{x}} du \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} e^{-\pi k^2/x} \\
&= \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)
\end{aligned}
$$

using the substitution $y\sqrt{x} = u$. Replacing with ω,

$$\sqrt{x}(2\omega(x) + 1) = 2\omega\left(\frac{1}{x}\right) + 1 \implies \omega\left(\frac{1}{x}\right) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}.$$

We thus write, using $y = 1/x$,

$$
\begin{aligned}
\xi(s) &= \int_0^1 \omega(x) x^{s/2-1} dx + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \omega(1/y) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\sqrt{y}\omega(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{1-s} + \frac{1}{s} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{s(1-s)} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx.
\end{aligned}
$$

ω decays exponentially in x, so the integral converges and the last expression is well defined on $\mathbb{C}$ with simple poles at $s = 1$ or $s = 0$. Finally, notice that the last expression is symmetric when s is replaced with $(1 - s)$, so proves equation 1.3. □

Finally, we extend to $\zeta(0)$ by noticing that the poles of the functional equation from $\zeta(1)$ and $\Gamma(0)$ cancel out, so the Riemann Extension Theorem can be applied.

From the functional equation, we get 'trivial' zeros of the zeta function from the poles of Γ.

Corollary 1.5. *On $\Re(s) > 1$ or $\Re(s) < 0$, $\zeta(s) \neq 0$, except $\forall n \in \mathbb{N}, \zeta(-2n) = 0$.*

Proof. Using the product representation of ζ where it converges, none of $(1 - p^{-s})^{-1} = 0$, so $\zeta(s) \neq 0$ on $\Re(s) > 1$. Γ has no zeros and has a simple pole at $-n$ for all $n \in \mathbb{N}$, so by equation 1.3 we get the zeros for $\Re(s) > 0$ are exactly at the negative even integers. □

These zeros are known as the trivial zeros of ζ. The remaining zeros lie between $0 \leq \Re(s) \leq 1$.

Definition 1.6 (Critical Strip and Critical Line)**.** *We denote the region $0 \leq \Re(s) \leq 1$ as the* ***critical strip****. We denote the line $\Re(s) = 1/2$ as the* ***critical line****.*

Corollary 1.7. *On the critical strip, if $\zeta(s) = 0$, $\zeta(\overline{s}) = \zeta(1-s) = \zeta(1-\overline{s}) = 0$.*

Proof. This follows from equation 1.3, and $\zeta(\overline{s}) = \overline{\zeta(s)}$ holds where the Dirichlet series converges, thus holds everywhere. □

The number of zeros in the critical strip can be calculated using the argument principle applied to the function ξ over the box with corners $-1+iT$,$-1-iT$,$2-iT$,$2+iT$. Applying the functional equation, we get the following result.

Theorem 1.8 (Number of zeros of ζ)**.** *The number of zeros up to height T*

$$\#\{\sigma+it \mid \zeta(\sigma+it)=0, 0\leq\sigma\leq 1, |t|\leq T\} = \frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T).$$

Theorem 1.8 is obtained by Riemann in his famous 1859 paper on the Zeta function, where he analytically continued the Zeta function and conjectured that all the zeros lie on the critical line [9].

Conjecture 1.9 (Riemann Hypothesis)**.** *The* ***Riemann Hypothesis*** *(RH) asserts that on the critical strip,*

$$\zeta(s)=0 \implies \Re(s)=\frac{1}{2}.$$

1.2 The Prime Number Theorem

Theorem 1.10 (Prime Number Theorem)**.** *Let $\Pi(N)=\sum_{p\leq N}1$. Then*

$$\Pi(N)=(1+o(1))\frac{N}{\log N}.$$

In this section we will prove the Prime Number Theorem. This proof is heavily inspired by one of Terence Tao's lectures, with modifications to fit the techniques used in this thesis. The Prime Number theorem serves as a starting point for studying primes in short intervals, and sets the stage for zero-density theorems.

Definition 1.11 (Von Mangoldt Function)**.** *The* ***Von Mangoldt function*** *Λ is defined as follows:*

$$\Lambda(n)=\begin{cases}\log p, \text{ if } n=p^k \text{ for some } k\in\mathbb{N}\\ 0, \text{ else}\end{cases}$$

The sum of the Von Mangoldt function $\sum\Lambda(n)$ is a more natural way to express a prime counting function in the language of ζ. To see why, consider the expression

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= (\log\zeta(s))' \\ &= \left[-\sum_p \log\left(1-p^{-s}\right)\right]' \\ &= -\sum_p \frac{p^s\log p}{1-p^{-s}} \\ &= -\sum_p \log p\sum_{k\in\mathbb{N}} p^{-ks} \\ &= -\sum_{n\in\mathbb{N}}\frac{\Lambda(n)}{n^s}\end{aligned}$$

on $\Re(s)>1$ where the sum and products are absolutely convergent.

Proposition 1.12. *$\sum_{n\leq N}\Lambda(n)=(1+o(1))N$ implies the Prime Number Theorem.*

Proof. On one hand, we have

$$\begin{aligned}\sum_{n\leq N}\Lambda(n) &\leq \sum_{p\leq N}\Lambda(p) \\ &\leq \Pi(N)\log N.\end{aligned}$$

And for $\epsilon > 0$,

$$\begin{aligned}\sum_{n\le N}\Lambda(n) &\ge \sum_{N^{1-\epsilon}\le n\le N}\Lambda(n)\\ &\ge \sum_{N^{1-\epsilon}\le p\le N}(1-\epsilon)\log(N)\\ &= (1-\epsilon)(\Pi(N)\log(N)+O(N^{1-\epsilon}\log N)).\end{aligned}$$

□

Moreover, the sum of the Von Mangoldt function can be related to the zeros of the zeta function. Let ϕ be smooth and rapidly decaying at infinity, and $\tilde{\phi}$ be its Mellin transform. Let $N \in \mathbb{N}$ and $c \ge 2$. Then

$$\begin{aligned}\sum_{n\in\mathbb{N}}\Lambda(n)\phi\left(\frac{n}{N}\right) &= \sum_{n\in\mathbb{N}}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\sum_{n\in\mathbb{N}}\Lambda(n)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)N^s\frac{\zeta'(s)}{\zeta(s)}ds\end{aligned} \tag{1.4}$$

By the rapid decay of $\tilde{\phi}$, we change the line of integration from c to $-\infty$, we get residue contributions from a pole at $s = 1$, $s = 0$, as well as all ρ such that $\zeta(\rho) = 0$ on the critical strip, and all the trivial zeros. Morally, we can take the indicator function $\phi = 1$ on $[0, 1]$.

$$\sum_{n\le N}\Lambda(n) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}N^s\frac{\zeta'(s)}{\zeta(s)}ds \tag{1.5}$$

If we move the line of integration across to $-\infty$, this gives

$$\begin{aligned}\sum_{n\le N}\Lambda(n) &= N - \sum_{\rho}\frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{k\in\mathbb{N}}\frac{N^{-2k}}{2k}\\ &= N - \sum_{\rho}\frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log\left(1-N^{-2}\right).\end{aligned} \tag{1.6}$$

This formula, due to von Mangoldt, can be derived with more care about the convergence in the sum: The sum over zeros ρ is not absolutely convergent, and is ordered in increasing $|\Im(\rho)|$.

Theorem 1.13 (Riemann-von Mangoldt explicit formula)**.** *Let $N > 1$ be not a prime power. Then*

$$\sum_{n\le N}\Lambda(n) = N - \lim_{T\to\infty}\sum_{|\Im(\rho)|\le T}\frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log\left(1-N^{-2}\right). \tag{1.7}$$

In practice, we truncate the integral in up to height T to obtained a truncated version of the explicit formula. This is obtained through integration along the choice of $c = 1 + 1/\log N$

Theorem 1.14. *Let $N > 1$. Then*

$$\sum_{n\le N}\Lambda(n) = N - \sum_{|\Im(\rho)|\le T}\frac{N^\rho}{\rho} + O(\frac{N}{T}(\log NT)^2) + O(\log N). \tag{1.8}$$

Theorems 1.13 and 1.14 are stated in [2].

The term N in the explicit formula is already suggestive of the Prime Number Theorem. The major error term comes from N^ρ in the sum. Bounding N^ρ is in turn equivalent to bounding $\Re(\rho)$, and the best case is when all the zeros have real part 1/2. This is the motivation behind the Riemann Hypothesis, which has not yet been proved.

Assuming the Riemann Hypothesis, we consider the sum over the non trivial zeros

$$\left|\sum_{|\Im(\rho)|\le T|}\frac{N^\rho}{\rho}\right| \le N^{1/2}\sum_{|\Im(\rho)|\le T}\left|\frac{1}{\rho}\right|.$$

We know there are $\sim \log T$ zeros of height $[T, T+1)$, thus the integral $\sum|\rho^{-1}|$ behaves as

$$\sum_{n\le T}\frac{\log n}{n} = O(\log^2 T).$$

Taking $N = T$ in the truncated explicit formula, we obtain

$$\sum_{n\le N}\Lambda(n) = N + O(N^{1/2}\log^2 N). \tag{1.9}$$

Which implies the prime number theorem.

Remark: The PNT stated in 1.9 (with this error bound) can be shown to be equivalent to the Riemann Hypothesis.

The prime number theorem is also true without assuming the strong Riemann Hypothesis. To show this, it is sufficient to show that there are no zeros with real part 1, so the terms in the sum contributes $O(N^{1-\epsilon})$ which will be dominated by N.

Theorem 1.15. *Let $t\in\mathbb{R}$. Then $\zeta(1+it)\neq 0$.*

Proof. Let $\sigma > 1$. We consider the expressions

$$\Re\left(\frac{\zeta'}{\zeta}(\sigma+it)\right) = -\sum_n \frac{\Lambda(n)}{n^\sigma}\cos(t\log n)$$

and

$$2(1+\cos\theta)^2 = 2+4\cos\theta+2\cos^2\theta = 3+4\cos\theta+\cos 2\theta.$$

So that

$$\begin{aligned}\Re\left(3\frac{\zeta'}{\zeta}(\sigma)+4\frac{\zeta'}{\zeta}(\sigma+it)+\frac{\zeta'}{\zeta}(\sigma+2it)\right) &= -\sum_n\frac{\Lambda(n)}{n^\sigma}(3+4\cos(t\log n)+\cos(2t\log n))\\ &= -\sum_n\frac{\Lambda(n)}{n^\sigma}2(1+\cos(t\log n))^2\\ &\le 0.\end{aligned}$$

Now for the sake of contradiction, we let $\zeta(1+it) = 0$ be a zero of order d, and since we know ζ has a pole of order 1 at $s = 1$, we can let $t\neq 0$. Consider the function $f(s) = \zeta(s)^3\zeta(s+it)^4\zeta(s+2it)$. By the computation above, $\Re(f'/f)\le 0$ when $\Re(s) > 1$. But we also have that f, by construction, has a zero of order $k \ge 4d-3 > 0$ at $s = 1$. So that $\Re(f'/f) = k/(s-1) +$ a holomorphic part. Now taking $s\to 1^+$, $\Re(f'/f)\to+\infty$, contradicting $\Re(f'/f)\le 0$. ☐

Proof of the Prime Number Theorem. Let $\phi = \phi_{N,T}$ be a bump function that equals 1 on the interval $[2, N]$ and supported on $[3/2, N + N/T]$, such that $\phi^{(j)}(x) = O_j(1)$ and $\phi^{(j)}(x) = O_j(T/x)^j$ on the intervals $[3/2, 2]$ and $[N, N + N/T]$ respectively. Then

$$\begin{aligned}\sum_{n\le N}\Lambda(n) &\le \sum_n \Lambda(n)\phi(n)\\ &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}\frac{\zeta'(s)}{\zeta(s)}ds\\ &= \tilde{\phi}(1) - \sum_\rho\tilde{\phi}(\rho) - \sum_n\tilde{\phi}(-2n)\end{aligned}$$

The first term

$$\begin{aligned}\tilde{\phi}(1) &= \int_0^\infty \phi(x)dx \\ &= N + O(N/T)\end{aligned}$$

gives the term we want from the PNT. In the third term, we rewrite by Monotone Convergence

$$\begin{aligned}\sum_n \tilde{\phi}(-2n) &= \sum_n \int_0^\infty \phi(x)x^{-2n-1}dx \\ &= \int_0^\infty \phi(x)\sum_n x^{-2n-1}dx \\ &= \int_0^\infty \phi(x)\frac{1}{x^3-x}dx \\ &= O(1)\end{aligned}$$

Finally, to bound the second term, we define a parameter $\bar{T} = \bar{T}(T)$ and split the sum into

$$\sum_{|\Im\rho|\le\bar{T}} \tilde{\phi}(\rho) + \sum_{|\Im\rho|>\bar{T}} \tilde{\phi}(\rho)$$

In the first summation, we let $\epsilon = \epsilon_{\bar{T}}$ such that there are no zeros in the region $\Re(s) > 1-\epsilon, |\Im(s)| \le \bar{T}$, then

$$\begin{aligned}\sum_{|\Im\rho|\le\bar{T}} \tilde{\phi}(\rho) &= \sum_{|\Im\rho|\le\bar{T}} \int_0^\infty \phi(x)x^{\rho-1}dx \\ &= O_T(N^{1-\epsilon}).\end{aligned}$$

In the second summation, we apply integration by parts to show that

$$\begin{aligned}\left|\int_0^\infty \phi(x)x^{\rho-1}dx\right| &= \left|\frac{1}{\rho(\rho+1)}\int_0^\infty \phi''(x)x^{\rho+1}dx\right| \\ &= O\left(\frac{1}{|\rho|^2}\frac{T^2}{N^2}\frac{N}{T}N^2\right) \\ &= O\left(\frac{1}{|\rho|^2}TN\right)\end{aligned}$$

The sum over $|\rho|^{-2}$ behaves as $\sum_n n^{-2}\log n$, so we can pick $\bar{T}$ large enough depending on T to make the contribution of $\sum_{\Im(\rho)>\bar{T}} |\rho|^{-2}$ to be $O(T^{-2})$. So that

$$\begin{aligned}\sum_{n\le N} \Lambda(n) &\le N + O(N/T) + O_T(N^{1-\epsilon}) \\ &= N + O(N/T)\end{aligned}$$

for $N = N(T)$ sufficiently large. Similarly, repeating the same argument on $\phi = \phi_{N,T}$ equals 1 on the interval $[2, N - N/T]$ and supported on $[3/2, N]$ gives

$$\sum_{n\le N} \Lambda(n) \ge N - O(N/T).$$

Sending $T \to \infty$ gives the PNT. $\square$

Chapter 2

Primes in Short Intervals and Zero Density Results

2.1 Primes in Short Intervals

We would like to answer the following question about primes in short intervals. Let $y := y(x)$. What is the smallest asymptotic behavior of y such that

$$\sum_{x \le n \le x+y} \Lambda(n) = (1 + o(1))y \tag{2.1}$$

for large enough x? That is, what is the shortest interval such that we have the behavior of the Prime Number Theorem? If 2.1 holds for some y, we say the Prime Number Theorem holds for intervals of y.

Remark: This question can be rephrased into finding primes in short intervals, by including a factor of $\log x$.

Proposition 2.1. *Assume the RH. Then the Prime Number Theorem holds in intervals of $x^{1/2+\epsilon}$.*

Proof. Assume the RH, then

$$\sum_{x \le n \le x+y} \Lambda(n) = y + O(x^{1/2} \log^2 x) = x^{1/2+\epsilon} + o(x^{1/2+\epsilon}),$$

so that the sum is non-zero for large enough x. □

Recalling that the error term is related to the real part of the zeros of the Zeta function, we motivate the following definition of zero-density:

Definition 2.2. *Let $N(\sigma, T)$ denote the number of zeros of the zeta function with real part greater than σ and imaginary part between $-T$ and T. That is,*

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \mid \beta \ge \sigma, |\gamma| \le T\}.$$

Remark: The ideal scenario is that $N(\sigma, T) = 0$ for all $\sigma > 1/2$.

Theorem 2.3 (Littlewood [7])**.** *There exists a constant A such that $\zeta(\sigma + iT) \ne 0$ in the region*

$$\sigma > 1 - A\frac{\log\log|T|}{\log|T|}.$$

Theorem 2.4 (Hoheisel)**.** *Let A be defined as in the previous theorem. Suppose that $N(\sigma, T) \ll T^{a(1-\sigma)} \log^b T$ uniformly in $1/2 \le \sigma < 1$ and in T. Then for all*

$$\theta > 1 - \frac{1}{a + b/A},$$

the Prime Number Theorem holds in intervals of $y = x^\theta$.

This following proof is expanded from a sketch of proof in Sagun Chanillo's lecture notes [1].

Proof. First notice that $N(1/2, T)$ gets at least half of the zeros of height T, so $a \geq 2$. Let $y \ll x$. We consider the expression

$$S = S(x,y) = \frac{1}{y} \sum_{x \leq n \leq x+y} \Lambda(n).$$

By the truncated version of the explicit formula in Theorem 1.14, we get

$$S = 1 - \sum_{|\Im(\rho)| \leq T} \frac{(x+y)^\rho - x^\rho}{\rho y} + O(\frac{x}{yT}(\log xT)^2) + O(\frac{\log x}{y}).$$

We want to show that except for the constant 1 term, the remaining parts are $o(1)$. We focus on the sum over the non-trivial zeros with height less than T, and enumerate them ρ_j. For each $\rho_j = \sigma_j + it_j$, we apply the Mean Value Theorem on the function $f(x) = x_j^\rho$ to get

$$\begin{aligned}
\left| \sum_{\rho_j} \frac{(x+y)^\rho - x^\rho}{\rho y} \right| &\leq \sum_{\rho_j} \left| \frac{(x+y)^{\rho_j} - x^{\rho_j}}{\rho_j y} \right| \\
&\ll \sum_{\rho_j} x^{\sigma_j - 1} \\
&= \sum_{\rho_j} x^{\sigma_j - 1} - x^{-1} + x^{-1} \\
&= O\left(\frac{T \log T}{x}\right) + \sum_{\rho_j} x^{\sigma_j - 1} - x^{-1}.
\end{aligned}$$

And by replacing $x^{\sigma_j} - 1$ by an integral,

$$\begin{aligned}
\sum_{\rho_j} x^{\sigma_j - 1} - x^{-1} &= \sum_{\rho_j} \int_0^{\sigma_j} x^{u-1} \log x \ du \\
&= \int_0^{1 - A\frac{\log\log T}{\log T}} \sum_{\rho_j} \mathbb{1}_{u \leq \sigma_j} x^{u-1} \log x \ du \\
&= \int_0^{1 - A\frac{\log\log T}{\log T}} N(u,T) x^{u-1} \log x \ du
\end{aligned}$$

Where in the penulitimate step we made use of Littlewood's bound and exchanged the order of integration and summation. Now we can apply the hypothesis that $N(\sigma, T) \ll T^{a(1-\sigma)} \log^b T$ for $\sigma > 1/2$ and trivially $N(\sigma, T) \ll T \log T \ll T^{a(1-\sigma)} \log^b T$ for $\sigma \leq 1/2$. This evaluates to

$$\begin{aligned}
\sum_{\rho_j} x^{\sigma_j - 1} - x^{-1} &\ll \int_0^{1 - A\frac{\log\log T}{\log T}} T^{a(1-u)} \ \log^b T \ x^{u-1} \log x \ du \\
&= \log^b T \int_0^{1 - A\frac{\log\log T}{\log T}} \left(\frac{T^a}{x}\right)^{1-u} \log x \ du \\
&= \frac{\log x \log^b T}{a \log T - \log x} \left[\frac{T^a}{x} - \left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}} \right]
\end{aligned}$$

Combined with the previous bounds, we have

$$S = 1 + O\left(\frac{T \log T}{x}\right) + O\left(\frac{\log x \log^b T}{a \log T - \log x} \left[\frac{T^a}{x} - \left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}} \right]\right) + O(\frac{x}{yT}(\log xT)^2) + O(\frac{\log x}{y}).$$

To make all terms (except for the first) to be $o(1)$, we want to set $y = x^\theta$, $T = x^k$, such that θ, k satisfy

$$k < 1, \ k + \theta > 1,$$

so that the second, fourth and fifth terms are $o(1)$ in x. For the third term, we require the denominator to be non zero, so we add the constraint

$$ak < 1.$$

We can simplify

$$\begin{aligned}\frac{\log x \log^b T}{a\log T - \log x}\left[\frac{T^a}{x} - \left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}}\right] &= \frac{k^b \log^b x}{ak-1}\left[x^{ak-1} - x^{(ak-1)A\frac{\log(k\log x)}{k\log x}}\right]\\ &\le \frac{k^b\log^b x}{1-ak}x^{ak-1} + \frac{k^b\log^b x}{1-ak}\exp\left((ak-1)A\frac{\log(k\log x)}{k}\right)\\ &\le \frac{k^b\log^b x}{1-ak}x^{ak-1} + \frac{k^b\log^b x}{1-ak}\exp\left((ak-1)A\frac{\log(k\log x)}{k}\right)\\ &= O(x^{ak-1}) + O\left((\log x)^{b+\frac{(ak-1)A}{k}}\right).\end{aligned}$$

We require that the last term decays in x, and this happens when

$$b + \frac{(ak-1)A}{k} < 0 \implies (aA+b)k < A \implies k < \frac{1}{a+\frac{b}{A}}$$

We had $a \ge 2 > 1$, so this k satisfies $k < 1$ and $ak < 1$. Finally, for $k = 1/(a + bA^{-1}) - \delta/2$ we let $\theta = 1 - k + \delta$ to satisfy $\theta + k > 1$, so we can find any $1/(a + bA^{-1}) + \delta > \theta > 1 - 1/(a + bA^{-1})$, and

$$\frac{1}{y}\sum_{x\le n\le x+y}\Lambda(n) = S = 1 + o(1)$$

for $y = x^\theta$. This completes the proof.

□

Theorem 2.4 gives the classical way to relate the distribution of primes in short intervals to the density of zeros away from the real-half line. The long-standing bound for zero density is due to separate proofs of Ingham and Huxley in 1940 and 1971 respectively:

Theorem 2.5 (Ingham bound for zero density [5]). *Let $1/2 \le \sigma \le 3/4$. We have*

$$N(\sigma,t) \lesssim T^{\frac{3(1-\sigma)}{2-\sigma}}.$$

Theorem 2.6 (Huxley bound for zero density [4]). *Let $3/4 \le \sigma \le 1$. We have*

$$N(\sigma,t) \lesssim T^{\frac{3(1-\sigma)}{3\sigma-1}}.$$

Combining these two bounds, we get the following zero density theorem.

Theorem 2.7 (Ingham-Huxley bound for zero density). *We have*

$$N(\sigma,t) \lesssim T^{\frac{12}{5}(1-\sigma)},$$

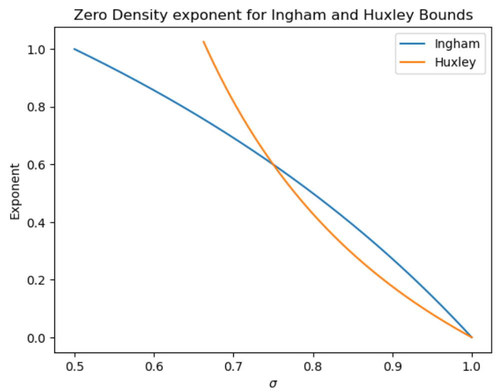
uniformly for $1/2 \le \sigma \le 1$.

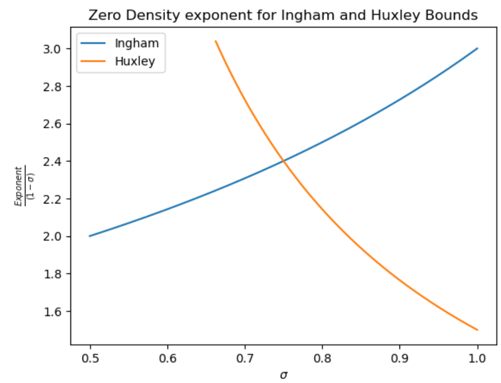
Notice that $12/5$ comes from $\sigma = 3/4$. In May 2024, Guth and Maynard published a proof that improves the Ingham-Huxley bound at $\sigma \in [7/10, 8/10]$, thus improving the result of primes in short intervals (as well as many other number theoretic results). The following sections will be dedicated to Huxley's proof of zero density, as well as Guth-Maynard's ideas in the proof. Finally, adapting from Guth and Maynard, we will provide a proof of the analogous zero-density result for L-functions.

Theorem 2.8 (Guth-Maynard bound for zero density [3]). *We have*

$$N(\sigma,t) \lesssim T^{\frac{30}{13}(1-\sigma)},$$

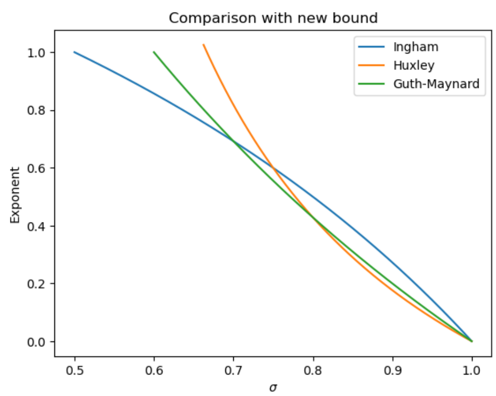
uniformly for $1/2 \le \sigma \le 1$.

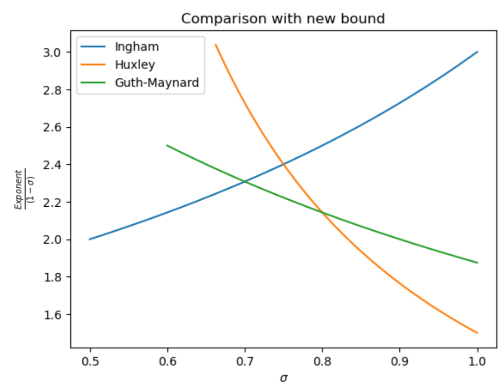




(a) The bounds for the exponent coincide at $\sigma = 3/4$

(b) $\sigma = 3/4$ is also the bottleneck when written in Hoheisel's form.





(c) Guth-Maynard's result improves in the range at $\sigma \in [7/10, 8/10]$.

(d) The exponent is reduced around the bottleneck region.

2.2 Huxley's Proof of Zero Density

We shall prove a weaker statement of Huxley's zero density bound.

Theorem 2.9 (Huxley)**.** *Let $\sigma \in [3/4, 1]$. We have*

$$N(\sigma, t) \lesssim T^{\frac{5\sigma-3}{\sigma^2+\sigma-1}(1-\sigma)}.$$

This also gives $T^{12/5(1-\sigma)}$ when combined with Ingham's bound, as both give the same exponent $6/5$ when evaluated at $\sigma = 3/4$, but the bound given in the previous section is tighter away from $3/4$. Huxley's methodology for detecting zeros as follows. Let $M_x(s) \coloneqq \sum_{n=1}^{x} \mu(n) n^{-s}$. Since this also converges absolutely on $\Re(s) > 1$, we can write the dirichlet series of $\zeta(s) M_x(s)$ as

$$\zeta(s) M_x(s) \coloneqq \sum_n a_n n^{-s}$$

for some choice of $a_n = a_n(x)$. The zeros of its analytic continuation will contain the zeros of ζ. This may look inefficient as we may have introduced extra zeros from M_x, but the tradeoff is that we can bound these a_n's.

Proposition 2.10. *We have*

$$\begin{cases} a_1 = 1, & \\ a_n = 0, & \text{if } 1 < n \le x, \\ |a_n| \le d(n), & \text{if } n > x. \end{cases}$$

Proof. For all $n \le x$, this follows from Möbius inversion. For $n > x$, we just apply the trivial bound $|\mu(d)| \le 1$ on

$$a_n = \sum_{d|n} \mu(d).$$

□

Let $y > x$ a parameter to be choosen later, and $y \le T^A$ for an absolute constant A. We apply the Mellin transform to

$$\begin{aligned}\sum_n a_n n^{-s} e^{-n/y} =& \frac{1}{2\pi i} \sum_n a_n n^{-s} \int_{2-i\infty}^{2+i\infty} \Gamma(w) y^w n^{-w} dw \\ =& \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s+w) M_x(s+w) \Gamma(w) y^w dw.\end{aligned}$$

If we move the line of integration to $\Re(w) = 1/2 - \Re(s)$, we get simple pole residue contributions from ζ and Γ

$$\begin{aligned}e^{-1/y} + \sum_{n>x} a_n n^{-s} e^{-n/y} = \sum_n a_n n^{-s} e^{-n/y} =& \zeta(s) M_x(s) + M_x(1)\Gamma(1-s) y^{1-s} \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(\frac{1}{2} + i\Im(s) + it) M_x(\frac{1}{2} + i\Im(s) + it) \\ &\cdot \Gamma\left(\frac{1}{2} - \Re(s) + it\right) y^{\frac{1}{2} - \Re(s) + it} dt.\end{aligned} \tag{2.2}$$

We take y large enough so that $e^{-1/y}$ is close to 1. Since $M_x(s)$ is an approximation of $1/\zeta$, we should expect that the term $\zeta(s)M_x(s)$ is about 1 most of the time and the other terms are small. However, if s is a zero of ζ, then $\zeta(s)M_x(s) = 0$, so at least one of the following things need to happen

(i) $|\sum_{n>x} a_n n^{-s} e^{-n/y}|$ is large.

(ii) The integral in t is large.

(iii) $|M_x(1)\Gamma(1-s)y^{1-s}|$ is large.

We thus transform the problem of detecting zeros to counting the number of occurences of extreme values. We will later see that type (iii) zeros are negligible, so we need to bound the number of type (i) and type (ii) zeros. To bound the occurences of extreme values, we introduce the following two lemmas.

Lemma 2.11. *Let $W = \{t_j\} \subseteq [0, T]$ be a one-separated set such that*

$$\left|\zeta\left(\frac{1}{2} + it_j\right)\right| > V \ \ \forall j,$$

then

$$|W| \lesssim TV^{-4}.$$

Proof. By corollary 10.4 of [8], we have

$$WV^4 \le \sum_{t \in W} \left|\zeta\left(\frac{1}{2} + it_j\right)\right|^4 \ll T(\log T)^5.$$

The result follows immediately. □

Lemma 2.12 (Halász Inequality)**.** *Let a be an arithmetic function, and $D_N(s) = \sum_{n \le N} a(n) n^s$, and $G = \sum_{n \le N} |a(n)|^2$. If $W = \{t_j\} \subseteq [0, T]$ is a one-separated set such that*

$$|D_N(it_j)| > V \ \ \forall j,$$

then

$$|W| \lesssim GNV^{-2} + G^3 NTV^{-6}.$$

This is equation 2.9 from [4].

Proof of Huxley's Zero Density Theorem. From equation 2.2, we take $y > 6$ so that $e^{-1/y} > 5/6$. We also truncate the sum in $n > x$ to $x < n \le y^2$ with an error of $1/6$ for large enough y. Finally, by the rapid decay of Γ, we truncate the integral in t to the range $|t| \le B \log T$ with an error of $1/6$. Thus, s is a zero only if

(i) $|\sum_{x<n\le y^2} a_n n^{-s} e^{-n/y}| \ge \frac{1}{6}$, or

(ii) $\frac{1}{2\pi}|\int_{-B\log T}^{B\log T} \zeta(\frac{1}{2} + i\Im(s) + it) M_x(\frac{1}{2} + i\Im(s) + it)\Gamma\left(\frac{1}{2} - \Re(s) + it\right) y^{\frac{1}{2}-\Re(s)+it} dt| \ge \frac{1}{6}$, or

(iii) $|M_x(1)\Gamma(1-s)y^{1-s}| \ge \frac{1}{6}$.

Of the zeros $\rho = \beta + i\gamma$ of ζ in the region, at the cost of a factor of $\log^2 T$, we take representatives such that if $\rho_1 \ne \rho_2$ then $|\rho_1 - \rho_2| \gtrsim 1$.

For Class (iii) zeros, we use Stirling's formula on Γ to get

$$\log \Gamma(1-s) = \left(\frac{1}{2} - s\right)\log(1-s) + s - \frac{1}{2}\log 2\pi + O(|s|^{-1}).$$

. Therefore, the $\Gamma(1-s)$ decays at the rate of $O(|s|^{1/2-\sigma})$ Therefore, the large values can only happen in the range $\Im(s) \lesssim 1$ and so we can bound the occurances of these zeros to be $O(T^\epsilon)$. Therefore, this is negligible.

For Class (i) zeros, we split the sum dyadically to get

$$\left|\sum_{n\sim U, n\le y^2} a(n)n^{-\rho}e^{n/y}\right| \ge O((\log T)^{-1}), \tag{2.3}$$

for some $x \le U = 2^k \le y$. Currently, the zeros do not have the same real part. However, we can remove the dependence of β by introducing a bump function $\phi(x)$ that equals $e^{x(\beta-\sigma)}$ on $\log x \sim U$, so that

$$\begin{aligned}\sum_{n\sim U} a(n)n^{-\rho}e^{n/y} &= \sum_{n\sim U} a(n)n^{-\sigma-i\gamma}e^{n/y}\phi(\log n)\\ &= \sum_{n\sim U} a(n)n^{-\sigma-i\gamma}e^{n/y}\int \hat{\phi}(\xi)e(\xi\log n)d\xi\\ &= \int \hat{\phi}(\xi)\sum_{n\sim U} a(n)n^{-\sigma-i\gamma-2\pi i\xi}e^{n/y}d\xi\\ &= \int_{|\xi|\lesssim 1} \hat{\phi}(\xi)\sum_{n\sim U} a(n)n^{-\sigma-i\gamma-2\pi i\xi}e^{n/y}d\xi + O_{\epsilon,A}(T^{-A}).\end{aligned}$$

$\hat{\phi}$ is bounded by $O(\log U)$, so this log factor is negligible compared to T^ϵ. Applying the ML estimate on the integral gives there is a value $|\xi| \lesssim 1$ such that

$$\sum_{n\sim U} a(n)n^{-\sigma-i\gamma-2\pi i\xi}e^{n/y}d\xi \gg T^{-\epsilon}.$$

Therefore, we can assume that all the zeros have real part σ without affecting the argument. Applying Lemma 2.12, we get that the number of times that equation 2.3 can happen for each U is

$$\lesssim U^{2-2\sigma} + U^{4-6\sigma}T \lesssim y^{2-2\sigma} + x^{4-6\sigma}T.$$

Note that $\sigma \ge 3/4$, so that $4 - 6\sigma < 0$, so $x^{4-6\sigma}$ is used instead of $y^{4-6\sigma}$. For Class (ii) zeros, we apply ML estimate to see that for each zero $\rho_j = \beta_j + i\gamma_j$ there $|t_j - \gamma_j| \ll \log T$ such that

$$\zeta(\frac{1}{2} + i\gamma_j + it_j)M_x(\frac{1}{2} + i\gamma_j + it_j) \gg y^{\Re(s)-\frac{1}{2}}/\log T \gg y^{\sigma-\frac{1}{2}}/\log T,$$

Therefore, we can consider a new set of 'zeros' $\sigma + i(\gamma_j + t_j)$, and make them 1-separated at a cost of a factor of $O(\log T)$. We separate these 'zeros' into two cases

(a) $\zeta(\frac{1}{2} + i\gamma_j + it_j) \geq A$,

(b) $M_x(\frac{1}{2} + i\gamma_j + it_j) \geq B$.

where $AB \gg y^{\sigma-\frac{1}{2}}/\log T$. For case (a) zeros, this is bounded by corollary 2.11 to be

$$\lesssim TA^{-4}.$$

For case (b) zeros, we reapply lemma 2.12 to bound this by

$$\lesssim xB^{-2} + xTB^{-6}.$$

Here, we took G to be $O(\sum_{n\leq x} d(n)n^{-1})$. The divisor function grows slower than any n^{ϵ}, so $G \lesssim 1$. Combined with type 1 zeros, we have

$$N(\sigma,T) \lesssim y^{2-2\sigma} + x^{4-6\sigma}T + TA^{-4} + xB^{-2} + xTB^{-6}.$$

We first consider the terms with A and B. We take $B^4 \ll T$, so that $xB^{-2} \ll xTB^{-6}$. We can also replace TA^{-4} with $TB^4y^{2-4\sigma}$ to get

$$N(\sigma,T) \lesssim y^{2-2\sigma} + x^{4-6\sigma}T + TB^4y^{2-4\sigma} + xTB^{-6}.$$

We take $B = (xy^{4\sigma-2})^{1/10}$, so that the last two terms are of the same magnitude

$$O(Tx^{2/5}y^{(6-12\sigma)/5}).$$

In our new bound

$$N(\sigma,T) \lesssim y^{2-2\sigma} + x^{4-6\sigma}T + Tx^{2/5}y^{(6-12\sigma)/5},$$

we remove the dependence on x by setting

$$x = y^{\frac{(6-12\sigma)/5}{4-2/5-6\sigma}} = y^{\frac{1-2\sigma}{3-5\sigma}},$$

so that the last two terms are of the same magnitude. Finally,

$$N(\sigma,T) \lesssim y^{2-2\sigma} + y^{\frac{(1-2\sigma)(4-6\sigma)}{3-5\sigma}}T,$$

we set

$$y = T^{\frac{5\sigma-3}{2(\sigma^2+\sigma-1)}}$$

so that the last two terms combine into one bound

$$N(\sigma,T) \lesssim T^{\frac{(5\sigma-3)(1-\sigma)}{(\sigma^2+\sigma-1)}}.$$

When $\sigma = 3/4$, we have

$$\frac{(5\sigma-3)}{(\sigma^2+\sigma-1)} = \frac{12}{5},$$

and the first derivative test gives that this function is decreasing in σ in the range $\sigma \in [3/4, 1]$. Finally, we have to check the conditions

$$x < y,\ B^4 \ll T.$$

The first condition is true as $(1-2\sigma)/(3-5\sigma) < 1$ in this range. The second condition is true as

$$B^4 = (xy^{4\sigma-2})^{2/5} = (y^{4\sigma-2+\frac{(6-12\sigma)/5}{4-2/5-6\sigma}})^{2/5} = T^{\frac{2}{5}\frac{5\sigma-3}{2(\sigma^2+\sigma-1)}(4\sigma-2+\frac{(6-12\sigma)/5}{4-2/5-6\sigma})} = T^{\frac{(3\sigma-2)(\sigma-1)}{\sigma^2+\sigma-1}} \ll T$$

in this range. □

Chapter 3

Guth-Maynard's Large Values Estimate near $\sigma = 3/4$

In May 2024, Guth and Maynard published an improvement to the zero density bound at $\sigma \in [7/10, 8/10]$. This result is in fact a corollary of their main theorem - an improvement on the Halász inequality. Since the zero detection methodology is the same as in Huxley's proof, we omit their proof of zero density and focus on the main result instead.

Theorem 3.1 (Guth-Maynard Large Values Estimate)**.** *Let (b_n) be a sequence of complex numbers such that $|b_n| \le 1$ for all n, and $W = \{t_j\}_{j=1}^{|W|}$ be a 1-separated set $\subseteq [0, T]$, such that*

$$\left| \sum_{n \sim N} b_n n^{it_j} \right| \ge V$$

for each $t_j \in W$. Then

$$|W| \lesssim N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4}.$$

Let us compare this bound to Lemma 2.12, which states

$$|W| \lesssim N^2 V^{-2} + T N^4 V^{-6}.$$

In the critical case $V = N^{3/4}, N \le T^{5/6-\epsilon}$, the original bound will give

$$|W| \lesssim N^2 N^{-3/2} + T N^4 N^{-9/2} \lesssim N^{1/2} + T N^{-1/2} \lesssim T N^{-1/2},$$

while the bound by Guth and Maynard gives

$$|W| \lesssim N^2 N^{-3/2} + N^{18/5} N^{-3} + T N^{12/5} N^{-3} \lesssim N^{1/2} + T N^{-3/5} \lesssim T N^{-3/5}.$$

This new theorem, when applied in Huxley's proof in the previous chapter, gives an improvement in the bound of zero density in the range $\sigma \in [7/10, 8/10]$.

3.1 Outline and Sketch of proof

The structure of the proof can be broken down as follows: We first notice that $|W|$ is bounded by the operator norm of a matrix M. This operator norm, using results from linear algebra, is bounded by the trace. Applying Poisson summation on the trace gives 4 terms that are separately handled, which we will name S_0 to S_3. We will see that S_0 gives the 'main term' that is consistent with the density hypothesis, S_1 is negligible, S_2 is bounded by a theorem by Heath-Brown, which we state below.

Theorem 3.2 (Heath-Brown). *Let $\mathcal{S} = \{(t_j, \chi_j)\}$, such that $t_j \in [0,T]$, $T \geq 1$, and each χ_j is a primitive character of modulus q. Further assume that the t_j's are one separated for the same character. Then*

$$\sum_{\substack{(t_1,\chi_1)\\(t_2,\chi_2)}} \left|\sum_{n=1}^{N} b_n n^{-1/2-i(t_1-t_2)}\chi_1\bar{\chi}_2(n)\right|^2 \lesssim (|\mathcal{S}|N + |\mathcal{S}|^2 + |\mathcal{S}|^{5/4}(qT)^{1/2}) \max_{n\leq N}|b_n|.$$

The most tricky term, S_3 is a summation over a three-dimensional lattice. We will see that S_3 is bounded by what is known as the *additive energy* of the set W, defined by

$$E(W) := \#\{t_1, t_2, t_3, t_4 \in W : |t_1 + t_2 - t_3 - t_4| \ll T^{\epsilon}\}.$$

This term describes the 'additive structure' of W. We see that $E(W)$ is bounded below by $|W|^2$, as the condition is satisfied when $t_1 = t_3$ and $t_2 = t_4$. Moreover, since W is $1-$*separated*, the choice of t_1, t_2, t_3 fixes $O(1)$ choices for t_4, so $E(W)$ is bounded above by $|W|^3$. In the extreme case that the additive structure of W is high, such as when $t_j = j\alpha$ for a constant α, the energy of the set is $O(|W|^3)$. This definition naturally arrises from taking the fourth moment of the function

$$R(v) := \sum_{t\in W} v^{it}.$$

This gives us

$$R(v)^4 = \sum_{t_1,t_2,t_3,t_4\in W} v^{i(t_1+t_2-t_3-t_4)}.$$

The naive choice $E(W) \leq W^3$ is slightly too loose to beat the Ingham-Huxley bound. However, an orthogonal bound can be found for $E(W)$ based on Heath-Brown's theorem. Finally, the bound in 7 and 8 combined is enough to give an improvement in most cases, a further refinement of the S_3 bound was required. This relies on the averaging over the affine summations of R.

$$\sup_{0<M_1,M_2,M_3<M} \int \left(\sum_{\substack{|m_1|\sim M_1\\|m_2|\sim M_2\\|m_3|\ll M_3}} \left| R\left(\frac{m_1 u + m_3}{m_2}\right)\right|^2 \right)^2 du \lesssim M^6 \|R\|_{L_2}^4 + M^4 \|R\|_{L_4}^4.$$

We give a quick sketch of the whole proof below. In the next section, we will give a full proof of the generalized statement of the theorem that considers primitive Dirichlet characters mod q. The proof of Guth-Maynard can be recovered by using the special case $q = 1$.

0. Setup

First, as in the theorem, we let (b_n) be a sequence of complex numbers such that $|b_n| \leq 1$ for all n,

$$D_n(t) := \sum_{n\sim N} b_n n^{it},$$

$W = \{t_j\}_{j=1}^{|W|}$ be a T^{ϵ}-separated set $\subseteq [0,T]$, such that

$$|D_n(t_j)| \geq V$$

for each $t_j \in W$. Notice that we now let the set be T^{ϵ} separated for $\epsilon > 0$. This means that we will give up a factor of T^{ϵ} in the final bound, but this makes many computations cleaner as this T^{ϵ} dominates the log factors. Moreover, we can introduce a bump function ω with support in $[1,2]$ to localize the summation, and rewrite

$$D_n(t_j) = \sum_n \omega\left(\frac{n}{N}\right) b_n n^{it_j}.$$

This is added for the Poisson summation in step 3.

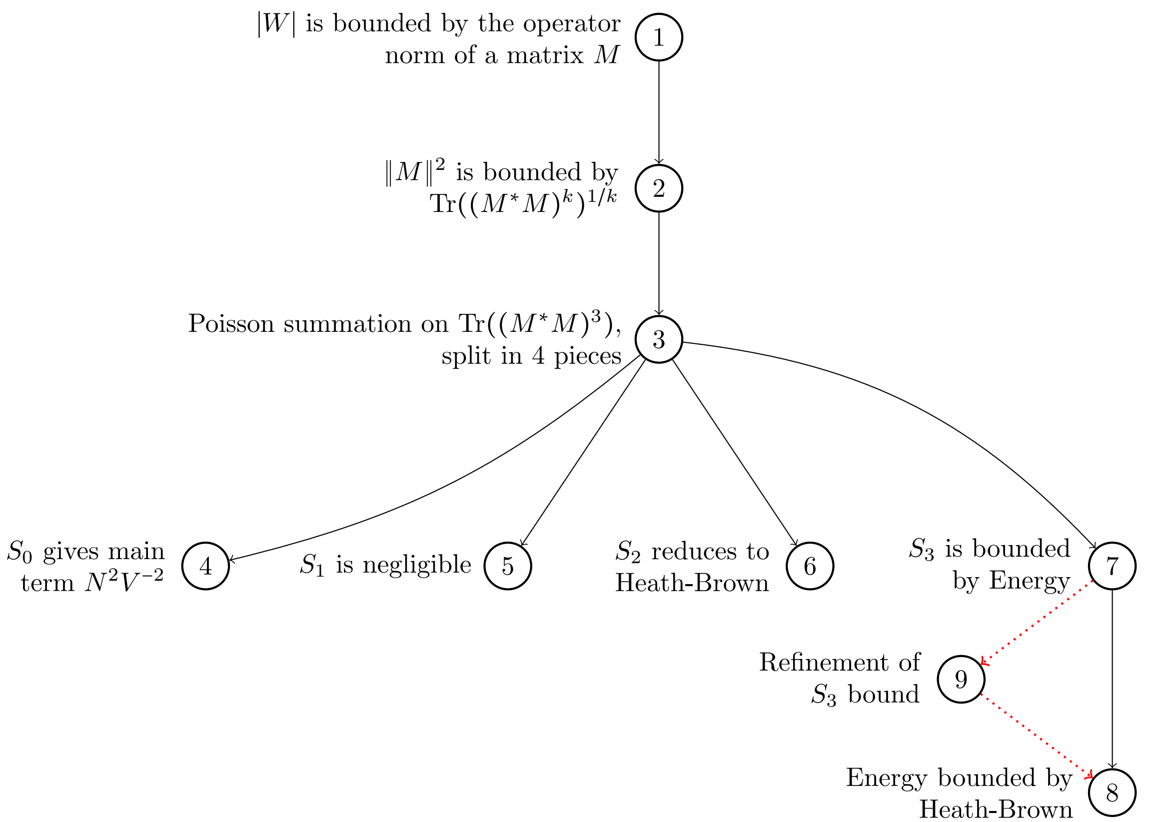


Figure 3.1: Graphical representation of Guth-Maynard proof outline

1. Bounding $|W|$ with operator norm

We view $\vec{b} = (b_n)_{n\sim N}$ as a N-dimensional vector, and consider the $|W| \times N$ matrix, indexed by j from 1 to $|W|$ and $n \sim N$,

$$M_{j,n} = n^{it_j} = \omega\left(\frac{n}{N}\right)n^{it_j}.$$

Then we can view the j-th entry of the product $M\vec{b}$ as $D_n(t_j)$. In other words

$$|M\vec{b}|^2 \geq V^2|W|.$$

However, we can bound $|M\vec{b}|$ using the operator norm of M and $|b_n| \leq 1$ to get

$$|M\vec{b}|^2 \leq \|M\|^2|\vec{b}|^2 \leq \|M\|^2 N.$$

Combined with the previous inequality, we get

$$|W| \leq \|M\|^2 NV^{-2}. \tag{3.1}$$

2. Bounding $\|M\|$

An immediate way to proceed is to note that $\|M\|^2$ is the largest eigenvalue of MM^*, which in turn is bounded by sum of eigenvalues which is the trace of M^*M. However, this is somewhat inefficient. Consider N-dimensional vector that enumerates through the eigenvalues (λ_n) of MM^*, so that the trace will be the L_1 norm of this vector. In principle, we would like the L_∞ norm of this vector, so we can try to take L_k norms of this vector for big k to get close to L_∞. Using an eigenbasis for MM^*, we can see that the L_k norm is represented by

$$\left(\sum_{n\sim N}\lambda_n^k\right)^{1/k} = \mathrm{Tr}((MM^*)^k)^{1/k}.$$

We take $k = 3$, which is the highest power we can afford given the tools at our disposal. This gives

$$|W| \le \mathrm{Tr}((MM^*)^3)^{1/3}NV^{-2}. \tag{3.2}$$

3. Expansion of $\mathrm{Tr}((\mathrm{MM}^*)^3)$

We first compute

$$(MM^*)_{n_1,n_2} = \sum_{t\in W}\omega\left(\frac{n_1}{N}\right)\omega\left(\frac{n_2}{N}\right)n_1^{-it_j}n_2^{it_j}$$

so that

$$\begin{aligned}\mathrm{tr}((M^*M)^3) &= \sum_{t_1,t_2,t_3\in W}\sum_{n_1,n_2,n_3\sim N}\omega\left(\frac{n_1}{N}\right)^2\omega\left(\frac{n_2}{N}\right)^2\omega\left(\frac{n_3}{N}\right)^2 n_1^{i(t_1-t_3)}n_2^{i(t_2-t_1)}n_3^{i(t_3-t_2)}\\ &= \sum_{t_1,t_2,t_3\in W}\sum_{n_1,n_2,n_3}\omega\left(\frac{n_1}{N}\right)^2\omega\left(\frac{n_2}{N}\right)^2\omega\left(\frac{n_3}{N}\right)^2\left(\frac{n_1}{N}\right)^{i(t_1-t_3)}\left(\frac{n_2}{N}\right)^{i(t_2-t_1)}\left(\frac{n_3}{N}\right)^{i(t_3-t_2)}.\end{aligned}$$

Let $h_t(u) := \omega(u)^2u^{it}$, we can apply Poisson summation in the inner integral over n_1, n_2, n_3 to get

$$\mathrm{tr}((\mathrm{M^*M})^3) = \mathrm{N}^3\sum_{\mathrm{t_1,t_2,t_3\in W}}\sum_{\mathrm{m_1,m_2,m_3}}\hat{\mathrm{h}}_{\mathrm{t_1-t_3}}(\mathrm{Nm_1})\hat{\mathrm{h}}_{\mathrm{t_2-t_1}}(\mathrm{Nm_2})\hat{\mathrm{h}}_{\mathrm{t_3-t_2}}(\mathrm{Nm_3}). \tag{3.3}$$

What we can gain here is that $\hat{h}_tm$ has decay in t or m based on the principle of non-stationary phase. The proof is very standard and uses repeated integration by parts.

Lemma 3.3 (Non-stationary phase). *We have for any integer $A > 0$*

$$|\hat{h}_t(\xi)| \ll_A \frac{1+|t|^A}{|\xi|^A},$$
$$|\hat{h}_t(\xi)| \ll_A \frac{1+|\xi|^A}{|t|^A}.$$

Proof. We have

$$\hat{h}_t(\xi) = \int\omega(u)^2u^{it}e^{2\pi i\xi u}du.$$

By repeated integration by parts on $\omega(u)^2u^{it}$ and $e^{2\pi i\xi u}$, we get

$$|\hat{h}_t(\xi)| = \left|\int(2\pi i\xi)^{-A}e^{2\pi i\xi u}\frac{d^A}{(du)^A}\left(\omega^2(u)u^{it}\right)du\right| \ll_A \frac{1+|t|^A}{|\xi|^A}.$$

A similar argument for integration by parts on $\omega(u)^2e^{2\pi i\xi u}$ and u^{it} gives

$$|\hat{h}_t(\xi)| = \left|\int\frac{1}{(it+1)(it+2)\dots(it+A)}u^{it+A}\frac{d^A}{(du)^A}\left(\omega^2(u)e^{2\pi i\xi u}\right)du\right| \ll_A \frac{1+|\xi|^A}{|t|^A}.$$

□

This means that we can handle terms in equation 3.3 if m_i is small and $t_j - t_k$ is big, or m_i is big and $t_j - t_k$ is small. With this in mind, we split the sum over m_1, m_2, m_3 in the equation into four parts. S_0, where all three m terms are zero, S_1, where exactly one of the m terms is non-zero, S_2, where exactly two of the m terms are non-zero, and S_3, where all three m terms are non-zero. That is,

$$\mathrm{tr}((\mathrm{M}^*\mathrm{M})^3) = \mathrm{S}_0 + \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3,$$

where

$$S_j = N^3 \sum_{m_1,m_2,m_3,\#\{m_k=0\}=j} I_m,$$

$$I_m = I_{(m_1,m_2,m_3)} := N^3 \sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(Nm_1)\hat{h}_{t_2-t_1}(Nm_2)\hat{h}_{t_3-t_2}(Nm_3).$$

4. Bounding S_0

S_0 only has one term in the sum.

$$S_0 = N^3 \sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(0)\hat{h}_{t_2-t_1}(0)\hat{h}_{t_3-t_2}(0)$$

Now we can apply that W is T^ϵ separated, so there is a trivial bound $|W| \le T$ and $\hat{h}_{t_j-t_k}$ is negligible by the principle of non-stationary phase. So we can only consider

$$S_0 = N^3 \sum_{t\in W} \hat{h}_0(0) + O(T^{-100}) = N^3|W|\|\omega\|_{L_2}^6.$$

Taking the cube root, this term gives $O(N^2V^{-2}|W|)$ in equation 3.2. This is strikingly similar to the N^2V^{-2} term that the density hypothesis conjectures. Guth and Maynard isolates this term by introducing the following lemma.

Lemma 3.4. *Let A be an $m \times n$ matrix. Then*

$$\|A\| \le 2\left(\mathrm{tr}((\mathrm{AA}^*)^3) - \frac{\mathrm{tr}(\mathrm{AA}^*)^3}{\mathrm{m}^2}\right)^{1/6} + 2\left(\frac{\mathrm{tr}(\mathrm{AA}^*)}{m}\right)^{1/2}.$$

Proof. This is Lemma 4.2 from Guth-Maynard. [3] □

Applying this lemma, we can compute that

$$\mathrm{tr}(\mathrm{MM}^*) = \sum_{\mathrm{n}\sim\mathrm{N}} \sum_{\mathrm{t}\in\mathrm{W}} \omega\left(\frac{\mathrm{n}}{\mathrm{N}}\right)^2 \mathrm{n}^{-\mathrm{it}}\mathrm{n}^{\mathrm{it}} = |\mathrm{W}| \sum_{\mathrm{n}} \omega\left(\frac{\mathrm{n}}{\mathrm{N}}\right)^2.$$

Applying Poisson summation, this equals

$$|W| \sum_m N\hat{h}_0(mN).$$

By non-stationary phase use the rapid decay of $\hat{h}_0(\xi)$ in ξ to only consider the term $m = 0$ at the cost of N^{-100}. Therefore, $\mathrm{tr}(\mathrm{MM}^*) = |\mathrm{W}|\mathrm{N}\|\omega\|_{\mathrm{L}_2}^2 + \mathrm{O}(\mathrm{N}^{-100})$. Lemma 3.4 gives

$$|W| \ll NV^{-2}(N + (S_0 + S_1 + S_2 + S_3 - N^3|\omega|_{L_2}^6|W|)^{1/3}) \ll N^2V^{-2} + NV^{-2}(S_1 + S_2 + S_3)^{1/3}.$$

5. Bounding S_1

By symmetry in m_1, m_2, m_3, we can consider the terms where $m_3 \neq 0$ at a cost of a factor of 3. Then

$$S_1 = 3N^3 \sum_{m\neq 0} \sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(0)\hat{h}_{t_2-t_1}(0)\hat{h}_{t_3-t_2}(mN).$$

This term is bounded by non-stationary phase. If $|m| > T^{1+\epsilon}/N$, then $|m|/|t_3 - t_2| < T^\epsilon$, so we can truncate the sum to $|m| \le T^{1+\epsilon}/N$ with an error of $O_\epsilon(T^{-100})$. In this range, if $t_1 \neq t_3$ or $t_2 \neq t_1$, then they are T^ϵ apart, then we get rapid decay in $\hat{h}_{t_1-t_3}(0)$ or $\hat{h}_{t_2-t_1}(0)$ to be $O_\epsilon T^{-100}$. But when $t_1 = t_2 = t_3$, we get decay in the last term $\hat{h}_0(mN)$. Combining all cases, this term is negligible.

6. Bounding S_2

By symmetry again we can consider the terms where $m_1, m_2 \neq 0, m_3 = 0$.Then

$$S_2 = 3N^3 \sum_{m_1,m_2\neq 0} \sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(m_1N)\hat{h}_{t_2-t_1}(m_2N)\hat{h}_{t_3-t_2}(0).$$

Due to decay of the last term in $|t_3 - t_2|$, we can only consider the terms $t_3 = t_2$ with error $O_{\epsilon,A}(T^{-A})$. Then we can rewrite

$$\begin{aligned}3N^3\hat{h}_0(0) \sum_{m_1,m_2\neq 0} \sum_{t_1,t_2\in W} \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_1}(m_2N) &= 3N^3\hat{h}_0(0) \sum_{m_1,m_2\neq 0} \sum_{t_1,t_2\in W} \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_1-t_2}(-m_2N)\\ &= 3N^3\hat{h}_0(0) \sum_{t_1,t_2\in W} \left(\sum_{m\neq 0} \hat{h}_{t_1-t_2}(mN)\right)^2.\end{aligned}$$

Poisson summation gives

$$N\sum_m \hat{h}_{t_1-t_2}(mN) = \sum_n h_{t_1-t_2}\Big(\frac{n}{N}\Big) = \sum_n \omega\left(\frac{n}{N}\right) n^{i(t_1-t_2)}.$$

Therefore, a direct application of Heath Brown's theorem 3.2 gives a bound for S_2. Here we have added in the terms for $m = 0$, which is insignificant when t_1 and t_2 are T^ϵ separated. However, this is somewhat lossy for terms $t_1 = t_2$. However, we can ignore the terms with $t_1 = t_2$ at a small error due to the rapid decay in $\hat{h}_{t_1-t_2}(mN)$, apply Poisson summation, then add the terms $t_1 = t_2$ again.

7. Bounding S_3

S_3 sums over most points on the 3-dimensional lattice. By symmetry, we can consider only the terms with $|m_1| \le |m_2| \le |m_3|$ with an error factor of 6. Recall that

$$I_m = N^3 \sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(Nm_1)\hat{h}_{t_2-t_1}(Nm_2)\hat{h}_{t_3-t_2}(Nm_3).$$

By the the principle of non-stationary phase, we can truncate the sum across I_m to $|m_1|, |m_2|, |m_3| \lesssim T/N$, at the cost of $O\epsilon(T^{-100})$, as $t_j - t_k = O(T)$. We expand $\hat{h}$ in integral form, so that

$$\begin{aligned}I_{\vec{m}} &= N^3 \sum_{t_1,t_2,t_3\in W} \int_{\mathbb{R}^3} \omega(u_1)^2\omega(u_2)^2\omega(u_3)^2 u_1^{i(t_1-t_3)} u_2^{i(t_2-t_1)} u_3^{i(t_3-t_2)} e(-N\vec{m}\cdot\vec{u})d\vec{u}\\ &= N^3 \sum_{t_1,t_2,t_3\in W} \int_{\mathbb{R}^3} \tilde{\omega}(\vec{u})\Big(\frac{u_1}{u_2}\Big)^{it_1}\Big(\frac{u_2}{u_3}\Big)^{it_2}\Big(\frac{u_3}{u_1}\Big)^{it_3} e(-N\vec{m}\cdot\vec{u})d\vec{u}\end{aligned}$$

For $\tilde{\omega}(\vec{u}) = \omega(u_1)^2\omega(u_2)^2\omega(u_3)^2$. Because $\tilde{\omega}$ is supported away from $u_3 = 0$, we can introduce the change of variables $v_1 = u_1/u_3$, $v_2 = u_2/u_3$. The Jacobian is u_3^2, and $u_1/u_2 = v_1/v_2$, so that

$$\begin{aligned}I_{\vec{m}} &= N^3 \sum_{t_1,t_2,t_3\in W} \int_{\mathbb{R}^3} u_3^2\tilde{\omega}(v_1u_3, v_2u_3, u_3)\Big(\frac{v_1}{v_2}\Big)^{it_1} v_2^{it_2}\Big(\frac{1}{v_1}\Big)^{it_3} e(-Nu_3(m_1v_1 + m_2v_2 + m_3))dv_1\ dv_2\ du_3\\ &= N^3 \sum_{t_1,t_2,t_3\in W} \int_{\mathbb{R}^2}\int_{\mathbb{R}} u_3^2\tilde{\omega}(v_1u_3, v_2u_3, u_3)e(-Nu_3(m_1v_1 + m_2v_2 + m_3))du_3 \ \Big(\frac{v_1}{v_2}\Big)^{it_1} v_2^{it_2}\Big(\frac{1}{v_1}\Big)^{it_3} dv_1\ dv_2\end{aligned}$$

The inner integral in u_3 places restrictions on the domain of integration. First, the support of $\tilde{\omega}$ is $[1,2] \times [1,2] \times [1,2]$. Thus, if it is non-zero, we have $v_1u_3, v_2u_3, u_3 \in [1,2] \implies v_1, v_2 \in [1/2, 2]$. Therefore, we can restrict the outer integral in v_1 and v_2 to this range. Next, since $v_1, v_2 = O(1)$, the chain rule gives

$$\left(\frac{\partial}{\partial u_3}\right)^j \omega(v_1u_3, v_2u_3, u_3) \ll_j 1.$$

Therefore, we can apply the repeated integration by parts to get rapid decay of the integral in $|N(m_1v_1 + m_2v_2 + m_3)|$. In particular, we can truncate the integral to the range $|N(m_1v_1 + m_2v_2 + m_3)| \ll T^{\epsilon}$ at an error of $O_\epsilon(T^{-100})$, and use

$$\int_{\mathbb{R}} u_3^2 \tilde{\omega}(v_1u_3, v_2u_3, u_3) e(-Nu_3(m_1v_1 + m_2v_2 + m_3)) du_3 = O(1)$$

in this range by the compact support of $\tilde{\omega}$. This gives us

$$|I_{\vec{m}}| \le \left| N^3 \sum_{t_1,t_2,t_3 \in W} \int_{\substack{|v_1m_1+v_2m_2+m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} \Big(\frac{v_1}{v_2}\Big)^{it_1} v_2^{it_2} \Big(\frac{1}{v_1}\Big)^{it_3} dv_1 \ dv_2 \right| + O_\epsilon(T^{-100}). \tag{3.4}$$

Recall that in the outline we defined

$$R(v) \coloneqq \sum_{t \in W} v^{it}.$$

Exchanging the summation and integral, we get the term

$$\begin{aligned} &\left| N^3 \int_{\substack{|v_1m_1+v_2m_2+m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} R\Big(\frac{v_1}{v_2}\Big) R\Big(v_2\Big) R\Big(\frac{1}{v_1}\Big) dv_1 \ dv_2 \right| \\ \le & N^3 \int_{\substack{|v_1m_1+v_2m_2+m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} \left| R\Big(\frac{v_1}{v_2}\Big) R\Big(v_2\Big) R\Big(\frac{1}{v_1}\Big) \right| dv_1 \ dv_2 \\ = & N^3 \int_{\substack{|v_1m_1+v_2m_2+m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} \left| R\Big(\frac{v_2}{v_1}\Big) R\Big(v_2\Big) R(v_1) \right| dv_1 \ dv_2, \end{aligned}$$

where the last step, we used

$$|R(v^{-1})| = \Big| \sum_{t \in W} v^{-it} \Big| = \Big| \sum_{t \in W} v^{it} \Big| = |R(v)|.$$

Now we fix v_1, and consider the integral in v_2 in the range

$$|v_1m_1 + v_2m_2 + m_3| \lesssim \frac{1}{N} \implies \Big| v_2 - \frac{v_1m_1 + m_3}{-m_2} \Big| \lesssim \frac{1}{|m_2|N}.$$

If we enforce the conditions $|m_1| \le |m_2| \le |m_3|$ and $v_2 \asymp 1$, we see that the domain of integration is empty unless $|m_2| \asymp |m_3|$. Thus, we can break the sum across

$$\sum_{|m_1|,|m_2|,|m_3| \lesssim T/N}$$

to be

$$\log T^{1+\epsilon}/N \ \mathrm{sup}_{\mathrm{U}=2^{\mathrm{j}},\mathrm{V}\le\mathrm{U}} \sum_{\substack{|m_1| \sim V \\ |m_2|,|m_3| \sim U}} .$$

Moreover, we integrate in v_2 over a very small neighborhood of width $\lesssim 1/(|m_2|N)$ around $v_2 = (v_1m_1 + m_3)/(-m_2)$. In principle, we can estimate this integral by taking the value of R at this point to get

$$\approx \frac{1}{|m_2|N} \int_{\frac{1}{2} \le v_1 \le 2} \left| R\Big(\frac{v_1m_1 + m_3}{-v_1m_2}\Big) R\Big(\frac{v_1m_1 + m_3}{-m_2}\Big) R(v_1) \right| dv_1.$$

This is made precise by apply to a $(1/N|m_2|)$-smoothened version of R, stated in Proposition 4.24. Finally, Hölder's inequality gives a bound of

$$\left\|R\Big(\frac{vm_1+m_3}{-vm_2}\Big)\right\|_{L^4_{v\asymp 1}}\left\|R\Big(\frac{vm_1+m_3}{-m_2}\Big)\right\|_{L^4_{v_1\asymp 1}}\left\|R(v)\right\|_{L^2_{v_1\asymp 1}}$$

The second moment of R is bounded by the size of W. Indeed, we have

$$\int_{v\asymp 1}|R(v)|^2dv = \sum_{t_1,t_2\in W}\int_{v\asymp 1}v^{i(t_1-t_2)}dv.$$

If $t_1-t_2\neq 0$, then $|t_1-t_2|>T^\epsilon$, so the oscillatory integral will be negligible by the fast decay. There are $|W|$ terms satisfying $t_1=t_2$, and each contributes $O(1)$ to the sum. Similarly, the fourth moment of R is bounded by the energy, recalling its definition

$$E(W) := \#\{t_1,t_2,t_3,t_4\in W : |t_1+t_2-t_3-t_4|\ll T^\epsilon\}.$$

We get

$$\int_{v\asymp 1}|R(v)|^4dv = \sum_{t_1,t_2,t_3,t_4\in W}\int_{v\asymp 1}v^{i(t_1+t_2-t_3-t_4)}dv,$$

and the terms in the summation are negligble unless $|t_1+t_2-t_3-t_4|\ll T^\epsilon$. Assembling everything together gives an initial bound of

$$S_3\lesssim N^3\sum_{|m_1|\sim V,|m_2|\sim V,|m_3|\sim V}\frac{1}{|m_2|N}E(W)^{1/2}|W|^{1/2}\lesssim T^2E(W)^{1/2}|W|^{1/2},$$

recalling that $U,V\lesssim T/N$.

9. Refinement of S_3 bound

Recall that in the previous section, we summed across

$$\sum_{\substack{|m_1|\sim V\\|m_2|,|m_3|\sim U}}\frac{N}{|m_2|}\int_{\frac{1}{2}\le v_1\le 2}\left|R\Big(\frac{v_1m_1+m_3}{-v_1m_2}\Big)R\Big(\frac{v_1m_1+m_3}{-m_2}\Big)R(v_1)\right|dv_1$$

By repeatedly applying Cauchy-Schwartz, we can move the summation into the integral and obtain terms that resembles the form

$$\int_{v_1\asymp 1}\Bigg(\sum_{\substack{|m_1|\sim V\\|m_2|,|m_3|\sim U}}\left|R\Big(\frac{v_1m_1+m_3}{-m_2}\Big)\right|^2\Bigg)^2dv_1.$$

The term $(v_1m_1+m_3)/(-m_2)$ describes an affine transformation in v_1. Intuitively, thus averaging over all the affine transformations is can be expressed as the expected value of a random variable squared, where the random variable

$$X := X(v_1) := \sum_{\substack{|m_1|\sim V\\|m_2|,|m_3|\sim U}}\left|R\Big(\frac{v_1m_1+m_3}{-m_2}\Big)\right|^2,$$

and the measure is uniform across $v_1\asymp 1$. This can be split up to get

$$E[X^2] = E[X]^2 + \mathrm{Var(X)}.$$

Indeed, we have Lemma 3.5 that gives us this estimate. By a change of variables, the expected value of X is represented by $M^3\int_{u\asymp 1}R(u)^2du$.

Lemma 3.5. *Let $M > 0$. Let $f(u) \geq 0$, supported on $u \asymp 1$, and $|\hat{f}(\xi)| \lesssim_j (|\xi|/T)^j$ for all j. Then*

$$\begin{aligned} J(f) :=& \sup_{0<M_1,M_2,M_3<M} \int \left(\sum_{|m_1|\sim M_1, |m_2|\sim M_2, |m_3| \ll M_3} f\left(\frac{m_1 u + m_3}{m_2}\right)\right)^2 du \\ \lesssim& M^6 \Big(\int f(u)du \Big)^2 + M^4 \int f(u)^2 du. \end{aligned}$$

This is Proposition 9.1 in [3]. This improves the bound on S_3 to

$$S_3 \lesssim T^2 |W|^{3/2} + TN|W|^{1/2} E(W)^{1/2}.$$

This gives an improvement from the previous bound for the case with high energy and $N < T^{1-\delta}$.

Remark: The proof to Lemma 3.5 does not use probability. Instead, it applies Plancherel's theorem on $\sum_{m_1,m_2,m_3} f$ and bounds in the Fourier domain. At low frequencies, this can be bounded by the L_1 norm of f. At higher frequencies, the integral relates to

$$(M^4 \int f^2 du)^{1/2} J(f)^{1/2},$$

by Cauchy Schwarz and a change of variables to decouple m_1 and m_2, providing the required bound by backward induction. The proof of the hybrid statement requires an extension of this theorem which we will prove.

8. Bound on $E(W)$

Finally, we give an orthogonal bound on the energy of the set W. The idea is that if $|t_1 + t_2 - t_3 - t_4|$ is small, then we can approximate

$$D_N(t_4) \approx D_N(t_1 + t_2 - t_3).$$

This is made precise by applying a smoothing of D_N over a width of $\lesssim 1$. Therefore, since the choice of t_4 is fixed by the choice of t_1, t_2, t_3, we have

$$E(W)V^2 \leq \sum_{|t_1 - t_2 - t_3 - t_4| \lesssim 1} |D_N(t_4)|^2 \approx \sum_{|t_1 - t_2 - t_3 - t_4| \lesssim 1} |D_N(t_1 + t_2 - t_3)|^2 \leq \sum_{t_1,t_2,t_3 \in W} |D_N(t_1 + t_2 - t_3)|^2.$$

Now we can expand $|D_N(t_1 + t_2 - t_3)|^2$ to get

$$\begin{aligned} \sum_{t_1,t_2,t_3 \in W} |D_N(t_1 + t_2 - t_3)|^2 &= \sum_{t_1,t_2,t_3 \in W} \sum_{n_1,n_2 \sim N} b_{n_1} \bar{b}_{n_2} \left(\frac{n_1}{n_2}\right)^{i(t_1+t_2-t_3)} \\ &= \sum_{n_1,n_2\sim N} b_{n_1}\bar{b}_{n_2} \sum_{t_1,t_2,t_3 \in W} \left(\frac{n_1}{n_2}\right)^{i(t_1+t_2-t_3)} \\ &\leq \sum_{n_1,n_2\sim N} \left| R\Big(\frac{n_1}{n_2}\Big) R\Big(\frac{n_1}{n_2}\Big) R\Big(\frac{n_2}{n_1}\Big) \right| \\ &\leq \sum_{n_1,n_2\sim N} \left| R\Big(\frac{n_1}{n_2}\Big)\right|^3. \end{aligned}$$

Now we can apply the trivial bound $|R| \leq |W|$ to get that

$$E(W) \lesssim V^{-2} |W| \sum_{n_1,n_2\sim N} \left| R\Big(\frac{n_1}{n_2}\Big)\right|^2.$$

This is in turn bounded by Heath-Brown's result as

$$\sum_{n_1,n_2\sim N} \left| R\Big(\frac{n_1}{n_2}\Big)\right|^2 = \sum_{n_1,n_2\sim N} \sum_{t_1,t_2 \in W} \left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)}$$

$$
\begin{aligned}
&= \sum_{t_1, t_2 \in W} \Big| \sum_{n \sim N} n^{i(t_1 - t_2)} \Big|^2 \\
&\lesssim |W| N^2 + |W|^2 N + |W|^{5/4} T^{1/2} N.
\end{aligned}
$$

This is enough to give an improvement on Ingham-Huxley's result, but can be further improved using Cauchy Schwartz on the third moment

$$
\sum_{n_1, n_2 \sim N} \left| R\Big(\frac{n_1}{n_2}\Big) \right|^3 \leq \left(\sum_{n_1, n_2 \sim N} \left| R\Big(\frac{n_1}{n_2}\Big) \right|^2 \right)^{1/2} \left(\sum_{n_1, n_2 \sim N} \left| R\Big(\frac{n_1}{n_2}\Big) \right|^4 \right)^{1/2}.
$$

The fourth moment can be reduced back to the second moment by taking representative classes of $\lfloor t_1 - t_2 \rfloor$, thus can also be bounded using Heath-Brown's result.

Chapter 4

Towards a Hybrid Large Values Estimate Result

We would like to generalize Guth and Maynard's result to L-functions. Specifically, let χ be a Dirichlet character. we are interested in the the zeros of the function defined by

$$L(s,\chi) := \sum_n \frac{\chi(n)}{n^{-s}}$$

on $\Re(s) > 1$ and its analytic continuation on the whole complex plane. The zeta function is a special case of an L-function with the Dirichlet character 1 everywhere. The structure of the arguments for analytic continuation of an L-function, its line of symmetry along $\Re(s) = 1/2$, and the locations of zeros are very similar to that of the zeta function. This motivates the Generalized Riemann Hypothesis.

Conjecture 4.1 (Generalized Riemann Hypothesis). *The Generalized Riemann Hypothesis asserts that on the critical strip,*

$$L(s,\chi) = 0 \implies \Re(s) = \frac{1}{2},$$

for any Dirichlet character.

The Generalized Riemann Hypothesis leads to even stronger for primes in short intervals. Namely, fix an integer q, we have that

$$\sum_{\substack{n \le N \\ n \equiv a \bmod q}} \Lambda(n) = \begin{cases} \frac{1}{\phi(q)} N + O(x^{2+o(1)}), & \text{if } \gcd(a,q) = 1 \\ o(n), & \text{otherwise.} \end{cases}$$

This means that not only that the Prime Number Theorem holds in intervals of $x^{2+\epsilon}$, the distribution of primes in each of the residual classes (coprime to q) are uniform at this scale too. Noticing that we can modify Huxley's proof with

$$M_{x,\chi} = \sum_{n \le x} \chi(n)\mu(n)n^{-s},$$

we have

$$L(s,\chi)M_{x,\chi} = \sum_n a_n \chi(n)\mu(n)n^{-s}.$$

Thus, we can reproduce a similar proof on the zero density of L-functions.

Definition 4.2 (Zero Density for L-functions). *Let $N(\sigma,\chi,T)$ denote the number of zeros of the L-function $L(-,\chi)$ with real part greater than σ and imaginary part between $-T$ and $-T$. That is*

$$N(\sigma,\chi,T) := \#\{\rho = \beta + i\gamma \mid \beta \ge \sigma, |\gamma| \le T\}.$$

For backwards compatibility with our previous definition, we take $N(\sigma,T) := N(\sigma,1,T)$.

The hybrid analogs of the zero density bounds of Ingham and Huxley are known.

Theorem 4.3 (Hybrid Ingham bound for zero density). *Let $1/2 \le \sigma \le 3/4$. We have*

$$\sum_{\chi^*} N(\sigma,\chi^*,t) \lesssim (qT)^{\frac{3(1-\sigma)}{2-\sigma}},$$

where $\sum_{\chi^*}$ sums over all the primitive characters χ^* of modulus q.

Theorem 4.4 (Hybrid Huxley bound for zero density). *Let $3/4 \le \sigma \le 1$. We have*

$$\sum_{\chi^*} N(\sigma,\chi^*,t) \lesssim (qT)^{\frac{3(1-\sigma)}{3\sigma-1}},$$

where $\sum_{\chi^*}$ sums over all the primitive characters χ^* of modulus q.

The method for detecting zeros is very similar to Huxley's proof above with the slight change in definition of $M_{x,\chi}$. This argument then reduces to bounding the number of times large values of Dirichlet polynomials can occur. Therefore we want a result in the form of the Halász inequality, this time with twisted Dirchlet polynomials:

> Let $W = \{(t_j,\chi_j)\}$ be a set such that each χ_j is a primitive Dirichlet character of modulus q, and $|t_j - t_k| \ge 1$ if $j \ne k$ and $\chi_j = \chi_k$. (That is, the t's are 1-separated if the characters are the same.)Let $|b_n| \le 1$ be a sequence of numbers indexed in n, and suppose also that
>
> $$\left|\sum_{n\sim N} b_n \chi_j(n) n^{it_j}\right| > V.$$
>
> We want to find a bound on $|W|$.

4.1 Statement of Main Theorem and its Reduction

Notation

From this point onward, we interpret Big O and little o asymptotic quantities as $qT \to \infty$. $T \ge 1$ is assumed in the statements of theorems and propositions.

Our result is as follows:

Theorem 4.5 (Generalization of Guth-Maynard Halász Inequality). *Let $W = \{(t_j,\chi_j)\}$ be a set such that each χ_j is a primitive Dirichlet character of modulus q, and $|t_j - t_k| \ge 1$ if $j \ne k$ and $\chi_j = \chi_k$. (That is, the t's are 1-separated if the characters are the same.)Let $|b_n| \le 1$ be a sequence of numbers indexed in n, and suppose also that*

$$\left|\sum_{n\sim N} b_n \chi_j(n) n^{it_j}\right| > V.$$

Then we have, for $N \ge q^{5/6}$,

$$|W| \le (qT)^{o(1)}(N^2V^{-2} + N^{18/5}V^{-4} + (qT)^{12/5}V^{-4}).$$

Remark: This is a restatement of Guth-Maynard's inequality with all T's changed to qT.

This theorem is incomplete, the case $N < q^{5/6}$ is still in progress. The place where the proof breaks down is going from 4.6 to this statement. If $qT > N^{6/5}$, we subdivide T into smaller intervals of $T' = N^{6/5}/q$, and apply the reduction on each of these intervals. This breaks down when q is too large, as we would have $T' < 1$, which affects the application of Heath Brown's theorem.

Proposition 4.6 (Reduction of Main Theorem)**.** *Let $W = \{(t_j, \chi_j)\}$ be a set such that each χ_j is a primitive Dirichlet character of modulus q, and $t_j \in [0, T]$. Further assume that $|t_j - t_k| \geq (qT)^\epsilon$ if $j \neq k$ and $\chi_j = \chi_k$. Let $|b_n| \leq 1$ be a sequence of numbers indexed in n, ω be a smooth bump function that equals 1 on $[6/5, 9/5]$ and has support in $[1, 2]$ (thus $\omega^{(A)} \ll_A 1$ for all A). Let $V = N^\sigma$, where $[7/10, 8/10]$, and $N = (qT)^{5/6}$. Suppose also that*

$$|D_N(t_j, \chi_j)| := \left| \sum_{n\sim N} \omega\left(\frac{n}{N}\right) b_n \chi_j(n) n^{it_j} \right| > V.$$

for all $(t_j, \chi_j) \in W$. Then

$$|W| \ll (qT) N^{(12-20\sigma)/5+o_\epsilon(1)}.$$

Proof of Theorem 4.5 assuming Proposition 4.6. The argument is essentially identical to the Guth and Maynard's proof, except checking the condition that T is not too small compared to q.

The result is given by Ingham and Huxley when $V \leq N^{7/10+o(1)}$ or $V \geq N^{8/10+o(1)}$. Now let $V \in [4N^{7/10}, N^{8/10}]$. Then we have $N^2 V^{-2} \ll N^{18/5} V^{-4}$. We split

$$\sum_{n\sim N} b_n \chi_j(n) n^{it_j} = \sum_{N \leq n < 6N/5} b_n \chi_j(n) n^{it_j} + \sum_{6N/5 \leq n < 9N/5} b_n \chi_j(n) n^{it_j} + \sum_{9N/5 \leq n < 2N} b_n \chi_j(n) n^{it_j}.$$

So that if

$$\left| \sum_{n\sim N} b_n \chi_j(n) n^{it_j} \right| > V,$$

then at least one of the three summations in the right has magnitude at least $V/3$. But then each of these sums can be written as dirichlet sums in the form of the lemma:

$$\sum_{N \leq n < 6N/5} b_n \chi_j(n) n^{it_j} = \sum_{N \leq n < 6N/5} \omega\left(\frac{n}{11N/15}\right) b_n \chi_j(n) n^{it_j},$$

$$\sum_{6N/5 \leq n < 9N/5} b_n \chi_j(n) n^{it_j} = \sum_{6N/5 \leq n < 9N/5} \omega\left(\frac{n}{N}\right) b_n \chi_j(n) n^{it_j},$$

$$\sum_{9N/5 \leq n < 2N} b_n \chi_j(n) n^{it_j} = \sum_{9N/5 \leq n < 2N} \omega\left(\frac{n}{19N/15}\right) b_n \chi_j(n) n^{it_j}.$$

Applying the lemma on each of these three pieces and summing the three bounds gives the same bound for $\sum_{n\sim N} b_n \chi_j(n) n^{it_j}$ up to a constant factor, so it suffices to show the result assuming that $b_n = 0$ for $6N/5 \leq n < 9N/5$, in which case $b_n = b_n \omega(n/N)$. Now suppose that $qT \leq N^{6/5}$. Then the t_j's lie in an interval of length $N^{6/5}/q \geq T$. Thus in this case at a cost of $(qT)^\epsilon$ we consider the $(qT)^\epsilon$ disjoint subsets of $|W|$ each being $(qT)^\epsilon$ separated in the t_j's if they share a character. Setting $V = N^\sigma$, the proposition gives

$$|W| \leq (qT)^\epsilon N^{(18-20\sigma)/5+o_\epsilon(1)} \leq (qT)^\epsilon N^{18/5+o_\epsilon(1)} V^{-4}.$$

Letting $\epsilon \to 0$ gives the result. If $qT \geq N^{6/5}$, we divide $|W|$ into $[qT/N^{6/5}]$ subsets each supported on a length of $N^{6/5}/q$. Therefore, we can apply the proposition (on the $(qT)^\epsilon$ disjoint subsets) to get

$$|W| \leq (qT)^{1+\epsilon} N^{-6/5} (N^{18/5+o_\epsilon(1)} V^{-4}) \leq (qT)^{1+\epsilon+o_\epsilon(1)} N^{12/5} V^{-4}.$$

Letting $\epsilon \to 0$ sufficiently slowly gives the result in this case too. □

The idea of the proof of Proposition 4.6 is similar to Guth and Maynard's proof. We can define a $|W| \times N$ matrix M with entries

$$M_{t_j, \chi_j, n} = \chi_j(n) n^{it_j}$$

for $(t_j, \chi_j) \in W$ and $n \sim N$, and bound its operator norm in the exact same way: taking it to the $M^* M$ to the third power and calculating its trace. We will highlight similar ideas when they come up.

4.2 The matrix M and its trace expansion

We define M a $|W| \times N$ matrix with entries

$$M_{(t_j,\chi_j),n} = \omega\Big(\frac{n}{N}\Big)\chi_j(n)n^{it_j}$$

for $(t_j,\chi_j) \in W$ and $n \sim N$. Thus by the same reasoning that $(M\vec{b})_j = D_N(t_j,\chi_j)$, we want to bound the size of W by trace of the matrix

$$\operatorname{tr}((M^*M)^3).$$

. Our key intermediate result this section is as follows:

Proposition 4.7. *Let*

$$P(n,(t_1,\chi_1),(t_2,\chi_2)) \coloneqq \omega\left(\frac{n}{N}\right)^2\left(\frac{n}{N}\right)^{i(t_2-t_1)}\chi_2\bar{\chi}_1(n),$$

and

$$S \coloneqq \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in W\\(*)}} \sum_{n_1,n_2,n_3\sim N} P(n_1,(t_1,\chi_1),(t_3,\chi_3))P(n_2,(t_2,\chi_2),(t_1,\chi_1))P(n_3,(t_3,\chi_3),(t_2,\chi_2)),$$

where the condition $(*)$ denotes that the three pairs are not all identical. We have

$$|W| \ll N^2V^{-2} + NV^{-2}S^{1/3}.$$

Lemma 4.8. *Let A be an $m\times n$ matrix. Then*

$$\|A\| \le 2\left(\operatorname{tr}((\mathrm{AA}^*)^3) - \frac{\operatorname{tr}(\mathrm{AA}^*)^3}{\mathrm{m}^2}\right)^{1/6} + 2\left(\frac{\operatorname{tr}(\mathrm{AA}^*)}{m}\right)^{1/2}.$$

Proof. This is Lemma 4.2 from Guth-Maynard. [3] ☐

Corollary 4.9. *We have*

$$|W| \ll NV^{-2}\left(\frac{\operatorname{tr}(\mathrm{AA}^*)}{|W|}\right) + NV^{-2}\left(\operatorname{tr}((\mathrm{MM}^*)^3) - \frac{\operatorname{tr}(\mathrm{MM}^*)^3}{|\mathrm{W}|^2}\right)^{1/3}.$$

Proof. Consider the product $M\vec{b}$. The j-th entry is exactly $D_n(t_j,\chi_j)$. Thus we have

$$V^2|W| \le |M\vec{b}|^2 \le \|M\|^2|\vec{b}|^2 \le \|M\|^2N.$$

Applying Lemma 4.8 gives the desired result. ☐

We now compute the expression for the traces.

Lemma 4.10 (Explicit Expression of Traces)**.** *Let*

$$P(n,(t_1,\chi_1),(t_2,\chi_2)) \coloneqq \omega\left(\frac{n}{N}\right)^2\left(\frac{n}{N}\right)^{i(t_2-t_1)}\chi_2\bar{\chi}_1(n).$$

We have

$$\operatorname{tr}(\mathrm{M}^*\mathrm{M}) = |W|\sum_{n\sim N,\gcd(n,q)=1}\omega\left(\frac{n}{N}\right)^2,$$

$$\operatorname{tr}((\mathrm{M}^*\mathrm{M})^3) = \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in W}} \sum_{n_1,n_2,n_3\sim N} P(n_1,(t_1,\chi_1),(t_3,\chi_3))P(n_2,(t_2,\chi_2),(t_1,\chi_1))P(n_3,(t_3,\chi_3),(t_2,\chi_2)).$$

Proof. We see that

$$(MM^*)_{(t_j,\chi_j),(t_k,\chi_k)} = \sum_{n\sim N} \omega\left(\frac{n}{N}\right)^2 n^{i(t_k-t_j)}\bar{\chi}_j\chi_k(n) = \sum_{n\sim N} P(n,(t_k,\chi_k),(t_j,\chi_j))N^{i(t_k-t_j)},$$

so that

$$\begin{aligned}\mathrm{tr}(MM^*) =& \sum_{(t_j,\chi_j)}\sum_{n\sim N} P(n,(t_j,\chi_j),(t_j,\chi_j))\\ =&|W|\sum_{n\sim N}|\chi_j(n)|\omega\left(\frac{n}{N}\right)^2.\end{aligned}$$

For the third power, we have

$$(MM^*)^3_{(t_j,\chi_j),(t_k,\chi_k)} = \sum_{(t_1,\chi_1),(t_2,\chi_2)\in W}(MM^*)_{(t_j,\chi_j),(t_1,\chi_1)}(MM^*)_{(t_1,\chi_1),(t_2,\chi_2)}(MM^*)_{(t_2,\chi_2),(t_k,\chi_k)},$$

so that the trace is given by

$$\begin{aligned}&\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in W}}(MM^*)_{(t_3,\chi_3),(t_1,\chi_1)}(MM^*)_{(t_1,\chi_1),(t_2,\chi_2)}(MM^*)_{(t_2,\chi_2),(t_3,\chi_3)}\\ =&\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in W}}\sum_{n_1,n_2,n_3\sim N}P(n_1,(t_1,\chi_1),(t_3,\chi_3))P(n_2,(t_2,\chi_2),(t_1,\chi_1))P(n_3,(t_3,\chi_3),(t_2,\chi_2)).\end{aligned}$$

□

Consider the sum

$$\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in W}}$$

in the expression of $\mathrm{tr}((\mathrm{M}^*\mathrm{M})^3)$. We isolate the terms where $(t_1,\chi_1) = (t_2,\chi_2) = (t_3,\chi_3)$ in the sum, and call it S_0.

Lemma 4.11 (Explicit Calculation of S_0)**.** *We have*

$$S_0 = |W|^2\Bigg(\sum_{n\sim N,\mathrm{gcd}(n,q)=1}\omega\Big(\frac{n}{N}\Big)^2\Bigg)^2.$$

In particular,

$$S_0 = \Big(\mathrm{tr}(\mathrm{M}^*\mathrm{M})\Big)^3/|\mathrm{W}|^2.$$

Proof. Notice that

$$P(n_1,(t_1,\chi_1),(t_1,\chi_1)) = \omega\Big(\frac{n_1}{N}\Big)^2\mathbb{I}_{\mathrm{gcd}(n_1,q)=1}.$$

Thus we have

$$S_0 = \sum_{(t_1,\chi_1)\in W}\sum_{n_1,n_2,n_3\sim N}\omega\Big(\frac{n_1}{N}\Big)^2\omega\Big(\frac{n_2}{N}\Big)^2\omega\Big(\frac{n_3}{N}\Big)^2\mathbb{I}_{\mathrm{gcd}(n_1,q)=1}.\mathbb{I}_{\mathrm{gcd}(n_2,q)=1}.\mathbb{I}_{\mathrm{gcd}(n_3,q)=1}$$

which gives the result. □

Thus assembling this lemma with Proposition 4.9 gives Proposition 4.7.

4.3 Poisson summation on S

We now bound S using Poisson summation. We would like to apply Poisson summation directly to P in n_1, n_2, n_3, respectively. However, we do not have a good way to express the Dirichlet character $\chi_j\bar{\chi}_k$ as a C^∞ function. To get around this, we just split the sum in n into q pieces.

Lemma 4.12 (Poisson Summation with Periodic Arithmetic Function). *Let $f:\mathbb{R}\to\mathbb{C}$ be Schwartz, and $g:\mathbb{N}\to\mathbb{C}$ be an arithmetic with period q. Then*

$$\sum_n g(n)f(n) = \frac{1}{q}\sum_\xi \hat{g}(\xi)\hat{f}\left(\frac{\xi}{q}\right).$$

Proof. By absolute convergence, we have

$$\sum_n g(n)f(n) = \sum_n \sum_{x \bmod q} g(qn+x)f(qn+x) = \sum_{x \bmod q} g(x)\sum_n f(qn+x).$$

Applying Poisson summation to the last expression in n gives

$$\frac{1}{q}\sum_{x \bmod q} g(x)\sum_\xi e\left(\frac{-x\xi}{q}\right)\hat{f}\left(\frac{\xi}{q}\right) = \frac{1}{q}\sum_\xi \hat{g}(\xi)\hat{f}\left(\frac{\xi}{q}\right).$$

□

Corollary 4.13 (Poisson summation of Trace). *Let $h_t(u) := \omega(u)^2u^{it}$, Then*

$$S = \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in W\\(*)}} \frac{N^3}{q^3}\sum_{m\in\mathbb{Z}^3}\widehat{\chi_1\overline{\chi_3}}(m_1)\widehat{\chi_2\overline{\chi_1}}(m_2)\widehat{\chi_3\overline{\chi_2}}(m_3)\hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right).$$

In the summation in m, we separate this into two terms. S_2, when at least one of the m_i's are zero, and S_3, when all three m_i's are non-zero.

Finally, we restate the key result that allows us to bound $\hat{h}$ in the Fourier domain. We have not modified the $\hat{h}$ function, so the proof is the same as in Lemma 3.3.

Lemma (Non-stationary phase). *We have for any integer $A>0$*

$$|\hat{h}_t(\xi)| \ll_A \frac{1+|t|^A}{|\xi|^A},$$
$$|\hat{h}_t(\xi)| \ll_A \frac{1+|\xi|^A}{|t|^A}.$$

4.4 S_2 Bound

Proposition 4.14 (Bound on S_2). *We have*

$$S_2 \lesssim_k N^2|W|^2 + qTN|W|^{2-1/k} + N^2|W|^2\left(\frac{(qT)^{1/2}}{|W|^{3/4}}\right)^{1/k}.$$

This bound relies on Heath-Brown's theorem. We recall its statement here.

Theorem (Heath-Brown). *Let $\mathcal{S}=\{(t_j,\chi_j)\}$ be one-separate, primitive characters of modulus q. Then*

$$\sum_{\substack{(t_1,\chi_1)\\(t_2,\chi_2)}}\left|\sum_{n=1}^N b_n n^{-1/2-i(t_1-t_2)}\chi_1\bar{\chi}_2(n)\right|^2 \lesssim (|\mathcal{S}|N + |\mathcal{S}|^2 + |\mathcal{S}|^{5/4}(qT)^{1/2})\max_{n\le N}|b_n|.$$

With a loss of a factor of 3, we can set $m_3 = 0$. Thus

$$S_2 \ll \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in W\\(*)}} \frac{N^3}{q^3} \sum_{m_1,m_2\in\mathbb{Z}} \widehat{\chi_1\overline{\chi}_3}(m_1)\widehat{\chi_2\overline{\chi}_1}(m_2)\widehat{\chi_3\overline{\chi}_2}(0)\hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}(0).$$

By the orthogonality of characters, the terms when $\chi_2 \neq \chi_3$ vanish as $\widehat{\chi_3\overline{\chi_2}}(0) = 0$. But when $\chi_2 = \chi_3$ and $t_3 \neq t_2$, then they are $(qT)^\epsilon$ apart and $\hat{h}_{t_3-t_2}(0) \ll (qT)^{-100}$. Since we have a trivial bound on the $|W|$ to be qT, the terms for which $(t_2,\chi_2) \neq (t_3,\chi_3)$ are negligible by Lemma 4.3 and we have

$$S_2 \ll \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W\\(t_1,\chi_1)\neq(t_2,\chi_2)}} \frac{N^3}{q^2} \sum_{m_1,m_2\in\mathbb{Z}} \widehat{\chi_1\overline{\chi}_2}(m_1)\widehat{\chi_2\overline{\chi}_1}(m_2)\hat{h}_{t_1-t_2}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right) + O((qT)^{-100}).$$

Noticing that $\hat{h}_t(\xi) = \overline{\hat{h}_{-t}(-\xi)}$, we rewrite

$$S_2 \ll \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W\\(t_1,\chi_1)\neq(t_2,\chi_2)}} \frac{N^3}{q^2}\left|\sum_{m\in\mathbb{Z}} \widehat{\chi_1\overline{\chi}_2}(m)e\left(\frac{-xm}{q}\right)\hat{h}_{t_1-t_2}\left(\frac{Nm}{q}\right)\right|^2. \tag{4.1}$$

Lemma 4.15 (Approximate Functional Equation)**.** *Let $|t| \le T_0$, and χ be a Dirichlet character mod q. Further assume that if χ is principal then $|t| \ge (qT)^\epsilon$. Then we have*

$$\left|\sum_{m\in\mathbb{Z}} \hat{\bar{\chi}}(m)\hat{h}_{t_1-t_2}\left(\frac{Nm}{q}\right)\right| \lesssim qN^{1/2}\int_{|u|\lesssim 1}\left|\sum_{n\lesssim\frac{qT_0}{N}} n^{-1/2-i(t+u)}\bar{\chi}(n)\right|du + O((qT)^{-100}).$$

The proof to this approximate functional equation is due to Vishal Gupta, who had been collaborating with me on this problem. The proof relies on the reflection method for L-functions akin to Theorem 9.15 in [6].

Proof of Proposition 4.14. Beginning from equation 4.1, we apply the approximate functional equation to get

$$\begin{aligned} S_2 \lesssim & N^2 \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W\\(t_1,\chi_1)\neq(t_2,\chi_2)}} \left|\int_{u\lesssim 1}\left|\sum_{n\lesssim\frac{qT}{N}} n^{-1/2-i(t_1-t_2+u)}\bar{\chi}_2\chi_1(n)\right|du\right|^2 \\ \overset{\text{CS}}{\lesssim} & \sup_{|u|\lesssim 1} N^2 \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W}} \left|\sum_{n\lesssim\frac{qT_0}{N}} n^{-1/2-i(t_1-t_2+u)}\bar{\chi}_2\chi_1(n)\right|^2. \end{aligned}$$

We now apply Hölder's inequality on

$$\begin{aligned} & \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W}} \left|\sum_{n\lesssim\frac{qT_0}{N}} n^{-1/2-i(t_1-t_2+u)}\bar{\chi}_2\chi_1(n)\right|^2 \\ \overset{\text{Hölder}}{\le} & \left(\sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W}} 1\right)^{(k-1)/k}\left(\sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W}}\left|\sum_{n\lesssim\frac{qT_0}{N}} n^{-1/2-i(t_1-t_2+u)}\bar{\chi}_2\chi_1(n)\right|^{2k}\right)^{1/k} \\ = & |W|^{2-2/k}\left(\sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in W}}\left|\sum_{n\lesssim\left(\frac{qT_0}{N}\right)^k} b_n n^{-1/2-i(t_1-t_2)}\bar{\chi}_2\chi_1(n)\right|^2\right)^{1/k} \end{aligned}$$

where each $|b_n| \lesssim 1$ by the divisor bound. We can thus apply Heath-Brown's theorem to get

$$S_2 \lesssim N^2|W|^{2-2/k}(|W|M^k + |W|^2 + W^{5/4}(qT)^{1/2})^{1/k},$$

where $M = qT/N$. This gives the proposition upon reduction. □

4.5 Summations over affine transformations result

To bound S_3, we first introduce a result that exploits equidistribution over the affine transformations. Morally, the two terms in the following proposition represents the average and the variance of the distribution of f respectively.

Proposition 4.16. *Let $M > 0$. Let $f(u,y) : \mathbb{R} \times \mathbb{Z}/d \to \mathbb{R}$ be non-negative, supported on $u \asymp 1$ uniformly in y, and $|\hat{f}(\xi,x)| \lesssim_j (|\xi|/(qT)^2)^{-j}$ for all j. Then*

$$J(f) := \sup_{0<M_1,M_2,M_3<M} \sum_{y \bmod d} \int \Bigg(\sum_{\substack{|m_1|\sim M_1, |m_2|\sim M_2, |m_3| \ll M_3 \\ (m_1,d)=(m_2,d)=1}} f\left(\frac{m_1 u + m_3}{m_2}, (m_1 y + m_3)m_2^{-1}\right)\Bigg)^2 du$$
$$\lesssim \frac{M^6}{d}\Big(\sum_{y \bmod d} \int f(u,y)du\Big)^2 + M^4 \sum_{y \bmod d} \int f(u,y)^2 du.$$

Lemma 4.17 (Iterative bound for $J(f)$)**.** *There exists a bump function $\phi(x)$ supported on $|x| \lesssim 1$, such that for*

$$\tilde{f}(u,y) := \int (qT)^2\phi((qT)^2(u'-u))f(u',y)du'$$

we have

$$J(f) \lesssim \frac{M^6}{d}\|f\|_{L_1}^2 + M^2\|f\|_{L_2}J(\tilde{f}).$$

Proof of Proposition 4.16 from Lemma 4.17. The argument is exactly the same as Lemma 9.2 in [3]. □

Let ψ be a smooth bump on $|x| \lesssim 1$, such that $\psi(m_3/M_3) = 1$ where $m_3 \lesssim M_3$ in the summation conditiion. For the choice of M_1, M_2, M_3 that the supremum in the lemma is achieved, we define

$$g(u,y) := \sum_{\substack{m_1\sim M_1, m_2 \sim M_2 \\ (m_1,d)=(m_2,d)=1}} \sum_{m_3 \in \mathbb{Z}} \psi\left(\frac{m_3}{M_3}\right) f\left(\frac{m_1 u + m_3}{m_2}, (m_1 y + m_3)m_2^{-1}\right),$$

so that we have

$$J(f) \le \sum_{y \text{ mod } d} \int |g(u,y)|^2 du \overset{\text{Plancheral}}{=} \frac{1}{d} \sum_{x \text{ mod } d} \int |\hat{g}(\xi,x)|^2 d\xi.$$

We now compute, using the change of variables $\tilde{u} = u + m_3/m_1$ and $\tilde{y} \equiv y + m_3 m_1^{-1}(d)$,

$$\begin{aligned}
\hat{g}(\xi,x) &= \sum_{y=0}^{d-1} \int e\left(\frac{-xy}{d}\right) e(-u\xi) \sum_{\substack{m_1\sim M_1, m_2 \sim M_2 \\ (m_1,d)=(m_2,d)=1}} \sum_{m_3 \in \mathbb{Z}} \psi\left(\frac{m_3}{M_3}\right) f\left(\frac{m_1 u + m_3}{m_2}, (m_1 y + m_3)m_2^{-1}\right) du \\
&= \sum_{\substack{m_1\sim M_1, m_2 \sim M_2 \\ (m_1,d)=(m_2,d)=1}} \sum_{m_3 \in \mathbb{Z}} \sum_{\tilde{y}=0}^{d-1} \int e\left(\frac{-x\tilde{y}}{d}\right) e\left(\frac{[m_3 x {m_1}^{-1}]}{d}\right) e(-\tilde{u}\xi) e\left(\frac{m_3\xi}{m_1}\right) \psi\left(\frac{m_3}{M_3}\right) f\left(\frac{m_1\tilde{u}}{m_2}, (m_1\tilde{y})m_2^{-1}\right) d\tilde{u} \\
&= \sum_{\substack{m_1\sim M_1, m_2 \sim M_2 \\ (m_1,d)=(m_2,d)=1}} \sum_{\tilde{y}=0}^{d-1} \int e\left(\frac{-x\tilde{y}}{d}\right) e(-\tilde{u}\xi) f\left(\frac{m_1\tilde{u}}{m_2}, (m_1\tilde{y})m_2^{-1}\right) d\tilde{u} \sum_{m_3 \in \mathbb{Z}} \psi\left(\frac{m_3}{M_3}\right) e\left(\frac{[m_3 x {m_1}^{-1}]}{d}\right) e\left(\frac{m_3\xi}{m_1}\right) \\
&= \sum_{\substack{m_1\sim M_1, m_2 \sim M_2 \\ (m_1,d)=(m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, m_2 m_1^{-1} x\right) \sum_{m_3 \in \mathbb{Z}} \psi\left(\frac{m_3}{M_3}\right) e\left(\frac{[m_3 x {m_1}^{-1}]}{d}\right) e\left(\frac{m_3\xi}{m_1}\right)
\end{aligned}$$

$$\overset{\text{Poisson}}{=} \sum_{\substack{m_1 \sim M_1, m_2 \sim M_2 \\ (m_1,d)=(m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, m_2 m_1^{-1} x\right) \sum_{l \in \mathbb{Z}} M_3 \hat{\psi}\left(M_3\left(l - \frac{\xi}{m_1} - \frac{[m_1^{-1}x]}{d}\right)\right)$$

By non-stationary phase, $\hat{\psi}$ has rapid decay and thus we can truncate the summation in l to

$$M_3\left(l - \frac{\xi}{m_1} - \frac{[m_1^{-1}x]}{d}\right) \lesssim 1 \implies m_1\left(l - \frac{[m_1^{-1}x]}{d}\right) - \xi \lesssim \frac{M_1}{M_3}$$

with negligible error. Thus we have

$$|\hat{g}(\xi, x)| \lesssim \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \sum_{l : m_1\left(l - \frac{[m_1^{-1}x]}{d}\right) - \xi \lesssim \frac{M_1}{M_3}} M_3 \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, m_2 m_1^{-1} x\right) \right| + O(T^{-100})$$

where the choice of the representative $[m_1^{-1}x]$ is arbitrary, as it only induces a shift in the the range of l by an integer.

Let $\eta > 0$. We break

$$\sum_{x \bmod d} \int |\hat{g}(\xi, x)|^2 d\xi$$

into five pieces.

Ai $\int_{|\xi| < \frac{M_1}{M_3}(qT)^\eta} |\hat{g}(\xi, 0)|^2 d\xi$

Aii $\sum_{x \neq 0 \bmod d} \int_{|\xi| < \frac{M_1}{M_3}(qT)^\eta} |\hat{g}(\xi, x)|^2 d\xi$

Bi $\int_{\frac{M_1}{M_3}(qT)^\eta \leq |\xi| < (qT)^4} |\hat{g}(\xi, 0)|^2 d\xi$

Bii $\sum_{x \neq 0 \bmod d} \int_{\frac{M_1}{M_3}(qT)^\eta \leq |\xi| < (qT)^4} |\hat{g}(\xi, x)|^2 d\xi$

C $\sum_{x \bmod d} \int_{|\xi| \geq (qT)^4} |\hat{g}(\xi, x)|^2 d\xi$

By the decay in $\hat{f}$ and $T > d^{1+\epsilon}, M^{1+\epsilon}$, C is negligible.

Lemma 4.18 (Ai bound)**.** *We have*

$$\mathrm{Ai} \lesssim (\mathrm{qT})^{3\eta} \mathrm{M}^6 \left(\sum_{\mathrm{y}} \int \mathrm{f(u,y)du}\right)^2$$

Proof. We have

$$|\hat{g}(\xi, 0)| \lesssim \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \sum_{l : m_1 l - \xi \lesssim \frac{M_1}{M_3}} M_3 \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, 0\right) \right| + O((qT)^{-100}),$$

so the possible non-zero contributions of l satisfy $l \lesssim (qT)^\eta / M_3$. Therefore, for $\xi \leq (qT)^\eta M_1/M_3$

$$\begin{aligned} |\hat{g}(\xi, 0)|^2 \lesssim & \left| \left(M_3 + (qT)^\eta\right) \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, 0\right) \right| \right|^2 \\ \overset{\text{CS}}{\ll} & M_3^2 (qT)^{2\eta} M_1 \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, 0\right) \right|^2 \\ \lesssim & (qT)^{2\eta} M_3^2 M_2^4 |\sup_{\xi'} \hat{f}(\xi', 0)|^2. \end{aligned}$$

We have $|\hat{f}(\xi', 0)| \leq \sum_{y \bmod d} \int |f(\xi, y)|$, so integrating over the range $\xi \leq M_1/M_3(qT)^\eta$ and taking $M_1, M_2, M_3 \leq M$ gives the result. □

Lemma 4.19 (Aii, Bi, Bii bound). *We have*

$$\mathrm{Aii} + \mathrm{Bi} + \mathrm{Bii} \lesssim \mathrm{d}\mathrm{T}^{2\eta}\left(\mathrm{M}^4 \sum_{\mathrm{y}\ mod\ \mathrm{d}} \int \mathrm{f}(\mathrm{u},\mathrm{y})^2 \mathrm{du}\right)^{1/2} (\mathrm{J}(\tilde{\mathrm{f}}))^{1/2}$$

Proof. As in Ai bound, the non-zero contribution from l occurs only if

$$\left| m_1 \left(l - \frac{[m_1^{-1}x]}{d} \right) - \xi \lesssim \frac{M_1}{M_3} \right| \implies \left| m_1 \left(dl - [m_1^{-1}]x \right) - d\xi \right| \lesssim \frac{dM_1}{M_3}$$

Let $s = m_1\left(dl - [m_1^{-1}]x\right)$, so that s must be an integer within $\lesssim dM_1/M_3$ of $d\xi$. Moreover, $s \equiv -x \bmod d$, so there are $\lesssim (1 + M_1/M_3)$ candidates of s. Since $\xi \geq T^\eta M_1/M_3$ or $x \neq 0$, we also have $s \neq 0$. Thus noticing the choice of m_1 and s fixes the value of l we have by the divisor bound

$$\lesssim 1 + \frac{M_1}{M_3}$$

pairs of (m_1, l) that satisfy these conditions. So in this range

$$\begin{aligned}&|\hat{g}(\xi,x)|^2 \\ &\overset{\text{CS}}{\ll} (M_1M_3 + M_3^2) \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \sum_{l: m_1\left(l - \frac{[m_1^{-1}x]}{d}\right) - \xi \lesssim \frac{M_1}{M_3}} \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, m_2 m_1^{-1} x\right)\right|^2,\end{aligned}$$

giving

$$\begin{aligned}\mathrm{Aii} + \mathrm{Bi} + \mathrm{Bii} \leq &\sum_{x \bmod d} \int_{|\xi| \leq (qT)^4} (1 - \mathbb{I}_{x=0} * \mathbb{I}_{\xi \leq M_1 T^\eta / M_3}) |\hat{g}(\xi,x)|^2 d\xi \\ \lesssim &(M_1M_3 + M_3^2) \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \sum_{x \bmod d} \int_{|\xi| < (qT)^4} \sum_{l: m_1\left(l - \frac{[m_1^{-1}x]}{d}\right) - \xi \lesssim \frac{M_1}{M_3}} \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, m_2 m_1^{-1} x\right)\right|^2 d\xi \\ \leq &(M_1M_3 + M_3^2) \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \sum_{x \bmod d} \sum_l \int_{\left| m_1\left(l - \frac{[m_1^{-1}x]}{d}\right) - \xi \right| \lesssim \frac{M_1}{M_3}} \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(\frac{m_2}{m_1}\xi, m_2 m_1^{-1} x\right)\right|^2 d\xi.\end{aligned}$$

We now remove the dependence on m_1. First $\tilde{x} \equiv x m_1^{-1}$, so that the summation condition in x and l becomes

$$\sum_{\tilde{x} \bmod d} \sum_{l: m_1\left(l - \frac{\tilde{x}}{d}\right) - \xi \lesssim \frac{M_1}{M_3}}.$$

Moreover, for each m_1, we now make the change of variables

$$\xi = m_1\left(l - \frac{\tilde{x}}{d}\right) + \tau \frac{m_1}{M_3}$$

so by slightly extending the domain of integration we can find uniformly $|\tau| \lesssim 1$ with Jacobian $\asymp M_1/M_3$. Combined we get

$$\begin{aligned}&\mathrm{Aii} + \mathrm{Bi} + \mathrm{Bii} \lesssim \frac{M_1}{M_3}(M_3^2 + M_3M_1) \sum_{\substack{m_1 \sim M_1 \\ (m_1,d)=1}} \sum_{\tilde{x} \bmod d} \sum_l \int_{|\tau| \lesssim 1} \left| \sum_{\substack{m_2 \sim M_2 \\ (m_2,d)=1}} \frac{m_2}{m_1} \hat{f}\left(m_2\left(l - \frac{\tilde{x}}{d}\right) + \frac{m_2}{M_3}\tau, m_2\tilde{x}\right)\right|^2 d\tau \\ &\lesssim (M_1 + M_3)\Sigma_1,\end{aligned}$$

where

$$\Sigma_1 := \sum_{\tilde{x} \bmod d} \sum_l \int_{|\tau|\lesssim 1} \Big| \sum_{\substack{m_2\sim M_2\\(m_2,d)=1}} m_2 \hat{f}\left(m_2\Big(l-\frac{\tilde{x}}{d}\Big)+\frac{m_2}{M_3}\tau, m_2\tilde{x}\right)\Big|^2 d\tau.$$

Since $\hat{f}(\xi,x)$ is negligible unless $|\xi|\lesssim (qT)^2$, we can introduce a bump function $\psi_2(x)$ supported on $|x|\lesssim 1$ to get

$$\Sigma_1 = \sum_{\tilde{x} \bmod d} \sum_l \psi_2\left(\frac{M_2(l-\tilde{x}/d)}{(qT)^2}\right) \int_{|\tau|\lesssim 1} \Big| \sum_{\substack{m_2\sim M_2\\(m_2,d)=1}} m_2 \hat{f}\left(m_2\Big(l-\frac{\tilde{x}}{d}\Big)+\frac{m_2}{M_3}\tau, m_2\tilde{x}\right)\Big|^2 d\tau + O((qT)^{-100}).$$

Expanding the square, we have

$$\begin{aligned}\Sigma_1 = &\sum_{\tilde{x} \bmod d} \sum_l \psi_2\left(\frac{M_2(l-\tilde{x}/d)}{(qT)^2}\right) \int_{|\tau|\lesssim 1} \int\int \sum_{\substack{m_2,m_2'\sim M_2\\(m_2,d)=(m_2',d)=1}} \sum_{y,y' \bmod d} m_2 m_2' f(u,y) f(u',y')\\ &\times e\left(\frac{\tilde{x}(m_2'y'-m_2y)}{d}\right) e\left((m_2'u'-m_2u)\Big(l-\frac{\tilde{x}}{d}\Big)\right) e\left((m_2'u'-m_2u)\Big(\frac{\tau}{M_3}\Big)\right) du du' d\tau\\ = &\sum_{\tilde{x} \bmod d} \int\int \sum_{y,y' \bmod d} \sum_{\substack{m_2,m_2'\sim M_2\\(m_2,d)=(m_2',d)=1}} m_2 m_2' f(u,y) f(u',y') e\left(\frac{\tilde{x}(m_2'y'-m_2y)}{d}\right) \Sigma_2\Sigma_3 du du'.\end{aligned}$$

where

$$\begin{aligned}\Sigma_2 &:= \int_{|\tau|\lesssim 1} e\left((m_2'u'-m_2u)\Big(\frac{\tau}{M_3}\Big)\right) d\tau\\ \Sigma_3 &:= \sum_l \psi_2\left(\frac{M_2(l-\tilde{x}/d)}{(qT)^2}\right) e\left((m_2'u'-m_2u)l\right) e\left((m_2u-m_2'u')\Big(\frac{\tilde{x}}{d}\Big)\right)\end{aligned}$$

On Σ_3, we apply Poisson summation to get

$$\begin{aligned}\Sigma_3 &\overset{\text{Poisson}}{=} \frac{(qT)^2}{M_2} \sum_k \hat{\psi}_2\left(\frac{k+m_2-m_2'u'}{M_2/(qT)^2}\right) e\left(-\frac{k\tilde{x}}{d}\right)\\ &= \frac{(qT)^2}{M_2} \sum_{\substack{k\\ |k+m_2u-m_2'u'|\lesssim M_2/(qT)^2}} \hat{\psi}_2\left(\frac{k+m_2u-m_2'u'}{M_2/T}\right) + O(T^{-100}),\end{aligned}$$

where we truncated in the last step by the rapid decay of $\hat{\psi}_2$. We can directly compute the remaining dependecies on $\tilde{x}$

$$\sum_{\tilde{x} \bmod d} e\left(\frac{\tilde{x}(m_2'y'-m_2y)}{d}\right) e\left(-\frac{k\tilde{x}}{d}\right) = d\mathbb{I}_{d|(m_2'y'-m_2y-k)}.$$

Finally, we trivially bound

$$|\Sigma_2| \lesssim 1.$$

This gives us

$$\Sigma_1 \lesssim d \int M_2(qT)^2 \sum_{y \bmod d} \sum_{\substack{m_2,m_2'\sim M_2\\(m_2,d)=(m_2',d)=1}} \sum_k f(u,y) \sum_{\substack{y' \bmod d\\ d|(m_2'y'-m_2y-k)}} \int_{|k+m_2u-m_2'u'|\lesssim M_2/(qT)^2} f(u',y') du' du.$$

The condition $d|(m_2'y'-m_2y-k)$ forces $y'=(m_2y+k)m_2'^{-1}$. For the second condition, we notice that u' is in a $\lesssim 1/(qT)^2$ neighborhood of $(m_2u+k)/m_2'$, so let $\phi(x)$ be a bump function supported on $|x|\lesssim 1$, and

$$\tilde{f}(u,y) := \int (qT)^2 \phi((qT)^2(u'-u)) f(u',y) du'$$

if needed, we extend the support of ϕ by a constant such that

$$\Sigma_1 \lesssim d \int M_2 \sum_{y \bmod d} \sum_{\substack{m_2, m_2' \sim M_2 \\ (m_2,d)=(m_2',d)=1}} \sum_k f(u,y) \tilde{f}\left(\frac{m_2 u + k}{m_2'}, (m_2 y + k) m_2'^{-1}\right) du.$$

Thus putting this bound for Σ_1 we obtain

$$\begin{aligned}\mathrm{Aii} + \mathrm{Bi} + \mathrm{Bii} \lesssim &(M_1 + M_3) d \int M_2 \sum_{y \bmod d} \sum_{\substack{m_2, m_2' \sim M_2 \\ (m_2,d)=(m_2',d)=1}} \sum_k f(u,y) \tilde{f}\left(\frac{m_2 y + k}{m_2'}, (m_2 y + k) m_2'^{-1}\right) du \\ \overset{\mathrm{CS}}{\ll} &M^2 d \left(\sum_{y \bmod d} \int f(u,y)^2 du\right)^{1/2} \left(\sum_{y \bmod d} \int \Big(\sum_{\substack{m_2, m_2' \sim M_2 \\ (m_2,d)=(m_2',d)=1}} \sum_k \tilde{f}\left(\frac{m_2 u + k}{m_2'}, (m_2 y + k) m_2'^{-1}\right)\Big)^2 du\right)^{1/2}\end{aligned}$$

Finally, noticing that $\tilde{f}$ has support on $y \asymp 1$ we can restrict the summation in k to be $|k| \ll M_2$, so that

$$\mathrm{Aii} + \mathrm{Bi} + \mathrm{Bii} \lesssim \mathrm{M}^2 \mathrm{d} \left(\sum_{\mathrm{y} \bmod \mathrm{d}} \int \mathrm{f}(\mathrm{u},\mathrm{y})^2 \mathrm{du}\right)^{1/2} \left(\mathrm{J}(\tilde{\mathrm{f}})\right)^{1/2}.$$

□

Proof of Lemma 4.17. Recall that

$$J(f) \leq \frac{1}{d} \sum_{x \bmod d} \int |\hat{g}(\xi, x)|^2 d\xi.$$

By the decay in $\hat{f}$, the range $|\xi| \geq T^2$ is negligble. The remaining parts are bounded by lemmas 4.18 and 4.19 to get

$$J(f) \lesssim \frac{1}{d} T^{3\eta} M^6 \left(\sum_y \int f(u,y) du\right)^2 + M^2 \left(\sum_{y \bmod d} \int f(u,y)^2 du\right)^{1/2} \left(J(\tilde{f})\right)^{1/2}.$$

Letting $\eta \to 0$ sufficiently slowly gives the result. □

Corollary 4.20. *Let f be as in Proposition 4.16, except f is now supported on $u \asymp L$, and $\tilde{M} = ML$. Then*

$$\begin{aligned} J_M(f) := &\sup_{\substack{0 < M_1, M_3 < \tilde{M}, \\ 0 < M_2 < M}} \sum_{y \bmod d} \int \left(\sum_{\substack{|m_1| \sim M_1, |m_2| \sim M_2, |m_3| \ll M_3 \\ (m_1,d)=(m_2,d)=1}} f\left(\frac{m_1 u + m_3}{m_2}, (m_1 y + m_3) m_2^{-1}\right)\right)^2 du \\ \lesssim &\frac{M^4 \tilde{M}^2}{d} \Big(\sum_{y \bmod d} \int f(u,y) du\Big)^2 + M^2 \tilde{M}^2 \sum_{y \bmod d} \int f(u,y)^2 du \\ = &\frac{\tilde{M}^6}{L^4 d} \Big(\sum_{y \bmod d} \int f(u,y) du\Big)^2 + \frac{\tilde{M}^4}{L^2} \sum_{y \bmod d} \int f(u,y)^2 du. \end{aligned}$$

Proof. By the same argument as above, we keep track of the M_2's to get

$$J_M(f) \lesssim \frac{M^4 \tilde{M}^2}{d} \Big(\sum_{y \bmod d} \int f(u,y) du\Big)^2 + \Big(M^2 \tilde{M}^2 \sum_{y \bmod d} \int f(u,y)^2 du\Big)^{1/2} (J_M(\tilde{f}))^{1/2},$$

where the summation in the iterative bound sums over $m_1, m_2 \sim M_2, |m_3| \ll LM_2 \leq \tilde{M}$. The bound on m_3 is due to the restriction that f is now supported on length $\ll L$. □

4.6 S_3 bound

Proposition 4.21 (S_3 bound)**.** *We have*

$$S_3 \lesssim (qT)^2|W|^{3/2} + N(qT)|W|^{1/2}E(W)^{1/2}.$$

Recall that

$$S_3 = \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in W\\(*)}} \frac{N^3}{q^3} \sum_{m\in(\mathbb{Z}\setminus\{0\})^3} \widehat{\chi_1\overline{\chi}_3}(m_1)\widehat{\chi_2\overline{\chi}_1}(m_2)\widehat{\chi_3\overline{\chi}_2}(m_3)\hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right).$$

Therefore, we define

$$I_m := \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in W}} \widehat{\chi_1\overline{\chi}_3}(m_1)\widehat{\chi_2\overline{\chi}_1}(m_2)\widehat{\chi_3\overline{\chi}_2}(m_3)\hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right).$$

By non-stationary phase, I_m is negligible for the terms $qT/N \lesssim |m|$, so (by relaxing the condition $(*)$ in the sum over pairs in W and exchanging the summations)

$$S_3 \le \sum_{0<|m_1|,|m_2|,|m_3|\lesssim qT/N} I_m + O((qT)^{-100}). \tag{4.2}$$

We first consider the triple summation in $\mathbb{Z}/q$. Define

$$A_{m,\chi_1,\chi_2,\chi_3} := \widehat{\chi_1\overline{\chi}_3}(m_1)\widehat{\chi_2\overline{\chi}_1}(m_2)\widehat{\chi_3\overline{\chi}_2}(m_3) = \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3)e\left(\frac{-x\cdot m}{q}\right),$$

which is the inner summation in I_m. Notice that $\chi(0)=0$, so we can define the summation in A to run over $(\mathbb{Z}/q\mathbb{Z})^\times$ to get

$$A_{m,\chi_1,\chi_2,\chi_3} = \sum_{x\in((\mathbb{Z}/q\mathbb{Z})^\times)^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3)e\left(\frac{-x\cdot m}{q}\right)$$

We now make the substitution $y_1 = x_1x_3^{-1}, y_2 = x_2x_3^{-1} \bmod q$ for the summation over x. We thus rewrite the sum over x as

$$\begin{aligned}&\sum_{y_1,y_2,x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi_1(y_1y_2^{-1})\chi_2(y_2)\chi_3(y_1^{-1})e\left(\frac{-(y_1m_1+y_2m_2+m_3)x_3}{q}\right)\\ &= \sum_{y_1,y_2\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi_1(y_1)\bar{\chi}_1(y_2)\chi_2(y_2)\bar{\chi}_3(y_1) \sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{-(y_1m_1+y_2m_2+m_3)x_3}{q}\right)\end{aligned}$$

Now we can apply Möbius inversion on the summation in x_3 to get

$$\begin{aligned}\sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{-(y_1m_1+y_2m_2+m_3)x_3}{q}\right) &= \sum_{q_0|q}\mu\left(\frac{q}{q_0}\right)\sum_{\tilde{x}_3\in\mathbb{Z}/q_0} e\left(\frac{-(y_1m_1+y_2m_2+m_3)\tilde{x}_3}{q_0}\right)\\ &= \sum_{q_0|q}\mu\left(\frac{q}{q_0}\right)q_0\mathbb{I}_{q_0|(y_1m_1+y_2m_2+m_3)}.\end{aligned}$$

With this in mind, we break $I_m = \sum_{q_0|q}\mu(q/q_0)I_{m,q_0}$,

$$\begin{aligned}I_{m,q_0} :=& \frac{N^3q_0}{q^2} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in W}} \sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z}\\ y_1m_1+y_2m_2+m_3\equiv 0 \bmod q_0}} \chi_1(y_1)\bar{\chi}_1(y_2)\chi_2(y_2)\bar{\chi}_3(y_1)\\ &\times \hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right).\end{aligned}$$

Proposition 4.22 (Divisor reduction of S_3)**.** *We have*

$$S_3 \lesssim \sup_{q_0|q} \sum_{\substack{0<|m_1|,|m_2|,\\ |m_3|\lesssim qT/N}} I_{m,q_0}.$$

Proof. We split the sum in I_m to be across I_{m,q_0} for each divisor of q. The divisor function grows slower than any power of q thus slower than any $(qT)^\epsilon$, so gives the proposition. ☐

We then consider the subproduct in I_{m,q_0}:

$$\hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right).$$

Expanding the Fourier transform as an integral, this expression equals

$$\begin{aligned}&\int_{\mathbb{R}^3}\tilde{\omega}(\mathbf{u})u_1^{i(t_1-t_3)}u_2^{i(t_2-t_1)}u_3^{i(t_3-t_2)}e\left(\frac{-N\mathbf{m}\cdot\mathbf{u}}{q}\right)d\mathbf{u}\\ &=\int_{\mathbb{R}^3}\tilde{\omega}(\mathbf{u})\left(\frac{u_1}{u_2}\right)^{it_1}\left(\frac{u_2}{u_3}\right)^{it_2}\left(\frac{u_3}{u_1}\right)^{it_3}e\left(\frac{-N\mathbf{m}\cdot\mathbf{u}}{q}\right)d\mathbf{u}\end{aligned}$$

where $\tilde{\omega}(u_1,u_2,u_3) \coloneqq \omega(u_1)^2\omega(u_2)^2\omega(u_3)^2$ is compactly supported. The observation is that the choice of u_1/u_2 and u_2/u_3 fixes u_3/u_1, so this triple integral can be rewritten in two variables. We change variables $v_1 = u_1/u_3, v_2 = u_2/u_3$ for the integral, which is well defined on the support of $\tilde{\omega}$. This gives us a Jacobian of u_3^2 and equals

$$\begin{aligned}&\int_{\mathbb{R}^3}\tilde{\omega}(v_1u_3,v_2u_3,u_3)\left(\frac{v_1}{v_2}\right)^{it_1}(v_2)^{it_2}\left(\frac{1}{v_1}\right)^{it_3}u_3^2\, e\left(\frac{-N(v_1m_1+v_2m_2+m_3)u_3}{q}\right)\,dv_1\,dv_2\,du_3\\ &=\int_{\mathbb{R}^2}\int_{\mathbb{R}}u_3^2\,\tilde{\omega}(v_1u_3,v_2u_3,u_3)e\left(\frac{-N(v_1m_1+v_2m_2+m_3)u_3}{q}\right)du_3\left(\frac{v_1}{v_2}\right)^{it_1}(v_2)^{it_2}\left(\frac{1}{v_1}\right)^{it_3}\,dv_1\,dv_2.\end{aligned}\tag{4.3}$$

Let ψ be a bump function with support in $v \asymp 1$. We define

$$\begin{aligned}R(v,n_1,n_2) &\coloneqq \psi(v)\sum_{(t,\chi)\in W}\chi(n_1)\bar{\chi}(n_2)v^{it},\\ R(v,n) &\coloneqq R(v,n,1).\end{aligned}$$

Therefore,

$$I_{m,q_0} = \frac{N^3q_0}{q^3}\sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z}\\ y_1m_1+y_2m_2+m_3\equiv 0 \bmod q_0}}\int_{\mathbb{R}^2}\tilde{I}_{u_3}(v_1,v_2)R\left(\frac{v_1}{v_2},y_1,y_2\right)R(v_2,y_2)R\left(\frac{1}{v_1},y_1^{-1}\right)\,dv_1\,dv_2,$$

where

$$\tilde{I}_{u_3}(v_1,v_2) \coloneqq \tilde{I}_{u_3}(v_1,v_2,m) \coloneqq \int_{\mathbb{R}}u_3^2\,\tilde{\omega}(v_1u_3,v_2u_3,u_3)e\left(\frac{-N(v_1m_1+v_2m_2+m_3)u_3}{q}\right)du_3.$$

The innermost integral $\tilde{I}_{u_3}(v_1,v_2)$ has cancellation property. By the principle of non-stationary phase through repeated integration by parts, this integral is $O_{\epsilon,A}((qT)^{-A})$ for any $|v_1m_1+v_2m_2+m_3| > (qT)^\epsilon q/N$. Therefore, we can truncate the domain of the integrals in v_1 and v_2 to $|v_1m_1+v_2m_2+m_3| \lesssim q/N$ with negligible error. On this domain, the innermost integral in u_3 is $O(1)$ by the trivial bound. Moreover, by the compact support of $\tilde{\omega}$ on $[1,2]\times[1,2]\times[1,2]$ the integrand of innermost integral is non-zero only if

$$v_1u_3, v_2u_3, u_3 \sim N.$$

Importantly, this requires $1/2 \le v_1, v_2 \le 2$, so we can further restrict the outermost integrals to this region. Let $d_1 \coloneqq \gcd(m_1,q_0), d_2 \coloneqq \gcd(m_2,q_0), d_3 \coloneqq \gcd(m_3,q_0)$.

Proposition 4.23 (Simplification of I_{m,q_0} domain)**.**

$$|I_{m,q_0}| \ll \frac{N^3 q_0}{q^3} \sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z} \\ y_1m_1+y_2m_2+m_3\equiv 0 \mod q_0}} \int_{\substack{|v_1m_1+v_2m_2+m_3|\lesssim \frac{q}{N} \\ \frac{1}{2}\le v_1,v_2\le 2}} R\Big(\frac{v_1}{v_2}, y_1, y_2\Big) R(v_2,y_2) R\Big(\frac{1}{v_1}, y_1^{-1}\Big)\, dv_1\, dv_2 + O_\epsilon((qT)^{-100}).$$

Furthermore, $I_{m,q_0} = 0$ *unless* $\gcd(d_1,d_2) = \gcd(d_2,d_3) = \gcd(d_3,d_1)$.

Proof. It suffices to show the second statement, as the first statement is the result of our work so far. Suppose $|I_{m,q_0}| \neq 0$. Then the summation condition $y_1m_1 + y_2m_2 + m_3 \equiv 0 \bmod q_0$ is fulfilled for some choice of y_1, y_2 which are coprime to q_0, as the characters vanish when y_1, y_2 are not coprime. We want to show that there exists a value d such that $d_1/d, d_2/d, d_3/d$ are mutually coprime. We proceed by contradiction. Let $d = \gcd(d_1, d_2, d_3)$, and suppose without lost of generality (because y_1, y_2 are coprime to q_0) that $(d_1/d, d_2/d) = k \neq 1$. Naturally $k|(q_0/d)$ Then for some choice of y_1, y_2, α, we have

$$y_1m_1 + y_2m_2 = -m_3 + \alpha q_0 \implies y_1\Big(\frac{m_1}{d}\Big) + y_2\Big(\frac{m_1}{d}\Big) = -\frac{m_3}{d} + \frac{q_0}{d} \implies (m_3/d) \equiv 0 \bmod k.$$

□

We now further consider the domain of integration for v_1, v_2. It can be written as

$$|v_1m_1 + v_2m_2 + m_3| \lesssim \frac{q}{N} \implies \left|v_2 - \frac{v_1m_1 + m_3}{-m_2}\right| \lesssim \frac{q}{|m_2|N}.$$

Thus, if we fix v_1, the integration in v_2 is in a $q/(|m_2|N)$-small neighborhood of $\frac{v_1m_1+m_3}{-m_2}$. In principle, we can estimate the value of the integral by evaluating at $v_2 = v_1m_1 + m_3/-m_2$ and multiplying it by $q/(|m_2|N)$. This is made precise by splitting the sum across m_2 into dyadic interals $\sim M_2$, and smoothing over ranges of $q/(M_2N)$.

Let $\tilde{\phi}$ be a smooth bump function such that equals $\tilde{\phi} = 1$ on $|x| \lesssim 1$ and is supported in $|x| \lesssim 1$ (with a larger constant), so that $\|\tilde{\phi}^{(j)} \lesssim_j 1\|$ for all j. We define

$$\tilde{R}_M(v, y_1, y_2) := \left(\int \frac{NM}{q}\tilde{\phi}\left(\frac{NM}{q}(v - v')\right)\tilde{\phi}(v)\,|R(v', y_1, y_2)|^2 dv'\right)^{1/2}.$$

Proposition 4.24. *Let*

$$\tilde{I}_{m,M_2,y_1,y_2,q_0} := \tilde{I}_{m,q_0} := \int_{v_1\asymp 1}\left|\tilde{R}_{M_2}\left(\frac{m_1v_1 + m_3}{-m_2v_1}, y_2, y_1\right)\tilde{R}_{M_2}\left(\frac{m_1v_1 + m_3}{-m_2}, y_2\right) R(v_1, y_1)\right| dv_1.$$

There is a choice of $q_0|q$ *and* $0 < M_1 \le M_3 \le M_2 \lesssim qT/N$ *such that*

$$S_3 \lesssim \frac{N^2 q_0}{M_2 q^2} \sum_{\substack{|m_1|\sim M_1, \\ |m_2|\sim M_2, \\ |m_3|\sim M_3}} \sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z} \\ y_1m_1+y_2m_2+m_3\equiv 0 \\ \mod q_0}} \tilde{I}_{m,q_0} + O(T^{-100}).$$

Proof. By Proposition 4.22 and Proposition 4.23, we expand the sum over m_1, m_2, m_3 dyadically. Thus for some value $M_1, M_2, M_3 \lesssim qT/N$ and for the value of q_0 that achieves supremum.

$$S_3 \lesssim \sum_{\substack{|m_1|\sim M_1, \\ |m_2|\sim M_2, \\ |m_3|\sim M_3}} |I_{m,q_0}| + O(T^{-100}).$$

By symmetry in the arguments, we can consider the case when $M_1 \le M_3 \le M_2$. We now consider each I_{m,q_0} where $|m_2| \sim M_2$,

$$\int_{\substack{|v_1m_1+v_2m_2+m_3|\lesssim \frac{q}{N} \\ \frac{1}{2}\le v_1,v_2\le 2}} \left|R\left(\frac{v_2}{v_1}, y_2, y_1\right) R(v_2, y_2) R\left(\frac{1}{v_1}, y_1^{-1}\right)\right| dv_1\, dv_2$$

$$
\begin{aligned}
&= \int_{\substack{|v_1 m_1 + v_2 m_2 + m_3| \lesssim \frac{q}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} \left| R\left(\frac{v_2}{v_1}, y_2, y_1\right) R(v_2, y_2) R(v_1, y_1) \right| dv_1 \, dv_2 \\
&\ll \int_{v_1 \asymp 1} |R(v_1, y_1)| \int_{\left|v_2 - \frac{v_1 m_1 + m_3}{-m_2}\right| \lesssim \frac{q}{|m_2| N}} \left| R\left(\frac{v_2}{v_1}, y_2, y_1\right) R(v_2, y_2) \right| dv_2 \, dv_1 \\
&\ll \int_{v_1 \asymp 1} |R(v_1, y_1)| \int_{\left|v_2 - \frac{v_1 m_1 + m_3}{-m_2}\right| \lesssim \frac{q}{M_2 N}} \left| R\left(\frac{v_2}{v_1}, y_2, y_1\right) R(v_2, y_2) \right| dv_2 \, dv_1.
\end{aligned}
$$

The inner integral, by Cauchy-Schwarz, is

$$
\begin{aligned}
&\le \left(\int_{\left|v_2 - \frac{v_1 m_1 + m_3}{-m_2}\right| \lesssim \frac{q}{M_2 N}} \left| R\left(\frac{v_2}{v_1}, y_2, y_1\right) \right|^2 dv_2 \right)^{1/2} \left(\int_{\left|v_2 - \frac{v_1 m_1 + m_3}{-m_2}\right| \lesssim \frac{q}{M_2 N}} |R(v_2, y_2)|^2 \, dv_2 \right)^{1/2} \\
&\ll \frac{q}{M_2 N} \tilde{R}_{M_2}\left(\frac{v_1 m_1 + m_3}{-m_2 v_1}, y_2, y_1\right) \tilde{R}_{M_2}\left(\frac{v_1 m_1 + m_3}{-m_2}, y_2\right)
\end{aligned}
$$

where in the last step, we used $v_1 \asymp 1$, so the Jacobian from the change of variables is $\asymp 1$. Moreover, since we have $M_1, M_3 \le M_2$, we can restrict $\tilde{R}_{M_2}$ to be supported in the range where the first argument is $\ll 1$. Thus, for $|m_2| \sim M$,

$$
|I_{m,q_0}| \lesssim \frac{N^3 q_0}{q^3} \frac{q}{N M_2} \sum_{\substack{y_1, y_2 \in \mathbb{Z}/q\mathbb{Z} \\ y_1 m_1 + y_2 m_2 + m_3 \equiv 0 \bmod q_0}} \tilde{I}_{m,q_0}.
$$

The proposition follows from this claim. □

To apply Hölder's inequality, we need to find bounds on the second and fourth moments of $R(v, y_1, y_2)$. This is analogous to the second and fourth moments of R in Guth and Maynard's proof.

Lemma 4.25 (Second and fourth moments of R)**.** *Let $W = \{(t_j, \chi_j)\}$, such that χ_j is a character mod q, and the t's are contained in an interval of length T, and are T^ϵ-separated for the same character. Then uniformly in $y_2 \in \mathbb{Z}/q\mathbb{Z}$,*

$$
\sum_{y_1 \in (\mathbb{Z}/q\mathbb{Z})} \int_{|v| \ll 1} |R(v, y_1, y_2)|^2 \, dv \ll_\epsilon \phi(q) |W|,
$$

and

$$
\sum_{y_1 \in (\mathbb{Z}/q\mathbb{Z})} \int_{|v| \ll 1} |R(v, y_1, y_2)|^4 \, dv \lesssim \phi(q) E(W).
$$

Proof. For the second moment, we have

$$
|R(v, y_1, y_2)|^2 = \sum_{(t_1, \chi_1), (t_2, \chi_2) \in W} \chi_1 \bar{\chi}_2(y_1) \bar{\chi}_1 \chi_2(y_2) v^{i(t_1 - t_2)}.
$$

By the orthogonality of characters,

$$
\sum_{y_1 \in (\mathbb{Z}/q\mathbb{Z})} |R(v, y_1, y_2)|^2 = \phi(q) \sum_{(t_1, \chi_1), (t_2, \chi_2) \in W} \delta_{\chi_1 \chi_2} \bar{\chi}_1 \chi_2(y_2) v^{i(t_1 - t_2)} = \phi(q) \sum_{(t_1, \chi_1), (t_2, \chi_2) \in W} \delta_{\chi_1 \chi_2} \mathbb{I}_{(y_2, q) = 1} v^{i(t_1 - t_2)},
$$

so it is enough to consider the second moment of $R(v, y, 1)$ (and y_2 is coprime to q). Let ψ be a bump function supported on $|v| \ll 1$ and equals 1 on the domain of integration in the lemma. Then,

$$
\sum_{y \in (\mathbb{Z}/q\mathbb{Z})^\times} \int_{|v| \ll 1} |R(v, y)|^2 \, dv \le \sum_{y \in (\mathbb{Z}/q\mathbb{Z})^\times} \int \psi(v) |R(v, y)|^2 \, dv
$$

$$=\phi(q)\int \psi(v) \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W\\ \chi_1=\chi_2}} v^{i(t_1-t_2)}dv$$
$$=\phi(q)\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W\\ \chi_1=\chi_2}}\int \psi(v)v^{i(t_1-t_2)}dv.$$

In the sum, the terms $t_1=t_2$ contribute $O(|W|)$. If $t_1\neq t_2$, then $|t_1-t_2|\geq (qT)^\epsilon$. The integral in this case is $O_\epsilon((qT)^{-100})$ and is negligible.

Similarly for the fourth moment, it is enough to consider $R(v,y)$. We have

$$|R(v,y)|^4=\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\ (t_3,\chi_3),(t_4,\chi_4)\in W}}\chi_1\chi_2\bar{\chi_3}\bar{\chi_4}(y)v^{i(t_1+t_2-t_3-t_4)}.$$

So again by the orthogonality of characters,

$$\sum_{y\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v\asymp 1}|R(v,y)|^4\,dv=\phi(q)\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\ (t_3,\chi_3),(t_4,\chi_4)\in W\\ \chi_1\chi_2=\chi_3\chi_4}}\int_{v\asymp 1}v^{i(t_1+t_2-t_3-t_4)}dv.$$

Similar to the previous proof, we can introduce the bump function ψ for the integral, and restrict the summation to the terms $|t_1+t_2-t_3-t_4|\leq (qT)^\epsilon$ with an error of $O_\epsilon((qT)^{-100})$. The remaining terms in the summation contribute $O(E(W))$. ☐

Lemma 4.26. *Let $E(W)=\#\{(t_1,\chi_1),(t_2,\chi_2),(t_3,\chi_3),(t_4,\chi_4)\in W:|t_1+t_2-t_3-t_4|\leq 1,\chi_1\chi_2=\chi_3\chi_4\}$. Then*

$$\sum_{y_1\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v\asymp 1}\left|\tilde{R}_M(v,y_1,y_2)\right|^4 dv\lesssim\phi(q)E(W).$$

Proof. It is enough to show the case where $y_2=1$. We apply Cauchy-Schwarz to

$$\int_{|v|\ll 1}\left|\tilde{R}_M(v,y)\right|^4 dv\lesssim\int_{|v|\ll 1}\left(\int_{|u-v|\lesssim q/NM}\frac{NM}{q}|R(u,y)|^2du\right)^2dv$$
$$\overset{\text{CS}}{\lesssim}\frac{NM}{q}\int_{|v|\ll 1}\int_{|u-v|\lesssim q/NM}|R(u,y)|^4du\ dv$$
$$\lesssim\int_{|u|\ll 1}|R(u,y)|^4du.$$

Lemma 4.25 completes the proof. ☐

We repeat a similar process in the $\bmod q$ summations. That is, we want to express y_2 in terms of y_1. We now set $d=\gcd(d_1,d_2,d_3)$, and set $k_i:=m_i/d$, $\tilde{d}_i:=d_i/d$

$$y_1m_1+y_2m_2+m_3\equiv 0 \bmod q_0\implies y_1k_1+y_2k_2+k_3\equiv 0\bmod\frac{q_0}{d}.$$

We can solve the last equation in y_2 in terms of y_1 as

$$y_2\quad\begin{cases}\equiv\left[\frac{y_1k_1+k_3}{-k_2}\right]_{q_0/d_2}, & \text{if } \tilde{d}_2|y_1k_1+k_3,\\ \text{has no solutions}, & \text{otherwise.}\end{cases}$$

One caveat is that $k_2/\tilde{d}_2$ might not be coprime to q_0/d, for instance if $k_1=k_3=1,k_2=q^2$ and $q_0=q$. Therefore, we cannot naively evaluate $\left[\frac{y_1k_1+k_3}{-k_2}\right]_{q_0/d}$ to find the class of $\left[\frac{y_1k_1+k_3}{-k_2}\right]_{q_0/d_2}$. However, we can isolate the non-coprime parts by setting $k_2=\tilde{k}_2\tilde{d}_2\omega$, where $\tilde{k}_2$ is coprime to q_0/d. Then the solution set for y_2 for when $\tilde{d}_2|y_1k_1+k_3$ is

$$\left[\frac{y_1k_1+k_3}{-\tilde{k}_2\tilde{d}_2}\right]_{q_0/d_2}\left[\frac{1}{\omega}\right]_{q_0/d_2}+\alpha\frac{q_0}{d_2}$$

for integer α. If we set y_2 mod q then we can pick $0 \le \alpha < qd_2/q_0$.

We are now ready to state the new definition for the smoothed function $\tilde{R}$, which intuitively sums up the solutions in y_2.

Proposition 4.27 (Extension of Proposition 4.24)**.** *Let*

$$\tilde{R}_M^{(q_0,d,d_2,\omega_2)}(u,y) := \tilde{R}_M^{(d)}(u,y) := \mathbb{I}_{\tilde{d}_2|y}\left(\sum_{l=0}^{(qd_2/q_0)-1} \frac{q_0}{qd_2}\left|\tilde{R}_M\left(u,\left[\frac{y}{\tilde{d}_2}\right]_{q_0/d_2}\left[\frac{1}{\omega_2}\right]_{q_0/d_2}+l\frac{q_0}{d_2}\right)\right|^2\right)^{1/2}.$$

Then the following hold:

1. *$\tilde{R}_M^{(d)}$ is q_0/d-periodic in the second argument.*
2. *$\sum_{a \bmod q_0/d}\int_{u\asymp 1}|\tilde{R}_M^{(d)}(u,a)|^2du \lesssim \frac{q_0}{d_2}|W|$.*
3. *$\sum_{a \bmod q_0/d}\int_{u\asymp 1}|\tilde{R}_M^{(d)}(u,a)|^4du \lesssim \frac{q_0}{d_2}E(W)$.*

Moreover, there is a choice of $d_1,d_2,d_3|q_0, q_0|q$, $0 < M_1,M_2,M_3 \lesssim qT/N$ such that

$$S_3 \lesssim \frac{N^2d_2}{M_2q}\sum_{\substack{\omega_1,\omega_2,\omega_3\\(**)}}\sum_{\substack{|m_1|\sim M_1,\\|m_2|\sim M_2,\\|m_3|\sim M_3\\(***)}}\sum_{\substack{y_1 \bmod q\\(y_1,q)=1}}\int_{v_1\asymp 1}\tilde{R}_{M_2}^{(d)}\left(\frac{k_1v_1+k_3}{-k_2v_1},\frac{k_1y_1+k_3}{-\tilde{k}_2y_1}\right)\tilde{R}_{M_2}^{(d)}\left(\frac{k_1v_1+k_3}{-k_2},\frac{k_1y_1+k_3}{-\tilde{k}_2}\right)|R(v_1,y_1)|dv_1,$$

where the summation conditions are$(**)$ on ω_i's are that $\omega_i < qT/N$ only has prime factors that are also prime factors of d_i, $(***)$ on m_i's are that $m_i = d_i\tilde{k}_i\omega_i$ where $(\tilde{k}_i,q_0)=1$.

Remark: As a recall, we defined $d := \gcd(d_1,d_2,d_3)$, $k_i := m_i/d$, $\tilde{d}_i = \gcd(k_i,q_0/d)$, $\tilde{k}_i = k_i/(\tilde{d}_i\omega_i) = m_i/(d_i\omega_i)$.

Proof. The first statement is evident from construction. We have $[(y+q_0/d)/\tilde{d}_2] \equiv [y/\tilde{d}_2+q_0/d_2] \equiv [y/\tilde{d}_2]$.

For the second statement, we have

$$\begin{aligned}\sum_{a \bmod q_0/d}\int_{u\asymp 1}|\tilde{R}_M^{(d)}(u,a)|^2du &= \sum_{a \bmod q_0/d}\int_{u\asymp 1}\mathbb{I}_{\tilde{d}_2|y}\sum_{l=0}^{(qd_2/q_0)-1}\frac{q_0}{qd_2}\left|\tilde{R}_M\left(u,\left[\frac{a}{\tilde{d}_2}\right]_{q_0/d_2}\left[\frac{1}{\omega_2}\right]_{q_0/d_2}+l\frac{q_0}{d_2}\right)\right|^2\\ &= \sum_{a \bmod q_0/d_2}\int_{u\asymp 1}\sum_{l=0}^{(qd_2/q_0)-1}\frac{q_0}{qd_2}\left|\tilde{R}_M\left(u,[a]_{q_0/d_2}\left[\frac{1}{\omega_2}\right]_{q_0/d_2}+l\frac{q_0}{d_2}\right)\right|^2\\ &= \sum_{b \bmod q}\int_{u\asymp 1}\frac{q_0}{qd_2}\left|\tilde{R}_M\left(u,b\right)\right|^2\\ &\lesssim \frac{q_0}{d_2}|W|.\end{aligned}$$

Similarly, the third statement follows from

$$\begin{aligned}\sum_{a \bmod q_0/d}\int_{u\asymp 1}|\tilde{R}_M^{(d)}(u,a)|^4 &= \sum_{a \bmod q_0/d}\int_{u\asymp 1}\mathbb{I}_{\tilde{d}_2|y}\left(\sum_{l=0}^{(qd_2/q_0)-1}\frac{q_0}{qd_2}\left|\tilde{R}_M\left(u,\left[\frac{a}{\tilde{d}_2}\right]_{q_0/d_2}\left[\frac{1}{\omega_2}\right]_{q_0/d_2}+l\frac{q_0}{d_2}\right)\right|^2\right)^2\\ &\overset{\text{CS}}{\le} \sum_{a \bmod q_0/d}\int_{u\asymp 1}\mathbb{I}_{\tilde{d}_2|y}\sum_{l=0}^{(qd_2/q_0)-1}\frac{q_0}{qd_2}\left|\tilde{R}_M\left(u,\left[\frac{a}{\tilde{d}_2}\right]_{q_0/d_2}\left[\frac{1}{\omega_2}\right]_{q_0/d_2}+l\frac{q_0}{d_2}\right)\right|^4\\ &\lesssim \frac{q_0}{d_2}E(W).\end{aligned}$$

Finally, from Proposition 4.24, we choose $q_0|q$, $0 < M_1 \le M_3 \le M_2 = M \lesssim qT/N$ such that

$$S_3 \lesssim \frac{N^2 q_0}{Mq^2} \sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3}} \sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z}\\ y_1m_1+y_2m_2+m_3\equiv 0\\ \bmod q_0}} \tilde{I}_{m,q_0} + O(T^{-100}).$$

We split the summation in m_1, m_2, m_3 according to the gcd's d_1, d_2, d_3. By the divisor bound, we have for a choice of $d_1, d_2, d_3|q_0$,

$$S_3 \lesssim \frac{N^2 q_0}{Mq^2} \sum_{\substack{|m_1|\sim M_1, |m_2|\sim M_2, |m_3|\sim M_3\\ (m_1,q_0)=d_1,(m_2,q_0)=d_2,(m_3,q_0)=d_3}} \sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z}\\ y_1m_1+y_2m_2+m_3\equiv 0 \bmod q_0/d}} \tilde{I}_{m,q_0} + O(T^{-100}).$$

In the summation of m_i's we split the m_i's according to ω_i's. Such that we can write $m_i = d_i\tilde{k}_i\omega_i$ with $\tilde{k}_i$ coprime to q_0. Therefore we have

$$S_3 \lesssim \frac{N^2 q_0}{Mq^2} \sum_{\substack{\omega_1,\omega_2,\omega_3\\ (**)}} \sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3\\ (***)}} \sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z}\\ y_1m_1+y_2m_2+m_3\equiv 0 \bmod q_0/d}} \tilde{I}_{m,q_0} + O(T^{-100}),$$

where the conditions $(**)$ on ω_i's are that $\omega_i < qT/N$ only has prime factors that are also prime factors of d_i, $(***)$ on m_i's are that $m_i = d_i\tilde{k}_i\omega_i$ where $(\tilde{k}_i, q_0) = 1$. Recall that

$$\tilde{I}_{m,q_0} := \int_{v_1\asymp 1} \left| \tilde{R}_{M_2}\left(\frac{m_1v_1+m_3}{-m_2v_1}, y_2, y_1\right) \tilde{R}_{M_2}\left(\frac{m_1v_1+m_3}{-m_2}, y_2\right) R(v_1,y_1) \right| dv_1.$$

We now rewrite the summation in y, using $m = dk$, as

$$\begin{aligned}\sum_{\substack{y_1,y_2\in\mathbb{Z}/q\mathbb{Z}\\ y_1k_1+y_2k_2+k_3\equiv 0 \bmod q_0/d}} \tilde{I}_{dk,q_0} &= \sum_{y_1 \bmod q} \sum_{\substack{y_2 \bmod q,\\ y_2\equiv y_1k_1+y_2k_2+k_3\equiv 0 \bmod q_0/d}} \tilde{I}_{dk,q_0}\\ &= \sum_{y_1 \bmod q} \sum_{\substack{y_2 \bmod q,\\ y_1k_1+y_2k_2+k_3\equiv 0 \bmod q_0/d}} \int_{v_1\asymp 1} \Bigg| \tilde{R}_{M_2}\left(\frac{dk_1v_1+dk_3}{-dk_2v_1}, y_2, y_1\right)\\ &\qquad \tilde{R}_{M_2}\left(\frac{dk_1v_1+dk_3}{-dk_2}, y_2\right) R(v_1,y_1) \Bigg| dv_1\end{aligned}$$

Keeping in mind that R only has support in y_1, y_2 coprime to q, we apply Cauchy-schwarz to the summation in y_2 to get (and simplifying notation by setting $M = M_2$)

$$\begin{aligned}&\sum_{\substack{y_2 \bmod q,\\ y_1k_1+y_2k_2+k_3\equiv 0 \bmod q_0/d}} \left| \tilde{R}_M\left(\frac{dk_1v_1+dk_3}{-dk_2v_1}, [y_2y_1^{-1}]_q\right) \tilde{R}_M\left(\frac{dk_1v_1+dk_3}{-dk_2}, [y_2]_q\right) \right|\\ \le&\left(\sum_{\substack{y_2 \bmod q,\\ y_1k_1+y_2k_2+k_3\equiv 0 \bmod q_0/d}} \left| \tilde{R}_M\left(\frac{k_1v_1+k_3}{-k_2v_1}, [y_2y_1^{-1}]_q\right) \right|^2 \right)^{1/2} \left(\sum_{\substack{y_2 \bmod q,\\ y_1k_1+y_2k_2+k_3\equiv 0 \bmod q_0/d}} \left| \tilde{R}_M\left(\frac{k_1v_1+k_3}{-k_2}, [y_2]_q\right) \right|^2 \right)^{1/2}\\ =&\frac{qd_2}{q_0} \tilde{R}_M^{(d)}\Big(\frac{k_1v_1+k_3}{-k_2v_1}, \frac{k_1y_1+k_3}{-\tilde{k}_2y_1}\Big) \tilde{R}_M^{(d)}\Big(\frac{k_1v_1+k_3}{-k_2}, \frac{k_1y_1+k_3}{-\tilde{k}_2}\Big),\end{aligned}$$

Combined, we have

$$S_3 \lesssim \frac{N^2 d_2}{Mq} \sum_{\substack{\omega_1,\omega_2,\omega_3\\ (**)}} \sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3\\ (***)}} \sum_{\substack{y_1 \bmod q\\ (y_1,q)=1}} \int_{v_1\asymp 1} \tilde{R}_M^{(d)}\Big(\frac{k_1v_1+k_3}{-k_2v_1}, \frac{k_1y_1+k_3}{-\tilde{k}_2y_1}\Big) \tilde{R}_M^{(d)}\Big(\frac{k_1v_1+k_3}{-k_2}, \frac{k_1y_1+k_3}{-\tilde{k}_2}\Big) |R(v_1,y_1)| dv_1$$

□

Corollary 4.28. *If $\omega(q) = o(\log q/\log\log q)$, there is a choice of $d_1, d_2, d_3|q_0, q_0|q$, $0 < M_1 \leq M_3 \leq M_2 = M \lesssim qT/N$, $\omega_1, \omega_2, \omega_3$ such that*

$$S_3 \lesssim \frac{N^2 d_2}{Mq} \sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3\\ (***)}} \sum_{\substack{y_1 \; mod \; q\\ (y_1,q)=1}} \int_{v_1 \asymp 1} \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2 v_1}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2 y_1}\Big) \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2}\Big) |R(v_1, y_1)| dv_1,$$

where each $\omega_i < qT/N$ consists of only factors that are also prime factors of d_i, and the summation condition $(***)$ on m_i's are that $m_i = d_i \tilde{k}_i \omega_i$ where $(\tilde{k}_i, q_0) = 1$.

Proof. It suffices to show that

$$\Sigma := \sum_{\omega_1 (**)} 1 \lesssim 1.$$

The conditions on ω_2, ω_3 are the same. Since ω_1 consists of only prime factors of d_i, the number of choices of ω_1 is bounded above by

$$\prod_{p|d_1, p \leq qT/N} \log_p \frac{qT}{N} \leq \prod_{p|q, p\leq qT/N} \log_p \frac{qT}{N} \ll \prod_{p|q} \log \frac{qT}{N}.$$

Therefore we have

$$\Sigma \ll (\log qT)^{\omega(q)} = \exp(\omega(q) \times \log\log qT) \ll_\epsilon \exp(\epsilon \log qT)$$

for any value of ϵ. □

With this bound, we can prove the statement for almost all q with a small number of distinct prime factors. This is because the Erdös Kac theorem implies almost all numbers q satisfy $\omega(q) \asymp \log\log q$. With some careful handling, we can prove this bound for all q.

Proposition 4.29. *There is a choice of $d_1, d_2, d_3|q_0, q_0|q$, $0 < M_1 \leq M_3 \leq M_2 = M \lesssim qT/N$, $\omega_1, \omega_2, \omega_3$ such that*

$$S_3 \lesssim \frac{N^2 d_2}{Mq} \sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3\\ (***)}} \sum_{\substack{y_1 \; mod \; q\\ (y_1,q)=1}} \int_{v_1 \asymp 1} \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2 v_1}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2 y_1}\Big) \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2}\Big) |R(v_1, y_1)| dv_1,$$

where each $\omega_i < qT/N$ consists of only factors that are also prime factors of d_i, and the summation condition $(***)$ on m_i's are that $m_i = d_i \tilde{k}_i \omega_i$ where $(\tilde{k}_i, q_0) = 1$.

Proof. Similar to the above corollary, it suffices to show that

$$\Sigma \lesssim 1.$$

We shall show that $\log \Sigma = o(\log qT)$, thus Σ grows slower than any power of qT.

Fix d_1, and let $p_1, p_2, \ldots, p_k$ the prime factors of d_1, not counting multiplicity. Let $s = d_1\omega_1$. Since ω_1 only consists of prime factors that are prime factors of d_1, and $d_1|\omega_1$, the number of s is equal to the number of positive integer lattice points $(x_1, \ldots, x_k)$ that satisfy

$$\sum_{i=1}^{k} x_i \log p_k \leq \log M,$$

recalling that $M \leq qT/N = (qT)^{1/6}$. Without loss of generality, we can also assume that p_i is the i-th smallest prime, as this leads to the upper bound. Moreover, since the product of the first k primes is of order $\exp((1 + o(1))k \log k)$, we have

$$(1 + o(1))k \log k \leq \log q$$

and

$$p_k = (1 + o(1))k \log k.$$

We now bound the number of lattice points. Notice that each positive integer lattice point that satisfies the inequality can be identified uniquely with a unit cube completely contained inside the simplex

$$\Delta = \{\sum_{i=1}^{k} x_i \log p_i \leq \log M, x_i \geq 0\},$$

and the interiors of each of the cubes are pairwise disjoint. Thus, the number of ω_1 is bounded above by the volume of the simplex

$$\mathrm{Vol}\Delta = \frac{1}{\mathrm{k}!} \prod_{\mathrm{i}=1}^{\mathrm{k}} \frac{\log \mathrm{M}}{\log \mathrm{p_i}}.$$

Taking the logarithm,

$$\begin{aligned} \log \Sigma &\leq \log \mathrm{Vol}\Delta \\ &= \int_{3/2}^{p_k} (\log\log M - \log\log y) d\pi(y) - \log k!. \end{aligned}$$

We apply integration by parts to the integral. This gives us

$$\begin{aligned} &\pi(p_k)(\log\log M - \log\log p_k) + \int_{3/2}^{p_k} \frac{\pi(y)}{y \log y} dy \\ =&k \log\log M - k \log\log p_k + O\Big(\int_{3/2}^{p_k} \frac{1}{(\log y)^2} dy\Big) \end{aligned}$$

by the Prime Number Theorem. Since we have $p_k \ll \log q$ the integral behaves as $\log q/(\log\log q)^2 = o(\log qT)$. Similarly the second term is negligible by $k \ll \log q/\log\log q$. Therefore we have

$$\int_{3/2}^{p_k} (\log\log M - \log\log y) d\pi(y) = k \log\log M + o(\log qT).$$

We apply Stirling's approximation to get

$$\log k! = k \log k + o(\log qT).$$

Finally, we need to show that

$$k \log\log M - k \log k = o(\log qT).$$

Taking the first derivative in k gives $\log\log M - \log k - 1$, we see that this function is increasing in k when $k \ll \log q/\log\log q$ for sufficiently large q, as $\log q/\log\log q = o(\log qT)$. So we can bound this by evaluating at $k = (1 + o(1)) \log q/\log\log q$. We also simplify using $M = q^{\alpha}$. (so that $qT = q^{6\alpha}$)

$$\begin{aligned} k \log\log M - k \log k &= k(\log\log M - \log k) \\ &\leq (1 + o(1)) \frac{\log q}{\log\log q} \left(\log \frac{\alpha \log q \log\log q}{\log q} \right) \\ &= (1 + o(1)) \frac{\log q}{\log\log q} \log \alpha + (1 + o(1)) \frac{\log q \log\log\log q}{\log\log q} \\ &= (1 + o(1)) \frac{\log q \log \alpha}{\log\log q} + o(\log qT). \end{aligned}$$

To show this is $o(\log qT)$, we consider

$$\frac{\log q \log \alpha}{\log\log q} (\log qT)^{-1} = \frac{\log \alpha}{6\alpha \log\log q} \to 0.$$

As $\log qT = 6\alpha \log q \to \infty$.

□

Proof of Proposition 4.21. From Proposition 4.29, we have

$$S_3 \lesssim \frac{N^2 d_2}{Mq} \sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2, |m_3|\sim M_3\\ (***)}} \sum_{\substack{y_1 \bmod q\\ (y_1,q)=1}} \int_{v_1 \asymp 1} \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2 v_1}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2 y_1}\Big) \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2}\Big) |R(v_1, y_1)| dv_1.$$

We apply Cauchy Schwarz repeatedly to get

$$S_3 \lesssim \frac{N^2 d_2}{Mq} \Big(\sum_{y_1 \bmod q} \int_{v_1 \asymp 1} |R(v_1, y_1)| dv_1 \Big)^{1/2} (S_3^{(1)})^{1/2},$$

where

$$\begin{aligned} S_3^{(1)} &= \sum_{\substack{y_1 \bmod q\\ (y_1,q)=1}} \int_{v_1 \asymp 1} \left(\sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3\\ (***)}} \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2 v_1}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2 y_1}\Big) \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2}\Big) \right)^2 dv_1 \\ &\overset{\text{CS}}{\lesssim} (S_3^{(2,1)})^{1/2} (S_3^{(2,2)})^{1/2}. \end{aligned}$$

In the last step we define

$$S_3^{(2,1)} := \sum_{\substack{y_1 \bmod q\\ (y_1,q)=1}} \int_{v_1 \asymp 1} \left(\sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3\\ (***)}} \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2 v_1}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2 y_1}\Big)^2 \right)^2 dv_1,$$

$$S_3^{(2,2)} := \sum_{\substack{y_1 \bmod q\\ (y_1,q)=1}} \int_{v_1 \asymp 1} \left(\sum_{\substack{|m_1|\sim M_1,\\ |m_2|\sim M_2,\\ |m_3|\sim M_3\\ (***)}} \tilde{R}_M^{(d)}\Big(\frac{k_1 v_1 + k_3}{-k_2}, \frac{k_1 y_1 + k_3}{-\tilde{k}_2}\Big)^2 \right)^2 dv_1.$$

We shall bound $S_3^{(2,1)}$ The second term is similar. Recall that

$$k_i = m_i/d = \tilde{k}_i \tilde{d}_i \omega_i,$$

where $\tilde{k}_i$ is coprime to q_0/d, ω_i is coprime to $q_0/d_i = q_0/(d\tilde{d}_i)$. Thus

$$\begin{aligned} S_3^{(2,1)} &= \sum_{\substack{y_1 \bmod q\\ (y_1,q)=1}} \int_{v_1 \asymp 1} \left(\sum_{\substack{|\tilde{k}_1|\sim M_1/(d_1\omega_1),\\ |\tilde{k}_2|\sim M_2/(d_2\omega_2),\\ |\tilde{k}_3|\sim M_3/(d_3\omega_3)\\ (***)}} \tilde{R}_M^{(d)}\Big(\frac{\omega_1 \tilde{d}_1 \tilde{k}_1 v_1 + \omega_3 \tilde{d}_3 \tilde{k}_3}{-\omega_2 \tilde{d}_2 \tilde{k}_2 v_1}, \frac{\omega_1 \tilde{d}_1 \tilde{k}_1 y_1 + \omega_3 \tilde{d}_3 \tilde{k}_3}{-\tilde{k}_2 y_1}\Big)^2 \right)^2 dv_1 \\ &\asymp \sum_{\substack{x \bmod q\\ (x,q)=1}} \int_{u \asymp 1} \left(\sum_{\substack{|\tilde{k}_1|\sim M_1/(d_1\omega_1),\\ |\tilde{k}_2|\sim M_2/(d_2\omega_2),\\ |\tilde{k}_3|\sim M_3/(d_3\omega_3)\\ (***)}} \tilde{R}_M^{(d)}\Big(\frac{\omega_1 \tilde{d}_1 \tilde{k}_1 + \omega_3 \tilde{d}_3 \tilde{k}_3 u}{-\omega_2 \tilde{d}_2 \tilde{k}_2}, \frac{\omega_1 \tilde{d}_1 \tilde{k}_1 + \omega_3 \tilde{d}_3 \tilde{k}_3 x}{-\tilde{k}_2}\Big)^2 \right)^2 du \end{aligned}$$

by the change of variables $x \equiv y^{-1} \bmod q$, $u = 1/v_1$. Since the second argument is q_0/d periodic, we can fold in x to get

$$S_3^{(2,1)} \ll \frac{qd}{q_0} \sum_{x \bmod q_0/d} \int_{u \asymp 1} \left(\sum_{\substack{|\tilde{k}_1|\sim M_1/(d_1\omega_1),\\ |\tilde{k}_2|\sim M_2/(d_2\omega_2),\\ |\tilde{k}_3|\sim M_3/(d_3\omega_3)\\ (***)}} \tilde{R}_M^{(d)}\Big(\frac{\omega_1 \tilde{d}_1 \tilde{k}_1 + \omega_3 \tilde{d}_3 \tilde{k}_3 u}{-\omega_2 \tilde{d}_2 \tilde{k}_2}, \frac{\omega_1 \tilde{d}_1 \tilde{k}_1 + \omega_3 \tilde{d}_3 \tilde{k}_3 x}{-\tilde{k}_2}\Big)^2 \right)^2 du.$$

We now make a change in variables $\tilde{u} = \omega_3\tilde{d}_3 u$, and $\tilde{x} \equiv \omega_3\tilde{d}_3 x$. The Jacobian factor is $1/(\omega_3\tilde{d}_3)$, and noticing that $\gcd(\omega_3\tilde{d}_3, q_0/d) = \tilde{d}_3$ by the condition $\gcd(m_3, q_0) = d_3$, $\tilde{x}$ counts the multiples of $\tilde{d}_3$ by $\tilde{d}_3$ times.

$$S_3^{(2,1)} \ll \frac{qd}{q_0}\frac{1}{\omega_3\tilde{d}_3}\tilde{d}_3 \sum_{\tilde{x} \bmod q_0/d} \int \Bigg(\sum_{\substack{|\tilde{k}_1|\sim M_1/(d_1\omega_1),\\ |\tilde{k}_2|\sim M_2/(d_2\omega_2),\\ |\tilde{k}_3|\sim M_3/(d_3\omega_3),\\ \text{all coprime to } q_0/d}} \tilde{R}_M^{(d)}\Big(\frac{\omega_1\tilde{d}_1\tilde{k}_1 + \tilde{k}_3\tilde{u}}{-\omega_2\tilde{d}_2\tilde{k}_2}, \frac{\omega_1\tilde{d}_1\tilde{k}_1 + \tilde{k}_3\tilde{x}}{-\tilde{k}_2}\Big)^2\Bigg)^2 d\tilde{u}.$$

We now apply corollary 4.20 to the function $f(u,x) \coloneqq R_M^{(d)}(u/(-\tilde{d}_2\omega_2), x)^2$, and setting $L \coloneqq \omega_2\tilde{d}_2$ gives a bound, and we have bounds that $\tilde{M} = \max(ML/(d_2\omega_2), M_1/(d_1\omega_1), M_3/(d_3\omega_3)) \le M/d$. First we confirm that f has the desired Fourier decay. Recall that $R_M^{(d)}$ is obtained by folding and averaging R_M in the second argument, so it actually suffices to check the Fourier decay in R_M. Since R_M^2 is obtained from a convolution with a bump function of length $NM/q \lesssim T$, its Fourier transform in the first argument is rapidly decreasing when $|\xi| \gtrsim T$. So that the scaling of $1/L$ means that f has Fourier transform rapidly decreasing when $|\xi| \gtrsim TL$. Since $L = \tilde{d}_2\omega_2 \le qT/(Nd)$, we have the desired Fourier transform decay of f when $|\xi| \gtrsim (qT)^2$.

$$\begin{aligned}
S_3^{(2,1)} \lesssim& \frac{qd}{q_0\omega_3}\left[\frac{M^6 d}{d^6L^4q_0}\Big(\sum_{y \bmod q_0/d}\int f(u,y)du\Big)^2 + \frac{M^4}{d^4L^2}\sum_{y \bmod q_0/d}\int f(u,y)^2du\right]\\
\lesssim& \frac{qd}{q_0\omega_3}\left[\frac{M^6 d}{d^6L^4q_0}\Big(\frac{q_0}{d_2}|W|L\Big)^2 + \frac{M^4}{d^4L^2}\frac{q_0}{d_2}LE(W)\right]\\
\lesssim& \frac{qd}{q_0}\left[\frac{M^6 d}{d^6L^2q_0}\Big(\frac{q_0}{d_2}|W|\Big)^2 + \frac{M^4}{d^4L}\frac{q_0}{d_2}E(W)\right].
\end{aligned}$$

Where we had to introduce a factor of L in the second and fourth moments as we defined $f(u,y) = \tilde{R}_M^{(d)}(-u/L, y)^2$. The same bound holds for $S_3^{(2,2)}$. Therefore we have

$$\begin{aligned}
S_3 \lesssim& \frac{N^2d_2}{Mq}(q|W|)^{1/2}\left\{\frac{qd}{q_0}\left[\frac{M^6 d}{L^2d^6q_0}\Big(\frac{q_0}{d_2}|W|\Big)^2 + \frac{M^4}{d^4L}\frac{q_0}{d_2}E(W)\right]\right\}^{1/2}\\
\lesssim& \frac{N^2M^2}{Ld^2}|W|^{3/2} + N^2M|W|^{1/2}E(W)^{1/2}\frac{d_2^{1/2}}{d^{3/2}\tilde{d}_2^{1/2}\omega_2^{1/2}}\\
\lesssim& N^2M^2|W|^{3/2} + N^2M|W|^{1/2}E(W)^{1/2}\\
\lesssim& (qT)^2|W|^{3/2} + N(qT)|W|^{1/2}E(W)^{1/2}.
\end{aligned}$$

Where in the last two steps, we used $L = \tilde{d}_2\omega_2$, so that $d\tilde{d}_2 = d_2$ cancels in the the last term, and that $M \lesssim qT/N$. □

4.7 Energy bound

Here we provide the generalization for the orthogonal energy bound for Guth and Maynard's result.

Proposition 4.30. *Let $(qT)^{3/4} \le N \le (qT)$. Then*

$$E(W) \lesssim |W|N^{4-4\sigma} + |W|^{21/8}(qT)^{1/4}N^{1-2\sigma} + |W|^3N^{1-2\sigma}.$$

The idea for bounding energy is similar; if $\chi_1\chi_2 = \chi_3\chi_4$ and $|t_1 + t_2 - t_3 - t_4|$ is small, we should expect $|D_N(t_1 + t_2 - t_3, \chi_1\chi_2\bar{\chi}_3)| \approx |D_N(t_4, \chi_4)| > N^\sigma$.

Lemma 4.31.

$$D_N(t,\chi) \lesssim \int_{|u-t|\lesssim 1}|D_N(u,\chi)|du + O((qT)^{-100}),$$

uniformly in χ.

Proof. This is Lemma 11.3 is [3], with the restriction $\lesssim 1$ in terms of $(qT)^\epsilon$ instead of just T^ϵ. $\square$

Lemma 4.32. *We have*

$$E(W) \lesssim N^{-2\sigma} \sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^3.$$

Proof. We have

$$E(W) = \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in W\\|t_1+t_2-t_3-t_4|\lesssim 1\\ \chi_1\chi_2=\chi_3\chi_4}} 1 \leq N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in W\\|t_1+t_2-t_3-t_4|\lesssim 1\\ \chi_1\chi_2=\chi_3\chi_4}} |D_N(t_4,\chi_4)|^2.$$

Now we apply the previous lemma and Cauchy-Schwarz to get

$$|D_N(t_4,\chi_4)|^2 \lesssim \int_{|u-t_4|\lesssim 1} |D_N(u,\chi_4)|^2 du \lesssim \int_{|u-t_1-t_2+t_3|\lesssim 1} |D_N(u,\chi_1\chi_2\bar{\chi}_3)|^2 du,$$

Since χ_1,χ_2,χ_3 fixes χ_4, and the t's within the same character are T^ϵ separated, there is $O(1)$ possible pairs of (t_4,χ_4) for each choice of $(t_1,\chi_1),(t_2,\chi_2),(t_3,\chi_3)$, so

$$\begin{aligned}
E(W) \lesssim& N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in W\\|t_1+t_2-t_3-t_4|\lesssim 1\\ \chi_1\chi_2=\chi_3\chi_4}} \int_{|u-t_1-t_2+t_3|\lesssim 1} |D_N(u,\chi_1\chi_2\bar{\chi}_3)|^2 du\\
\lesssim& N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in W}} \int_{|u-t_1-t_2+t_3|\lesssim 1} |D_N(u,\chi_1\chi_2\bar{\chi}_3)|^2 du\\
=& N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in W}} \int_{|u|\lesssim 1} |D_N(t_1+t_2-t_3+u,\chi_1\chi_2\bar{\chi}_3)|^2 du\\
=& N^{-2\sigma} \sum_{n_1,n_2} b_{n_1}\bar{b}_{n_2}\omega\left(\frac{n_1}{N}\right)\omega\left(\frac{n_2}{N}\right) \int_{|u|\lesssim 1} \left(\frac{n_1}{n_1}\right)^{iu} R\left(\frac{n_1}{n_2},n_1,n_2\right)^2 R\left(\frac{n_2}{n_1},n_2,n_1\right) du\\
\lesssim& N^{-2\sigma} \sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2},n_1,n_2\right)^2 R\left(\frac{n_2}{n_1},n_2,n_1\right)\right|\\
\lesssim& N^{-2\sigma} \sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^3.
\end{aligned}$$

$\square$

Lemma 4.33. *We have*

$$\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^2 \lesssim |W|N^2 + |W|^2N + |W|^{5/4}(qT)^{1/2}N.$$

Proof. From the definition of R,

$$\begin{aligned}
\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^2 &= \sum_{n_1,n_2\sim N} \sum_{(t_1,\chi_1),(t_2,\chi_2)\in W} \chi_1(n_1)\bar{\chi}_1(n_2)\left(\frac{n_1}{n_2}\right)^{it_1} \bar{\chi}_2(n_1)\chi_2(n_2)\left(\frac{n_1}{n_2}\right)^{-it_2}\\
&= \sum_{(t_1,\chi_1),(t_2,\chi_2)\in W} \left| \sum_{n\sim N} \chi_1(n_1)\bar{\chi}_2(n_1) n^{i(t_1-t_2)}\right|.
\end{aligned}$$

A direct application of Heath Brown's Theorem 3.2 gives

$$\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^2 \lesssim |W|N^2 + |W|^2N + |W|^{5/4}(qT)^{1/2}N.$$

$\square$

The trivial bound for $R \le |W|$ gives

$$\begin{aligned}E(W) &\lesssim N^{-2\sigma} \sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^3 \\ &\lesssim |W| N^{-2\sigma} \sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^2 \\ &\lesssim |W|^2 N^{2-2\sigma} + |W|^3 N^{1-2\sigma} + |W|^{9/4}(qT)^{1/2} N^{1-2\sigma}.\end{aligned}$$

The arguments beyond will be adaptations from GM. These involve the refinement of the energy bound through the use of Cauchy-Schwarz

$$\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^3 \le \Bigg(\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^2 \Bigg)^{1/2} \Bigg(\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^4 \Bigg)^{1/2}.$$

Lemma 4.34. *We have*

$$\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^4 \lesssim E(W)N^2 + |W|^4 N + E(W)^{3/4}|W|(qT)^{1/2} N.$$

Proof. Let

$$U_B = \{(u,\chi) \,:\, u \in \mathbb{Z}, \chi \text{ character modulus } q,\ \#\{((t_1,\chi_1),(t_2,\chi_2)) \in W \,:\, \lfloor t_1 - t_2 \rfloor = u, \chi_1\bar{\chi}_2 = \chi\} \sim B\}.$$

We split the sum in R as

$$\begin{aligned}\left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^4 &= \Bigg| \sum_{(t_1,\chi_1),(t_2,\chi_2)\in W} \chi_1\bar{\chi}_2(n_1)\bar{\chi}_1\chi_2(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)} \Bigg|^2 \\ &= \Bigg| \sum_{j=0}^{\lfloor \log_2 |W| \rfloor} \sum_{(u,\chi)\in U_{2^j}} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W \\ \lfloor t_1-t_2\rfloor = u, \\ \chi_1\bar{\chi}_2=\chi}} \chi_1\bar{\chi}_2(n_1)\bar{\chi}_1\chi_2(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)} \Bigg|^2 \\ &\overset{\text{CS}}{\lesssim} \sum_{j=0}^{\lfloor \log_2 |W| \rfloor} \Bigg| \sum_{(u,\chi)\in U_{2^j}} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W \\ \lfloor t_1-t_2\rfloor = u, \\ \chi_1\bar{\chi}_2=\chi}} \chi(n_1)\bar{\chi}(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)} \Bigg|^2.\end{aligned}$$

Therefore,

$$\sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^4 \lesssim \sup_{j\le \lfloor \log_2|W|\rfloor} \sum_{n_1,n_2\sim N} \Bigg| \sum_{(u,\chi)\in U_{2^j}} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W \\ \lfloor t_1-t_2\rfloor = u, \\ \chi_1\bar{\chi}_2=\chi}} \chi(n_1)\bar{\chi}(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)} \Bigg|^2.$$

For the value of j that achieves supremum,

$$\begin{aligned}&\sum_{n_1,n_2\sim N} \Bigg| \sum_{(u,\chi)\in U_{2^j}} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W \\ \lfloor t_1-t_2\rfloor = u, \\ \chi_1\bar{\chi}_2=\chi}} \chi(n_1)\bar{\chi}(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)} \Bigg|^2 \\ &\le \sum_{n_1,n_2\sim N} \sum_{\substack{(u_\alpha,\chi_\alpha), \\ (u_\beta,\chi_\beta)\in U_{2^j}}} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W \\ \lfloor t_1-t_2\rfloor = u_\alpha, \\ \chi_1\bar{\chi}_2=\chi_\alpha}} \sum_{\substack{(t_3,\chi_3),(t_4,\chi_4)\in W \\ \lfloor t_3-t_4\rfloor = u_\beta, \\ \chi_3\bar{\chi}_4=\chi_\beta}} \chi_\alpha\bar{\chi}_\beta(n_1)\bar{\chi}_\alpha\chi_\beta(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2-t_3+t_4)}\end{aligned}$$

$$= \sum_{\substack{(u_\alpha,\chi_\alpha),\\(u_\beta,\chi_\beta)\in U_{2^j}}} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W\\ \lfloor t_1-t_2\rfloor=u_\alpha,\\ \chi_1\bar\chi_2=\chi_\alpha}} \sum_{\substack{(t_3,\chi_3),(t_4,\chi_4)\in W\\ \lfloor t_3-t_4\rfloor=u_\beta,\\ \chi_3\bar\chi_4=\chi_\beta}} \Big|\sum_{n\sim N}\chi_\alpha\bar\chi_\beta(n)n^{i(t_1-t_2-t_3+t_4)}\Big|^2$$

$$\le \sum_{\substack{(u_\alpha,\chi_\alpha),\\(u_\beta,\chi_\beta)\in U_{2^j}}} \Bigg(\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in W\\ \lfloor t_1-t_2\rfloor=u_\alpha,\\ \chi_1\bar\chi_2=\chi_\alpha}} 1\Bigg)\Bigg(\sum_{\substack{(t_3,\chi_3),(t_4,\chi_4)\in W\\ \lfloor t_3-t_4\rfloor=u_\beta,\\ \chi_3\bar\chi_4=\chi_\beta}} 1\Bigg) \sup_{|s|\ll 1}\Big|\sum_{n\sim N}\chi_\alpha\bar\chi_\beta(n)n^{i(u_\alpha-u_\beta+s)}\Big|^2$$

$$\lesssim 2^{2j} \sum_{\substack{(u_\alpha,\chi_\alpha),\\(u_\beta,\chi_\beta)\in U_{2^j}}} \left(\int_{|s|\lesssim 1}\Big|\sum_{n\sim N}\chi_\alpha\bar\chi_\beta(n)n^{i(u_\alpha-u_\beta+s)}\Big|ds\right)^2$$

$$\overset{\mathrm{CS}}{\lesssim} 2^{2j} \sum_{\substack{(u_\alpha,\chi_\alpha),\\(u_\beta,\chi_\beta)\in U_{2^j}}} \int_{|s|\lesssim 1}\Big|\sum_{n\sim N}\chi_\alpha\bar\chi_\beta(n)n^{i(u_\alpha-u_\beta+s)}\Big|^2 ds$$

by Lemma 4.31. After swapping the order of summation and integration, for each fixed s, we can use Heath-Brown's theorem 3.2 to bound

$$\sum_{\substack{(u_\alpha,\chi_\alpha),\\(u_\beta,\chi_\beta)\in U_{2^j}}} \Big|\sum_{n\sim N}\chi_\alpha\bar\chi_\beta(n)n^{i(u_\alpha-u_\beta+s)}\Big|^2 \lesssim |U_{2^j}|N^2 + |U_{2^j}|^2 N + |U_{2^j}|^{5/4}(qT)^{1/2}N.$$

Therefore,

$$\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^4 \lesssim 2^{2j}(|U_{2^j}|N^2 + |U_{2^j}|^2 N + |U_{2^j}|^{5/4}(qT)^{1/2}N).$$

Recall U_B contains (u,χ) that have is representated $\sim B$ times in $(\lfloor t_1-t_2\rfloor, \chi_1\bar\chi_2)$ for $(t_1,\chi_1),(t_2,\chi_2)\in W$. Therefore,

$$B|U_B| \ll |W|^2.$$

Moreover, each tuple $((t_1,\chi_1),(t_2,\chi_2),(t_3,\chi_3),(t_4,\chi_4))$ satisfying $(\lfloor t_1-t_2\rfloor,\chi_1\bar\chi_2) = (\lfloor t_3-t_4\rfloor,\chi_3\bar\chi_4)$ also satisfies the condition for energy

$$\chi_1\bar\chi_2=\chi_3\bar\chi_4, |t_1-t_2+t_3-t_4|\lesssim 1.$$

Therefore,

$$B^2|U_B| \ll E(W).$$

For $B^2|U_B|^{5/4}$, we can bound this by

$$B^2|U_B|^{5/4} = (B^{3/2}U_B^{3/4})(B^{1/2}U_B^{1/2}) \ll E(W)^{3/4}|W|.$$

So we have

$$\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^4 \lesssim E(W)N^2 + |W|^4 N + E(W)^{3/4}|W|(qT)^{1/2}N.$$

□

We split the sum in $\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^3$ according to the GCD of n_1 and n_2. Let d be this GCD, and noticing that when $q|d$, we have $R(n_1/n_2,n_1,n_2)=0$,

$$\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^3 \le \sum_{d\le D}\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1}}\left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3 + \sum_{d>D}\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1}}\left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3,$$

where D is a parameter to be decided.

We first take care of the large GCD terms using CS. By our two bounds on the second and fourth moments,

$$\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1}}\left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3\leq\sum_{n_1',n_2'\sim(N/d)}\left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3$$

$$\lesssim\left(|W|\left(\frac{N}{d}\right)^2+|W|^2\frac{N}{d}+|W|^{5/4}(qT)^{1/2}\frac{N}{d}\right)^{1/2}$$

$$\left(E(W)\left(\frac{N}{d}\right)^2+|W|^4\frac{N}{d}+E(W)^{3/4}|W|(qT)^{1/2}\frac{N}{d}\right)^{1/2}.$$

Summing over $d>D$ (and using Cauchy-Schwarz to move the summation into the square root) gives the following result.

Proposition 4.35. *We have*

$$\sum_{d>D}\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1}}\left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3\lesssim\left(|W|\frac{N^2}{D}+|W|^2N+|W|^{5/4}(qT)^{1/2}N\right)^{1/2}$$

$$\left(E(W)\frac{N^2}{D}+|W|^4N+E(W)^{3/4}|W|(qT)^{1/2}N\right)^{1/2}.$$

When d is small, we expect fractions n_1/n_2 for $n_1,n_2\sim N/d$ to be well-distributed across $\asymp 1$, so we would like to estimate the summation in terms of an integral.

Lemma 4.36. *Let n_1,n_2 be coprime to q. We have for $v\asymp 1$,*

$$|R(v,n_1,n_2)|\lesssim T\int_{|s|\lesssim 1/T}|R(v+s)|ds.$$

Remark: qT is not required here. As we will see in the proof, the $1/T$ factor comes from the fact that the t_j's are contained in an interval of length T.

Proof. Recall

$$R(v,n_1,n_2)=\sum_j\chi_j(n_1)\chi_j(n_2)v^{i(t_j)},$$

so it suffices to prove the relation for any

$$f(v)=\sum_j c_jv^{i(t_j)},$$

uniformly for $|c_j|\leq 1$. Let ψ be a smooth bump that equals 1 on $[0,1]$, then

$$f(v)=\sum_j c_jv^{i(t_j)}=\sum_j c_je(t_j\log v/(2\pi))\psi(t_j/T)=\sum_j c_je\left(\frac{t_j\log v}{2\pi}\right)\int\hat{\psi}(\xi)e\left(\frac{t_j\xi}{T}\right)d\xi$$

The right hand side equals

$$\int\sum_j c_j\hat{\psi}(\xi)e\left(t_j\left(\frac{\xi}{T}+\frac{\log v}{2\pi}\right)\right)d\xi.$$

By the rapid decay in $\hat{\psi}$, we can truncate the integral to the range $|\xi|\lesssim 1$ at a cost of $O((qT)^{-100})$, and use the trivial bound $\hat{\psi}(\xi)\ll 1$. Therefore,

$$|f(v)|\ll\int_{|\xi|\lesssim 1}\left|\sum_j c_je\left(t_j\left(\frac{\xi}{T}+\frac{\log v}{2\pi}\right)\right)\right|d\xi+O((qT)^{-100}).$$

We now make the change of variables

$$\log s = \log v + \frac{2\pi\xi}{T},$$

so this gives a Jacobian factor of T, and

$$|f(v)| \ll T \int_{|s-v|\lesssim 1/T} \left|\sum_j c_j e\left(\frac{t_j s}{2\pi}\right)\right| ds + O((qT)^{-100})$$

which gives the bound in the lemma upon making a second change of variables $v' = s - v$. □

Returning to the small GCD terms, we have for a not divisible by q,

$$\begin{aligned}\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1\\n_1'/n_2'\cong a \bmod q}} \left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3 &\overset{\text{Hölder}}{\ll} T \sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1\\n_1'/n_2'\cong a \bmod q}} \int_{|s-\frac{n_1'}{n_2'}|\lesssim 1/T} |R(s,a)|^3\, ds\\ &\le T\int_{s\asymp 1} |R(s,a)|^3 \left(\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1\\n_1'/n_2'\cong a \bmod q\\|s-\frac{n_1'}{n_2'}|\lesssim 1/T}} 1\right) ds.\end{aligned}$$

We now consider for two distinct pairs $(n_1,n_2),(n_3,n_4)$ satisfying $\gcd(n_1,n_2)=\gcd(n_3,n_4)=1, n_1/n_2 \cong n_3/n_4 \cong a$,

$$\left|\frac{n_1}{n_2}-\frac{n_3}{n_4}\right| = \left|\frac{n_1n_4-n_2n_3}{n_2n_4}\right| \ge \frac{qd^2}{N^2}.$$

Therefore, the sum is $O(1+N^2/(qd^2T))$. So we have

$$\begin{aligned}\sum_{d\le D}\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1}} \left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3 &= \sum_{a \bmod q}\sum_{d\le D}\sum_{\substack{n_1',n_2'\sim(N/d)\\(n_1',n_2')=1\\n_1'/n_2'\cong a \bmod q}} \left|R\left(\frac{n_1'}{n_2'},n_1',n_2'\right)\right|^3\\ &\ll T\sum_{d\le D}\left(1+\frac{N^2}{qd^2T}\right)\sum_{a \bmod q}\int_{s\asymp 1}|R(s,a)|^3\,ds\\ &\overset{\text{CS}}{\ll} T\left(D+\frac{N^2}{qT}\right)\left(\sum_{a \bmod q}\int_{s\asymp 1}|R(s,a)|^2\,ds\right)^{1/2}\left(\sum_{a \bmod q}\int_{s\asymp 1}|R(s,a)|^4\,ds\right)^{1/2}\\ &\lesssim (qTD+N^2)|W|^{1/2}E(W)^{1/2}.\end{aligned}$$

Proof of Proposition 4.30. We have

$$\begin{aligned}E(W) &\lesssim N^{-2\sigma}\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^3\\ &\lesssim (qTD+N^2)|W|^{1/2}E(W)^{1/2} + \left(|W|\frac{N^2}{D}+|W|^2N+|W|^{5/4}(qT)^{1/2}N\right)^{1/2}\\ &\quad\left(E(W)\frac{N^2}{D}+|W|^4N+E(W)^{3/4}|W|(qT)^{1/2}N\right)^{1/2}.\end{aligned}$$

Picking the choice of $D = N^2/(qT)$, this is the analog for Guth and Maynard's result in bounding energy. The remaining part is to reduce our bounds. The computations are exactly the same except replacing T with

qT. Following the same steps by considering cases when $|W| > T^{2/3}$ and $|W| < T^{2/3}$ (the explicit computation are in [3], Proposition 11.1, Lemma 11.9). This reduces to

$$E(W) \lesssim |W|N^{4-4\sigma} + |W|^{21/8}(qT)^{1/4}N^{1-2\sigma} + |W|^3N^{1-2\sigma}.$$

□

4.8 Proof of reduction of theorem

Proof of Proposition 4.6. By Proposition 4.7,

$$|W| \ll N^2V^{-2} + NV^{-2}S^{1/3}.$$

Recall we have $S := S_2 + S_3$, where

$$S_2 \lesssim_k N^2|W|^2 + qTN|W|^{2-1/k} + N^2|W|^2\Big(\frac{(qT)^{1/2}}{|W|^{3/4}}\Big)^{1/k}.$$

by Proposition 4.14, and

$$S_3 \lesssim (qT)^2|W|^{3/2} + N(qT)|W|^{1/2}E(W)^{1/2}$$

by Proposition 4.21. Finally, by Proposition 4.30, we have

$$E(W) \lesssim |W|N^{4-4\sigma} + |W|^{21/8}(qT)^{1/4}N^{1-2\sigma} + |W|^3N^{1-2\sigma}.$$

These are analgous statements to Proposition 4.6, Propositions 6.1 (summed with Proposition 5.1), Proposition 10.1, and Proposition 11.1 in [3] respectively with T swapped with qT. Thus the reduction is the same as in the proof of Proposition 3.1. □

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