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November 27, 2024

Abstract

Notation

Below are the notational preferences of the author.

1. The set of natural numbers $\mathbb{N}$ does not contain 0.
2. p always denotes a prime, and by extension p_j, p_n etc.
3. s denotes a complex number, and is written in real and imaginary parts $s = \sigma + it$.
4. We denote $e(x) := \exp(2\pi i x)$.

1 Introduction to the Riemann Zeta Function

Note to self: For now, give a zeta function proof of huxley and show how to beat huxley at 3/4 using guth maynard (no need for total optimization but just at 3/4) short account of guth maynard

In this section, we give a quick introduction to the zeta function, including its product representation and analytic continuation, as well as conjectures regarding the zeta function.

Definition 1.1 (Zeta Function)**.** *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$. The properties of the zeta function as they relate to the distribution primes. In particular, the Dirichlet series can be represented as a product of primes.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

Remark: This expresion also converges absolutely for $\Re(s) > 1$. Since

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{p^s}{p^s - 1} = 1 + \frac{1}{p^s - 1}$$

and $\sum (p^s - 1)^{-1}$ converges absolutely by comparison to the zeta function Dirichlet series.

Sketch of proof. Write $s = \sigma + it$. For each p,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p. We thus take for $m > N$

$$\prod_{p \leq N} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \leq N} \left(\sum_{k=1}^{m} \frac{1}{p^{ks}} + O(2^{-m\sigma})\right)$$

$$
\begin{aligned}
&\overset{(*)}{=} \sum_{n=1}^{N} \frac{1}{n^s} + O_1\Big(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}\Big) + O(2^{-m\sigma}) \\
&= \zeta(s) + O_1\Big(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}\Big) + O(2^{-m\sigma})
\end{aligned}
$$

Where we apply to Fundemental Theorem of Arithmetic in (*) to show that each term n^{-s} has coefficient 1 determined by the unique prime factorization. As $m \to \infty$, $2^{-m\sigma} \to 0$. Then we take $N \to \infty$, the tail of the infinite sum converges to zero too. □

Proposition 1.3. *ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too.*

Extension of ζ to $\sigma > 0$. We apply integration by parts on Dirichlet series when $s > 1$

$$
\begin{aligned}
\zeta(s) &= \int_{1/2}^{\infty} x^{-s} d\lfloor x \rfloor \\
&= s \int_{1/2}^{\infty} \lfloor x \rfloor x^{-s-1} dx \\
&= s \int_{1}^{\infty} x^{-s} - \frac{\{x\}}{x^{-s-1}} dx \\
&= \frac{s}{s-1} - s \int_{1}^{\infty} \frac{\{x\}}{x^{-s-1}} dx
\end{aligned}
$$

where in the last expression, the integral converges when $s > 0$, and the pole at $s = 1$ arises from the first term. □

The continuation of the zeta function to the whole complex plane is best described by its functional equation: ζ has a line of symmetry across $\Re(s) = 1/2$.

Proposition 1.4. *Let $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then*

$$\xi(s) = \xi(1-s). \tag{3}$$

Proof. Using

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx,$$

we make the substitution $x = \pi n^2 y$ to get

$$
\begin{aligned}
\Gamma(s) &= \int_0^{\infty} e^{-\pi n^2 y} (\pi n^2 y)^{s-1} \pi n^2 dy \\
\implies \frac{\Gamma(s)}{\pi^s n^{2s}} &= \int_0^{\infty} e^{-\pi n^2 y} y^{s-1} dy
\end{aligned}
$$

So that by the Monotone Convergence Theorem,

$$
\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n=1}^{\infty} \frac{\Gamma(s/2)}{\pi^{s/2} n^s} \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 x} x^{s/2-1} dx \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} \left(e^{-\pi n^2 x}\right) x^{s/2-1} dx.
\end{aligned}
$$

We now let

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}, \quad \theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 2\omega(x) + 1,$$

and apply Poisson Summation to

$$\begin{aligned}
\theta(x) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i k u/\sqrt{x}} du \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} e^{-\pi k^2/x} \\
&= \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)
\end{aligned}$$

using the substitution $y\sqrt{x} = u$. Replacing with ω,

$$\sqrt{x}(2\omega(x)+1) = 2\omega\left(\frac{1}{x}\right) + 1 \implies \omega\left(\frac{1}{x}\right) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}.$$

We thus write, using $y = 1/x$,

$$\begin{aligned}
\xi(s) &= \int_0^1 \omega(x) x^{s/2-1} dx + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \omega(1/y) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\sqrt{y}\omega(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{1-s} + \frac{1}{s} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{s(1-s)} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx.
\end{aligned}$$

ω decays exponentially in x, so the integral converges and the last expression is well defined on $\mathbb{C}$ with simple poles at $s = 1$ or $s = 0$. Finally, notice that the last expression is symmetric when s is replaced with $(1-s)$, so proves equation 3. □

From the functional equation, we get 'trivial' zeros of the zeta function from the poles of Γ.

Corollary 1.5. *On $\Re(s) > 1$ or $\Re(s) < 0$, $\zeta(s) \neq 0$, except $\forall n \in \mathbb{N}, \zeta(-2n) = 0$.*

Proof. Using the product representation of ζ where it converges, none of $(1-p^{-s})^{-1} = 0$, so $\zeta(s) \neq 0$ on $\Re(s) > 1$. Γ has no zeros and has a simple pole at $-n$ for all $n \in \mathbb{N}$, so by equation 3 we get the zeros for $\Re(s) > 0$ are exactly at the negative even integers. □

These zeros are known as the trivial zeros of ζ. The remaining zeros lie between $0 \leq \Re(s) \leq 1$.

Definition 1.6 (Critical Strip and Critical Line)**.** *We denote the region $0 \leq \Re(s) \leq 1$ as the **critical strip**. We denote the line $\Re(s) = 1/2$ as the **critical line**.*

Corollary 1.7. *On the critical strip, if $\zeta(s) = 0$, $\zeta(\overline{s}) = \zeta(1-s) = \zeta(1-\overline{s}) = 0$.*

Proof. This follows from equation 3, and $\zeta(\overline{s}) = \overline{\zeta(s)}$ holds where the Dirichlet series converges, thus holds everywhere. □

The number of zeros in the critical strip can be calculated using the argument principle applied to the function ξ over the box with corners $-1+iT$,$-1-iT$,$2-iT$,$2+iT$. Applying the functional equation, we get the following result.

Theorem 1.8 (Number of zeros of ζ)**.** *The number of zeros up to height T*

$$\#\{\sigma + it \mid \zeta(\sigma + it) = 0, 0 \leq \sigma \leq 1, |t| \leq T\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Finally, we introduce some hypotheses regarding the zeta function, these statements have not been proved, but are supported by a great amount of heuristic evidence.

Definition 1.9 (Riemann Hypothesis)**.** *The **Riemann Hypothesis** (RH) asserts that on the critical strip,*

$$\zeta(s) = 0 \implies \Re(s) = \frac{1}{2}.$$

Definition 1.10 (Lindelöf Hypothesis)**.** *Let $\epsilon > 0$. The **Lindelöf Hypothesis** (LH) asserts that on the critical line,*

$$\zeta\left(\frac{1}{2} + it\right) = O(t^\epsilon).$$

2 The Prime Number Theorem

Theorem 2.1 (Prime Number Theorem)**.** *Let $\Pi(N) = \sum_{p \leq N} 1$. Then*

$$\Pi(N) = (1 + o(1))\frac{N}{\log N}.$$

In this section we will prove the Prime Number Theorem. This result is a minor goal of this paper. The Prime Number theorem serves as a starting point for studying primes in short intervals, and sets the stage for zero-density theorems.

Definition 2.2 (Von Mangoldt Function)**.** *The **Von Mangoldt function** Λ is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p, \text{ if } n = p^k \text{ for some } k \in \mathbb{N} \\ 0, \text{ else} \end{cases}$$

The sum of the Von Mangoldt function $\sum \Lambda(n)$ is a more natural way to express a prime counting function in the language of ζ. To see why, consider the expression

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= (\log \zeta(s))' \\ &= \left[-\sum_p \log\left(1 - p^{-s}\right)\right]' \\ &= -\sum_p \frac{p^s \log p}{1 - p^{-s}} \\ &= -\sum_p \log p \sum_{k \in \mathbb{N}} p^{-ks} \\ &= -\sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \end{aligned}$$

on $\Re(s) > 1$ where the sum and products are absolutely convergent.

Proposition 2.3. *$\sum_{n \leq N} \Lambda(n) = (1 + o(1))N$ implies the Prime Number Theorem.*

Proof. On one hand, we have

$$\sum_{n \leq N} \Lambda(n) \leq \sum_{p \leq N} \Lambda(N)$$

$$\leq \Pi(N) \log N.$$

And for $\epsilon > 0$,

$$\begin{aligned}\sum_{n\leq N}\Lambda(n) &\geq \sum_{N^{1-\epsilon}\leq n\leq N}\Lambda(n)\\ &\geq \sum_{N^{1-\epsilon}\leq p\leq N}(1-\epsilon)\log(N)\\ &= (1-\epsilon)(\Pi(N)\log(N)+O(N^{1-\epsilon}\log N)).\end{aligned}$$

□

Moreover, the sum of the Von Mangoldt function can be related to the zeros of the zeta function. Let ϕ be smooth and rapidly decaying at infinity, and $\tilde{\phi}$ be its Mellin transform. Let $N \in \mathbb{N}$ and $c \geq 2$. Then

$$\begin{aligned}\sum_{n\in\mathbb{N}}\Lambda(n)\phi\left(\frac{n}{N}\right) &= \sum_{n\in\mathbb{N}}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\sum_{n\in\mathbb{N}}\Lambda(n)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)N^s\frac{\zeta'(s)}{\zeta(s)}ds\end{aligned} \tag{4}$$

By the rapid decay of $\tilde{\phi}$, we change the line of integration from c to $-\infty$, we get residue contributions from a pole at $s = 1$, $s = 0$, as well as all ρ such that $\zeta(\rho) = 0$ on the critical strip, and all the trivial zeros. Morally, we can take the indicator function $\phi = 1$ on $[0, 1]$.

$$\sum_{n\leq N}\Lambda(n) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}N^s\frac{\zeta'(s)}{\zeta(s)}ds \tag{5}$$

This gives

$$\begin{aligned}\sum_{n\leq N}\Lambda(n) &= N - \sum_{\rho}\frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{k\in\mathbb{N}}\frac{N^{-2k}}{2k}\\ &= N - \sum_{\rho}\frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log\left(1-N^{-2}\right).\end{aligned} \tag{6}$$

This formula, due to von Mangoldt, can be derived with more care about the convergence in the sum: The sum over zeros ρ is not absolutely convergent, and is ordered in increasing $|\Im(\rho)|$.

Theorem 2.4 (Riemann-von Mangoldt explicit formula)**.** *Let $N > 1$ be not a prime power. Then*

$$\sum_{n\leq N}\Lambda(n) = N - \lim_{T\to\infty}\sum_{|\Im(\rho)|\leq T}\frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log\left(1-N^{-2}\right). \tag{7}$$

In practice, we truncate the integral in up to height T to obtained a truncated version of the explicit formula. This is obtained through integration along the choice of $c = 1 + 1/\log N$

Theorem 2.5. *Let $N > 1$. Then*

$$\sum_{n\leq N}\Lambda(n) = N - \sum_{|\Im(\rho)|\leq T}\frac{N^\rho}{\rho} + O(\frac{N}{T}(\log NT)^2) + O(\log N). \tag{8}$$

The term N in the explicit formula is already suggestive of the Prime Number Theorem. The major error term comes from N^ρ in the sum, so bounding the $\Re(\rho)$ becomes the most important part in reducing the error term in the prime number theorem. This in turn is equivalent to bounding $\Re(\rho)$, and the best case is when all the zeros have real part 1/2. Assuming the Riemann Hypothesis, we consider the sum over the non trivial zeros

$$\left| \sum_{|\Im(\rho)| \leq T|} \frac{N^\rho}{\rho} \right| \leq N^{1/2} \sum_{|\Im(\rho)| \leq T} \left| \frac{1}{\rho} \right|.$$

We know there are $\sim \log T$ zeros of height $[T, T+1)$, thus the integral $\sum |\rho^{-1}|$ behaves as

$$\sum_{n \leq T} \frac{\log n}{n} = O(\log^2 T).$$

Taking $N = T$ in the truncated explicit formula, we obtain

$$\sum_{n \leq N} \Lambda(n) = N + O(N^{1/2} \log^2 N). \tag{9}$$

Which implies the prime number theorem.

Remark: The PNT stated in 9 (with this error bound) can be shown to be equivalent to the Riemann Hypothesis.

The prime number theorem is also true without assuming the strong Riemann Hypothesis. To show this, it is sufficient to show that there are no zeros with real part 1, so the terms in the sum contributes $O(N^{1-\epsilon})$ which will be dominated by N.

Theorem 2.6. *Let $t \in \mathbb{R}$. Then $\zeta(1+it) \neq 0$.*

Proof of the Prime Number Theorem. Condition on proving Theorem 2.6, we use it to derive the prime number theorem. Let $\phi = \phi_{N,T}$ be a bump function that equals 1 on the interval $[2, N]$ and supported on $[3/2, N + N/T]$. By construction we can also make $\phi^{(j)}(x) = O_j(1)$ and $\phi^{(j)}(x) = O_j(T/x)^j$ on the intervals $[3/2, 2]$ and $[N, N + N/T]$ respectively. Then

$$\begin{aligned}
\sum_{n \leq N} \Lambda(n) &\leq \sum_n \Lambda(n)\phi(n) \\
&= \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi} \frac{\zeta'(s)}{\zeta(s)} ds \\
&= \tilde{\phi}(1) - \sum_\rho \tilde{\phi}(\rho) - \sum_n \tilde{\phi}(-2n)
\end{aligned}$$

The first term

$$\begin{aligned}
\tilde{\phi}(1) &= \int_0^\infty \phi(x) dx \\
&= N + O(N/T)
\end{aligned}$$

gives the term we want from the PNT. In the third term, we rewrite by Monotone Convergence

$$\begin{aligned}
\sum_n \tilde{\phi}(-2n) &= \sum_n \int_0^\infty \phi(x) x^{-2n-1} dx \\
&= \int_0^\infty \phi(x) \sum_n x^{-2n-1} dx \\
&= \int_0^\infty \phi(x) \frac{1}{x^3 - x} dx \\
&= O(1)
\end{aligned}$$

Finally, to bound the second term, we define a parameter $\bar{T} = \bar{T}(T)$ and split the sum into

$$\sum_{|\Im\rho| \leq \bar{T}} \tilde{\phi}(\rho) + \sum_{|\Im\rho| > \bar{T}} \tilde{\phi}(\rho)$$

In the first summation, we let $\epsilon = \epsilon_{\bar{T}}$ such that there are no zeros in the region $\Re(s) > 1-\epsilon, |\Im(s)| \leq \bar{T}$, then

$$\begin{aligned}\sum_{|\Im\rho|\leq\bar{T}} \tilde{\phi}(\rho) &= \sum_{|\Im\rho|\leq\bar{T}} \int_0^\infty \phi(x)x^{\rho-1}dx \\ &= O_T(N^{1-\epsilon}).\end{aligned}$$

In the second summation, we apply integration by parts to show that

$$\begin{aligned}\left|\int_0^\infty \phi(x)x^{\rho-1}dx\right| &= \left|\frac{1}{\rho(\rho+1)}\int_0^\infty \phi''(x)x^{\rho+1}dx\right| \\ &= O\left(\frac{1}{|\rho|^2}\frac{T^2}{N^2}\frac{N}{T}N^2\right) \\ &= O\left(\frac{1}{|\rho|^2}TN\right)\end{aligned}$$

The sum over $|\rho|^{-2}$ behaves as $\sum_n n^{-2}\log n$, so we can pick $\bar{T}$ large enough depending on T to make the contribution of $\sum_{\Im(\rho)>\bar{T}} |\rho|^{-2}$ to be $O(T^{-2})$. So that

$$\begin{aligned}\sum_{n\leq N} \Lambda(n) &\leq N + O(N/T) + O_T(N^{1-\epsilon}) \\ &= N + O(N/T)\end{aligned}$$

for $N = N(T)$ sufficiently large. Similarly, repeating the same argument on $\phi = \phi_{N,T}$ equals 1 on the interval $[2, N - N/T]$ and supported on $[3/2, N]$ gives

$$\sum_{n\leq N} \Lambda(n) \geq N + O(N/T).$$

Sending $T \to \infty$ gives the PNT. ☐

Proof of Theorem 2.6. Let $\sigma > 1$. We consider the expressions

$$\Re\left(\frac{\zeta'}{\zeta}(\sigma+it)\right) = -\sum_n \frac{\Lambda(n)}{n^\sigma}\cos(t\log n)$$

and

$$2(1+\cos\theta)^2 = 2 + 4\cos\theta + 2\cos^2\theta = 3 + 4\cos\theta + \cos 2\theta.$$

So that

$$\begin{aligned}\Re\left(3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma+it) + \frac{\zeta'}{\zeta}(\sigma+2it)\right) &= -\sum_n \frac{\Lambda(n)}{n^\sigma}(3 + 4\cos(t\log n) + \cos(2t\log n)) \\ &= -\sum_n \frac{\Lambda(n)}{n^\sigma} 2(1+\cos(t\log n))^2 \\ &\leq 0.\end{aligned}$$

Now for the sake of contradiction, we let $\zeta(1+it) = 0$ be a zero of order d, and since we know ζ has a pole of order 1 at $s = 1$, we can let $t \neq 0$. Consider the function $f(s) = \zeta(s)^3\zeta(s+it)^4\zeta(s+2it)$. By the computation above, $\Re(f'/f) \leq 0$ when $\Re(s) > 1$. But we also have that f, by construction, has a zero of order $k \geq 4d-3 > 0$ at $s = 1$. So that $\Re(f'/f) = k/(s-1) +$ a holomorphic part. Now taking $s \to 1^+$, $\Re(f'/f) \to +\infty$, contradicting $\Re(f'/f) \leq 0$. ☐

3 Primes in Short Intervals

We would like to answer the following question about primes in short intervals. Let $y = y(x)$. What is the smallest asymptotic behavior of y such that

$$\sum_{x\leq n\leq x+y} \Lambda(n) = (1+O(1))y \tag{10}$$

for large enough x? That is, what is the shortest interval such that we have the behavior of the Prime Number Theorem? If 10 holds for some y, we say the Prime Number Theorem holds for intervals of y.

Remark: This question can be rephrased into finding primes in short intervals, by including a factor of $\log x$.

Proposition 3.1. *Assume the RH. Then the Prime Number Theorem holds in intervals of* $x^{1/2+\epsilon}$.

Proof. Assume the RH, then

$$\sum_{x\leq n\leq x+y} \Lambda(n) = y + O(x^{1/2}\log^2 x) = x^{1/2+\epsilon} + o(x^{1/2+\epsilon}),$$

so that the sum is non-zero for large enough x. □

Recalling that the error term is related to the real part of the zeros of the Zeta function, we motivate the following definition of zero-density:

Definition 3.2. *Let $N(\sigma, T)$ denote the number of zeros of the zeta function with real part greater than σ and imaginary part between $-T$ and T. That is,*

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \mid \beta \geq \sigma, |\gamma| \leq T\}.$$

Remark: The ideal scenario is that $N(\sigma, T) = 0$ for all $\sigma \geq 1/2$.

Theorem 3.3 (Chudakov)**.** *There exists a constant A such that $\zeta(\sigma + iT) \neq 0$ in the region*

$$\sigma > 1 - A\frac{\log\log T}{\log T}.$$

add reference

Theorem 3.4 (Hoheisel)**.** *Let A be defined as in the previous theorem. Suppose that $N(\sigma, T) \ll T^{a(1-\sigma)}\log^b T$ uniformly in $1/2 \leq \sigma < 1$ and in T. Then for all*

$$\theta > 1 - \frac{1}{a + b/A},$$

the Prime Number Theorem holds in intervals of $y = x^\theta$.

Proof. First notice that $N(1/2, T)$ gets at least half of the zeros of height T, so $a \geq 2$. Let $y \ll x$. We consider the expression

$$S = S(x, y) = \frac{1}{y}\sum_{x\leq n\leq x+y} \Lambda(n).$$

By the truncated version of the explicit formula in Theorem 2.5, we get

$$S = 1 - \sum_{|\Im(\rho)|\leq T} \frac{(x+y)^\rho - x^\rho}{\rho y} + O(\frac{x}{yT}(\log xT)^2) + O(\frac{\log x}{y}).$$

We focus on the sum over the non-trivial zeros with height less than T, and enumerate them ρ_j. For each $\rho_j = \sigma_j + it_j$, we apply Mean Value

$$\begin{aligned}\left|\sum_{\rho_j} \frac{(x+y)^\rho - x^\rho}{\rho y}\right| &\leq \sum_{\rho_j}\left|\frac{(x+y)^{\rho_j} - x^{\rho_j}}{\rho_j y}\right| \\ &\ll \sum_{\rho_j} x^{\sigma_j - 1} \\ &= \sum_{\rho_j} x^{\sigma_j-1} - x^{-1} + x^{-1} \\ &= O\left(\frac{T\log T}{x}\right) + \sum_{\rho_j} x^{\sigma_j-1} - x^{-1}.\end{aligned}$$

And by replacing $x^{\sigma_j}-1$ by an integral,

$$\begin{aligned}\sum_{\rho_j} x^{\sigma_j-1}-x^{-1} &= \sum_{\rho_j}\int_0^{\sigma_j} x^{u-1}\log x\ du \\ &= \int_0^{1-A\frac{\log\log T}{\log T}} \sum_{\rho_j} \mathbb{1}_{u\le\sigma_j} x^{u-1}\log x\ du \\ &= \int_0^{1-A\frac{\log\log T}{\log T}} N(u,T)x^{u-1}\log x\ du\end{aligned}$$

Where in the penulitimate step we made use of Chudaokov's bound and exchanged the order of integration and summation. Now we can apply the hypothesis that $N(\sigma,T)\ll T^{a(1-\sigma)}\log^b T$ for $\sigma>1/2$ and trivially $N(\sigma,T)\ll T\log T \ll T^{a(1-\sigma)}\log^b T$ for $\sigma\le 1/2$. This evaluates to

$$\begin{aligned}\sum_{\rho_j} x^{\sigma_j-1}-x^{-1} &\ll \int_0^{1-A\frac{\log\log T}{\log T}} T^{a(1-u)}\ \log^b T\ x^{u-1}\log x\ du \\ &= \log^b T\int_0^{1-A\frac{\log\log T}{\log T}} \left(\frac{T^a}{x}\right)^{1-u} \log x\ du \\ &= \frac{\log x\log^b T}{a\log T-\log x}\left[\frac{T^a}{x}-\left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}}\right]\end{aligned}$$

Combined with the previous bounds, we have

$$S = 1+O\left(\frac{T\log T}{x}\right)+O\left(\frac{\log x\log^b T}{a\log T-\log x}\left[\frac{T^a}{x}-\left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}}\right]\right)+O(\frac{x}{yT}(\log xT)^2)+O(\frac{\log x}{y}).$$

To make all terms (except for the first) to be $o(1)$, we want to set $y=x^\theta$, $T=x^k$, such that θ,k satisfy

$$k<1,\ ak<1,\ k+\theta>1,$$

so that the second, fourth and fifth terms are $o(1)$ in x. For the second term, we can simplify

$$\begin{aligned}\frac{\log x\log^b T}{a\log T-\log x}\left[\frac{T^a}{x}-\left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}}\right] &= \frac{k^b\log^b x}{ak-1}\left[x^{ak-1}-x^{(ak-1)A\frac{\log(k\log x)}{k\log x}}\right] \\ &\le \frac{k^b\log^b x}{1-ak}x^{ak-1}+\frac{k^b\log^b x}{1-ak}\exp\left((ak-1)A\frac{\log(k\log x)}{k}\right) \\ &\le \frac{k^b\log^b x}{1-ak}x^{ak-1}+\frac{k^b\log^b x}{1-ak}\exp\left((ak-1)A\frac{\log(k\log x)}{k}\right) \\ &= O(x^{ak-1})+O\left((\log x)^{b+\frac{(ak-1)A}{k}}\right).\end{aligned}$$

To have decay in the last term in x, we need

$$b+\frac{(ak-1)A}{k}<0 \implies (aA+b)k<A \implies k<\frac{1}{a+\frac{b}{A}}$$

We had $a>1$, so this k satisfies $k<1$ and $ak<1$. Finally, for $k=1/(a+bA^{-1})+\delta/2$ we let $\theta=1-k+\delta/2$ to satisfy $\theta+k>1$, so we can find any $1/(a+bA^{-1})+\delta>\theta>1-1/(a+bA^{-1})$, and for $y=x^\theta$, we get

$$\frac{1}{y}\sum_{x\le n\le x+y}\Lambda(n)=S=1+o(1).$$

□

Theorem 3.4 gives the classical way to relate the distribution of primes in short intervals to the density of zeros away from the real-half line. *todo: huxley zeta, then generalize guth maynard to hybridized*

4 Towards a Hybrid Zero Density Result

Here I will document some progress with working with the hybrid version of Guth-Maynard. This section will be removed when I send the draft to Prof. Wunsch. Let ω be a bump function supported on *[placeholder]*. Let $N = T^{[placeholder]}$. We denote

$$D_N(t,\chi) = \sum_{n\sim N} \omega\left(\frac{n}{N}\right) b_n \chi(n) n^{it}$$

and $\mathcal{S} = \{(t_j,\chi_j)\}_{j\leq|\mathcal{S}|}$ such that

$$|D_N(t_j,\chi_j)| \geq V$$

for all $(t_j,\chi_j) \in \mathcal{S}$. We also let $0 \leq t_j \leq T$ to be T^ϵ separated, so if $j \neq k$ and $\chi_j = \chi_k$, $|t_j - t_k| \geq T^\epsilon$. We can also let χ_j's be Dirichlet characters modulo q (potentially primitive if we need this hypothesis later).

Here we write M a $|\mathcal{S} \times N|$ matrix with entries

$$M_{t_j,\chi_j,n} = \chi_j(n) n^{it_j}$$

for $(t_j,\chi_j) \in \mathcal{S}$ and $n \sim N$. Thus by the same reasoning that $(M\vec{b})_j = D_N(t_j,\chi_j)$, we want to bound the trace of the matrix

$$\mathrm{tr}((M^*M)^3)$$

which is formalized in section 4 of Guth and Maynard's proof. We see that

$$(MM^*)_{t_j,t_k} = \sum_{n\sim N} \omega\left(\frac{n}{N}\right)^2 n^{i(t_k-t_j)} \bar{\chi}_j \chi_k(n),$$

so that

$$\mathrm{tr}(MM^*) = |\mathcal{S}| \sum_{n\sim N} \omega\left(\frac{n}{N}\right)^2$$

and Lemmas 4.1-4.4 (Large values controlled by singular values, bound for singular values in terms of trace, Principle of non-stationary phase, Hilbert-Schmidt Norm estimate) can be adapted directly from the paper.

For the expansion of the trace (analog to lemma 4.5),

$$(M^*M)_{n_1,n_2} = \sum_{(t_j,\chi_j)\in\mathcal{S}} \omega\left(\frac{n_1}{N}\right) \omega\left(\frac{n_2}{N}\right) \bar{\chi}_j(n_1)\chi_j(n_2) n_1^{-it_j} n_2^{it_j}$$

so that

$$\begin{aligned}\mathrm{tr}((M^*M)^3) = \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{n_1,n_2,n_3\sim N} &\omega\left(\frac{n_1}{N}\right)^2 \left(\frac{n}{N}\right)^{i(t_1-t_3)} \chi_1\bar{\chi}_3(n_1) \\ &\times \omega\left(\frac{n_2}{N}\right)^2 \left(\frac{n}{N}\right)^{i(t_2-t_1)} \chi_2\bar{\chi}_1(n_2) \\ &\times \omega\left(\frac{n_3}{N}\right)^2 \left(\frac{n}{N}\right)^{i(t_3-t_2)} \chi_3\bar{\chi}_2(n_3).\end{aligned}$$

Let $h_t(u) = \omega(u)u^{it}$, so by applying Poisson summation to the inner sum, we have

$$\begin{aligned}\mathrm{tr}((M^*M)^3) = \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \frac{N^3}{q^3} \sum_{m\in\mathbb{Z}^3} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} &\chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3) e\left(\frac{-x\cdot m}{q}\right) \\ &\times \hat{h}_{t_1-t_3}\left(\frac{Nm_1x_1}{q}\right) \hat{h}_{t_2-t_1}\left(\frac{Nm_2x_2}{q}\right) \hat{h}_{t_3-t_2}\left(\frac{Nm_3x_3}{q}\right)\end{aligned}$$

Finally, we exchange the 2 outermost integrals. And split the outermost sum into four parts (same as Guth Maynard) $S_0 + S_1 + S_2 + S_3$, where S_j runs over the values of m with exactly j non-zero entries.

4.1 S_0 bound

S_0 only has one term corresponding to $m = 0$. By the principle of non-stationary phase, $\hat{h}_t(0)$ has rapid decay in t, so contributes $O(T^{-1000})$ except possibly when t_1, t_2, t_3 are not T^ϵ separated. Moreover, by the orthogonality of characters, all three terms $\chi_a\bar{\chi}_b$ must be principal to have non-zero contribution, so this fixes the sum to be across $(t_1, \chi_1) = (t_2, \chi_2) = (t_3, \chi_3)$ to give a $N^3\phi(q)^3|\mathcal{S}|\|\omega\|_{L_2}^6/q^3$ term. So that

$$\mathrm{tr}((M^*M)^3) = \frac{N^3\phi(q)^3}{q^3}|\mathcal{S}|\|\omega\|_{L_2}^6 + \sum_{m\in\mathbb{Z}^3-\{0\}} I_m + O(T^{-100}),$$

where

$$I_m = \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3)e\left(\frac{-x\cdot m}{q}\right)$$
$$\times \hat{h}_{t_1-t_3}\left(\frac{Nm_1x_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2x_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3x_3}{q}\right).$$

which gives the analogous Lemmas 4.5 and 4.6.

4.2 S_1 bound

By symmetry, we sum I_m across all $m = (0, 0, m_3 \neq 0)$ at a cost of a factor of 3. We then have

$$I_m = \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3)e\left(\frac{-x_3m_3}{q}\right)$$
$$\times \hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3x_3}{q}\right)$$

Again by the orthogonality of characters, the only way to get non-zero contribution is when $\chi_1 = \chi_2$ and $\chi_2 = \chi_3$. So this reduces to

$$I_m = \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \phi(q)^2 \sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{-x_3m_3}{q}\right)$$
$$\times \hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3x_3}{q}\right)$$

So we trivially bound S_1 by

$$|S_1| \ll \frac{N^3}{q^3}\phi(q)^2 \sum_{m_3\neq 0} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} \left|\hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3x_3}{q}\right)\right|$$

By the quick decay and boundedness of $\hat{h}_t(\xi)$ in both ξ and t (decays in terms of $\langle\xi\rangle^{-A}\langle t\rangle^A$ or $\langle\xi\rangle^A\langle t\rangle^{-A}$), we can bound the terms when summed across all $|m_3| > qT^{1+\epsilon}/(Nx_3)$. For the remaining terms, $t_1 \neq t_2$, $t_1 \neq t_3$ can be bounded by $O(T^{-10})$. Finally, when $t_1 = t_2 = t_3$ and $|m_3|$ is small, we get decay in terms of $(N/q)^{-100}$.

The sum over terms $|m_3| > qT^{1+\epsilon}/N$,

$$\sum_{|m_3|>qT^{1+\epsilon}/N} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} \left|\hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3x_3}{q}\right)\right|$$

$$\ll_{\epsilon,A} \sum_{|m_3|>qT^{1+\epsilon}/N} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} \left|T^A\left(\frac{Nm_3}{q}\right)^{-A}\right|$$
$$\ll |\mathcal{S}|^3\phi(q)T^{-10}.$$

For $|m_3| \le qT^{1+\epsilon}/N$, we use the same bound for terms $t_1 \neq t_2$ or $t_1 \neq t_3$ by the $\hat{h}_t(0)$ terms. When $t_1 = t_2 = t_3$, we use $m_3 \neq 0$ to bound

$$\left|\hat{h}_{t_3-t_2}\left(\frac{Nm_3x_3}{q}\right)\right| \ll \left(\frac{q}{N}\right)^{-100}$$

since $|\mathcal{S}| \ll T\phi(q)$, we can set $T \gg q$ to get the contribution of S_1 to be $O\epsilon(T^{-10})$.

4.3 S_2 bound

We write by symmetry

$$S_2 = 3\frac{N^3}{q^3} \sum_{m_1,m_2\neq 0} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3)e\left(\frac{-x_1m_1 - x_2m_2}{q}\right)$$
$$\times\, \hat{h}_{t_1-t_3}\left(\frac{Nm_1x_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2x_2}{q}\right)\hat{h}_{t_3-t_2}(0)$$

Removing zero contributions from $\chi_2 \neq \chi_3$ by orthogonality, we have

$$= 3\frac{N^3}{q^3}\phi(q) \sum_{m_1,m_2\neq 0} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3=\chi_2)\in\mathcal{S}}} \sum_{x_1,x_2\in\mathbb{Z}/q\mathbb{Z}} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)e\left(\frac{-x_1m_1 - x_2m_2}{q}\right)$$
$$\times\, \hat{h}_{t_1-t_3}\left(\frac{Nm_1x_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2x_2}{q}\right)\hat{h}_{t_3-t_2}(0)$$

5 Preliminaries

Here we give some supplementary definitions and statements of theorems.

Theorem 5.1 (Möbius Inversion)**.** *The Möbius function μ is defined for $n \in \mathbb{N}$,*

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1p_2...p_k \text{ for distinct } p\text{'s} \\ 0, & \text{otherwise} \end{cases}$$

Suppose we have arithmetic functions f, g, and that

$$f(n) = \sum_{d|n} g(d)$$

Then the Möbius Inversion formula gives

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

Example 5.2. *On $\Re(s) > 1$, let $M_N(s) = \sum_{n\le N}\mu(n)n^{-s}$. Then setting $f(n) = 1$ for all n, $g(1) = 1$, $g(n) = 0$ for $n \ge 2$, we multiply M_N by ζ in Dirichlet series to get*

$$\zeta(s)M_N(s) = \sum_n \frac{a_n}{n^{-s}},$$

where $a_n = g(n)$ *for all* $n \leq N$*. Similarly, letting* $M_N(s) = \sum_{n \leq N} \chi(n)\mu(n)n^{-s}$ *for some Dirichlet character* χ*, we get*

$$L(s,\chi)M_N(s) = \sum_n \frac{a_n \chi(n)}{n^{-s}}$$

with the same a_n *as in the previous equation.*

Theorem 5.3 (Fourier Inversion)**.** *In Schwartz space, the Fourier transform of* $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ *of* $f \in \mathcal{S}(\mathbb{R}^d)$ *is given by*

$$\hat{f}(\xi) := \mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e(-\xi \cdot \mathbf{x}) f(\mathbf{x}) \; d\mathbf{x}$$

has inverse given by

$$f(\mathbf{x}) = \mathcal{F}^{-1}\hat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e(\xi \cdot \mathbf{x}) \hat{f}(\xi) \; d\xi.$$

Theorem 5.4 (Mellin Inversion)**.** *The Mellin transform of a function* $f : (0,\infty) \to \mathbb{C}$

$$\tilde{f}(s) := \mathcal{M}f(s) := \int_0^\infty f(x) x^{s-1} \; dx$$

has inverse

$$\mathcal{M}^{-1}\tilde{f}(x) = \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) x^{-s} ds$$

on $a < c < b$ *provided that the integral* $\tilde{f}$ *is absolute convergent on the strip* $a < \Re(s) < b$*.*

Theorem 5.5 (Poisson Summation)**.**