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Notation

We denote summation over the natural numbers $\mathbb{N}$ to be over the positive integers. We always denote p as a prime, and by extension p_j, p_n etc. We denote $e(x) := \exp(2\pi i x)$. For asymptotic behaviors, we write $A \ll B$ if there is an absolute constant c such that $A < cB$, and $A \ll_\epsilon B$ if $A < cB$ with c possibly depending on ϵ. Similar to the notation in Guth-Maynard's paper, we write $A \asymp B$ if $A \ll B$ and $B \ll A$ and $A \sim B$ for $B < A \leq 2B$. We also write $A \lesssim B$ if $A \ll_\epsilon T^\epsilon B$ for any $\epsilon > 0$.

1 Introduction to the Riemann Zeta Function

Note to self: For now, give a zeta function proof of huxley and show how to beat huxley at 3/4 using guth maynard (no need for total optimization but just at 3/4) short account of guth maynard

In this section, we give a quick introduction to the zeta function, including its product representation and analytic continuation, as well as conjectures regarding the zeta function.

Definition 1.1 (Zeta Function)**.** *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$. The properties of the zeta function as they relate to the distribution primes. In particular, the Dirichlet series can be represented as a product of primes.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

Remark: This expresion also converges absolutely for $\Re(s) > 1$. Since

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{p^s}{p^s - 1} = 1 + \frac{1}{p^s - 1}$$

and $\sum (p^s - 1)^{-1}$ converges absolutely by comparison to the zeta function Dirichlet series.

Sketch of proof. Write $s = \sigma + it$. For each p,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p. We thus take for $m > N$

$$\prod_{p \leq N} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \leq N} \left(\sum_{k=1}^{m} \frac{1}{p^{ks}} + O(2^{-m\sigma})\right)$$

$$\begin{aligned}
&\overset{(*)}{=} \sum_{n=1}^{N} \frac{1}{n^s} + O_1\Big(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}\Big) + O(2^{-m\sigma}) \\
&= \zeta(s) + O_1\Big(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}\Big) + O(2^{-m\sigma})
\end{aligned}$$

Where we apply to Fundemental Theorem of Arithmetic in (*) to show that each term n^{-s} has coefficient 1 determined by the unique prime factorization. As $m \to \infty$, $2^{-m\sigma} \to 0$. Then we take $N \to \infty$, the tail of the infinite sum converges to zero too. □

Proposition 1.3. *ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too.*

We will prove Proposition 1.3 in two steps. First, we will extend ζ to $\sigma > 0$. Then, we will describe the continuation of the zeta function to the whole plane using by its functional equation: ζ has a line of symmetry across $\Re(s) = 1/2$.

Proposition 1.4. *Let $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then*

$$\xi(s) = \xi(1-s). \tag{3}$$

Extension of ζ to $\sigma > 0$. We apply integration by parts on Dirichlet series when $\sigma > 1$

$$\begin{aligned}
\zeta(s) &= \int_{1/2}^{\infty} x^{-s} d\lfloor x \rfloor \\
&= s\int_{1/2}^{\infty} \lfloor x \rfloor x^{-s-1} dx \\
&= s\int_{1}^{\infty} x^{-s} - \frac{\{x\}}{x^{-s-1}} dx \\
&= \frac{s}{s-1} - s\int_{1}^{\infty} \frac{\{x\}}{x^{-s-1}} dx
\end{aligned}$$

where in the last expression, the integral converges when $\sigma > 0$, and the pole at $s = 1$ arises from the first term. □

Proof of Proposition 1.4. Using

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx,$$

we make the substitution $x = \pi n^2 y$ to get

$$\begin{aligned}
\Gamma(s) &= \int_0^{\infty} e^{-\pi n^2 y} (\pi n^2 y)^{s-1} \pi n^2 dy \\
\implies \frac{\Gamma(s)}{\pi^s n^{2s}} &= \int_0^{\infty} e^{-\pi n^2 y} y^{s-1} dy
\end{aligned}$$

So that by the Monotone Convergence Theorem,

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n=1}^{\infty} \frac{\Gamma(s/2)}{\pi^{s/2} n^s} \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 x} x^{s/2-1} dx \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} \left(e^{-\pi n^2 x}\right) x^{s/2-1} dx.
\end{aligned}$$

We now let

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}, \quad \theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 2\omega(x) + 1,$$

and apply Poisson Summation to

$$\begin{aligned}
\theta(x) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i k u/\sqrt{x}} du \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} e^{-\pi k^2/x} \\
&= \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)
\end{aligned}$$

using the substitution $y\sqrt{x} = u$. Replacing with ω,

$$\sqrt{x}(2\omega(x) + 1) = 2\omega\left(\frac{1}{x}\right) + 1 \implies \omega\left(\frac{1}{x}\right) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}.$$

We thus write, using $y = 1/x$,

$$\begin{aligned}
\xi(s) &= \int_0^1 \omega(x) x^{s/2-1} dx + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \omega(1/y) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\sqrt{y}\omega(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{1-s} + \frac{1}{s} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{s(1-s)} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx.
\end{aligned}$$

ω decays exponentially in x, so the integral converges and the last expression is well defined on $\mathbb{C}$ with simple poles at $s = 1$ or $s = 0$. Finally, notice that the last expression is symmetric when s is replaced with $(1 - s)$, so proves equation 3. □

Finally, we extend to $\zeta(0)$ by noticing that the poles of the functional equation from $\zeta(1)$ and $\Gamma(0)$ cancel out, so the Riemann Extension Theorem can be applied.

From the functional equation, we get 'trivial' zeros of the zeta function from the poles of Γ.

Corollary 1.5. *On* $\Re(s) > 1$ *or* $\Re(s) < 0$, $\zeta(s) \neq 0$, *except* $\forall n \in \mathbb{N}, \zeta(-2n) = 0$.

Proof. Using the product representation of ζ where it converges, none of $(1 - p^{-s})^{-1} = 0$, so $\zeta(s) \neq 0$ on $\Re(s) > 1$. Γ has no zeros and has a simple pole at $-n$ for all $n \in \mathbb{N}$, so by equation 3 we get the zeros for $\Re(s) > 0$ are exactly at the negative even integers. □

These zeros are known as the trivial zeros of ζ. The remaining zeros lie between $0 \leq \Re(s) \leq 1$.

Definition 1.6 (Critical Strip and Critical Line). *We denote the region* $0 \le \Re(s) \le 1$ *as the* ***critical strip****. We denote the line* $\Re(s) = 1/2$ *as the* ***critical line****.*

Corollary 1.7. *On the critical strip, if* $\zeta(s) = 0$, $\zeta(\overline{s}) = \zeta(1-s) = \zeta(1-\overline{s}) = 0$.

Proof. This follows from equation 3, and $\zeta(\overline{s}) = \overline{\zeta(s)}$ holds where the Dirichlet series converges, thus holds everywhere. ☐

The number of zeros in the critical strip can be calculated using the argument principle applied to the function ξ over the box with corners $-1+iT$,$-1-iT$,$2-iT$,$2+iT$. Applying the functional equation, we get the following result.

Theorem 1.8 (Number of zeros of ζ). *The number of zeros up to height* T

$$\#\{\sigma + it \mid \zeta(\sigma+it) = 0, 0 \le \sigma \le 1, |t| \le T\} = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

2 The Prime Number Theorem

Theorem 2.1 (Prime Number Theorem). *Let* $\Pi(N) = \sum_{p \le N} 1$. *Then*

$$\Pi(N) = (1+o(1))\frac{N}{\log N}.$$

In this section we will prove the Prime Number Theorem. This result is a minor goal of this paper. The Prime Number theorem serves as a starting point for studying primes in short intervals, and sets the stage for zero-density theorems.

Definition 2.2 (Von Mangoldt Function). *The* ***Von Mangoldt function*** Λ *is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p, \text{ if } n = p^k \text{ for some } k \in \mathbb{N} \\ 0, \text{ else} \end{cases}$$

The sum of the Von Mangoldt function $\sum \Lambda(n)$ is a more natural way to express a prime counting function in the language of ζ. To see why, consider the expression

$$\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} &= (\log \zeta(s))' \\
&= \left[-\sum_p \log\left(1-p^{-s}\right)\right]' \\
&= -\sum_p \frac{p^s \log p}{1-p^{-s}} \\
&= -\sum_p \log p \sum_{k\in\mathbb{N}} p^{-ks} \\
&= -\sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s}
\end{aligned}$$

on $\Re(s) > 1$ where the sum and products are absolutely convergent.

Proposition 2.3. $\sum_{n \le N} \Lambda(n) = (1+o(1))N$ *implies the Prime Number Theorem.*

Proof. On one hand, we have

$$\begin{aligned}
\sum_{n\le N} \Lambda(n) &\le \sum_{p \le N} \Lambda(p) \\
&\le \Pi(N)\log N.
\end{aligned}$$

And for $\epsilon > 0$,

$$\begin{aligned}\sum_{n\leq N}\Lambda(n) &\geq \sum_{N^{1-\epsilon}\leq n\leq N}\Lambda(n)\\ &\geq \sum_{N^{1-\epsilon}\leq p\leq N}(1-\epsilon)\log(N)\\ \\ &= (1-\epsilon)(\Pi(N)\log(N)+O(N^{1-\epsilon}\log N)).\end{aligned}$$

□

Moreover, the sum of the Von Mangoldt function can be related to the zeros of the zeta function. Let ϕ be smooth and rapidly decaying at infinity, and $\tilde{\phi}$ be its Mellin transform. Let $N\in\mathbb{N}$ and $c\geq 2$. Then

$$\begin{aligned}\sum_{n\in\mathbb{N}}\Lambda(n)\phi\left(\frac{n}{N}\right) &= \sum_{n\in\mathbb{N}}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\sum_{n\in\mathbb{N}}\Lambda(n)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)N^s\frac{\zeta'(s)}{\zeta(s)}ds\end{aligned}\tag{4}$$

By the rapid decay of $\tilde{\phi}$, we change the line of integration from c to $-\infty$, we get residue contributions from a pole at $s=1$, $s=0$, as well as all ρ such that $\zeta(\rho)=0$ on the critical strip, and all the trivial zeros. Morally, we can take the indicator function $\phi=1$ on $[0,1]$.

$$\sum_{n\leq N}\Lambda(n) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}N^s\frac{\zeta'(s)}{\zeta(s)}ds\tag{5}$$

If we move the line of integration across to $-\infty$, this gives

$$\begin{aligned}\sum_{n\leq N}\Lambda(n) &= N-\sum_{\rho}\frac{N^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}+\sum_{k\in\mathbb{N}}\frac{N^{-2k}}{2k}\\ &= N-\sum_{\rho}\frac{N^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2}\log\left(1-N^{-2}\right).\end{aligned}\tag{6}$$

This formula, due to von Mangoldt, can be derived with more care about the convergence in the sum: The sum over zeros ρ is not absolutely convergent, and is ordered in increasing $|\Im(\rho)|$.

Theorem 2.4 (Riemann-von Mangoldt explicit formula)**.** *Let $N>1$ be not a prime power. Then*

$$\sum_{n\leq N}\Lambda(n) = N-\lim_{T\to\infty}\sum_{|\Im(\rho)|\leq T}\frac{N^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2}\log\left(1-N^{-2}\right).\tag{7}$$

In practice, we truncate the integral in up to height T to obtained a truncated version of the explicit formula. This is obtained through integration along the choice of $c=1+1/\log N$

Theorem 2.5. *Let $N>1$. Then*

$$\sum_{n\leq N}\Lambda(n) = N-\sum_{|\Im(\rho)|\leq T}\frac{N^\rho}{\rho}+O(\frac{N}{T}(\log NT)^2)+O(\log N).\tag{8}$$

The term N in the explicit formula is already suggestive of the Prime Number Theorem. The major error term comes from N^ρ in the sum, so bounding the $\Re(\rho)$ becomes the most important part in reducing the error term in the prime number theorem. This in turn is equivalent to bounding $\Re(\rho)$, and the best case is when all the zeros have real part 1/2. This assumption is known as the Riemann Hypothesis, and has not yet been proved.

Conjecture 2.6 (Riemann Hypothesis). *The **Riemann Hypothesis** (RH) asserts that on the critical strip,*

$$\zeta(s) = 0 \implies \Re(s) = \frac{1}{2}.$$

Assuming the Riemann Hypothesis, we consider the sum over the non trivial zeros

$$\left| \sum_{|\Im(\rho)| \le T|} \frac{N^\rho}{\rho} \right| \le N^{1/2} \sum_{|\Im(\rho)| \le T} \left| \frac{1}{\rho} \right|.$$

We know there are $\sim \log T$ zeros of height $[T, T+1)$, thus the integral $\sum |\rho^{-1}|$ behaves as

$$\sum_{n \le T} \frac{\log n}{n} = O(\log^2 T).$$

Taking $N = T$ in the truncated explicit formula, we obtain

$$\sum_{n \le N} \Lambda(n) = N + O(N^{1/2} \log^2 N). \tag{9}$$

Which implies the prime number theorem.

Remark: The PNT stated in 9 (with this error bound) can be shown to be equivalent to the Riemann Hypothesis.

The prime number theorem is also true without assuming the strong Riemann Hypothesis. To show this, it is sufficient to show that there are no zeros with real part 1, so the terms in the sum contributes $O(N^{1-\epsilon})$ which will be dominated by N.

Theorem 2.7. *Let $t \in \mathbb{R}$. Then $\zeta(1+it) \neq 0$.*

Proof. Let $\sigma > 1$. We consider the expressions

$$\Re\left(\frac{\zeta'}{\zeta}(\sigma + it)\right) = -\sum_n \frac{\Lambda(n)}{n^\sigma} \cos(t \log n)$$

and

$$2(1+\cos\theta)^2 = 2 + 4\cos\theta + 2\cos^2\theta = 3 + 4\cos\theta + \cos 2\theta.$$

So that

$$\begin{aligned} \Re\left(3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma+it) + \frac{\zeta'}{\zeta}(\sigma+2it)\right) &= -\sum_n \frac{\Lambda(n)}{n^\sigma}(3 + 4\cos(t\log n) + \cos(2t\log n)) \\ &= -\sum_n \frac{\Lambda(n)}{n^\sigma} 2(1+\cos(t\log n))^2 \\ &\le 0. \end{aligned}$$

Now for the sake of contradiction, we let $\zeta(1+it) = 0$ be a zero of order d, and since we know ζ has a pole of order 1 at $s = 1$, we can let $t \neq 0$. Consider the function $f(s) = \zeta(s)^3\zeta(s+it)^4\zeta(s+2it)$. By the computation above, $\Re(f'/f) \le 0$ when $\Re(s) > 1$. But we also have that f, by construction, has a zero of order $k \ge 4d-3 > 0$ at $s = 1$. So that $\Re(f'/f) = k/(s-1) +$ a holomorphic part. Now taking $s \to 1^+$, $\Re(f'/f) \to +\infty$, contradicting $\Re(f'/f) \le 0$. □

Proof of the Prime Number Theorem. Let $\phi = \phi_{N,T}$ be a bump function that equals 1 on the interval $[2, N]$ and supported on $[3/2, N + N/T]$, such that $\phi^{(j)}(x) = O_j(1)$ and $\phi^{(j)}(x) = O_j(T/x)^j$ on the intervals $[3/2, 2]$ and $[N, N + N/T]$ respectively. Then

$$\sum_{n \le N} \Lambda(n) \le \sum_n \Lambda(n)\phi(n)$$

$$
\begin{aligned}
&= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}\frac{\zeta'(s)}{\zeta(s)}ds \\
&= \tilde{\phi}(1) - \sum_{\rho}\tilde{\phi}(\rho) - \sum_{n}\tilde{\phi}(-2n)
\end{aligned}
$$

The first term

$$
\begin{aligned}
\tilde{\phi}(1) &= \int_0^{\infty}\phi(x)dx \\
&= N + O(N/T)
\end{aligned}
$$

gives the term we want from the PNT. In the third term, we rewrite by Monotone Convergence

$$
\begin{aligned}
\sum_n \tilde{\phi}(-2n) &= \sum_n \int_0^{\infty}\phi(x)x^{-2n-1}dx \\
&= \int_0^{\infty}\phi(x)\sum_n x^{-2n-1}dx \\
&= \int_0^{\infty}\phi(x)\frac{1}{x^3-x}dx \\
&= O(1)
\end{aligned}
$$

Finally, to bound the second term, we define a parameter $\bar{T} = \bar{T}(T)$ and split the sum into

$$
\sum_{|\Im\rho|\le\bar{T}}\tilde{\phi}(\rho) + \sum_{|\Im\rho|>\bar{T}}\tilde{\phi}(\rho)
$$

In the first summation, we let $\epsilon = \epsilon_{\bar{T}}$ such that there are no zeros in the region $\Re(s) > 1-\epsilon, |\Im(s)| \le \bar{T}$, then

$$
\begin{aligned}
\sum_{|\Im\rho|\le\bar{T}}\tilde{\phi}(\rho) &= \sum_{|\Im\rho|\le\bar{T}}\int_0^{\infty}\phi(x)x^{\rho-1}dx \\
&= O_T(N^{1-\epsilon}).
\end{aligned}
$$

In the second summation, we apply integration by parts to show that

$$
\begin{aligned}
\left|\int_0^{\infty}\phi(x)x^{\rho-1}dx\right| &= \left|\frac{1}{\rho(\rho+1)}\int_0^{\infty}\phi''(x)x^{\rho+1}dx\right| \\
&= O\left(\frac{1}{|\rho|^2}\frac{T^2}{N^2}\frac{N}{T}N^2\right) \\
&= O\left(\frac{1}{|\rho|^2}TN\right)
\end{aligned}
$$

The sum over $|\rho|^{-2}$ behaves as $\sum_n n^{-2}\log n$, so we can pick $\bar{T}$ large enough depending on T to make the contribution of $\sum_{\Im(\rho)>\bar{T}}|\rho|^{-2}$ to be $O(T^{-2})$. So that

$$
\begin{aligned}
\sum_{n\le N}\Lambda(n) &\le N + O(N/T) + O_T(N^{1-\epsilon}) \\
&= N + O(N/T)
\end{aligned}
$$

for $N = N(T)$ sufficiently large. Similarly, repeating the same argument on $\phi = \phi_{N,T}$ equals 1 on the interval $[2, N - N/T]$ and supported on $[3/2, N]$ gives

$$
\sum_{n\le N}\Lambda(n) \ge N - O(N/T).
$$

Sending $T \to \infty$ gives the PNT. □

3 Primes in Short Intervals

We would like to answer the following question about primes in short intervals. Let $y = y(x)$. What is the smallest asymptotic behavior of y such that

$$\sum_{x \le n \le x+y} \Lambda(n) = (1 + o(1))y \tag{10}$$

for large enough x? That is, what is the shortest interval such that we have the behavior of the Prime Number Theorem? If 10 holds for some y, we say the Prime Number Theorem holds for intervals of y.

Remark: This question can be rephrased into finding primes in short intervals, by including a factor of $\log x$.

Proposition 3.1. *Assume the RH. Then the Prime Number Theorem holds in intervals of $x^{1/2+\epsilon}$.*

Proof. Assume the RH, then

$$\sum_{x \le n \le x+y} \Lambda(n) = y + O(x^{1/2} \log^2 x) = x^{1/2+\epsilon} + o(x^{1/2+\epsilon}),$$

so that the sum is non-zero for large enough x. □

Recalling that the error term is related to the real part of the zeros of the Zeta function, we motivate the following definition of zero-density:

Definition 3.2. *Let $N(\sigma, T)$ denote the number of zeros of the zeta function with real part greater than σ and imaginary part between $-T$ and T. That is,*

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \mid \beta \ge \sigma, |\gamma| \le T\}.$$

Remark: The ideal scenario is that $N(\sigma, T) = 0$ for all $\sigma > 1/2$.

Theorem 3.3 (Chudakov)**.** *There exists a constant A such that $\zeta(\sigma + iT) \ne 0$ in the region*

$$\sigma > 1 - A\frac{\log\log T}{\log T}.$$

add reference

Theorem 3.4 (Hoheisel)**.** *Let A be defined as in the previous theorem. Suppose that $N(\sigma, T) \ll T^{a(1-\sigma)} \log^b T$ uniformly in $1/2 \le \sigma < 1$ and in T. Then for all*

$$\theta > 1 - \frac{1}{a + b/A},$$

the Prime Number Theorem holds in intervals of $y = x^\theta$.

Proof. First notice that $N(1/2, T)$ gets at least half of the zeros of height T, so $a \ge 2$. Let $y \ll x$. We consider the expression

$$S = S(x, y) = \frac{1}{y} \sum_{x \le n \le x+y} \Lambda(n).$$

By the truncated version of the explicit formula in Theorem 2.5, we get

$$S = 1 - \sum_{|\Im(\rho)| \le T} \frac{(x+y)^\rho - x^\rho}{\rho y} + O(\frac{x}{yT}(\log xT)^2) + O(\frac{\log x}{y}).$$

We want to show that except for the constant 1 term, the remaining parts are $o(1)$. We focus on the sum over the non-trivial zeros with height less than T, and enumerate them ρ_j. For each $\rho_j = \sigma_j + it_j$, we apply the Mean Value Theorem on the function $f(x) = x_j^\rho$ to get

$$\left|\sum_{\rho_j} \frac{(x+y)^\rho - x^\rho}{\rho y}\right| \le \sum_{\rho_j} \left|\frac{(x+y)^{\rho_j} - x^{\rho_j}}{\rho_j y}\right|$$

$$
\begin{aligned}
&\ll \sum_{\rho_j} x^{\sigma_j - 1} \\
&= \sum_{\rho_j} x^{\sigma_j - 1} - x^{-1} + x^{-1} \\
&= O\left(\frac{T \log T}{x}\right) + \sum_{\rho_j} x^{\sigma_j - 1} - x^{-1}.
\end{aligned}
$$

And by replacing $x^{\sigma_j} - 1$ by an integral,

$$
\begin{aligned}
\sum_{\rho_j} x^{\sigma_j - 1} - x^{-1} &= \sum_{\rho_j} \int_0^{\sigma_j} x^{u-1} \log x \ du \\
&= \int_0^{1 - A\frac{\log\log T}{\log T}} \sum_{\rho_j} \mathbb{1}_{u \le \sigma_j} x^{u-1} \log x \ du \\
&= \int_0^{1 - A\frac{\log\log T}{\log T}} N(u, T) x^{u-1} \log x \ du
\end{aligned}
$$

Where in the penulitimate step we made use of Chudaokov's bound and exchanged the order of integration and summation. Now we can apply the hypothesis that $N(\sigma, T) \ll T^{a(1-\sigma)} \log^b T$ for $\sigma > 1/2$ and trivially $N(\sigma, T) \ll T \log T \ll T^{a(1-\sigma)} \log^b T$ for $\sigma \le 1/2$. This evaluates to

$$
\begin{aligned}
\sum_{\rho_j} x^{\sigma_j - 1} - x^{-1} &\ll \int_0^{1 - A\frac{\log\log T}{\log T}} T^{a(1-u)} \ \log^b T \ x^{u-1} \log x \ du \\
&= \log^b T \int_0^{1 - A\frac{\log\log T}{\log T}} \left(\frac{T^a}{x}\right)^{1-u} \log x \ du \\
&= \frac{\log x \log^b T}{a \log T - \log x} \left[\frac{T^a}{x} - \left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}} \right]
\end{aligned}
$$

Combined with the previous bounds, we have

$$
S = 1 + O\left(\frac{T \log T}{x}\right) + O\left(\frac{\log x \log^b T}{a \log T - \log x} \left[\frac{T^a}{x} - \left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}} \right] \right) + O\left(\frac{x}{yT} (\log xT)^2\right) + O\left(\frac{\log x}{y}\right).
$$

To make all terms (except for the first) to be $o(1)$, we want to set $y = x^\theta$, $T = x^k$, such that θ, k satisfy

$$
k < 1, \ \ ak < 1, \ \ k + \theta > 1,
$$

so that the second, fourth and fifth terms are $o(1)$ in x. For the third term, we can simplify

$$
\begin{aligned}
\frac{\log x \log^b T}{a \log T - \log x} \left[\frac{T^a}{x} - \left(\frac{T^a}{x}\right)^{A\frac{\log\log T}{\log T}} \right] &= \frac{k^b \log^b x}{ak - 1} \left[x^{ak-1} - x^{(ak-1)A\frac{\log(k \log x)}{k \log x}} \right] \\
&\le \frac{k^b \log^b x}{1 - ak} x^{ak-1} + \frac{k^b \log^b x}{1 - ak} \exp\left((ak-1)A \frac{\log(k \log x)}{k} \right) \\
&\le \frac{k^b \log^b x}{1 - ak} x^{ak-1} + \frac{k^b \log^b x}{1 - ak} \exp\left((ak-1)A \frac{\log(k \log x)}{k} \right) \\
&= O(x^{ak-1}) + O\left((\log x)^{b + \frac{(ak-1)A}{k}} \right).
\end{aligned}
$$

We require that the last term decays in x, and this happens when

$$
b + \frac{(ak-1)A}{k} < 0 \implies (aA + b)k < A \implies k < \frac{1}{a + \frac{b}{A}}
$$

We had $a \geq 2 > 1$, so this k satisfies $k < 1$ and $ak < 1$. Finally, for $k = 1/(a + bA^{-1}) - \delta/2$ we let $\theta = 1 - k + \delta$ to satisfy $\theta + k > 1$, so we can find any $1/(a + bA^{-1}) + \delta > \theta > 1 - 1/(a + bA^{-1})$, and

$$\frac{1}{y} \sum_{x \leq n \leq x+y} \Lambda(n) = S = 1 + o(1)$$

for $y = x^\theta$. This completes the proof.

$\square$

Theorem 3.4 gives the classical way to relate the distribution of primes in short intervals to the density of zeros away from the real-half line. The long-standing bound for zero density is due to separate proofs of Ingham and Huxley:

Theorem 3.5 (Ingham bound for zero density)**.** *Let* $1/2 \leq \sigma \leq 3/4$. *We have*

$$N(\sigma, t) \lesssim T^{\frac{3(1-\sigma)}{2-\sigma}}.$$

Theorem 3.6 (Huxley bound for zero density)**.** *Let* $3/4 \leq \sigma \leq 1$. *We have*

$$N(\sigma, t) \lesssim T^{\frac{3(1-\sigma)}{3\sigma-1}}.$$

Combining these two bounds, we get the following zero density theorem.

Theorem 3.7 (Ingham-Huxley bound for zero density)**.** *We have*

$$N(\sigma, t) \lesssim T^{\frac{12}{5}(1-\sigma)},$$

uniformly for $1/2 \leq \sigma \leq 1$

Notice that $12/5$ comes from $\sigma = 3/4$. In June 2024, Guth and Maynard published a proof that improves the Ingham-Huxley bound at $\sigma \in [7/10, 8/10]$, thus improving the result of primes in short intervals (as well as many other number theoretic results). The following sections will be dedicated to Huxley's proof of zero density, as well as Guth-Maynard's ideas in the proof. Finally, adapting from Guth and Maynard, we will provide a proof of the analogous zero-density result for L-functions.

Theorem 3.8 (Guth-Maynard bound for zero density)**.** *We have*

$$N(\sigma, t) \lesssim T^{\frac{30}{13}(1-\sigma)},$$

uniformly for $1/2 \leq \sigma \leq 1$

4 Huxley's Proof of Zero Density

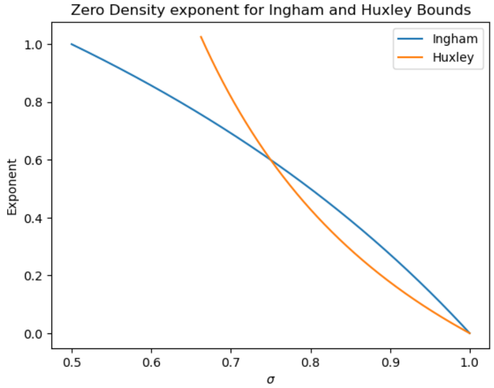
Theorem 4.1 (Huxley)**.** *We have*

$$N(\sigma, t) \lesssim T^{\frac{12}{5}(1-\sigma)}.$$

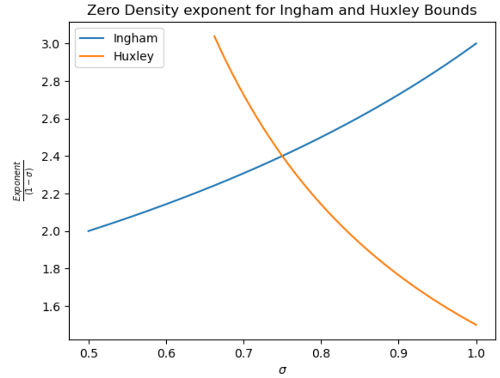
Huxley's methodology for detecting zeros as follows. Let $M_x(s) \coloneqq \sum_{n=1}^{x} \mu(n) n^{-s}$. Since this also converges absolutely on $\Re(s) > 1$, we can write the dirichlet series of $\zeta(s)M_x(s)$ as

$$\zeta(s)M_x(s) \coloneqq \sum_n a_n n^{-s}$$

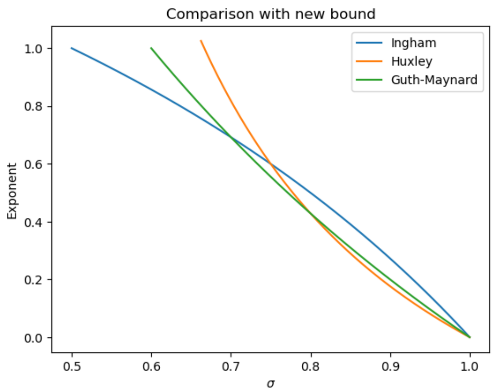
for some choice of $a_n = a_n(x)$. The zeros of its analytic continuation will contain the zeros of ζ. This may look inefficient as we may have introduced extra zeros from M_x, but the tradeoff is that we can bound these a_n's.



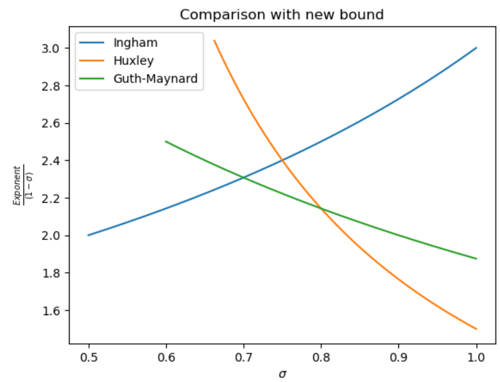
(a) The bounds for the exponent coincide at $\sigma = 3/4$



(b) $\sigma = 3/4$ is also the bottleneck when written in Hoheisel's form.



(c) Guth-Maynard's result improves in the range at $\sigma \in [7/10, 8/10]$.



(d) The exponent is reduced around the bottleneck region.

Proposition 4.2. *We have*

$$\begin{cases} a_1 = 1, & \\ a_n = 0, & \text{if } 1 < n \le x, \\ |a_n| \le d(n), & \text{if } n > x. \end{cases}$$

Proof. For all $n \le x$, this follows from Möbius inversion. For $n > x$, we just apply the trivial bound $|\mu(d)| \le 1$ on

$$a_n = \sum_{d|n} \mu(d).$$

□

Let $y > x$ a parameter to be choosen later, and $y \le T^A$ for an absolute constant A. We apply the Mellin transform to

$$\begin{aligned} \sum_n a_n n^{-s} e^{-n/y} =& \frac{1}{2\pi i} \sum_n a_n n^{-s} \int_{2-i\infty}^{2+i\infty} \Gamma(w) y^w n^{-w} dw \\ =& \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s+w) M_x(s+w) \Gamma(w) y^w dw. \end{aligned}$$

If we move the line of integration to $\Re(w) = 1/2 - \Re(s)$, we get simple pole residue contributions from ζ and

Γ

$$
\begin{aligned}
e^{-1/y}+\sum_{n>x}a_n n^{-s}e^{-n/y}=\sum_n a_n n^{-s}e^{-n/y} =&\zeta(s)M_x(s)+M_x(1)\Gamma(1-s)y^{1-s}\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta(\frac{1}{2}+i\Im(s)+iw)M_x(\frac{1}{2}+i\Im(s)+iw)\\
&\cdot\Gamma\left(\frac{1}{2}-\Re(s)+it\right)y^{\frac{1}{2}-\Re(s)+it}dt.
\end{aligned} \tag{11}
$$

We take y large enough so that $e^{-1/y}$ is close to 1. Since $M_x(s)$ is an approximation of $1/\zeta$, we should expect that the term $\zeta(s)M_x(s)$ is about 1 most of the time and the other terms are small. However, if s is a zero of ζ, then $\zeta(s)M_x(s)=0$, so at least one of the following things need to happen

(i) $|\sum_{n>x}a_n n^{-s}e^{-n/y}|$ is large.

(ii) The integral in t is large.

(iii) $|M_x(1)\Gamma(1-s)y^{1-s}|$ is large.

We thus transform the problem of detecting zeros to counting the number of occurences of extreme values. We will later see that type (iii) zeros are negligible, so we need to bound the number of type (i) and type (ii) zeros.

Lemma 4.3. *Let a be an arithmetic function, and $D_N(s)=\sum_{n\le N}a(n)n^s$. If $W=\{t_j\}\subseteq[0,T]$ is a one-separated set such that*

$$|D_N(it_j)|>V\ \ \forall j,$$

then

$$|W|\ll\frac{\log^2 T}{V^\alpha}\int_{-(\log N)^{-1}}^{(\log N)^{-1}}\int_0^T|D_N(x+it)|^\alpha dt\ dx$$

for $\alpha>0$.

Proof. With a cost of $O(1)$ we can consider $W\subseteq[(\log N)^{-1},T-(\log N)^{-1}]$. Since D_N is analytic, $|D_N|^\alpha$ is subharmonic. Let $B(t_j)$ describe a square-box of side length $(\log N)^-1$ centered at it_j in the complex plane, then

$$V^\alpha\sum_j 1\le\sum_j|D(it_j)|^\alpha\le\log^2 N\sum_j\int_{B(t_j)}|D(s)|^\alpha dA\le\log^2 T\int_{-(\log N)^{-1}}^{(\log N)^{-1}}\int_0^T|D_N(x+it)|^\alpha dt\ dx.$$

□

Corollary 4.4. *Let $W=\{t_j\}\subseteq[0,T]$ be a one-separated set such that*

$$\left|\zeta\left(\frac{1}{2}+it_j\right)\right|>V\ \ \forall j,$$

then

$$|W|\ll TV^{-4}\log^{O(1)}T.$$

Proof. We have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right|\ll\sum_{n\le\sqrt{T}}n^{-1/2-it},$$

so applying the previous lemma on $D(s)=\sum_{n\le\sqrt{T}}n^{-s}$ with $\alpha=4$ gives

$$|W|\ll\frac{\log^2 T}{V^4}\int_{-2(\log T)^{-1}}^{2(\log T)^{-1}}\int_0^T|D_N(x+it)|^4 dt\ dx$$

□

Lemma 4.5 (Halász Inequality)**.** *Let a be an arithmetic function, and $D_N(s) = \sum_{n\le N} a(n)n^s$, and $G = \sum_{n\le N} |a(n)|^2$. If $W = \{t_j\} \subseteq [0,T]$ is a one-separated set such that*

$$|D_N(it_j)| > V \ \ \forall j,$$

then

$$|W| \lesssim GNV^{-2} + G^3 NTV^{-6}.$$

Proof. Let $-\theta_j$ be the argument of $D(it_j)$, then we have

$$V|W| \le \sum_j |D(it_j)| = \sum_j e^{\theta_j} D(it_j) = \sum_{n\le N}\sum_j e^{i\theta_j} a(n) n^{-it_j}.$$

By Cauchy-Schwarz, this summation is

$$\le \left(\sum_{n\le N} |a(n)|^2\right)^{1/2} \left(\sum_{n\le N}\left|\sum_j e^{i\theta_j} n^{-it_j}\right|^2\right)^{1/2}.$$

The first summation in this expression is G, so we want to bound the latter nested summation. Expanding the summation gives

$$\begin{aligned}\sum_{n\le N}\left|\sum_j e^{i\theta_j} n^{-it_j}\right|^2 &= \sum_{n\le N}\sum_{j_1,j_2} e^{i\theta_{j_1}-i\theta_{j_2}} n^{it_{j_1}-it_{j_2}} \\ &\le |W|N + \sum_{j_1,j_2}\left|\sum_{n\le N} n^{it_{j_1}-it_{j_2}}\right|.\end{aligned}$$

TODO ☐

Proof of Huxley's Zero Density Theorem. From equation 11, we take $y > 6$ so that $e^{-1/y} > 5/6$. We also truncate the sum in $n > x$ to $x < n \le y^2$ with an error of $1/6$ for large enough y. Finally, we truncate the integral in t to the range $|t| \le B\log T$ with an error of $1/6$. Thus, s is a zero only if

(i) $|\sum_{x<n\le y^2} a_n n^{-s} e^{-n/y}| \ge \frac{1}{6}$, or

(ii) $\frac{1}{2\pi}|\int_{-B\log T}^{B\log T} \zeta(\frac{1}{2} + i\Im(s) + iw) M_x(\frac{1}{2} + i\Im(s) + iw)\Gamma\left(\frac{1}{2} - \Re(s) + it\right) y^{\frac{1}{2}-\Re(s)+it} dt| \ge \frac{1}{6}$, or

(iii) $|M_x(1)\Gamma(1-s)y^{1-s}| \ge \frac{1}{6}$.

Of the zeros $\rho = \beta + i\gamma$ of ζ in the region, at the cost of a factor of $\log T$, we take representatives such that if $\rho_1 \ne \rho_2$ then $|\rho_1 - \rho_2| \ge 1$. For Class (i) zeros, we split the sum dyadically to get

$$\left|\sum_{n\sim U, n\le y^2} a(n) n^{-\rho} e^{n/y}\right| \ge O((\log T)^{-1}), \tag{12}$$

for some $x \le U = 2^k \le y$. Applying Lemma 4.5, we get that the number of times that equation 12 can happen for each U is

$$\lesssim U^{2-2\sigma} + U^{4-6\sigma} T \lesssim$$

(Note that the log factors are dominated by any choice of T^ϵ) ☐

5 Guth-Maynard's proof of Large Values of Dirichlet Polynomials

In June 2024, Guth and Maynard published an improvement of the large values of Dirichlet polynomails estimate at $\sigma \in [7/10, 8/10]$.

Theorem 5.1 (Guth-Maynard Large Values Estimate)**.** *Let* (b_n) *be a sequence of complex numbers such that* $|b_n| \le 1$ *for all* n*, and* $W = \{t_j\}_{j=1}^{|W|}$ *be a* 1*-separated set* $\subseteq [0, T]$*, such that*

$$\left| \sum_{n \sim N} b_n n^{it_j} \right| \ge V$$

for each $t_j \in W$*. Then*

$$|W| \lesssim N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4}.$$

Let us compare this bound to Lemma 4.5, which states

$$|W| \lesssim N^2 V^{-2} + T N^4 V^{-6}.$$

In the critical case $V = N^{3/4}, N \le T^{5/6-\epsilon}$, the original bound will give

$$|W| \lesssim N^2 N^{-3/2} + T N^4 N^{-9/2} \lesssim N^{1/2} + T N^{-1/2} \lesssim T N^{-1/2},$$

while the bound by Guth and Maynard gives

$$|W| \lesssim N^2 N^{-3/2} + N^{18/5} N^{-3} + T N^{12/5} N^{-3} \lesssim N^{1/2} + T N^{-3/5} \lesssim T N^{-3/5}.$$

5.1 Outline and Sketch of proof

The structure of the proof can be broken down as follows: We first notice that $|W|$ is bounded by the operator norm of a matrix M. This operator norm, using results from linear algebra, is bounded by the trace. Applying Poisson summation on the trace gives 4 terms that are separately handled, which we will name S_0 to S_3. We will see that S_0 gives the 'main term' that is consistent with the density hypothesis, S_1 is negligible, S_2 is bounded by a theorem by Heath-Brown, which we state below.

Theorem 5.2 (Heath-Brown)**.** *Let* $\mathcal{S} = \{(t_j, \chi_j)\}$ *be one-separate, primitive characters of modulus* q*. Then*

$$\sum_{\substack{(t_1,\chi_1)\\(t_2,\chi_2)}} \left| \sum_{n=1}^{N} b_n n^{-i(t_1 - t2)} \chi_1 \bar{\chi}_2(n) \right|^2 \lesssim |\mathcal{S}| N^2 + |\mathcal{S}|^2 N + |\mathcal{S}|^{5/4} (qT)^{1/2} N.$$

The most tricky term, S_3 is a summation over a three-dimensional lattice. We will see that S_3 is bounded by what is known as the *additive energy* of the set W, defined by

$$E(W) := \#\{t_1, t_2, t_3, t_4 \in W : |t_1 + t_2 - t_3 - t_4| \ll T^{\epsilon}\}.$$

This term describes the 'additive structure' of W. We see that $E(W)$ is bounded below by $|W|^2$, as the condition is satisfied when $t_1 = t_3$ and $t_2 = t_4$. Moreover, since W is $1-$*separated*, the choice of t_1, t_2, t_3 fixes $O(1)$ choices for t_4, so $E(W)$ is bounded above by $|W|^3$. In the extreme case that the additive structure of W is high, such as when $t_j = j\alpha$ for a constant α, the energy of the set is $O(|W|^3)$. This definition naturally arrises from taking the fourth moment of the function

$$R(v) := \sum_{t \in W} v^{it}.$$

This gives us

$$R(v)^4 = \sum_{t_1, t_2, t_3, t_4 \in W} v^{i(t_1 + t_2 - t_3 - t_4)}.$$

The naive choice $E(W) \le W^3$ is slightly too loose to beat the Ingham-Huxley bound. However, an orthogonal bound can be found for $E(W)$ based on Heath-Brown's theorem. Finally, the bound in 7 and 8 combined is enough to give an improvement in most cases, a further refinement of the S_3 bound was required. This relies on the averaging over the affine summations of R.

$$\sup_{0 < M_1, M_2, M_3 < M} \int \left(\sum_{\substack{|m_1| \sim M_1\\ |m_2| \sim M_2\\ |m_3| \ll M_3}} \left| R\left(\frac{m_1 u + m_3}{m_2} \right) \right| \right)^2 du \lesssim M^6 \|R\|_{L_2}^4 + M^4 \|f\|_{L_4}^4.$$

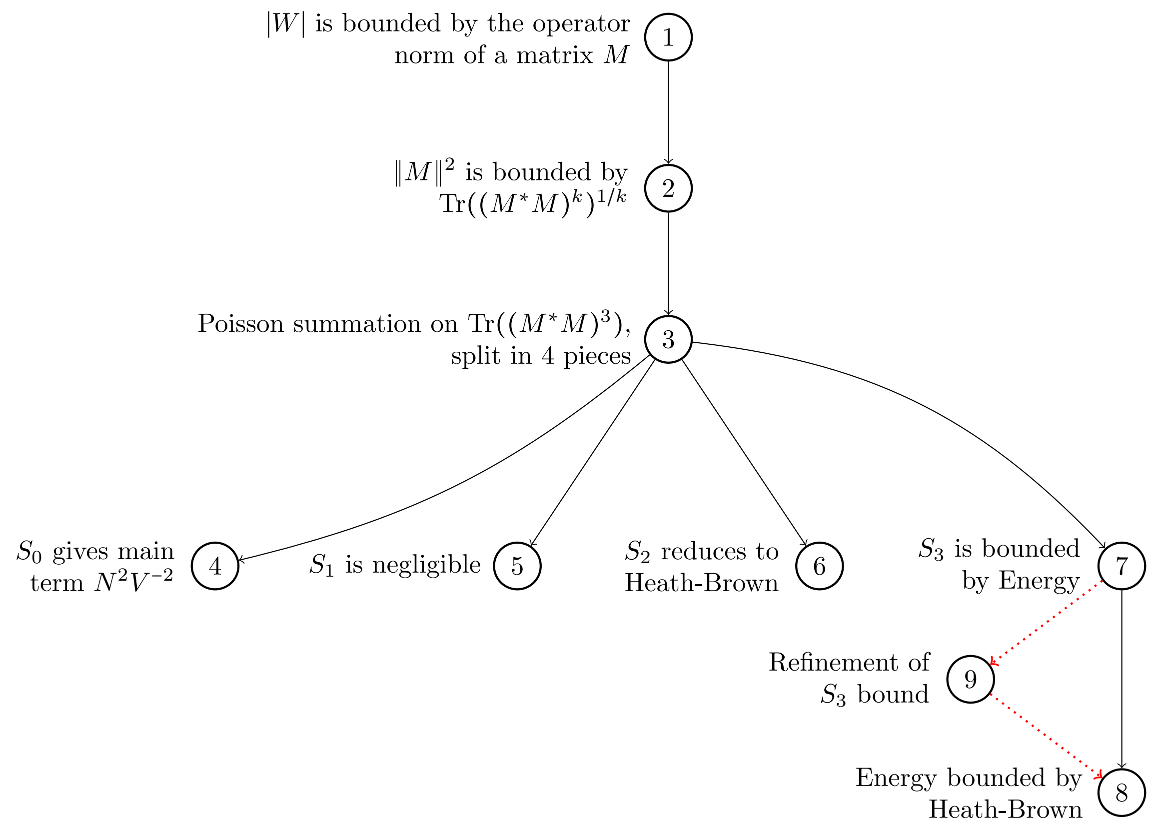


Figure 2: Graphical representation of Guth-Maynard proof outline

We give a quick sketch of the whole proof below. In the next section, we will give a full proof of the generalized statement of the theorem that considers primitive Dirichlet characters mod q. The proof of Guth-Maynard can be recovered by using the special case $q = 1$.

0. Setup

First, as in the theorem, we let (b_n) be a sequence of complex numbers such that $|b_n| \le 1$ for all n,

$$D_n(t) := \sum_{n\sim N} b_n n^{it},$$

$W = \{t_j\}_{j=1}^{|W|}$ be a T^ϵ-separated set $\subseteq [0,T]$, such that

$$|D_n(t_j)| \ge V$$

for each $t_j \in W$. Notice that we now let the set be T^ϵ separated for $\epsilon > 0$. This means that we will give up a factor of T^ϵ in the final bound, but this makes many computations cleaner as this T^ϵ dominates the log factors. Moreover, we can introduce a bump function ω with support in $[1,2]$ to localize the summation, and rewrite

$$D_n(t_j) = \sum_n \omega\left(\frac{n}{N}\right) b_n n^{it_j}.$$

This is added for the Poisson summation in step 3.

1. Bounding $|W|$ with operator norm

We view $\vec{b} = (b_n)_{n\sim N}$ as a N-dimensional vector, and consider the $|W| \times N$ matrix, indexed by j from 1 to $|W|$ and $n \sim N$,

$$M_{j,n} = n^{it_j} = \omega\left(\frac{n}{N}\right) n^{it_j}.$$

Then we can view the j-th entry of the product $M\vec{b}$ as $D_n(t_j)$. In other words

$$|M\vec{b}|^2 \geq V^2|W|.$$

However, we can bound $|M\vec{b}|$ using the operator norm of M and $|b_n| \leq 1$ to get

$$|M\vec{b}|^2 \leq \|M\|^2|\vec{b}|^2 \leq \|M\|^2 N.$$

Combined with the previous inequality, we get

$$|W| \leq \|M\|^2 N V^{-2}. \tag{13}$$

2. Bounding $\|M\|$

An immediate way to proceed is to note that $\|M\|^2$ is the largest eigenvalue of MM^*, which in turn is bounded by sum of eigenvalues which is the trace of M^*M. However, this is somewhat inefficient. Consider N-dimensional vector that enumerates through the eigenvalues (λ_n) of MM^*, so that the trace will be the L_1 norm of this vector. In principle, we would like the L_∞ norm of this vector, so we can try to take L_k norms of this vector for big k to get close to L_∞. Using an eigenbasis for MM^*, we can see that the L_k norm is represented by

$$\left(\sum_{n\sim N} \lambda_n^k\right)^{1/k} = \mathrm{Tr}((MM^*)^k)^{1/k}.$$

We take $k = 3$, which is the highest power we can afford given the tools at our disposal. This gives

$$|W| \leq \mathrm{Tr}((MM^*)^3)^{1/3} N V^{-2}. \tag{14}$$

3. Expansion of $\mathrm{Tr}((\mathrm{MM}^*)^3)$

We first compute

$$(MM^*)_{n_1,n_2} = \sum_{t\in W} \omega\left(\frac{n_1}{N}\right)\omega\left(\frac{n_2}{N}\right) n_1^{-it_j} n_2^{it_j}$$

so that

$$\begin{aligned}\mathrm{tr}((M^*M)^3) &= \sum_{t_1,t_2,t_3\in W}\ \sum_{n_1,n_2,n_3\sim N} \omega\left(\frac{n_1}{N}\right)^2 \omega\left(\frac{n_2}{N}\right)^2 \omega\left(\frac{n_3}{N}\right)^2 n_1^{i(t_1-t_3)} n_2^{i(t_2-t_1)} n_3^{i(t_3-t_2)} \\ &= \sum_{t_1,t_2,t_3\in W}\ \sum_{n_1,n_2,n_3} \omega\left(\frac{n_1}{N}\right)^2 \omega\left(\frac{n_2}{N}\right)^2 \omega\left(\frac{n_3}{N}\right)^2 \left(\frac{n_1}{N}\right)^{i(t_1-t_3)} \left(\frac{n_2}{N}\right)^{i(t_2-t_1)} \left(\frac{n_3}{N}\right)^{i(t_3-t_2)}.\end{aligned}$$

Let $h_t(u) := \omega(u)^2 u^{it}$, we can apply Poisson summation in the inner integral over n_1, n_2, n_3 to get

$$\mathrm{tr}((\mathrm{M^*M})^3) = \mathrm{N}^3 \sum_{\mathrm{t_1,t_2,t_3\in W}}\ \sum_{\mathrm{m_1,m_2,m_3}} \hat{\mathrm{h}}_{\mathrm{t_1-t_3}}(\mathrm{Nm_1})\hat{\mathrm{h}}_{\mathrm{t_2-t_1}}(\mathrm{Nm_2})\hat{\mathrm{h}}_{\mathrm{t_3-t_2}}(\mathrm{Nm_3}). \tag{15}$$

What we can gain here is that $\hat{h}_t m$ has decay in t or m based on the principle of non-stationary phase.

Lemma 5.3 (Non-stationary phase)**.** *We have for any integer $A > 0$*

$$|\hat{h}_t(\xi)| \ll_A \frac{1+|t|^A}{|\xi|^A},$$
$$|\hat{h}_t(\xi)| \ll_A \frac{1+|\xi|^A}{|t|^A}.$$

Proof. We have

$$\hat{h}_t(\xi) = \int \omega(u)^2 u^{it} e^{2\pi i\xi u} du.$$

By repeated integration by parts on $\omega(u)^2 u^{it}$ and $e^{2\pi i\xi u}$, we get

$$|\hat{h}_t(\xi)| = \left| \int (2\pi i\xi)^{-A} e^{2\pi i\xi u} \frac{d^A}{(du)^A} \left(\omega^2(u) u^{it}\right) du \right| \ll_A \frac{1+|t|^A}{|\xi|^A}.$$

A similar argument for integration by parts on $\omega(u)^2 e^{2\pi i\xi u}$ and u^{it} gives

$$|\hat{h}_t(\xi)| = \left| \int \frac{1}{(it+1)(it+2)\dots(it+A)} u^{it+A} \frac{d^A}{(du)^A} \left(\omega^2(u) e^{2\pi i\xi u}\right) du \right| \ll_A \frac{1+|\xi|^A}{|t|^A}.$$

□

This means that we can handle terms in equation 15 if m_i is small and $t_j - t_k$ is big, or m_i is big and $t_j - t_k$ is small. With this in mind, we split the sum over n_1, n_2, n_3 in the equation into four parts. S_0, where all three m terms are zero, S_1, where exactly one of the m terms is non-zero, S_2, where exactly two of the m terms are non-zero, and S_3, where all three m terms are non-zero. That is,

$$\mathrm{tr}((\mathrm{M}^*\mathrm{M})^3) = \mathrm{S}_0 + \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3,$$

where

$$S_j = N^3 \sum_{m_1, m_2, m_3, \#\{m_k = 0\} = j} I_m,$$
$$I_m = I_{(m_1,m_2,m_3)} := N^3 \sum_{t_1,t_2,t_3 \in W} \hat{h}_{t_1-t_3}(Nm_1)\hat{h}_{t_2-t_1}(Nm_2)\hat{h}_{t_3-t_2}(Nm_3).$$

4. Bounding S_0

S_0 only has one term in the sum.

$$S_0 = N^3 \sum_{t_1,t_2,t_3 \in W} \hat{h}_{t_1-t_3}(0)\hat{h}_{t_2-t_1}(0)\hat{h}_{t_3-t_2}(0)$$

Now we can apply that W is T^ϵ separated, so there is a trivial bound $|W| \le T$ and $\hat{h}_{t_j-t_k}$ is negligible by the principle of non-stationary phase. So we can only consider

$$S_0 = N^3 \sum_{t \in W} \hat{h}_0(0) + O(T^{-100}) = N^3 |W| \|\omega\|_{L_2}^6.$$

Taking the cube root, this term gives $O(N^2 V^{-2}|W|)$ in equation 14. This is strikingly similar to the N^2V^{-2} term that the density hypothesis conjectures. Guth and Maynard isolates this term by introducing the following lemma.

Lemma 5.4. *Let A be an $m \times n$ matrix. Then*

$$\|A\| \le 2\left(\mathrm{tr}((\mathrm{AA}^*)^3) - \frac{\mathrm{tr}(\mathrm{AA}^*)^3}{\mathrm{m}^2}\right)^{1/6} + 2\left(\frac{\mathrm{tr}(\mathrm{AA}^*)}{m}\right)^{1/2}.$$

Proof. This is Lemma 4.2 from Guth-Maynard. [?] □

Applying this lemma, we can compute that

$$\mathrm{tr}(\mathrm{MM}^*) = \sum_{\mathrm{n}\sim\mathrm{N}}\sum_{\mathrm{t}\in\mathrm{W}} \omega\left(\frac{\mathrm{n}}{\mathrm{N}}\right)^2 \mathrm{n}^{-\mathrm{t}}\mathrm{n}^{\mathrm{t}} = |\mathrm{W}|\sum_{\mathrm{n}} \omega\left(\frac{\mathrm{n}}{\mathrm{N}}\right)^2.$$

Applying Poisson summation, this equals

$$|W|\sum_m N\hat{h}_0(mN).$$

By non-stationary phase use the rapid decay of $\hat{h}_0(\xi)$ in ξ to only consider the term $m = 0$ at the cost of N^{-100}. Therefore, $\mathrm{tr}(\mathrm{MM}^*) = |\mathrm{W}|\mathrm{N}\|\omega\|^2_{\mathrm{L}_2} + \mathrm{O}(\mathrm{N}^{-100})$. Lemma 5.4 gives

$$|W| \ll NV^{-2}(N + (S_0 + S_1 + S_2 + S_3 - N^3|\omega|^6_{L_2}|W|)^{1/3}) \ll N^2V^{-2} + NV^{-2}(S_1 + S_2 + S_3)^{1/3}.$$

5. Bounding S_1

By symmetry in m_1, m_2, m_3, we can consider the terms where $m_3 \neq 0$ at a cost of a factor of 3. Then

$$S_1 = 3N^3 \sum_{m\neq 0}\sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(0)\hat{h}_{t_2-t_1}(0)\hat{h}_{t_3-t_2}(mN).$$

This term is bounded by non-stationary phase. If $|m| > T^{1+\epsilon}/N$, then $|m|/|t_3 - t_2| < T^\epsilon$, so we can truncate the sum to $|m| \leq T^{1+\epsilon}/N$ with an error of $O_\epsilon(T^{-100})$. In this range, if $t_1 \neq t_3$ or $t_2 \neq t_1$, then they are T^ϵ apart, then we get rapid decay in $\hat{h}_{t_1-t_3}(0)$ or $\hat{h}_{t_2-t_1}(0)$ to be $O_\epsilon T^{-100}$. But when $t_1 = t_2 = t_3$, we get decay in the last term $\hat{h}_0(mN)$. Combining all cases, this term is negligible.

6. Bounding S_2

By symmetry again we can consider the terms where $m_1, m_2 \neq 0, m_3 = 0$.Then

$$S_2 = 3N^3 \sum_{m_1,m_2\neq 0}\sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(m_1N)\hat{h}_{t_2-t_1}(m_2N)\hat{h}_{t_3-t_2}(0).$$

Due to decay of the last term in $|t_3 - t_2|$, we can only consider the terms $t_3 = t_2$ with error $O_{\epsilon,A}(T^{-A})$. Then we can rewrite

$$\begin{aligned} 3N^3\hat{h}_0(0)\sum_{m_1,m_2\neq 0}\sum_{t_1,t_2\in W} \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_2-t_1}(m_2N) &= 3N^3\hat{h}_0(0)\sum_{m_1,m_2\neq 0}\sum_{t_1,t_2\in W} \hat{h}_{t_1-t_2}(m_1N)\hat{h}_{t_1-t_2}(-m_2N) \\ &= 3N^3\hat{h}_0(0)\sum_{t_1,t_2\in W}\left(\sum_{m\neq 0}\hat{h}_{t_1-t_2}(mN)\right)^2. \end{aligned}$$

Poisson summation gives

$$N\sum_m \hat{h}_{t_1-t_2}(mN) = \sum_n h_{t_1-t_2}\left(\frac{n}{N}\right) = \sum_n \omega\left(\frac{n}{N}\right) n^{i(t_1-t_2)}.$$

Remark: Here we have added in the terms for $m = 0$. This is somewhat lossy for terms $t_1 = t_2$. However, a variation of the argument (and applying Heath-Brown's result in the end) gives a tighter bound, which will be adapted for the general proof.

7. Bounding S_3

S_3 sums over most points on the 3-dimensional lattice. By symmetry, we can consider only the terms with $m_1 \leq m_2 \leq m_3$ with an error factor of 6. Recall that

$$I_m = N^3 \sum_{t_1,t_2,t_3\in W} \hat{h}_{t_1-t_3}(Nm_1)\hat{h}_{t_2-t_1}(Nm_2)\hat{h}_{t_3-t_2}(Nm_3).$$

By the the principle of non-stationary phase, we can truncate the sum across I_m to $|m_1|, |m_2|, |m_3| \lesssim T/N$, at the cost of $O\epsilon(T^{-100})$, as $t_j - t_k = O(T)$. We expand $\hat{h}$ in integral form, so that

$$\begin{aligned} I_{\vec{m}} &= N^3 \sum_{t_1,t_2,t_3 \in W} \int_{\mathbb{R}^3} \omega(u_1)^2 \omega(u_2)^2 \omega(u_3)^2 u_1^{i(t_1-t_3)} u_2^{i(t_2-t_1)} u_3^{i(t_3-t_2)} e(-N\vec{m}\cdot\vec{u}) d\vec{u} \\ &= N^3 \sum_{t_1,t_2,t_3 \in W} \int_{\mathbb{R}^3} \tilde{\omega}(\vec{u}) \Big(\frac{u_1}{u_2}\Big)^{it_1} \Big(\frac{u_2}{u_3}\Big)^{it_2} \Big(\frac{u_3}{u_1}\Big)^{it_3} e(-N\vec{m}\cdot\vec{u}) d\vec{u} \end{aligned}$$

For $\tilde{\omega}(\vec{u}) = \omega(u_1)^2\omega(u_2)^2\omega(u_3)^2$. Because $\tilde{\omega}$ is supported away from $u_3 = 0$, we can introduce the change of variables $v_1 = u_1/u_3$, $v_2 = u_2/u_3$. The Jacobian is u_3^2, and $u_1/u_2 = v_1/v_2$, so that

$$\begin{aligned} I_{\vec{m}} &= N^3 \sum_{t_1,t_2,t_3 \in W} \int_{\mathbb{R}^3} u_3^2 \tilde{\omega}(v_1u_3, v_2u_3, u_3) \Big(\frac{v_1}{v_2}\Big)^{it_1} v_2^{it_2} \Big(\frac{1}{v_1}\Big)^{it_3} e(-Nu_3(m_1v_1 + m_2v_2 + m_3)) dv_1\ dv_2\ du_3 \\ &= N^3 \sum_{t_1,t_2,t_3 \in W} \int_{\mathbb{R}^2} \int_{\mathbb{R}} u_3^2 \tilde{\omega}(v_1u_3, v_2u_3, u_3) e(-Nu_3(m_1v_1 + m_2v_2 + m_3)) du_3\ \Big(\frac{v_1}{v_2}\Big)^{it_1} v_2^{it_2} \Big(\frac{1}{v_1}\Big)^{it_3} dv_1\ dv_2 \end{aligned}$$

The inner integral in u_3 places restrictions on the domain of integration. First, the support of $\tilde{\omega}$ is $[1,2] \times [1,2] \times [1,2]$. Thus, if it is non-zero, we have $v_1u_3, v_2u_3, u_3 \in [1,2] \implies v_1, v_2 \in [1/2, 2]$. Therefore, we can restrict the outer integral in v_1 and v_2 to this range. Next, since $v_1, v_2 = O(1)$, the chain rule gives

$$\Big(\frac{\partial}{\partial u_3}\Big)^j \omega(v_1u_3, v_2u_3, u_3) \ll_j 1.$$

Therefore, we can apply the repeated integration by parts to get rapid decay of the integral in $|N(m_1v_1 + m_2v_2 + m_3)|$. In particular, we can truncate the integral to the range $|N(m_1v_1 + m_2v_2 + m_3)| \ll T^\epsilon$ at an error of $O_\epsilon(T^{-100})$, and use

$$\int_{\mathbb{R}} u_3^2 \tilde{\omega}(v_1u_3, v_2u_3, u_3) e(-Nu_3(m_1v_1 + m_2v_2 + m_3)) du_3 = O(1)$$

in this range by the compact support of $\tilde{\omega}$. This gives us

$$|I_{\vec{m}}| \le \left| N^3 \sum_{t_1,t_2,t_3 \in W} \int_{\substack{|v_1m_1 + v_2m_2 + m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} \Big(\frac{v_1}{v_2}\Big)^{it_1} v_2^{it_2} \Big(\frac{1}{v_1}\Big)^{it_3} dv_1\ dv_2 \right| + O_\epsilon(T^{-100}). \tag{16}$$

Recall that in the outline we defined

$$R(v) := \sum_{t \in W} v^{it}.$$

Exchanging the summation and integral, we get the term

$$\begin{aligned} &\left| N^3 \int_{\substack{|v_1m_1 + v_2m_2 + m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} R\Big(\frac{v_1}{v_2}\Big) R\big(v_2\big) R\Big(\frac{1}{v_1}\Big) dv_1\ dv_2 \right| \\ \le &N^3 \int_{\substack{|v_1m_1 + v_2m_2 + m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} \left| R\Big(\frac{v_1}{v_2}\Big) R\big(v_2\big) R\Big(\frac{1}{v_1}\Big) \right| dv_1\ dv_2 \\ = &N^3 \int_{\substack{|v_1m_1 + v_2m_2 + m_3| \lesssim \frac{1}{N} \\ \frac{1}{2} \le v_1, v_2 \le 2}} \left| R\Big(\frac{v_2}{v_1}\Big) R\big(v_2\big) R(v_1) \right| dv_1\ dv_2, \end{aligned}$$

where the last step, we used

$$|R(v^{-1})| = \Big| \sum_{t \in W} v^{-it} \Big| = \Big| \sum_{t \in W} v^{it} \Big| = |R(v)|.$$

Now we fix v_1, and consider the integral in v_2 in the range

$$|v_1 m_1 + v_2 m_2 + m_3| \lesssim \frac{1}{N} \implies \left|v_2 - \frac{v_1 m_1 + m_3}{-m_2}\right| \lesssim \frac{1}{|m_2| N}.$$

If we enforce the conditions $|m_1| \le |m_2| \le |m_3|$ and $v_2 \asymp 1$, we see that the domain of integration is empty unless $|m_2| \asymp |m_3|$. Thus, we can break the sum across

$$\sum_{|m_1|,|m_2|,|m_3| \lesssim T/N}$$

to be

$$\log T^{1+\epsilon}/N \ \sup_{\mathrm{U}=2^{\mathrm{j}},\mathrm{V}\le\mathrm{U}} \sum_{\substack{|m_1|\sim V \\ |m_2|,|m_3|\sim U}} .$$

Moreover, we integrate in v_2 over a very small neighborhood of width $\lesssim 1/N$ around $v_2 = (v_1 m_1 + m_3)/(-m_2)$. In principle, we can estimate this integral by taking the value of R at this point to get

$$\approx \frac{N}{|m_2|} \int_{\frac{1}{2} \le v_1 \le 2} \left| R\Big(\frac{v_1 m_1 + m_3}{-v_1 m_2}\Big) R\big(\frac{v_1 m_1 + m_3}{-m_2}\big) R(v_1) \right| dv_1.$$

This is made precise by apply to a $(1/N|m_2|)$-smoothened version of R, stated in Proposition 7.5. Finally, Hölder's inequality gives a bound of

$$\Big\| R\Big(\frac{v m_1 + m_3}{-v m_2}\Big)\Big\|_{L^4_{v\asymp 1}} \Big\| R\Big(\frac{v m_1 + m_3}{-m_2}\Big)\Big\|_{L^4_{v_1\asymp 1}} \Big\| R(v) \Big\|_{L^2_{v_1 \asymp 1}}$$

The second moment of R is bounded by the size of W. Indeed, we have

$$\int_{v\asymp 1} |R(v)|^2 dv = \sum_{t_1, t_2 \in W} \int_{v \asymp 1} R(v)^{i(t_1 - t_2)} dv.$$

If $t_1 - t_2 \ne 0$, then $|t_1 - t_2| > T^\epsilon$, so the oscillatory integral will be negligible by the fast decay. There are $|W|$ terms satisfying $t_1 = t_2$, and each contributes $O(1)$ to the sum. Similarly, the fourth moment of R is bounded by the energy, recalling its definition

$$E(W) := \#\{t_1, t_2, t_3, t_4 \in W : |t_1 + t_2 - t_3 - t_4| \ll T^\epsilon\}.$$

We get

$$\int_{v\asymp 1} |R(v)|^2 dv = \sum_{t_1, t_2, t_3, t_4 \in W} \int_{v\asymp 1} R(v)^{i(t_1 + t_2 - t_3 - t_4)} dv,$$

and the terms in the summation are negligble unless $|t_1 + t_2 - t_3 - t_4| \ll T^\epsilon$. Assembling everything together gives an initial bound of

$$S_3 \lesssim T^2 E(W)^{1/2} |W|^{1/2}.$$

9. Refinement of S_3 bound

Recall that in the previous section, we summed across

$$\sum_{\substack{|m_1|\sim V \\ |m_2|,|m_3|\sim U}} \frac{N}{|m_2|} \int_{\frac{1}{2} \le v_1 \le 2} \left| R\Big(\frac{v_1 m_1 + m_3}{-v_1 m_2}\Big) R\Big(\frac{v_1 m_1 + m_3}{-m_2}\Big) R(v_1) \right| dv_1$$

By repeatedly applying Cauchy-Schwartz, we can move the summation into the integral and obtain terms that resembles the form

$$\int_{v_1 \asymp 1} \sum_{\substack{|m_1|\sim V \\ |m_2|,|m_3|\sim U}} \left| R\Big(\frac{v_1 m_1 + m_3}{-m_2}\Big) \right|^2 dv_1.$$

The term $(v_1m_1+m_3)/(-m_2)$ describes an affine transformation in v_1. Therefore, averaging over all the affine transformations, we may expect that this estimates some L^k norm of R. Indeed, we have Lemma 7.3 that gives us this estimate. This improves the bound on S_3 to

$$S_3 \lesssim T^2|W|^{3/2} + TN|W|^{1/2}E(W)^{1/2}.$$

This gives an improvement from the previous bound for the case $N \ll T^{1-\delta_1}$ and $E(W) \gg |W|^{3-\delta_2}$ for some small δ's.

8. Bound on $E(W)$

Finally, we give an orthogonal bound on the energy of the set W. The idea is that if $|t_1+t_2-t_3-t_4|$ is small, then we can approximate

$$D_N(t_4) \approx D_N(t_1+t_2-t_3).$$

This is made precise by applying a smoothing of D_N over a width of $\lesssim 1$. Therefore, since the choice of t_4 is fixed by the choice of t_1,t_2,t_3, we have

$$E(W)V^2 \le \sum_{|t_1-t_2-t_3-t_4|\lesssim 1} |D_N(t_4)|^2 \approx \sum_{|t_1-t_2-t_3-t_4|\lesssim 1} |D_N(t_1+t_2-t_3)|^2 \le \sum_{t_1,t_2,t_3\in W} |D_N(t_1+t_2-t_3)|^2.$$

Now we can expand $|D_N(t_1+t_2-t_3)|^2$ to get

$$\begin{aligned}\sum_{t_1,t_2,t_3\in W} |D_N(t_1+t_2-t_3)|^2 &= \sum_{t_1,t_2,t_3\in W}\sum_{n_1,n_2\sim N} b_{n_1}\bar{b}_{n_2}\Big(\frac{n_1}{n_2}\Big)^{i(t_1+t_2-t_3)} \\ &= \sum_{n_1,n_2\sim N} b_{n_1}\bar{b}_{n_2}\sum_{t_1,t_2,t_3\in W}\Big(\frac{n_1}{n_2}\Big)^{i(t_1+t_2-t_3)} \\ &\le \sum_{n_1,n_2\sim N}\Big|R\Big(\frac{n_1}{n_2}\Big)R\Big(\frac{n_1}{n_2}\Big)R\Big(\frac{n_2}{n_1}\Big)\Big| \\ &\le \sum_{n_1,n_2\sim N}\Big|R\Big(\frac{n_1}{n_2}\Big)\Big|^3.\end{aligned}$$

Now we can apply the trivial bound $|R| \le |W|$ to get that

$$E(W) \lesssim V^{-2}|W|\sum_{n_1,n_2\sim N}\Big|R\Big(\frac{n_1}{n_2}\Big)\Big|^2.$$

This is in turn bounded by Heath-Brown's result as

$$\begin{aligned}\sum_{n_1,n_2\sim N}\Big|R\Big(\frac{n_1}{n_2}\Big)\Big|^2 &= \sum_{n_1,n_2\sim N}\sum_{t_1,t_2\in W}\Big(\frac{n_1}{n_2}\Big)^{i(t_1-t_2)} \\ &= \sum_{t_1,t_2\in W}\Big|\sum_{n\sim N} n^{i(t_1-t_2)}\Big|^2.\end{aligned}$$

This is enough to give an improvement on Ingham-Huxley's result, but can be further improved using Cauchy Schwartz on the third moment

$$\sum_{n_1,n_2\sim N}\Big|R\Big(\frac{n_1}{n_2}\Big)\Big|^3 \le \Bigg(\sum_{n_1,n_2\sim N}\Big|R\Big(\frac{n_1}{n_2}\Big)\Big|^2\Bigg)^{1/2}\Bigg(\sum_{n_1,n_2\sim N}\Big|R\Big(\frac{n_1}{n_2}\Big)\Big|^4\Bigg)^{1/2}.$$

The fourth moment can be reduced back to the second moment by taking representative classes of $\lfloor t_1-t_2\rfloor$, thus can also be bounded using Heath-Brown's result.

6 Towards a Hybrid Zero Density Result

We would like to generalize Guth and Maynard's result to L-functions. Specifically, let χ be a Dirichlet character. we are interested in the the zeros defined by

$$L(s,\chi) := \sum_n \frac{\chi(n)}{n^{-s}}$$

on $\Re(s) > 1$ and its analytic continuation on the whole complex plane. The structure of the arguments for analytic continuation, its line of symmetry along $\Re(s) = 1/2$, and the locations of trivial zeros are very similar to that of the zeta function. This motivates the Generalized Riemann Hypothesis.

Conjecture 6.1 (Generalized Riemann Hypothesis). *The Generalized Riemann Hypothesis asserts that on the critical strip,*

$$L(s,\chi) = 0 \implies \Re(s) = \frac{1}{2},$$

for any Dirichlet character.

This leads to even stronger for primes in short intervals. Namely, fix an integer q, we have that

$$\sum_{\substack{n \le N \\ n \equiv a \bmod q}} \Lambda(n) = \begin{cases} \frac{1}{\phi(q)} N + O(x^{2+o(1)}), & \text{if } \gcd(a,q) = 1 \\ o(n), & \text{otherwise.} \end{cases}$$

This means that not only that the Prime Number Theorem holds in intervals of $x^{2+\epsilon}$, the distribution of primes in each of the residual classes (coprime to q) are uniform at this scale too. Noticing that we can modify Huxley's proof with

$$M_{x,\chi} = \sum_{n \le x} \chi(n)\mu(n)n^{-s},$$

we have

$$L(s,\chi)M_{x,\chi} = \sum_n a_n \chi(n)\mu(n)n^{-s}.$$

Thus, we can reproduce a similar proof on the zero density of L-functions.

Definition 6.2 (Zero Density for L-functions). *Let $N(\sigma,\chi,T)$ denote the number of zeros of the L-function $L(-,\chi)$ with real part greater than σ and imaginary part between $-T$ and $-T$. That is*

$$N(\sigma,\chi,T) := \#\{\rho = \beta + i\gamma \mid \beta \ge \sigma, |\gamma| \le T\}.$$

For backwards compatibility with our previous definition, we take $N(\sigma,T) := N(\sigma,1,T)$.

The hybrid analogs of the zero density bounds of Ingham and Huxley are known.

Theorem 6.3 (Hybrid Ingham bound for zero density). *Let $1/2 \le \sigma \le 3/4$. We have*

$$\sum_{\chi^*} N(\sigma,\chi^*,t) \lesssim (qT)^{\frac{3(1-\sigma)}{2-\sigma}},$$

where $\sum_{\chi^*}$ sums over all the primitive characters χ^* of modulus q.

Theorem 6.4 (Hybrid Huxley bound for zero density). *Let $3/4 \le \sigma \le 1$. We have*

$$\sum_{\chi^*} N(\sigma,\chi^*,t) \lesssim (qT)^{\frac{3(1-\sigma)}{3\sigma-1}},$$

where $\sum_{\chi^*}$ sums over all the primitive characters χ^* of modulus q.

The method for detecting zeros is very similar to Huxley's proof above with the slight change in definition of $M_{x,\chi}$. This argument then reduces to bounding the number of times large values of Dirichlet polynomials can occur. Therefore we want a result in the form:

Let $\mathcal{S} = \{(t_j, \chi_j)\}$ be a set such that each χ_j is a primitive Dirichlet character of modulus q, and $|t_j - t_k| \geq 1$ if $j \neq k$ and $\chi_j = \chi_k$. (That is, the t's are 1-separated if the characters are the same.)Let $|b_n| \leq 1$ be a sequence of numbers indexed in n, and suppose also that

$$\left| \sum_{n \sim N} b_n \chi_j(n) n^{it_j} \right| > V.$$

We want to find a bound on $|\mathcal{S}|$.

Our result is as follows:

Theorem 6.5. *To be determined ☹.*

The idea of the proof is very similar to Guth and Maynard's proof. We can define a $|\mathcal{S} \times N|$ matrix M with entries

$$M_{t_j, \chi_j, n} = \chi_j(n) n^{it_j}$$

for $(t_j, \chi_j) \in \mathcal{S}$ and $n \sim N$, and bound its operator in the exact same way: taking it to the M^*M to the third power and calculating its trace. The Dirichlet characters are not nice to handle when extended to a function $\mathbb{R} \to \mathbb{C}$, so we break the sum into q summations across each residue class $\mod q$, and apply Poisson summation on each piece of the sum. This gives us

$$\begin{aligned} \operatorname{tr}((M^*M)^3) = \sum_{\substack{(t_1,\chi_1),\\ (t_2,\chi_2),\\ (t_3,\chi_3) \in \mathcal{S}}} \frac{N^3}{q^3} \sum_{m \in \mathbb{Z}^3} \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^3} \chi_1 \bar{\chi}_3(x_1) \chi_2 \bar{\chi}_1(x_2) \chi_3 \bar{\chi}_2(x_3) e\left(\frac{-x \cdot m}{q} \right) \\ \times \hat{h}_{t_1 - t_3}\left(\frac{N m_1}{q} \right) \hat{h}_{t_2 - t_1}\left(\frac{N m_2}{q} \right) \hat{h}_{t_3 - t_2}\left(\frac{N m_3}{q} \right), \end{aligned}$$

where $h_t(x) \coloneqq \omega(u)^2 x^{it}$ has the same definition, thus its fourier transform has the same properties in decay. Similarly, we find that when we break the sum in $\mathbb{Z}^3$ into S_0 to S_3 in the same way, S_0 gives the main term, S_1 is negligible, and S_2 can be bounded by Heath-Brown's theorem. For the S_3 bound, we redefine

$$R(v, n_1, n_2) \coloneqq \sum_{(t,\chi) \in \mathcal{S}} \chi(n_1) \bar{\chi}(n_2) v^{it}.$$

This additional structure in χ may look complicated when taking the L_2 or L_4 norm of R in $v \asymp 1$, but this cancels out when taking

$$\sum_{n_1 \in \mathbb{Z}/q\mathbb{Z}} R(v, n_1, n_2)^2 = \sum_{n_1 \in \mathbb{Z}/q\mathbb{Z}} \sum_{(t_1,\chi_1),(t_2,\chi_2) \in \mathcal{S}} \chi_1(n_1) \bar{\chi}_2(n_1) \bar{\chi}_1(n_2) \chi_2(n_2) v^{i(t_1 - t_2)}.$$

By the orthogonality of Dirichlet characters, the sum in n_1 vanishes except when $\chi_1 = \chi_2$, in which case (assuming $\gcd(n_2, q) = 1$) we can reduce

$$\sum_{n_1 \in \mathbb{Z}/q\mathbb{Z}} R(v, n_1, n_2)^2 = \phi(q) \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2) \in \mathcal{S} \\ \chi_1 = \chi_2}} v^{i(t_1 - t_2)}.$$

When $\chi_1 = \chi_2$, we have $t_1 = t_2$ or they are T^ϵ separated, giving us decay in the calculation of the second moment of R to be $\phi(q)|\mathcal{S}|$. Similarly, we see the fourth power of R cancels in the sum

$$\sum_{n_1 \in \mathbb{Z}/q\mathbb{Z}} R(v, n_1, n_2)^4 = \sum_{n_1 \in \mathbb{Z}/q\mathbb{Z}} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\ (t_3,\chi_3),(t_4,\chi_4) \\ \in \mathcal{S}}} \chi_1 \chi_2 \bar{\chi}_3 \bar{\chi}_4(n_1) \bar{\chi}_1 \bar{\chi}_2 \chi_3 \chi_4(n_2) v^{i(t_1 + t_2 - t_3 - t_4)}.$$

Only terms where $\chi_1 \chi_2 = \chi_3 \chi_4$ can have non-zero contribution, giving us a natural definition for the energy of the set to be

$$E(\mathcal{S}) \coloneqq \#\{(t_1,\chi_1),(t_2,\chi_2),(t_3,\chi_3),(t_4,\chi_4) \in \mathcal{S} \mid \chi_1 \chi_2 = \chi_3 \chi_4, |t_1 + t_2 - t_3 - t_4| \lesssim 1\}.$$

This in turn can be bounded by a third moment of R:

$$E(\mathcal{S}) \lesssim N^{-2\sigma} \sum_{n_1,n_2\sim N} \left| R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^3.$$

This expression is then bounded by Heath-Brown's theorem.

Things beyond are slightly more disorganized.

6.1 Setup and Reduction of Theorem

Lemma 6.6. *Let $\mathcal{S} = \{(t_j, \chi_j)\}$ be a set such that each χ_j is a primitive Dirichlet character of modulus q, and $|t_j - t_k| \geq T^\epsilon$ if $j \neq k$ and $\chi_j = \chi_k$. Let $|b_n| \leq 1$ be a sequence of numbers indexed in n, ω be a smooth bump function that equals 1 on $[6/5, 9/5]$ and has support in $[1,2]$ (thus $\omega^{(A)} \ll_A 1$ for all A). Let $V = N^\sigma$, where $\sigma \in [placeholder]$, and $N = T^\alpha$, where $\alpha \in [placeholder]$ Suppose also that*

$$|D_N(t_j,\chi_j)| := \left| \sum_{n\sim N} b_n \chi_j(n) n^{it_j} \right| > V.$$

for all $(t_j,\chi_j) \in \mathcal{S}$. Then

$$|\mathcal{S}| \lesssim ☺.$$

We define M a $|\mathcal{S}\times N|$ matrix with entries

$$M_{t_j,\chi_j,n} = \omega\left(\frac{n}{N}\right)\chi_j(n) n^{it_j}$$

for $(t_j,\chi_j) \in \mathcal{S}$ and $n \sim N$. Thus by the same reasoning that $(M\vec{b})_j = D_N(t_j,\chi_j)$, we want to bound the trace of the matrix

$$\operatorname{tr}((M^*M)^3)$$

. We see that

$$(MM^*)_{t_j,t_k} = \sum_{n\sim N} \omega\left(\frac{n}{N}\right)^2 n^{i(t_k-t_j)} \bar{\chi}_j \chi_k(n),$$

so that

$$\operatorname{tr}(MM^*) = |\mathcal{S}| \sum_{n\sim N} \omega\left(\frac{n}{N}\right)^2$$

and Lemmas 4.1-4.4 (Large values controlled by singular values, bound for singular values in terms of trace, Principle of non-stationary phase, Hilbert-Schmidt Norm estimate) can be adapted directly from the paper.

For the expansion of the trace (analog to lemma 4.5),

$$(M^*M)_{n_1,n_2} = \sum_{(t_j,\chi_j)\in\mathcal{S}} \omega\left(\frac{n_1}{N}\right)\omega\left(\frac{n_2}{N}\right)\bar{\chi}_j(n_1)\chi_j(n_2) n_1^{-it_j} n_2^{it_j}$$

so that

$$\begin{aligned}\operatorname{tr}((M^*M)^3) = \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{n_1,n_2,n_3\sim N} &\omega\left(\frac{n_1}{N}\right)^2 \left(\frac{n}{N}\right)^{i(t_1-t_3)} \chi_1\bar{\chi}_3(n_1) \\ &\times \omega\left(\frac{n_2}{N}\right)^2 \left(\frac{n}{N}\right)^{i(t_2-t_1)} \chi_2\bar{\chi}_1(n_2) \\ &\times \omega\left(\frac{n_3}{N}\right)^2 \left(\frac{n}{N}\right)^{i(t_3-t_2)} \chi_3\bar{\chi}_2(n_3).\end{aligned}$$

Let $h_t(u) = \omega(u)^2 u^{it}$, so by applying Poisson summation to the inner sum, we have

$$\operatorname{tr}((M^*M)^3) = \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \frac{N^3}{q^3} \sum_{m\in\mathbb{Z}^3} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3) e\left(\frac{-x\cdot m}{q}\right)$$
$$\times \hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right)$$

Finally, we exchange the 2 outermost integrals. And split the sum over m into four parts (same as Guth Maynard) $S_0 + S_1 + S_2 + S_3$, where S_j runs over the values of m with exactly j non-zero entries.

6.2 S_0 bound

S_0 only has one term corresponding to $m = 0$. By the principle of non-stationary phase, $\hat{h}_t(0)$ has rapid decay in t, so contributes $O(T^{-1000})$ except possibly when t_1, t_2, t_3 are not T^ϵ separated. Moreover, by the orthogonality of characters, all three terms $\chi_a\bar{\chi}_b$ must be principal to have non-zero contribution, so this fixes the sum to be across $(t_1,\chi_1) = (t_2,\chi_2) = (t_3,\chi_3)$ to give a $N^3\phi(q)^3|\mathcal{S}|\|\omega\|_{L_2}^6/q^3$ term. So that

$$\operatorname{tr}((M^*M)^3) = \frac{N^3\phi(q)^3}{q^3}|\mathcal{S}|\|\omega\|_{L_2}^6 + \sum_{m\in\mathbb{Z}^3-\{0\}} I_m + O(T^{-100}),$$

where

$$I_m = \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3) e\left(\frac{-x\cdot m}{q}\right)$$
$$\times \hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right).$$

which gives the analogous Lemmas 4.5 and 4.6.

6.3 S_1 bound

Proposition 6.7. *$S_1 = O\epsilon(T^{-10})$.*

By symmetry, we sum I_m across all $m = (0,0,m_3 \neq 0)$ at a cost of a factor of 3. We then have

$$I_m = \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3) e\left(\frac{-x_3m_3}{q}\right)$$
$$\times \hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right)$$

Again by the orthogonality of characters, the only way to get non-zero contribution is when $\chi_1 = \chi_2$ and $\chi_2 = \chi_3$. So this reduces to

$$\begin{aligned}
I_m &= \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \phi(q)^2 \sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{-x_3m_3}{q}\right)\\
&\times \quad h_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right)\\
&= \frac{N^3}{q^3}\phi(q)^2\frac{\phi(q)}{\phi\left(\frac{q}{\gcd(m_3,q)}\right)}\mu\left(\frac{q}{\gcd(m_3,q)}\right) \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right)
\end{aligned}$$

if So we trivially bound S_1 by

$$|S_1| \ll \frac{N^3}{q^3}\phi(q)^3 \sum_{m_3\neq 0} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \left|\hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right)\right|$$

By the quick decay and boundedness of $\hat{h}_t(\xi)$ in both ξ and t (decays in terms of $\langle\xi\rangle^{-A}\langle t\rangle^{A}$ or $\langle\xi\rangle^{A}\langle t\rangle^{-A}$), we can bound the terms when summed across all $|m_3| > qT^{1+\epsilon}/N$. For the remaining terms, $t_1 \neq t_2$, $t_1 \neq t_3$ can be bounded by $O(T^{-10})$. Finally, when $t_1 = t_2 = t_3$ and $|m_3|$ is small, we get decay in terms of $(N/q)^{-100}$.

The sum over terms $|m_3| > qT^{1+\epsilon}/N$,

$$\begin{aligned}&\sum_{|m_3|>qT^{1+\epsilon}/N} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \left|\hat{h}_{t_1-t_3}(0)\,\hat{h}_{t_2-t_1}(0)\,\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right)\right|\\ \ll_{\epsilon,A}&\sum_{|m_3|>qT^{1+\epsilon}/N} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2=\chi_1),\\(t_3,\chi_3=\chi_1)\in\mathcal{S}}} \left|T^A\left(\frac{Nm_3}{q}\right)^{-A}\right|\\ \ll&|\mathcal{S}|^3T^{-10}.\end{aligned}$$

For $|m_3| \leq qT^{1+\epsilon}/N$, we use the same bound for terms $t_1 \neq t_2$ or $t_1 \neq t_3$ by the $\hat{h}_t(0)$ terms. When $t_1 = t_2 = t_3$, we use $m_3 \neq 0$ to bound

$$\left|\hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right)\right| \ll \left(\frac{q}{N}\right)^{100}$$

since $|\mathcal{S}| \ll T\phi(q)$, we can set $T \gg q$ to get the contribution of S_1 to be $O\epsilon(T^{-10})$.

6.4 S_2 bound

We write by symmetry

$$\begin{aligned}S_2 = 3\frac{N^3}{q^3} \sum_{m_1,m_2\neq 0} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3)e\left(\frac{-x_1m_1-x_2m_2}{q}\right)\\ \times \hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}(0)\end{aligned}$$

Removing zero contributions from $\chi_2 \neq \chi_3$ by orthogonality, we have

$$\begin{aligned}= 3\frac{N^3}{q^3}\phi(q) \sum_{m_1,m_2\neq 0} \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2),\\(t_3,\chi_3=\chi_2)\in\mathcal{S}}} \sum_{x_1,x_2\in\mathbb{Z}/q\mathbb{Z}} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)e\left(\frac{-x_1m_1-x_2m_2}{q}\right)\\ \times \hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right)\hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right)\hat{h}_{t_3-t_2}(0)\end{aligned}$$

Here, we can isolate contributions from the terms where $t_2 \neq t_3$ (hence since $\chi_2 = \chi_3$, are T^ϵ separated) to be $O(T^{-10})$. For the other terms, we can write

$$\hat{h}_t(\xi) = \overline{\hat{h}_{-t}(-\xi)}$$

to get

$$S_2 = 3\frac{N^3}{q^3}\phi(q)\hat{h}_0(0) \sum_{\substack{(t_1,\chi_1),\\(t_2,\chi_2)\in\mathcal{S}}} \left| \sum_{m\neq 0} \sum_{x\in\mathbb{Z}/q\mathbb{Z}} \chi_1\bar{\chi}_2(x) e\left(\frac{-mx}{q}\right) \hat{h}_{t_1-t_2}\left(\frac{Nm}{q}\right) \right|^2 + O(T^{-10}).$$

By the principle of non-stationary phase we can move the terms where $|t_1-t_2| < T^\epsilon$ into $O(T^{-10})$ by decay in Nm/q. *We also used the fact that there are at most $\phi(q)$ characters mod q, so the $O(q^2)$ factor is negligible compared to N^{-100}.*

For the other terms where t_1 and t_2 are T^ϵ separated, we want to apply Heath Brown's theorem.

rough work

At the cost of $O_\epsilon(T^{-100})$ we can add in the term $\hat{h}_{t_1-t_2}(0)$ in when t_1, t_2 are at T^ϵ separated. Let W be the Mellin transform of the function $\omega(x)^2$.

$$\begin{aligned}
&N^{1+it} \sum_{m\in\mathbb{Z}} \sum_{x \bmod q} \chi(x) e\left(\frac{-mx}{q}\right) \hat{h}_t\left(\frac{Nm}{q}\right)\\
=&\sum_n n^{it}\chi(n)\omega\left(\frac{n}{N}\right)^2\\
=&\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} W(s)N^s L(s-it,\chi)ds\\
=&\frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} W(s)N^s L(s-it,\chi)ds + \varepsilon(\chi)\frac{\phi(q)}{q}N^{1+it}W(1+it)
\end{aligned}$$

where ε detects if χ is principal or not. The second term arising from the (potential) pole at 1 decays rapidly in $t > T^\epsilon$. For the first term, we let χ be induced by the primitive χ^* with modulus r, so

$$L(s-it,\chi) = L(s-it,\chi^*)\prod_{p|q}\left(1-\frac{\chi^*(p)}{p^s}\right).$$

We also let

$$G(s) = \frac{\tau(\chi^*)}{i^\delta\sqrt{r}} r^{s-1/2}\pi^{1/2-s}\frac{\Gamma(\frac{1-s+\delta}{2})}{\Gamma(\frac{s+\delta}{2})},$$

so that $L(s-it,\chi^*)(s) = G(s-it)L(1-s+it,\overline{\chi^*})$. The integral becomes

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} W(s)N^s L(s-it,\chi)ds\\
=&\frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} W(s)N^s G(s-it)L(1-s+it,\overline{\chi^*})\prod_{p|q}\left(1-\frac{\chi^*(p)}{p^s}\right)ds\\
=&\frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} W(s)N^s G(s-it)\left(\sum_{n\le M}\frac{\overline{\chi^*}(n)}{n^{1-s+it}} + \sum_{n>M}\frac{\overline{\chi^*}(n)}{n^{1-s+it}}\right)\prod_{p|q}\left(1-\frac{\chi^*(p)}{p^s}\right)ds
\end{aligned}$$

Where M is a parameter to be determined. The summation is convergent as the real part is larger than 1. We thus break up the integral into two pieces according to the two summations $I_1 + I_2$. Moving the line of integration of I_1 to $\Re(s) = 1$ and I_2 to $\Re(s) = -2k$,

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(1+iu)N^{1+iu}G(1+iu-it)\sum_{n\le M}\overline{\chi^*}(n)n^{-i(u-t)}\prod_{p|q}\left(1-\frac{\chi^*(p)}{p^{1+iu}}\right)du,\\
I_2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(-2k+iu)N^{-2k+iu}G(-2k+iu-it)\sum_{n>M}\overline{\chi^*}(n)n^{-2k-1-i(u-t)}\prod_{p|q}\left(1-\frac{\chi^*(p)}{p^{-2k+iu}}\right)du.
\end{aligned}$$

By the decay of W, we can truncate both integrals to the region $|u| \ll T^\epsilon$. Moreover, *decay of gamma - might have a typo in GM paper?*

7 S_3 bound

Proposition 7.1. *We have*
$$S_3 \lesssim \phi(q)^3 T^2 E(\mathcal{S})^{1/2} |\mathcal{S}|^{1/2}.$$

This bound has a refinement, which uses one of the propositions from Guth and Maynard's proof. This refinement of the S_3 bound is based on the same ideas as the first S_3 bound, so the first bound will be the main focus of the section.

Proposition 7.2 (Refinement of S_3)**.** *We have*

$$S_3 \lesssim \phi(q)^{7/2} T^2 |\mathcal{S}|^{3/2} + \phi(q)^3 \frac{NT}{q} |\mathcal{S}|^{1/2} E(\mathcal{S})^{1/2}.$$

The proof of Proposition 7.2 relies on the result for summation over affine transformation by Guth and Maynard.

Lemma 7.3. *Let $M > 0$. Let $f(u) \geq 0$, supported on $u \asymp 1$, and $|\hat{f}(\xi)| \lesssim_j (|\xi|/T)^j$ for all j. Then*

$$\sup_{0<M_1,M_2,M_3<M} \int \left(\sum_{\substack{|m_1|\sim M_1 \\ |m_2|\sim M_2 \\ |m_3|\ll M_3}} f\left(\frac{m_1 u + m_3}{m_2}\right)\right)^2 du \lesssim M^6 \|f\|_{L_1}^2 + M^4 \|f\|_{L_2}^2.$$

This is Proposition 9.1 from [GM].

By non-stationary phase, I_m is negligible for the terms $qT/N \lesssim |m|$, so

$$S_3 = \sum_{0<|m_1|,|m_2|,|m_3| \lesssim qT/N} I_m + O(T^{-100}). \tag{17}$$

We define

$$R(v, n_1, n_2) := \sum_{(t,\chi)\in\mathcal{S}} \chi(n_1)\bar{\chi}(n_2) v^{it},$$
$$R(v, n) := R(v, n, 1).$$

Proposition 7.4.

$$|I_m| \ll \phi(q) \frac{N^3}{q^3} \sum_{y_1,y_2\in(\mathbb{Z}/q\mathbb{Z})^\times} \int_{\substack{|v_1 m_1 + v_2 m_2 + m_3| \lesssim \frac{q}{N} \\ \frac{1}{2}\leq v_1,v_2\leq 2}} \left| R\left(\frac{v_2}{v_1}, y_2, y_1\right) R(v_2, y_2) R(v_1, y_1)\right| dv_1\ dv_2 + O(T^{-100}).$$

Moreover, if $|m_1| \leq |m_2| \leq |m_3|$, $|I_m| = O(T^{-100})$ unless $|m_2| \asymp |m_3|$.

Proof. Recall

$$I_m = \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1), \\ (t_2,\chi_2), \\ (t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3) e\left(\frac{-x\cdot m}{q}\right)$$
$$\times \hat{h}_{t_1-t_3}\left(\frac{Nm_1}{q}\right) \hat{h}_{t_2-t_1}\left(\frac{Nm_2}{q}\right) \hat{h}_{t_3-t_2}\left(\frac{Nm_3}{q}\right).$$

Expanding the integrals,

$$I_m = \frac{N^3}{q^3} \sum_{\substack{(t_1,\chi_1), \\ (t_2,\chi_2), \\ (t_3,\chi_3)\in\mathcal{S}}} \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^3} \chi_1\bar{\chi}_3(x_1)\chi_2\bar{\chi}_1(x_2)\chi_3\bar{\chi}_2(x_3) e\left(\frac{-x\cdot m}{q}\right)$$

$$\times \int_{\mathbb{R}^3} \tilde{\omega}(\mathbf{u}) u_1^{i(t_1-t_3)} u_2^{i(t_2-t_1)} u_3^{i(t_3-t_2)} e\left(\frac{-N\mathbf{m}\cdot\mathbf{u}}{q}\right) d\mathbf{u},$$

where $\tilde{\omega}(\mathbf{u}) = \omega(u_1)^2\omega(u_2)^2\omega(u_3)^2$ is compactly supported. We now make the substitution $y_1 = x_1x_3^{-1}, y_2 = x_2x_3^{-}1 \bmod q$ for the summation over x, and $v_1 = u_1/u_3, v_2 = u_2/u_3$ for the integral on the support of $\tilde{\omega}$. We thus rewrite the sum over x as

$$\sum_{y_1,y_2,x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi_1(y_1y_2^{-1})\chi_2(y_2)\chi_3(y_1^{-1}) e\left(\frac{-(y_1m_1+y_2m_2+m_3)x_3}{q}\right)$$
$$= \sum_{y_1,y_2\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi_1(y_1)\bar{\chi}_1(y_2)\chi_2(y_)\bar{\chi}_3(y_1) \sum_{x_3\in(\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{-(y_1m_1+y_2m_2+m_3)x_3}{q}\right),$$

where we can use the trivial bound $\phi(q)$ for the innermost sum. We also rewrite triple integral as

$$\int_{\mathbb{R}^3} \tilde{\omega}(v_1u_3, v_2u_3, u_3)\left(\frac{v_1}{v_2}\right)^{it_1} (v_2)^{it_2}\left(\frac{1}{v_1}\right)^{it_3} u_3^2\; e\left(\frac{-N(v_1m_1+v_2m_2+m_3)u_3}{q}\right)\; dv_1\; dv_2\; du_3$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}} u_3^2\; \tilde{\omega}(v_1u_3, v_2u_3, u_3) e\left(\frac{-N(v_1m_1+v_2m_2+m_3)u_3}{q}\right) du_3 \left(\frac{v_1}{v_2}\right)^{it_1} (v_2)^{it_2}\left(\frac{1}{v_1}\right)^{it_3}\; dv_1\; dv_2.$$

The integrand of the innermost integral is non-zero only if

$$v_1u_3, v_2u_3, u_3 \sim N.$$

Importantly, this requires $1/2 \le v_1, v_2 \le 2$, so we can truncate the outermost integrals to these regions. Moreover, by repeated integration by parts, this integral is $O_{\epsilon,A}(T^{-A})$ for any $|v_1m_1+v_2m_2+m_3| > qT^\epsilon/N$. So

$$|I_m| \ll \phi(q)\frac{N^3}{q^3} \sum_{y_1,y_2\in(\mathbb{Z}/q\mathbb{Z})^\times} \left| \int_{\substack{|v_1m_1+v_2m_2+m_3|\lesssim \frac{q}{N} \\ \frac{1}{2}\le v_1,v_2\le 2}} R\left(\frac{v_1}{v_2}, y_1, y_2\right) R(v_2, y_2) R\left(\frac{1}{v_1}, 1, y_1\right) dv_1\; dv_2 \right|$$
$$+ O(T^{-100}).$$

Since $|R(v_1^{-1}, 1, y_1)| = |R(v_1, y_1)|$, $|R\left(\frac{v_1}{v_2}, y_1, y_2\right)| = R\left(\frac{v_2}{v_1}, y_2, y_1\right)|$, we have the first part of the proposition. The second part of the proposition follows from the integral bounds $|v_1m_1+v_2m_2+m_3| \lesssim q/N$ and $v_1, v_2 \asymp 1$. These force $|m_2| \asymp |m_3|$, or else the integral will be zero.

$\square$

Adapting from Guth and Maynard, when $|m_2| \asymp |m_3|$, the domain of integration can be written as

$$|v_1m_1+v_2m_2+m_3| \lesssim \frac{q}{N} \implies \left|v_2 - \frac{v_1m_1+m_3}{-m_2}\right| \lesssim \frac{q}{|m_2|N} \asymp \frac{q}{|m_3|N}.$$

Thus, the integration in v_2 is in a small neighborhood of $\frac{v_1m_1+m_3}{-m_2}$.

Let $\tilde{\phi}$ be a smooth bump function such that equals $\tilde{\phi} = 1$ on $|x| \lesssim 1$ and is supported in $|x| \lesssim 1$, with a larger constant, so that $\|\tilde{\phi}^{(j)} \lesssim_j 1\|$ for all j. We define

$$\tilde{R}_M(v, y_1, y_2) := \left(\int \frac{NM}{q}\tilde{\phi}\left(\frac{NM}{q}(v-v')\right)|R(v', y_1, y_2)|^2 dv'\right)^{1/2}.$$

Proposition 7.5. *There is a choice of* $0 < M_1 \leq M \lesssim qT/N$ *such that*

$$S_3 \lesssim \phi(q)\frac{N^2}{Mq^2}\sum_{|m_1|\sim M_1, |m_2|,|m_3|\sim M} \tilde{I}_m + O(T^{-100}).$$

where

$$\tilde{I}_m := \sum_{y_1,y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v_1\asymp 1}\left|R(v_1,y_1)\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2},y_2\right)\right|dv_1.$$

Proof. By Proposition 7.4, we consider the terms $|m_1| \leq |m_2| \leq |m_3|$ at the cost of a factor of 6, and $|m_2| \asymp |m_3|$. Expanding the sum over m_1, m_2, m_3 dyadically, we get for some $M_1 \leq M \lesssim qT/N$

$$S_3 \lesssim \sum_{|m_1|\sim M_1, |m_2|,|m_3|\sim M}|I_m| + O(T^{-100})$$

We now consider

$$\begin{aligned}
&\int_{\substack{|v_1m_1+v_2m_2+m_3|\lesssim \frac{q}{N}\\ \frac{1}{2}\leq v_1,v_2\leq 2}}\left|R\left(\frac{v_2}{v_1},y_2,y_1\right)R(v_2,y_2)R(v_1,y_1)\right|dv_1\,dv_2\\
&\ll \int_{v_1\asymp 1}|R(v_1,y_1)|\int_{\left|v_2-\frac{v_1m_1+m_3}{-m_2}\right|\lesssim\frac{q}{|m_2|N}}\left|R\left(\frac{v_2}{v_1},y_2,y_1\right)R(v_2,y_2)\right|dv_2\,dv_1\\
&\ll \int_{v_1\asymp 1}|R(v_1,y_1)|\int_{\left|v_2-\frac{v_1m_1+m_3}{-m_2}\right|\lesssim\frac{q}{MN}}\left|R\left(\frac{v_2}{v_1},y_2,y_1\right)R(v_2,y_2)\right|dv_2\,dv_1
\end{aligned}$$

when $|m_2| \asymp M$. The inner integral, by Cauchy-Schwarz, is

$$\begin{aligned}
&\leq\left(\int_{\left|v_2-\frac{v_1m_1+m_3}{-m_2}\right|\lesssim\frac{q}{MN}}\left|R\left(\frac{v_2}{v_1},y_2,y_1\right)\right|^2\,dv_2\right)^{1/2}\left(\int_{\left|v_2-\frac{v_1m_1+m_3}{-m_2}\right|\lesssim\frac{q}{MN}}|R(v_2,y_2)|^2\,dv_2\right)^{1/2}\\
&\ll\frac{q}{MN}\tilde{R}_M\left(\frac{v_1m_1+m_3}{-m_2v_1},y_2,y_1\right)\tilde{R}_M\left(\frac{v_1m_1+m_3}{-m_2},y_2\right)
\end{aligned}$$

where in the last step, we used $v_1 \asymp 1$. Thus, for $|m_2| \sim M$,

$$|I_m| \lesssim \phi(q)\frac{N^2}{Mq^2}\tilde{I}_m.$$

The proposition follows from this claim. ☐

Lemma 7.6. *Let* $\mathcal{S} = \{(t_j,\chi_j)\}$*, and the* t*'s are contained in an interval of length* T*, and are* T^ϵ*-separated for the same character. Then*

$$\sum_{y\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v\asymp 1}|R(v,y)|^2\,dv \ll_\epsilon \phi(q)|\mathcal{S}|.$$

Proof. We have

$$|R(v,y)|^2 = \sum_{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}}\chi_1\bar{\chi}_2(y)v^{i(t_1-t_2)}.$$

Let ψ be a bump function supported on $v \asymp 1$ and equals 1 on the domain of integration in the lemma. By orthogonality of characters,

$$\sum_{y\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v\asymp 1}|R(v,y)|^2\,dv \leq \sum_{y\in(\mathbb{Z}/q\mathbb{Z})^\times}\int\psi(v)\,|R(v,y)|^2\,dv$$

$$
\begin{aligned}
&=\phi(q)\int\psi(v)\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}\\ \chi_1=\chi_2}} v^{i(t_1-t_2)}dv\\
&=\phi(q)\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}\\ \chi_1=\chi_2}}\int\psi(v)v^{i(t_1-t_2)}dv.
\end{aligned}
$$

In the sum, the terms $t_1=t_2$ contribute $O(|\mathcal{S}|)$. If $t_1\neq t_2$, then $|t_1-t_2|\geq T^{\epsilon}$. The integral in this case is $O_\epsilon(T^-100)$ and is negligible. □

Lemma 7.7. *Let* $E(\mathcal{S})=\#\{(t_1,\chi_1),(t_2,\chi_2),(t_3,\chi_3),(t_4,\chi_4)\in\mathcal{S}:|t_1+t_2-t_3-t_4|\leq 1,\chi_1\chi_2=\chi_3\chi_4\}$. *Then*

$$
\sum_{y\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v\asymp 1}|R(v,y)|^4\,dv\lesssim\phi(q)E(\mathcal{S}).
$$

Proof. We have

$$
|R(v,y)|^4=\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in\mathcal{S}}}\chi_1\chi_2\bar{\chi_3}\bar{\chi_4}(y)v^{i(t_1+t_2-t_3-t_4)}.
$$

So again by the orthogonality of characters,

$$
\sum_{y\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v\asymp 1}|R(v,y)|^4\,dv=\phi(q)\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in\mathcal{S}\\ \chi_1\chi_2=\chi_3\chi_4}}\int_{v\asymp 1}v^{i(t_1+t_2-t_3-t_4)}dv.
$$

Similar to the previous proof, we can introduce a bump function for the integral, and restrict the summation to the terms $|t_1+t_2-t_3-t_4|\leq T^{\epsilon}$ with an error of $O_\epsilon(T^{-100})$. The remaining terms in the summation contribute $O(E(\mathcal{S}))$. □

Lemma 7.8. *Let* $E(\mathcal{S})=\#\{(t_1,\chi_1),(t_2,\chi_2),(t_3,\chi_3),(t_4,\chi_4)\in\mathcal{S}:|t_1+t_2-t_3-t_4|\leq 1,\chi_1\chi_2=\chi_3\chi_4\}$. *Then*

$$
\sum_{y\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v\asymp 1}\left|\tilde{R}_M(v,y)\right|^4dv\lesssim\phi(q)E(\mathcal{S}).
$$

Proof. We apply Cauchy-Schwarz to

$$
\begin{aligned}
\int_{v\asymp 1}\left|\tilde{R}_M(v,y)\right|^4dv&\lesssim\int_{v\asymp 1}\left(\int_{|u-v|\lesssim q/NM}\frac{NM}{q}|R(u)|^2du\right)^2dv\\
&\lesssim\frac{NM}{q}\int_{v\asymp 1}\int_{|u-v|\lesssim q/NM}|R(u)|^4du\;dv\\
&\lesssim\int_{u\asymp 1}|R(u)|^4du.
\end{aligned}
$$

Lemma 7.7 completes the proof. □

Proof of Proposition 7.1. We first appler Hölder's inequality on the integral to get

$$
\begin{aligned}
\tilde{I}_m\leq\sum_{y_1,y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}&\left(\int_{v_1\asymp 1}|R(v_1,y_1)|^2dv_1\right)^{1/2}\left(\int_{v_1\asymp 1}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\right|^4dv_1\right)^{1/4}\\
&\left(\int_{v_1\asymp 1}\left|\tilde{R}_M(\frac{m_1v_1+m_3}{-m_2},y_2)\right|^4dv_1\right)^{1/4},
\end{aligned}
$$

Notice the first integral is independent of y_2, for sum of the second and third integrals over y_2, we apply Cauchy-Schwarz to get

$$
\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\left(\int_{v_1\asymp 1}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\right|^4dv_1\right)^{1/4}\left(\int_{v_1\asymp 1}\left|\tilde{R}_M(\frac{m_1v_1+m_3}{-m_2},y_2)\right|^4dv_1\right)^{1/4}
$$

$$
\begin{aligned}
&\le\Bigg(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\bigg(\int_{v_1\asymp 1}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\right|^4dv_1\bigg)^{1/2}\Bigg)^{1/2}\\
&\qquad\Bigg(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\bigg(\int_{v_1\asymp 1}\left|\tilde{R}_M(\frac{m_1v_1+m_3}{-m_2},y_2)\right|^4dv_1\bigg)^{1/2}\Bigg)^{1/2}\\
&\le\phi(q)^{\frac12}\Bigg(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v_1\asymp 1}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\right|^4dv_1\Bigg)^{1/4}\\
&\qquad\Bigg(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v_1\asymp 1}\left|\tilde{R}_M(\frac{m_1v_1+m_3}{-m_2},y_2)\right|^4dv_1\Bigg)^{1/4}\\
&\le\phi(q)^{\frac12}\Bigg(\sum_{y_3\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v_1\asymp 1}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_3\right)\right|^4dv_1\Bigg)^{1/4}\\
&\qquad\Bigg(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v_1\asymp 1}\left|\tilde{R}_M(\frac{m_1v_1+m_3}{-m_2},y_2)\right|^4dv_1\Bigg)^{1/4}\\
&\lesssim\phi(q)E(\mathcal{S})^{\frac12}(M/M_1)^{1/4}\le\phi(q)E(\mathcal{S})^{\frac12}M/M_1.
\end{aligned}
$$

where in the penultimate step, we made a change of variables $y_3=y_2y_1^{-1}$. In the last step we change variables of integration $u=(m_1v_1+m_3)/(-m_2v_1)$ and $u=(m_1v_1+m_3)/(-m_2)$ with a Jacobian factor of $\asymp 1$ and $\sim M/M_1$ respectively. For the first integral, applying Cauchy Schwarz gives

$$
\sum_{y_1\in(\mathbb{Z}/q\mathbb{Z})^\times}\bigg(\int_{v_1\asymp 1}|R(v_1,y_1)|^2dv_1\bigg)^{1/2}\le\phi(q)^{\frac12}\Bigg(\sum_{y_1\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v_1\asymp 1}|R(v_1,y_1)|^2dv_1\Bigg)^{1/2}\ll_\epsilon\phi(q)|\mathcal{S}|^{1/2}.
$$

Combined with Proposition 7.5, this gives

$$
S_3\lesssim\phi(q)^3\frac{N^2M^2}{q^2}E(\mathcal{S})^{1/2}|\mathcal{S}|^{1/2}\lesssim\phi(q)^3T^2E(\mathcal{S})^{1/2}|\mathcal{S}|^{1/2}.
$$

□

Proof of Proposition 7.2. Recall that

$$
\begin{aligned}
S_3\lesssim\phi(q)\frac{N^2}{Mq^2}&\sum_{|m_1|\sim M_1,|m_2|,|m_3|\sim M}\ \sum_{y_1,y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}\int_{v_1\asymp 1}\\
&\left|R(v_1,y_1)\,\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2},y_2\right)\right|dv_1.
\end{aligned}
$$

We apply Cauchy Schwarz repeatedly to get

$$
\begin{aligned}
S_3&\lesssim\phi(q)\frac{N^2}{Mq^2}\sum_{y_1\in(\mathbb{Z}/q\mathbb{Z})^\times}S_{3,1}^{1/2}\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}S_{3,2}^{1/2}\\
&\lesssim\phi(q)^{3/2}\frac{N^2}{Mq^2}\sum_{y_1\in(\mathbb{Z}/q\mathbb{Z})^\times}S_{3,1}^{1/2}\Bigg(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})^\times}S_{3,2}\Bigg)^{1/2},
\end{aligned}
$$

where

$$
S_{3,1}=S_{3,1}(y_1)=\int_{v_1\asymp 1}|R(v_1,y_1)|^2dv_1,
$$

$$S_{3,2}=S_{3,2}(y_1,y_2)=\int_{v_1\asymp 1}\left(\sum_{|m_1|\sim M_1,|m_2|,|m_3|\sim M}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2},y_2\right)\right|\right)^2 dv_1,$$

and

$$\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})}S_{3,2}\lesssim\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})}(S_{3,3}S_{3,4})^{1/2}\lesssim\left(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})}S_{3,5}\right)^{1/2}\left(\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})}S_{3,4}\right)^{1/2}$$

where

$$\begin{aligned}
S_{3,3}&=\int_{v_1\asymp 1}\left(\sum_{|m_1|\sim M_1,|m_2|,|m_3|\sim M}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2,y_1\right)\right|^2\right)^2 dv_1,\\
S_{3,4}&=\int_{v_1\asymp 1}\left(\sum_{|m_1|\sim M_1,|m_2|,|m_3|\sim M}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2},y_2\right)\right|^2\right)^2 dv_1,\\
S_{3,3}&=\int_{v_1\asymp 1}\left(\sum_{|m_1|\sim M_1,|m_2|,|m_3|\sim M}\left|\tilde{R}_M\left(\frac{m_1v_1+m_3}{-m_2v_1},y_2\right)\right|^2\right)^2 dv_1.
\end{aligned}$$

The remaining arguments are similar with GM to get

$$\begin{aligned}
\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})}S_{3,4}\lesssim&\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})}M^6\left(\int_{v\asymp 1}|R(v,y)|^2\,dv\right)^2+\sum_{y_2\in(\mathbb{Z}/q\mathbb{Z})}M^4\left(\int_{v\asymp 1}|R(v,y)|^4\,dv\right)\\
\lesssim&M^6\phi(q)^2|\mathcal{S}|^2+M^4\phi(q)E(\mathcal{S}).
\end{aligned}$$

So we get

$$\begin{aligned}
S_3\lesssim&\phi(q)^{3/2}\frac{N^2}{Mq^2}\phi(q)|\mathcal{S}|^{1/2}\left(M^6\phi(q)^2|\mathcal{S}|^2+M^4\phi(q)E(\mathcal{S})\right)^{1/2}\\
\lesssim&\phi(q)^{7/2}\frac{N^2}{q^2}M^2|\mathcal{S}|^{3/2}+\phi(q)^3\frac{N^2}{q^2}M|\mathcal{S}|^{1/2}E(\mathcal{S})^{1/2}.
\end{aligned}$$

Taking $M\lesssim qT/N$,

$$S_3\lesssim\phi(q)^{7/2}T^2|\mathcal{S}|^{3/2}+\phi(q)^3\frac{NT}{q}|\mathcal{S}|^{1/2}E(\mathcal{S})^{1/2}.$$

□

8 Energy bound

Here we provide the generalization for the orthogonal energy bound for Guth and Maynard's result.

Proposition 8.1.

$$E(\mathcal{S})\lesssim|\mathcal{S}|^2N^{2-2\sigma}+|\mathcal{S}|^3N^{1-2\sigma}+|\mathcal{S}|^{9/4}(qT)^{1/2}N^{1-2\sigma}.$$

The idea for bounding energy is similar; if $\chi_1\chi_2=\chi_3\chi_4$ and $|t_1+t_2-t_3-t_4|$ is small, we should expect $|D_N(t_1+t_2-t_3,\chi_1\chi_2\bar{\chi}_3)|\simeq|D_N(t_4,\chi_4)|>N^\sigma$.

Lemma 8.2.

$$D_N(t,\chi)\lesssim\int_{|u-t|\lesssim 1}|D_N(u,\chi)|du+O(T^{-100}),$$

uniformly in χ.

Proof. (GM)

$$D_N(t,\chi) = \sum_n \omega\left(\frac{n}{N}\right) b_n n^{it} \psi\left(\frac{\log n}{2\pi}\right)$$

For other characters, we can just redefine $b'_n = b_n \chi(n)$. $\square$

Lemma 8.3. *We have*

$$E(\mathcal{S}) \lesssim N^{-2\sigma} \sum_{n_1,n_2\sim N} \left|R\left(\frac{n_1}{n_2}, n_1, n_2\right)\right|^3.$$

Proof. We have

$$E(\mathcal{S}) = \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in\mathcal{S}\\|t_1+t_2-t_3-t_4|\le 1\\ \chi_1\chi_2=\chi_3\chi_4}} 1 \le N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in\mathcal{S}\\|t_1+t_2-t_3-t_4|\le 1\\ \chi_1\chi_2=\chi_3\chi_4}} |D_N(t_4,\chi_4)|^2.$$

Now we apply the previous lemma and Cauchy-Schwarz to get

$$|D_N(t_4,\chi_4)|^2 \lesssim \int_{|u-t_4|\lesssim 1} |D_N(u,\chi_4)|^2 du \lesssim \int_{|u-t_1-t_2+t_3|\lesssim 1} |D_N(u,\chi_1\chi_2\bar{\chi}_3)|^2 du,$$

Since χ_1,χ_2,χ_3 fixes χ_4, and the t's within the same character are T^ϵ separated, there is $O(1)$ possible pairs of (t_4,χ_4) for each choice of $(t_1,\chi_1),(t_2,\chi_2),(t_3,\chi_3)$, so

$$\begin{aligned}
E(\mathcal{S}) \lesssim& N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3),(t_4,\chi_4)\in\mathcal{S}\\|t_1+t_2-t_3-t_4|\le 1\\ \chi_1\chi_2=\chi_3\chi_4}} \int_{|u-t_1-t_2+t_3|\lesssim 1} |D_N(u,\chi_1\chi_2\bar{\chi}_3)|^2 du \\
\lesssim& N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \int_{|u-t_1-t_2+t_3|\lesssim 1} |D_N(u,\chi_1\chi_2\bar{\chi}_3)|^2 du \\
=& N^{-2\sigma} \sum_{\substack{(t_1,\chi_1),(t_2,\chi_2),\\(t_3,\chi_3)\in\mathcal{S}}} \int_{|u|\lesssim 1} |D_N(t_1+t_2-t_3+u,\chi_1\chi_2\bar{\chi}_3)|^2 du \\
=& N^{-2\sigma} \sum_{n_1,n_2} b_{n_1}\bar{b}_{n_2} \omega\left(\frac{n_1}{N}\right)\omega\left(\frac{n_2}{N}\right) \int_{|u|\lesssim 1} \left(\frac{n_1}{n_1}\right)^{iu} R\left(\frac{n_1}{n_2},n_1,n_2\right)^2 R\left(\frac{n_2}{n_1},n_2,n_1\right) du \\
\lesssim& N^{-2\sigma} \sum_{n_1,n_2\sim N} \left|R\left(\frac{n_1}{n_2},n_1,n_2\right)^2 R\left(\frac{n_2}{n_1},n_2,n_1\right)\right| \\
\lesssim& N^{-2\sigma} \sum_{n_1,n_2\sim N} \left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^3.
\end{aligned}$$

$\square$

Lemma 8.4. *We have*

$$\sum_{n_1,n_2\sim N} \left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^2 \lesssim |\mathcal{S}|N^2 + |\mathcal{S}|^2 N + |\mathcal{S}|^{5/4}(qT)^{1/2}N.$$

Proof. From the definition of R,

$$\begin{aligned}
\sum_{n_1,n_2\sim N} \left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^2 &= \sum_{n_1,n_2\sim N} \sum_{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}} \chi_1(n_1)\bar{\chi}_1(n_2)\left(\frac{n_1}{n_2}\right)^{it_1} \bar{\chi}_2(n_1)\chi_2(n_2)\left(\frac{n_1}{n_2}\right)^{-it_2} \\
&= \sum_{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}} \left|\sum_{n\sim N} \chi_1(n_1)\bar{\chi}_2(n_1) n^{i(t_1-t_2)}\right|.
\end{aligned}$$

A direct application of Heath Brown's Theorem 5.2 gives

$$\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^2 \lesssim |\mathcal{S}|N^2+|\mathcal{S}|^2N+|\mathcal{S}|^{5/4}(qT)^{1/2}N.$$

□

The trivial bound for $R\le|\mathcal{S}|$ gives

$$\begin{aligned}E(\mathcal{S}) \lesssim& N^{-2\sigma}\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^3\\ \lesssim&|\mathcal{S}|N^{-2\sigma}\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^2\\ \lesssim&|\mathcal{S}|^2N^{2-2\sigma}+|\mathcal{S}|^3N^{1-2\sigma}+|\mathcal{S}|^{9/4}(qT)^{1/2}N^{1-2\sigma}..\end{aligned}$$

TODO: Merge with previous part to find bound

The arguments beyond will be adaptations from GM.

Lemma 8.5. *We have*

$$\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^4 \lesssim ?$$

Proof. Let

$$U_B=\{u\in\mathbb{Z}\ :\ \#\{((t_1,\chi_1),(t_2,\chi_2))\in\mathcal{S}\ :\ \lfloor t_1-t_2\rfloor=u\}\sim B\},$$

so that we split the sum in R as

$$\begin{aligned}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^4 =&\left|\sum_{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}}\chi_1(n_1)\bar{\chi}_2(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)}\right|^2\\ =&\left|\sum_{j=0}^{\lfloor\log_2|\mathcal{S}|\rfloor}\sum_{u\in U_{2^j}}\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}\\ \lfloor t_1-t_2\rfloor=u}}\chi_1(n_1)\bar{\chi}_2(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)}\right|^2\\ \lesssim&\sum_{j=0}^{\lfloor\log_2|\mathcal{S}|\rfloor}\left|\sum_{u\in U_{2^j}}\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}\\ \lfloor t_1-t_2\rfloor=u}}\chi_1(n_1)\bar{\chi}_2(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)}\right|^2\end{aligned}$$

where we applied Cauchy-Schwarz in the last step. Therefore,

$$\begin{aligned}\sum_{n_1,n_2\sim N}\left|R\left(\frac{n_1}{n_2},n_1,n_2\right)\right|^4 \lesssim&\sup_{j\le\lfloor\log_2|\mathcal{S}|\rfloor}\sum_{n_1,n_2\sim N}\left|\sum_{u\in U_{2^j}}\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}\\ \lfloor t_1-t_2\rfloor=u}}\chi_1(n_1)\bar{\chi}_2(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)}\right|^2\\ \le&\sup_{j\le\lfloor\log_2|\mathcal{S}|\rfloor}\sum_{n_1,n_2\sim N}\left|\sum_{u\in U_{2^j}}\sum_{\substack{(t_1,\chi_1),(t_2,\chi_2)\in\mathcal{S}\\ \lfloor t_1-t_2\rfloor=u}}\chi_1(n_1)\bar{\chi}_2(n_2)\left(\frac{n_1}{n_2}\right)^{i(t_1-t_2)}\right|^2\end{aligned}$$

□

9 Preliminaries

Here we give some supplementary definitions and statements of theorems.

Number Theory

Definition 9.1 (Dirichlet Characters)**.** *Let $q \in \mathbb{N}$. A Dirichlet character $\chi : \mathbb{N} \to \mathbb{C}$ modulus q is an arithmetic function satisfying*

- *(Periodicity)* $\chi(n+q) = \chi(n) \forall n \in \mathbb{N}$.
- *(Complete multiplicativity)* $\chi(nm) = \chi(n)\chi(m) \forall n, m \in \mathbb{N}$.
- $|\chi(n)| = \begin{cases} 1, & \text{if gcd(n,q) = 1,} \\ 0, & \text{otherwise.} \end{cases}$

Proposition 9.2. *There are $\phi(q)$ Dirichlet characters of modulus q.*

Proof. Taking residual classes mod q, we see that Dirchlet characters are in one-to-one correspondence with one-dimensional representations of the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$. Since this group is abelian, all of its irreducible representations are one-dimensional. Therefore, the number of Dirichlet characters equals the number of irreducible representations of the $(\mathbb{Z}/q\mathbb{Z})^\times$. It is known that the sum of squares of the dimensions irreducible representations equals the order of the group, so we have

$$\phi(q) = |(\mathbb{Z}/q\mathbb{Z})^\times| = \sum_{\text{irreducible representations } \varphi} (\dim \varphi)^2 = \sum_{\text{irreducible representations } \varphi} 1.$$

□

Definition 9.3. *A Dirichlet character χ modulus q is induced by another character χ^* mod $m < q$ if they agree on all n such that $\gcd(q,n) = 1$. A Dirchlet character is primitive if it is not induced by another character. A Dirchlet character is principal if it is induced by the character $\chi_1(n) := 1(n) \equiv 1$, thus corresponds to the trivial representation.*

Theorem 9.4 (Möbius Inversion)**.** *The Möbius function μ is defined for $n \in \mathbb{N}$,*

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 p_2 \ldots p_k \text{ for distinct } p\text{'s} \\ 0, & \text{otherwise} \end{cases}$$

Suppose we have arithmetic functions f, g, and that

$$f(n) = \sum_{d|n} g(d)$$

Then the Möbius Inversion formula gives

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

Example 9.5. *On $\Re(s) > 1$, let $M_N(s) = \sum_{n \le N} \mu(n) n^{-s}$. Then setting $f(n) = 1$ for all n, $g(1) = 1$, $g(n) = 0$ for $n \ge 2$, we multiply M_N by ζ in Dirichlet series to get*

$$\zeta(s) M_N(s) = \sum_n \frac{a_n}{n^{-s}},$$

where $a_n = g(n)$ for all $n \le N$. Similarly, letting $M_N(s) = \sum_{n \le N} \chi(n)\mu(n) n^{-s}$ for some Dirichlet character χ, we get

$$L(s,\chi) M_N(s) = \sum_n \frac{a_n \chi(n)}{n^{-s}}$$

with the same a_n as in the previous equation.

Harmonic Analysis

Theorem 9.6 (Fourier Inversion)**.** *In Schwartz space, the Fourier transform of $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ of $f \in \mathcal{S}(\mathbb{R}^d)$ is given by*

$$\hat{f}(\xi) \coloneqq \mathcal{F}f(\xi) \coloneqq \int_{\mathbb{R}^d} e(-\xi \cdot \mathbf{x}) f(\mathbf{x}) \, d\mathbf{x}$$

has inverse given by

$$f(\mathbf{x}) = \mathcal{F}^{-1}\hat{f}(\mathbf{x}) \coloneqq \int_{\mathbb{R}^d} e(\xi \cdot \mathbf{x}) \hat{f}(\xi) \, d\xi.$$

Theorem 9.7 (Mellin Inversion)**.** *The Mellin transform of a function $f : (0, \infty) \to \mathbb{C}$*

$$\tilde{f}(s) \coloneqq \mathcal{M}f(s) \coloneqq \int_0^\infty f(x) x^{s-1} \, dx$$

has inverse

$$\mathcal{M}^{-1}\tilde{f}(x) = \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) x^{-s} ds$$

on $a < c < b$ provided that the integral $\tilde{f}$ is absolute convergent on the strip $a < \Re(s) < b$.

Theorem 9.8 (Poisson Summation)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be Schwartz. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{\xi \in \mathbb{Z}} \hat{f}(\xi).$$