Large values of Dirichlet Polynomials and Zero Density Results

Honors Thesis Presentation

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Overview

1. Motivation
2. Zeta function and PNT
3. Primes in short intervals and Zero density
4. Large Values Estimates results

Motivation

- We want to estimate the distribution of primes

$$\pi(x) \stackrel{\text{def}}{=} \sum_{p \leq x} 1.$$

- The Prime Number Theorem gives

$$\pi(x) = (1 + o(1)) \frac{x}{\log x}.$$

- What about

$$\sum_{x \leq p \leq x+y} 1$$

for $y = o(x)$? Is it still $\sim y / \log x$?

Zeta function

- For $\Re(s) > 1$ define
$$\zeta(s) \stackrel{\text{def}}{=} \sum_{n \geq 1} n^{-s}.$$
- This also has a product representation
$$\zeta(s) = \prod_p (1 - p^{-1})^{-1}.$$
- We can also analytically continue this to the whole complex plane.
$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

Idea of proof: Use Integral representation of Γ, exchange summation and intgration and apply Poisson summation.

Relation to PNT

1. We scale everything by $\log x$. Let

$$\Lambda(n) \stackrel{\text{def}}{=} \begin{cases} \log p, & \text{if } n = p^k, \\ 0, & \text{else,} \end{cases}$$

   and

$$\psi(x) \stackrel{\text{def}}{=} \sum_{n \leq x} \Lambda(n).$$

2. Using product representation

$$\frac{\zeta'}{\zeta} = [\log \zeta]' = -\sum_{n} \frac{\Lambda(n)}{n^s}$$

Relation to PNT (cont.)

We can make this connection precise through the explicit formula

Theorem (Riemann-von Mangoldt explicit formula)

Let N be not a prime power. We have

$$\psi(N) = N - \lim_{T\to\infty} \sum_{|\Im(\rho)|\le T, \zeta(\rho)=0} \frac{N^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + \frac{1}{2}\log(1-N^{-2}).$$

Theorem (Truncated Riemann-von Mangoldt explicit formula)

We have

$$\psi(N) = N - \sum_{|\Im(\rho)|\le T, \zeta(\rho)=0} \frac{N^\rho}{\rho} + O(N(\log NT)^2/T) + O(\log N).$$

Assume Riemann Hypothesis, then $|N^\rho| = N^{1/2}$, so we have $\psi(N) = N + O(N^{1/2+\epsilon})$.

Prime numbers in short intervals

What about $\sum_{x \leq n \leq x+y} \Lambda(n) \sim y$? If RH holds then we can take $y = x^{1/2+\epsilon}$. Show enough zeros have 'small' real part. Use a zero counting function for zeros with large real part.

Definition (Zero density)

$$N(\sigma, T) \stackrel{\text{def}}{=} \#\{\rho : \zeta(\rho) = 0, |\Im(\rho)| \leq T\}.$$

Theorem (Hoheisel)

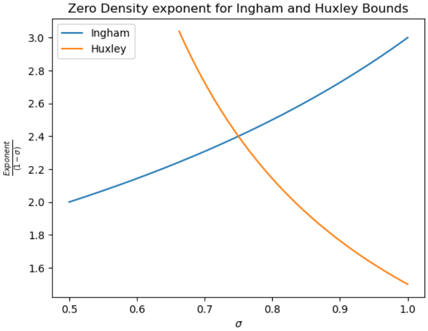
If $N(\sigma, T) \ll T^{a(1-\sigma)} \log^b T$ uniformly for $\sigma \in [1/2, T]$ then we can take $y = x^\theta$, for

$$\theta > 1 - \frac{1}{a + \frac{b}{A}},$$

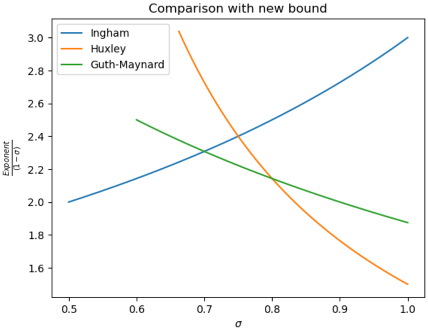
and A is a (large) absolute constant.

Zero density

Before 2023: $N(\sigma, T) \ll T^{12(1-\sigma)/5+o(1)}$ (Ingham-Huxley)
2024: $N(\sigma, T) \ll T^{30(1-\sigma)/13+o(1)}$ (Guth-Maynard).



(a) $\sigma = 3/4$ is also the bottleneck when written in Hoheisel's form.



(b) The exponent is reduced around the bottleneck region.

Idea of Zero density proof

1. $\frac{1}{\zeta}(s)$ can be represented as
$$\sum_n \frac{\mu(n)}{n^s}.$$

2. Let $M_x(s) \stackrel{\text{def}}{=} \sum_{n \leq x} \frac{\mu(n)}{n^s}$, which is an approximation of $1/\zeta$. Then we have by Möbius inversion
$$\zeta(s)M_x(s) = 1 + \sum_{n>x} a_{n,x}n^{-s} \approx 1 + \sum_{n>x} a_{n,x}n^{-s}e^{-n/y} \approx 1 + \sum_{y^2>n>x} a_{n,x}n^{-s}e^{-n/y}$$
with some error decreasing in y.

3. If s is a zero of ζ then the left hand side is zero, which means the Dirichlet series on the right hand side has magnitude close to 1.

Guth-Maynard zero density result

Theorem (Guth-Maynard)

Let $D_N(t) \stackrel{\text{def}}{=} \sum_{N \le x < 2N} b_n n^{it}$. *If* $W \subset [0, T]$ *is a set of* 1*-separated points such that*

$$|D_N(t_i)| > V \ \forall t_i \in W$$

then

$$|W| \le T^{o(1)} \left(\max_{N \le n < 2N} |b_n| \right) \left(N^2 V^{-2} + N^{18/5} V^{-4} + T N^{12/5} V^{-4} \right).$$

Hybrid Zero Density and Dirichlet Value Estimate

There are generalizations for zero densities of Dirichlet-L functions $L(s, \chi) \stackrel{\text{def}}{=} \sum_n \chi(n) n^{-s}$.

Theorem

Let $D_N(t, \chi) \stackrel{\text{def}}{=} \sum_{N \leq x < 2N} b_n \chi(n) n^{it}$. If $W = \{t_i, \chi_i\}$ such that χ_i is a primitive character mod q, $t_i \in [0, T]$ are 1-separated for the same character,

$$|D_N(t_i, \chi_i)| > V \ \forall (t_i, \chi_i) \in W.$$

Then we have for $N \geq q^{5/6}$

$$|W| \leq (qT)^{o(1)} (\max_{N \leq n < 2N} |b_n|)(N^2 V^{-2} + N^{18/5} V^{-4} + qTN^{12/5} V^{-4}).$$