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Abstract

Notation

Below are the notational preferences of the author.

1. The set of natural numbers \mathbb{N} does not contain 0.
2. p always denotes a prime, and by extension p_j, p_n etc.

Preliminaries

Number Theory Results

Theorem 0.1 (Möbius Inversion). *The Möbius function μ is defined for $n \in \mathbb{N}$,*

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 p_2 \dots p_k \text{ for distinct } p\text{'s} \\ 0, & \text{otherwise} \end{cases}$$

1 Introduction to the Riemann Zeta Function

Definition 1.1 (Zeta Function). *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

Remark: This expression also converges absolutely for $\Re(s) > 1$. Since

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{p^s}{p^s - 1} = 1 + \frac{1}{p^s - 1}$$

and $\sum (p^s - 1)^{-1}$ converges absolutely by comparison to the zeta function Dirichlet series.

Sketch of proof. Write $s = \sigma + it$. For each p ,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p . We thus take for $m > N$

$$\begin{aligned} \prod_{p \leq N} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \leq N} \left(\sum_{k=1}^m \frac{1}{p^{ks}} + O(2^{-m\sigma})\right) \\ &\stackrel{(*)}{=} \sum_{n=1}^N \frac{1}{n^s} + O_1\left(\sum_{n=N+1}^{\infty} \frac{1}{n^{\sigma}}\right) + O(2^{-m\sigma}) \\ &= \zeta(s) + O_1\left(\sum_{n=N+1}^{\infty} \frac{1}{n^{\sigma}}\right) + O(2^{-m\sigma}) \end{aligned}$$

Where we apply to Fundamental Theorem of Arithmetic in (*) to show that each term n^{-s} has coefficient 1 determined by the unique prime factorization. As $m \rightarrow \infty$, $2^{-m\sigma} \rightarrow 0$. Then we take $N \rightarrow \infty$, the tail of the infinite sum converges to zero too. \square

Proposition 1.2 shows an inherent connection of the zeta function with primes. To further see this connection, we need to extend the zeta function.

Theorem 1.3. ζ extends to a meromorphic function on \mathbb{C} with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too.

Theorem 1.4. Let $\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$. Then

$$\xi(s) = \xi(1-s). \quad (3)$$

Proof. Using

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx,$$

we make the substitution $x = \pi n^2 y$ to get

$$\begin{aligned} \Gamma(s) &= \int_0^{\infty} e^{-\pi n^2 y} (\pi n^2 y)^{s-1} \pi n^2 dy \\ \implies \frac{\Gamma(s)}{\pi^s n^{2s}} &= \int_0^{\infty} e^{-\pi n^2 y} y^{s-1} dy \end{aligned}$$

So that by the Monotone Convergence Theorem,

$$\begin{aligned} \pi^{-s/2} \Gamma(s/2) \zeta(s) &= \sum_{n=1}^{\infty} \frac{\Gamma(s/2)}{\pi^{s/2} n^s} \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 x} x^{s/2-1} dx \\ &= \int_0^{\infty} \sum_{n=1}^{\infty} \left(e^{-\pi n^2 x}\right) x^{s/2-1} dx. \end{aligned}$$

We now let

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}, \quad \theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 2\omega(x) + 1,$$

and apply Poisson Summation to

$$\begin{aligned} \theta(x) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi i k y} dy \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i k u / \sqrt{x}} du \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} e^{-\pi k^2 / x} \\
&= \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)
\end{aligned}$$

using the substitution $y\sqrt{x} = u$. Replacing with ω ,

$$\sqrt{x}(2\omega(x) + 1) = 2\omega\left(\frac{1}{x}\right) + 1 \implies \omega\left(\frac{1}{x}\right) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}.$$

We thus write, using $y = 1/x$,

$$\begin{aligned}
\xi(s) &= \int_0^1 \omega(x) x^{s/2-1} dx + \int_1^\infty \omega(x) x^{s/2-1} dx \\
&= \int_1^\infty \omega(1/y) y^{-s/2-1} dy + \int_1^\infty \omega(x) x^{s/2-1} dx \\
&= \int_1^\infty \left(\sqrt{y}\omega(y) + \frac{\sqrt{y}}{2} - \frac{1}{2} \right) y^{-s/2-1} dy + \int_1^\infty \omega(x) x^{s/2-1} dx \\
&= \int_1^\infty \left(\frac{\sqrt{y}}{2} - \frac{1}{2} \right) y^{-s/2-1} dy + \int_1^\infty \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2} \right) dx \\
&= \frac{1}{1-s} + \frac{1}{s} + \int_1^\infty \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2} \right) dx \\
&= \frac{1}{s(1-s)} + \int_1^\infty \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2} \right) dx.
\end{aligned}$$

ω decays exponentially in x , so the integral converges and the last expression is well defined on \mathbb{C} except when $s = 1$ or $s = 0$. Finally, notice that the last expression is symmetric when s is replaced with $(1-s)$, so proves equation 3. \square

Remark: This also gives an analytic continuation of ζ in Theorem 1.3, excluding some minor details on the treatment of $\zeta(0)$ and $\zeta(1)$.

Corollary 1.5. *On $\Re(s) > 1$ or $\Re(s) < 0$, $\zeta(s) \neq 0$, except $\forall n \in \mathbb{N}, \zeta(-2n) = 0$.*

Proof. Using the product representation of ζ where it converges, none of $(1-p^{-s})^{-1} = 0$, so $\zeta(s) \neq 0$ on $\Re(s) > 1$. Γ has no zeros and has a simple pole at $-n$ for all $n \in \mathbb{N}$, so by equation 3 we get the zeros for $\Re(s) > 0$ are exactly at the negative even integers. \square

Definition 1.6 (Critical Strip and Critical Line). *We denote the region $0 \leq \Re(s) \leq 1$ as the **critical strip**. We denote the line $\Re(s) = 1/2$ as the **critical line**.*

Corollary 1.7. *On the critical strip, if $\zeta(s) = 0$, $\zeta(\bar{s}) = \zeta(1-s) = \zeta(1-\bar{s}) = 0$.*

Proof. This follows from equation 3, and $\zeta(\bar{s}) = \overline{\zeta(s)}$ holds where the Dirichlet series converges, thus holds everywhere. \square

We now introduce some hypotheses regarding the zeta function, these statements have not been proved, but are supported by a great amount of heuristic evidence.

Definition 1.8 (Riemann Hypothesis). *The **Riemann Hypothesis** (RH) asserts that on the critical strip,*

$$\zeta(s) = 0 \implies \Re(s) = \frac{1}{2}.$$

Definition 1.9 (Lindelöf Hypothesis). *Let $\epsilon > 0$. The **Lindelöf Hypothesis** (LH) asserts that on the critical line,*

$$\zeta\left(\frac{1}{2} + it\right) = O(t^\epsilon).$$

2 The Prime Number Theorem

Theorem 2.1 (Prime Number Theorem). *Let $\Pi(x) = \sum_{p \leq x} 1$. Then*

$$\Pi(x) = (1 + o(1)) \frac{x}{\log x}.$$

In this section we will prove the Prime Number Theorem. However, note that this result is a minor goal of this paper. The Prime Number Theorem provides an illustration of multiple ideas and techniques of manipulating the zeta function. Furthermore, this theorem serves as a starting point for studying primes in short intervals, and demonstrates why the proof of RH (or the weaker LH) is widely sought after, thus sets the stage for zero-density theorems.

Definition 2.2 (Von Mangoldt Function). *The **Von Mangoldt function** Λ is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some } k \in \mathbb{N} \\ 0, & \text{else} \end{cases}$$

The sum of the Von Mangoldt function $\sum \Lambda(n)$ is a more natural way to express a prime counting function in the language of ζ . To see why, consider the expression

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= (\log \zeta(s))' \\ &= \left[- \sum_p \log(1 - p^{-s}) \right]' \\ &= - \sum_p \frac{p^s \log p}{1 - p^{-s}} \\ &= - \sum_p \log p \sum_{k \in \mathbb{N}} p^{-ks} \\ &= - \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \end{aligned}$$

On $\Re(s) > 1$ where the sum and products are absolutely convergent. We thus motivate the following technique: Let ϕ be smooth and rapidly decaying at infinity, and $\tilde{\phi}$ be its Mellin transform. Let $N \in \mathbb{N}$ and $c \geq 2$. Then

$$\begin{aligned} \sum_{n \in \mathbb{N}} \Lambda(n) \phi\left(\frac{n}{N}\right) &= \sum_{n \in \mathbb{N}} \Lambda(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi}(s) \left(\frac{n}{N}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi}(s) \sum_{n \in \mathbb{N}} \Lambda(n) \left(\frac{n}{N}\right)^{-s} ds \\ &= \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi}(s) N^s \frac{\zeta'(s)}{\zeta(s)} ds \end{aligned} \tag{4}$$

Now take $\phi = 1$ on $[0, 1]$ and 0 outside $[0 - \epsilon, 1 + \epsilon]$. Applying Dominated Convergence theorem on equation gives for $\epsilon \downarrow 0$,

$$\sum_{n \leq N} \Lambda(n) = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} N^s \frac{\zeta'(s)}{\zeta(s)} ds$$