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Abstract

Notation

Below are the notational preferences of the author.

1. The set of natural numbers $\mathbb{N}$ does not contain 0.
2. p always denotes a prime, and by extension p_j, p_n etc.

Preliminaries

Number Theory Results

Theorem 0.1 (Möbius Inversion)**.** *The Möbius function μ is defined for $n \in \mathbb{N}$,*

$$\mu(n) = \begin{cases} 1, & \textit{if } n = 1 \\ (-1)^k, & \textit{if } n = p_1 p_2 ... p_k \textit{ for distinct } p\textit{'s} \\ 0, & \textit{otherwise} \end{cases}$$

1 Introduction to the Riemann Zeta Function

Definition 1.1 (Zeta Function)**.** *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

Remark: This expresion also converges absolutely for $\Re(s) > 1$. Since

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{p^s}{p^s - 1} = 1 + \frac{1}{p^s - 1}$$

and $\sum (p^s - 1)^{-1}$ converges absolutely by comparison to the zeta function Dirichlet series.

Sketch of proof. Write $s = \sigma + it$. For each p,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ...\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p. We thus take for $m > N$

$$\begin{aligned}
\prod_{p\le N}\left(1-\frac{1}{p^s}\right)^{-1} &= \prod_{p\le N}\left(\sum_{k=1}^{m}\frac{1}{p^{ks}}+O(2^{-m\sigma})\right)\\
&\overset{(*)}{=} \sum_{n=1}^{N}\frac{1}{n^s}+O_1(\sum_{n=N+1}^{\infty}\frac{1}{n^\sigma})+O(2^{-m\sigma})\\
&= \zeta(s)+O_1(\sum_{n=N+1}^{\infty}\frac{1}{n^\sigma})+O(2^{-m\sigma})
\end{aligned}$$

Where we apply to Fundemental Theorem of Arithmetic in (*) to show that each term n^{-s} has coefficient 1 determined by the unique prime factorization. As $m \to \infty$, $2^{-m\sigma} \to 0$. Then we take $N \to \infty$, the tail of the infinite sum converges to zero too. $\square$

Proposition 1.2 shows an inherent connection of the zeta function with primes. To further see this connection, we need to extend the zeta function.

Theorem 1.3. *ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too.*

Theorem 1.4. *Let $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then*

$$\xi(s) = \xi(1-s). \tag{3}$$

Proof. Using

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx,$$

we make the substitution $x = \pi n^2 y$ to get

$$\begin{aligned}
\Gamma(s) &= \int_0^\infty e^{-\pi n^2 y}(\pi n^2 y)^{s-1}\pi n^2 dy\\
\implies \frac{\Gamma(s)}{\pi^s n^{2s}} &= \int_0^\infty e^{-\pi n^2 y}y^{s-1}dy
\end{aligned}$$

So that by the Monotone Convergence Theorem,

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n=1}^{\infty}\frac{\Gamma(s/2)}{\pi^{s/2}n^s}\\
&= \sum_{n=1}^{\infty}\int_0^\infty e^{-\pi n^2 x}x^{s/2-1}dx\\
&= \int_0^\infty \sum_{n=1}^{\infty}\left(e^{-\pi n^2 x}\right)x^{s/2-1}dx.
\end{aligned}$$

We now let

$$\omega(x) = \sum_{n=1}^{\infty}e^{-\pi n^2 x}, \quad \theta(x) = \sum_{n=-\infty}^{\infty}e^{-\pi n^2 x} = 2\omega(x)+1,$$

and apply Poisson Summation to

$$\begin{aligned}
\theta(x) &= \sum_{n=-\infty}^{\infty}e^{-\pi n^2 x}\\
&= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\pi y^2 x}e^{-2\pi iky}dy\\
&= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\pi y^2 x}e^{-2\pi iky}dy
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi iku/\sqrt{x}} du \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} e^{-\pi k^2/x} \\
&= \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)
\end{aligned}
$$

using the substitution $y\sqrt{x} = u$. Replacing with ω,

$$
\sqrt{x}(2\omega(x)+1) = 2\omega\left(\frac{1}{x}\right) + 1 \implies \omega\left(\frac{1}{x}\right) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}.
$$

We thus write, using $y = 1/x$,

$$
\begin{aligned}
\xi(s) &= \int_0^1 \omega(x) x^{s/2-1} dx + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \omega(1/y) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\sqrt{y}\omega(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{1-s} + \frac{1}{s} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{s(1-s)} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx.
\end{aligned}
$$

ω decays exponentially in x, so the integral converges and the last expression is well defined on $\mathbb{C}$ except when $s = 1$ or $s = 0$. Finally, notice that the last expression is symmetric when s is replaced with $(1 - s)$, so proves equation 3. □

Remark: This also gives an analytic continuation of ζ in Theorem 1.3, excluding some minor details on the treatment of $\zeta(0)$ and $\zeta(1)$.

Corollary 1.5. *On $\Re(s) > 1$ or $\Re(s) < 0$, $\zeta(s) \neq 0$, except $\forall n \in \mathbb{N}, \zeta(-2n) = 0$.*

Proof. Using the product representation of ζ where it converges, none of $(1-p^{-s})^{-}1 = 0$, so $\zeta(s) \neq 0$ on $\Re(s) > 1$. Γ has no zeros and has a simple pole at $-n$ for all $n \in \mathbb{N}$, so by equation 3 we get the zeros for $\Re(s) > 0$ are exactly at the negative even integers. □

Definition 1.6 (Critical Strip and Critical Line)**.** *We denote the region $0 \leq \Re(s) \leq 1$ as the **critical strip**. We denote the line $\Re(s) = 1/2$ as the **critical line**.*

Corollary 1.7. *On the critical strip, if $\zeta(s) = 0$, $\zeta(\overline{s}) = \zeta(1-s) = \zeta(1-\overline{s}) = 0$.*

Proof. This follows from equation 3, and $\zeta(\overline{s}) = \overline{\zeta(s)}$ holds where the Dirichlet series converges, thus holds everywhere. □

We now introduce some hypotheses regarding the zeta function, these statements have not been proved, but are supported by a great amount of heuristic evidence.

Definition 1.8 (Riemann Hypothesis)**.** *The **Riemann Hypothesis** (RH) asserts that on the critical strip,*

$$
\zeta(s) = 0 \implies \Re(s) = \frac{1}{2}.
$$

Definition 1.9 (Lindelöf Hypothesis)**.** *Let $\epsilon > 0$. The **Lindelöf Hypothesis** (LH) asserts that on the critical line,*

$$
\zeta\left(\frac{1}{2} + it\right) = O(t^{\epsilon}).
$$

2 The Prime Number Theorem

Theorem 2.1 (Prime Number Theorem)**.** *Let* $\Pi(x) = \sum_{p\le n} 1$. *Then*

$$\Pi(x) = (1 + o(1))\frac{x}{\log x}.$$

In this section we will prove the Prime Number Theorem. However, note that this result is a minor goal of this paper. The Prime Number Theorem provides an illustration of multiple ideas and techniques of manipulating the zeta function. Furthermore, this theorem serves as a starting point for studying primes in short intervals, and demonstrates why the proof of RH (or the weaker LH) is widely sought after, thus sets the stage for zero-density theorems.

Definition 2.2 (Von Mangoldt Function)**.** *The* ***Von Mangoldt function*** Λ *is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p, \text{ if } n = p^k \text{ for some } k \in \mathbb{N} \\ 0, \text{ else} \end{cases}$$

The sum of the Von Mangoldt function $\sum \Lambda(n)$ is a more natural way to express a prime counting function in the language of ζ. To see why, consider the expression

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= (\log \zeta(s))' \\ &= \left[-\sum_p \log\left(1 - p^{-s}\right)\right]' \\ &= -\sum_p \frac{p^s \log p}{1 - p^{-s}} \\ &= -\sum_p \log p \sum_{k\in\mathbb{N}} p^{-ks} \\ &= -\sum_{n\in\mathbb{N}} \frac{\Lambda(n)}{n^s} \end{aligned}$$

on $\Re(s) > 1$ where the sum and products are absolutely convergent. We thus motivate the following technique: Let ϕ be smooth and rapidly decaying at infinity, and $\tilde{\phi}$ be its Mellin transform. Let $N \in \mathbb{N}$ and $c \ge 2$. Then

$$\begin{aligned} \sum_{n\in\mathbb{N}} \Lambda(n)\phi\left(\frac{n}{N}\right) &= \sum_{n\in\mathbb{N}} \Lambda(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi}(s) \left(\frac{n}{N}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi}(s) \sum_{n\in\mathbb{N}} \Lambda(n) \left(\frac{n}{N}\right)^{-s} ds \\ &= \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi}(s) N^s \frac{\zeta'(s)}{\zeta(s)} ds \end{aligned} \tag{4}$$

Now take a bump function $\phi = 1$ on $[0, 1]$ and 0 outside $[0-\epsilon, 1+\epsilon]$. Applying Dominated Convergence theorem on equation 4 gives for $\epsilon \downarrow 0$,

$$\sum_{n\le N} \Lambda(n) = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} N^s \frac{\zeta'(s)}{\zeta(s)} ds \tag{5}$$

Changing the line of integration from c to $-\infty$ **need to give bounds on log' zeta**, we get residue contributions from a pole at $s = 1$, $s = 0$, as well as all ρ such that $\zeta(\rho) = 0$ on the critical strip, and all the trivial zeros. This gives

$$\begin{aligned} \sum_{n\le N} \Lambda(n) &= N - \sum_\rho \frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{k\in\mathbb{N}} \frac{N^{-2k}}{2k} \\ &= N - \sum_\rho \frac{N^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log\left(1 - N^{-2}\right). \end{aligned} \tag{6}$$