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October 28, 2024

Abstract

Notation

Below are the notational preferences of the author.

1. p always denotes a prime, and by extension p_j, p_n etc.

1 Introduction to the Riemann Zeta Function

Definition 1.1 (Zeta Function)**.** *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

Remark: This expresion also converges absolutely for $\Re(s) > 1$.

Sketch of proof. Write $s = \sigma + it$. For each p,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ...\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p. We thus take

$$\begin{aligned}
\prod_{p \le N} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \le N} \left(\sum_{k=1}^{m} \frac{1}{p^{ks}} + O(\epsilon)\right) \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + O(\epsilon) \\
&= \zeta(s) + O(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}) + O(\epsilon)
\end{aligned}$$

As $m \to \infty$, $epsilon \to 0$. Then we take $N \to \infty$, the tail of the infinite sum converges to zero too. □

Proposition 1.2 shows an inherent connection of the zeta function with primes. To further see this connection, we need to extend the zeta function.

Theorem 1.3. *ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too.*

Riemann's Extension of ζ (Adapted). Using

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx,$$

We make the substitution $x = ny$ to get

$$\begin{aligned} \Gamma(s) &= \int_0^\infty e^{-ny} (ny)^{s-1} n dy \\ \implies \frac{\Gamma(s)}{n^s} &= \int_0^\infty e^{-ny} y^{s-1} dy \end{aligned}$$

So that by the Monotone Convergence Theorem,

$$\begin{aligned} \Gamma(s)\zeta(s) &= \sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} \\ &= \sum_{n=1}^\infty \int_0^\infty e^{-ny} y^{s-1} dy \\ &= \int_0^\infty \sum_{n=1}^\infty \left(e^{-ny}\right) y^{s-1} dy \\ &= \int_0^\infty \left(\frac{e^{-y}}{1 - e^{-y}}\right) y^{s-1} dy \\ &= \int_0^\infty \frac{y^{s-1}}{e^y - 1} dy \end{aligned}$$

□