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Abstract

Notation

Below are the notational preferences of the author.

1. The set of natural numbers $\mathbb{N}$ does not contain 0.
2. p always denotes a prime, and by extension p_j, p_n etc.

1 Introduction to the Riemann Zeta Function

Definition 1.1 (Zeta Function)**.** *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

Remark: This expresion also converges absolutely for $\Re(s) > 1$.

Sketch of proof. Write $s = \sigma + it$. For each p,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ...\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p. We thus take for $m > N$

$$\begin{aligned}
\prod_{p \le N} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \le N} \left(\sum_{k=1}^{m} \frac{1}{p^{ks}} + O(2^{-m\sigma})\right) \\
&\overset{(*)}{=} \sum_{n=1}^{N} \frac{1}{n^s} + O_1\left(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}\right) + O(2^{-m\sigma}) \\
&= \zeta(s) + O_1\left(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}\right) + O(2^{-m\sigma})
\end{aligned}$$

Where we apply to Fundemental Theorem of Number Theory in (*) to show that each term n^{-s} has coefficient 1 determined by the unique prime factorization. As $m \to \infty$, $2^{-m\sigma} \to 0$. Then we take $N \to \infty$, the tail of the infinite sum converges to zero too. □

Proposition 1.2 shows an inherent connection of the zeta function with primes. To further see this connection, we need to extend the zeta function.

Theorem 1.3. *ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too. Moreover, $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfies*

$$\xi(s) = \xi(1-s). \tag{3}$$

Proof. Using

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx,$$

We make the substitution $x = \pi n^2 y$ to get

$$\begin{aligned}
\Gamma(s) &= \int_0^\infty e^{-\pi n^2 y}(\pi n^2 y)^{s-1}\pi n^2 dy \\
\implies \frac{\Gamma(s)}{\pi^s n^{2s}} &= \int_0^\infty e^{-\pi n^2 y}y^{s-1}dy
\end{aligned}$$

So that by the Monotone Convergence Theorem,

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n=1}^\infty \frac{\Gamma(s/2)}{\pi^{s/2}n^s} \\
&= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x}x^{s/2-1}dx \\
&= \int_0^\infty \sum_{n=1}^\infty \left(e^{-\pi n^2 x}\right)x^{s/2-1}dx.
\end{aligned}$$

We now let

$$\omega(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}, \quad \theta(x) = \sum_{n=-\infty}^\infty e^{-\pi n^2 x} = 2\omega(x) + 1.$$

We apply Poisson Summation to

$$\begin{aligned}
\theta(x) &= \sum_{n=-\infty}^\infty e^{-\pi n^2 x} \\
&= \sum_{k=-\infty}^\infty \int_{-\infty}^\infty e^{-\pi y^2 x}e^{-2\pi iky}dy \\
&= \sum_{k=-\infty}^\infty \int_{-\infty}^\infty e^{-\pi y^2 x}e^{-2\pi iky}dy \\
&= \sum_{k=-\infty}^\infty \frac{1}{\sqrt{x}}\int_{-\infty}^\infty e^{-\pi u^2}e^{-2\pi iku/\sqrt{x}}du \\
&= \sum_{k=-\infty}^\infty \frac{1}{\sqrt{x}}e^{-\pi k^2/x} \\
&= \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right)
\end{aligned}$$

using the substitution $y\sqrt{x} = u$. Replacing with ω,

$$\sqrt{x}(2\omega(x)+1) = 2\omega\left(\frac{1}{x}\right) + 1 \implies \omega\left(\frac{1}{x}\right) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}$$

We thus write, using $y = 1/x$,

$$\begin{aligned}
\xi(s) &= \int_0^1 \omega(x)x^{s/2-1}dx + \int_1^\infty \omega(x)x^{s/2-1}dx \\
&= \int_1^\infty \omega(1/y)y^{-s/2-1}dy + \int_1^\infty \omega(x)x^{s/2-1}dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty \left(\sqrt{y}\omega(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^\infty \omega(x) x^{s/2-1} dx \\
&= \int_1^\infty \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^\infty \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= -\frac{1}{1-s} + \frac{1}{s} + \int_1^\infty \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^\infty \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{s(1-s)} + \int_1^\infty \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^\infty \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx
\end{aligned}
$$

ω decays exponentially in x, so the integral converges and the last is well defined on $\mathbb{C}$ except when $s = 1$ or $s = 0$. Finally, notice that the last expression is symmetric when s is replaced with $(1 - s)$, so shows equation 3. $\square$

Corollary 1.4. $\forall n \in \mathbb{N}, \zeta(-2n) = 0$.

Proof. Γ has a simple pole at $-n$. $\square$

Corollary 1.5.