Yung Chi Li
Advised by Maksym Radziwiłł

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Abstract

Notation

Below are the notational preferences of the author.

1. The set of natural numbers $\mathbb{N}$ does not contain 0.
2. p always denotes a prime, and by extension p_j, p_n etc.

Preliminaries

Number Theory Results

Theorem 0.1 (Möbius Inversion)**.** *The Möbius function μ is defined for $n \in \mathbb{N}$,*

$$\mu(n) = \begin{cases} 1, \ \textit{if } n = 1 \\ (-1)^k, \ \textit{if } n = p_1 p_2 ... p_k \textit{ for distinct } p\textit{'s} \\ 0, \textit{otherwise} \end{cases}$$

1 Introduction to the Riemann Zeta Function

In this section, we give a quick introduction to the zeta function, including its product representation and analytic continuation, as well as conjectures regarding the zeta function.

Definition 1.1 (Zeta Function)**.** *Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

The zeta function converges absolutely on $\Re(s) > 1$ by comparing to the integral $\int x^{-\Re(s)} dx$. The properties of the zeta function as they relate to the distribution primes. In particular, the Dirichlet series can be represented as a product of primes.

Proposition 1.2. *On $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \in \mathbb{N}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

Remark: This expresion also converges absolutely for $\Re(s) > 1$. Since

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{p^s}{p^s - 1} = 1 + \frac{1}{p^s - 1}$$

and $\sum (p^s - 1)^{-1}$ converges absolutely by comparison to the zeta function Dirichlet series.

Sketch of proof. Write $s = \sigma + it$. For each p,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ...\right)$$

converges absolutely for $\Re(s) > 1$ and uniformly across all p. We thus take for $m > N$

$$\begin{aligned}\prod_{p \leq N} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \leq N} \left(\sum_{k=1}^{m} \frac{1}{p^{ks}} + O(2^{-m\sigma})\right) \\ &\overset{(*)}{=} \sum_{n=1}^{N} \frac{1}{n^s} + O_1(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}) + O(2^{-m\sigma}) \\ &= \zeta(s) + O_1(\sum_{n=N+1}^{\infty} \frac{1}{n^\sigma}) + O(2^{-m\sigma})\end{aligned}$$

Where we apply to Fundemental Theorem of Arithmetic in (*) to show that each term n^{-s} has coefficient 1 determined by the unique prime factorization. As $m \to \infty$, $2^{-m\sigma} \to 0$. Then we take $N \to \infty$, the tail of the infinite sum converges to zero too. □

Proposition 1.3. *ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. By abuse of notation, we identify the extension of the zeta function with ζ too.*

The continuation of the zeta function is best described by its functional equation: ζ has a line of symmetry across $\Re(s) = 1/2$.

Proposition 1.4. *Let $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then*

$$\xi(s) = \xi(1 - s). \tag{3}$$

Proof. Using

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx,$$

we make the substitution $x = \pi n^2 y$ to get

$$\begin{aligned}\Gamma(s) &= \int_0^\infty e^{-\pi n^2 y} (\pi n^2 y)^{s-1} \pi n^2 dy \\ \implies \frac{\Gamma(s)}{\pi^s n^{2s}} &= \int_0^\infty e^{-\pi n^2 y} y^{s-1} dy\end{aligned}$$

So that by the Monotone Convergence Theorem,

$$\begin{aligned}\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n=1}^{\infty} \frac{\Gamma(s/2)}{\pi^{s/2} n^s} \\ &= \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 x} x^{s/2-1} dx \\ &= \int_0^\infty \sum_{n=1}^{\infty} \left(e^{-\pi n^2 x}\right) x^{s/2-1} dx.\end{aligned}$$

We now let

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}, \quad \theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 2\omega(x) + 1,$$

and apply Poisson Summation to

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

$$\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi iky} dy \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi iky} dy \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi iku/\sqrt{x}} du \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{x}} e^{-\pi k^2/x} \\
&= \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)
\end{aligned}$$

using the substitution $y\sqrt{x} = u$. Replacing with ω,

$$\sqrt{x}(2\omega(x) + 1) = 2\omega\left(\frac{1}{x}\right) + 1 \implies \omega\left(\frac{1}{x}\right) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}.$$

We thus write, using $y = 1/x$,

$$\begin{aligned}
\xi(s) &= \int_0^1 \omega(x) x^{s/2-1} dx + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \omega(1/y) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\sqrt{y}\omega(y) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) x^{s/2-1} dx \\
&= \int_1^{\infty} \left(\frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-s/2-1} dy + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{1-s} + \frac{1}{s} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx \\
&= \frac{1}{s(1-s)} + \int_1^{\infty} \omega(x) \left(x^{s/2-1} + x^{-s/2-1/2}\right) dx.
\end{aligned}$$

ω decays exponentially in x, so the integral converges and the last expression is well defined on $\mathbb{C}$ except when $s = 1$ or $s = 0$. Finally, notice that the last expression is symmetric when s is replaced with $(1 - s)$, so proves equation 3. □

Remark: This also gives an analytic continuation of ζ in Theorem 1.3, excluding some minor details on the treatment of $\zeta(0)$ and $\zeta(1)$.

From the functional equation, we get 'trivial' zeros of the zeta function from the poles of Γ.

Corollary 1.5. *On $\Re(s) > 1$ or $\Re(s) < 0$, $\zeta(s) \neq 0$, except $\forall n \in \mathbb{N}, \zeta(-2n) = 0$.*

Proof. Using the product representation of ζ where it converges, none of $(1-p^{-s})^{-}1 = 0$, so $\zeta(s) \neq 0$ on $\Re(s) > 1$. Γ has no zeros and has a simple pole at $-n$ for all $n \in \mathbb{N}$, so by equation 3 we get the zeros for $\Re(s) > 0$ are exactly at the negative even integers. □

Definition 1.6 (Critical Strip and Critical Line)**.** *We denote the region $0 \leq \Re(s) \leq 1$ as the* ***critical strip****. We denote the line $\Re(s) = 1/2$ as the* ***critical line****.*

Corollary 1.7. *On the critical strip, if $\zeta(s) = 0$, $\zeta(\overline{s}) = \zeta(1-s) = \zeta(1-\overline{s}) = 0$.*

Proof. This follows from equation 3, and $\zeta(\overline{s}) = \overline{\zeta(s)}$ holds where the Dirichlet series converges, thus holds everywhere. □

The number of zeros in the critical strip can be calculated using the argument principle applied to the function ξ over the box with corners $-1 + iT$,$-1 - iT$,$2 - iT$,$2 + iT$. Applying the functional equation, we get the following result.

Theorem 1.8 (Number of zeros of ζ)**.** *The number of zeros up to height T*

$$\#\{\sigma + it \mid \zeta(\sigma + it) = 0, 0 \le \sigma \le 1, |t| \le T\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Finally, we introduce some hypotheses regarding the zeta function, these statements have not been proved, but are supported by a great amount of heuristic evidence.

Definition 1.9 (Riemann Hypothesis)**.** *The* ***Riemann Hypothesis*** *(RH) asserts that on the critical strip,*

$$\zeta(s) = 0 \implies \Re(s) = \frac{1}{2}.$$

Definition 1.10 (Lindelöf Hypothesis)**.** *Let $\epsilon > 0$. The* ***Lindelöf Hypothesis*** *(LH) asserts that on the critical line,*

$$\zeta\left(\frac{1}{2} + it\right) = O(t^\epsilon).$$

2 The Prime Number Theorem

Theorem 2.1 (Prime Number Theorem)**.** *Let $\Pi(N) = \sum_{p \le N} 1$. Then*

$$\Pi(N) = (1 + o(1))\frac{N}{\log N}.$$

In this section we will prove the Prime Number Theorem. This result is a minor goal of this paper. The Prime Number theorem serves as a starting point for studying primes in short intervals, and sets the stage for zero-density theorems.

Definition 2.2 (Von Mangoldt Function)**.** *The* ***Von Mangoldt function*** *Λ is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p, \text{ if } n = p^k \text{ for some } k \in \mathbb{N} \\ 0, \text{ else} \end{cases}$$

The sum of the Von Mangoldt function $\sum \Lambda(n)$ is a more natural way to express a prime counting function in the language of ζ. To see why, consider the expression

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= (\log \zeta(s))' \\ &= \left[-\sum_p \log\left(1 - p^{-s}\right) \right]' \\ &= -\sum_p \frac{p^s \log p}{1 - p^{-s}} \\ &= -\sum_p \log p \sum_{k \in \mathbb{N}} p^{-ks} \\ &= -\sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s} \end{aligned}$$

on $\Re(s) > 1$ where the sum and products are absolutely convergent.

Proposition 2.3. *$\sum_{n \le N} \Lambda(n) = (1 + o(1))N$ implies the Prime Number Theorem.*

Proof. On one hand, we have

$$\begin{aligned} \sum_{n \le N} \Lambda(n) &\le \sum_{p \le N} \Lambda(N) \\ &\le \Pi(x) \log x. \end{aligned}$$

And for $\epsilon > 0$,

$$\begin{aligned}\sum_{n\leq N}\Lambda(n) &\geq \sum_{N^{1-\epsilon}\leq n\leq N}\Lambda(n)\\ &\geq \sum_{N^{1-\epsilon}\leq p\leq N}(1-\epsilon)\log(N)\\ &= \Pi(N)\log(N)+O(N^{1-\epsilon}logN).\end{aligned}$$

□

Moreover, the sum of the Von Mangoldt function can be related to the zeros of the zeta function. Let ϕ be smooth and rapidly decaying at infinity, and $\tilde{\phi}$ be its Mellin transform. Let $N\in\mathbb{N}$ and $c\geq 2$. Then

$$\begin{aligned}\sum_{n\in\mathbb{N}}\Lambda(n)\phi\left(\frac{n}{N}\right) &= \sum_{n\in\mathbb{N}}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)\sum_{n\in\mathbb{N}}\Lambda(n)\left(\frac{n}{N}\right)^{-s}ds\\ &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(s)N^s\frac{\zeta'(s)}{\zeta(s)}ds\end{aligned}\tag{4}$$

Morally, we can take a bump function $\phi = 1$ on $[0,1]$ and supported in $[0-\epsilon, 1+\epsilon]$.

$$\sum_{n\leq N}\Lambda(n) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}N^s\frac{\zeta'(s)}{\zeta(s)}ds\tag{5}$$

Temporarily ignoring issues of convergence, we change the line of integration from c to $-\infty$, we get residue contributions from a pole at $s=1$, $s=0$, as well as all ρ such that $\zeta(\rho)=0$ on the critical strip, and all the trivial zeros. This gives

$$\begin{aligned}\sum_{n\leq N}\Lambda(n) &= N-\sum_{\rho}\frac{N^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}+\sum_{k\in\mathbb{N}}\frac{N^{-2k}}{2k}\\ &= N-\sum_{\rho}\frac{N^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2}\log\left(1-N^{-2}\right).\end{aligned}\tag{6}$$

The sum over zeros ρ is not absolutely convergent, and is ordered in increasing $|\Im(\rho)|$. This formula is in fact true.

Theorem 2.4 (Riemann-von Mangoldt explicit formula)**.** *Let $N > 1$ be not a prime power. Then*

$$\sum_{n\leq N}\Lambda(n) = N-\lim_{T\to\infty}\sum_{|\Im(\rho)|\leq T}\frac{N^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2}\log\left(1-N^{-2}\right).\tag{7}$$

In practice, we truncate the integral in up to height T, *is the proof needed?* to obtained a truncated version of the explicit formula.

Theorem 2.5. *Let $N > 1$. Then*

$$\sum_{n\leq N}\Lambda(n) = N-\sum_{|\Im(\rho)|\leq T}\frac{N^\rho}{\rho}+O\left(\frac{N\log N\log T}{T}+\log T\right).\tag{8}$$

Assuming the Riemann Hypothesis, we consider the sum over the non trivial zeros

$$\left|\sum_{|\Im(\rho)\leq T|}\frac{N^\rho}{\rho}\right|\leq N^{1/2}\sum_{\rho}\left|\frac{1}{\rho}\right|.$$

We know there are $\sim \log T$ zeros of height $[T, T+1)$, thus the integral $\sum |\rho^{-1}|$ behaves as

$$\sum_{n \leq T} \frac{\log n}{n} = O(\log^2 T).$$

Taking $N = T$ in the truncated explicit formula, we obtain

$$\sum_{n \leq N} \Lambda(n) = N + O(N^{1/2} \log^2 N). \tag{9}$$

Which implies the prime number theorem.

Remark: The prime number theorem with the error term in can be shown to be equivalent to the Riemann Hypothesis.

The major error term comes from N^ρ in the sum, so bounding the $\Re(\rho)$ becomes the most important part in reducing the error term in the prime number theorem.

Theorem 2.6. *Let $t \in \mathbb{R}$. Then $\zeta(1+it) \neq 0$.*

Let us assume Theorem 2.6 and use it to derive the prime number theorem.

3 Primes in Short Intervals

We would like to answer the following question about primes in short intervals. Let $y = y(x)$. What is the smallest asymptotic behavior of y such that

$$\sum_{x \leq n \leq x+y} \Lambda(n) \sim y \tag{10}$$

for large enough x? That is, what is the shortest interval such that we have the behavior of the Prime Number Theorem? If 10 holds for some y, we say the Prime Number Theorem holds for intervals of y. We remark that this question can be rephrased into finding primes in short intervals,

Proposition 3.1. *Assume the RH. Then the Prime Number Theorem holds in intervals of $x^{1/2+\epsilon}$.*

Proof. Assume the RH, then

$$\sum_{x \leq n \leq x+y} \Lambda(n) = y + O(x^{1/2} \log^2 x) = x^{1/2+\epsilon} + o(x^{1/2+\epsilon}),$$

so that the sum is non-zero for large enough x. The argument is similar without assuming the RH. □

Recalling that the error term is related to the real part of the zeros of the Zeta function, we motivate the following definition of zero-density:

Definition 3.2. *Let $N(\sigma, T)$ denote the number of zeros of the zeta function with real part greater than σ and imaginary part between $-T$ and T. That is,*

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \mid \beta > \sigma, |\gamma| \leq T\}.$$

Remark: The ideal scenario is that $N(\sigma, T) = 0$ for all $\sigma \geq 1/2$.

Theorem 3.3 (Chudakov)**.** *There exists a constant A such that $\zeta(\sigma + iT) \neq 0$ in the region*

$$\sigma > 1 - A \frac{\log \log T}{\log T}.$$

add reference

Theorem 3.4 (Hoheisel)**.** *Let A be defined as in the previous theorem. Suppose that $N(\sigma, T) \ll T^{a(1-\sigma)} \log^b T$ uniformly in $1/2 < \sigma < 1$ and in T. Then for all*

$$\theta > 1 - \frac{1}{a + b/A},$$

the Prime Number Theorem holds in intervals of $y = x^\theta$.

add reference Theorem 3.4 gives the classic estimate for