EE 495 Game Theory and Networked Systems

0 Syllabus

All logistics are on canvas.

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There are no required texts for the class. There are some references (4) that are suggested on Canvas.

Prereqs

1. No prior knowledge on game theory is required.
2. Mathematical background is required. Linear algebra, probability, optimization, mathematical maturity.

Grades

- Midterm: in class, 35%
- Final project: 35%
- Problemsets: 30%

Theorem 0.1

Randy is a chill professor.

1 Lecture 1: Introduction to Game Theory

Definition 1.1 (Game Theory)

Game theory is the study of interactions of *multiple strategic agents*.

Features of game theory:

- More than one decision makers.
- Each agent makes decisions to maximize self-interest.
- These **agents** are players of the 'game', and can be people, firms, countries (in political science), AI-agents etc.

Example 1.2

The following are examples of 'games':

- 2 people playing chess. Their incentives do not align because each player wants to checkmate the other.
- 2 firms competing in a market. They are selling the similar items and are trying to price their items to get a larger market share.
- 4 countries competing to maximize GDP.

The other component of this course is network systems.

Example 1.3

The following are examples of network systems:

- Communication network.
- Electricity network.
- Transportation network.

We will use these as examples to illustrate the concepts in game theory. However, the same theory can extend into the other games. We want to model and analyze games. People are complicated to model, and our models are simplifications of reality. We need to understand what assumptions are made for each models to apply analysis.

Basic Game Model

Definition 1.4 (Basic Game Model)

A **strategic form game** G consists of the following elements:

1. The set of agents/players R, usually enumerated $R = \{1, 2, 3, \ldots, n\}$.
2. For every $r \in R$, the action set of player r S_r. If $|\bigsqcup_r S_r| < \infty$, we call the game a **finite game**.
3. For every $r \in R$, a payoff function $\pi_r : \otimes_{r'} S_{r'} \to \mathbb{R}$. Each agent r wants to maximize π_r.

That means, there is only one round of this game, and everyone makes the same decision all at once.

Remark. *The action set can also be called the strategic set.*

Notation. *The ordered set of everyone's actions except r is*

$$\overline{s_{-r}} = (s_1, s_2, \ldots, s_{r-1}, s_{r+1}, \ldots, s_n).$$

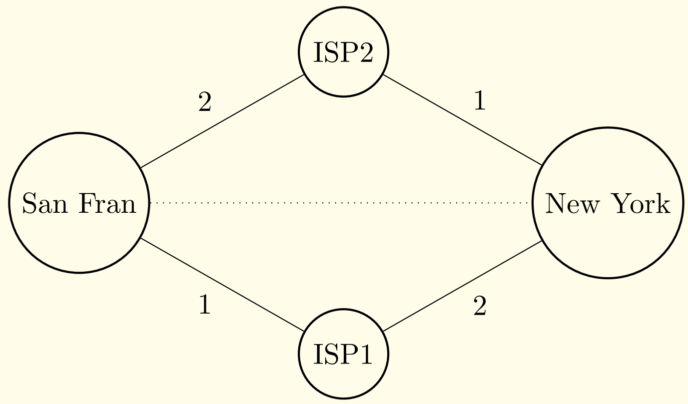
Therefore player r is actually maximizing

$$\pi_r(s_r, \overline{s_{-r}}).$$

The second part of this function is outside of r's control.

Example 1.5

Consider two Internet Service Providers (ISPs)



They have peered i.e. no charge to send traffic to each other. There is 1MB of traffic to ISP 1 customers in NY. There is 1mB of traffic to ISP 2 in SF. Each ISP will incur the cost of usage of the two (respective) edges connected to it, per mB of traffic. Here,

$$R = \{1, 2\}$$
$$S_r = \{\text{near}, \text{far}\},$$

that is, each ISP can decide whether to send the traffic directly or through the other ISP.

We can represent this as a matrix

ISP 1\ISP2	near	far
near	(-4,-4)	(-1,-5)
far	(-5,-1)	(-2,-2)

Definition 1.6 (Dominant action)

An Action is s_r^* is **weakly dominant** if

$$\pi_r(s_r^*, \overline{s_{-r}}) \geq \pi_r(s_r, \overline{s_{-r}})$$

for all $s_r \neq s_r^*$, and any $\overline{s_{-r}}$. If inequality is strict, then the action is **strictly dominant**.

Corollary 1.7: If the game has a strictly dominant action for each player, this will give a unique dominant strategy equilibrium.

Example 1.8

Consider the game with the reward matrix:

1\2	L	M	R
U	(1,0)	(2,-1)	(1,2)
D	(0,3)	(1,4)	(0,1)

There is no dominant action for player 2, but there is a dominant action (U) for player 1. Therefore, player 2 can rationally assume that player 1 does not play D. Under this assumption, player 2 has a dominant strategy of (R).

Example 1.9

Alice and Bob want to get lunch together Consider the game with the reward matrix:

1\2	Tech	Kellog
Tech	(2,3)	(1,1)
Kellog	(1,1)	(3,2)

There is no dominant action for each player. This is known as a coordination game, it is better to follow what the other player chooses.

Definition 1.10 (Nash Equilibrium)

A strategy profile $(s_1^*, \ldots, s_n^*)$ is a **Nash equilibrium** if

$$\pi_r(s_r^*, s_{-r}^*) \geq \pi_r(s_r, s_{-r}^*)$$

for each agent r, $s_r \in S_r$.
In other words, no player benefits from unilateral deviations.

In the case of Example 1.9, the two Nash equilibria are when both players pick the same place to eat. However, there are no dominant strategies! Even if we made the matrix

1\2	Tech	Kellog
Tech	(5,5)	(1,1)
Kellog	(1,1)	(2,2)

there will still be two Nash equilibria, even though the (5,5) outcome is much better than the other (2,2) - some are better than others.

Proposition 1.11

A dominant strategy equilibrium is a Nash equilibrium.

Proof. By definition.

Example 1.12

Consider the reward matrix

Attacker\User	A	B
A	(1,0)	(0,1)
B	(0,1)	(1,0)

The attacker always wants to choose the channel with the user, but the user wants a different channel. In this case, there is no Nash equilibrium (either the attacker can move to the user channel or the user move away from the attacker channel).

Next Time

Mixed strategies, thm: finite game with randomized strategies always have nash equilibria.

2 Lecture 2: Games with Continuous Strategies

Recap of last lecture

1. Definition of Game
2. Dominant Strategies
3. Nash Equilibrium

Example 2.1 (First Price Auction)

A single object is to be assigned to one player from $1-n$ in exchange for payment. Each player values the object at $v_1 > v_2 > \ldots > v_n > 0$ respectively. The players submit bids $b_i \geq 0$. The player with the highest bid gets the item and pays its bid. If there is a tie, the object goes to the player with the lowest index.

In this case, the payoff function would be $v_i - b_i$ for the winning player i, and 0 for everyone else.

$$\pi_r(b_r, \overline{b_{-r}}) = \begin{cases} v_r - b_r, & \text{if } b_r > b_s \forall s < r,\ b_r \geq b_s \forall s \geq r \\ 0, & \text{otherwise.} \end{cases}$$

There is no dominant strategy. Each bidder will always want to bid slightly higher than everyone else. There is a Nash Equilibrium. Namely, $b_r = v_r$ for all $r \geq 2$ and $b_1 = v_2$.

Remark. *Nash equilibrium here is not unique. But the outcome is always the same. The item always goes to player* 1.

Example 2.2 (Cournot Competition)

There are two firms. They produce the same good (indistiguishable). Each firm chooses a quantity q_i to produce at a cost of c_i per good. They sell it at a market price $p(q_1 + q_2) = 1 - q_1 - q_2$. Find the Nash equilibrium of the game.

In this case, the payoff function is

$$\pi_i(q_i, q_{-i}) = (1 - q_1 - q_2)q_i - c_i q_i.$$

To solve for the equilibrium, we motivate the following definition.

Definition 2.3 (Best Response Correspondence)

For each player i, let

$$B_i(q_{-i}) \stackrel{\text{def}}{=} \text{argmax}_{q_i} \pi_i(q_i, q_{-i})$$

be the **best response correspondence** of player i. This function can be multi-valued.

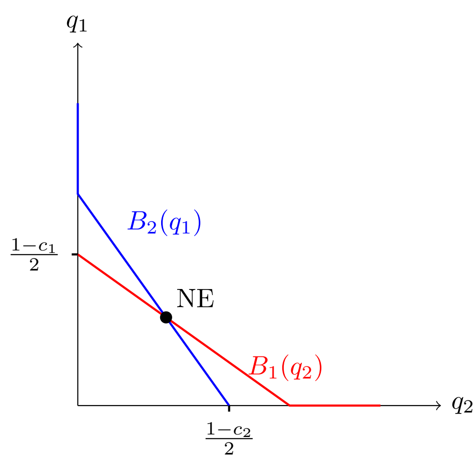
Proposition 2.4

At a Nash Equilibrium profile $\overline{q^*}$, we would have $q_i^* \in B_i(q*_{-i})$.

Solving for the previous example, we would get, by completing the square

$$B_1(q_2) = \max\left(\frac{1 - q_2 - c_1}{2}, 0\right)$$
$$B_2(q_1) = \max\left(\frac{1 - q_1 - c_2}{2}, 0\right)$$

We can thus plot B_1 and B_2 on the axes q_1, q_2. The intersection of the two graphs will be the nash equilibrium, as they are playing the best response to the other player.



Solving the two equations give

$$q_1 = \frac{1 - 2c_1 + c_2}{3},$$
$$q_2 = \frac{1 - 2c_2 + c_1}{3},$$

provided that c_1, c_2 are small enough that these do not into the negatives.

Sanity check

The solution is symmetric. If the cost for both firms are the same, they will each produce $(1-c_1)/3$. If c_1 increases, q_1 decreases and q_2 increases.

Notation. *Given an action profile $\overline{s}$, we set the best response correspondence for all players to be the vector-valued function*

$$B(\overline{s}) = \begin{bmatrix} B_1(\overline{s}) \\ \vdots \\ B_n(\overline{s}) \end{bmatrix}$$

Thus a Nash equilibrium profile $\overline{s^*}$ satisfies

$$B(\overline{s^*}) = \overline{s^*}.$$

Therefore, we are interested in the fixed points of $B : \prod_r^{S_r} \to \prod_r^{S_r}$. We introduce two main fixed point results.

Theorem 2.5 (Brower)

Let V be a compact convex set. Then any continuous function $f : V \to V$ has a fixed point.

Theorem 2.6 (Kakutani)

TBD

Definition 2.7 (Concave Game)

A game is said to be **concave** if for each player r:

1. S_r is a non-empty compact convex subset of $\mathbb{R}^n$.
2. The payoff $\pi_r(s_r, \overline{s_{-r}})$ is continuous in s_r for each $\overline{s_{-r}}$.
3. The payoff $\pi_r(s_r, \overline{s_{-r}})$ is concave in s_r for each $\overline{s_{-r}}$.

Theorem 2.8

Every concave game has a Nash equilibrium.

Idea of proof. Assume that $B(\overline{s})$ is single-valued. We want to show that B is continuous on the convex set $\prod S_r$. So we cheat and apply the following lemma (Maximum thoerem) to get our result. 🌸

Lemma 2.9 (Maximum Theorem)

If $f(\overline{x}, \overline{\theta})$ is continuous in $\overline{x}$ and $\overline{\theta}$, then

$$x^* = \text{argmax}_{\overline{x} \in A} f(\overline{x}, \overline{\theta})$$

is continuous in θ.

We can apply this theorem to the Cournot game. First, we notice that it is not profitable to produce $q_i > 1$ goods for each player, so we can restrict the strategy space to the convex set $[0, 1]$. Next, the function is quadratic and is continuous and concave. Thus, there is a Nash equilibrium.

Remark. *You can generalize the existence of Nash equilibrium to 'quasi-concave games'. You can also put futher restrictions to the guarantee the uniqueness of the Nash equilibrium (Rosen).*

Justification

Nash equilibrium assumes that players are rational interspective agents. That is, there is a **common knowledge of rationality**, which means that each player is not only rational and knows that each other player is rational, and that each other player is aware that each other player is aware is each other player is rational, inductively ad infinitum.

There is also no existence of binding arguments.

Focal points. A Nash equilibrium is a self-enforcing outcome. If the same game is played multiple times, then through a best-response dynamic, (supposing it converges), it converes to a Nash equilibrium.

Taken from Randy's notes:

We know that action of an agent in a Nash equilibria is a rational response to the equilibrium profile of the other users, but how do we coordinate Nash equilibrium with other players if we have just met. In other words, How do we know that other agents will play this profile and why they chose this profile to play? There are couple of justifications that can be used based on the problem we are solving. Here we talk about some possible ones:

- **Nash equilibrium as a self-enforcing outcome.** This first justification works for so-called one-shot games that players just meet and play a game just once. In this setting, what will lead players to a NE? One possible approach of justification is a non-binding agreement between players. If this non-binding agreement is a NE, none of these players have any incentive to break the rule and deviate from it. In this sense, each player enforce himself to follow the agreement. Note that this justification assumes rationality of players.

- **Nash equilibrium as the outcome of long-run learning.** One other idea of justification NE comes as a re-sult of learning process of players when they have the chance to play one game many times. We assume that players can experiment with different actions to seek possible actions to improve their payoff functions. Such a process might not reach a NE necessarily, but if it reaches a steady state where players can't improve their actions given what others are playing, then this steady-state is necessarily a NE. In this justification, we can assume that each player does not have full information about payoff function and rationality of other players and may learn enough about them by playing the game repeatedly and reaching a NE. One possible downside of this justification is that players might deviate from their learning in order to fool other players.

- **Nash equilibrium as a result of lots of thinking.** Nash equilibrium can also be justified when players put a lot of effort to compute how other people might play the game before actually starting the game.

Note that although these justifications might work in some certain settings, justification becomes moresophisticated when there are more than one NE, where equilibrium selection is needed.

3 Lecture 3: Finite Games

Recap of last lecture

1. Games with continuous strategy spaces.
2. Best responses
3. Existence of Nash Equilibria for concave games.

Recall example 1.12, we do not have a Nash equilibrium for this. However we can introduce mixed strategies.

Definition 3.1 (Mixed Strategy)

A **mixed strategy** is a probability distribution over S_i. Let σ_i denote a mixed strategy for player i. $\sigma_i(s_i) \stackrel{\text{def}}{=}$ probability of playing s_i. If there is $\sigma_i(s_i) = 1$, this is called a pure strategy.

Given a set of mixed strategies in a game, let

$$\bar{\sigma}(\bar{s}) = \prod_{r=1}^{n} \sigma_r(s_r).$$

That is, the decisions of each player are independent. Thus, the payoff of player i (by abuse of notation)

$$\pi_i(\bar{\sigma}) = \sum_{\bar{s} \in \prod S_i} \bar{\sigma}(\bar{s}) \pi_i(\bar{s})$$

Example 3.2

Return to the game in example 1.12 with reward matrix

Attacker\User	A	B
A	(1,0)	(0,1)
B	(0,1)	(1,0)

Suppose the attacker (player 1) chooses A or B with 0.5 probability each, while the defender (player 2) chooses A at 0.25 probability and B with 0.75 probability. Calculate the expected payoff of the game.

We have

$$\pi_1(\bar{\sigma}) = 0.5 \cdot 0.25 \cdot 1 + 0.5 \cdot 0.75 \cdot 1 + \text{terms involving 0's} = 0.5.$$

Let Σ_i denote the space of all mixed strategies for player i. For a finite game, we can plot this as a k-dimensional simplex $x_1 + \cdots + x_k = 1$. We then view games with mixed strategies as a game where strategy set is now Σ_i with payoff $\pi_i(\bar{\sigma})$. We can thus define Nash Equilibrium for a finite game with mixed strategies to be the Nash Equilibrium of this game.

Definition 3.3 (Nash Equilibrium w/ Mixed Strategies)

A Nash Equilibrium $\sigma_1^*, \dots, \sigma_n^*$ is a set of probability distribution such that

$$\pi_i(\sigma_i^*, \bar{\sigma}_{-i}^*) \geq (\sigma_i, \bar{\sigma}_{-i}^*)$$

for all σ_i.

Theorem 3.4

Nash equilibria exists for finite games with mixed strategies.

Proof. We show that the game with continuous strategy sets is a concave game.
Σ_i is a closed, bounded, convex set.
$\pi_i(\bar{\sigma})$ is a continuous function in $\bar{\sigma}$
$\pi_i(\sigma_i, \bar{\sigma}_{-i}^i)$ is concave (linear) in σ_i for all σ_{-i}. Since this game is concave, the Nash equilibrium exists by theorem 2.8.

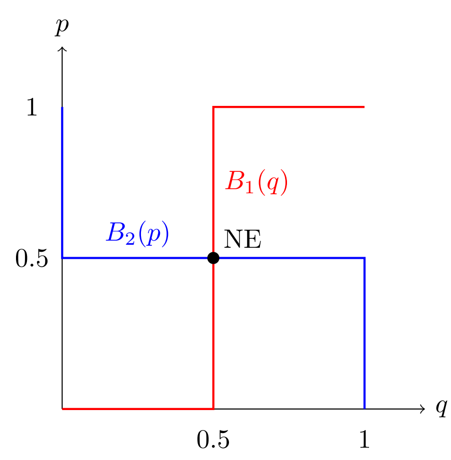
What is the Nash equilibrium for the attacker-defender scenario? Let us say that 1 and 2 chooses channel A with probability p and q respectively. Then 1 seeks to maximize

$$pq + (1-p)(1-q) = p(2q-1) + (1-q).$$

in p. Obviously, the attacker wants to choose the channel such that the defender has 1/2 chance of choosing, or any random distribution if the defender choose each channel with 1/2. 2 seeks to maximize

$$q(1-p) + p(1-q)$$

in q.



Proposition 3.5

In any mixed strategy nash equilibrium, each player must be getting the same payoff from any action it plays with positive probability.

Proof. The best response to any strategy is to put all the weight on the response with highest payoff.

Example 3.6 (Public Good Game)

Suppose N people, each receives a value of $v > 0$ if any one of them provides a 'good' at cost of $c > 0$ and get 0 payoff otherwise. Call the strategy set "y" and "n". To keep things interesting, assume $c > v$.

For example, filing a public report, or community networks (set up some kind of wireless access point). In this game the payoff is

$$\pi_i(s_i, \bar{s}_{-i}) = \begin{cases} v - c, & \text{if } s_i = \mathrm{y}, \\ v, & \text{if } s_i = \mathrm{n}, \exists \mathrm{s_j} = \mathrm{y}, \\ 0, & \text{if } s_j = \mathrm{n} \forall \mathrm{j}. \end{cases}$$

There are n different Nash equilibria. Each corresponds to one person providing the good and no one else does. However, there is no clear reason to pick one person to offer the good than another. We wonder if there is a symmetric mixed strategy equilibrium. I.e.

A strategy where each player chooses "y" with probability p.

Therefore, we are solving for a p such that for each agent alone, the payoff for any person switching from y to n does not matter. The payoff for choosing y is $v - c$. The payoff for choosing n is

$$v - v(1-p)^{n-1}.$$

Therefore we solve for

$$v - c = v - v(1-p)^{n-1} \implies c = v(1-p)^{n-1} \implies 1 - \sqrt[n-1]{\frac{c}{v}} = p.$$

At this equilibrium, notice that the probability that the good is provided is $1 - (1-p)^n = 1 - (c/v)^{1+1/n} \to 1 - c/v$ as $n \to \infty$. This is known as a 'free-rider problem', since as more people enter the game, they all want to contribute less.

Mixed strategies can apply to games with infinite strategy spaces.

Example 3.7

$S_r = [0, 1]$, then the strategy set (with mixed strategies) are all probability measures on $[0, 1]$.

Definition 3.8 (Continuous game)

A **continuous game** is a game for which the strategy spaces are non-empty, compact subsets of $\mathbb{R}^n$ and payoffs that are continuous in $\bar{s}$.

Theorem 3.9 (Glicksberg)

A continuous game has a mixed strategy Nash equilibrium.

Remark. *Recall we have a pure strategy Nash equilibrium for concave games. If we remove the concave assumption, we will need mixed strategies to reduce the game into a concave game.*

Defences and critiques of Mixed strategies

- **A mixed strategy equilibrium is not as predictive as a pure strategy.** In our attacker-defender game, the mixed-strategy equilibrium gives no additional information (entropically speaking) on which channel the players will play.
- **There are no 'mixed strategies' for a single play of a game.** If we just do the game once, we will not be able to observe the proabiltiy distribution. If the game is played multiple times over, we can get statistics to infer probability distributions.
- **In some games, mixed strategies can seem natural.** e.g. attacker defender, rock paper scissors.
- **Pure strategies might be prefered.** E.g. Cournot competitions.

4 Lecture 4: Zero sum games and infinite population games

Definition 4.1 (Zero-sum game)

A two player game is said to be **zero-sum** if for any pure action profile $\bar{s}$,

$$\pi_1(\bar{s}) + \pi_2(\bar{s}) = k$$

for a constant k. WLOG we can assume $k = 0$.

Remark. *This dates back to 1928 (von Neumann).*

Example 4.2

The following are examples of zero-sum games. These are "strictly competitive" settings.

- Attacker/defender
- Chess
- Futures/options contracts in finance
- Cake-cutting
- Worst case analysis in computer science.

Example 4.3 (Cake Cutting)

cutter\chooser	Larger	Smaller
equal	(50,50)	(50,50)
unequal	(40,60)	(60,40)

Example 4.4

Consider the zero-sum game with reward matrix

(1,-1)	(-1,1)
(-1,1)	(1,-1)

We can represent it as a single matrix

1	-1
-1	1

Now the row player wants to maximize the value, and the column player wants to minimze the value.

Now we suppose

$$\bar{\sigma}_1 = \begin{bmatrix} \sigma_{1,1} \\ \sigma_{1,2} \\ \vdots \\ \sigma_{1,m} \end{bmatrix}$$

be a mixed strategy for player 1. Then $\bar{\sigma}_1^T A$ denotes the expected payoff of player 1 if player 2 chooses the corresponding column (pure strategy). Therefore if $\bar{\sigma}_2$ denotes the mixed strategy of player 2, we have the expected value of player 1 is $\bar{\sigma}_1^T A \bar{\sigma}_2$.

Definition 4.5 (Maxi-min/min-max strategies)

We define

$$V_1^* \stackrel{\text{def}}{=} \max_{\bar{\sigma}_1} \min_{\bar{\sigma}_2} \bar{\sigma}_1^T A \bar{\sigma}_2.$$

This denotes the largest worst-case payoff for player 1. Similarly we define

$$V_2^* \stackrel{\text{def}}{=} \min_{\bar{\sigma}_2} \max_{\bar{\sigma}_1} \bar{\sigma}_1^T A \bar{\sigma}_2.$$

Proposition 4.6

We have

$$V_1^* \leq V_2^*.$$

Proof. We have $V_1^* \leq \min_{\sigma_2} B_1(\bar{\sigma}_2)^T A \bar{\sigma}_2 = V_2^*$.

Theorem 4.7 (Minimax)

For any 2 player zero-sum game,

- $V_1^* = V_2^*$
- The set of Nash equilibria is $\{\bar{\sigma}_1^*, \bar{\sigma}_2^*\}$ for $\bar{\sigma}_1^*$ maximin and $\bar{\sigma}_2^*$ minimax.

Proof. Let $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ be a Nash equilibrium. Then

$$\tilde{\sigma}_1 = \arg\max_{\sigma_1} \sigma_1^T A \tilde{\sigma}_2$$

and

$$\tilde{\sigma}_2 = \arg\min_{\sigma_2} \tilde{\sigma}_1^T A \sigma_2.$$

So that

$$V_2^* = \min_{\sigma_2} \max_{\sigma_1} \sigma_1^T A \sigma_2 \leq \max_{\sigma_1} \sigma_1^T A \tilde{\sigma}_2 \overset{\text{nash}}{=} \min_{\sigma_2} \tilde{\sigma}_1^T A \sigma_2 \leq \max_{\sigma_1} \min_{\sigma_2} \sigma_1^T A \sigma_2 = V_1^*.$$

The proposition $V_1^* \leq V_2^*$ completes the proof.

The compute this equilibrium, we find

$$\max_{\sigma_1} v,$$

such that $[\sigma_1^T A]_j \geq v \forall j$, subject to $\sigma_{1,1} + \sigma_{1,2} + \cdots + \sigma_{1,m} = 1$ and $\sigma_{1,j} \geq 0$. This is a linear program, so we can compute this in polynomial time. All hail Dinic and push relabel.

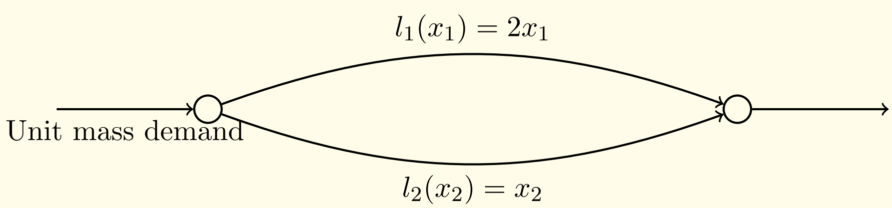
Definition 4.8 (Infinite Population Games)

An **infintie population game** is a game where the set of players is infinite.

Example 4.9 (Infinite Population Game)

Consider a game where the set of players is the continuum $[0, 1]$, such that each player is infinitesimal. This is known as 'non-atomic player with a mass of 1'.

Example 4.10 (Traffic Equilibrium)



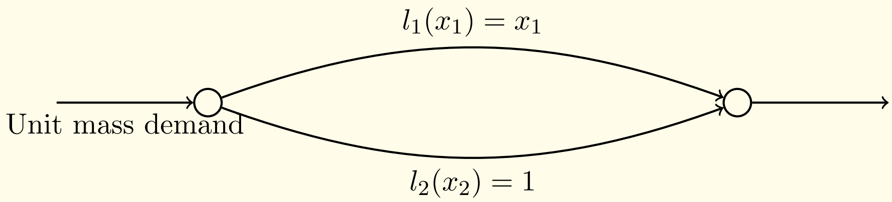
Suppose we have a unit mass of traffic going through two routes with l_1 or l_2 cost each. We find the nash equilibrium subject to $x_1 + x_2 = 1$, $x_1, x_2 \geq 0$.

This is very intuitive. Relating $l_1(x_1) = l_2(x_2)$ gives $x_1 = 1/3, x_2 = 2/3$.

Remark. *This is also known as a* ***wardrop equilibrium****. The cost on all the paths used is the same. The cost on all the paths that are not used is greater than on the used paths.*

Example 4.11 (Pigon)

Consider the following 'traffic diagram'.



The wardrop equilibrium here is $x_1 = 1, x_2 = 0$. This is interesting, as we can consider the social cost, defined as the expected value of the cost for the players. In this case, everyone needs to pay a cost of 1.

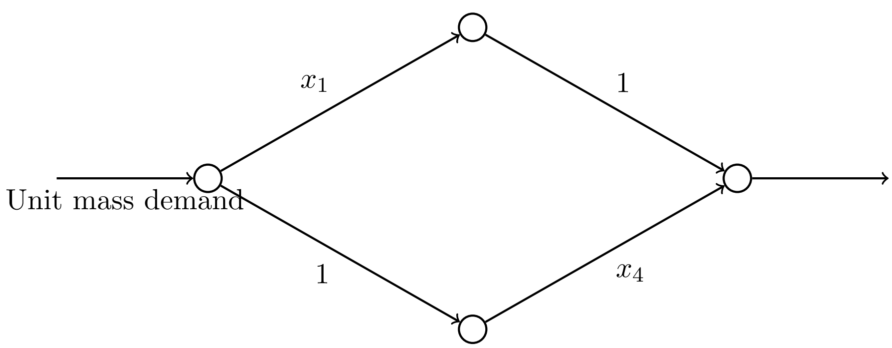
However, let us suppose that there is a planner who would like to minimize this total cost

$$\min_{x_1} x_1 l(x_1) + x_2 l(x_2) = \min x_1^2 + (1 - x_1).$$

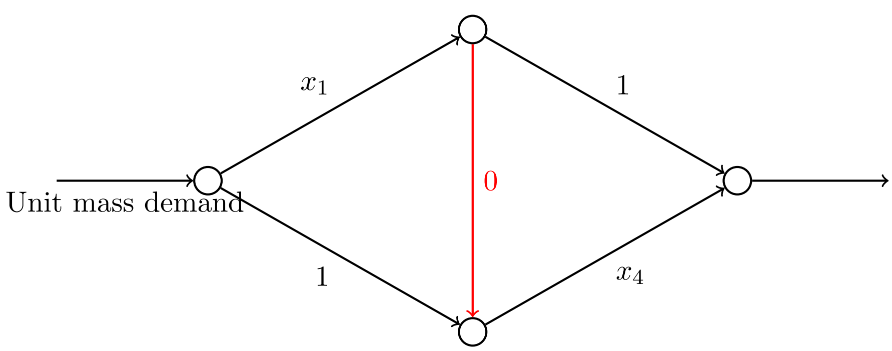
Completing the square gives a minimal cost when $x_1 = 1/2$ with a total cost of $3/4$. However, this is not a stable minima. There is an incentive to deviate from the assigned route. Let us define the price of anarchy

$$P \stackrel{\text{def}}{=} \frac{\text{cost with equilibrium}}{\text{cost with planner}}.$$

$P = 1$ is the best outcome. But in this case we have a price of $1/(3/4) = 4/3$. Consider the following diagram.



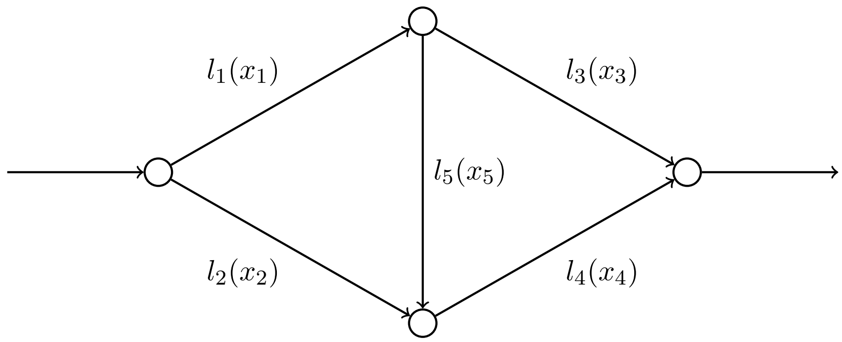
Now, the equilibrium is when $x_1 = x_4 = 1/2$, with a total social cost of $3/2$. Suppose we now create a new road.



Now, the best road is to go up then through the red edge. This means the new equilibrium now is $x_1 = x_4 = 1$ with a total social cost of 2. This is known as **Braess's paradox**, where adding more resources to the system causes the social cost to increase.

Existence of Wardrop Equilibria

. We want to determine when wardrop equilibria exist in the following graph.



> **Theorem 4.12**
>
> Assume that each $l_i(x_i)$ is non-negative, non-decreasing, and differentiable. Then a unique Wardrop Equilibrium exists.

Sketch of proof. Let P be the set of all paths from the source to the destination. Let x_p be the number of users on path i, and E be the set of edges. We now have an optimization

$$\min_{\{x_p : p \in P\}} \sum_{i \in E} \int_0^{\text{total traffic on } i} l_i(z)dz$$

subject to $\sum_{p \in P} x_p = 1, x_p \geq 0 \forall p \in P$.

We claim that this optimization problem has a unique solution, and this solution is the Wardrop equilibrium. The argument boils down to showing that the function is convex,thus has a unique minimum.

Let λ be a Lagrange Multiplier for the condition $\sum_{p \in P} x_p = 1$. Then the first order optimality shows that

$$\sum_{\text{edges in path } p} l_i(\sum_{\text{paths } p \text{ using } i} x_p) \geq \lambda$$

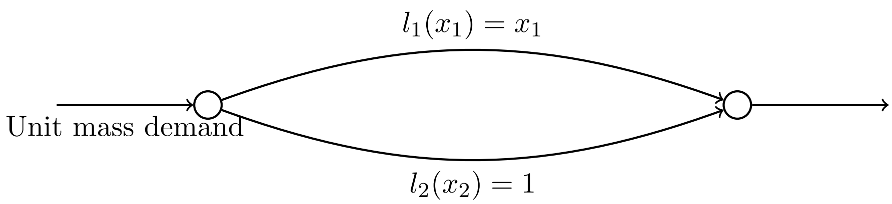
with equality when $\sum x_p > 0$.

5 Lecture 5: Pricing

Recap

1. Zero sum games
2. Infinite population games - non-atomic traffic flow games.

Recall in the last lecture we had in example **??**:



Which had conflicting social optimal and equilibrium. However, we can introduce a price of $t_1 = 1/2$ to the upper route. i.e. $l_1(x_1) = x_1 + 1/2$ In this case, the equilibrium is when $x_1 = x_2 = 1/2$. From an observer standpoint, we can disregard the tolls and get the social optimum strategy. However, for the drivers' standpoint, they still incur the full cost of 1.

From an economic standpoint, we can disregard the cost of the toll. This is because the planner gets the toll fee of $(1/2)(1/2) = 1/4$, but then this can be redistributed to everyone.

Remark. *You can think of it 'transfering' part of the cost of the bottom road to the top road to have* $3/4$ *cost each.*

We revisit the original model without tolls. Suppose that x_1 drivers are taking the top road and we consider the next driver $(x_1 + \epsilon?)$.

We define the social cost of the driver taking the top road as

$$\frac{d}{dx_1} x_1 l_1(x_1) = x_1 l_1'(x_1) + l_1(x_1).$$

Previously each driver in the selfish equilibrium compares the costs $l_1(x_1)$ and $l_2(x_2)$. This is the second term in the cost. The first term is called **externality**.

Theorem 5.1 (Pigovian tax)

Let $\tilde{l}_j(x) = x l_j'(x) + l_j(x)$. Under the new graph with costs $\tilde{l}_j$, the Wardrop eqiulibrium is the social optimal strategy with original costs.

Corollary 5.2: The optimal toll cost for each road is $xl'(x)$. (up to a constant difference, but that is redistributed anyway.)

Finite number of users

These leads to models that are known as congestion games. (Rosenthal, 1973)

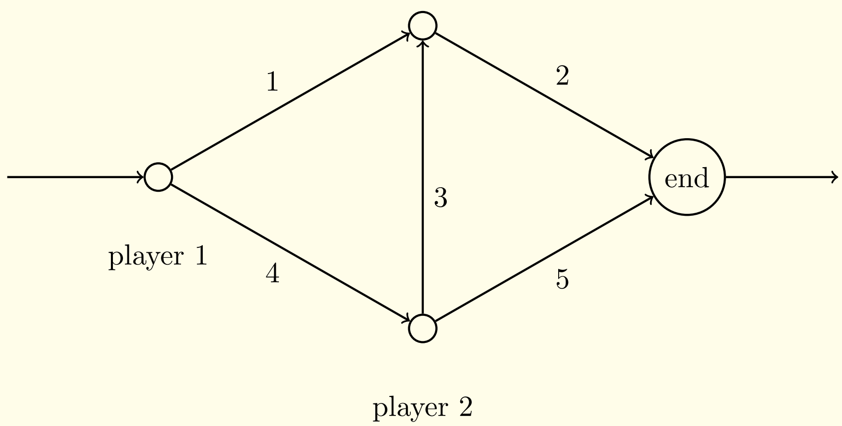
Remark. *Rosenthal studied IEMS in Northwestern.*

Definition 5.3 (Congestion Game)

A **congestion game** is a game with

- A set of players $R = \{1, 2, \ldots, n\}$
- A set of resources $M = \{1, 2, \ldots, m\}$
- For each $i \in R$, a set $S_i \subseteq M$ denoting the set of resources i can use.
- Each resource has a cost of $c^j(x_j)$, where x_j is the total number of players using the resource. The cost for player i is $c_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(x_j)$.

Example 5.4



Consider this game where both players want to reach the end.The roads are numbered 1 to 4. We have

$$
\begin{aligned}
M =& \{1, 2, 3, 4, 5\}, \\
S_1 =& \{(1, 2), (4, 5), (4, 3, 2)\}, \\
S_2 =& \{(3, 2), 5\}.
\end{aligned}
$$

Example 5.5 (El Farol Bar Problem)

10 people decide to go to the bar or go home. If 6 or fewer go they have more fun than staying home. If 7 or more go, the bar is too crowded and they have less fun.

How do we model this as a congestion game? We can model the resources as

$$\{\text{bar}, \text{home}\}.$$

The strategy set of each player is

$$S_i = \{\text{bar}, \text{home}\}$$

Under this, the payoff is

$$\pi_i(s_i) = \begin{cases} 1, & \text{if } x_{\text{bar}} \leq 6 \text{ and } s_i = \text{bar} \\ -1, & \text{if } x_{\text{bar}} \geq 7 \text{ and } s_i = \text{bar} \\ 0, & \text{if } s_i = \text{home} \end{cases}$$

Theorem 5.6 (Rosenthal)

Every congestion game has a pure strategy Nash equilibirum.

We will prove this result later.

Definition 5.7 (Potential Game)

A game $G = (R, \{S_r\}, \{\pi_r\})$ is a **potential game** if there is a function

$$P : \prod_{r \in R} S_r \to \mathbb{R}$$

such that for all $r \in R$, $\bar{s}_{-r}$, and $x, z \in S_r$,

$$\pi_r(x, \bar{s}_{-r}) - \pi_r(z, \bar{s}_{-r}) \geq 0 \iff P(x, \bar{s}_{-r}) - P(z, \bar{s}_{-r}) \geq 0$$

with equality on the left iff equality on the right. In this case, we call P a(an) **(ordinal) potential**. If

$$\pi_r(x, \bar{s}_{-r}) - \pi_r(z, \bar{s}_{-r}) = P(x, \bar{s}_{-r}) - P(z, \bar{s}_{-r}),$$

we can P an **exact potential**.

That is, a player benefits from changing strategies if and only if they can increase the potential. Potential games exist.

Example 5.8 (Cournot competition potential)

R firms choose $q_i \in [\epsilon, q_{\max}]$. The payoff for each company is

$$\pi_i = q_i(P(Q) - c)$$

where $Q = \sum q_r$. We let

$$\Psi(q_1, \ldots, q_R) = \left(\prod_r q_r\right)(P(Q) - c).$$

Then

$$\begin{aligned}\Psi(q_i, \bar{q}_{-i}) - \Psi(\tilde{q}_i, \bar{q}_{-i}) &= \left(\prod_{r \neq i} q_r\right)(q_i(P(Q_{-i} + q_i) - c) - \tilde{q}_i(P(Q_{-i} + \tilde{q}_i) - c)) \\ &= \left(\prod_{r \neq i} q_r\right)(\pi_i(q_i, \bar{q}_{-i}) - \pi_i(\tilde{q}_i, \bar{q}_{-i})).\end{aligned}$$

This is *almost* a potential game for $q_i \in [0, q_{\max}]$. Stricter conditions on P can give potentials.

Lemma 5.9

Every congestion game is a potential game

Sketch. We need to construct a potential for the congestion game. We claim

$$P(\bar{s}) = \sum_{j \in M} \sum_{1}^{x_j} C^j(k)$$

is a potential. Intuitively, take each resource j, then sum C^j all the way from 1 user to x_j users.

Suppose user i using one s_i and wants to switch towards using $\tilde{s}_i$. The change in cost is

$$\pi_i(s_i, \bar{s}_{-i}) - \pi_i(\tilde{s}_i, \bar{s}_{-i}) = C^{s_i}(x_{s_i}) - C^{\tilde{s}_i}(x_{s_i}).$$

This is exactly the change in potential. (Apply an inductive argument).

Theorem 5.10

Every potential game has a Nash equilibrium if the potential obtains its maximum for some strategy profile.

Proof. There is a maximal value for the potential function. Each player picks the strategy that maximize this potential function. This must be a Nash equilibirum.

Corollary 5.11: Every finite potential game has at least one pure strategy Nash equilibrium. Every game with a compact strategy set has at least one pure strategy Nash equilibrium.

Proof of Rosenthal. A congestion game has finite players and finite resources. Therefore it is a finite game. Since all congestion games are potential games, we are done. 🌸

Nash equilibriums can be reached by using best response in potential games. I.e. choose one player and change his strategy to be the best response. The potential is strictly increasing, so has to be reached at some point.