Game theory midterm notes :

Def (concave game) : $\forall r, \begin{cases} S_r \subseteq \mathbb{R}^n \text{ convex compact} \\ \pi_r(S_r, S_{-r}) \text{ continuous in } S_r \\ \pi_r(S_r, S_{-r}) \text{ concave game} \end{cases}$

mixed strategy: must be indifferent to all actions w/ positive probability.

Def (minmax / maxmin) $V_1^* \overset{def}{=} \max_{\bar\sigma_1} \min_{\bar\sigma_2} \bar\sigma_1^T A \sigma_2$ ↙ player 1 choose first

$V_2^* \overset{def}{=} \min_{\bar\sigma_2} \max_{\bar\sigma_1} \bar\sigma_1^T A \sigma_2$ ↙ player 2 choose first

so $V_1^* \le V_2^*$

Def (Price of anarchy) $P \overset{def}{=} \dfrac{\text{Cost with equilibrium}}{\text{Cost with planner}} \ge 1$

Thm (Pigovian tax)

$\tilde l_j(x) = x l_j'(x) + l_j(x)$ gives NE equal to socially optimal strategy w/ $l_j$ weights.

pf. the social cost for a driver choosing the $j^{th}$ road is $\frac{d}{dx} x\, l_j(x) = x l_j'(x) + l_j(x)$.

Def. (Congestion game) has
$\begin{cases} \text{set of resources } M, \text{ set of players } R \\ \forall i \in R, \quad S_i \subseteq 2^M \\ \text{Cost is shared: } \sum_{j \in S_i} c^j(x_j) = C_i(s_i, s_{-i}). \end{cases}$

Def (potential game) a potential

$P: \prod_{r \in R} S_r \to \mathbb{R}$ s.t.

$\pi_r(x, \bar S_r) - \pi_r(z, \bar S_r) \ge 0 \Leftrightarrow P(x, \bar S_r) - P(z, \bar S_r) \ge 0$

if equality $\Delta\pi = \Delta P$, $P$ is an exact potential.

Def (Increasing differences) $X \subseteq \mathbb{R}$, $T$ partially ordered

$f: X \times T \to \mathbb{R}$ has increasing differences

if $f(x', t') - f(x, t') \ge f(x', t) - f(x, t)$.

for $x' \ge x$, $t' \ge t$.

if $T \subseteq \mathbb{R}^k$ w/ order $\le$ in every entry, $f \in C^2$,

condition equivalent to $\dfrac{\partial^2 f}{\partial x \partial t_i} \ge 0$.

Def (Supermodular game)

$S_r$ compact.

$\pi_r(S_r, S_{-r})$ has increasing diff in $S_r, \bar S_{-r}$.

Def. fictitious play.

Parallel best response to the estimated mixed strategy.

Converges if it stabilizes.

or

Converges in time average if distribution of plays converges

Thm

Concave game has Nash equilibrium: Sketch:

Brouwer fixed pt $f: V \to V$ on $B_r = \operatorname{argmax}_{z_r} \pi_r(z_r, s_{-r})$

mixed strategy Nash equilibrium exists for finite games: sketch forms a concave game

mixed strategy Nash equilibrium exists for continuous games.

minimax thm : 2 player zero sum game

$V_1^* = V_2^*$, set of nash eq. is $\{\bar\sigma_1^*, \bar\sigma_2^*\}$ (maximin, minimax)

pf. Let $(\tilde\sigma_1, \tilde\sigma_2)$ a NE

then $V_2^* = \min_{\sigma_2} \max_{\sigma_1} \sigma_1^T A \sigma_2 \le \max_{\sigma_1} \sigma_1^T A \tilde\sigma_2$ ← one example of $\tilde\sigma_2$.

$\overset{\text{nash}}{=} \min_{\sigma_2} \tilde\sigma_1^T A \sigma_2 \le \max_{\sigma_1} \min_{\sigma_2} \sigma_1^T A \sigma_2 = V_1^*$.

Existence of wardrop equilibrium when each cost of road $l_i(x_i)$ non-negative, non-decreasing, differentiable. Eq. is unique too.

sketch : optimize

$$\min_{\{x_p : p \in P\}} \sum_{i \in E} \int_0^{\text{total traffic on } i} l_i(z)\,dz. \quad \text{subject to } \sum_{p \in P} x_p = 1. \quad x_p \ge 0.$$

$P$ = set of all paths to destination

Existence of pure Nash equilibria for potential game w/ maximal potential.

usage:

Congestion game: $P(\bar z) = \sum_{j \in M} \sum_{k=1}^{x_j} c^j(k)$

changing resource = change in potential

=> Exact potential.

Cournot competition: $P(\bar q) = (\prod_r q_r)(P(Q) - c)$

Cost of deviation

$P(q_i, \bar q_{-i}) - P(\tilde q_i, \bar q_{-i}) = (\prod_{r \ne i} q_r)\big(\underbrace{q_i(P(Q_{-i} + q_i) - c) - \tilde q_i(P(Q_{-i} + \tilde q_i) - c)}_{\text{payoff change}}\big)$

Thm (Topkis)

iff cont. in $X$,

$\begin{cases} \operatorname{argmax}_{x \in X} f(x,t) \text{ non empty.} \\ \max, \min \arg_{x \in X} f(x,t) \text{ are both increasing in } t. \end{cases}$

Thm Parallel best response converges for supermodular game.

Start with $S_r = \sup S_r$.

run parallel best response, then $S_i^{k+1} = B_i(S_{-i}^k) \overset{\text{Topkis}}{\le} B_i(S_{-i}^{k-1}) = S_i^k$.

MCT. constant with min instead.

Thm. 1. If profile converges, must be a nash equilibrium

2. if a nash equilibrium is played at any stage, the play stabilizes.

3. if profile converges in time average to $\sigma$, $\sigma$ is a mixed strategy eq.

RMK fictitious play converges in T.A. for
- two player zero sum
- two player with at most 2 strategies for each
- iterated strict dominance solvable
- potential game.

Def Evolutionary game thy. Let finite symmetric game

fitness of strategy $s_i$ is expected payoff of $s_i$ against a random member of population

$s_j$ invades $s_i$ at level $\varepsilon \in [0,1]$ if $\varepsilon$ of population is type j and $1-\varepsilon$ is type $s_i$.

Evolutionary stable strategy (ESS): $\exists \bar{\varepsilon} > 0$ s.t. any strategy invading at $\varepsilon < \bar{\varepsilon}$ has lower fitness. strict*

Can be used for mixed strategies too. Solve for symmetric mixed strategy eq. and show ESS.

ESS is a Nash eq.

pf. for small $\varepsilon$, any other strategy $\tilde{s}$ satisfies

$$(1-\varepsilon)\pi(s^*,s^*) + \varepsilon\pi(s^*,\tilde{s}) > (1-\varepsilon)\pi(\tilde{s},s^*) + \varepsilon\pi(\tilde{s},\tilde{s})$$

& $\varepsilon \downarrow 0 \Rightarrow \pi(s^*,s^*) \geq \pi(\tilde{s},s^*)$.

Strict Nash $\subset$ ESS $\subset$ Nash.

examples

| | A | B |
|---|---|---|
| A | (0,0) | (6,0) |
| B | (0,6) | (3,3) |

(AA) ~~is ESS~~ is weak Nash, (not strict) but is ESS because B benefits A.

| | A | B |
|---|---|---|
| A | (4,4) | (0,4) |
| B | (4,0) | (3,3) |

(AA) weak Nash but not ESS.

Thm: if $x^*$ ESS $\Rightarrow$ it is asymptotically stable for replicator dynamics.

Def. Replicator dynamics.

evolve

$$\frac{d}{dt}\frac{x_i}{x_j} = \frac{x_i}{x_j}[\underbrace{f_i}_{\text{fitness}}(\vec{x}(t)) - f_j(\vec{x}(t))],$$

or

$$\frac{d}{dt}x_i(t) = x_i(t)[f_i(\vec{x}(t)) - \phi(\vec{x}(t))],$$

$\phi = \sum_i x_i f_j(\vec{x})$ is average fitness.

Def (Dynamic game)

Sequentially rational / subgame perfect Nash Eq. is one that is NE in each subgame.

use backwards induction on extensive form representation.

Thm. yields exactly all the SPNE.

Def (information set)

a collection of ~~distinct~~ decision nodes that are indistinguishable for the player making the decision at the nodes.


circle, or dotted line to denote.

Work on all the proper subgames when working w/ information sets. i.e. starts at singleton & do not break information sets.

If all information sets are singletons, it is a perfect information game.

Def (Repeated games)

games repeated with a future payoff discount $\delta$

for infinitely repeating games, payoff $= (1-\delta)\sum_{t=0}^{\infty}\delta^t \pi_i(\vec{a}^t)$, $(1-\delta)$ normalization as $1+\delta+\dots = \frac{1}{1-\delta}$.

Thm. If stage game has unique NE and finite repetitions, the only SPNE is to play NE at every stage

Thm (folk)

an outcome $\vec{v}$ is feasible if $\vec{v} \in$ Convex hull of $\{\pi(S)\}$.

individually rational if $v_i \geq \min_{a_{-i}}\max_{a_i}\pi_i(a_i,a_{-i})$

there is a NE with payoff $\vec{v}$ if $\vec{v}$ is feasible & strictly individually rational.

Def. Bayesian game

types $\Theta$ w/ joint distributions $P$, $\pi$ depending on types

Finite strategy set Bayesian game has mixed strategy Bayes-Nash eq.

i.e. function $\sigma_r : \Theta_r \to S_r$

$$\sigma_r(\theta_r) \in \arg\max_{s_r} \sum_{\theta_{-r}} p(\theta_{-r}|\theta_r)\,\pi_r(\theta_r, \theta_{-r}, s_r, \underbrace{s_{-r}}_{=\sigma_{-r}(\theta_{-r})})$$

Justifications for models:

NE: self enforcing outcome, result of long run learning, result of lots of thinking

mixed strategies: natural in some games e.g. rock paper scissors, attacker/defender

no mixed strategies in a single play of game

not as predictive as pure strategy

Bayes: (Harsanyi)

| | A | B |
|---|---|---|
| A | $(2+t_A, 1)$ | 0,0 |
| B | (0,0) | $(1, 2+t_B)$ |

$t_A, t_B$ unif $[0,x]$, $\xrightarrow{x \downarrow 0}$

| | A | B |
|---|---|---|
| A | (2,1) | (0,0) |
| B | (0,0) | (1,2) |

p1 play A with $p = 1 - \frac{-3+\sqrt{9+4x}}{2x} \to \frac{2}{3}$.

Counter example of fictitious play not converging to mixed eq. in T.A.

Rock paper scissors, winner gets 1, loser/tie gets 0.

Start w/ (R,S).

Common NE.

Cournot: cost of manufacture $c_1, c_2$

price $= (1-q_1-q_2)$

$q_1^* = \frac{1-2c_1+c_2}{3}$, $q_2^* = \frac{1-2c_2+c_1}{3}$.

Public good game: N people, provide good at cost c, shared payoff of v.

N pure equilibria of 1 person provide good.

Symmetric eq: provide at probability $p = 1 - \sqrt[N-1]{\frac{c}{v}}$

Hawk dove:

| | D | H |
|---|---|---|
| D | (3,3) | (1,5) |
| H | (5,1) | (0,0) |

solving $P(\text{hawk}) = p$, need $3(p) + 1(1-p) = 5p + 0(1-p)$ $\Rightarrow p = \frac{1}{3}$.

Infinite horizon bargaining: 2 players alternate proposing splitting 1 dollar ~~split~~ w/ time discount $\delta$.

let a = payoff of 1st player. then 3rd round = 1st round.

$1 - \delta(1-\delta a) = a \Rightarrow a = \frac{1}{1+\delta}$.