EE 495 Game Theory and Networked Systems

Contents

Syllabus

All logistics are on canvas.

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There are no required texts for the class. There are some references (4) that are suggested on Canvas.

Prereqs

1. No prior knowledge on game theory is required.

2. Mathematical background is required. Linear algebra, probability, optimization, mathematical maturity.

Grades

- Midterm: in class, 35%

- Final project: 35%

- Problemsets: 30%

> **Theorem 0.1**
>
> Randy is a chill professor.

Lecture 1: Introduction to Game Theory

> **Definition 1.1 (Game Theory)**
>
> Game theory is the study of interactions of *multiple strategic agents.*

Features of game theory:

- More than one decision makers.

- Each agent makes decisions to maximize self-interest.

- These **agents** are players of the 'game', and can be people, firms, countries (in political science), AI-agents etc.

> **Example 1.2**
>
> The following are examples of 'games':
>
> - 2 people playing chess. Their incentives do not align because each player wants to checkmate the other.
>
> - 2 firms competing in a market. They are selling the similar items and are trying to price their items to get a larger market share.
>
> - 4 countries competing to maximize GDP.

The other component of this course is network systems.

> **Example 1.3**
>
> The following are examples of network systems:
>
> - Communication network.
>
> - Electricity network.
>
> - Transportation network.

We will use these as examples to illustrate the concepts in game theory. However, the same theory can extend into the other games. We want to model and analyze games. People are complicated to model, and our models are simplifications of reality. We need to understand what assumptions are made for each models to apply analysis.

Basic Game Model

> **Definition 1.4 (Basic Game Model)**
>
> A **strategic form game** G consists of the following elements:
>
> 1. The set of agents/players R, usually enumerated $R = \{1, 2, 3, \ldots, n\}$.
>
> 2. For every $r \in R$, the action set of player r S_r. If $|\bigsqcup_r S_r| < \infty$, we call the game a **finite game**.
>
> 3. For every $r \in R$, a payoff function $\pi_r : \otimes_{r'} S_{r'} \to \mathbb{R}$. Each agent r wants to maximize π_r.

That means, there is only one round of this game, and everyone makes the same decision all at once.

Remark. *The action set can also be called the strategic set.*

Notation. *The ordered set of everyone's actions except r is*

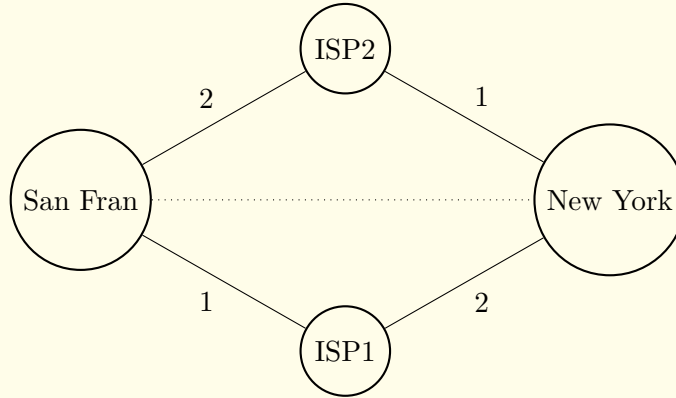$$\overline{s_{-r}} = (s_1, s_2, \ldots, s_{r-1}, s_{r+1}, \ldots, s_n).$$

Therefore player r is actually maximizing

$$\pi_r(s_r, \overline{s_{-r}}).$$

The second part of this function is outside of r's control.

Consider two Internet Service Providers (ISPs)

They have peered i.e. no charge to send traffic to each other. There is 1MB of traffic to ISP 1 customers in NY. There is 1mB of traffic to ISP 2 in SF. Each ISP will incur the cost of usage of the two (respective) edges connected to it, per mB of traffic. Here,

$$R = \{1, 2\}$$
$$S_r = \{\text{near}, \text{far}\},$$

that is, each ISP can decide whether to send the traffic directly or through the other ISP.

We can represent this as a matrix

ISP 1\ISP2	near	far
near	(-4,-4)	(-1,-5)
far	(-5,-1)	(-2,-2)

Definition 1.6 (Dominant action)

An Action is s_r^* is **weakly dominant** if

$$\pi_r(s_r^*, \overline{s_{-r}}) \geq \pi_r(s_r, \overline{s_{-r}})$$

for all $s_r \neq s_r^*$, and any $\overline{s_{-r}}$. If inequality is strict, then the action is **strictly dominant**.

Corollary 1.7: If the game has a strictly dominant action for each player, this will give a unique dominant strategy equilibrium.

Consider the game with the reward matrix:

1\2	L	M	R
U	(1,0)	(2,-1)	(1,2)
D	(0,3)	(1,4)	(0,1)

There is no dominant action for player 2, but there is a dominant action (U) for player 1. Therefore, player 2 can rationally assume that player 1 does not play D. Under this assumption, player 2 has a dominant strategy of (R).

Alice and Bob want to get lunch together Consider the game with the reward matrix:

1\2	Tech	Kellog
Tech	(2,3)	(1,1)
Kellog	(1,1)	(3,2)

There is no dominant action for each player. This is known as a coordination game, it is better to follow what the other player chooses.

Definition 1.10 (Nash Equilibrium)

A strategy profile $(s_1^*, \ldots, s_n^*)$ is a **Nash equilibrium** if

$$\pi_r(s_r^*, s_{-r}^*) \geq \pi_r(s_r, s_{-r}^*)$$

for each agent r, $s_r \in S_r$. If equality is strict, we say that the Nash ewquilibrium is **strict**. In other words, no player benefits from unilateral deviations.

In the case of Example 1.9, the two Nash equilibria are when both players pick the same place to eat. However, there are no dominant strategies! Even if we made the matrix

1\2	Tech	Kellog
Tech	(5,5)	(1,1)
Kellog	(1,1)	(2,2)

there will still be two Nash equilibria, even though the (5,5) outcome is much better than the other (2,2) - some are better than others.

Proposition 1.11

A dominant strategy equilibrium is a Nash equilibrium.

Proof. By definition. ✿

Example 1.12

Consider the reward matrix

Attacker\User	A	B
A	(1,0)	(0,1)
B	(0,1)	(1,0)

The attacker always wants to choose the channel with the user, but the user wants a different channel. In this case, there is no Nash equilibrium (either the attacker can move to the user channel or the user move away from the attacker channel).

Next Time

Mixed strategies, thm: finite game with randomized strategies always have nash equilibria.

Lecture 2: Games with Continuous Strategies

Recap of last lecture

1. Definition of Game

2. Dominant Strategies

3. Nash Equilibrium

Example 2.1 (First Price Auction)

A single object is to be assigned to one player from $1 - n$ in exchange for payment. Each player values the object at $v_1 > v_2 > \ldots > v_n > 0$ respectively. The players submit bids $b_i \geq 0$. The player with the highest bid gets the item and pays its bid. If there is a tie, the object goes to the player with the lowest index.

In this case, the payoff function would be $v_i - b_i$ for the winning player i, and 0 for everyone else.

$$\pi_r(b_r, \overline{b_{-r}}) = \begin{cases} v_r - b_r, & \text{if } b_r > b_s \forall s < r, \ b_r \geq b_s \forall s \geq r \\ 0, & \text{otherwise.} \end{cases}$$

There is no dominant strategy. Each bidder will always want to bid slightly higher than everyone else. There is a Nash Equilibrium. Namely, $b_r = v_r$ for all $r \geq 2$ and $b_1 = v_2$.

Remark. *Nash equilibrium here is not unique. But the outcome is always the same. The item always goes to player 1.*

Example 2.2 (Cournot Competition)

There are two firms. They produce the same good (indistiguishable). Each firm chooses a quantity q_i to produce at a cost of c_i per good. They sell it at a market price $p(q_1 + q_2) = 1 - q_1 - q_2$. Find the Nash equilibrium of the game.

In this case, the payoff function is

$$\pi_i(q_i, q_{-i}) = (1 - q_1 - q_2)q_i - c_i q_i.$$

To solve for the equilibrium, we motivate the following definition.

Definition 2.3 (Best Response Correspondence)

For each player i, let
$$B_i(q_{-i}) \stackrel{\text{def}}{=} \operatorname{argmax}_{q_i} \pi_i(q_i, q_{-i})$$
be the **best response correspondence** of player i. This function can be multi-valued.
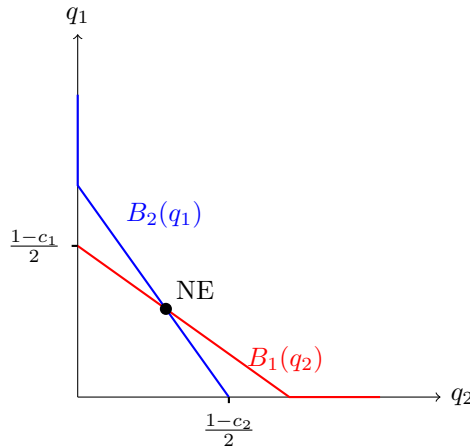

Proposition 2.4

At a Nash Equilibrium profile $\overline{q^*}$, we would have $q_i^* \in B_i(q*_{-i})$.

Solving for the previous example, we would get, by completing the square

$$B_1(q_2) = \max\left(\frac{1 - q_2 - c_1}{2}, 0\right)$$
$$B_2(q_1) = \max\left(\frac{1 - q_1 - c_2}{2}, 0\right)$$

We can thus plot B_1 and B_2 on the axes q_1, q_2. The intersection of the two graphs will be the nash equilibrium, as they are playing the best response to the other player.

Solving the two equations give

$$q_1 = \frac{1 - 2c_1 + c_2}{3},$$
$$q_2 = \frac{1 - 2c_2 + c_1}{3},$$

provided that c_1, c_2 are small enough that these do not into the negatives.

Sanity check

The solution is symmetric. If the cost for both firms are the same, they will each produce $(1 - c_1)/3$. If c_1 increases, q_1 decreases and q_2 increases.

Notation. *Given an action profile $\bar{s}$, we set the best response correspondence for all players to be the vector-valued function*

$$B(\bar{s}) = \begin{bmatrix} B_1(\bar{s}) \\ \vdots \\ B_n(\bar{s}) \end{bmatrix}$$

Thus a Nash equilibrium profile $\overline{s^*}$ satisfies

$$B(\overline{s^*}) = \overline{s^*}.$$

Therefore, we are interested in the fixed points of $B : \prod_r^{S_r} \to \prod_r^{S_r}$. We introduce two main fixed point results.

Theorem 2.5 (Brower)

Let V be a compact convex set. Then any continuous function $f : V \to V$ has a fixed point.

Theorem 2.6 (Kakutani)

TBD

Definition 2.7 (Concave Game)

A game is said to be **concave** if for each player r:

1. S_r is a non-empty compact convex subset of $\mathbb{R}^n$.

2. The payoff $\pi_r(s_r, \overline{s_{-r}})$ is continuous in s_r for each $\overline{s_{-r}}$.

3. The payoff $\pi_r(s_r, \overline{s_{-r}})$ is concave in s_r for each $\overline{s_{-r}}$.

Theorem 2.8

Every concave game has a Nash equilibrium.

Idea of proof. Assume that $B(\bar{s})$ is single-valued. We want to show that B is continuous on the convex set $\prod S_r$. So we cheat and apply the following lemma (Maximum thoerem) to get our result. ✿

Lemma 2.9 (Maximum Theorem)

If $f(\bar{x}, \bar{\theta})$ is continuous in $\bar{x}$ and $\bar{\theta}$, then

$$x^* = \mathrm{argmax}_{\bar{x} \in A} f(\bar{x}, \bar{\theta})$$

is continuous in θ.

We can apply this theorem to the Cournot game. First, we notice that it is not profitable to produce $q_i > 1$ goods for each player, so we can restrict the strategy space to the convex set $[0, 1]$. Next, the function is quadratic and is continuous and concave. Thus, there is a Nash equilibrium.

Remark. *You can generalize the existence of Nash equilibrium to 'quasi-concave games'. You can also put futher restrictions to the guarantee the uniqueness of the Nash equilibrium (Rosen).*

Justification

Nash equilibrium assumes that players are rational interspective agents. That is, there is a **common knowledge of rationality**, which means that each player is not only rational and knows that each other player is rational, and that each other player is aware that each other player is aware is each other player is rational, inductively ad infinitum.

There is also no existence of binding arguments.

Focal points. A Nash equilibrium is a self-enforcing outcome. If the same game is played multiple times, then through a best-response dynamic, (supposing it converges), it converes to a Nash equilibrium.

Taken from Randy's notes:

We know that action of an agent in a Nash equilibria is a rational response to the equilibrium profile of the other users, but how do we coordinate Nash equilibrium with other players if we have just met. In other words, How do we know that other agents will play this profile and why they chose this profile to play? There are couple of justifications that can be used based on the problem we are solving. Here we talk about some possible ones:

- **Nash equilibrium as a self-enforcing outcome.** This first justification works for so-called one-shot games that players just meet and play a game just once. In this setting, what will lead players to a NE? One possible approach of justification is a non-binding agreement between players. If this non-binding agreement is a NE, none of these players have any incentive to break the rule and deviate from it. In this sense, each player enforce himself to follow the agreement. Note that this justification assumes rationality of players.

- **Nash equilibrium as the outcome of long-run learning.** One other idea of justification NE comes as a re-sult of learning process of players when they have the chance to play one game many times. We assume that players can experiment with different actions to seek possible actions to improve their payoff functions. Such a process might not reach a NE necessarily, but if it reaches a steady state where players can't improve their actions given what others are playing, then this steady-state is necessarily a NE. In this justification, we can assume that each player does not have full information about payoff function and rationality of other players and may learn enough about them by playing the game repeatedly and reaching a NE. One possible downside of this justification is that players might deviate from their learning in order to fool other players.

- **Nash equilibrium as a result of lots of thinking.** Nash equilibrium can also be justified when players put a lot of effort to compute how other people might play the game before actually starting the game.

Note that although these justifications might work in some certain settings, justification becomes moresophisticated when there are more than one NE, where equilibrium selection is needed.

Lecture 3: Finite Games

Recap of last lecture

1. Games with continuous strategy spaces.

2. Best responses

3. Existence of Nash Equilibria for concave games.

Recall example 1.12, we do not have a Nash equilibrium for this. However we can introduce mixed strategies.

Definition 3.1 (Mixed Strategy)

A **mixed strategy** is a probability distribution over S_i. Let σ_i denote a mixed strategy for player i. $\sigma_i(s_i) \stackrel{\text{def}}{=}$ probability of playing s_i. If there is $\sigma_i(s_i) = 1$, this is called a pure strategy.

Given a set of mixed strategies in a game, let

$$\bar{\sigma}(\bar{s}) = \prod_{r=1}^{n} \sigma_r(s_r).$$

That is, the decisions of each player are independent. Thus, the payoff of player i (by abuse of notation)

$$\pi_i(\bar{\sigma}) = \sum_{\bar{s} \in \prod S_i} \bar{\sigma}(\bar{s}) \pi_i(\bar{s})$$

Example 3.2

Return to the game in example 1.12 with reward matrix

Attacker\User	A	B
A	(1,0)	(0,1)
B	(0,1)	(1,0)

Suppose the attacker (player 1) chooses A or B with 0.5 probability each, while the defender (player 2) chooses A at 0.25 probability and B with 0.75 probability. Calculate the expected payoff of the game.

We have

$$\pi_1(\bar{\sigma}) = 0.5 \cdot 0.25 \cdot 1 + 0.5 \cdot 0.75 \cdot 1 + \text{terms involving 0's} = 0.5.$$

Let Σ_i denote the space of all mixed strategies for player i. For a finite game, we can plot this as a k-dimensional simplex $x_1 + \cdots + x_k = 1$. We then view games with mixed strategies as a game where strategy set is now Σ_i with payoff $\pi_i(\bar{\sigma})$. We can thus define Nash Equilibrium for a finite game with mixed strategies to be the Nash Equilibrium of this game.

Definition 3.3 (Nash Equilibrium w/ Mixed Strategies)

A Nash Equilibrium $\sigma_1^*, \ldots, \sigma_n^*$ is a set of probability distribution such that

$$\pi_i(\sigma_i^*, \bar{\sigma}_{-i}^*) \geq (\sigma_i, \bar{\sigma}_{-i}^*)$$

for all σ_i.

Theorem 3.4

Nash equilibria exists for finite games with mixed strategies.

Proof. We show that the game with continuous strategy sets is a concave game.
Σ_i is a closed, bounded, convex set.
$\pi_i(\bar{\sigma})$ is a continuous function in $\bar{\sigma}$
$\pi_i(\sigma_i, \bar{\sigma}_{-i}^i)$ is concave (linear) in σ_i for all σ_{-i}. Since this game is concave, the Nash equilibrium exists by theorem 2.8. ✿

What is the Nash equilibrium for the attacker-defender scenario? Let us say that 1 and 2 chooses channel A with probability p and q respectively. Then 1 seeks to maximize

$$pq + (1-p)(1-q) = p(2q-1) + (1-q).$$

in p. Obviously, the attacker wants to choose the channel such that the defender has $1/2$ chance of choosing, or any random distribution if the defender choose each channel with $1/2$. 2 seeks to maximize

$$q(1-p) + p(1-q)$$

in q.
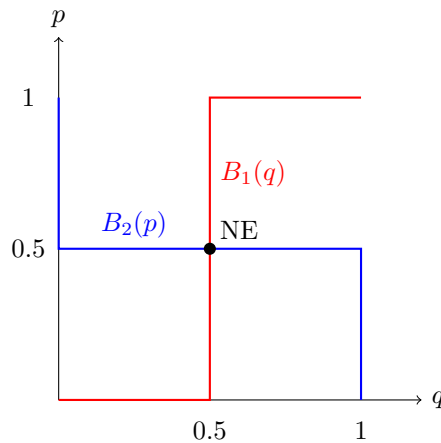
Proposition 3.5

In any mixed strategy nash equilibrium, each player must be getting the same payoff from any action it plays with positive probability.

Proof. The best response to any strategy is to put all the weight on the response with highest payoff. ✿

Example 3.6 (Public Good Game)

Suppose N people, each receives a value of $v > 0$ if any one of them provides a 'good' at cost of $c > 0$ and get 0 payoff otherwise. Call the strategy set "y" and "n". To keep things interesting, assume $c > v$.

For example, filing a public report, or community networks (set up some kind of wireless access point). In this game the payoff is

$$\pi_i(s_i, \bar{s}_{-i}) = \begin{cases} v - c, & \text{if } s_i = \text{y}, \\ v, & \text{if } s_i = \text{n}, \exists \text{s}_\text{j} = \text{y}, \\ 0, & \text{if } s_j = \text{n} \forall \text{j}. \end{cases}$$

There are n different Nash equilibria. Each corresponds to one person providing the good and no one else does. However, there is no clear reason to pick one person to offer the good than another. We wonder if there is a symmetric mixed strategy equilibrium. I.e.

A strategy where each player chooses "y" with probability p.

Therefore, we are solving for a p such that for each agent alone, the payoff for any person switching from y to n does not matter. The payoff for choosing y is $v - c$. The payoff for choosing n is

$$v - v(1 - p)^{n-1}.$$

Therefore we solve for

$$v - c = v - v(1-p)^{n-1} \implies c = v(1-p)^{n-1} \implies 1 - \sqrt[n-1]{\frac{c}{v}} = p.$$

At this equilibrium, notice that the probability that the good is provided is $1 - (1 - p)^n = 1 - (c/v)^{1+1/n} \to 1 - c/v$ as $n \to \infty$. This is known as a 'free-rider problem', since as more people enter the game, they all want to contribute less.

Mixed strategies can apply to games with infinite strategy spaces.

Example 3.7

$S_r = [0, 1]$, then the strategy set (with mixed strategies) are all probability measures on $[0, 1]$.

Definition 3.8 (Continuous game)

A **continuous game** is a game for which the strategy spaces are non-empty, compact subsets of $\mathbb{R}^n$ and payoffs that are continuous in $\bar{s}$.

Theorem 3.9 (Glicksberg)

A continuous game has a mixed strategy Nash equilibrium.

Remark. *Recall we have a pure strategy Nash equilibrium for concave games. If we remove the concave assumption, we will need mixed strategies to reduce the game into a concave game.*

Defences and critiques of Mixed strategies

- **A mixed strategy equilibrium is not as predictive as a pure strategy.** In our attacker-defender game, the mixed-strategy equilibrium gives no additional information (entropically speaking) on which channel the players will play.

- **There are no 'mixed strategies' for a single play of a game.** If we just do the game once, we will not be able to observe the proabiltiy distribution. If the game is played multiple times over, we can get statistics to infer probability distributions.

- **In some games, mixed strategies can seem natural.** e.g. attacker defender, rock paper scissors.

- **Pure strategies might be prefered.** E.g. Cournot competitions.

Lecture 4: Zero sum games and infinite population games

Definition 4.1 (Zero-sum game)

A two player game is said to be **zero-sum** if for any pure action profile $\bar{s}$,

$$\pi_1(\bar{s}) + \pi_2(\bar{s}) = k$$

for a constant k. WLOG we can assume $k = 0$.

Remark. *This dates back to 1928 (von Neumann).*

Example 4.2

The following are examples of zero-sum games. These are "strictly competitive" settings.

- Attacker/defender

- Chess

- Futures/options contracts in finance

- Cake-cutting

- Worst case analysis in computer science.

Example 4.3 (Cake Cutting)

cutter\chooser	Larger	Smaller
equal	(50,50)	(50,50)
unequal	(40,60)	(60,40)

Now we suppose

$$\bar{\sigma}_1 = \begin{bmatrix} \sigma_{1,1} \\ \sigma_{1,2} \\ \vdots \\ \sigma_{1,m} \end{bmatrix}$$

be a mixed strategy for player 1. Then $\bar{\sigma}_1^T A$ denotes the expected payoff of player 1 if player 2 chooses the corresponding column (pure strategy). Therefore if $\bar{\sigma}_2$ denotes the mixed strategy of player 2, we have the expected value of player 1 is $\bar{\sigma}_1^T A \bar{\sigma}_2$.

Definition 4.5 (Maxi-min/min-max strategies)

We define

$$V_1^* \overset{\text{def}}{=} \max_{\bar{\sigma}_1} \min_{\bar{\sigma}_2} \bar{\sigma}_1^T A \bar{\sigma}_2.$$

This denotes the largest worst-case payoff for player 1. Similarly we define

$$V_2^* \overset{\text{def}}{=} \min_{\bar{\sigma}_2} \max_{\bar{\sigma}_1} \bar{\sigma}_1^T A \bar{\sigma}_2.$$

Proposition 4.6

We have

$$V_1^* \leq V_2^*.$$

Proof. We have $V_1^* \leq \min_{\sigma_2} B_1(\bar{\sigma}_2)^T A \bar{\sigma}_2 = V_2^*.$ ❀

Theorem 4.7 (Minimax)

For any 2 player zero-sum game,

- $V_1^* = V_2^*$

- The set of Nash equilibria is $\{\bar{\sigma}_1^*, \bar{\sigma}_2^*\}$ for $\bar{\sigma}_1^*$ maximin and $\bar{\sigma}_2^*$ minimax.

Proof. Let $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ be a Nash equilibrium. Then

$$\tilde{\sigma}_1 = \arg\max_{\sigma_1} \sigma_1^T A \tilde{\sigma}_2$$

and

$$\tilde{\sigma}_2 = \arg\min_{\sigma_2} \tilde{\sigma}_1^T A \sigma_2.$$

So that

$$V_2^* = \min_{\sigma_2} \max_{\sigma_1} \sigma_1^T A \sigma_2 \leq \max_{\sigma_1} \sigma_1^T A \tilde{\sigma}_2 \stackrel{\text{nash}}{=} \min_{\sigma_2} \tilde{\sigma}_1^T A \sigma_2 \leq \max_{\sigma_1} \min_{\sigma_2} \sigma_1^T A \sigma_2 = V_1^*.$$

The proposition $V_1^* \leq V_2^*$ completes the proof. ❀

The compute this equilibrium, we find

$$\max_{\sigma_1} v,$$

such that $[\sigma_1^T A]_j \geq v \forall j$, subject to $\sigma_{1,1} + \sigma_{1,2} + \cdots + \sigma_{1,m} = 1$ and $\sigma_{1,j} \geq 0$. This is a linear program, so we can compute this in polynomial time. All hail Dinic and push relabel.
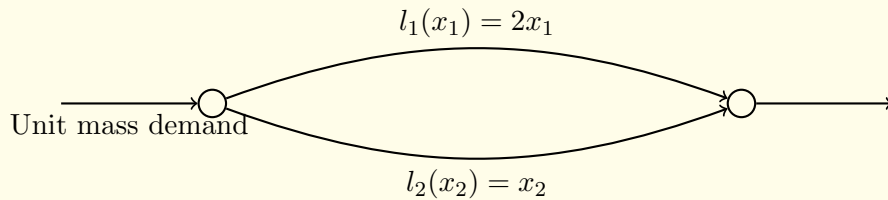

Definition 4.8 (Infinite Population Games)

An **infinite population game** is a game where the set of players is infinite.

Example 4.9 (Infinite Population Game)

Consider a game where the set of players is the continuum $[0, 1]$, such that each player is infinitesimal. This is known as 'non-atomic player with a mass of 1'.


Example 4.10 (Traffic Equilibrium)

Suppose we have a unit mass of traffic going through two routes with l_1 or l_2 cost each. We find the nash equilibrium subject to $x_1 + x_2 = 1$, $x_1, x_2 \geq 0$.

This is very intuitive. Relating $l_1(x_1) = l_2(x_2)$ gives $x_1 = 1/3, x_2 = 2/3$.

Remark. *This is also known as a **wardrop equilibrium**. The cost on all the paths used is the same. The cost on all the paths that are not used is greater than on the used paths.*

Consider the following 'traffic diagram'.

$$l_1(x_1) = x_1$$

Unit mass demand

$$l_2(x_2) = 1$$
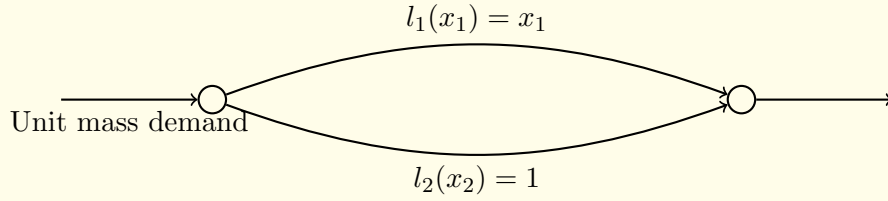
The wardrop equilibrium here is $x_1 = 1, x_2 = 0$. This is interesting, as we can consider the social cost, defined as the expected value of the cost for the players. In this case, everyone needs to pay a cost of 1.

However, let us suppose that there is a planner who would like to minimize this total cost

$$\min_{x_1} x_1 l(x_1) + x_2 l(x_2) = \min x_1^2 + (1 - x_1).$$

Completing the square gives a minimal cost when $x_1 = 1/2$ with a total cost of 3/4. However, this is not a stable minima. There is an incentive to deviate from the assigned route. Let us define the price of anarchy

$$P \stackrel{\text{def}}{=} \frac{\text{cost with equilibrium}}{\text{cost with planner}}.$$

$P = 1$ is the best outcome. But in this case we have a price of $1/(3/4) = 4/3$. Consider the following diagram.


x_1

1

Unit mass demand

1

x_4
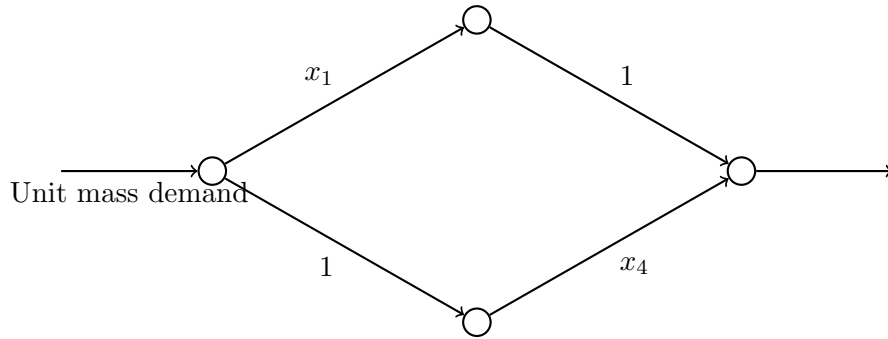
Now, the equilibrium is when $x_1 = x_4 = 1/2$, with a total social cost of 3/2. Suppose we now create a new road.

x_1

1

Unit mass demand

0

1

x_4
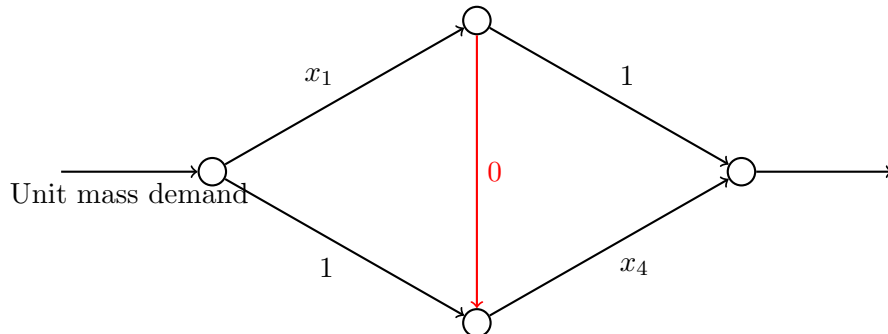
Now, the best road is to go up then through the red edge. This means the new equilibrium now is $x_1 = x_4 = 1$ with a total social cost of 2. This is known as **Braess's paradox**, where adding more resources to the system causes the social cost to increase.

Existence of Wardrop Equilibria

. We want to determine when wardrop equilibria exist in the following graph.
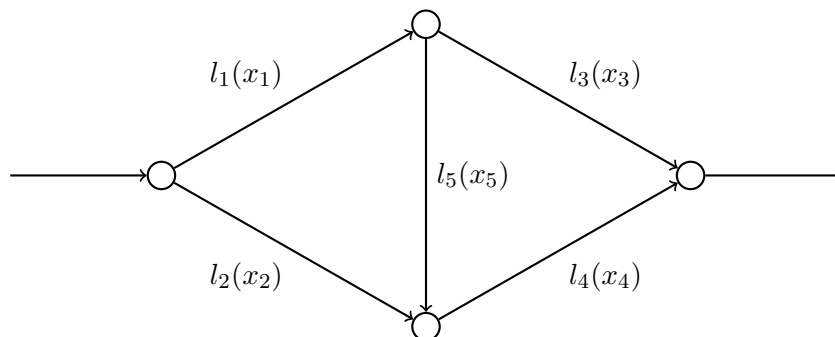
Theorem 4.12

Assume that each $l_i(x_i)$ is non-negative, non-decreasing, and differentiable. Then a unique Wardrop Equilibrium exists.

Sketch of proof. Let P be the set of all paths from the source to the destination. Let x_p be the number of users on path i, and E be the set of edges. We now have an optimization

$$\min_{\{x_p : p \in P\}} \sum_{i \in E} \int_0^{\text{total traffic on } i} l_i(z)dz$$

subject to $\sum_{p \in P} x_p = 1, x_p \geq 0 \forall p \in P$.

We claim that this optimization problem has a unique solution, and this solution is the Wardrop equilibrium. The argument boils down to showing that the function is convex, thus has a unique minimum.

Let λ be a Lagrange Multiplier for the condition $\sum_{p \in P} x_p = 1$. Then the first order optimality shows that

$$\sum_{\text{edges in path } p} l_i(\sum_{\text{paths } p \text{ using } i} x_p) \geq \lambda$$
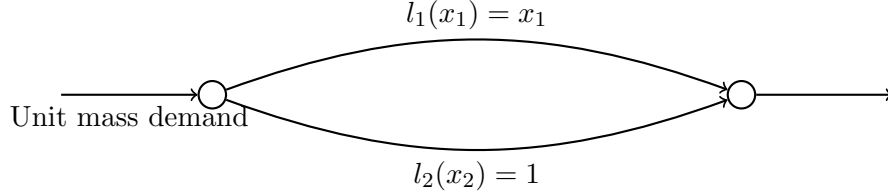
with equality when $\sum x_p > 0$. ❀

Lecture 5: Pricing

Recap

1. Zero sum games

2. Infinite population games - non-atomic traffic flow games.

Recall in the last lecture we had in example **??**:

$$l_1(x_1) = x_1$$

Unit mass demand

$$l_2(x_2) = 1$$

Which had conflicting social optimal and equilibrium. However, we can introduce a price of $t_1 = 1/2$ to the upper route. i.e. $l_1(x_1) = x_1 + 1/2$ In this case, the equilibrium is when $x_1 = x_2 = 1/2$. From an observer standpoint, we can disregard the tolls and get the social optimum strategy. However, for the drivers' standpoint, they still incur the full cost of 1.

From an economic standpoint, we can disregard the cost of the toll. This is because the planner gets the toll fee of $(1/2)(1/2) = 1/4$, but then this can be redistributed to everyone.

Remark. *You can think of it 'transfering' part of the cost of the bottom road to the top road to have 3/4 cost each.*

We revisit the original model without tolls. Suppose that x_1 drivers are taking the top road and we consider the next driver $(x_1 + \epsilon?)$.

We define the social cost of the driver taking the top road as

$$\frac{d}{dx_1} x_1 l_1(x_1) = x_1 l_1'(x_1) + l_1(x_1).$$

Previously each driver in the selfish equilibrium compares the costs $l_1(x_1)$ and $l_2(x_2)$. This is the second term in the cost. The first term is called **externality**.

Theorem 5.1 (Pigovian tax)

Let $\tilde{l}_j(x) = x l_j'(x) + l_j(x)$. Under the new graph with costs $\tilde{l}_j$, the Wardrop eqiulibrium is the social optimal strategy with original costs.

Corollary 5.2: The optimal toll cost for each road is $x l'(x)$. (up to a constant difference, but that is redistributed anyway.)

Finite number of users

These leads to models that are known as congestion games. (Rosenthal, 1973)

Remark. *Rosenthal studied IEMS in Northwestern.*

Definition 5.3 (Congestion Game)

A **congestion game** is a game with

- A set of players $R = \{1, 2, \ldots, n\}$

- A set of resources $M = \{1, 2, \ldots, m\}$

- For each $i \in R$, a set $S_i \subseteq M$ denoting the set of resources i can use.

- Each resource has a cost of $c^j(x_j)$, where x_j is the total number of players using the resource. The cost for player i is $c_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(x_j)$.

Example 5.4
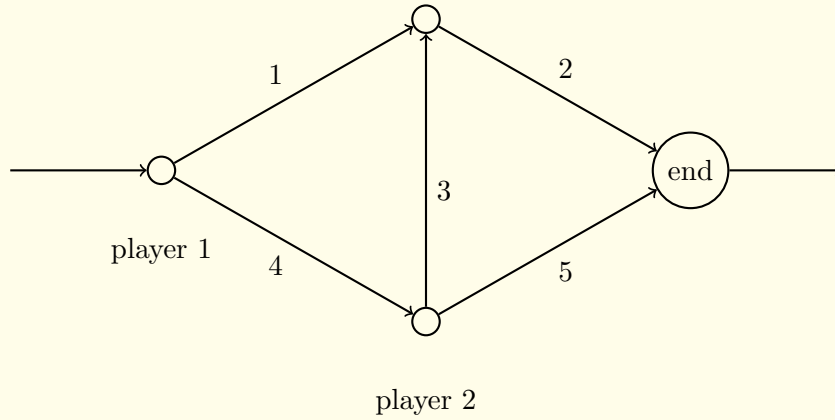
player 2

Consider this game where both players want to reach the end. The roads are numbered 1 to 4. We have

$$
\begin{aligned}
M &= \{1, 2, 3, 4, 5\}, \\
S_1 &= \{(1, 2), (4, 5), (4, 3, 2)\}, \\
S_2 &= \{(3, 2), 5\}.
\end{aligned}
$$

Example 5.5 (El Farol Bar Problem)

10 people decide to go to the bar or go home. If 6 or fewer go they have more fun than staying home. If 7 or more go, the bar is too crowded and they have less fun.

How do we model this as a congestion game? We can model the resources as

$$\{\text{bar}, \text{home}\}.$$

The strategy set of each player is

$$S_i = \{\text{bar}, \text{home}\}$$

Under this, the payoff is

$$\pi_i(s_i) = \begin{cases} 1, & \text{if } x_{\text{bar}} \le 6 \text{ and } s_i = \text{bar} \\ -1, & \text{if } x_{\text{bar}} \ge 7 \text{ and } s_i = \text{bar} \\ 0, & \text{if } s_i = \text{home} \end{cases}$$

Theorem 5.6 (Rosenthal)

Every congestion game has a pure strategy Nash equilibirum.

We will prove this result later.

Definition 5.7 (Potential Game)

A game $G = (R, \{S_r\}, \{\pi_r\})$ is a **potential game** if there is a function

$$P : \prod_{r \in R} S_r \to \mathbb{R}$$

such that for all $r \in R$, $\bar{s}_{-r}$, and $x, z \in S_r$,

$$\pi_r(x, \bar{s}_{-r}) - \pi_r(z, \bar{s}_{-r}) \ge 0 \iff P(x, \bar{s}_{-r}) - P(z, \bar{s}_{-r}) \ge 0$$

with equality on the left iff equality on the right. In this case, we call P a(an) **(ordinal) potential**. If

$$\pi_r(x, \bar{s}_{-r}) - \pi_r(z, \bar{s}_{-r}) = P(x, \bar{s}_{-r}) - P(z, \bar{s}_{-r}),$$

we can P an **exact potential**.

That is, a player benefits from changing strategies if and only if they can increase the potential. Potential games exist.

Example 5.8 (Cournot competition potential)

R firms choose $q_i \in [\epsilon, q_{\max}]$. The payoff for each company is

$$\pi_i = q_i(P(Q) - c)$$

where $Q = \sum q_r$. We let

$$\Psi(q_1, \ldots, q_R) = \left(\prod_r q_r\right)(P(Q) - c).$$

Then

$$\Psi(q_i, \bar{q}_{-i}) - \Psi(\tilde{q}_i, \bar{q}_{-i}) = \left(\prod_{r \neq i} q_r\right)(q_i(P(Q_{-i} + q_i) - c) - \tilde{q}_i(P(Q_{-i} + \tilde{q}_i) - c))$$

$$= \left(\prod_{r \neq i} q_r\right)(\pi_i(q_i, \bar{q}_{-i}) - \pi_i(\tilde{q}_i, \bar{q}_{-i})).$$

This is *almost* a potential game for $q_i \in [0, q_{\max}]$. Stricter conditions on P can give potentials.

Lemma 5.9

Every congestion game is a potential game

Sketch. We need to construct a potential for the congestion game. We claim

$$P(\bar{s}) = \sum_{j \in M} \sum_{1}^{x_j} C^j(k)$$

is a potential. Intuitively, take each resource j, then sum C^j all the way from 1 user to x_j users.

Suppose user i using one s_i and wants to switch towards using $\tilde{s}_i$. The change in cost is

$$\pi_i(s_i, \bar{s}_{-i}) - \pi_i(\tilde{s}_i, \bar{s}_{-i}) = C^{s_i}(x_{s_i}) - C^{\tilde{s}_i}(x_{s_i}).$$

This is exactly the change in potential. (Apply an inductive argument). ✿

Theorem 5.10

Every potential game has a Nash equilibrium if the potential obtains its maximum for some strategy profile.

Proof. There is a maximal value for the potential function. Each player picks the strategy that maximize this potential function. This must be a Nash equilibirum. ✿

Corollary 5.11: Every finite potential game has at least one pure strategy Nash equilibrium. Every game with a compact strategy set has at least one pure strategy Nash equilibrium.

Proof of Rosenthal. A congestion game has finite players and finite resources. Therefore it is a finite game. Since all congestion games are potential games, we are done. ❀

Nash equilibriums can be reached by using best response in potential games. I.e. choose one player and change his strategy to be the best response. The potential is strictly increasing, so has to be reached at some point.

Lecture 6: Best Response Dynamics

Recap

1. Pigovian tax

2. Congestion games

3. Potential game

Definition 6.1 (Sequential Best Response Dynamics)

Given a game G. We consider the following algorithm:

- set $k = 0$

- pick an initial profile $(s_1^0, \ldots, s_R^0)$

- pick one player i, update $s_i^{k+1} \in B_i(\bar{s}_{-i}^k)$.

- For every $j \neq i$ set $s_j^{k+1} = s_j^k$.

- This is the new strategy profile. $k{+}=1$ and repeat.

Proposition 6.2

If this converges, we have a Nash Equilibrium.

Example 6.3

Consider the game with the following payoff:

1\2	L	M	R
U	(10,10)	(2,2)	(1,1)
M	(7,8)	(6,3)	(5,9)
D	(6,6)	(5,2)	(8,4)

Starting from (M,M), we alternate between players 1 and 2 for best response to go to (M,R), (D,R), (D,L), (U,L), which is the Nash equilibrium.

In general, this forms a directed graph with $R\#$number of states edges. So this is the bound on the number of iterations. If you have N players and 2 strategies each, you can have $N2^N$ iterations.

Example 6.4

Consider the game with the following payoff:

1\2	L	M	R
U	(10,10)	(0,0)	(0,0)
M	(0,0)	(0,1)	(1,0)
D	(0,0)	(1,0)	(0,1)

We embedded the attacker defender payoff in the payoff, so this is actually a loop; no way to leave to get the Nash equilibrium.

Definition 6.5 (Parallel Best Response)

We consider the the best response dynamic, except for all i, we update $s_i^{k+1} = B_i(\bar{s}_{-i}^k)$ simutaneously.
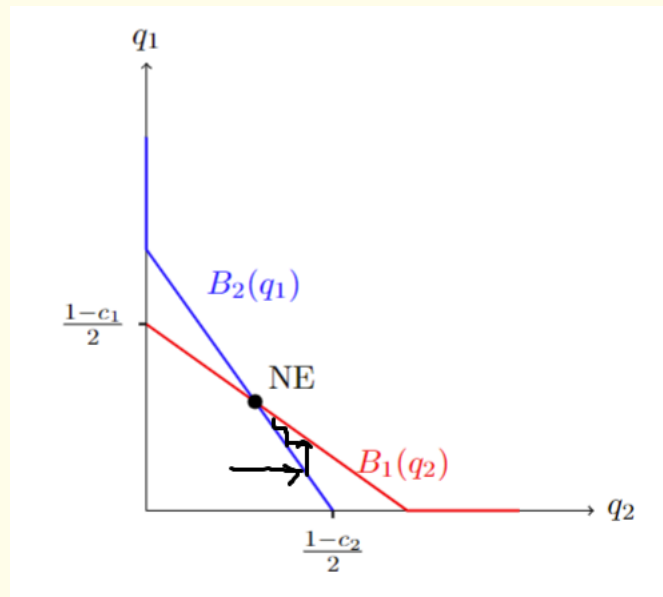
Example 6.6

This is not guaranteed to converge (even if iteratively the sequential best response dyanmic gives convergence). The game with payoff
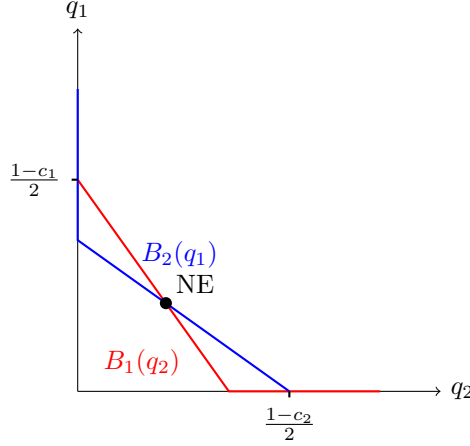
(10,10)	(0,0)
(0,0)	(10,10)

does not converge with parallel response dynamic if you start in the wrong corner.

Example 6.7 (Cournot competition convergence)

However, if we swap

The middle nash equilibrium point becomes unstable and the ends become stable.

Definition 6.8 (Stable and unstable equilibria)

We call the equilibrium stable best response dynamic converges to the equilibrium and unstable otherwise.

Definition 6.9 (Strategic complements)

This refers to a game where the value of a given action increases as other agents take that action.

Example 6.10 (Language game)

We have N people and two languages, French and English. The payoff of choosing a language is equal to the number of people choosing the language.

Example 6.11 (Power control)

There are 2 transmitters. Each transmitter chooses the power level $P_i \in [0,1]$. The payoff function

$$\pi_i(P_i, P_{-i}) = \begin{cases} 1 - cP_i, & \text{if } \frac{P_i}{P_{-i}+N} > \gamma \\ 0, & \text{otherwise.} \end{cases}$$

With c chosen to be small enough that it is interesting.

Definition 6.12 (Increasing differences)

Let $X \subseteq \mathbb{R}$, and T a partially ordered set. Then $f : X \times T \to \mathbb{R}$ has **increasing differences** if $\forall x' \geq x, t' \geq t$

$$f(x', t') - f(x, t') \geq f(x, t) - f(x, t).$$

Remark. *If $T \subseteq \mathbb{R}^k$ (partial order is $\geq$ in every entry of the vector) and f is C^2, this condition is equivalent to*

$$\frac{\partial^2 f}{\partial x \partial t_i} \geq 0$$

for all x, t_i.

> **Theorem 6.13 (Topkis)**
>
> Let $X \subseteq \mathbb{R}$ compact and T partially ordered. Let $f : X \times T \to \mathbb{R}$ be continuous in X and has increasing differences. Then
>
> - $x(t) \stackrel{\text{def}}{=} \arg\max_{x \in X} f(x, t)$ is non-empty.
>
> - $x(t)$ has a largest and smallest element $\bar{x}(t)$ and $\underline{x}(t)$ respectively.
>
> - $\bar{x}(t)$ and $\underline{x}(t)$ are increasing in t.

Proof. The first two statements are a consequence of continuity and compactness, and the extreme value theorem. The last statement probably uses some kind of measure theory. ✿

> **Definition 6.14 (Super Modularity)**
>
> A game $G = \{R, \{S_r\}, \{\pi_r\}\}$ is **supermodular** if for all $r \in R$,
>
> - S_r is a compact subset of $\mathbb{R}$.
>
> - π_r is continuous in s_i and s_{-i}
>
> - π_r has increasing differences in $s_i, \bar{s}_{-i}$.

Remark. *This is not the most general definition. But this is good enough.*

Given a supermodular game, we get that the best response $B_i \stackrel{\text{def}}{=} \arg\max_{s_i \in S_i} \pi_i(s_i, \bar{s}_{-i})$ is non empty, has a largest and smallest element $\bar{B}_i$, $\underline{B}_i$, and these are increasing in $\bar{s}_{-i}$. This captures the idea of complementarity. If other agents increase their strategies, the best response is to also increase.

> **Example 6.15**
>
> Let $s^0 = (s_1^0, \ldots, s_R^0)$ where s_i^0 is the largest element in S_i. Starting here, we run parallel best response using $\bar{B}_i$ for all i. We show that this converges to a Nash equilibrium.

Proof. We show that $s^k \leq s^{k-1}$ for all k. Then, by MCT, this converges to a Nash equilibrium. To show this, we proceed by induction. The case $s^1 \leq s^0$ is trivial by the definition of s^0. Then if $s^{k-1} \leq s^k$, then

$$s_i^{k+1} = \bar{B}_i(s_{-i}^k) \leq \bar{B}_i(s_{-i}^{k-1}) = s_i^k$$

by Topkis's theorem. ✿

Remark. *We can also start from the smallest element, then work our way up to a (possibly different) Nash equilibrium. If all the strategy sets are intervals, we can consider the box that has the two vertices with the two Nash equilibria. It is not hard to see that any other Nash equilibrium has to be inside the box, by the increasing difference property. Hence, if the two equilibria are the same there is a unique equilibria.*

We have N firms, each firm chooses their price p_i. The demand for firm i is

$$D_i(p_i, \bar{p}_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j,$$

where $a_i, b_i, d_{ij} \geq 0$. The payoff of the firm is

$$\pi_i(p_i, \bar{p}_{-i}) = p_i D_i(p_i, \bar{p}_{-i}).$$

Then we have

$$\frac{\partial \pi_i}{\partial p_i} = a_i - 2b_i p_i + \sum_{j \neq i} d_{ij} p_j,$$

$$\frac{\partial \pi_i}{\partial p_i \partial p_j} = d_{ij} \geq 0.$$

Subject to some technicalities on the prices the firms can choose, this is a supermodular game.

Lecture 7: Fictitious play

Logistics

- Posted project description
- HW will be posted tonight

Recap

1. Best response dynamics
2. Supermodular games

The idea of fictitious play is that each player assumes the opponents are using a 'stationary mixed strategy'. Instead of best response to last action, we estimate the mixed strategy and respond to the estimated strategy.

Definition 7.1

Let $\{G_t\}_{t \in \mathbb{N}}$ be a sequence of the same games (i.e same players) indexed in t (and played in this order). Let S_i^t be the (pure) action of player i at time t. Let $\gamma_i^t(s_i)$ be the count $\#\{s_i^{t' < t} = s_i\}$. Therefore, we can approximate the probability distribution

$$p_i^t(s_i) = \frac{\gamma_i^t(s_i)}{t}.$$

Each player in fictitious play chooses $s_i^t \in B_i(\bar{p}_{-i}^t)$.

Remark. *The trajectory is not unique.*

Remark. *At $t = 0$ we can assume that the strategy is arbitrary, or each player has an initial 'belief' of the other agents' mixed strategies and update accordingly.*

Example 7.2

Consider the game with payoff

	L	R
U	(3,3)	(0,0)
D	(4,0)	(1,1)

There is a Nash equilibrium of (1,1). The game progresses as

$t = 0$	$p_1(U, D)$	$p_2(L, R)$	(U, L)
$t = 1$	(1,0)	(1,0)	(D, L)
$t = 2$	(1/2,1/2)	(1,0)	(D, L)
$t = 3$	(1/3,2/3)	(1,0)	(D, L)
$t = 4$	(1/4,3/4)	(1,0)	(D, R)

And (D, R) continues after $t > 4$.

This equilibrium is found without requiring knowledge of the other's payoffs.

Definition 7.3 (Convergence of strategy profiles)

A sequence of strategy profiles $\{\bar{s}^t\}$ **converges** to a profile $\bar{s}$ if there exists T such that $\bar{s}^t = \bar{s}$ for all $t \geq T$. We denote this $\bar{s}^t \to \bar{s}$.

Proposition 7.4

Under fictitious play,

1. A strategy profile that converges must converge to a pure strategy Nash Equilibrium.

2. If for some time T, $\bar{s}^T = \bar{s}^*$, a strict Nash equilibrium, then $\bar{s}^t = \bar{s}^*$ for all $t > T$.

Proof. For the first statement, we play the game enough so that the empirical probability distribution concentrates approaches a pure strategy. Then by definition this is a pure strategy Nash equilibrium.

For the second statement, we note that

$$p_i^{T+1}(s_i) = \frac{\gamma_i^T(s_i) + I_{s_i = s_i^*}}{T + 1}$$
$$= (1 - \alpha)p_i^T(s_i) + \alpha I_{s_i = s_i^*},$$

where $\alpha = t/(t + 1)$. That means the new distribution is a convex (linear) mixture of the previous distribution and the Nash equilibrium. But for the Nash equilibrium and the previous distribution, the best response is s_i^* for each player i. Since the best response for the Nash equilibrium has a strict inequality, we can conclude that the best response for time $T + 1$ must also lead to the Nash equilibrium. Inductively, the strategy profile has to be the Nash equilibrium for all $t > T$. ✿

Definition 7.5 (Convergence of Empirical Beliefs)

We say that $\{\bar{s}^t\}$ converges to σ in the time average sense if

$$\lim_{t \to \infty} p_i^t(s_i) = \sigma(s_i)$$

for all i and $s_i \in S_i$.

Example 7.6

For the attacker defender problem, we can show that this converges to $(1/2, 1/2)$ for each player.

$t = 0$	$p_1(A, B)$	$p_2(A, B)$	(A, A)
$t = 1$	(1,0)	(1,0)	(A, B)
$t = 2$	(1,0)	(1/2,1/2)	(B, B)
$t = 3$	(2/3,1/3)	(1/3,2/3)	(B, A)
$t = 4$	(1/2,1/2)	(1/2,1/2)	(A, A)

and the cycle repeats. The empirical beliefs converges to (1/2,1/2), (1/2,1/2).

Theorem 7.7

Suppose that fictitious play converges in time average to σ. Then σ is a mixed-strategy Nash equilibrium.

Theorem 7.8

Fictitious play converges in time average for

1. A two player zero-sum game.

2. A two player game with at most two strategies for each player.

3. A game that is solvable by iterated strict dominance.

4. A potential game.

5. Some supermodular games.

Example 7.9 (Miscoordination)

Consider the game with payoff matrix

	A	B
A	(1,1)	(0,0)
B	(0,0)	(1,1)

There is a mixed strategy equilibrium where the players choose A and B with half probability each. There is a sequence

$t = 0$	$p_1(A, B)$	$p_2(A, B)$	(A, B)
$t = 1$	(1,0)	(0,1)	(B, A)
$t = 2$	(1/2,1/2)	(1/2,1/2)	(A, B)
$t = 3$	(1/3,2/3)	(2/3,1/3)	(B, A)
$t = 4$	(1/2,1/2)	(1/2,1/2)	(A, B)

In the limit, this indeed gives us the time average empirical beliefs. However, they are not getting the correct payoffs. (0 each time).

Example 7.10 (Non-convergence)

Consider the game with the reward matrix

	R	P	S
R	(0,0)	(1,0)	(0,1)
P	(0,1)	(0,0)	(1,0)
S	(1,0)	(0,1)	(0,0)

The unique Nash equilibirum is $(1/3, 1/3, 1/3)$. The sequence here starts with everyone best responding (R,S), (R,P), (S,P), (S,R), (P,R), (P,S), (R,S). Then the cycle length increases (R,S), (R,P), (R,P), (S,P), (S,P)...
So this does not converge (cycle lengths are too long).

Lecture 8: Evolutionary Game Theory

Logistics

- HW2 posted, due one week from today.

Evolutionary Game Theory dates back to 1973 (John Maynard Smith, George Price).

The idea is that we have a large population of potential players. For instance, different species/phenotypes of individuals. We pick two agents from this population to play a game. The winner reproduces faster.

Definition 8.1 (Evolutionary game theory terminology)

Let $G = (\{1,2\}, \{S_1, S_2\}, \pi_1, \pi_2)$ be a finite, symmetric game (i.e. $S_1 = S_2 = S, \pi_1 = \pi_2 = \pi$).
The **fitness** of a player playing strategy $s_i \in S$ is the expected payoff against a random member of the population.
A strategy s_j is said to **invade** strategy s_i at level $\epsilon \in [0,1]$ if ϵ of the population is type j and $1 - \epsilon$ of the population is type i.
An **evolutionary stable strategy** (ESS) is a strategy s_i if $\exists \bar\epsilon > 0$ such that if any other strategy s_j invades at level $\epsilon < \bar\epsilon$, the fitness of type $i <$ the fitness of type j.

Remark. *The fitness depends on the distribution of strategy 'types'.*

Remark. *Intuitively, an evolutionary stable strategy can withstand invation of another strategy up to $\bar\epsilon$.*

Example 8.2

Consider this payoff for beetle sizes.

	smol	beeg
smol	(5,5)	(1,8)
beeg	(8,1)	(3,3)

Two smol beetles share the food. Beeg beetle takes most of the food. Two beeg beetles fight too much and the food is not as useful.
With only smol beetles, the fitness is 5. Now let us introduce $\epsilon > 0$ beeg beetles into the population. The smol beetles now have payoff

$$5(1 - \epsilon) + \epsilon = 5 - 4\epsilon.$$

But the beeg beetles have payoff

$$8(1 - \epsilon) + 3\epsilon = 8 - 5\epsilon > 5 - 4\epsilon$$

for $0 \le \epsilon \le 1$. Therefore, being smol is not an evolutionary stable strategy.
We change variables $\epsilon \mapsto 1 - \epsilon$ to see that being beeg is an evolutionary stable strategy.

Proposition 8.3

An evolutionary stationary strategy is a Nash Equilibrium.

Proof. Suppose a strategy s^* is an evolutionary stable strategy, then

$$(1 - \epsilon)\pi(s^*, s^*) + \epsilon\pi(s^*, s') > (1 - \epsilon)\pi(s', s^*) + \epsilon\pi(s', s')$$

for some ϵ small enough and all $s' \ne s^*$. Now we take $\epsilon \to 0$,

$$\pi(s^*, s^*) \ge \pi(s', s^*).$$

Remark. *If the Nash equilibrium is weak, then the other inequality is strict:*

$$\pi(s^*, s') > \pi(s', s')$$

The converse is also true, a strict nash equilibrium is an evolutionary stable strategy.

$$\text{Strict Nash} \subset \text{ESS} \subset Nash$$

These are distinct sets.

Example 8.4 (ESS but not strict)

	A	B
A	(0,0)	(6,0)
B	(0,6)	(3,3)

There is a weak Nash equilibirum (A,A). But this is an evolutionary stable strategy.

$$(1 - \epsilon)(0) + \epsilon(6) > (1 - \epsilon)(0) + \epsilon(3).$$

The reasoning is that adding in B benefits A.

Example 8.5 (Nash but not ESS)

	A	B
A	(4,4)	(0,4)
B	(4,0)	(3,3)

There is a weak Nash equilibirum (A,A). But this is an not an evolutionary stable strategy.

$$(1 - \epsilon)(4) + \epsilon(0) < (1 - \epsilon)(4) + \epsilon(3).$$

Adding in any small amount of B to the population of A causes a decrease in payoff.

Definition 8.6 (Mixed evolutionary stable strategy)

A mixed strategy σ^* is an ESS if there exists $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$,

$$\pi_i(\sigma^*, \epsilon\sigma' + (1 - \epsilon)\sigma^*) > \pi_i(\sigma', \epsilon\sigma' + (1 - \epsilon)\sigma^*),$$

for any other mixed strategy σ'.

	D	H
D	(3,3)	(1,5)
H	(5,1)	(0,0)

There are two symmetric strong Nash equilibria (D,H) and (H,D). However, these are not candidates for a pure ESS. We find the mixed ESS by letting a strategy p chance of playing D and $(1-p)$ chance of playing H. We solve for

$$p(3) + (1-p)(1) = p(5) + (1-p)0 \implies p = \frac{1}{3}.$$

This is the only candidate for an evolutionary stable strategy. Indeed,for any other strategy playing $(1, 1-q)$ invading $(1/3, 2/3)$ at level ϵ, the probability of meeting a dove strategy is

$$r = (1 - \epsilon)\frac{1}{3} + \epsilon q$$

The payoffs are

$$(\pi_i(\sigma^*, \epsilon\sigma' + (1-\epsilon)\sigma^*)) - (\pi_i(\sigma', \epsilon\sigma' + (1-\epsilon)\sigma^*)) = \frac{1}{3}r3 + \frac{1}{3}(1-r)1 + \frac{2}{3}r3 + \frac{2}{3}(1-r)0 - \ldots > 0$$

Just trust me bro.

By the linearity of expectations,

$$(\pi_i(\sigma^*, \epsilon\sigma' + (1-\epsilon)\sigma^*)) - (\pi_i(\sigma', \epsilon\sigma' + (1-\epsilon)\sigma^*))$$
$$= \epsilon(\pi(\sigma^*, \sigma') - \pi(\sigma^*, \sigma^*)) + (1-\epsilon)\epsilon(\pi(\sigma', \sigma') - \pi(\sigma', \sigma^*))$$

Assume that σ^* plays all actions with some probability, and that σ^* is a Nash equilibrium. Then

$$\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*),$$

as a different mixing of the same strategies. Therefore, it suffices to check that

$$\pi(\sigma^*, \sigma') - \pi(\sigma^*, \sigma^*) > 0$$

to see that it is an ESS.

Replicator Dynamics

Definition 8.8 (Replicator Dynamics)

Let $x_i(t)$ be the fraction of type i in the population at time t, so that $\sum_i x_i(t) = 1$ for all t. Let $f_i(\bar{x}(t))$ be the fitness of type i at time t. Under **replicator dyanmics**, we evolve the system through the differential equation

$$\frac{d}{dt}\frac{x_i}{x_j} = \frac{x_i}{x_j}[f_i(\bar{x}(t)) - f_j(\bar{x}(t))].$$

> **Proposition 8.9**
>
> Under replicator dynamics,
> $$\frac{d}{dt}x_i(t) = x_i(t)[f_i(\bar{x}(t)) - \phi(\bar{x}(t))],$$
> where
> $$\phi(\bar{x}(t)) \stackrel{\text{def}}{=} \sum_j x_j(t)f_j(\bar{x}(t))$$
> is the average fitness of the population at time t.

> **Theorem 8.10**
>
> If x^* is an ESS, then it is asymptotically stable for replicator dynamics.

Lecture 9: Dynamic Games

There are different game types. Static or Dynamic, Complete info or incomplete info, perfect info and imperfect info.

> **Definition 9.1 (Dynamic Game)**
>
> Everybody knows what a dynamic game is. You know it when you see it.

> **Example 9.2 (Entry Game)**
>
> A company (A) can enter the market or decide not to enter the market. Another company (B) is incumbent (already in the market) and can decide to cut or hold prices. The payoff matrix is as follows.
>
> | A\B | cut | hold |
> |---|---|---|
> | Don't Enter | (0,2) | (0,3) |
> | Enter | (-1,1), | (1,1) |
>
> The Nash equilibrium is Enter and Hold.

However, we can model the game using an extensive form representation. Each B represents a subgame. There are two subgames depending on A's action.
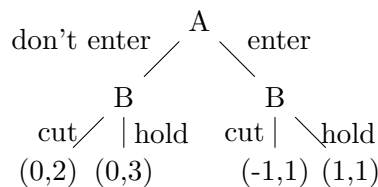
Therefore, we can expand the strategy set of B to be

| A\B	(cut, cut)	(cut, hold)	(hold, cut)	(hold, hold)
Don't Enter	(0,2)	(0,2)	(0,3)	(0,3)
Enter	(-1,1),	(1,1)	(-1,1),	(1,1)

Therefore, we get a Nash equilibria of (enter, (cut,hold)), (enter, (hold,hold)), (don't enter, (hold,cut)).

However, the first equilibrium is a bit weird. This is because we are implying that B will cut if A doesn't enter. We call this equilibrium **not sequentially rational**.

Definition 9.3 (Sequential rationality)

A pure strategy Nash equilibrium is said to be **sequentially rational**, or a **subgame perfect Nash equilibrium** if the agents' strategies continue to be a Nash equilibrium in every sub game.

We can check this using a backward induction algorithm. We first consider the strategies B can have at each subgame. If A doesnt enter, then B plays hold. If A enters, it does not matter which one. Now we propagate backwards. If B plays (hold,cut), then A's decision tree is

$$\text{don't enter} \diagup \overset{\text{A}}{} \diagdown \text{enter}$$
$$(0,3) \qquad (-1,1)$$
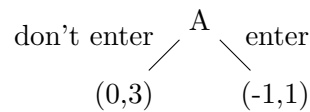
So (don't enter, (hold,cut)) is a subgame perfect Nash equilibrium.
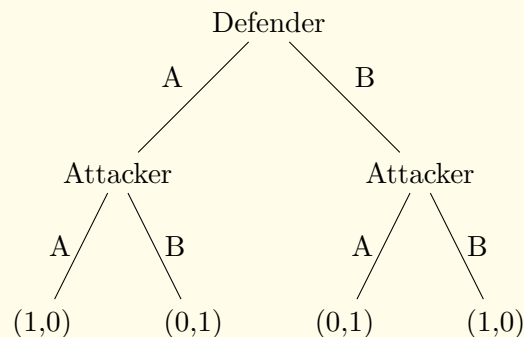

Theorem 9.4

For a finite dynamic game, the backwards induction algorithm yields exactly all the subgame perfect Nash equilibria.

Example 9.5

Return to example 1.12.

Attacker\User	A	B
A	(1,0)	(0,1)
B	(0,1)	(1,0)

We can change this into an extensive form -

It is obvious that the attacker has the advantage by choosing the strategy (A,B). The attacker is said to have a **second mover advantage** as the follower.

Example 9.6 (Stackelberg Competition)

Consider the sequential version of the Cournot competition. Suppose in addition that $c_1 = c_2 = c \in [0, 1]$. Now, let firm 1 plays first, then firm 2. Then firm 2's strategy is just to play $B_2(q_1) = \max\{(1 - q_1 - c)/2, 0\}$. Naturally we can define $q_2 = q_2(q_1) = B_2(q_1)$. Then q_1 is the best response to q_2 strategy

$$B_1(q_2) = \arg\max_{q_1 \geq 0}(1 - q_1 - q_2(q_1)) - cq_1.$$

solving this yields

$$q_1^* = \frac{1 - c}{2}, q_2^* = \frac{1 - c}{4}.$$

Let us compare this to the Nash equilibirum of the static Cournot competition $((1-c)/3, (1-c)/3)$. The total quantity produced is

$$3\frac{(1 - c)}{4} > 2\frac{(1 - c)}{3},$$

this is good for the consumer because the prices are lower. Firm one profits

$$\frac{1 - c}{8} > \frac{1 - c}{9}.$$

Form two profits

$$\frac{1 - c}{16} > \frac{1 - c}{9}.$$

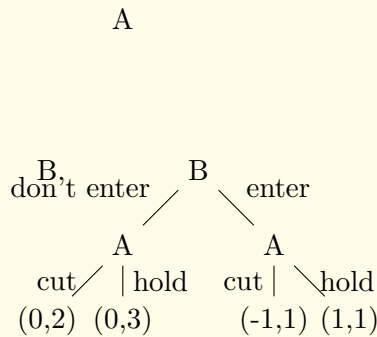This game has a first mover advantage.

TODO: too much work to draw these trees... but you probably understand the concept so whatever

Example 9.7 (Three player game)

The outcome at the subgame perfect Nash equilibrium is R,L,R, with a strategy

$$\{(R, L, (R, R, L))\}.$$

Example 9.8

A

B   B
don't enter /   \ enter
           A   A
      cut/ |hold   cut| \hold
     (0,2) (0,3) (-1,1) (1,1)

Lecture 10: Dynamic games (cont.)

Recap

1. Dynamic (Multi-stage) games.

2. Game tree (extensive form game)

3. Sequential rationality - subgame-perfect Nash equilibrium using backward induction.

Example 10.1 (3 period Bargaining)

Two players are splitting one dollar. They follow an alternating offer bargaining model. The game is terminated when one player accepts.

- Period 1: Player 1 makes an offer $(s_1, 1 - s_1)$. Player 2 can accept or reject the offer.

- Period 2: Player 2 makes an offer $(s_2, 1 - s_2)$. Player 1 can accept or reject.

- Period 3: The players get $(s, 1 - s)$.

Let $\delta \in (0, 1)$. This is the discounted payoff, i.e. how much they discount the future payoff. Therefore, we rewrite it as:

1: Player 1 makes an offer $(s_1, 1 - s_1)$. Player 2 can accept or reject the offer.

2: Player 2 makes an offer $(\delta s_2, \delta(1 - s_2))$. Player 1 can accept or reject.

3: The players get $(\delta^2 s, \delta^2(1 - s))$.

Work out the SPNE of the game.

Using backward induction, for period 2, player 1's response is

$$\begin{cases} \text{accept,} & \text{if } \delta s_2 \geq \delta^2 s \implies s_2 \geq \delta s, \\ \text{reject,} & \text{otherwise.} \end{cases}$$

So player 2 can offer $(\delta s, 1 - \delta s)$ with an intention to get $\delta(1 - \delta s)$. Therefore, player 2 rejects player 1's offer in period 1 if

$$1 - s_1 < \delta(1 - \delta s)$$

Player 1 thus makes, in period 1, an offer of

$$s_1 = 1 - \delta(1 - \delta s) = (1 - \delta) + \delta^2 s.$$

There is an extension of this model.

> **Example 10.2 (Infinite horizon alternate bargaining)**
>
> Two players are splitting one dollar. Let $\delta \in (0,1)$. Therefore, we rewrite it as:
>
>   1: Player 1 makes an offer $(s_1, 1-s_1)$. Player 2 can accept or reject the offer.
>
>   2: Player 2 makes an offer $(s_2, (1-s_2))$. Player 1 can accept or reject.
>
>   3: return to period 1.
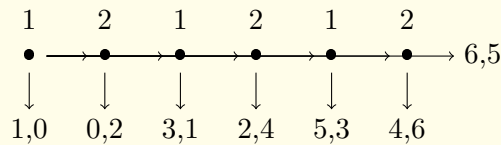>
> Work out the SPNE of the game.

There is no way to use backward induction of the game. However, there is a (cheesy) recursive way. Notice at period 3 onwards, the game is identical to the game starting at period one with splitting a δ^2 dollar bill.

Now suppose that there is an SPNE in period 3 that gives a payoff (a, b), then period 1 player 1 can get a payoff of

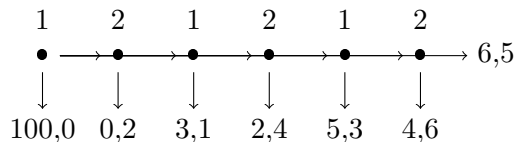$$1 - \delta(1 - \delta a) = a \implies a = \frac{1}{1+\delta}.$$

> **Example 10.3 (Centipede)**
>
> Consider:
>
>
>
> Intuitively, the sequence should continue sometime into the right. However, the SPNE actually results in (1,0)!

Or

In this case, we do not even consider what player 1 would do in the right, because the first choice dominates all possiblities. It is not certain that people think of subgames.

> **Definition 10.4 (Information set)**
>
> An information set is a collection of decision nodes that are indistinguishable for the player making the decision at the nodes.

We can also adjust the extensive form representation by drawing a circle/dashed line across indistinguishable subgames.
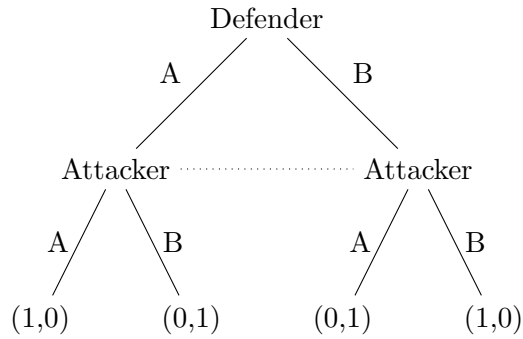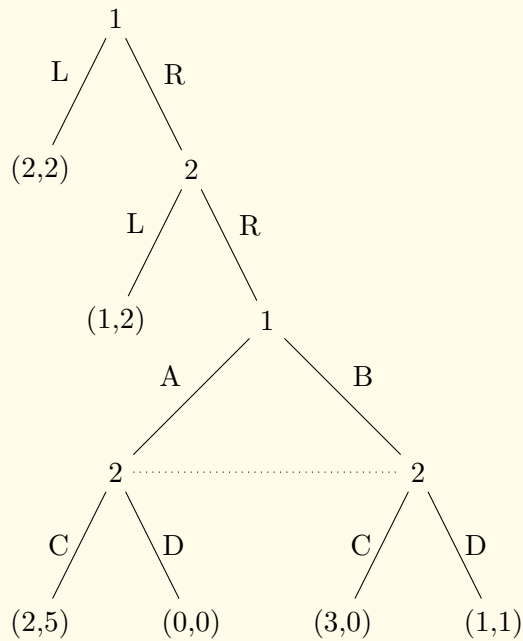
In this case, the Attacker cannot distinguish the subgames depending on what the Defender picked. The defender's information set is a singleton.

Definition 10.5 (Perfect information game)

If every information set is a singleton, we call the game a **game with perfect information.**

Example 10.6

We need to work on **proper subgames**, games that start at a singleton and does not break any information set. Starting with the second decision tree of 1, payoff matrix of

(2,5)	(0,0)
(3,0)	(1,1)

has a Nash equilibrium of (B,D). Then 2 chooses between (1,2) and (1,1), so would decide L in the tree. Finally 1's first decision between (2,2) and (1,2) will lead to a SPNE result of (2,2).

Example 10.7 (Bank Runs)

There are two investors. Each puts D dollars in the bank. This money is invested in a long term project.

- Period 1: The investors can withdraw or stay invested. Pulling out now will get a return of $2r$, $r \in (D, D/2)$.

- Period 2: If investment matures, the return is $2R$, $R > D$.

period 1	W	S
W	(r,r)	(D,2r-D)
S	(2r-D,D)	next stage

period 2	W	S
W	(R,R)	(D,2R-D)
S	(2R-D,D)	(R,R)

Putting (R,R) into the next stage, there are two nash equilibria (r,r) and (R,R).

Example 10.8 (Investment and Price Competition)

2 Firms first decide to invest and enter a new market at a cost of c_i. If both enter, they compete on price (Betrand competition). If one enters, they get to be a monopolist. If no one enters, the payoffs are zero.
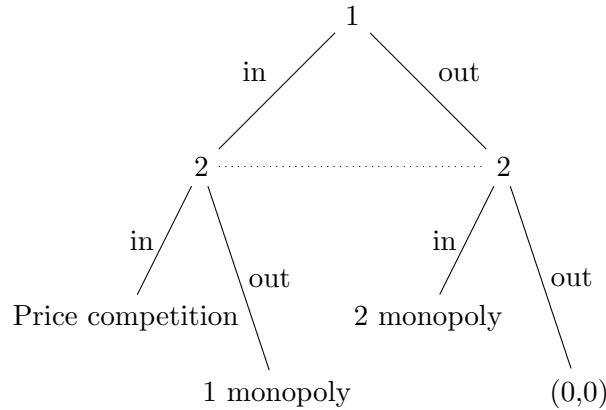
The price competition results in a payoff of $(-c_1, -c_2)$. The monopolist gains $(1/4 - c_i)$. We therefore have a payoff matrix in the first round:

period 1	in	out
in	$(-c_1, -c_2)$	$(1/4 - c_1, 0)$
out	$(0, 1/4 - c_2)$	$(0,0)$

If both $c_1, c_2 > 1/4$, then both will want to withdraw the market. If both of them are smaller, the nash equilibrium is that one of the firms enter the market as a Nash equilibrium. If exactly one of them has $c_i \leq 1/4$, then that firm enters the market.

Lecture 11: Repeated Games

Logistics

1. Midterm (three-quarter term) will be on Feb 25 2025.

2. One more homework will be posted before the midterm.

Recap

- Dynamic games with complete information

- Perfect information (knowing every action taken before the turn)

- Imperfect information (where there are simutaneous moves)

- Subgame perfect nash equilibria (SPNE) found through backward induction, assuming finite stages.

- A game with perfect information and finite strategies has an SPNE. All the SPNE can be found through backward induction.

We now work with repeated games. These are 'stage games' that players play repeatedly. The player's objective is their discounted payoff over time.

$$\sum_{t=0}^{T} \delta^t \pi_i(\bar{a}^t),$$

where $\delta \in (0, 1)$.

> **Example 11.1 (Prisoner's dilemna)**
>
> Consider the payoff matrix
>
> | | confess | deny |
> |---------|---------|--------|
> | confess | (0,0) | (4,-1) |
> | deny | (-1,4) | (3,3) |
>
> The dominant strategy is to confess (strictly dominant). However, there is a cooperative solution to both deny.

The cooperative solution does not occur in a single game. However, we might find a way to reach this stage through repeating this game multiple times.

Suppose we play the game N times. In the N-th stage, the payoff matrix is

	confess	deny
confess	(0,0)	$(4\delta^N,\text{-}1\delta^N)$
deny	$(\text{-}1\delta^N,4\delta^N)$	$(3\delta^N,3\delta^N)$

So both of them will confess. Then, in the previous stage, their payoff is whatever the game gives +0. So for any finitely repeated game of the prisoner's dilemna, the only SPNE is to confess every stage.

In a finite repeated game where the stage game has a unique Nash Equilibrium, the only SPNE is to repeat the Nash equilibrium at every stage.

Therefore, we should look at an infinitely repeated game, or a game without a unique Nash equilibirum.

Example 11.3

	D	C	N
D	(3,3)	(-1,4)	(0,0)
C	(-4,-1)	(0,0)	(0,0)
N	(0,0)	(0,0)	(2,2)

There are two Nash equilibria here. (CC) and (NN).
However, there is a strategy where the second play is contingent on the play of stage 1.

1. Play D.

2. If opponent plays D, play N in stage 2. Otherwise play C.

Assume that both players play this. We want to check that this is a subgame perfect Nash equilibrium. Suppose one person deviates, then playing C the first round gives a payoff of 4, opposed to $3 + 2\delta$. Then as long as $\delta > 1/2$ there will be no deviation.

These strategies that punish uncooperative behaviour tend to only work for big δ. If you don't care about the future punish then the game becomes more and more like a one-stage game.

Definition 11.4 (Infinitely repeated games)

We define the payoff of an infinite game to be

$$(1 - \delta) \sum_{t=0}^{\infty} \delta^t \pi_i(\bar{a}^t).$$

Remark. *The $(1 - \delta)$ is just a normalization constant. If you play the same strategy every game then the payoff is exactly the payoff in one game.*

Why model it infinitely?

A long time feels infinite. If you tak T to be large enough then δ^t becomes insignificant after some stage. (another argument: you do not know when the last stage of a game)

Proposition 11.5

Playing a Nash equilibrium of stage games repeated is still a Nash equilibrium in an infinitely repeated game.

Recall that the payoff matrix is

	confess	deny
confess	(0,0)	(4,-1)
deny	(-1,4)	(3,3)

Assume our opponent is playing the Grim trigger. We play D for $n-1$ turns and confess on the n-th turn. Starting from the $n+1$-st turn, our (best) payoff is 0. Our total payoff is thus

$$(1-\delta)\left(\sum_{t=0}^{n-1} 3\delta^t + 4\delta^n\right).$$

To make this better than not deviating, we require the (discounted)payoff starting in the n-st round to be larger than 3. That is,

$$4(1-\delta)\delta^n > 3\delta^n \implies 4(1-\delta) > 3 \implies \delta > \frac{1}{4}.$$

Remark. *This is also a subgame perfect Nash equilibrium. The argument is a bit more complex, as you have to show that constantly playing C works, even if your opponent repents!*

Remark. *There are other strategies such as punishing k-turns. Punishing for 1 turn is known as tit for tat.*

Theorem 11.7 (Folk theorem for repeated games)

Let V be the convex hull of all the (vectorized) payoffs

$$V \stackrel{\text{def}}{=} \text{conv}\{\tilde{v} \in \mathbb{R}^n | \text{there is } \bar{s} \text{ such that } \pi(\bar{s}) = \vec{v}\}.$$

An outcome $\vec{v}$ is **feasible** if $\vec{v} \in V$. We also say that an outcome $\vec{v}$ is **individually rational** if

$$v_i \geq \min_{a_{-1}} \max(a_i)\pi_i(a_i, a_{-i}).$$

Let $\vec{v}$ be feasible and strictly individually rational. Then there is $\tilde{\delta} \leq 1$ such that for all $\delta \geq \tilde{\delta}$, there is a Nash equilibirum of infinitely repeated game with payoff $\vec{v}$.

Remark. *The 'folk theorem' for repeated game is named this way because it is "common knowledge" and we cannot attribute it to one person.*

The proof is to use some sort of Grim trigger strategy. If someone defaults, then punish with the minimal strategy.

This theorem is both very useful and not useful. On one hand, this gives us a sufficient condition for a Nash equilibrium. On the other hand, we need to justify why each of the Nash equilibirum can show up in real life.

We can model 'societal norms' as these repeated games. Under this modelling, we can expect cooperative strategies that keep society functioning. (e.g. if you don't give up your seat we will throw you off the bus)

> **Example 11.8 (Repeated Betrand)**
>
> Recall that the Betrand competition is a price competition. The Nash equilibrium is both prices are 0. The feasible strategies lie within the triangle on the lower left half of $[0, 1/4] \times [0, 1/4]$, depending on who gets the lowest price.
> Theoretically, there is a good place where both firms 'collude' implicitly and declare the price of $1/2$, and each gets a payoff of $1/8$.

Lecture 12: Static Games with Incomplete Information

Logistics

1. Exam is on Feb 25. Can bring two double sided handwritten notes.

2. Hw 3 posted (shorter than usual homework), practice on dynamic games, due next monday so you can read the solutions.

Recap

1. Dyanmic games with complete information.

Static games with incomplete information are also known as Bayesian Games.

> **Example 12.1**
>
> There are two firms. Firm 1 is the incumbent and firm 2 is the entrant.
>
> - Firm 1 can decide to build a new factory or not.
>
> - Firm 2 can decide to enter or not.
>
> The cost of building is either 3 or 0. Firm 1 knows the cost but not firm 2. There are two possible scenarios for firm 2.
>
> | High cost | enter | not enter |
> |:---:|:---:|:---:|
> | build | (0,-1) | (2,0) |
> | not build | (2,1) | (3,0) |
>
> | Low cost | enter | not enter |
> |:---:|:---:|:---:|
> | build | (3,-1) | (5,0) |
> | not build | (2,1) | (3,0) |

The incumbent always knows whether to build or not build, as there is a dominant strategy. The entrant needs to estimate the probability of high cost as p. The payoff of entering is $p - (1-p) = 2p-1$.

The payoff of not entering is 0. So the threshold probability is 1/2. Enter if above 1/2, don't enter if below.

Now suppose we edit the game so that the payoffs are now

High cost	enter	not enter
build	(0,-1)	(2,0)
not build	(2,1)	(3,0)

Low cost	enter	not enter
build	(1.5,-1)	(3.5,0)
not build	(2,1)	(3,0)

Now the incumbent does not have a dominant strategy in the low cost situation, so the incumbent needs to have a belief on the entrant's belief...? and the entrant needs to have a belief on the incumbent's belief on the entrant's belief...

Harsanyi

Idea: There is an intiial move in game by 'nature' that determines the player's types. Moreover, everyone knows the common prior used by Nature.

<center>

Nature

High cost, probability p   Low cost, probability (1-p)

1   1

</center>
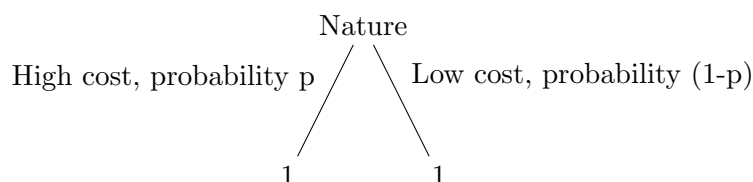
Let x be the probability of building if the cost is low. Let y be player 2's probability of entering. Player 2's expected payoff for entering is

$$p + (1-p)(x)(-1) + (1-p)(1-x) = (1-p)(1-2x) + p.$$

Not entering payoff is 0. SO player 2 should set $y = 1$ if $x \le 1/2(1-p)$. Similarly for player 1, want $x = 1$ if $y < 0.5$ and $x = 0$ if $y > 0.5$.

So if $1/2(1-p) \le 1 \iff p < 1/2$ we have two extra equilibirum (one mixed, one (build, don't enter)).

Definition 12.2 (Bayesian Game)

A **Bayesian game** (Normal form) consists of

- A set of players R.

- A set of actions S_r for each $r \in R$.

- A set of types for each player $\theta_r \in \Theta_r$.

- A payoff function $\pi_r(s_1, \ldots, s_R, \theta_1, \ldots, \theta_R)$.

- A joint distribution $p(\theta_1, \ldots, \theta_R)$.

- Optional: A signal to each player that randomly depends on types.

You might not know about your type. The 'signal' gives some information about the type.

Remark. *The joint distirbution is known to all players.*

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Example 12.3

Nature can choose high or low cost with probability p_1 and $(1 - p_1)$. Player 1 does not know it, but instead gets a signal with $(1 - \epsilon)$ accuracy. Probability of being high given a high signal is

$$\frac{p_1(1 - \epsilon)}{\epsilon(1 - p_1) + (1 - \epsilon)p_1}.$$

</div>

Definition 12.4

A pure strategy in a Bayesian game where players can see their own type is a map

$$\sigma_r : \Theta_r \to S_r$$

i.e. an action for every type.
A Bayes-Nash equilibrium (BNE) is

$$\sigma_r(\theta_r) \in \arg \max_{s_r' \in S_r} \sum_{\theta_{-r}} p(\theta_{-r} | \theta_r) \pi_r(s_r, s_{-r}, \theta_r, \theta_{-r}).$$

That is no one can unilaterally improve their payoff by changing the response (to a type).

Theorem 12.5

A bayesian game with finite strategy set and types has a mixed strategy Nash equilibrium.

Proof. Expand the strategy set to be functions

$$\sigma_r : \Theta_r \to S_r.$$

This is finite. Thus this is a finite game and has a nash equilibrium. ✿

Example 12.6 (Harsanyi mixed strategy justification)

	A	B
A	(2,1)	(0,0)
B	(0,0)	(1,2)

We have a mixed Nash equilibrium (2/3,1/3), (1/3,2/3). Now we preturb the payoffs by a bit.

	A	B
A	$(2+t_A,1)$	(0,0)
B	(0,0)	$(1,2+t_B)$

where t_A, t_B are iid, chosen uniformly on the interval $[0, x]$. t_A and t_B are known by each player respectivey. Notice player 1's payoff for playing A is monotonically increasing in t_A. Thus we would expect that player 1 plays A if $t_A > c_A$. Similarly, player 2 plays B if $t_B > c_B$. So player 1's payoff for choosing A is

$$(2 + t_A)c_B/x$$

Payoff choosing B is
$$1 - c_B/x$$

so is indifferent when

$$(2 + t_A)c_B/x = 1 - c_B/x \implies t_A = x/c_B - 3$$

If $c_A = c_B$, will get

$$c_A = c_B = \frac{-3 + \sqrt{9 + 4x}}{2}.$$

In this case, player 1 plays A with probability

$$1 - \frac{c_A}{x} = 1 - \frac{-3 + \sqrt{9 + 4x}}{2x} \to \frac{2}{3}$$

as $x \to 0$. This gives us the mixed strategy equilibria. That is, mixed strategies are just pure strategies in the Bayesian game with very small perturbations.

Lecture 13: Auctions

Logistics

1. HW2 solutions posted

2. Exam next Tuesday

3. HW3 due Monday

Recap

We started with static games with complete information.

- Dominant strategies

- Nash equilibria - concave games, mixed strategies, Nash's theorem

We then moved on to classes of games.

- Zero-sum games - minimax and maximin

- Infinite population games - wardrop equilibria

- Congestion games - as a potential game

- Supermodular game

Learning in games.

- Best response dynamics

- Fictitious play

- Evolutionary game

Dynamic games with complete information

- Extensive form of a game - 'game tree model'

- Subgame perfect equilibria - a refinement of Nash equilibria

- Perfect and imperfect information (might have information set)

- Repeated games - finite and infinitely repeated games with discounting - trigger strategies can lead to cooperative play

- Folk theorem - characterizes the equilibria to repeated games

Finally, in the last lecture we focused on static game with incomplete information (Bayesian game).

Auctions

Auctions are mechanisms to allocate an item to a set of bidders. Some examples include FCC auctions over wireless spectrum. In 2021, there was a C-band auction that allocated frequencies 3.7-3.98GHz for 4G and 5G wireless systems. It sold over 5684 licenses and resulted in 81 billion dollars in revenue. The Google ad platform operates like an auction, for allocating ad spots. There are two common information strategies for auctions.

1. Private value model: each agent has and independently drawn value for the item.

2. Common value: the object has a value that is the same for all agents, drawn at random from some distribution. Agents only get independent signals about the value.

There are different types of Auctions too.

We now focus on first price auctions.

Example 13.2 (First Price Auction)

Two bidders each have private values v_i drawn uniformly from $[0, 1]$. If there is a tie in bids, the winner is chosen at random. The winner pays his bid. Thus,

$$\pi_1(b_1, b_2, v_1, v_2) = \begin{cases} v_1 - b_1, & \text{if } b_1 > b_2 \\ \frac{v_1 - b_1}{2}, & \text{if } b_1 = b_2 \\ 0, & \text{otherwise} \end{cases}.$$

We want to find the Bayesian Nash equilibrium.

A pair of bidding functions $b_1(v_1), b_2(v_2)$ is a BNE when

$$b_i(v_i) = \arg\max_{b_i} (v_i - b_i) P(b_i > b_{-i}(v_{-i})) + \frac{1}{2}(v_i - b_i) P(b_i = b_{-i}(v_{-i})).$$

This is horrible to compute. Let us conjecture that b_1, b_2 are linear (affine) functions in v_1, v_2 respectively. i.e.

$$b_1(v_1) = a_1 + c_1 v_1,$$
$$b_2(v_2) = a_2 + c_2 v_2.$$

Let us fix a linear bid for bidder 2. We consider 1's best response. Since $P(b_1 = b_2(v_2)) = 0$ for any b_1, we can disregard the second term in the expected value, and want to maximize

$$(v_1 - b_1) P(b_1 > b_2(v_2)) = (v_1 - b_1) P(b_1 > a_2 + c_2 v_2)$$

A rational bid would lie between $[a_2, a_2 + c_2]$, so let us assume that. So that

$$P(b_1 > a_2 + c_2 v_2) = \frac{b_1 - a_2}{c_2}$$

Maximizing the quadratic function in b_1 leads to

$$b_1^* = \frac{v_1 + a_2}{2},$$

which is linear in v_1. A same argument would yield

$$b_2^* = \frac{v_2 + a_1}{2}$$

If both players are best responding to each other, we must get

$$a_1 = a_2/2, a_2 = a_1/2 \implies a_1 = a_2 = 0.$$

So the equilibrium is they each bid half their value. This means in a first price auction, there usually is some bid-shading where people bid not as high.

What is the expected payment of player one in terms of $v = v_1$? It turns out to be

$$P(1 \text{ wins})\frac{v}{2} = P(v > v_2)\frac{v}{2} = \frac{v^2}{2}.$$

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Example 13.3 (Second Price Auction)

Two bidders each have private values v_i drawn uniformly from $[0,1]$. If there is a tie in bids, the winner is chosen at random. The winner pays the loser's (second highest) bid. The payoff is now

$$\pi_1(b_1, b_2, v_1, v_2) = \begin{cases} v_1 - b_2, & \text{if } b_1 > b_2 \\ \frac{v_1 - b_2}{2}, & \text{if } b_1 = b_2 \\ 0, & \text{otherwise} \end{cases}$$

We want to find the Bayesian Nash equilibrium.

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We claim that there is a weakly dominant strategy to set $b_i = v_i$. This is known as a truthful bid. It is apparent from

$$E[\text{bidder 1 winnings}|v_1, b_1] = P(b_1 > b_2)(v_1 - b_2) + \frac{1}{2}P(b_1 = b_2)(v_1 - b_2).$$

The expected payment of 1 given $v = v_1$ is

$$E[v_2; v > v_2] = \frac{v^2}{2}.$$

This is the same payment as the first price auction!

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Theorem 13.4 (Revenue Equivalence)

Consider any auction where bidders submit single bids and the auction assigns the object to the highest bidder. Assume that the payments are non-decreasing in bids. Then if the values are IID, any symmetric equilibria yields the same revenue.

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Suppose that the bidders are honest. An English auction is similar to a second price auction. A Dutch auction is similar to a first price auction.