Equilibrium in Markov Games

EE495 Final Project

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Motivation

- We want to merge together repeated games and Bayesian games.
- Simplest: repeated Bayesian games
- Add a new criterion that the previous game influences the next game - the games played in each stage represents a Markov process.

Introduction to Markov Games

Definition

A **Markov game** $G = \{R, \mathcal{S}, \{A_r\}, \{\pi_{S,r}\}, q\}$ consists of:

- The set of players R and the state space $\mathcal{S}$.
- For each state $S \in \mathcal{S}$ and each player $r \in R$, a action (strategy) set $A_r(S)$ for the player in the game S.
- For each state $S \in \mathcal{S}$ and each player $r \in R$, a payoff function $\pi_{S,r} : \prod_{r' \in R} A_{r'}(S) \to \mathbb{R}$ for player r in game S.
- A transition function q for the next state given by$q(S^{t+1}|S^t, a^t)$ where S^t is the current state at time t and a^t is the set of actions choosen by each player.

If required, we can also introduce a discount rate δ.

Markov Perfect Equilirium

- A natural question to ask is if there exists a strategy that depends on state.

Definition

A **Markov strategy** for player r is $\Theta : \mathcal{S} \to \sqcup_{S \in \mathcal{S}} A_r(S)$ that determines the action at time t

$$a_r^t = \Theta(S^t).$$

A **Markov Perfect Equilibirum** is a Nash equilibirum such that each player's strategy is Markov.

- If everyone else is playing a Markov strategy, the best response is also a Markov strategy.

Markov Perfect Equilirium - existence

Theorem [3]

Let G be a finite Markov game. Then G has a mixed-strategy Markov perfect equilibrium.

Proof.

Expand the strategy set of each player to the set of all Markov strategies, and the payoff to be the expected discounted payoff in the long run. This new game is still finite, so has a mixed equilibrium by Nash. $\square$

Corollary

Monopoly has a pure strategy Markov Perfect Equilibirium.

Other Markov games

- Bob has a cake. The enjoyment out of eating the cake is determined by a Markov process with countable states. When should Bob eat the cake?
- Intuitive strategy: at each day, choose whether to eat or save it for later (and get the expected value of enjoyment)
- Let $v(i)$ be the best payoff if the cake starts in state i, then solve

$$v(i) = \max(\text{enjoyment today}, \delta E[v(\text{state tomorrow})]).$$

- Optimal stopping solution exists for irreducible aperiodic Markov chains.

Other Markov games [1]

- Alice has a cake and Bob wants to buy it (to eat using the optimal stopping strategy)
- Alice and Bob take turns each day to propose a price for the cake (Alice offers day 1, Bob bids day 2...)
- Once agreement is reached, Bob can choose to eat it any time.

Theorem

Markov Perfect equilibrium exists for when:

- The underlying Markov Process is aperiodic and irreducible
- Bob is more patient than Alice

Idea of proof: same as the infinite horizon game, can solve for recurrance relations. (just more annoying)

Justification for Markov games

- Describes Monopoly behavior reasonably well: No trading in 2 player games, but 3+ player. Trading in 2+ players only to expedite the game.
- Does not describe state of mind well. How many states would we need to compute the MPE of monopoly?
$$(40)^N * (28)^{N+1} * 6^8 * (M)^N$$
- Competitive programmer heuristic: 1 second is around 10^{9-10} computations.
- In Markov Bargaining - the solved equilibria resembles that of a fixed bargaining with discount
- Except: seller is more patient than buyer. In that case, the buyer might not obtain the cake before he wants to eat it.
- Non-cooperative solution.

Folk theorem for Markov Games

Folk Theorem for Markov Games [2]

Suppose that the long term payoff and min-max (for punishment) for all agents do not depend on the initial state. Then any individually rational[a] and feasible[b] outcome is attainable[c] for sufficiently patient players.

[a] Each agent gets at least the min-max payoff

[b] Lies within the convex hull of payoffs for all pure Markov strategies

[c] For any $\epsilon > 0$ the probability of attaining an outcome within ϵ of the desired outcome

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