AM 212 Exam 1

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Problem 1: This problem is based on 5.3.9 from Haberman (cf. Lecture 5), followed by the Green's function problem (cf Lecture 8).

Part 1: Consider the equation

$$x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} = -\lambda f \tag{1}$$

with boundary conditions $f(1) = 0, f(2) = 0$.

1. Show that this is a regular Sturm-Liouville problem and put it in Sturm-Liouville form

   Proof.

   $$(1) \to x\frac{d^2 f}{dx^2} + \frac{df}{dx} = -\lambda \frac{f}{x}$$
   $$\frac{d}{dx}\left(x\frac{df}{dx}\right) = -\lambda\frac{f}{x}$$

   This is the S-L form for this differential equation. We find that $p(x) = x$, $q(x) = 0$, and $w(x) = 1/x$ which are continuous in $(1, 2)$ and strictly positive. Furthermore, the coefficients on the Boundaries are nonzero. Thus, this problem is a regular Sturm-Liouville problem. □

2. Find the eigenfunctions and corresponding eigenvalues. Hint: solutions are of the form x^α where α is complex. Recall that $a^b = e^{b\ln(a)}$.

   Proof. We solve the eigenvalue problem using an ansatz of the form $f = x^\alpha$:

   $$(1) \to x^2\left(\alpha(\alpha-1)x^{\alpha-2}\right) + \alpha x^\alpha + \lambda x^\alpha = 0$$
   $$x^\alpha(\alpha^2 + \lambda) = 0 \implies \alpha = \sqrt{-\lambda}$$

   We have found that the eigenfunctions for this problem take the form of $f = x^{\sqrt{-\lambda}}$. Here it is important to identify the sign of the eigenvalues. If the eigenvalues are negative, the solution is a sum of real valued polynomials. If the eigenvalues are positive, the solution is a sum of cosines and sines which vary with respect to $\ln(x)$. We notice that the boundary conditions for this problem require that our eigenfunctions have zeroes at $x = 1$ and $x = 2$. The eigenfunctions with negative eigenvalues are not able to satisfy these boundayr conditions. Thus we limit the eigenvalues of this problem to be positive. In fact, we can show these eigenvalues to be $\lambda_n = (n\pi/\ln(2))^2$:

   $$cx^{i\beta} = ce^{i\beta\ln(x)} = c\cos(\beta\ln(x)) + ic\sin(\beta\ln(x))$$
   $$c\cos(\beta\ln(1)) + ic\sin(\beta\ln(1)) = c \implies \mathrm{Re}(c) = 0$$
   $$ic\sin(\beta\ln(2)) = 0 \implies \beta\ln(2) = n\pi$$
   $$\beta_n = \frac{n\pi}{\ln(2)}$$

   where $\beta = \sqrt{\lambda}$. Thus we have eigenfunctions $\phi_n(x) = \sin(n\pi\ln(x)/\ln(2))$ with corresponding eigenvalues $\lambda_n = (n\pi/\ln(2))^2$. □

3. What is the orthogonality relationship associated with these eigenfunctions? (note: you can find the normalization constant by using the change of variable $y = \ln(x)$.

Proof. In order to demonstrate orthogonality, we use the change in variable $y = \ln(x)$ when taking the inner product. Let us begin with eigenfunctions ϕ_n and ϕ_m.

$$\begin{aligned}\langle \phi_n, \phi_m \rangle &= \int_1^2 \sin\left(\frac{n\pi \ln(x)}{\ln(2)}\right) \sin\left(\frac{m\pi \ln(x)}{\ln(2)}\right) \frac{dx}{x} \\ &= \int_0^{\ln(2)} \sin\left(\frac{n\pi y}{\ln(2)}\right) \sin\left(\frac{m\pi y}{\ln(2)}\right) dy \\ &= \int_0^L \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right) dy, \quad L = \ln(2)\end{aligned}$$

This integral is related to the orthogonality relationship for sines and is the basis on which Fourier series are built. We have that

$$\langle \phi_n, \phi_m \rangle = \begin{cases} 0, & m \neq n \\ \ln(2)/2, & m = n \end{cases}$$

□

Part 2: Consider the equation

$$x^2 \frac{d^2G}{dx^2} + x\frac{dG}{dx} = \delta(x - x') \tag{0.1}$$

with boundary conditions $G(1, x') = 0, G(2, x') = 0$.

1. Find the Green's function $G(x, x')$ using the method of eigenfunction expansion.

Proof. To begin we first expand our eigenfunctions into the PDE.

$$-a_n \left(\frac{n\pi}{\ln(2)}\right)^2 \phi_n(x) = \delta(x - x')$$

Then we use the orthogonality condition to find the coefficients.

$$\begin{aligned}\int_1^2 \frac{\phi_m(x)}{x} \left(-a_n \left(\frac{n\pi}{\ln(2)}\right)^2 \phi_n(x)\right) dx &= \int_1^2 \delta(x - x') \frac{\phi_m(x)}{x} dx \\ -a_m \left(\frac{m\pi}{\ln(2)}\right)^2 \frac{\ln(2)}{2} &= \frac{\phi_m(x')}{x'} \\ a_m &= -\frac{2\ln(2)}{m^2\pi^2} \frac{\phi_m(x')}{x'} \\ G(x, x') &= \sum_n -\frac{2\ln(2)}{n^2\pi^2} \frac{\phi_n(x')}{x'} \phi_n(x)\end{aligned}$$

□

2. Find the Green's function $G(x, x')$ directly using the 'patching' method.

Proof. We begin by decompolsing the green's function into a piecewise funciton along two subdomains. We have,

$$G(x, x') = \begin{cases} G_L(x), & x < x' \\ G_R(x), & x > x' \end{cases}$$

We then solve the homogeneous differential equation in each subdomain.

$$\begin{gathered}
x^2G'' + xG' = 0 \\
h = G' \to x^2h' + xh = 0 \\
h' = -\frac{h}{x} \\
\frac{dh}{h} = -\frac{dx}{x} \\
\ln|h| = -\ln|x| + c \\
h = \frac{c}{x} \to G = a\ln(x) + b
\end{gathered}$$

We now look to solve the boundary conditions,

$$\begin{aligned}
G_L(1) &= a_L(0) + b_L = 0 & \qquad G_R(2) &= a_R\ln(2) + b_R = 0 \\
G_L(x') &= a_L\ln(x') + b_L = G_R(x') & \qquad G_R(x') &= a_R\ln(x') + b_R = G_L(x')
\end{aligned}$$

We also look at the integral condition to close the system.

$$\begin{gathered}
xG'' + G' = \frac{\delta(x-x')}{x} \\
\int_{x'-\epsilon}^{x'+\epsilon} xG'' + G'dx = \int_{x'-\epsilon}^{x'+\epsilon} \frac{\delta(x-x')}{x}dx \\
xG'\Big|_{x'-\epsilon}^{x'+\epsilon} = \frac{1}{x'} \\
x'\left(\frac{a_R}{x'} - \frac{a_L}{x'}\right) = \frac{1}{x'} \\
a_R - a_L = \frac{1}{x'}
\end{gathered}$$

Some immediate consequences of these conditions show that $b_L = 0$ due to the left boundary condition. The condition at $x = x'$ and enforcing continuity show that $b_R = \ln(x')(a_L - a_R)$ and using the integral condition we find that $b_R = -\ln(x')/x'$. Then using the right boundary condition we find that $a_R = -b_R/\ln(2) = \ln(x')/(x'\ln(2))$. Finally, using the integral condition we have that $a_L = a_R - 1/x' = \ln(x')/(x'\ln(2)) - 1/x'$. All that we need is a value for x' and we have a green's function.

$$G(x,x') = \begin{cases} \left(\frac{\ln(x')}{x'\ln(2)} - \frac{1}{x'}\right)\ln(x), & x < x' \\ \left(\frac{\ln(x')}{x'\ln(2)}\right)\ln(x) - \frac{\ln(x')}{x'}, & x > x' \end{cases}$$

□

3. Create a code that plots on the same figure the function $G(x, 5/4)$ using the two different meth- ods. Hand in the code and the figure.

Note: you can that your Green's function is correct by solving numerically

$$x^2\frac{d^2G}{dx^2} + x\frac{dG}{dx} = F(x) \tag{0.2}$$

for some forcing F (x) of your choice, with homogeneous boundary conditions, and comparing the answer with

$$\int_1^2 G(x,x')F(x')dx' \tag{0.3}$$

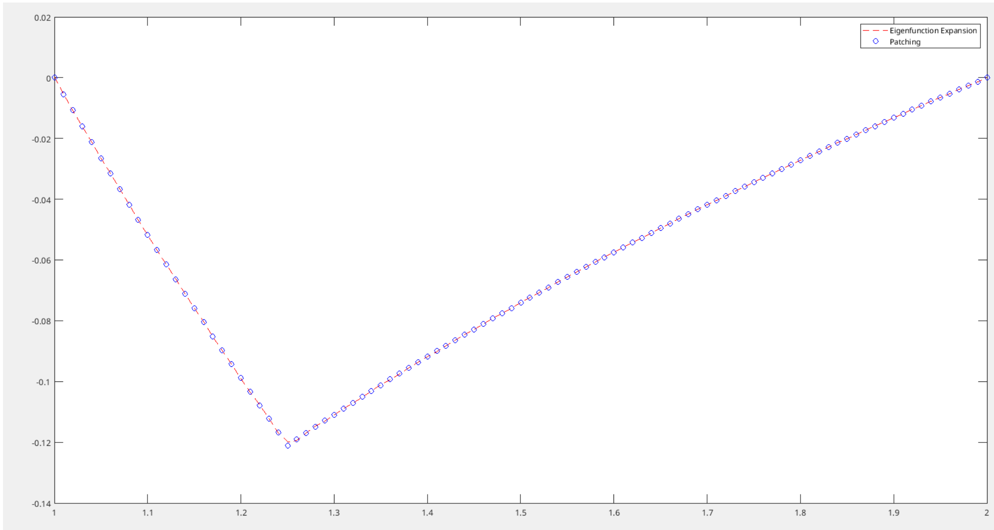

Figure 1: Plot of the Eigenfunction Expansion (red) and the Patching (blue) methods on the domain (1,2) for $x' = 1.25$

as we have done in class.

Problem 3:

Part 1: Consider the equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f \tag{0.4}$$

in a disk of radius a satisfying $f(a, \theta, t) = 0$, with initial conditions $f(r, \theta, 0) = h(r)\cos(5\theta)$.

1. Find all the 2D eigenfunctions solutions of $\nabla^2 f = -\lambda f$ subject to $f(a, \theta, t) = 0$ and a regularity condition at the origin.

   Proof. We begin by assuming a separation of variables for our function f. We have,

   $$f(r, \theta, t) = \tau(t)h(r)g(\theta)$$

   We then look to solve the eigenvalue problem for the steady state.

   $$\nabla^2 f = -\lambda f$$
   $$\tau(t)h''(r)g(\theta) + \frac{1}{r}\tau(t)h'(r)g(\theta) + \frac{1}{r^2}\tau(t)h(r)g''(\theta) = -\lambda\tau(t)h(r)g(\theta)$$
   $$r^2h''(r) + rh'(r) + h(r)\frac{g''(\theta)}{g(\theta)} = -\lambda r^2 h(r)$$

   One naturally assumes that $g(\theta)$ will have periodic boundary conditions due to the nature of the problem, i.e. $g(0) = g(2\pi)$ (or from $(-\pi, \pi)$) depending on how you define the domain. This allows us to suppose an (informed) ansatz for the solution for $g(\theta)$, $g(\theta) = \cos(\mu\theta)$. Using this, we find that $g''/g = -\mu^2$ and $\mu = m\pi$. Thus the ODE for $h(r)$ now becomes,

   $$r^2h''(r) + rh'(r) = (-\lambda r^2 + \mu^2)h(r)$$

   This is the Bessel's equation and using Hongyun's old 212 lecture notes, I can find the solution pretty easily. First, we'll rescale the equation using $z = \sqrt{\lambda}r$. The equation becomes:

   $$z^2\frac{d^2h}{dz^2} + z\frac{dh}{dz} + (z^2 - \mu^2)h = 0$$

This is now a Bessel's equation of order μ. Since we have that this function be bounded at $r = 0$, the solutions are Bessel's functions of the first kind of order μ.

$$h(r) = J_\mu(\sqrt{\lambda} r)$$

In order to satisfy the Boundary conditions for $h(r)$, we have that $\sqrt{\lambda} a = z_{n,\mu}$, where $z_{n,\mu}$ is the n-th zero of J_μ. Therefore,

$$\sqrt{\lambda_{n,\mu}} = \frac{z_{n,\mu}}{a}$$

Finally, we have the general solution to this problem:

$$f(r, \theta, t) = \sum_{n,\mu} a_{n,\mu} \cos\left(ct\frac{z_{n,\mu}}{a}\right) J_\mu\left(\frac{z_{n,\mu} r}{a}\right) \cos(\mu\theta)$$

where $\tau(t)$ was chosen to be $\cos(c\sqrt{\lambda}t)$ in order to satisfy $\tau''/\tau = c^2\lambda$ while also being nonzero @ $t = 0$. ☐

2. Explain why we only need to keep the subset of eigenfunctions proportional to $\cos(5\theta)$.

   Essentially, since $\cos(5\theta)$ is one of the eigenfunctions of the ODE for θ and the eigenfunctions for this problem are orthogonal, when we project our solution onto the initial condition, only one mode will will have a non-zero coefficient, that mode being for $\mu = 5$.

3. Project the initial condition on the remaining eigenfunctions to find the formal solution to the original problem for any h(r).

Proof. We begin by taking the inner product:

$$a_{n,\mu} = \frac{2}{a^2 \pi J_{\mu+1}^2(z_{n,\mu})} \int_0^a \int_0^{2\pi} J_\mu\left(\frac{z_{n,\mu} r}{a}\right) \cos(\mu\theta) h(r) \cos(5\theta) r dr d\theta$$

$$a_n = \frac{2}{a^2 \pi J_6^2(z_{n,5})} \int_0^a J_5\left(\frac{z_{n,5} r}{a}\right) h(r) r dr$$

$$f(r, \theta, t) = \sum_n a_n \cos\left(ct\frac{z_{n,5}}{a}\right) J_5\left(\frac{z_{n,5} r}{a}\right) \cos(5\theta)$$

Note that here the normalization factor for the inner product for the Bessel's function is taken Wolfram Alpha's website (Weisstein)(https://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html). ☐

References

Weisstein, Eric W.MathWorld