Take Home Midterm 1, AM212, October 26-27 2024.

You **must** do problem 1, and then pick **either** problem 2 **or** problem 3.

Problem 1: This problem is based on 5.3.9 from Haberman (cf. Lecture 5), followed by the Green's function problem (cf Lecture 8).

Part 1: Consider the equation

$$x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} = -\lambda f \tag{1}$$

with boundary conditions $f(1) = 0$, $f(2) = 0$.

(1) Show that this is a regular Sturm-Liouville problem and put it in Sturm-Liouville form

(2) Find the eigenfunctions and corresponding eigenvalues. Hint: solutions are of the form x^α where α is complex. Recall that $a^b = e^{b\ln(a)}$.

(3) What is the orthogonality relationship associated with these eigenfunctions? (note: you can find the normalization constant by using the change of variable $y = \ln(x)$.

Part 2: Consider the equation

$$x^2 \frac{d^2 G}{dx^2} + x\frac{dG}{dx} = \delta(x - x') \tag{2}$$

with boundary conditions $G(1, x') = 0$, $G(2, x') = 0$.

(1) Find the Green's function $G(x, x')$ using the method of eigenfunction expansion.

(2) Find the Green's function $G(x, x')$ directly using the 'patching' method.

(3) Create a code that plots on the same figure the function $G(x, 5/4)$ using the two different methods. Hand in the code and the figure.

Note: you can that your Green's function is correct by solving numerically

$$x^2 \frac{d^2 G}{dx^2} + x\frac{dG}{dx} = F(x) \tag{3}$$

for some forcing $F(x)$ of your choice, with homogeneous boundary conditions, and comparing the answer with

$$\int_1^2 G(x, x')F(x')dx' \tag{4}$$

as we have done in class.

Problem 2: (based on the magnetic diffusion problem from Lecture 3)

Consider the diffusion equation

$$\frac{\partial B}{\partial t} = D\frac{\partial^2 B}{\partial x^2} \tag{5}$$

in a domain $(0, L)$ subject the boundary condition that $B(0, t) = F(t)$ and $B(L, t) = 0$, and initial conditions $B(x, 0) = 0$. The forcing function $F(t)$ is generic, but you may assume that it satisfies $F(0) = 0$.

(1) Recast this problem into one that has homogeneous boundary conditions

(2) Find the solution for unspecified $F(t)$. Note that you may end up with integrals that involve dF/dt and may be tempted at that point to integrate by parts. Resist the temptation.

(3) Create a code that plots the solution $B(x, t)$ for $F(t) = \sin(4\pi t)$, $L = 1$, $D = 1$, at representative timesteps. Hand in the code and the figure.

Problem 3: (based on 6.7.1. of Haberman)

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Consider the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2\nabla^2 f \tag{6}$$

in a disk of radius a satisfying $f(a, \theta, t) = 0$, with initial conditions

$$f(r, \theta, 0) = h(r)\cos(5\theta), \frac{df}{dt}(r, \theta, 0) = 0. \tag{7}$$

(1) Find all the 2D eigenfunctions solutions of

$$\nabla^2 f = -\lambda f \tag{8}$$

subject to $f(a, \theta, t) = 0$ and a regularity condition at the origin.

(2) Explain why we only need to keep the subset of eigenfunctions proportional to $\cos(5\theta)$

(3) Project the initial condition on the remaining eigenfunctions to find the formal solution to the original problem for any $h(r)$.