AM212 MIDTERM 1 (PDES) 2024

Problem 1

Part 1

$$x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} = -\lambda f$$

$$f(1) = f(2) = 0$$

① This is the same as

$$x \frac{d^2 f}{dx^2} + \frac{df}{dx} = -\frac{\lambda f}{x}$$

$$\hookrightarrow \frac{d}{dx}\left(x \frac{df}{dx}\right) = -\frac{\lambda f}{x} \quad \rightarrow \text{a regular SL pb}$$

bc $p(x) = x > 0 \quad \forall x \in [1,2]$

$w(x) = \frac{1}{x} > 0$

② original eq is equidimensional so try ansatz

$$f(x) = x^{\alpha}$$

$$x^2 \alpha(\alpha-1)x^{\alpha-2} + x\alpha x^{\alpha-1} = -\lambda x^{\alpha}$$

$$\Rightarrow \quad \alpha(\alpha-1) + \alpha = -\lambda$$

$$\alpha^2 = -\lambda \quad \Rightarrow \quad \alpha = \pm i\lambda^{1/2} \text{ if } \lambda > 0$$

$$\alpha = \pm \lambda^{1/2} \text{ if } \lambda < 0$$

$$\alpha = 0 \text{ if } \lambda = 0$$

if $\lambda \leq 0$ it is not possible to fit homogeneous BCs $\rightarrow$

need $\lambda > 0$, $\alpha = \pm i\sqrt{\lambda}$

$$x^{\alpha} = e^{\alpha \ln x} = e^{\pm i\sqrt{\lambda} \ln x}$$

$$= \begin{cases} \cos(\sqrt{\lambda} \ln x) \\ \sin(\sqrt{\lambda} \ln x) \end{cases}$$

To fit BCs: at $x = 1$ $f(1) = 0$ $\rightarrow$ no cosine

at $x = 2$ $\sin(\sqrt{\lambda}\ln 2) = 0$

$$\sqrt{\lambda}\ln 2 = n\pi$$

$$\lambda_n = \frac{n^2\pi^2}{(\ln 2)^2}$$

so finally $f_n(x) = \sin\left(n\pi \frac{\ln x}{\ln 2}\right)$ are eigenfunctions

③ Orthogonality: $\displaystyle\int_1^2 f_n(x) f_m(x)\frac{dx}{x} = \delta_{mn}\cdot\int_1^2 f_n^2(x)\frac{dx}{x}$

$$\int_1^2 f_n^2 \frac{dx}{x} = \int_1^2 \sin^2\left(n\pi\frac{\ln x}{\ln 2}\right)\frac{dx}{x}$$

at $y = \ln x$ $\quad dy = \frac{dx}{x}$

$$= \int_0^{\ln 2} \sin^2\left(\frac{n\pi y}{\ln 2}\right) dy = \frac{\ln 2}{2}$$

so $\displaystyle\langle f_n, f_m\rangle = \int_1^2 f_n(x) f_m(x)\frac{dx}{x} = \frac{\ln 2}{2}\delta_{mn}$

<u>Part 2</u> $\quad x^2\dfrac{d^2G}{dx^2} + x\dfrac{dG}{dx} = \delta(x - x')$

① Using eigenfunction expansion

$$\frac{d}{dx}\left(x\frac{dG}{dx}\right) = \frac{\delta(x-x')}{x}$$

let $\displaystyle G(x) = \sum_{n=1}^{\infty} g_n f_n(x)$

$$\sum_{n=1}^{\infty} -g_n\lambda_n\frac{f_n(x)}{x} = \frac{\delta(x-x')}{x}$$

Project $\displaystyle\int_1^2 \sum -g_n\lambda_n\frac{f_n f_m}{x}\,dx = \int_1^2 \frac{\delta(x-x')}{x} f_m\,dx$

$$-g_m \lambda_m \frac{\ln 2}{2} = \frac{f_m(x')}{x'}$$

$$\Rightarrow \quad g_m = -\frac{2}{\lambda_m \ln 2} \frac{f_m(x')}{x'}$$

$$\Rightarrow \quad G(x) = \sum_{n=1}^{\infty} -\frac{2}{\lambda_m \ln 2} \frac{f_n(x) f_m(x')}{x'}$$

② Using direct method

$$x^2 \frac{d^2G}{dx^2} + x\frac{dG}{dx} = 0$$

$$\rightarrow \quad \frac{d}{dx}\left(x\frac{dG}{dx}\right) = 0 \qquad x\frac{dG}{dx} = C$$

$$\rightarrow \quad \frac{dG}{dx} = \frac{C}{x} \quad \rightarrow \quad G = C\ln x + D$$

at $x \leq x'$ $\quad G_L(x) = C_L \ln x + D_L$

$x \geq x'$ $\quad G_R(x) = C_R \ln x + D_L$

- $G_L(1) = 0 \Rightarrow D_L = 0$
- $G_R(2) = 0 \Rightarrow C_R \ln 2 + D_R = 0 \qquad D_R = -C_R \ln 2$
- $G_L(x') = G_R(x') \Rightarrow C_L \ln x' = C_R \ln x' - C_R \ln 2$

$$= C_R \ln\left(\frac{x'}{2}\right)$$

- $$\int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{d^2G}{dx^2} + \frac{1}{x}\frac{dG}{dx} = \frac{\delta(x-x')}{x^2}\right]$$

$$\Rightarrow \quad \left.\frac{dG_R}{dx}\right|_{x'} - \left.\frac{dG_L}{dx}\right|_{x'} = \frac{1}{x'^2} \Rightarrow \frac{C_R}{x'} - \frac{C_L}{x'} = \frac{1}{x'^2}$$

$$\Rightarrow \quad C_R - C_L = \frac{1}{x'}$$

So
$$C_R = C_L + \frac{1}{x'}$$
$$\Rightarrow \quad C_L \ln x' = \left(C_L + \frac{1}{x'}\right)\ln\left(\frac{x'}{2}\right)$$
$$C_L\left(\ln x' - \ln\left(\frac{x'}{2}\right)\right) = \frac{1}{x'}\ln\left(\frac{x'}{2}\right)$$
$$C_L \ln 2 = \frac{1}{x'}\ln\left(\frac{x'}{2}\right) \Rightarrow C_L = \frac{1}{x'\ln 2}\ln\left(\frac{x'}{2}\right)$$
$$C_R = C_L + \frac{1}{x'} \qquad D_R = -C_R \ln 2 \qquad D_L = 0$$

Problem 2
$$\frac{\partial B}{\partial t} = D\frac{\partial^2 B}{\partial x^2} \qquad B(0,t) = F(t) \quad B(L,t) = 0$$
$$B(x,0) = 0$$

① let $B(x,t) = u(x,t) + F(t)\frac{(L-x)}{L}$
$$u(0,t) = B(x,t) - F(t) = 0 \qquad u(x,0) = B(x,0) - F(0)\frac{L-x}{L}$$
$$u(L,t) = B(L,t) = 0 \qquad = 0$$
$$\frac{\partial u}{\partial t} = \frac{\partial B}{\partial t} - \frac{L-x}{L}\frac{dF}{dt} = D\frac{\partial^2 B}{\partial x^2} - \frac{L-x}{L}\frac{dF}{dt}$$
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \frac{L-x}{L}\frac{dF}{dt}$$

② let $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{L}\right)$
$$\frac{du_n}{dt} = -D\frac{n^2\pi^2}{L^2}u_n - \underbrace{\frac{2}{L}\int_0^L \frac{L-x}{L}\sin\left(\frac{n\pi x}{L}\right)dx \cdot \frac{dF}{dt}}_{F_n(t)}$$
$$\lambda_n = D\frac{n^2\pi^2}{L^2}$$
$$\frac{d}{dt}\left(u_n e^{\lambda_n t}\right) = e^{\lambda_n t}F_n$$
$$u_n(t)e^{\lambda_n t} = \int_0^t e^{\lambda_n t'}F_n(t')\,dt' \qquad \text{using } u_n(0) = 0$$

$$\Rightarrow \quad u_n(t) = e^{-\lambda_n t} \int_0^t e^{\lambda_n t'} F_n(t')\,dt'$$

$$= -\int_0^t e^{\lambda_n(t'-t)} \frac{2}{L} \int_0^L \frac{L-x'}{L} \sin\left(\frac{n\pi x'}{L}\right) dx' \cdot \left.\frac{dF}{dt}\right|_{t=t'}$$

so finally

$$u(x,t) = \sum_{n=1}^{\infty} -\int_0^t \int_0^L \frac{2}{L} e^{\lambda_n(t'-t)} \left.\frac{dF}{dt}\right|_{t=t'} \frac{L-x'}{L} \sin\left(\frac{n\pi x'}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} -\left[\int_0^t e^{\lambda_n(t'-t)} \left.\frac{dF}{dt}\right|_{t=t'} dt'\right] a_n \sin\frac{n\pi x}{L}$$

where $\quad a_n = \frac{2}{L} \int_0^L \frac{L-x'}{L} \sin\left(\frac{n\pi x'}{L}\right) dx'$

and

$$B(x,t) = \left(1-\frac{x}{L}\right) F(t) + u(x,t)$$

↑ as given above

Problem 3

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f \qquad f(a,\theta,t)=0$$

$$f(r,\theta,0) = h(r,\theta) = h(r)\cos 5\theta$$

① $\nabla^2 f = -\lambda f$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = -\lambda f$$

Eigenfunctions in θ direction are $\sin, \cos(m\theta)$

$\Rightarrow$ in radial direction we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{m^2}{r^2}R = -\lambda R$$

or equiv. $\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{m^2}{r}R = -\lambda r R$

shows $w(r)=r$

or equiv. $r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (\lambda r^2 - m^2)R = 0$

$\hookrightarrow \quad R_{mn}(r) = J_m(\sqrt{\lambda}\, r)$

$\hookrightarrow \quad \sqrt{\lambda}\, a = z_{mn}$ nth zero of J_m

$\hookrightarrow \quad \lambda_{mn} = \frac{z_{mn}^2}{a^2}$

So 2D eigenfunctions are

if $m \neq 0$: $\quad \sin m\theta \cdot J_m\left(\frac{z_{mn}}{a}r\right)$

$\quad \cos m\theta \cdot J_m\left(\frac{z_{mn}}{a}r\right)$

if $m=0$ just $J_0\left(\frac{z_{0n}}{a}r\right)$

② Each eigenfunction oscillates with frequency

$$\omega_{mn} = c\frac{z_{mn}}{a}$$

$\Rightarrow$ solution is

$$f(r,\theta,t) = \sum_{n=1}^{\infty} J_0\left(\frac{z_{0n}}{a}r\right)\left(a_{0n}\cos\omega_{0n}t + b_{0n}\sin\omega_{0n}t\right)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{z_{mn}}{a}r\right)\left(a_{mn}\cos\omega_{mn}t + b_{mn}\sin\omega_{mn}t\right) \cdot \left(\alpha_{mn}\cos m\theta + \beta_{mn}\sin m\theta\right)$$

At $t=0$, $\frac{\partial f}{\partial t} = 0 \Rightarrow$ all bs are 0

$$h(r,\theta) = \sum_{n=1}^{\infty} J_0\left(\frac{z_{0n}}{a}r\right) a_{0n}$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{z_{mn}}{a}r\right)\left(a_{mn}\right)\left(\alpha_{mn}\cos m\theta + \beta_{mn}\sin m\theta\right)$$

$=$ $h(r)\cos 5\theta \rightarrow$ all $\alpha_{mn} = 0$

$\beta_m = 0$ if $m \neq 5$

③ Projecting on ICS:

$$h(r)\cancel{\cos 5\theta} = \sum_{n=1}^{\infty} J_5\left(\frac{z_{5n}}{a}r\right)\underbrace{c_{5n}}_{\text{this is } \beta_{mn}\times\alpha_{mn}} \cancel{\cos 5\theta}$$

Project on $J_5\left(\frac{z_{5m}r}{r}\right)$:

$$\int_0^a h(r) J_5\left(\frac{z_{5m}}{a}r\right) r\,dr = \int_0^a \sum_{n=1} J_5\left(\frac{z_{5n}}{a}r\right) c_{5n} J_5\left(\frac{z_{5m}}{a}r\right) r\,dr$$

$$= \int_0^a J_5^2\left(\frac{z_{5m}r}{a}\right) c_{5m} r\,dr$$

$$\Rightarrow c_{5n} = \int_0^a h(r) J_5\left(\frac{z_{5n}}{a}r\right) r\,dr \Big/ \int_0^a J_5^2\left(\frac{z_{5n}r}{a}\right) r\,dr$$