# Midterm 2 - AM 212

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## Problem 1

For each of the following 3 ODEs,

a. Plot the numerical solution for $\epsilon = 0.1$, $\epsilon = 0.01$, and $\epsilon = 0.001$

b. Explain in a few words of what method you plan to use the solve this asymptotically and why, based on the numerical solution

c. Find the lowest order uniformly convergent analytical approximation to the solution for small positive $\epsilon$.

d. Compare the numerical and analytical solutions for $\epsilon = 0.01$.

ODE A:

$$\frac{d^2 f}{dt^2} = -f - \epsilon f^2 \left(\frac{df}{dt}\right) \quad \text{with } f(0) = 1, \frac{df}{dt}(0) = 0$$

a. See Figure 1

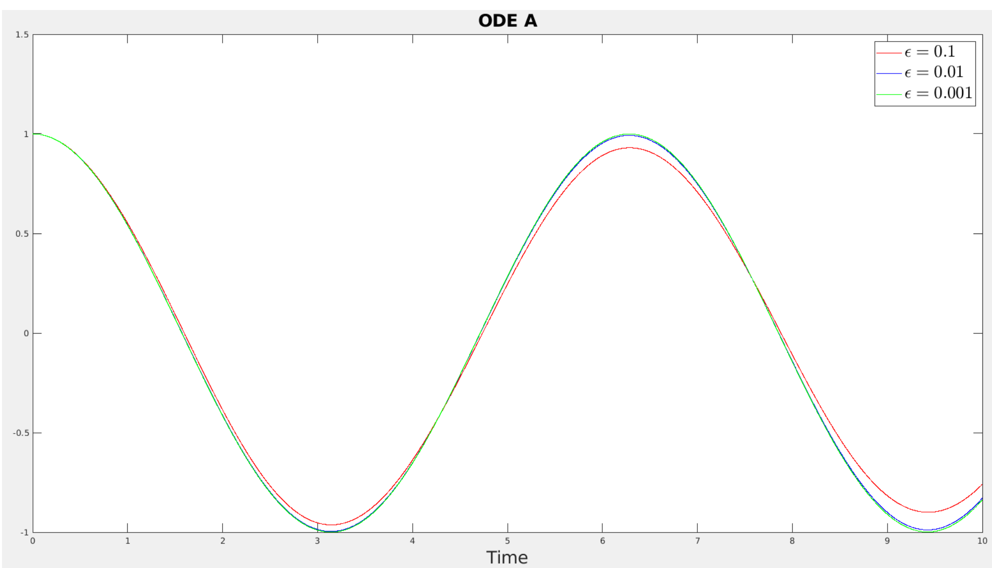


Figure 1: Numerical Solution for differing $\epsilon$ using 'ode45'

b. For this problem, we will use the multiscale method as it seems that the amplitude of the sine wave seems to decay on a timescale which is dependent on epsilon (on a longer time series it is clear that all three solutions decay).

c. To find the lowest order solution we first begin with the multiscale method.

$$t_s = \epsilon t, \quad t_f = t$$
$$\frac{\partial}{\partial t_s} = \frac{dt}{dt_s}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t_f} = \frac{\partial}{\partial t}$$

$$\left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)\left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)[f_0 + \epsilon f_1 + \cdots] =$$
$$-(f_0 + \epsilon f_1 + \cdots) - \epsilon(f_0 + \epsilon f_1 + \cdots)^2\left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)[f_0 + \epsilon f_1 + \cdots]$$
$$f_0(0) = 1, \quad f_i(0) = 0, \ \forall i \geq 1, \quad \frac{\partial f_i}{\partial t} = 0, \ \forall i \geq 0$$

$$O(\epsilon^0): \quad \frac{\partial^2 f_0}{\partial t_f^2} = -f_0 \implies f_0 = cos(t_f)g(t_s)$$
$$O(\epsilon): \quad \frac{\partial^2 f_1}{\partial t_f^2} - 2\frac{\partial g}{\partial t_s}\sin(t_f) = -f_1 + \cos^2(t_f)\sin(t_f)g^3(t_s)$$

The term $\cos^2(t_f)\sin(t_f)$ in the $O(\epsilon)$ equation produces a secular term. Using some trig identities we find that thie reduces to:

$$\cos^2(t_f)\sin(t_f) = \frac{1}{4}\sin(t_f) + \frac{1}{4}\sin(3t_f)$$

We solve for $g(t_s)$ in order to eliminate this secular term. (Notice that a partial derivative in this case becomes a regular derivative since g only depends on $t_s$).

$$-2\frac{dg}{dt_s}\sin(t_f) = \frac{1}{4}\sin(t_f)g^3(t_s)$$
$$\frac{dg}{g^3} = -\frac{dt_s}{8}$$
$$-\frac{1}{2g^2} = -\frac{t_s}{8} + c$$
$$g = \left(\frac{t_s}{4} + c\right)^{-1/2}$$
$$g = \left(\frac{t_s}{4} + 1\right)^{-1/2}, \quad \textbf{IC}$$
$$f_0 = \left(\frac{t_s}{4} + 1\right)^{-1/2}\sin(t_f)$$

This choice of $f_0$ successfully eliminates the secular term allowing $f_1$ to be bounded and therefore producing a uniform expansion. Thus the lowest order solution needed is $O(1)$.

d. The numerical and asymptotic solutions align very nicely for this problem (see Figure 2). Although the solution for $\epsilon = 0.01$ decays very slowly, its decline is visible within 50 time units.

ODE B:

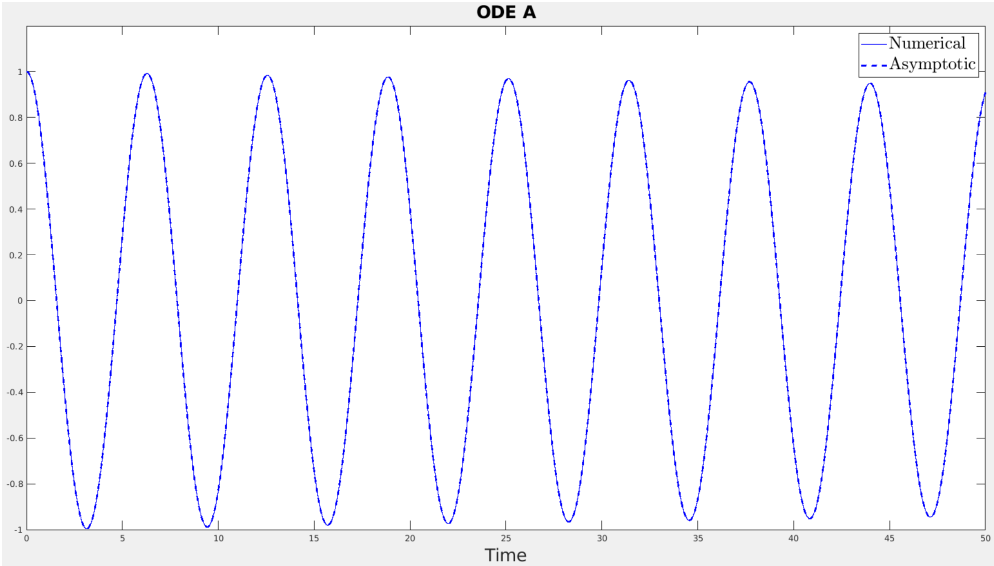


Figure 2: Comparison between numerical and asymptotic solutions for $\epsilon = 0.01$

$$\frac{d^2 f}{dt^2} = -f - \epsilon f \left(\frac{df}{dt}\right)^4 \quad \text{with } f(0) = 1, \frac{df}{dt}(0) = 0$$

a. See Figure 3

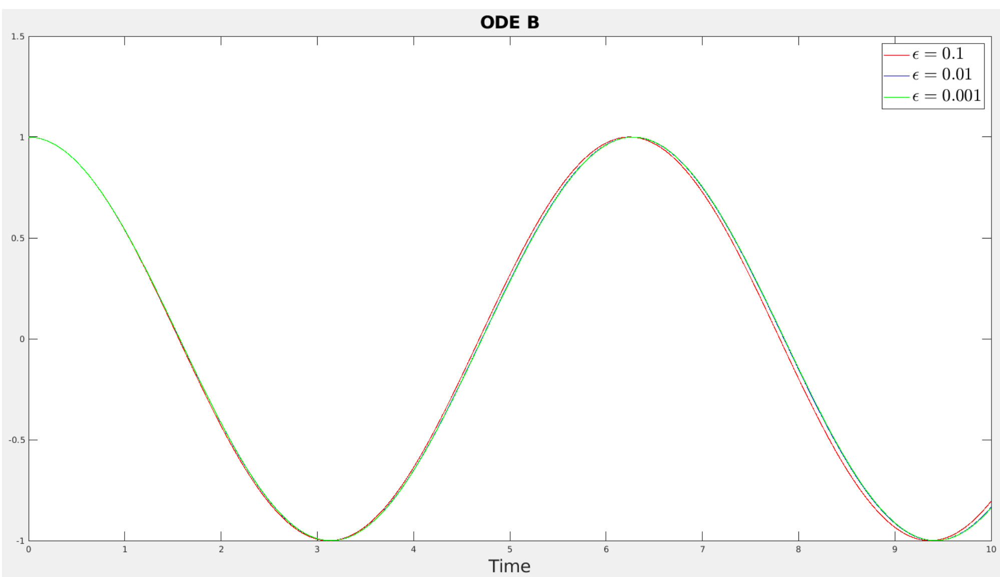


Figure 3: Numerical Solution for differing $\epsilon$ using ‘ode45’

b. I will solve this problem using the method of strained coordinates since the numerical solution suggests that $\epsilon$ will affect the periodicity of the solution rather than the amplitude.

c. In order to solve this problem we introduce $\tau = t + \epsilon w_1 t + O(\epsilon^2)$. We substitute this into the ODE,

$$\frac{d}{dt} = (1 + \epsilon w_1 + O(\epsilon^2))\frac{d}{d\tau}$$

$$(1 + 2\epsilon w_1 + \epsilon^2(w_1 + 2w_2) + O(\epsilon^3))\frac{d^2 f}{d\tau^2} =$$

$$-(f_0 + \epsilon f_1 + O(\epsilon^2) - \epsilon(f_0 + \epsilon f_1 + O(\epsilon^2))(1 + \epsilon w_1 + O(\epsilon^2))^4 \left(\frac{d}{d\tau}(f_0 + \epsilon f_1 + O(\epsilon^2))\right)$$

$$f_0(0) = 1, \quad f_i(0) = 0 \ \forall i \geq 1, \quad \frac{df_i}{d\tau} = 0 \ \forall i \geq 0$$

$$O(\epsilon^0): \quad \frac{d^2 f_0}{d\tau^2} = -f_0 \implies f_0 = \cos(\tau)$$

$$O(\epsilon^1): \quad \frac{d^2 f_1}{d\tau^2} + w_1 \frac{df_0}{d\tau} = -f_1 - \epsilon f_0 f_0'^4$$

$$O(\epsilon^1): \quad \frac{d^2 f_1}{d\tau^2} - 2w_1 \cos(\tau) = -f_1 - \epsilon \cos(\tau)\sin^4(\tau)$$

Here the term $\cos(\tau)\sin^4(\tau)$ will produce a secular term causing non-uniformity. In order to eliminate the secular term we use a trig identity and then fix $w_1$ in order to eliminate this term.

$$\cos(\tau)\sin^4(\tau) = \frac{1}{8}\cos(\tau) - \frac{3}{16}\cos(3\tau) + \frac{1}{16}\cos(5\tau)$$

$$-2w_1\cos(\tau) = -\frac{1}{8}\cos(\tau) \implies w_1 = \frac{1}{16}$$

$$\tau = t + \frac{\epsilon}{16}t + O(\epsilon^2)$$

We have now found $\tau$ to order $\epsilon$ in such a way that $f_1$ is bounded and therefore the first order expansion is uniform.

$$f_0 = \cos\left(t + \frac{\epsilon t}{16}\right)$$

d. As seen in Figure 4, the asymptotic solution follows the numerical solution.

ODE C:

$$\epsilon\frac{d^2 f}{dt^2} + \frac{df}{dt} + (t+1)f = 0 \quad \text{with } f(0) = 1, f(1) = 2$$

a. See Figure 5

b. For this problem, I will use the Boundary Layer method as this PDE has 2 (Boundary) conditions as well as the fact that the second order term in the ODE disappears as $\epsilon \to 0$ creating a Boundary Layer problem.

c. First we find the outer solution by setting $\epsilon = 0$,

$$\frac{df_{out}}{dt} = -(t+1)f_{out}, \quad f(1) = 2$$

$$\frac{df_{out}}{f} = -(t+1)dt$$

$$\ln|f_{out}| = -\frac{t^2}{2} - t + c$$

$$f_{out}(t) = Ce^{-t^2/2 - t}$$

$$\textbf{IC} \to f_{out}(t) = 2e^{-t^2/2 - t + 3/2}$$

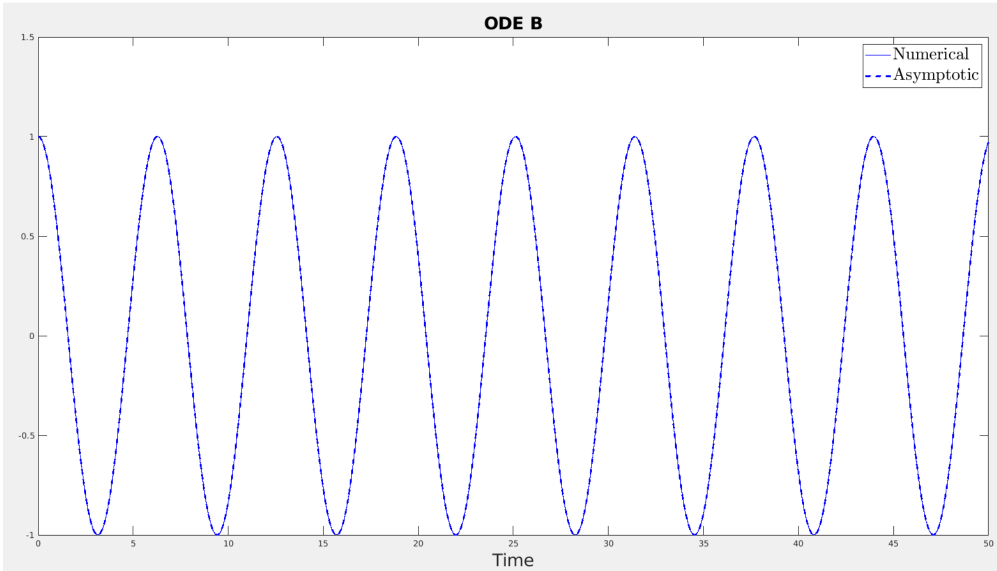


Figure 4: Comparison between numerical and asymptotic solutions for $\epsilon = 0.01$

This completes the outer solution. Next we must begin the inner solution by determining the dominant balance. First we set $t = s\epsilon^{\alpha}$.

$$\epsilon^{1-2\alpha}\frac{d^2 f_{in}}{ds^2} \approx -\epsilon^{-\alpha}\frac{df_{in}}{ds}$$
$$f_{in} = O(1) \implies \alpha = 1$$
$$(s\epsilon + 1)f_{in} \ll O(\epsilon^{-1})$$

This method is self-consistent.

$$\epsilon^{1-2\alpha}\frac{d^2 f_{in}}{ds^2} \approx -(se^{\alpha} + 1)f_{in}$$
$$f_{in} = O(1) \implies \alpha = \frac{1}{3}$$
$$\epsilon^{-1/3}\frac{df_{in}}{ds} \gg O(\epsilon^{1/3})$$

This method is not self consistent; the term it disregards becomes the most dominant term in the equation.

Thus we proceed with $\alpha = 1$ and disregard $(s\epsilon + 1)f$ from the equation.

$$\frac{d^2 f_{in}}{ds^2} = -\frac{df_{in}}{ds}$$
$$\frac{df_{in}}{ds} = c_1 e^{-s}$$
$$f_{in}(s) = -c_1 e^{-s} + c_2$$
$$f_{in}(0) = 1 \implies c_1 = 1 - c_2$$

From here we proceed with Prandtl's matching condition.

$$\lim_{s\to\infty} f_{in}(s) = \lim_{x\to 0} f_{out}(x)$$
$$c_2 = 2e^{3/2} \implies c_1 = 1 - 2e^{3/2}$$
$$f_{in} = (1 - 2e^{3/2})e^{-t/\epsilon} + 2e^{3/2}$$

Finally we write the composite solution.

$$f = f_{in} + f_{out} - L$$
$$f = (1 - 2e^{3/2})e^{-t/\epsilon} + 2e^{-t^2/2 - t + 3/2}$$

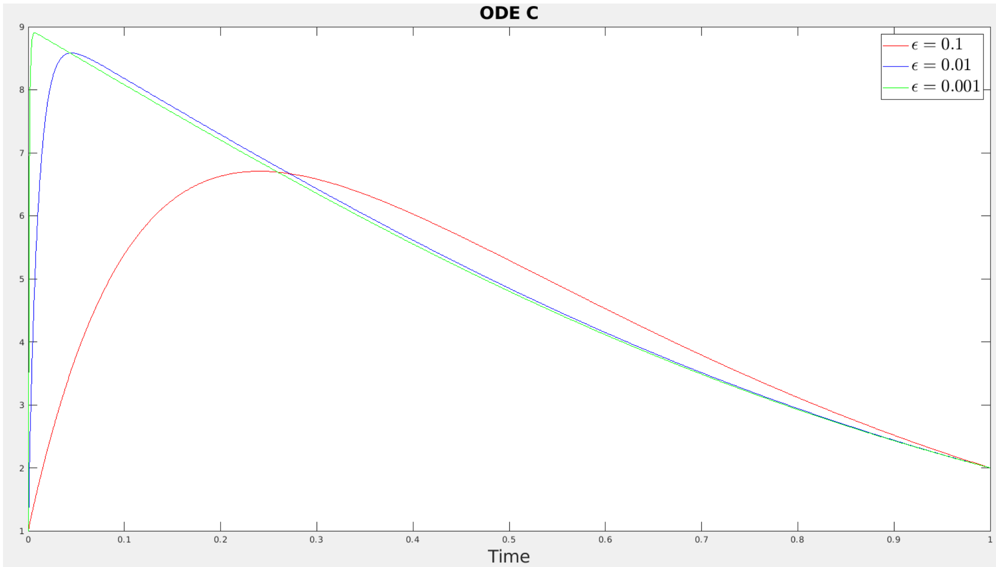


Figure 5: Numerical Solution for differing $\epsilon$ obtained using the shooting method and 'ode45'

d. As seen in Figure 6, the asymptotic solution approximates the numerical solution very closely. It should be noted that the Boundary Layer method often produces some degree of error which qualitatively decreases with $\epsilon$. For example, the error for $\epsilon = 0.1$ is quite large, however, the error for $\epsilon = 0.001$ is not noticeable to the eye on the plot.

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# Problem 2

Find the eigenvalues and eigenfunctions of this eigenvalue problem, in the limite where the eigenvalue $\lambda$ is very large and positive.

$$\frac{d^2 f}{dt^2} + \lambda(x+1)^2 f = 0 \quad \text{with } f(1) = 0, f(2) = 0$$

We begin solving this proble using WKB theory and following the procedure described in the notes. First we put this problem in SL form.

$$\frac{d}{dt}\left(\frac{df}{dt}\right) = -\lambda(x+1)^2 f, \quad p(x) = 1, \quad q(x) = 0, \quad w(x) = (x+1)^2$$

The first step is to take $\lambda = 1/\epsilon^2$ and introduce the WKB rescaling of $x_f = g(x)/\epsilon$.

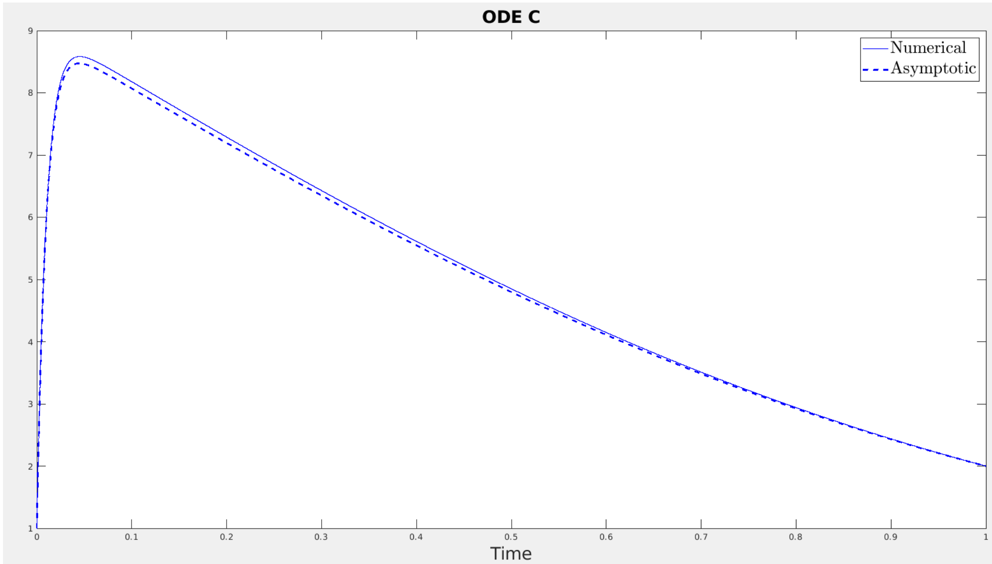


Figure 6: Comparison between numerical and asymptotic solutions for $\epsilon = 0.01$

$$\left(\frac{\partial}{\partial t_s} + \frac{g'(x_s)}{\epsilon}\frac{\partial}{\partial x_f}\right)\left(\frac{\partial f}{\partial x_s} + \frac{g'(x_s)}{\epsilon}\frac{\partial f}{\partial x_f}\right) = -\frac{(x_s+1)^2}{\epsilon^2} f$$

$$O(\epsilon^{-2}): \quad \frac{g'^2(x_s)}{\epsilon^2}\frac{\partial^2 f_0}{\partial x_f^2} = -\frac{(x_s+1)^2}{\epsilon^2} f_0 \implies g'(x_s) = (x_s+1)$$

$$f_0 = a(x_s)\cos(x_f) + b(x_s)\sin(xf), \quad g(x_s) = \frac{x_s^2}{2} + x_s - \frac{3}{2}$$

$$O(\epsilon^{-1}): \quad 2(x+1)\frac{\partial^2 f_0}{\partial x_s \partial x_f} + \frac{\partial f}{\partial xf} + (x_s+1)^2\frac{\partial^2 f_1}{\partial xf^2} = -(x_s+1)^2 f_1$$

$$2(x_s+1)\left(-a'(x_s)\sin(x_f) + b'(x_s)\cos(x_f)\right) + \left(-a(x_s)\sin(x_f) + b(x_s)\cos(x_f)\right) + (x_s+1)^2\frac{\partial^2 f_1}{\partial x_f^2} = -(x_s+1)^2 f_1$$

We have sources of secular terms in this equation and in order to eliminate those terms we require the following equations hold:

$$\begin{aligned} 2(x_s+1)a'(x_s) &= -a(x_s) \\ a'(x_s) &= -\frac{a(x_s)}{2(x_s+1)} \\ \frac{da}{a} &= -\frac{x_s}{2(x_s+1)} \\ \ln|a| &= -\frac{1}{2}\ln|x_s+1| + c \\ a &= A(x_s+1)^{-1/2} \end{aligned}$$

$$\begin{aligned} 2(x_s+1)b'(x_s) &= -b(x_s) \\ b'(x_s) &= -\frac{b(x_s)}{2(x+1)} \\ \frac{db}{b} &= -\frac{x_s}{2(x_s+1)} \\ \ln|b| &= -\frac{1}{2}\ln|x_s+1| + c \\ b &= B(x_s+1)^{-1/2} \end{aligned}$$

Thus we have

$$f_0 = v_n = (x+1)^{-1/2}\left(A\cos\left(\frac{x^2/2+x-3/2}{\epsilon}\right) + B\sin\left(\frac{x^2/2+x-3/2}{\epsilon}\right)\right)$$

$$\mathbf{BC} \to A = 0, \quad \epsilon = \frac{5}{2n\pi}$$

$$\lambda_n = \frac{4n^2\pi^2}{25}$$

For this problem we can approximate the eigenfunctions $v_n$ with $f_0$ and the corresponding eigenvalue with $\lambda_n$ when the eigenvalues are very large.