Midterm 2 - AM 212

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Problem 1

For each of the following 3 ODEs,

a. Plot the numerical solution for $\epsilon = 0.1$, $\epsilon = 0.01$, and $\epsilon = 0.001$

b. Explain in a few words of what method you plan to use the solve this asymptotically and why, based on the numerical solution

c. Find the lowest order uniformly convergent analytical approximation to the solution for small positive ϵ.

d. Compare the numerical and analytical solutions for $\epsilon = 0.01$.

ODE A:

$$\frac{d^2 f}{dt^2} = -f - \epsilon f^2 \left(\frac{df}{dt}\right) \quad \text{with } f(0) = 1, \frac{df}{dt}(0) = 0$$

a. See Figure 1

b. For this problem, we will use the multiscale method as it seems that the amplitude of the sine wave seems to decay on a timescale which is dependent on epsilon (on a longer time series it is clear that all three solutions decay).

c. To find the lowest order solution we first begin with the multiscale method.

$$t_s = \epsilon t, \quad t_f = t$$
$$\frac{\partial}{\partial t_s} = \frac{dt}{dt_s}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t_f} = \frac{\partial}{\partial t}$$

$$\left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)\left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)[f_0 + \epsilon f_1 + \cdots] =$$
$$-(f_0 + \epsilon f_1 + \cdots) - \epsilon(f_0 + \epsilon f_1 + \cdots)^2 \left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)[f_0 + \epsilon f_1 + \cdots]$$
$$f_0(0) = 1, \quad f_i(0) = 0, \ \forall i \geq 1, \quad \frac{\partial f_i}{\partial t} = 0, \ \forall i \geq 0$$

$$O(\epsilon^0): \quad \frac{\partial^2 f_0}{\partial t_f^2} = -f_0 \implies f_0 = cos(t_f)g(t_s)$$
$$O(\epsilon): \quad \frac{\partial^2 f_1}{\partial t_f^2} - 2\frac{\partial g}{\partial t_s}\sin(t_f) = -f_1 + \cos^2(t_f)\sin(t_f)g^3(t_s)$$

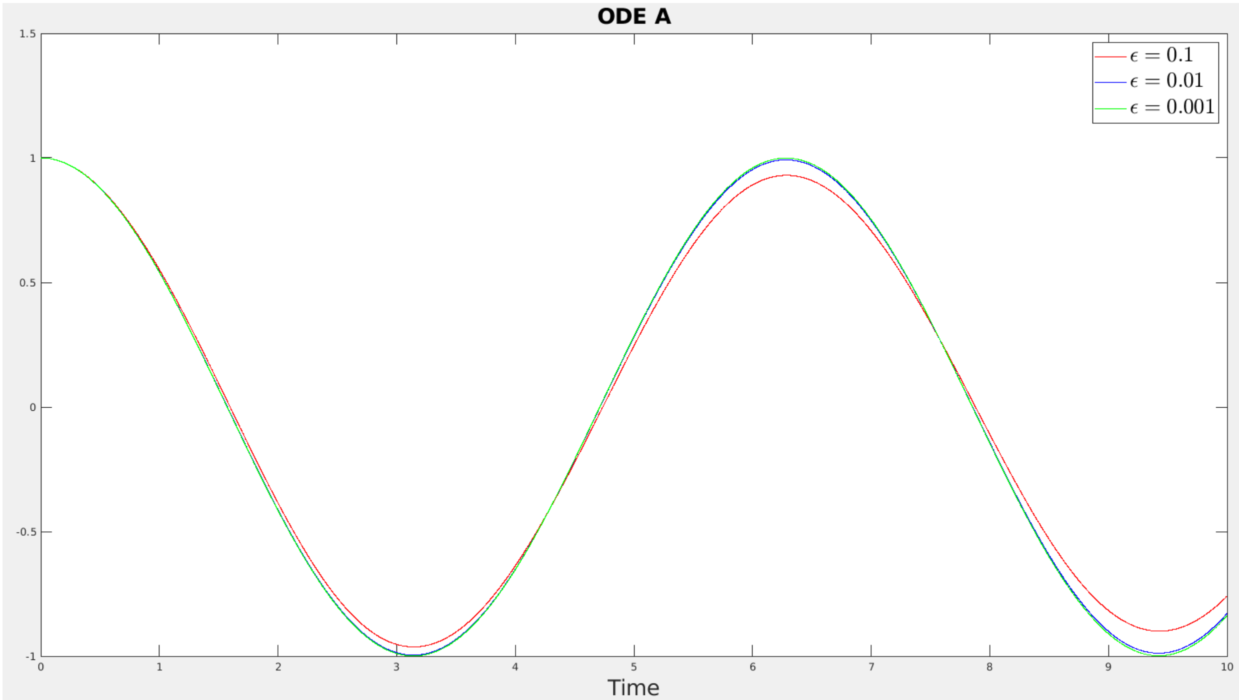


Figure 1: Numerical Solution for differing ϵ using 'ode45'

The term $\cos^2(t_f)\sin(t_f)$ in the $O(\epsilon)$ equation produces a secular term. Using some trig identities we find that thie reduces to:

$$\cos^2(t_f)\sin(t_f) = \frac{1}{4}\sin(t_f) + \frac{1}{4}\sin(3t_f)$$

We solve for $g(t_s)$ in order to eliminate this secular term. (Notice that a partial derivative in this case becomes a regular derivative since g only depends on t_s).

$$-2\frac{dg}{dt_s}\sin(t_f) = \frac{1}{4}\sin(t_f)g^3(t_s)$$

$$\frac{dg}{g^3} = -\frac{dt_s}{8}$$

$$-\frac{1}{2g^2} = -\frac{t_s}{8} + c$$

$$g = \left(\frac{t_s}{4} + c\right)^{-1/2}$$

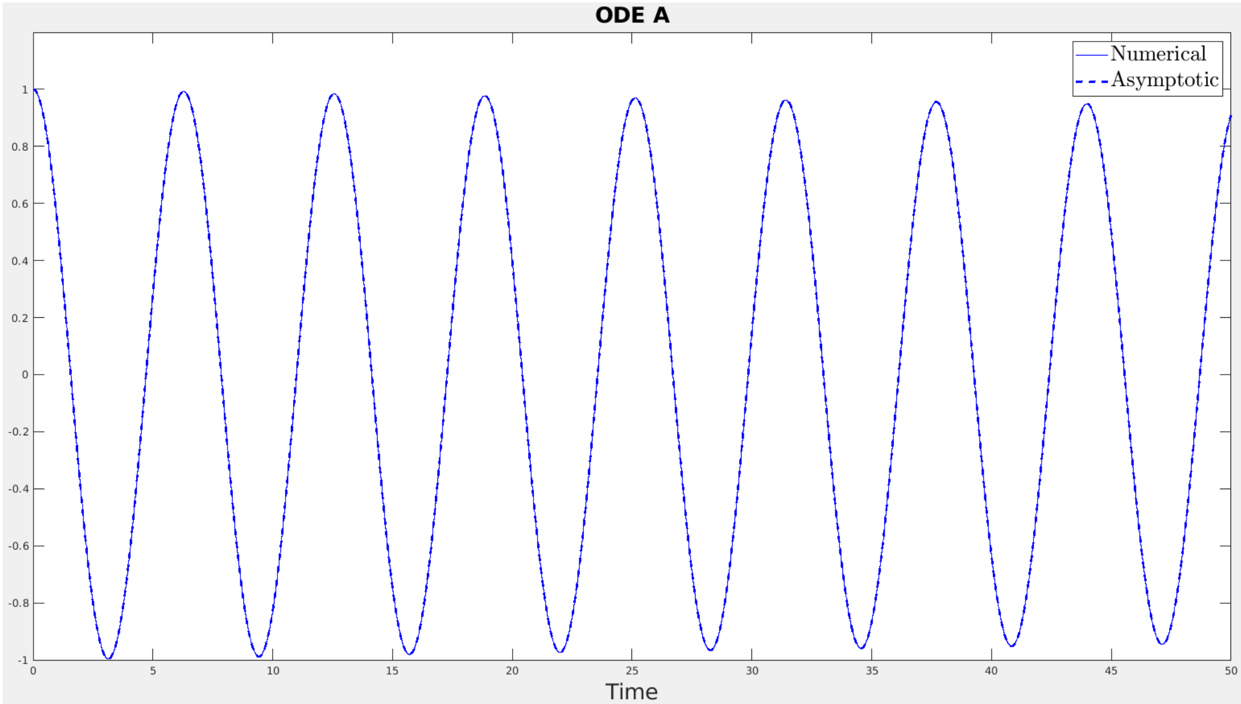
$$g = \left(\frac{t_s}{4} + 1\right)^{-1/2}, \quad \textbf{IC}$$

$$f_0 = \left(\frac{t_s}{4} + 1\right)^{-1/2}\sin(t_f)$$

This choice of f_0 successfully eliminates the secular term allowing f_1 to be bounded and therefore producing a uniform expansion. Thus the lowest order solution needed is $O(1)$.

d. Compare the numerical and analytical solutions for $\epsilon = 0.01$.

ODE B:



Comparison between numerical and asymptotic solutions for $\epsilon = 0.01$

$$\frac{d^2 f}{dt^2} = -f - \epsilon f \left(\frac{df}{dt}\right)^4 \quad \text{with } f(0) = 1, \frac{df}{dt}(0) = 0$$

a. See Figure 2

b. I will solve this problem using the method of strained coordinates since the numerical solution suggests that ϵ will affect the periodicity of the solution rather than the amplitude.

c. lowest order uniformly converging solution

d. Compare the numerical and analytical solutions for $\epsilon = 0.01$.

ODE C:

$$\epsilon \frac{d^2 f}{dt^2} + \frac{df}{dt} + (t+1) f = 0 \quad \text{with } f(0) = 1, f(1) = 2$$

a. See Figure 3

b. For this problem, I will use the Boundary Layer method as this PDE has 2 (Boundary) conditions as well as the fact that the second order term in the ODE disappears as $\epsilon \to 0$ creating a Boundary Layer problem.

c. lowest order uniformly converging solution

d. Compare the numerical and analytical solutions for $\epsilon = 0.01$.

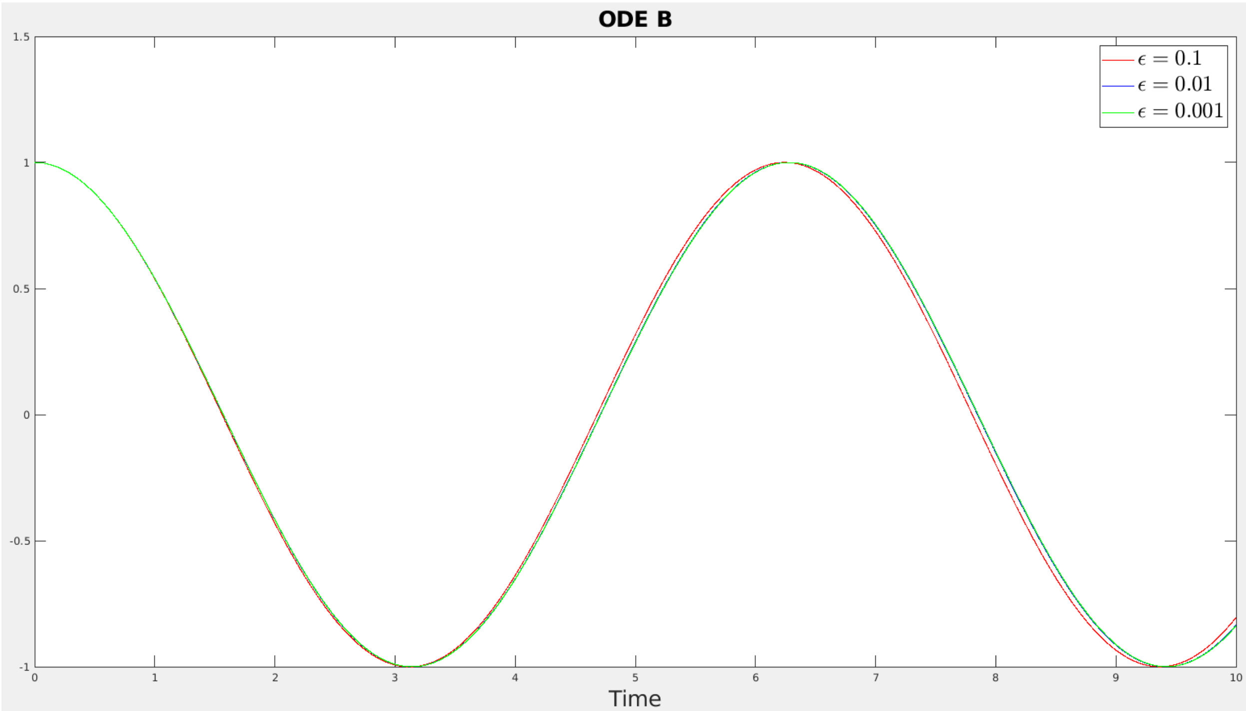


Figure 2: Numerical Solution for differing ϵ using 'ode45'

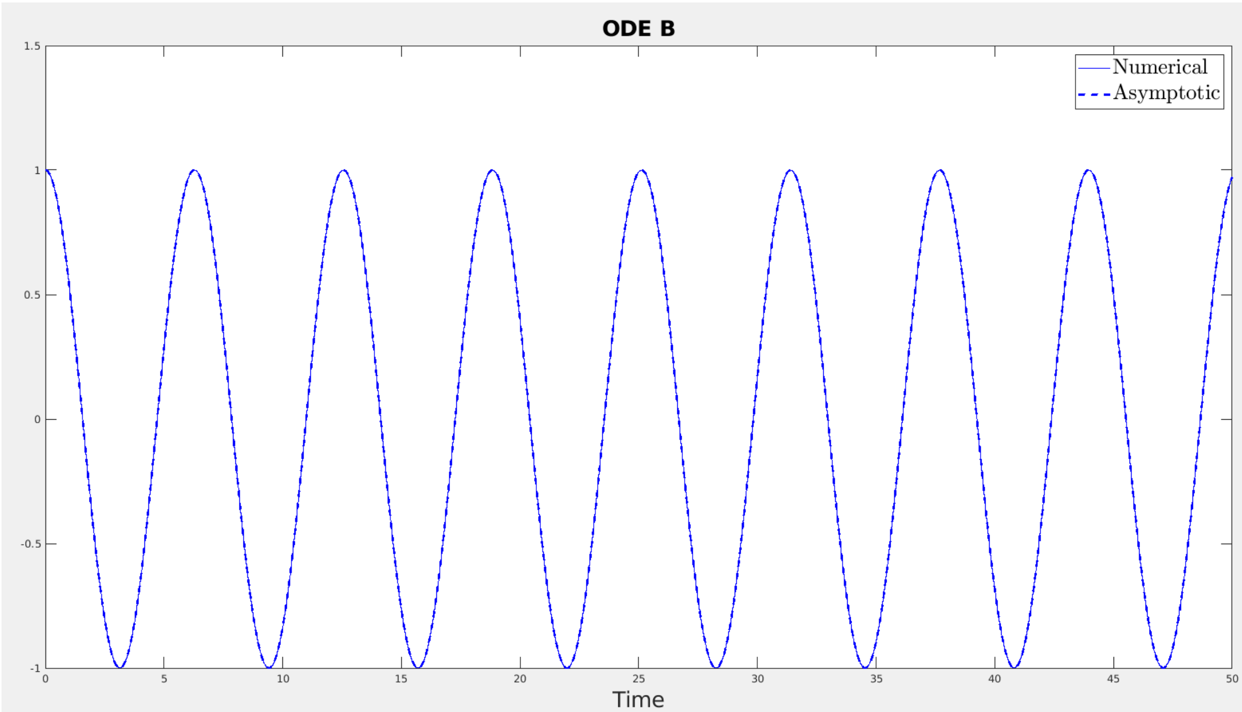
Problem 2

Find the eigenvalues and eigenfunctions of this eigenvalue problem, in the limite where the eigenvalue λ is very large and positive.

$$\frac{d^2 f}{dt^2} + \lambda(x+1)^2 f = 0 \quad \text{with } f(1) = 0, f(2) = 0$$

We begin solving this proble using WKB theory and following the procedure described in the notes. First we put this problem in SL form.

$$\frac{d}{dt}\left(\frac{df}{dt}\right) = -\lambda(x+1)^2 f, \quad p(x) = 1, \quad q(x) = 0, \quad w(x) = (x+1)^2$$



Comparison between numerical and asymptotic solutions for $\epsilon = 0.01$

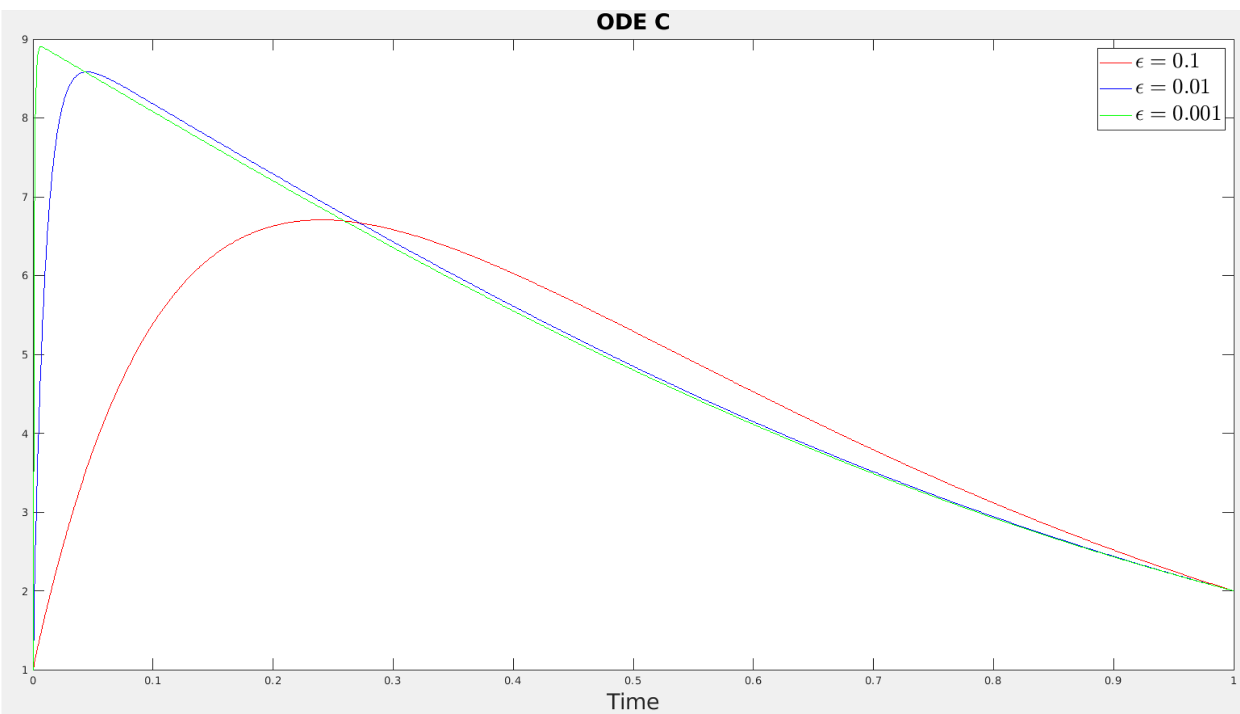
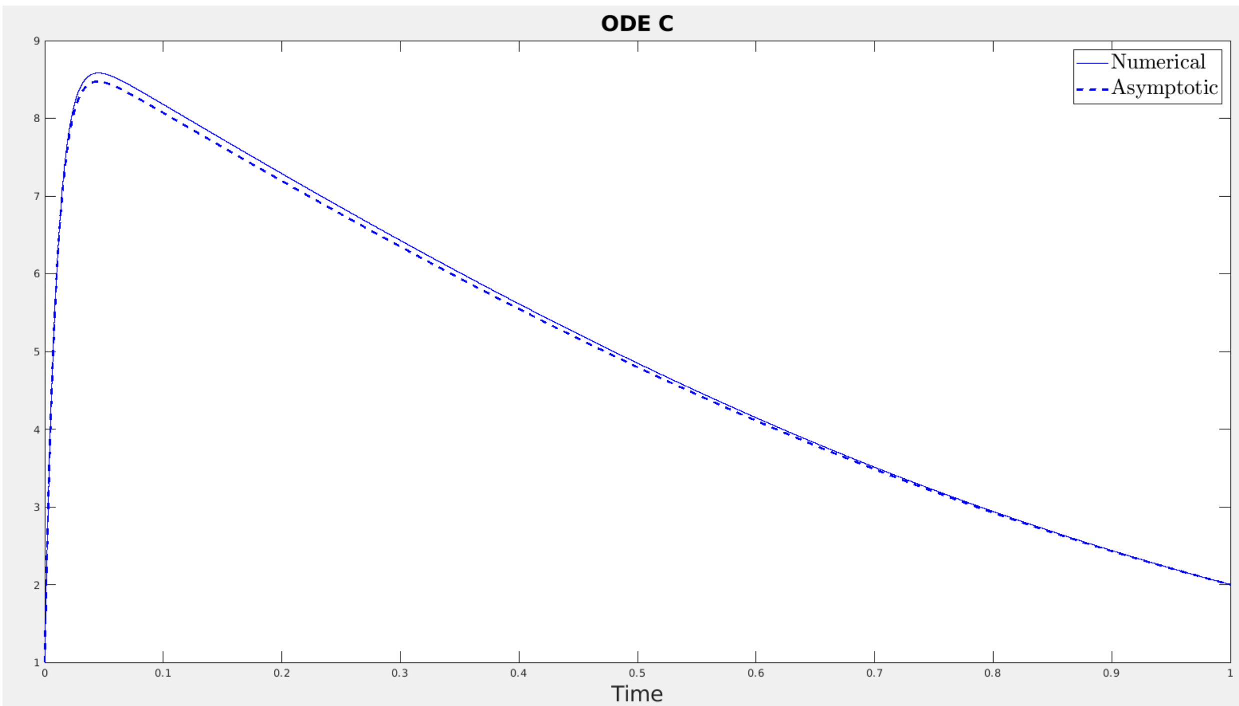


Figure 3: Numerical Solution for differing ϵ obtained using the shooting method and 'ode45'



Comparison between numerical and asymptotic solutions for $\epsilon = 0.01$