AM216 Stochastic Differential Equations

Lecture 12

List of topics in this lecture

- Forward/backward equations in terms of differential operators, inner product, adjoint operator
- An alternative derivation of the forward equation, average reward for a population
- Boundary conditions for the forward/backward equations
- Exit problem, probability of exit by time *t*, average exit time

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Recap

Autonomous SDE: $dX = b(X)dt + \sqrt{a(X)}\,dW$ (Ito)

IVP of the backward equation with a general initial condition

$$\begin{cases} u_t = b(z)u_z + \frac{1}{2}a(z)u_{zz} \\ u(z,t)\big|_{t=0} = u_0(z) \end{cases}$$

Meaning:

$u_0(z)$: reward function; reward is determined at the end time T as $u_0(X(T))$.

$u(z, t)$ = The average reward given starting at position z at time $(T-t)$.

Variable t in the backward equation corresponds to *real time* $(T-t)$.

$u(z, t)$ is (in general) not conserved; the backward equation is not conservative.

IVP of the forward equation with a general initial value

$$\begin{cases} p_t = -\big(b(x)p\big)_x + \frac{1}{2}\big(a(x)p\big)_{xx} \\ p(x,t)\big|_{t=0} = p_0(x) \end{cases}$$

Meaning:

$p_0(z)$: mass density at time 0.

$p(x, t)$ = mass density at time t.

Variable t in the forward equation = *real time* t.

$p(z, t)$ is conserved; the forward equation is conservative.

Forward equation and backward equation in terms of differential operators

We consider the autonomous SDE:

$$dX = b(X)dt + \sqrt{a(X)}\,dW, \qquad \text{(Ito)}$$

We introduce linear differential operator L_z.

$$L_z = b(z)\frac{\partial \bullet}{\partial z} + \frac{1}{2}a(z)\frac{\partial^2 \bullet}{\partial z^2}$$

which means $L_z[u] = b(z)\frac{\partial u}{\partial z} + \frac{1}{2}a(z)\frac{\partial^2 u}{\partial z^2}$

Short story:

1. Backward equation in terms of L_z:

$$u_t = L_z\left[u\right]$$

2. Forward equation in terms of L_z:

$$p_t = L_z^*\left[p\right]$$

where L_z^* is the adjoint operator of L_z, which we will introduce and discuss.

3. An alternative derivation of forward equation that is more intuitive and conceptually simpler than the method of test function.
4. Comments on ensemble average and boundary effect

Long story:

1. **Backward equation** can be written as

$$u_t = L_z\left[u\right].$$

This follows directly from the definition of operator L_z.

To write out the forward equation in terms of L_z, we need to define a few things.

Definition (inner product)

The inner product of two functions is defined as

$$\langle u_1, u_2 \rangle \equiv \int u_1(z) u_2(z) dz$$

<u>Definition</u> (adjoint operator)

The adjoint operator of L is denoted by L^*, and is defined by the condition

$\langle u, L^*[v] \rangle = \langle L[u], v \rangle$ for all functions $u(z)$ and $v(z)$ of compact support.

<u>Example:</u>

Let A be an $n \times n$ matrix. We view matrix A as a linear operator: $\mathbb{R}^n \to \mathbb{R}^n$

$$u \longrightarrow Au$$

For vectors, the inner product is

$$\langle u, v \rangle = \sum u_i v_i = u^T v$$

We use the definition of adjoint operator to find A^*.

$$\underbrace{\langle u, A^* v \rangle = \langle Au, v \rangle}_{\text{definition of } A^*} \to (Au)^T v \to u^T A^T v \to u^T (A^T v) \to \langle u, A^T v \rangle$$

==> $\langle u, A^* v \rangle = \langle u, A^T v \rangle$ for all vectors u and v.

==> $A^* = A^{\mathrm{T}}$

For matrix A, the adjoint operator of A is A^{T}.

<u>Example:</u>

Consider differential operator

$$D_x = a(x) \frac{\partial^2 \bullet}{\partial x^2}$$

We use the definition of adjoint operator to find D_x^*.

$$\underbrace{\langle u, D_x^*[v] \rangle = \langle D_x[u], v \rangle}_{\text{definition of } D_x^*} \to \int a(x) \frac{\partial^2 u}{\partial x^2} v(x) dx \to \int \frac{\partial^2 u}{\partial x^2} \big(a(x) v(x) \big) dx$$

Integrating by parts twice, we write the RHS as

$$\text{RHS} = -\int \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial x} \big(a(x) v(x) \big) dx = \int u(x) \cdot \frac{\partial^2}{\partial x^2} \big(a(x) v(x) \big) dx = \left\langle u, \frac{\partial^2}{\partial x^2} \big(a(x) v \big) \right\rangle$$

==> $\langle u, D_x^*[v] \rangle = \left\langle u, \frac{\partial^2}{\partial x^2} \big(a(x) v \big) \right\rangle$ for all functions $u(x)$ and $v(x)$.

==> $D_x^* = \frac{\partial^2}{\partial x^2}\left(a(x)\bullet\right)$

Example:

Consider differential operator

$$L_z = b(z)\frac{\partial \bullet}{\partial z} + \frac{1}{2}a(z)\frac{\partial^2 \bullet}{\partial z^2}$$

The adjoint operator of L_z is

$$L_z^* = -\frac{\partial}{\partial z}\left(b(z)\bullet\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(a(z)\bullet\right) \qquad \text{(homework problem)}$$

2. **Forward equation in terms of** L_Z:

   Comparing the expression of forward equation we derived previously and the expression of L_z^* in the example above, we write the forward equation as

   $p_t = L_z^*\left[p\right]$ where L_z^* is the adjoint operator of L_z.

   Note: We have changed variable x to variable z.

3. **A more intuitive derivation of the forward equation**

   Logically,

   - We discard the forward equation derived using the method of test function.
   - Instead we derive the governing equation for the mass density.

   First, we look at the IVP of the backward equation

   $$\begin{cases} u_t = A[u] \\ u(x,0) = u_0(x) \end{cases} \qquad \text{where } A = L_x.$$

   Note: We have changed variable z to variable x.

   Function $u(x, t)$ has the meaning:

   - The amount of reward is determined at the end time T as $u_0(X(T))$.
   - $u(x, t)$ = average reward given starting at position x at time $(T-t)$.

     $$u(x,t) = E\left(u_0(X(T)) \middle| X(T-t) = x\right)$$

   - Variable t in the backward equation corresponds to *real time* $(T-t)$.

   Next, we look at the IVP of the forward equation

$$\begin{cases} p_t = B[p] \\ p(x, T-t_0) = p_0(x) \end{cases}$$

We are going to show $B = A^*$.

Function $p(x, t)$ has the meaning:

- $p(x, T-t_0) = p_0(x)$ is the mass density at *real time* $(T-t_0)$.
- $p(x, t)$ = the mass density at *real time* $t > T-t_0$

<u>Key observation:</u>

Consider a population with mass density $p(x, T-t_0)$ at *real time* $(T-t_0)$.

We combine $u(\ )$ and $p(\ )$ to calculate the average reward for the population

$u(x, t_0)$, the conditional average reward given $X(T-t_0) = x$, and

$p(x, T-t_0)$ mass density of $X(T-t_0)$.

The law of total expectation gives

$$\text{Average reward} = \int u(x,t_0)p(x,T-t_0)dx \qquad \text{(Expression 1)}$$

On the other hand, by definition, the reward occurs at the end time T. The average reward for the population is determined by the reward function $u_0(x) \equiv u(x, 0)$ and the mass density $p(x, T)$ at time T.

$$\text{Average reward} = \int u_0(x)p(x,T)dx \qquad \text{(Expression 2)}$$

Equating the two expressions of average reward, we obtain

$$\int \underbrace{u(x,t_0)}_{\substack{\text{solved from the}\\ \text{backward Eq}}} \underbrace{p(x,T-t_0)}_{\substack{\text{mass density}\\ \text{at time } (T-t_0)}} dx = \int \underbrace{u_0(X)}_{\substack{\text{reward}\\ \text{function}}} \underbrace{p(x,T)}_{\substack{\text{solved from the}\\ \text{forward Eq}}} dx \quad \text{for all } t_0 > 0 \qquad \text{(E01)}$$

<u>Writing out operator B</u>

Given the mass density $p(x, T-t_0)$ at time $(T-t_0)$, we have two ways to calculate the average reward for the ensemble

- Solve the backward equation with the given $u_0(x)$ to calculate $u(x, t_0)$ …
- Solve the forward equation with the given $p(x, T-t_0)$ to calculate $p(x, T)$ …

This is how the backward equation and the forward equation are related in (E01). We use (E01) to write out operator B in terms of operator A.

Since (E01) is valid for all $t_0 > 0$, we set $t_0 = \Delta t$.

$$\int u(x,\Delta t)p(x,T-\Delta t)dx = \int u_0(x)p(x,T)dx \qquad \text{(E02)}$$

We set $p(x, T-\Delta t) = v(x)$.

We write $u(x, \Delta t)$ and $p(x, T)$ in terms of $u_0(x)$ and $v(x)$ as follows.

Backward equation: $u_t = A[u]$

$$u(x,\Delta t)=u(x,0)+\Delta t\, u_t(x,0)+o(\Delta t)$$
$$=u_0(x)+\Delta t\, A[u_0(x)]+o(\Delta t)$$

Forward equation: $p_t = B[p]$

$$p(x,T)=p(x,T-\Delta t)+\Delta t\, p_t(x,T-\Delta t)+o(\Delta t)$$
$$=v(x)+\Delta t\, B[v(x)]+o(\Delta t)$$

Substituting into (E02) leads to

$$\text{LHS}=\int\big(u_0(x)+\Delta t\, A[u_0(x)]+o(\Delta t)\big)v(x)dx$$
$$=\int u_0(x)v(x)dx+\Delta t\int A[u_0(x)]v(x)dx+o(\Delta t)$$

$$\text{RHS}=\int u_0(x)\big(v(x)+\Delta t\, B[v(x)]+o(\Delta t)\big)dx$$
$$=\int u_0(x)v(x)dx+\Delta t\int u_0(x)B[v(x)]dx+o(\Delta t)$$

Subtracting $\int u_0(x)v(x)dx$ from both LHS and RHS, dividing by Δt and taking the limit as $\Delta t \to 0$, we arrive at

$$\int A[u_0(x)]v(x)dx=\int u_0(x)B[v(x)]dx \qquad \text{for all } u_0(x) \text{ and } v(x)$$

In terms of inner product, it becomes

$$\big\langle u_0, B[v]\big\rangle=\big\langle A[u_0], v\big\rangle \qquad \text{for all } u_0(x) \text{ and } v(x)$$

which implies $B = A^*$.

The end of intuitive derivation of the forward equation

4. Comments on ensemble average and boundary effect

- Ensemble average

  On the RHS of (E01), the reward is averaged over all independent copies in the ensemble starting with mass density $p(x, T-t_0)$ at time $(T-t_0)$.

  $$\int u_0(X)p(x,T)dx\,, \qquad p(x,T)=\text{density based on the intact ensemble}$$

  If the ensemble is modified in $[(T-t_0), T]$, then (E01) is no longer valid.

  Example: Consider a call option of a stock.

  A call option is the right (not obligation) to buy a certain number of shares of the stock at a specified price at a preset time (expiration date). Let

  T = the expiration time

x_C = specified price in the call option

$X(t)$ = the stock price at time t.

t_0 = time until expiration;   corresponding to real time $(T-t_0)$

x_0 = the starting value of $X(T-t_0)$

The reward for the call option holder is realized at expiration and is determined by the market price of the stock at expiration. The reward function is

$$u_0\big(X(T)\big)=\begin{cases} X(T)-x_C\,, & X(T)>x_C \\ 0\,, & X(T)\le x_C \end{cases}$$

The expected reward given $X(T-t_0)$ is

$$E\Big(u_0(X(T))\Big|X(T-t_0)=x_0\Big)$$

This is the average over all independent copies in the ensemble starting with $X(T-t_0) = x_0$. The price of the call option reflects the expected reward. We will discuss the option pricing later (Black Scholes model).

Let $(T-t_2) > (T-t_0)$ be a later time. $t_2 < t_0$ means a shorter time until expiration. When the stock price $X(T-t_2) = x_2$ becomes known, the expected reward is updated to

$$E\Big(u_0(X(T))\Big|X(T-t_2)=x_2\Big)$$

This is the average over a modified ensemble, consisting of those copies in the original ensemble satisfying $X(T-t_2) = x_2$. When the original ensemble is modified, the expected reward is also modified.

$$E\Big(u_0(X(T))\Big|X(T-t_2)=x_2\Big) \ne E\Big(u_0(X(T))\Big|X(T-t_0)=x_0\Big)$$

<u>End of example</u>

Boundary effect is another way of modifying the original ensemble.

<u>Question:</u> In the derivation of forward equation above, if a boundary is present, how should we deal with the boundary effect?

- <u>Boundary effect in the derivation of forward equation</u>

Recall that in the derivation of backward equation, we start with $X(t) = x$ away from boundary; we select dt small enough such that

Pr(X hitting boundary in $[t, t+dt]$) = negligible …

In the derivation of forward equation above, we select $u_0(x)$ and $v(x)$ with supports not touching the boundary, and we select $t_0 = \Delta t$ small enough such that starting within the supports, the probability of touching boundary in $[t, t+dt]$ is negligible.

<u>Summary:</u>

Evolution equation itself is not affected by boundary effect.

The evolution of a given initial condition, however, is affected by boundary effect.

End of long story

Boundary conditions

SDE: $dX = b(X)dt + \sqrt{a(X)}\,dW$ (Ito)

Absorbing boundary at $x = L$

- When a particle gets to $x = L$, it is removed from the set of particles.
- When a game reaches $x = L$, it is ended (removed from the set of ongoing games).

For the forward equation, the absorbing boundary is described by

$$p(x,t)\Big|_{x=L} = 0$$

That is, the mass density at $x = L$ is zero.

For the backward equation,

$u(x, t)$ = average reward given starting at position x at time $(T-t)$

The absorbing boundary is described by

$$u(x,t)\Big|_{x=L} = 0$$

That is, starting at $x = L$, it is removed immediately and thus cannot get any reward.

Reflecting boundary at $x = L$

- When a particle tries to go through $x = L$, it is not allowed to pass through; it is not removed; instead it is "turned back".

For the forward equation, the reflecting boundary is described by

$$J(x,t)\Big|_{x=L} = 0$$

where $J(x,t) \equiv b(x)p - \frac{1}{2}\big(a(x)p\big)_x$ is the flux.

That is, the flux through $x = L$ is zero.

For the backward equation, the reflecting boundary is described by

$$\frac{\partial u(x,t)}{\partial x}\Big|_{x=L} = 0$$

Derivation:

At $x = L$, we set the increment $(dX \mid X(T-t) = L)$ as follows.

$$dX = -\left|b(L)dt + \sqrt{a(L)}dW\right| = -\sqrt{a(L)}\left|dW\right| + O(dt)$$

==> $E(dX) = -\sqrt{a(L)}\,E(|dW|) + O(dt) = O(\sqrt{dt})$ (This is the key)

$$u(L,t) = E\big(u(L+dX, t-dt)\big) = E\big(u(L,t) + u_x(L,t)dX + O(dt)\big)$$

==> $u_x(L,t)E(dX) = O(dt)$

==> $u_x(L,t) = \dfrac{O(dt)}{E(dX)} \to 0$

Next we look at applications of the forward and the backward equations.

Exit problem:

Suppose $X(t)$ is governed by the SDE

$$dX = b(X)dt + \sqrt{a(X)}\,dW \qquad \text{(Ito)}$$

Consider the problem of exiting (i.e., escaping from) a prescribed region.

We study the time until escape, also called the exit time or the escape time.

Probability of exit by time t

Let Y = the exit time (a random variable).

Let $u(x, t)$ = probability of exiting the region by time t given starting at x at time 0.

$$u(x,t) \equiv \Pr\big(Y \le t \,\big|\, X(0) = x\big)$$

Governing equation for $u(x, t)$

For x inside the region, when dt is small enough, we have

$$u(x,t) = E\big(u(x+dX, t-dt)\big) + o(dt)$$

Taylor expansion + moments of dX leads to the backward equation

$$u_t = b(x)u_x + \frac{1}{2}a(x)u_{xx}$$

Average exit time

Let $T(x)$ be the average exit time given that $X(0) = x$.

$$T(x) \equiv E\big(Y \,\big|\, X(0) = x\big)$$

Governing equation for $T(x)$

For x inside the region, when dt is small enough, we have

$$T(x) = E\big(T(x+dX)\big) + dt + o(dt)$$

Taylor expansion + moments of dX leads to an ODE for $T(x)$.

$$\frac{1}{2}a(x)T_{xx} + b(x)T_x = -1$$

We look at a few examples before discussing "escape of a Brownian particle".

Example: The particle undergoes pure diffusion with no net drift:

$a(x) = 1, \quad b(x) = 0$

The region is $[0, L]$. Exit can occur at either $x = 0$ or $x = L$.

We have seen this example in the Gambler's ruin problem (fair game) where

x: your initial cash; L: total cash of casino + you

$T(x)$ is the time until breaking the bank or bankrupt.

The boundary value problem (BVP) for $T(x)$ is

$$\begin{cases} T_{xx} = -2 \\ T(0) = 0, \quad T(L) = 0 \end{cases}$$

The solution is

$T(x) = x\,(L-x)$

In particular, we have

$T(L/2) = L^2/4$

Example: The particle undergoes diffusion with a net drift:

$a(x) = 1, \quad b(x) = b$

The region is $[L_1, L_2]$. Exit can occur at either L_1 or L_2.

This example is similar to the Gambler's ruin problem (biased game).

The boundary value problem (BVP) for $T(x)$ is

$$\begin{cases} T_{xx} + 2bT_x = -2 \\ T(L_1) = 0, \quad T(L_2) = 0 \end{cases}$$

The solution is

$$T(x) = \frac{1}{b}(L_2 - x) - \frac{1}{b}(L_2 - L_1)\cdot\frac{\exp(2b(L_2 - x)) - 1}{\exp(2b(L_2 - L_1)) - 1}$$

(homework problem)

For $b > 0$ (a net drift in the positive direction), we look at the limit as $L_1 \to -\infty$ (the lower boundary disappears).

$$T(x) \to \frac{1}{b}(L_2 - x) \qquad \text{as } L_1 \to -\infty .$$

This is consistent with the picture of a deterministic escape.

A model for the casino:

We model the situation in a casino as a two-player game:

- the casino is one player with $b > 0$ (positive net drift) and with initial cash x;
- all other gamblers are collectively viewed as the other player

Here the casino is the player in focus.

Suppose the casino starts with no cash ($x = 0$) but has a line of credit. It is solvent as long as the balance is above L_1 ($L_1 < 0$).

The end of game is defined as either the casino's balance dropping below L_1 (which is very unlikely) or the casino winning L_2 amount from the other player (getting L_2 amount of revenue). The average time until the end of game is

$$T(0) \to \frac{1}{b} L_2 \qquad \text{as } L_1 \to -\infty$$

This is the average time for the casino to get L_2 amount of revenue.

For the casino, the end of one game is also the start of a new game (transferring L_2 amount to the revenue account and resetting cash balance to 0).