AM216 Stochastic Differential Equations

Lecture 09
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List of topics in this lecture

- Convergence in probability, a sufficient condition for convergence in probability, a theorem for calculating the variance of sum of products
- Ito's interpretation and Stratonovich's interpretation of stochastic integrals, the relation between the two, proof of Ito's lemma
- Stochastic integrals based on axioms, the λ-chain rule

Recap

Going backward in time in equilibrium

Tool: Bayes theorem

Backward time evolution in equilibrium OU process

$$\Big(Y(-t)\Big|Y(0)=y_2\Big) \sim N\left(e^{-\beta t}y_2,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\right) \qquad \text{for } t>0$$

Forward time evolution in equilibrium OU process

$$\Big(Y(t)\Big|Y(0)=y_0\Big) \sim N\left(e^{-\beta t}y_0,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\right) \qquad \text{for } t>0$$

Time reversibility of equilibrium

Different interpretations of stochastic integrals

$$\int_0^t f(s,W(s))dW(s) = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j, W(\tilde{s}_j))\Delta W_j$$

where $\Delta s = \frac{t}{N}$, $s_j = j\Delta s$, $\Delta W_j = W_{j+1} - W_j$, $\tilde{s}_j \in [s_j\ s_{j+1}]$

Different choices of $\tilde{s}_j \in [s_j, s_{j+1}]$ lead to different results.

Ito's interpretation (Kiyosi Ito):

$\tilde{s}_j = s_j$

$$\int_0^t f(s,W(s))dW(s) = \lim_{N\to\infty}\sum_{j=0}^{N-1} f_j \Delta W_j\,, \qquad f_j \equiv f(s_j, W(s_j))$$

Stratonovich's interpretation (Ruslan Stratonovich):

$$\tilde{s}_j = \frac{1}{2}(s_j + s_{j+1})$$

$$\int_0^t f(s,W(s))dW(s) = \lim_{N\to\infty}\sum_{j=0}^{N-1} \frac{1}{2}(f_j + f_{j+1})\Delta W_j\,, \qquad f_j \equiv f(s_j, W(s_j))$$

Note: Stratonovich's interpretation is based on the trapezoidal rule; it is not exactly the Riemann sum with $\tilde{s}_j = (s_j + s_{j+1})/2$. The two are equivalent (see below).

Road map of the discussion:

1. We show that the Stratonovich's interpretation is equivalent to the Riemann sum with $\tilde{s}_j = s_{j+1/2} \equiv (s_j + s_{j+1})/2$.

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\left[\frac{1}{2}(f_j + f_{j+1}) - f_{j+1/2}\right]\Delta W_j = 0\,, \qquad f_{j+1/2} \equiv f(s_{j+1/2}, W(s_{j+1/2}))$$

2. We demonstrate the relation between the Ito interpretation and the Stratonovich interpretation in a simple example.
3. As a tool for connecting the Ito interpretation and the Stratonovich interpretation in the general case, we prove Ito's lemma.
4. We write out the relation between the Ito interpretation and the Stratonovich interpretation in the general case.

Preparation for discussion

Recall the convergence in probability.

Definition (convergence in probability)

Let $\{Q_N(\omega)\}$ be a sequence of random variables. We say that $\{Q_N(\omega)\}$ converges to q in probability as $N \to +\infty$, if for any $\varepsilon > 0$,

$$\lim_{N\to\infty} \Pr\left(|Q_N(\omega) - q| > \varepsilon\right) = 0$$

Theorem (a sufficient condition for convergence in probability)

Suppose $\lim_{N\to\infty} E\big(Q_N(\omega)\big)=q$ and $\lim_{N\to\infty} \mathrm{var}\big(Q_N(\omega)\big)=0$.

Then, $\{Q_N(\omega)\}$ converges to q in probability as $N \to +\infty$.

Proof: homework problem.

Theorem (a useful formula for calculating $\mathrm{var}\left(\sum_{j=0}^{N-1} Y_j X_j\right)$)

Suppose random variables $\{X_j, j = 0, 1, \ldots, N-1\}$ and $\{Y_k, k = 0, 1, \ldots, N-1\}$ satisfy

1. $E(X_j) = 0$ for all j,
2. X_j is independent of X_i for all $i \neq j$, and
3. X_j is independent of Y_k for all $k \leq j$.

Then we have
$$\mathrm{var}\left(\sum_{j=0}^{N-1} Y_j X_j\right)=\sum_{j=0}^{N-1} E(Y_j^2)E(X_j^2).$$

Remarks:

- Important note: the theorem does not require "Y_j is independent of Y_i for all $i \neq j$".
  Example: $X_j = (\Delta W_j)^2$, $Y_j = (W_j)^2$.
- We can write the conclusion as $\mathrm{var}\left(\sum_{j=0}^{N-1} Y_j X_j\right)=\mathrm{var}(Y_j X_j)$.

$$E(Y_j X_j)=E(Y_j)E(X_j)=0$$

$$\mathrm{var}(Y_j X_j)=E\big((Y_j X_j)^2\big)=E(Y_j^2)E(X_j^2)$$

Proof: homework problem.

Item 1 of the road map:

We state a general theorem that includes item 1 as a special case.

Theorem: (weighted average of two Riemann sums)

Let $f(s, w)$ be a smooth function of (s, w), and $\Delta s=\dfrac{t}{N}$, $s_j = j\Delta s$, $\Delta W_j = W_{j+1}-W_j$.

For any $0 \leq \beta \leq 1$, we have

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\Big[\big((1-\beta)f_j+\beta f_{j+1}\big)-f_{j+\beta}\Big]\Delta W_j=0$$

where $s_{j+\beta}\equiv s_j+\beta\Delta s$, $f_{j+\beta}\equiv f(s_{j+\beta},W(s_{j+\beta}))$

Proof: (Skip)

We look at the main steps in the proof for β = 1/2. Let

$$Q_N \equiv \sum_{j=0}^{N-1}\left(\frac{1}{2}(f_j + f_{j+1}) - f_{j+1/2}\right)\Delta W_j$$

We only need to show $\lim\limits_{N\to\infty} E\big(Q_N(\omega)\big) = q$ and $\lim\limits_{N\to\infty} \mathrm{var}\big(Q_N(\omega)\big) = 0$.

- Expand f_{j+1} and $f_{j+1/2}$ around s_j. Neglect O(Δs) terms in the expansions of f_{j+1} and $f_{j+1/2}$.

$$f_{j+1} = f_j + (f_w)_j(\Delta W_j) + O(\Delta s)$$

$$f_{j+1/2} = f_j + (f_w)_j(\Delta W_j^{(-)}) + O(\Delta s), \qquad \Delta W_j^{(-)} \equiv W_{j+1/2} - W_j$$

$$\frac{1}{2}(f_j + f_{j+1}) - f_{j+1/2} = (f_w)\frac{1}{2}\left(\Delta W_j^{(+)} - \Delta W_j^{(-)}\right) + O(\Delta s)$$

$$\Delta W_j^{(+)} \equiv W_{j+1} - W_{j+1/2}, \qquad \Delta W_j = \Delta W_j^{(-)} + \Delta W_j^{(+)}$$

- Multiply by $\Delta W_j = \Delta W_j^{(-)} + \Delta W_j^{(+)}$ and sum over j, we write Q_N as

$$Q_N = \frac{1}{2}\sum_{j=0}^{N-1}\underbrace{(f_w)_j}_{Y_j}\underbrace{\left((\Delta W_j^{(+)})^2 - (\Delta W_j^{(-)})^2\right)}_{X_j} + \underbrace{\sum_{j=0}^{N-1} O(\Delta s)\Delta W_j}_{o(1)}$$

- Use the theorem to show $\lim\limits_{N\to\infty} E\big(Q_N(\omega)\big) = q$ and $\lim\limits_{N\to\infty} \mathrm{var}\big(Q_N(\omega)\big) = 0$.

$$E(X_j) = E\left((\Delta W_j^{(+)})^2 - (\Delta W_j^{(-)})^2\right) = 0$$

$$E(X_j^2) = E\left((\Delta W_j^{(+)})^4 - 2(\Delta W_j^{(+)})^2(\Delta W_j^{(-)})^2 + (\Delta W_j^{(-)})^4\right) = (\Delta s)^2$$

Here we used $E\left((\Delta W_j^{(-)})^2\right) = (\Delta s)/2$, $\quad E\left((\Delta W_j^{(-)})^4\right) = 3(\Delta s)/2$.

Item 2 of the road map: (an example)

Key point: Ito interpretation and Stratonovich interpretation yield different values!

Example: $I = \int_0^t W(s)dW(s)$

Discretization: $\Delta s = \dfrac{t}{N}$, $\quad s_j = j\Delta s$, $\quad \Delta W_j = W_{j+1} - W_j$.

We first work out the Stratonovich interpretation:

$$I_{\text{Stratonovich}} = \lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(W_j + W_{j+1})(\Delta W_j) = \lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(W_j + W_{j+1})(W_{j+1} - W_j)$$

$$=\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}\left((W_{j+1})^2-(W_j)^2\right)=\frac{1}{2}\left((W_N)^2-(W_0)^2\right)=\frac{1}{2}W(t)^2$$

Ito interpretation:

$$I_{\text{Ito}}=\lim_{N\to\infty}\sum_{j=0}^{N-1}W_j(\Delta W_j)=\lim_{N\to\infty}\sum_{j=0}^{N-1}\left(\frac{1}{2}(W_j+W_{j+1})-\frac{1}{2}(W_{j+1}-W_j)\right)\Delta W_j$$

$$=\underbrace{\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(W_j+W_{j+1})\Delta W_j}_{\text{Stratonovich}}-\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(\Delta W_j)^2$$

In Lecture 4, as a special case of Ito's lemma, we showed $\lim_{N\to\infty}\sum_{j=0}^{N-1}\left((\Delta W_j)^2-\Delta s\right)=0$, which leads to $\lim_{N\to\infty}\frac{1}{2}\sum_{j=0}^{N-1}(\Delta W_j)^2=\lim_{N\to\infty}\frac{1}{2}\sum_{j=0}^{N-1}\Delta s=\frac{1}{2}\int_0^t ds=\frac{1}{2}t$. We arrive at

$$I_{\text{Ito}}=\frac{1}{2}W(t)^2-\frac{1}{2}t=I_{\text{Stratonovich}}-\frac{1}{2}t \qquad \text{for} \quad \int_0^t W(s)dW(s)$$

Item 3 of the road map:

Discretization: $\Delta s=\frac{t}{N}$, $s_j=j\Delta s$, $\Delta W_j=W_{j+1}-W_j$

Theorem (Ito's lemma)

$$\lim_{N\to\infty}\left(\sum_{j=0}^{N-1}g(s_j,W_j)(\Delta W_j)^2-\sum_{j=0}^{N-1}g(s_j,W_j)\Delta s\right)=0$$

Remark: The two Riemann sums and the corresponding integrals (after taking the limits) are still random variables.

Proof: Let

$$Q_N\equiv\sum_{j=0}^{N-1}\underbrace{g(s_j,W_j)}_{Y_j}\underbrace{\left((\Delta W_j)^2-\Delta s\right)}_{X_j}$$

$\{Q_N\}$ is a sequence of random variables. We only need to show

$$\lim_{N\to\infty}E\left(Q_N(\omega)\right)=0 \text{ and } \lim_{N\to\infty}\text{var}\left(Q_N(\omega)\right)=0$$

We use the theorem for calculating $\text{var}\left(\sum_{j=0}^{N-1}Y_jX_j\right)$. We first check the 3 conditions.

1. $E(X_j)=0$ for all j,
2. X_j is independent of X_i for all $i\neq j$, and

3. X_j is independent of Y_k for all $k \le j$.

Remark: We don't have and we don't need "Y_j is independent of Y_i for all $i \ne j$".

It follows that $E(Q_N) = 0$ and

$$\mathrm{var}(Q_N) = \mathrm{var}\left(\sum_{j=0}^{N-1} Y_j X_j\right) = \sum_{j=0}^{N-1} E(Y_j^{\,2}) E(X_j^{\,2})$$

$$E(X_j^{\,2}) = E\Big(((\Delta W_j)^2 - \Delta s)^2\Big) = \mathrm{var}\Big((\Delta W_j)^2\Big) = 2(\Delta s)^2 \qquad \text{(see homework)}$$

$$E(Y_j^{\,2}) = E\Big(g(s_j, W_j)^2\Big) = O(1)$$

$$\mathrm{var}(Q_N) = \sum_{j=0}^{N-1} E(Y_j^{\,2}) E(X_j^{\,2}) = 2(\Delta s)^2 \sum_{j=0}^{N-1} O(1)$$

$$= 2(\Delta s)^2 O(N) = O(\Delta s) \;\to\; 0 \text{ as } N \to +\infty$$

End of proof of Ito's lemma

Item 4 of the road map: (Relation between Ito and Stratonovich interpretations)

We look at the general case: $\int_0^t f(s, W(s))\, dW(s)$.

We start with the Stratonovich interpretation:

$$I_{\text{Stratonovich}} = \lim_{N\to\infty} \sum_{j=0}^{N-1} \frac{1}{2}(f_j + f_{j+1}) \Delta W_j\,, \qquad f_j \equiv f(s_j, W(s_j))$$

$$= \lim_{N\to\infty} \sum_{j=0}^{N-1} \left[f_j \Delta W_j + \frac{1}{2}(f_{j+1} - f_j) \Delta W_j \right]$$

Expand $(f_{j+1} - f_j)$ around s_j. Neglect $O(\Delta s)$ terms in the expansion of $(f_{j+1} - f_j)$.

$$f_{j+1} - f_j = (f_w)_j \Delta W_j + O(\Delta s)$$

Multiply by ΔW_j and sum over j, we get

$$I_{\text{Stratonovich}} = \underbrace{\lim_{N\to\infty} \sum_{j=0}^{N-1} f_j \Delta W_j}_{\text{Ito}} + \lim_{N\to\infty} \sum_{j=0}^{N-1} \frac{1}{2} (f_w)_j (\Delta W_j)^2 + \underbrace{\lim_{N\to\infty} \sum_{j=0}^{N-1} O(\Delta s) \Delta W_j}_{=0}$$

Ito's lemma gives

$$\lim_{N\to\infty} \sum_{j=0}^{N-1} \frac{1}{2} (f_w)_j (\Delta W_j)^2 = \lim_{N\to\infty} \frac{1}{2} \sum_{j=0}^{N-1} (f_w)_j \Delta s = \frac{1}{2} \int_0^t f_w(s, W(s))\, ds$$

Combining these results, we obtain the main theorem connecting the Stratonovich interpretation and the Ito interpretation in the general case.

Theorem:

For integral $\int_0^t f(s,W(s))dW(s)$, we have

$$I_{\text{Stratonovich}} = I_{\text{Ito}} + \frac{1}{2}\int_0^t f_w(s,W(s))ds$$

Stochastic integrals based on axioms

In the above, we interpreted stochastic integrals as limits of Riemann sums. Different choices of Riemann sums lead to different interpretations. Alternatively, we can calculate stochastic integrals based on a set of axioms.

Two axioms:

1) Fundamental theorem of calculus (FTC)

$$\int_a^b dH(t,W(t)) = H(t,W(t))\Big|_a^b$$

2) λ-chain rule

$$dH(t,W(t)) = H_t dt + H_w dW(t) + \left(\frac{1}{2}-\lambda\right)H_{ww}dt$$

Ito's interpretation: $\lambda = 0$

Stratonovich's interpretation: $\lambda = 0.5$

Meaning of the λ-chain rule

We compare the behaviors of increment $\Delta H \equiv H(t+\Delta t, u(t+\Delta t)) - \Delta H(t, u(t))$ respectively for $u(t)$ = smooth deterministic and for $u(t) = W(t)$.

Case 1: $u(t)$ = deterministic

- Expand around t to rewrite $\Delta H(t, u(t))$. Neglect $o(\Delta t)$ terms in the expansion.

$$\Delta H = H_t(t,u(t))\Delta t + H_u(t,u(t))\underbrace{\Delta u}_{O(\Delta t)} + o(\Delta t) \qquad \text{(E01A)}$$

Short notation: $\Delta H = H_t\big|_t \Delta t + H_u\big|_t \Delta u + o(\Delta t)$, $\quad H_u\big|_t \equiv H_u(t,u(t))$

- Expand around $(t+\Delta t)$ to rewrite $\Delta H(t, u(t))$. Neglect $o(\Delta t)$ terms in the expansion.

$$H\big|_t = H\big|_{t+\Delta t} + H_t\big|_{t+\Delta t}(-\Delta t) + H_u\big|_{t+\Delta t}(-\Delta u) + o(\Delta t), \qquad H_u\big|_{t+\Delta t} \equiv H_u(t+\Delta t, u(t+\Delta t))$$

$$\Delta H = H_t\big|_{t+\Delta t}\Delta t + H_u\big|_{t+\Delta t}\Delta u + o(\Delta t) \qquad \text{(E01B)}$$

(E01A) using $H_u|_t$ and (E01B) using $H_u|_{t+\Delta t}$ have the same form. We write it as a regular differential without specifying if H_u is $H_u|_t$ or $H_u|_{t+\Delta t}$.

$$dH = H_t dt + H_u du$$

Observation: For a smooth deterministic function, the differential has the same form whether we interpret H_u as $H_u|_t$ or as $H_u|_{t+\Delta t}$.

That is the beauty of deterministic calculus.

Case 2: $u(t) = W(t)$, the Wiener process

- Ito's interpretation:

  Expand around t to rewrite $\Delta H \equiv H(t+\Delta t, W(t+\Delta t)) - \Delta H(t, W(t))$.

  Neglect o(Δt) terms in the expansion.

$$\Delta H = H_t\big|_t \Delta t + H_w\big|_t \Delta W + \frac{1}{2}H_{ww}\big|_t (\Delta W)^2 + o(\Delta t), \qquad H_w\big|_t \equiv H_w(t, W(t)) \qquad \text{(E02A)}$$

  We replace $(\Delta W)^2$ by Δt to write out the differential. The differential has the form

$$dH = H_t dt + H_w dW + \frac{1}{2}H_{ww} dt$$

  with the understanding $H_w = H_w\big|_t$.

  This is the λ-chain rule with $\lambda = 0$.

- Stratonovich's interpretation:

  Expand around $(t+\Delta t)$ to rewrite ΔH. Neglect o(Δt) terms in the expansion.

$$H\big|_t = H\big|_{t+\Delta t} + H_t\big|_{t+\Delta t}(-\Delta t) + H_w\big|_{t+\Delta t}(-\Delta W) + \frac{1}{2}H_{ww}\big|_{t+\Delta t}(-\Delta W)^2 + o(\Delta t)$$

$$\text{where} \quad H_w\big|_{t+\Delta t} \equiv H_w(t+\Delta t, W(t+\Delta t))$$

$$\begin{aligned} \Delta H &= H_t\big|_{t+\Delta t}\Delta t + H_w\big|_{t+\Delta t}\Delta W - \frac{1}{2}H_{ww}\big|_{t+\Delta t}(\Delta W)^2 + o(\Delta t) \\ &= H_t\big|_t \Delta t + H_w\big|_{t+\Delta t}\Delta W - \frac{1}{2}H_{ww}\big|_t (\Delta W)^2 + o(\Delta t) \end{aligned} \qquad \text{(E02B)}$$

  Use the average of (E02A) and (E02B) to write ΔH as

$$\Delta H = H_t\big|_t \Delta t + \frac{1}{2}\left(H_w\big|_t + H_w\big|_{t+\Delta t}\right)\Delta W + o(\Delta t) \qquad \text{(E02C)}$$

  The differential has a diferent form

$$dH = H_t dt + H_w dW(t)$$

with the understanding $H_w = \frac{1}{2}\left(H_w\big|_t + H_w\big|_{t+\Delta t}\right)$.

This is the <u>λ-chain rule with $\lambda = 0.5$</u>.

<u>Remarks:</u>

- If we use the expansion <u>around</u> $(t + \Delta t)$, the differential has <u>another form</u>

  $$dH = H_t dt + H_w dW - \frac{1}{2} H_{ww} dt$$

  with the understanding $H_w = H_w\big|_{t+\Delta t}$.

  This is the <u>λ-chain rule with $\lambda = 1$</u>.

- <u>The λ-chain rule implicitly distinguishes</u> the choices of H_w in the differential.
  The differential $dH \equiv H(t+dt, W(t+dt)) - H(t, W(t))$ is always the same.
  dH takes different forms depending on whether we interpret H_w as

  $H_w|_t$ or $(H_w|_t + H_w|_{t+\Delta t})/2$ or $H_w|_{t+\Delta t}$

  $$dH = H_t dt + H_w dW + \frac{1}{2} H_{ww} dt \qquad H_w = H_w|_t$$

  $$dH = H_t dt + H_w dW \qquad H_w = (H_w|_t + H_w|_{t+\Delta t})/2$$

  $$dH = H_t dt + H_w dW - \frac{1}{2} H_{ww} dt \qquad H_w = H_w|_{t+\Delta t}$$

- To integrate $H_w dW$, we write $H_w dW = dH - (H_t + (0.5-\lambda)H_{ww})dt$.
  Different interpretations of $\int H_w dW$ are reflected in different values of λ (see below).

<u>Use the axioms to calculate</u> $\int_a^b f(t, W(t)) dW(t)$

<u>Strategy:</u>

- Write the λ-chain rule as $H_w dW = dH - (H_t + (0.5-\lambda)H_{ww})dt$.
- Solve $H_w = f(t, w)$ to find $H(t, w)$.
- Calculate the integrate as

$$\int_a^b f(t, W(t)) dW(t) = \underbrace{\int_a^b dH}_{\text{Fundamental theorem of calculus}} \underbrace{- \int_a^b \left(H_t + (\tfrac{1}{2} - \lambda) H_{ww}\right) dt}_{\text{See comments below}}$$

<u>Comments:</u>

The integrand ( ) varies with the value of λ. (reflecting different interpretations).

Given the integrand ( ), the integral is not affected by different interpretations.

Integral $\int_a^b g(t,W(t))dt$ is not affected by different interpretations.

$$\int_a^b g(t,W(t))dt = \lim_{N\to\infty}\sum_{j=0}^{N-1} g(\tilde{t}_j, W(\tilde{t}_j))\Delta t\,, \qquad \tilde{t}_j \in [t_j\ t_{j+1}]$$

The choice of $\tilde{t}_j \in [t_j\ t_{j+1}]$ does not matter (homework problem).

Procedure for calculating $\int_a^b f(t,W(t))dW(t)$

Step 1: Solve $H_w(t, w) = f(t, w)$ with $H(t, 0) = 0$ to define

$$H(t,w) = \int_0^w f(t,u)du \qquad \text{(this is a regular integral!)}$$

==> $f(t,W(t))dW(t) = H_w(t,W(t))dW(t)$

Step 2: Use the λ-chain rule to write $f(t, W(t))dW = dH - (\ )dt$.

λ-chain rule: $dH = H_t dt + H_w dW + \left(\frac{1}{2}-\lambda\right)H_{ww}dt$

==> $H_w dW = dH - \left(H_t + \left(\frac{1}{2}-\lambda\right)H_{ww}\right)dt$

==> $f(t,W(t))dW(t) = dH - \left(H_t + \left(\frac{1}{2}-\lambda\right)H_{ww}\right)dt$

Step 3: Differentiate $H(t, w)$ to calculate $H_t(t, w)$ and $H_{ww}(t, w)$.

Both are regular derivatives and regular functions.

Step 4: Use the fundamental theorem of calculus to calculate the integral

$$\int_a^b f(t,W(t))dW(t) = H(t,W(t))\Big|_a^b - \int_a^b \left(H_t + \left(\frac{1}{2}-\lambda\right)H_{ww}\right)dt$$

Example:

$$\int_a^b tW(t)^2 dW(t)$$

0. Identify function $f(t, w)$ in the integral

$$f(t, w) = t\,w^2$$

1. Solve $H_w(t, w) = f(t, w)$ with $H(t, 0) = 0$ to define

$$H(t,w) = \int_0^w t u^2 du = t\frac{w^3}{3}$$

$$\text{==>} \qquad tW(t)^2 dW(t) = H_w(t, W(t))dW(t)$$

2. Use the λ-chain rule to write $f(t, W(t))dW = dH - (\ )dt$.

$$dH = H_t dt + H_w dW + \left(\tfrac{1}{2} - \lambda\right) H_{ww} dt$$

$$\text{==>} \qquad H_w dW = dH - \left(H_t + \left(\tfrac{1}{2} - \lambda\right) H_{ww}\right) dt$$

$$\text{==>} \qquad tW(t)^2 dW(t) = dH - \left(H_t + \left(\tfrac{1}{2} - \lambda\right) H_{ww}\right) dt$$

3. Differentiate to calculate $H_t(t, w)$ and $H_{ww}(t, w)$.

$$H_t(t, w) = \frac{w^3}{3}, \qquad H_{ww}(t, w) = 2tw$$

4. Use the fundamental theorem of calculus

$$\int_a^b tW(t)^2 dW(t) = H(t, W(t))\Big|_a^b - \int_a^b \left(H_t + \left(\tfrac{1}{2} - \lambda\right) H_{ww}\right) dt$$

$$= t\frac{W(t)^3}{3}\bigg|_a^b - \int_a^b \left(\frac{W(t)^3}{3} + \left(\tfrac{1}{2} - \lambda\right) 2tW(t)\right) dt$$

Ito: λ = 0;

Stratonovich: λ = 0.5

All terms in the result are random variables.