AM 216 - Stochastic Differential Equations: Assignment

Dante Buhl

November 12, 2025

Problem 1: Adjoint of a differential operator

Proof.

$$\begin{aligned}
\int uL^*[v]dx &= \int L[u]vdx \\
&= \int \left(b(x)u_x + \frac{1}{2}au_{xx}\right) vdx \\
&= \int vbu_x + \frac{1}{2}avu_{xx}dx \\
&= \left(bvu + \frac{1}{2}avu_x\right)\Big|_D - \int \left(u(bv)_x + \frac{1}{2}u_x(av)_x\right) dx \\
&= \frac{1}{2}u(av)_x\Big|_D - \int \left(u(bv)_x - \frac{1}{2}u(av)_{xx}\right) dx \\
L_z^*[\star] &= -\left(b\star\right)_z + \frac{1}{2}\left(a\star\right)_{zz}
\end{aligned}$$

□

Problem 2: Backward Equation with Ito's interpretation

i)

$$\begin{aligned}
u(x,t) &= E(u + dXu_x - dtu_t + dX^2u_{xx}/2) \\
&= u + b(x)dtu_x - u_tdt(a(x)/2)u_{xx} \\
u_t &= b(x)u_x + \frac{a(x)}{2}u_{xx}
\end{aligned}$$

ii) We have the following boundary conditions which represent the exist condition and the initial condition which represents whether the exit condition is satisfied at $t = 0$

$$\begin{aligned}
u(0,t) &= 1 \\
u(L,t) &= 1 \\
u(x,0) &= \begin{cases} 0 & 0 < x < L \\ 1 & x \in \{0,1\} \end{cases}
\end{aligned}$$

iii) We begin with the definition of expectation

$$
\begin{aligned}
E(T) &= \int_0^\infty x \Pr(T = x) dx \\
&= \int_0^\infty x(\lim_{dt \to 0} \Pr(T < x) - \Pr(T < x - dt)) dx \\
&= \lim_{dt \to 0} \int_0^\infty x(u(x,t) - u(x, t-dt)) dx \\
&\approx \int_0^\infty x u_t(x,t) dx
\end{aligned}
$$

Problem 3: Solve the BVP

We begin with the method of integrating factor

$$
\begin{aligned}
m(x) &= e^{2bx} \\
mT_{xx} + 2bmT_x &= -2m \\
T_x &= \frac{1}{m} \int -2m dx \\
T_x &= -\frac{1}{b} + c_1 m^{-1} \\
T_x(L_1) = 0 \implies c_1 &= \frac{m(L_1)}{b} \\
T(x) &= -\frac{x}{b} - \frac{1}{2b^2} e^{2b(L_1 - x)} + c_2 \\
T(L_2) = 0 \implies c_2 &= \frac{L_2}{b} + \frac{1}{2b^2} e^{2b(L_1 - L_2)} \\
T(x) &= \frac{1}{b}(L_2 - x) + \frac{1 - e^{2b(L_2 - x)}}{2b^2 e^{2b(L_2 - L_1)}}
\end{aligned}
$$

Problem 4: Solving the backward equation with TPD

i)

$$
\begin{aligned}
u(x,t) &= E(A) \\
&= E(E(A|X(T - t + dt) = z + dX)) \\
&= E(u(z + dX, t - dt)) \\
&= E(u + dXu_z - dtu_t + \frac{1}{2} u_{zz} dX^2) \\
&= u - zu_z dt - u_t dt + \frac{1}{2} dt u_{zz} \\
u_t &= -zu_z + \frac{1}{2} u_{zz}
\end{aligned}
$$

ii) we can find an analytical expression for u since we know exactly how $X(T)$ is distributed given $X(T -$

$t) = z$. We have,

$$
\begin{aligned}
(X(T)|X(T-t)=z) &\sim N\left(e^{-t}z, \frac{1}{2}(1-e^{-2t})\right) \\
u(x,t;c_0) &= \frac{1}{\sqrt{\pi(1-e^{-2t})}} \int_{-\infty}^{\infty} H(x-c) e^{-\frac{\left(x-e^{-t}z\right)^2}{1-e^{-2t}}} dx \\
&= \frac{1}{\sqrt{\pi(1-e^{-2t})}} \int_{c_0}^{\infty} e^{-\frac{\left(x-e^{-t}z\right)^2}{1-e^{-2t}}} dx \\
&= \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x-e^{-t}z}{\sqrt{1-e^{-2t}}}\right)
\end{aligned}
$$

Problem 5: Linear scalings in an SDE

i) We begin with the necessary substitutions

$$
\begin{aligned}
\frac{1}{a} d\tilde{X}(\tilde{t}) &= dX(t) \\
&= -\frac{\mu}{ab}\tilde{X} d\tilde{t} + \sqrt{\frac{\sigma^2}{b}} dW(\tilde{t}) \\
d\tilde{X}(\tilde{t}) &= -\frac{\mu}{b}\tilde{X} d\tilde{t} + \sqrt{\frac{a^2\sigma^2}{b}} dW(\tilde{t}) \\
b = \mu \qquad a &= \sqrt{\frac{\mu}{\sigma^2}}
\end{aligned}
$$

ii)

$$
\begin{aligned}
u(x,t;c_0) &= \frac{1}{a} u^{(s)}(\tilde{x}, \tilde{t}; \tilde{c_0}) \\
&= \sqrt{\frac{\sigma^2}{\mu}} u^{(s)}(\tilde{x}, \tilde{t}; \tilde{c_0})
\end{aligned}
$$

Problem 6: More linear scalings in an SDE

i) We begin with the necessary substitutions

$$
\begin{aligned}
\frac{1}{a} d\tilde{X}(\tilde{t}) &= dX(t) \\
&= -\frac{\mu}{a^3 b}\tilde{X}^3 d\tilde{t} + \sqrt{\frac{\sigma^2}{b}} dW(\tilde{t}) \\
d\tilde{X}(\tilde{t}) &= -\frac{\mu}{a^2 b}\tilde{X} d\tilde{t} + \sqrt{\frac{a^2\sigma^2}{b}} dW(\tilde{t}) \\
b = \sqrt{\mu}\sigma \qquad a^2 &= \frac{\sqrt{\mu}}{\sigma}
\end{aligned}
$$

ii)

$$
\begin{aligned}
u(x,t;c_0) &= \frac{\alpha_0}{a^2} u^{(s)}(\tilde{x}, \tilde{t}; \tilde{c_0}) \\
&= \frac{\alpha_0 \sigma}{\sqrt{\mu}} u^{(s)}(\tilde{x}, \tilde{t}; \tilde{c_0})
\end{aligned}
$$

Problem 7: Periodic Steady State solution of the forward equation