AM 216 - Stochastic Differential Equations: Assignment 1

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Problem 1: Two Dice

Roll a set of two fair 6-sided dice, one colored red the other white. Assume that the two dice roll independently. Record the two numbers facing up as X_1 and X_2.

i) Mathematically describe the format of outcome. Describe the sample space.

Proof. The two quantities X_1 and X_2 are samples from a given discrete probability distribution. The sample space has six equally likely outcomes $1, 2, 3, 4, 5, 6$ with an expected value of $E(X) = 3.5$. The two samples X_1 and X_2 are independent and identically distributed samples. □

ii) Let X be the absolute difference between X_1 and X_2. Is X a random variable?

Proof. Yes, the sum/difference of two random variables is also a random variable, albeit with a different sample space and expected value. We can exam the sample space to find that there are 6 possibilities of different likelihood: $S = 0, 1, 2, 3, 4, 5$, with 5 being the least likely (only two outcomes where the result is 5) and 1 being the most likely (there are ten outcomes with a result of 1). □

iii) Find the PMF of X, $E(X)$

Proof. Specifically,

k	P(X = k)
0	0.166...
1	0.277...
2	0.222...
3	0.166...
4	0.111...
5	0.055...

and, therefore,

$$E(X) = 1(0.277...) + 2(0.222...) + 3(0.166...) + 4(0.111...) + 5(0.055...) = 1.944...$$

□

iv) Let $A = \{X_2^2 \geq 2X_1\}$. Find $Pr(A|X_1 = n)$ for $n = 1, 2, \ldots, 6$. Then use the law of total probability to find $Pr(A)$.

Proof. We will construct a table to show the values of $Pr(A|X_1 = n)$ for all possible n.

n	1	2	3	4	5	6
Pr(A)	0.833...	0.833...	0.666...	0.666...	0.5	0.5

From this, we can find the probability that $Pr(X_1 = n) = 0.166\ldots \forall n$, and therefore find that $Pr(A) = 0.666\ldots$. □

Problem 2: Variance

i) Show that $\text{Var}(\alpha X) = \alpha^2 \text{Var}(X)$.

Proof.

$$\begin{aligned}\text{Var}(\alpha X) &= E(\alpha X)^2 - E^2(\alpha X) \\ &= E(\alpha^2 X^2) - (\alpha E(X))^2 \\ &= \alpha^2 E(X^2) - \alpha^2 E^2(X) \\ &= \alpha^2 \text{Var}(X)\end{aligned}$$

□

ii) Show that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ if X and Y are independent.

Proof.

$$\begin{aligned}\text{Var}(X+Y) &= E(X+Y)^2 - E^2(X+Y) \\ &= E(X^2 + 2XY + Y^2) - (E^2(X) + 2E(X)E(Y) + E^2(Y)) \\ &= E(X^2) + E(2XY) + E(Y^2) - E^2(X) - 2E(X)E(Y) - E^2(Y) \\ &= E(X^2) + E(Y^2) - E^2(X) - E^2(Y) \\ &= \text{Var}(X) + \text{Var}(Y)\end{aligned}$$

where the change between the third and fourth lines follows from the fact that X and Y are independent, i.e. $E(XY) = E(X)E(Y)$. □

iii) Find an example where $\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y)$ does not imply that X and Y are independent.

Proof. To begin we shall imagine that we have some random variable X which is symmetrically distributed about its mean. To make it simpler, let us assume that its mean is zero. Let us assume that X is a uniform distribution centered on zero $X \in [-1, 1]$. We can then make Y ANY even function of X. We can consider $Y = X^2$, $Y = \cos(X)$, or $Y = \ln(|X| + 1)$. In any of these cases, we are guaranteed that the covariance of X and Y is zero. We can define the covariance to be $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \text{Var}(X+Y) - \text{Var}(X) - \text{Var}(Y)$. We have,

$$\begin{aligned}Y = X^2, \quad \text{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(XY) \\ &= \int_{-1}^{1} X X^2 \frac{1}{2} dX \\ &= \frac{1}{8} X^4 \Big|_{-1}^{1} \\ &= 0\end{aligned}$$

For the case of $\mu = 0$, this can be done for any PDF and $f(X)$ (such that $Y = f(X)$) which is even-symmetrically centered on the origin. □

Problem 3: Binomial and Normal Distributions

Calculate $E(X^2 + Y^2)$ where $X \sim \text{Bino}(n, p)$ and $Y \sim \mathcal{N}(\mu, \sigma)$.

Proof.

$$\begin{aligned}E(X^2+Y^2) &= E(X^2)+E(Y^2)\\ &= \mathrm{Var}(X)+(E(X))^2+\mathrm{Var}(Y)+(E(Y))^2\\ &= np(1-p+np)+\sigma^2-\mu^2\end{aligned}$$

□

Problem 4: Falling within some standard deviation

Proof.

$$\begin{aligned}Pr(\mu-\eta\sigma \le X \le \mu+\eta\sigma) = \cdots &= \frac{1}{\sigma\sqrt{2\pi}}\int_{\mu-\eta\sigma}^{\mu+\eta\sigma} e^{-(x-\mu)^2/2\sigma^2}dx\\ &= \frac{2}{\sigma\sqrt{2\pi}}\int_{\mu}^{\mu+\eta\sigma} e^{-(x-\mu)^2/2\sigma^2}dx\\ &= \frac{2}{\sqrt{\pi}}\int_{0}^{\eta/\sqrt{2}} e^{-z^2}dz, \quad z=(x-\mu)/\sigma\sqrt{2}\\ &= \mathrm{erf}\left(\frac{\eta}{\sqrt{2}}\right)\end{aligned}$$

□

Problem 5: Simple Convoluted Experiment

i) E(X)

Proof. We begin by first using the Law of Total Probability. We have,

$$Pr(X=k)=\sum_{i=0}^{n} Pr(X=k|Y=i)Pr(Y=i)$$

We notice some interesting things at first. We have as a given that $Pr(X=k|Y=m)=0$ if $k>m$. Then applying the law of total expectation that,

$$\begin{aligned}E(X) &= \sum_{i=0}^{n} E(X|Y=i)Pr(Y=i)\\ &= \sum_{i=0}^{n} \frac{i}{2}Pr(Y=i)\\ &= \sum_{i=0}^{n} \frac{n!}{2^{n+1}(i-1)!(n-i)!}\end{aligned}$$

□

ii) E(XY)

```python
mu = 0.6
sigma = 1.3

nr = 500000
nx = 10

act_in_95 = 0
est_in_95 = 0

for i in range(nr):
    x = np.random.normal(mu, sigma, nx)
    est_mu = x.mean()
    est_sigma = np.sqrt(np.sum((x-est_mu)**2)/(nx-1))
    if (abs(mu-est_mu) <= 2*sigma):
        act_in_95 += 1
    if (abs(mu-est_mu) <= 2*est_sigma):
        est_in_95 += 1

print('Fraction of time mean is in 95% confidence interval (actual sigma): ',\
    act_in_95/nr)
print('Fraction of time mean is in 95% confidence interval (estim. sigma): ',\
    est_in_95/nr)
```

Figure 1: Code to produce the results for this problem

Proof.

$$\begin{aligned} E(X) &= \sum_{i=0}^{n} E(XY|Y=i)Pr(Y=i) \\ &= \sum_{i=0}^{n} \frac{i^2}{2} Pr(Y=i) \\ &= \sum_{i=0}^{n} \frac{in!}{2^{n+1}(i-1)!(n-i)!} \end{aligned}$$

□

Problem 6: Confidence Interval

Here is a picture of my code for this problem, Figure 1.

i) 1.0

ii) 0.999858

Problem 7: Characteristic Functions

i) $E(X) = \mu$

Proof.

$$\begin{aligned} E(X) &= \frac{1}{i}\frac{\partial \phi_x}{\partial \xi}\Big|_{\xi=0} \\ &= \left(\mu - \frac{\sigma^2 \xi}{i}\right)\exp\left(i\mu\xi - \frac{\sigma^2\xi^2}{2}\right)\Big|_{\xi=0} \\ &= \mu \end{aligned}$$

□

ii) $\mathrm{Var}(X) = \sigma^2$

Proof.

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - (E(X))^2 \\ &= -\left(\frac{1}{i}\frac{\partial \phi_x}{\partial \xi}\Big|_{\xi=0}\right)^2 - \frac{\partial^2 \phi_X}{\partial \xi^2}\Big|_{\xi=0} \\ &= -\mu^2 - \frac{\partial}{\partial \xi}\left(i\mu - \sigma^2\xi\right)\exp\left(i\mu\xi - \frac{\sigma^2\xi^2}{2}\right)\Big|_{\xi=0} \\ &= -\mu^2 - \left(i\mu - \sigma^2\xi\right)^2\exp\left(i\mu\xi - \frac{\sigma^2\xi^2}{2}\right)\Big|_{\xi=0} + \sigma^2\exp\left(i\mu\xi - \frac{\sigma^2\xi^2}{2}\right)\Big|_{\xi=0} \\ &= \sigma^2 \end{aligned}$$

□

Problem 8: Making an experiment well-posed

i) No, currently we have no idea of the quantity, nor ratio of coins contained in the bin. For example, it could be the case that the bin contains 9 type 1 coins for every type 2 coin. If this were the case, then this would influence the conditional probability. We also don't know if, but would assume that, they are keeping the ratio of the coins inside the bins constant, i.e. they are placing the coin back into the bag and mixing it up before selecting the next coin.

ii) In order to make this well-posed, I suggest we include the changes to control the number/ratio of each coin in the bin, and make each sample identically distributed by placing the coin back into the bin and mixing the bin before sampling again.