AM216 Stochastic Differential Equations	Assignment 6
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Q1. The adjoint operator of a differential operator.

Consider the differential operator $L_z[\bullet] = b(z)\dfrac{\partial \bullet}{\partial z} + \dfrac{1}{2}a(z)\dfrac{\partial^2 \bullet}{\partial z^2}$.

Show that the adjoint operator of L_z is

$$L_z^*[\bullet] = -\frac{\partial}{\partial z}\big(b(z)\,\bullet\,\big) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\big(a(z)\,\bullet\,\big)$$

Q2. Backward equation and associated IBVP.

Let $X(t)$ be the stochastic process governed by the Ito interpretation of

$$dX = b(X)dt + \sqrt{a(X)}\,dW$$

Let $T(x;\omega)$ be the time of exiting $(0, L)$. Consider the probability of exiting by time t,

$$u(x,t) \equiv \Pr\big(T(x;\omega) \leqslant t \big| X(0) = x\big)$$

The law of total probability gives $u(x,t) = E\big(u(x+dX, t-dt)\big) + o(dt)$.

i) Expand the RHS and use moments of dX to derive the governing PDE of $u(x,t)$.

ii) Write out the IVP or IBVP for $u(x,t)$. Do we have an IVP or IBVP?

iii) Express $E\big(T(x;\omega)\big)$ in terms of $u(x,t)$.

Q3. Solve the BVP below

$$\begin{cases} T_{xx} + 2bT_x = -2 \\ T'(L_1) = 0, \quad T(L_2) = 0 \end{cases}$$

to derive

$$T(x) = \frac{1}{b}(L_2 - x) + \frac{1 - e^{2b(L_2 - x)}}{2b^2 e^{2b(L_2 - L_1)}}$$

<u>Remark:</u> This is an exit problem. $T(x) = E\big(T(x;\omega)\big)$. The left end ($x = L_1$) is a reflecting boundary; the right end ($x = L_2$) is an absorbing boundary.

Q4. Solution of backward equation using the transition probability density.

Let $X(t)$ be an Ornstein-Uhlenbeck process governed by $dX = -Xdt + dW$. Suppose the reward at the end time T is $u_0(X(T))$, defined as

$$u_0(X(T)) = H\big(X(T) - c_0\big), \qquad H(s) \equiv \begin{cases} 1, & s > 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $u(z,t;c_0)$ be the average reward given $X(T-t) = z$.

$$u(z,t;c_0) \equiv E\Big(u_0\big(X(T)\big)\Big| X(T-t) = z\Big)$$

i) Use the law of total probability to find the governing PDE and the IVP of $u(z,t;c_0)$.

ii) Find an analytical expression of $u(z,t;c_0)$. Write the answer in terms of erf( ).

Hint: Recall that for the Ornstein-Uhlenbeck process $dX = -\beta X dt + \gamma dW$, we obtained the exact solution of $\big(X(t_0+t)\big|X(t_0) = x_0\big)$. Use the exact solution to write out the transition probability density $q\big(x, T\big|z, (T-t)\big)$ as a normal probability density. Then use the transition probability density to calculate $u(z,t;c_0)$.

Q5. Linear scalings in a stochastic differential equation (SDE).

Consider two related variants of Q4. Let

$$u(z,t;c_0) = E\Big(u_0\big(X(T)\big)\Big|X(T-t) = z\Big), \qquad \underbrace{dX = -\mu X dt + \sqrt{\sigma^2} dW}_{\text{general case}} \tag{1}$$

$$u^{(s)}(z,t;c_0) = E\Big(u_0\big(X(T)\big)\Big|X(T-t) = z\Big), \qquad \underbrace{dX = -X dt + dW}_{\text{special case}} \tag{2}$$

where $u_0\big(X(T)\big)$ is the reward at the end time T, defined as

$$u_0(X(T)) = \text{ReLU}\big(X(T) - c_0\big), \qquad \text{ReLU}(s) \equiv \begin{cases} s, & s > 0 \\ 0, & \text{otherwise} \end{cases}$$

We want to express $u(z,t;c_0)$ in terms of the special case solution $u^{(s)}(\tilde{z},\tilde{t};\tilde{c}_0)$ where $(\tilde{z},\tilde{t};\tilde{c}_0)$ are scalings of $(z,t;c_0)$. Consider scalings $\tilde{t} \equiv bt$, $\tilde{X}(\tilde{t}) \equiv aX(t)$ in SDE (1).

$$t = \frac{1}{b}\tilde{t}, \quad dt = \frac{1}{b}d\tilde{t}, \qquad X(t) = \frac{1}{a}\tilde{X}(\tilde{t}), \quad dX(t) = \frac{1}{a}d\tilde{X}(\tilde{t})$$

$$\begin{aligned} u(z,t;c_0) &= E\Big(\text{ReLU}\big(X(T) - c_0\big)\Big|X(T-t) = z\Big), \quad c_0 = \frac{1}{a}\tilde{c}_0, \quad z = \frac{1}{a}\tilde{z} \\ &= E\Big(\text{ReLU}\big(\frac{1}{a}(\tilde{X}(\tilde{T}) - \tilde{c}_0)\big)\Big|\frac{1}{a}\tilde{X}(\tilde{T}-\tilde{t}) = \frac{1}{a}\tilde{z}\Big) \\ \tilde{u}(\tilde{z},\tilde{t};\tilde{c}_0) \equiv a u(z,t;c_0) &= E\Big(\text{ReLU}\big(\tilde{X}(\tilde{T}) - \tilde{c}_0\big)\Big|\tilde{X}(\tilde{T}-\tilde{t}) = \tilde{z}\Big) \end{aligned}$$

Note that $\tilde{u}(\tilde{z},\tilde{t};\tilde{c}_0)$ in scaled quantities $(\tilde{z},\tilde{t};\tilde{c}_0)$ has the same meaning of average reward as the original $u(z,t;c_0)$. After changing time variable to $\tilde{t}$, we write $dW(t)$ as

$$dW(t) = \sqrt{dt}\underbrace{\frac{dW(t)}{\sqrt{dt}}}_{\sim N(0,1) \text{ regardless of } t} = \frac{1}{\sqrt{b}}\sqrt{d\tilde{t}}\frac{dW(\tilde{t})}{\sqrt{d\tilde{t}}} = \frac{1}{\sqrt{b}}dW(\tilde{t})$$

i) Derive the governing SDE of $\tilde{X}(\tilde{t})$. Adjust coefficients (a,b) to make the SDE

$$d\tilde{X}(\tilde{t}) = -\tilde{X}(\tilde{t})d\tilde{t} + dW(\tilde{t}) \quad \leftarrow \text{ special case SDE (2)}$$

ii) Write $u(z,t;c_0)$ in terms of $u^{(s)}(\tilde{z},\tilde{t};\tilde{c}_0)$. Write out expressions of $(\tilde{z},\tilde{t};\tilde{c}_0)$.

Q6. Linear scalings in a stochastic differential equation (SDE).

Consider the two related SDEs given below.

$$u(z,t;c_0,\alpha_0) = E\Big(\alpha_0 u_0\big(X(T)\big)\Big|X(T-t)=z\Big), \qquad \underbrace{dX = -\mu X^3 dt + \sqrt{\sigma^2}dW}_{\text{general case}} \tag{3}$$

$$u^{(s)}(z,t;c_0) = E\Big(u_0\big(X(T)\big)\Big|X(T-t)=z\Big), \qquad \underbrace{dX = -X^3 dt + dW}_{\text{special case}} \tag{4}$$

where reward function $u_0\big(X(T)\big)$ is defined as

$$u_0(X(T)) = \text{ReLU}^2\big(X(T)-c_0\big), \qquad \text{ReLU}^2(s) \equiv \begin{cases} s^2, & s>0 \\ 0, & \text{otherwise} \end{cases}$$

Carry out scalings similar to what we did in Q5. Notice the coefficient α_0 in $u(z,t;c_0,\alpha_0)$ in (3). Be careful when defining function $\tilde{u}(\tilde{z},\tilde{t};\tilde{c}_0)$. Write $u(z,t;c_0,\alpha_0)$ in terms of $u^{(s)}(\tilde{z},\tilde{t};\tilde{c}_0)$. Write out expressions of $(\tilde{z},\tilde{t};\tilde{c}_0)$.

Q7. (Optional) Periodic steady state solution of forward equation.

Let $X(t)$ be governed by $dX = \dfrac{D}{k_BT}\big(F-\phi'(X)\big)dt + \sqrt{2D}dW$ where potential $\phi(x)$ is periodic with period L. The corresponding forward equation in the conservation form is

$$p_t = -\frac{\partial J(x,t)}{\partial x}, \qquad J(x,t) \equiv \frac{D}{k_BT}\big(F-\phi'(x)\big)p(x,t) - D\frac{\partial p(x,t)}{\partial x} \tag{5}$$

Let $p(x)$ be the steady state solution of (5) with periodic and normalization conditions.

$$p(x+L) = p(x), \qquad \int_0^L p(x)dx = 1$$

Solve for $p(x)$. Write the result in terms of $(F,\ \phi(x),\ D,\ k_BT)$.

Remark: This is a model for a Brownian particle driven by force F over a periodic potential $\phi(x)$. The average velocity and diffusion of the particle are calculated from the steady state $p(x)$.

Hint: At the steady state, $\dfrac{\partial J(x)}{\partial x} = 0$, which leads to

$$J(x) \equiv \frac{D}{k_BT}\big(F-\phi'(x)\big)p(x) - Dp'(x) = J_0 = const$$

Use the method of integrating factor to solve the ODE. After multiplying by the integrating factor, integrate from x to $x+L$. Then use the given conditions to determine J_0.