AM 216 - Stochastic Differential Equations: Assignment 8

Dante Buhl

November 26, 2025

Problem 1: Lambda-rule for stochastic derivatives

Proof. We begin this section by first taking $X(t) = H(t, W(t))$.

$$\begin{aligned} dX &= \left(X_t + \frac{1}{2}X_{WW}\right)dt + X_W dW \\ &= \left(2tX + \frac{1}{2}X\right)dt + XdW \end{aligned}$$

We notice that if this were true we must have several conditions with also follow with our original assumption. First and foremost,

$$\begin{aligned} X = X_W &\implies X_W = X_{WW} \\ X &\propto e^{W(t)} \\ X_t = 2tX &\implies X \propto e^{t^2} \\ X &= X_0 e^{t^2 + W(t)} \end{aligned}$$

This all follows if we take the λ-chain rule and the original SDE to be equivalent. What remains is to resolve the initial condition and this simply implies that $X_0 = 2$.

$$H(t, W(t)) = X(t) = e^{t^2 + W(t)}$$

$\square$

Problem 2: Time Reversal of an SDE

Proof.

$$\begin{aligned} \rho_{(X(t)|X(t+dt)=x_1)}(x_0) &\propto \exp\left(\frac{-1}{2\sigma^2 dt}\left[(x_0 - x_1)^2 + 2c_1 dt(x_0 - x_1) + c_2 dt(x_0 - x_1)^2 + \ldots\right]\right) \\ \hat{\sigma}^2 &= \frac{\sigma^2 dt}{1 + c_2 dt}, \quad \beta = \frac{c_1 dt}{1 + c_2 dt} \\ \rho_{(X(t)|X(t+dt)=x_1)}(x_0) &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2}\left[\Delta_x^2 + 2\beta\Delta_x + \ldots\right]\right) \\ &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2}\left[\Delta_x^2 + 2\beta\Delta_x + \beta^2\right]\right) \\ &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2}\left[\Delta_x + \beta\right]^2\right) \\ &\propto \exp\left(-\frac{1 + c_2 dt}{2\sigma^2 dt}\left[x_0 - x_1 + \frac{c_1 dt}{1 + c_2 dt}\right]^2\right) \\ (X(t)|X(t+dt) = x_1) &\sim N\left(\frac{-c_1 dt}{1 + c_2 dt}, \frac{\sigma^2 dt}{1 + c_2 dt}\right) \end{aligned}$$

$\square$

Problem 3: Feynman-Kac Formula

i) Write out the FVP

$$\begin{aligned} 0 &= u_t + \frac{1}{2}u_{xx} + xu \\ u(x,T,T) &= 1 \end{aligned}$$

ii) Verify the given solution

$$\begin{aligned} u(x,t,T) &= \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right) \\ u_t &= -\left(x + (T-t)^2/2\right)\exp\left(\frac{(T-t)^3}{6} + (T-t)x\right) \\ u_{xx} &= (T-t)^2\exp\left(\frac{(T-t)^3}{6} + (T-t)x\right) \\ 0 &= \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right)\left[-\left(x + \frac{(T-t)^2}{2}\right) + \frac{(T-t)^2}{2} + x\right] \\ &= 0 \\ u(x,T,T) &= \exp(0) = 1 \end{aligned}$$

Thus this solution satisfies the FVP.

iii) This specific problem is easy enough to directly integrate so I will do so.

Proof. We begin by expanding $X(s)$ (Note that, $\int_t^T \hat{W}(s)ds \sim N\left(0, \frac{(T-t)^3}{3}\right)$)

$$\begin{aligned} u(x,t,T) &= E\left[\exp\left(\int_t^T x + \hat{W}(s)ds\right)\Big|\hat{W}(t)=0\right] \\ &= E\left[\exp((T-t)x)\exp\left(\int_t^T \hat{W}(s)ds\right)\Big|\hat{W}(t)=0\right] \\ &= C\exp((T-t)x)\int_{-\infty}^{\infty} e^x e^{-x^2/2\sigma^2}dx \\ &= C\exp((T-t)x)\int_{-\infty}^{\infty} e^{-\left(x^2-2\sigma^2 x\right)/2\sigma^2}dx \\ &= C\exp((T-t)x)e^{\sigma^2/2}\int_{-\infty}^{\infty} e^{-\left(x-\sigma^2\right)^2/2\sigma^2}dx \\ &= \exp((T-t)x)e^{\sigma^2/2} \\ &= \exp\left(x(T-t) + \frac{(T-t)^3}{6}\right) \end{aligned}$$

$\square$

Problem 4: Integrating Factor to solve SDE

i)

$$
\begin{aligned}
dX - \frac{1}{1+t} X dt &= dW \\
d\left(\frac{X}{1+t}\right) &= \frac{dW(s)}{1+s} \\
X(t) &= (1+t)\left[c + \int_0^t \frac{dW(s)}{1+s}\right] \\
X(0) = 1 &\implies c = 1
\end{aligned}
$$

ii)

$$
\begin{aligned}
E(X(t)) &= (1+t)\left(1 + E\left(\int_0^t \frac{dW(s)}{1+s}\right)\right) \\
&= 1+t \\
\text{Var}(X(t)) &= (1+t)^2 \text{Var}\left(\int_0^t \frac{dW(s)}{1+s}\right) \\
&= (1+t)^2 \left(\int_t^T \frac{ds}{(1+s)^2}\right) \\
&= (1+t)^2 \left(-\frac{1}{1+s}\bigg|_0^t\right. \\
&= (1+t)^2 \left(1 - \frac{1}{1+t}\right) \\
&= t + t^2
\end{aligned}
$$

Problem 5: Simulating Feynman-Kac

i) We can verify this solution by simply plugging it into the original PDE and initial condition.

$$
\begin{aligned}
u(x,t,T) &= \exp\left(-\frac{(x-t)^2}{2}\right) \\
u_t &= (x-t)u \\
u_{xx} &= \frac{\partial}{\partial x}\left(-(x-t)u\right) \\
&= -u + (x-t)^2 u \\
0 &= u\left[(x-t) - \frac{1}{2} + \frac{(x-t)^2}{2} - \phi\right] \\
&= 0
\end{aligned}
$$

ii)

Problem 6: Nonlinear Function of an RV