AM 216 - Stochastic Differential Equations: Final Exam

Dante Buhl

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Problem 1:

i)

$$
\begin{aligned}
Y &= \int_0^1 \cos(\pi s) dW(s) \\
&= \int_0^1 \cos(\pi s) \sqrt{dt} X(s) \\
&\sim \mathcal{N}\left(0, \int_0^1 \cos^2(\pi s) dt\right) \\
&\sim \mathcal{N}\left(0, \frac{1}{2} \int_0^1 1 + \cos(2\pi s) dt\right) \\
&\sim \mathcal{N}\left(0, \frac{1}{2}\right) \\
E(Y) &= 0 \\
\mathrm{Var}(Y) &= \frac{1}{2}
\end{aligned}
$$

ii) Since Y is gaussian, its PDF can be rewritten simply using,

$$
\begin{aligned}
\rho_Y(y) &= \frac{1}{\sigma_Y \sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right) \\
&= \frac{1}{2\sqrt{\pi}} \exp\left(-y^2\right)
\end{aligned}
$$

Problem 2:

$$
\begin{aligned}
\frac{dX}{X} &= dt + 2dW \\
\ln|X| &= \exp(t + 2W(t)) \\
X(t) &= X_0 \exp(t + 2W(t)) \\
&= 3 \exp(t + 2W(t))
\end{aligned}
$$

Problem 3:

We begin by rephrasing this as a brownian bridge problem

$$\begin{aligned}
I &= \int_0^T W^2(t)dt \\
E(I|W(T)=w_1) &= \int_0^T E(W^2(t)|W(T)=w_1)dt \\
&= \int_0^T E(B^2(t))dt \\
&= \int_0^T E\left(W^2(t) + \frac{t_i^2}{T^2}\left(w_1 - W(T)\right)^2 - \frac{2t_i}{T}W(t)\left(w_1 - W(T)\right)\right) dt \\
&= \int_0^T t + \frac{t^2}{T^2}E\left(w_1^2 + W^2(T) - 2W(T)w_1\right) + \frac{2t}{T}E(W(t)W(T))dt \\
&= \int_0^T t + \frac{t^2}{T^2}\left(w_1^2 + T\right) + \frac{2t^2}{T}dt \\
&= \frac{1}{2}T^2 + \frac{w_1^2 T}{3} + \frac{T^2}{3} + \frac{2T^2}{3} \\
&= \frac{3}{2}T^2 + \frac{w_1^2}{3}T
\end{aligned}$$

Problem 4:

i)

$$\begin{aligned}
E(B(t)) &= t^{3/2}E(W(1/t^2) \\
&= 0 \\
\mathrm{Var}(B(t)) &= t^3\mathrm{Var}(W(1/t^2)) \\
&= t^3\frac{1}{t^2} \\
&= t
\end{aligned}$$

ii) In order to show that these are not the same stochastic process it sufficies to show that their SDEs are not the same. We have for example,

$$
\begin{aligned}
dW(t) &= dW(t) \\
dB(t) &= B(t+dt) - B(t) \\
&= (t+dt)^{3/2} W\left(\frac{1}{(t+dt)^2}\right) - t^{3/2} W(1/t^2) \\
&= \left(t^{3/2} + \frac{3dt}{2} t^{1/2} + h.o.t.\right) W\left(\frac{1}{(t+dt)^2}\right) - t^{3/2} W(1/t^2) \\
dt' &= \frac{1}{t^2} - \frac{1}{(t+dt)^2} \\
&= \frac{2dt}{t^2(t+dt)} \\
dB(t) &= \left(t^{3/2} + \frac{3dt}{2} t^{1/2} + h.o.t.\right) \left(W(1/t^2) + \sqrt{dt'}\mathcal{N}(0,1)\right) - t^{3/2} W(1/t^2) \\
&= \frac{3dt}{2} t^{1/2} W(1/t^2) + t^{3/2} dW' \\
&= \frac{3B(t)}{2t} dt + t^{3/2} dW' \\
&\neq dW
\end{aligned}
$$

Problem 5:

i) We have the given BVP for T is given by the following form,

$$
\begin{cases} \frac{a(x)}{2} T_{xx} + b(x) T_x = -1 \\ T(\varepsilon) = 0 \quad T'(1) = 0 \end{cases}
$$

$$
\begin{cases} \frac{9x^{4/3}}{2} T_{xx} + 3x^{1/3} T_x = -1 \\ T(\varepsilon) = 0 \quad T'(1) = 0 \end{cases}
$$

ii) Solving this BVP is a matter of the method of integrating factor.

$$
\begin{aligned}
T_{xx} + \frac{2}{3x} T_x &= -\frac{2}{9} x^{-4/3} \\
\mu(x) &= x^{2/3} \\
T_x &= -\frac{2}{9} x^{-2/3} \int x^{-2/3} dx \\
&= -\frac{2}{9} x^{-2/3} (3x^{1/3} + C_1) \\
&= -\frac{2}{3} x^{-1/3} + C_1 x^{-2/3} \\
T(x) &= -x^{2/3} + C_1 x^{1/3} + C_2 \\
T'(1) &= -\frac{2}{3} + \frac{C_1}{3} = 0 \\
T(\varepsilon) &= -\varepsilon^{2/3} + 2\varepsilon^{1/3} + C_2 = 0 \\
T(x) &= -x^{2/3} + 2x^{1/3} + \varepsilon^{2/3} - 2\varepsilon^{1/3}
\end{aligned}
$$