AM 216 - Stochastic Differential Equations: Assignment

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Problem 1: Time Reversability of Brownian Bridge

Proof. We begin by defining W_j. We have,

$$W_j = S_j = \sum_{k=0}^{j-1} \Delta W_k, \quad \Delta W_k \sim \sqrt{\Delta t} X_k, \quad X_k \sim N(0,1)$$

Thus we begin,

$$\begin{aligned}
E(W_j | W_n = 0) &= E(B_j) \\
&= E\left(W_j - \frac{t_j}{T} W_n\right) \\
&= \sum_{k=0}^{j-1} E(\Delta W_k) - \frac{t_j}{T} \sum_{k=0} n - 1 E(\Delta W_k) \\
&= 0
\end{aligned}$$

Next we look at the covariance between the two sets,

$$\begin{aligned}
\mathrm{Cov}(W_i, W_j | W_n = 0) &= \mathrm{Cov}(B_i, B_j) \\
&= E(B_i B_j) - E(B_i)E(B_j) \\
&= E\left[\left(S_i - \frac{t_i}{T} S_n\right)\left(S_j - \frac{t_j}{T} S_n\right)\right] \\
&= E(S_i S_j) - \frac{t_i}{T} E(S_j S_n) - \frac{t_j}{T} E(S_i S_n) + \frac{t_i t_j}{T^2} E(S_n^2)
\end{aligned}$$

We now look more specifically at the components of the sums S_i, S_j, and S_n. Take for example, $j > i$, we have then,

$$S_i = a, \quad S_j = a + b, \quad S_n = a + b + c$$
$$a = \sum_{k=0}^{i-1} \Delta W_k, \quad b = \sum_{k=i}^{j-1} \Delta W_k, \quad c = \sum_{k=j}^{n-1} \Delta W_k$$

Notice that each component a, b, c is independent of each other, $E(a) = E(b) = E(c) = 0$, and $E(a^2) = t_i$, $E((a+b)^2) = t_j$, and $E((a+b+c)^2) = T$. Thus,

$$\begin{aligned}
\mathrm{Cov}(W_i, W_j | W_n = 0) &= E(a^2 + ab) - \frac{t_i}{T} E\left((a+b)^2 + (a+b)c\right) - \frac{t_j}{T} E(a^2 + a(b+c)) + \frac{t_i t_j}{T^2} E\left((a+b+c)^2\right) \\
&= E(a^2) - \frac{t_i}{T} E\left((a+b)^2\right) - \frac{t_j}{T} E(a^2) + \frac{t_i t_j}{T^2} E\left((a+b+c)^2\right) \\
&= t_i - 2\frac{t_i t_j}{T} + \frac{t_i t_j}{T} \\
&= t_i \left(1 - \frac{t_j}{T}\right)
\end{aligned}$$

Now, in order to show time reversability we demonstrate that $E(W_{n-j}|W_n = 0) = E(W_j|W_n = 0)$ and $\text{Cov}(W_i, W_j|W_n = 0) = \text{Cov}(W_{n-i}, W_{n-j}, W_n = 0)$. The case for the expectation is trivial as the expectation is zero. For the covariance, we have,

$$\begin{aligned}
\text{Cov}(W_{n-i}, W_{n-j}|W_n = 0) &= \text{Cov}(B_{n-i}, B_{n-j}) \\
&= E(S_{n-i}S_{n-j}) - \frac{T - t_i}{T}E(S_{n-j}S_n) \\
&\quad - \frac{T - t_j}{T}E(S_{n-i}S_n) + \frac{(T - t_i)(T - t_j)}{T^2}E(S_n^2)
\end{aligned}$$

Since we had previously that $j > i$, we now have $n - i > n - j$. Thus, we have

$$S_{n-j} = d, \quad S_{n-i} = d + e, \quad S_n = d + e + f$$
$$d = \sum_{k=0}^{n-j-1} \Delta W_k, \quad b = \sum_{k=n-j}^{n-i-1} \Delta W_k, \quad c = \sum_{k=n-i}^{n-1} \Delta W_k$$
$$E(d) = E(e) = E(f) = 0, \quad E(d^2) = T - t_j, \quad E((d + e)^2) = T - t_i, \quad E((d + e + f)^2) = T$$

Therefore, we now can show,

$$\begin{aligned}
\text{Cov}(W_{n-i}, W_{n-j}|W_n = 0) &= (T - t_j) - \frac{(T - t_i)(T - t_j)}{T} - \frac{(T - t_j)(T - t_i)}{T} + \frac{(T - t_i)(T - t_j)}{T} \\
&= (T - t_j)\frac{(1 - T + t_i)}{T} \\
&= t_i\left(1 - \frac{t_j}{T}\right) = \text{Cov}(W_i, W_j|W_n = 0)
\end{aligned}$$

This concludes the proof that the brownian bridge is reversable in time, i.e. that moving backwards is distributed identically to moving forwards. This is attributable to the fact that there is no correlation between the direction of movement ΔW to the direction of time Δt. Only the amplitudes of the adjustments are correlated. $\square$

Problem 2: Convergence in Probability

We have,

$$\begin{aligned}
\lim_{n\to\infty} \text{Var}(Q_n(w)) &= 0 \\
&= \lim_{n\to\infty} \left(E(Q_n^2) - E^2(Q_n)\right) \\
\lim_{n\to\infty} E(Q_n^2) &= q^2 \\
\lim_{n\to\infty} E(|Q_n - q|^2) &\geq \varepsilon^2 \lim_{n\to\infty} Pr(|Q_n - q| \geq \varepsilon) \\
\lim_{n\to\infty} Pr(|Q_n - q| \geq \varepsilon) &\leq 0
\end{aligned}$$

thus we can state that $\{Q_n\}$ converges to q in probability as $n \to \infty$.

Problem 3: Gaussians

i)

$$
\begin{aligned}
I_2 &= \int_0^T \cos\left(n\pi\frac{t}{T}\right) dW(t) \\
&= \int_0^T \cos\left(n\pi\frac{t}{T}\right) \sqrt{dt}N(0,1) \\
&= N\left(0, \int_0^T \cos^2\left(n\pi\frac{t}{T}\right) dt\right) \\
&= N\left(0, \frac{1}{2}\int_0^T 1+\cos\left(2n\pi\frac{t}{T}\right) dt\right) \\
&= N\left(0, \frac{T}{2}+\frac{T}{2n\pi}\sin\left(2n\pi\frac{t}{T}\right)\Big|_0^T\right) \\
&= N\left(0, \frac{T}{2}\right)
\end{aligned}
$$

Thus we have, $E(I_2) = 0$ and $\mathrm{Var}(I_2) = T/2$

ii)

$$
\begin{aligned}
F_n &= \frac{2}{T}\int_0^T \sin\left(n\pi\frac{t}{T}\right)\left(W(t)-\frac{t}{T}W(T)\right) dt \\
&= \frac{2}{T}\left(\frac{T}{n\pi}\cos\left(n\pi\frac{t}{T}\right)\left(W(t)-\frac{t}{T}W(T)\right)\Big|_0^T - \frac{T}{n\pi}\int_0^T \cos\left(n\pi\frac{t}{T}\right)\left(dW(T)-\frac{dt}{T}W(T)\right)\right) \\
&= \frac{2}{n\pi}\int_0^T \cos\left(n\pi\frac{t}{T}\right)\left(dW(T)-\frac{dt}{T}W(T)\right)\frac{2}{n\pi}\int_0^T \\
&= \frac{2}{n\pi}\int_0^T \left(\cos\left(n\pi\frac{t}{T}\right)-\frac{1}{T}\int_0^T \cos\left(n\pi\frac{s}{T}\right) ds\right) dW \\
&= \frac{2}{n\pi}\int_0^T \cos\left(n\pi\frac{t}{T}\right) dW \\
&\sim N\left(0, \frac{4}{n^2\pi^2}\int_0^T \cos^2\left(n\pi\frac{t}{T}\right) dt\right) \\
&\sim N\left(0, \frac{2T}{n^2\pi^2}\right)
\end{aligned}
$$

Thus we have, $E(F_n) = 0$ and $\mathrm{Var}(F_n) = 2T/(n^2\pi^2)$

Problem 4: Variance of the sums of products of functions of independent variables lol

Proof. We begin by examining $\text{Var}(\cdot)$. We have,

$$\begin{aligned}
\text{Var}\left(\sum_{j=0}^{n-1} g(Y_j)f(X_j)\right) &= E(\star^2) - E^2(\star), \quad \star = \sum_{j=0}^{n-1} g(Y_j)f(X_j) \\
E(\star) &= \sum_{j=0}^{n-1} E\left(g(Y_j)f(X_j)\right) \\
&= \sum_{j=0}^{n-1} E(g(Y_j))E(f(X_j)) = 0 \\
\star^2 &= \sum_{j=0}^{n-1} g^2(Y_j)f^2(X_j) - 2\sum_{i=0, i\neq j}^{n-1} g(Y_j)f(X_j)g(Y_i)f(X_i) \\
E(\star^2) &= \sum_{j=0}^{n-1} E\left(g^2(Y_j)f^2(X_j)\right) - 2\sum_{j=0}^{n-1}\sum_{i=0, i\neq j}^{n-1} E\left(g(Y_j)f(X_j)g(Y_i)f(X_i)\right) \\
&= \sum(\triangleleft_j) - 2\sum\sum(\triangleright_{i,j})
\end{aligned}$$

Now consider the case for $\triangleright_{i,j}$ where $j > i$. We have that X_j is independent of X_i, Y_j and Y_i as givens for this problem. Thus we have,

$$\begin{aligned}
\triangleright_{i,j} &= E\left(g(Y_j)f(X_j)g(Y_i)f(X_i)\right) \\
&= E(f(X_j))E\left(g(Y_j)g(Y_i)f(X_i)\right) = 0 \\
\triangleleft_j &= E\left(g^2(Y_j)f^2(X_j)\right) \\
&= E\left(g^2(Y_j)\right)E\left(f^2(X_j)\right) \\
\text{Var}(\star) &= \sum_{j=0}^{n-1} \triangleleft_j \\
\text{Var}\left(\sum_{j=0}^{n-1} g(Y_j)f(X_j)\right) &= \sum_{j=0}^{n-1} E\left(g^2(Y_j)\right)E\left(f^2(X_j)\right)
\end{aligned}$$

The same result holds for $i > j$, as instead of X_j being independent of X_i, Y_i, Y_j it is instead X_i which is independent of X_j, Y_i, Y_j and the same arguement holds. $\square$

Problem 5: Ito's Lemma, again

Proof. We begin by determining the independence of W_j and ΔW_j. We have,

$$W_j = W_0 + \sum_{i=0}^{j-1} \Delta W_i, \quad W_i \sim N(0, \Delta t)$$

And so ΔW_j is completely independent of W_j as it is not contained inside the sum which comprises W_j. We have next to look at $E(Q_k - f_k)$.

$$\begin{aligned}
E(Q_k - f_k) &= E\left(\sum_{j=0}^{k-1} W_j^2\left(\Delta W_j^2 - \Delta t\right)\right) \\
&= \sum_{j=0}^{k-1} E\left(W_j^2\left(\Delta W_j^2 - \Delta t\right)\right) \\
&= \sum_{j=0}^{k-1} E(W_j^2) E\left(\Delta W_j^2 - \Delta t\right) \\
&= \sum_{j=0}^{k-1} E(W_j^2)\left[\Delta t - \Delta t\right] \\
&= 0
\end{aligned}$$

Where the expectation of the products in line 3 is seperabla as we have shown independence. Next we look at the variance, we have

$$\begin{aligned}
\mathrm{Var}(Q_k - f_k) &= E\left(Q_k - f_k\right)^2 - E^2(Q_k - f_k) \\
&= E\left(Q_k - f_k\right)^2 \\
&= E\left(\sum_{j=0}^{k-1} W_j^4(\Delta W_j^2 - \Delta t)^2 - 2\sum_{i=0, i\neq j}^{k-1} W_j^2 W_i^2(\Delta W_j^2 - \Delta t)(\Delta W_i^2 - \Delta t)\right) \\
&= \sum_{j=0}^{k-1} E\left(W_j^4(\Delta W_j^2 - \Delta t)^2\right) - 2\sum_{j=0}^{k-1}\sum_{i=0, i\neq j}^{k-1} E\left(W_j^2 W_i^2(\Delta W_j^2 - \Delta t)(\Delta W_i^2 - \Delta t)\right) \\
&= (1) + (2)
\end{aligned}$$

Here, we split this calculation into two parts, (1) and (2). Let us first look at (2). Notice that while $T_j = W_j^2(\Delta W_j^2 - \Delta t)$ is comprised of independently distributed products, we do not have that T_i is independent of T_j. That is, if $j > i$ we have that W_j is conditionally dependent on W_i and that ΔW_i is in the sum which comprises W_j. We do still have, however, that ΔW_j^2 is independent of W_j and T_i. Thus, we can write,

$$\begin{aligned}
(2) &= -2\sum_{j=0}^{k-1}\sum_{i=0, i\neq j}^{k-1} E\left((\Delta W_j^2 - \Delta t) W_j^2 T_i\right) \\
&= -2\sum_{j=0}^{k-1}\sum_{i=0, i\neq j}^{k-1} E(\Delta W_j^2 - \Delta t) E(W_j^2 T_i) \\
&= 0
\end{aligned}$$

This also holds for the case where $i > j$ and so this is true for all terms in the sum. Finally, we look at (1). We have,

$$\begin{aligned}
(1) &= \sum_{j=0}^{k-1} E(W_j^4)E(\Delta W_j^2 - \Delta t)^2 \\
&= \sum_{j=0}^{k-1} t_j^2 E(X_j^4)E(\Delta W_j^4 + \Delta t^2 - 2\Delta t \Delta W_j^2) \\
&= \sum_{j=0}^{k-1} t_j^2 (3)\left(3\Delta t^2 + \Delta t^2 - 2\Delta t^2\right) \\
&= \sum_{j=0}^{k-1} 6 t_j^2 \Delta t^2
\end{aligned}$$

□

Problem 6: PSD of Ornstein-Uhlenbeck process

$$\begin{aligned}
F\left[e^{-\beta|t|}\right] &\equiv \int_{-\infty}^{\infty} e^{-2\pi\xi t} e^{-\beta|t|} dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi\xi t - \beta|t|} dt \\
&= \int_{-\infty}^{0} e^{(-2\pi\xi + \beta)t} dt + \int_{0}^{\infty} e^{-(2\pi\xi + \beta)t} dt \\
&= \frac{1}{-2\pi\xi + \beta} e^{(-2\pi\xi + \beta)t}\Big|_{-\infty}^{0} - \frac{1}{2\pi\xi + \beta} e^{-(2\pi\xi + \beta)t}\Big|_{0}^{\infty} \\
&= \frac{1}{-2\pi\xi + \beta} + \frac{1}{2\pi\xi + \beta} \\
&= \frac{2\beta}{\beta^2 + 4\pi^2\xi^2}
\end{aligned}$$

Problem 7: Optional: Paley Wiener represation of Wiener process

Problem 8: Optional: Paley Wiener represation of Wiener process Continued