AM 216 - Stochastic Differential Equations: Assignment 3

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October 23, 2025

Problem 1: Fair Gambler's Ruin

Proof.

$$
\begin{aligned}
u(x,t) &= Pr(X(\tau) > 0, \tau \in [0,t] \& X(t) > c_0 | X(0) = x) \\
&= E(Pr(A|X(dt) = x + dW) + d(dt) \\
&= E(u(x + dW, t - dt)) \\
&= E\left[u(x,t) + u_x dW - u_t dt + u_{xx} dW^2/2 + o(dt^{3/2})\right] \\
&= u(x,t) - u_t dt + u_{xx} dt/2 \\
u_t &= \frac{1}{2} u_x x \\
u(0,t) = 0, \quad u(x,0) &= \begin{cases} 1 & x > c_0 \\ 0 & 0 < x < c_0 \end{cases}
\end{aligned}
$$

□

Problem 2: Solving the IBVP

i) We use the odd extension,

$$
u_t = \frac{1}{2} u_x x
$$

$$
u(0,t) = 0, \quad u(x,0) = \begin{cases} 1 & x > c_0 \\ 0 & -c_0 < x < c_0 \\ -1 & x < -c_0 \end{cases}
$$

ii)

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty} \infty e^{-\xi^2/2t} f(x - \xi) d\xi \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty} x - c_0 e^{-\xi^2/2t} d\xi - \int_{x+c_0} \infty e^{-\xi^2/2t} d\xi
\end{aligned}
$$

iii) By a goemetric argument we can state that $d = x + c_0$ and $b = |x - c_0|$, $a = d - b$ (note that we are guaraneed $d > 0$).

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi t}} \int_0^a e^{\xi^2/2t} d\xi \\
&= \frac{1}{2} \text{erf}\left(\frac{a}{\sqrt{2t}}\right) \\
&= \frac{1}{2} \text{erf}\left(\frac{x + c_0 - |x - c_0|}{\sqrt{2t}}\right)
\end{aligned}
$$

Problem 3: Solve the BVP

i) We have,

$$
\begin{aligned}
\frac{d}{dx}\left(T_x - 2mT\right) &= -2 \\
\left(T_x - 2mT\right) &= -2x + c \\
e^{-2mx}\left(T_x - 2mT\right) &= (-2x + c)e^{-2mx} \\
e^{-2mx}T &= \int (-2x + c)e^{-2mx}dx \\
T(x) &= e^{2mx}\int (-2x + c)e^{-2mx}dx \\
&= e^{2mx}\left(ce^{-2mx} + \frac{x}{m}e^{-2mx} + \frac{1}{2m^2}e^{-2mx} + c_2\right) \\
&= c + \frac{x}{m} + \frac{1}{2m^2} + c_2 e^{2mx} \\
&= \frac{x}{m} - \frac{C}{m}\left(\frac{e^{2mx} - 1}{e^{2mC} - 2}\right), \quad \text{(applying BC)}
\end{aligned}
$$

where solving for the BC looks like

$$
\begin{aligned}
T(0) &= c_1 + 0 + \frac{1}{2m^2} + c_2 = 0 \\
T(C) &= c_1 + \frac{C}{m} + \frac{1}{2m^2} + c_2 e^{2mC} = 0 \\
c_1 &= -c_2 - \frac{1}{2m^2} \\
0 &= \frac{C}{m} + c_2\left(e^{2mC} - 1\right) \\
c_2 &= -\frac{C}{m}\left(e^{2mC} - 1\right)^{-1} \\
T(x) &= \frac{x}{m} - \frac{C}{m}\frac{e^{2mx} - 1}{e^{2mC} - 1}
\end{aligned}
$$

Problem 4: Confirming Ito's Lemma

i)

$$
\begin{aligned}
E(X_j^4) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-x^2/2}dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} x^3\left(xe^{-x^2/2}\right)dx \\
&= \sqrt{\frac{2}{\pi}}3\int_0^{\infty} x\left(xe^{-x^2/2}\right)dx \\
&= \sqrt{\frac{2}{\pi}}3\int_0^{\infty} e^{-x^2/2}dx \\
&= 3 \\
E(dW_j^4) &= E((\sqrt{dt})^4 X_j) = 3dt^2
\end{aligned}
$$

ii) Since each dW_j is iid we have,

$$
\begin{aligned}
E(Q_n) &= \sum_{j=0}^{n-1} E(2t_j dW_j^2) \\
&= \sum_{j=0}^{n-1} 2t_j E(dW_j^2) \\
&= \sum_{j=0}^{n-1} 2t_j dt \\
\mathrm{Var}(Q_n) &= \sum_{j=0}^{n-1} \mathrm{Var}(2t_j dW_j^2) \\
&= \sum_{j=0}^{n-1} 4t_j^2 \mathrm{Var}(dW_j^2) \\
&= \sum_{j=0}^{n-1} 4t_j^2 dt^2 \mathrm{Var}(X_j^2) \\
&= \sum_{j=0}^{n-1} 8t_j^2 dt^2 \\
\lim_{n\to\infty} \sum_{j=0}^{n-1} 2t_j dt &= \lim_{n\to\infty} \frac{2t_f^2}{n^2} \sum_{j=0}^{n-1} j \\
&= \lim_{n\to\infty} \frac{2t_f^2}{n^2}(n-1)(n-2)/2 \\
&= t_f^2 = \int_0^{t_f} 2t dt \\
\lim_{n\to\infty} \mathrm{Var}(Q_n) &= \frac{8t_f^4}{n^4} \sum_{j=0}^{n-1} j^2 \\
&= \frac{8t_f^4}{n^4} O(n^3) = 0
\end{aligned}
$$

Problem 5: Showing brownian motion is iid

i) We begin by writing $X1 = W_3 - W_2$, $X2 = W_2 - W_1$ and $Y = W1$. We attempt to show that $P(X1|X2, Y) = P(X_1)$.

$$
\begin{aligned}
P(X1|X2, Y) &= \frac{P(X1, X2\&Y)}{P(X2\&Y} \\
&= \frac{P(X1)P(X2)P(Y)}{P(X2)P(Y)} = P(X1)
\end{aligned}
$$

where this simplification requires that we know that $X1$ and $X2$ are iid by the definition of brownian motion and that neither $X1$ and $X2$ depend on Y (since they are distributed as $N(0, dt)$). This shows that $X1$ is independent of $X2$ and Y and more specifically, $W_3 \sim W_2 + \sqrt{dt}N(0,1) = N(W_2, dt)$ Therefore we can make the statement,

$$
(W(t_3)|W(t_2) = w_2, W(t_1) = w_1) \sim (W(t_3)|W(t_2) = w_2)
$$

ii) The same argument holds for

$$(W(t_2)|W(t_3) = w_3, W(t_4) = w_4) \sim (W(t_2)|W(t_3) = w_3)$$

except here we define $X1 = W_2 - W_3$, $X2 = W_3 - W_4$, and $Y = W4$. The same exact argument holds and there is no sign change because the brownian increment is symmetrically distributed about zero (i.e. $dW_{32} \sim dW_{23}$)

Problem 6: Stationary Process

$$\begin{aligned} R(\tau) &= E(Z(s+\tau)Z(s)) \\ &= \frac{1}{\Delta t^2} E\left[(W(s+\tau+\Delta t) - W(s+\tau))(W(s+\Delta t) - W(s))\right] \\ &= \frac{1}{\Delta t^2} E\left[(a-b)(b-c)\right], \quad a = W(s+\tau+\Delta t) - W(s+\Delta t) \\ b = W(s+\Delta t) - W(s+\tau) \quad & c = W(s+\tau) - W(s) \\ &= \frac{1}{\Delta t^2} E\left[ab + ac + bb + bc\right] \\ &= \frac{1}{\Delta t^2} E\left[bb\right] = \frac{\Delta t - \tau}{\Delta t^2} \end{aligned}$$

This is, of course, for the case where $\Delta t > \tau$ and so the differences dW are not independent, i.e. there is some overlap in the two differences in time. In the case where $\tau > \Delta t$ there is no overlap and the two differences are independent and identically distributied as $N(0, dt)$. That is, $R(\tau) = 0$ when $\tau > \Delta t$.

Problem 7: Numerical Itos Lemma

i) See Figure 1

ii) See Figure 1

iii) See Figure 2

Problem 8: Refined Sampling for Wiener Process

See Figure 3

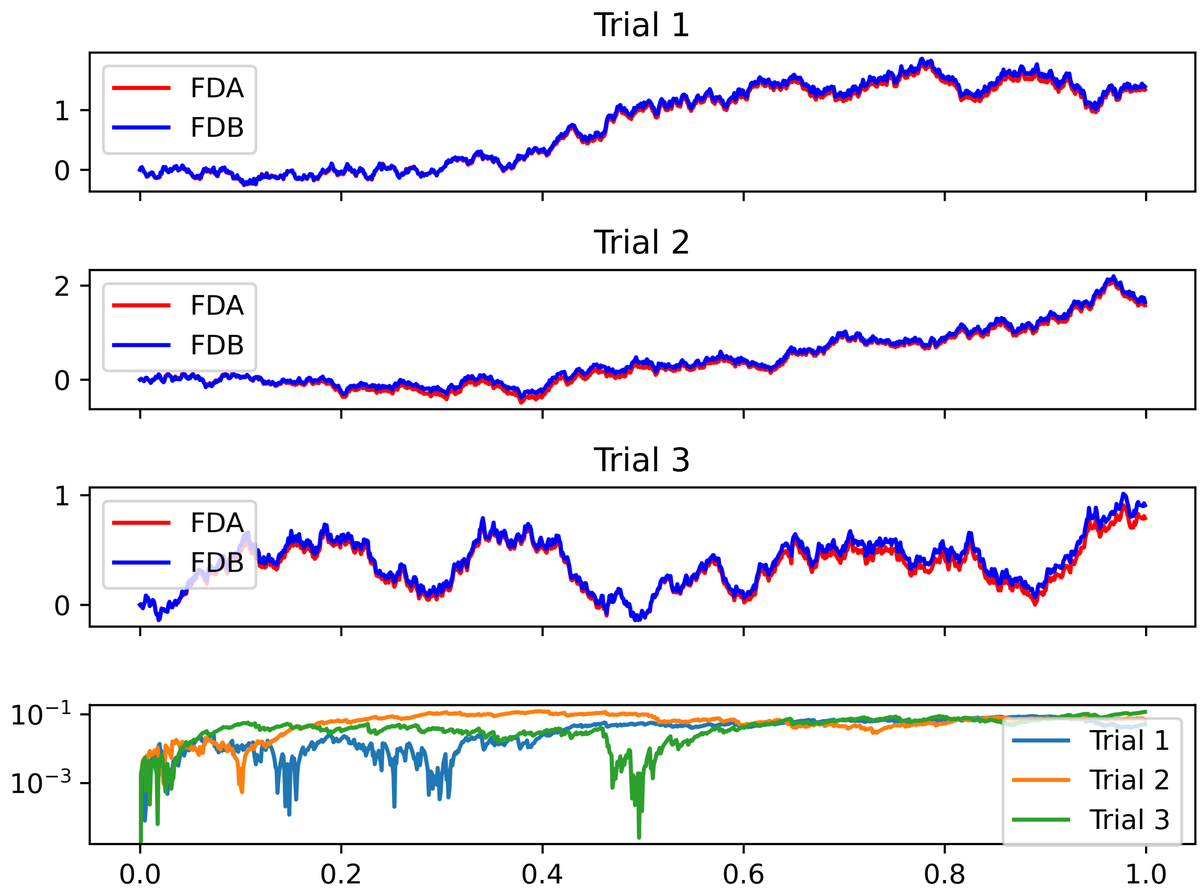

Figure 1: Numerical Results for problem 7 part b

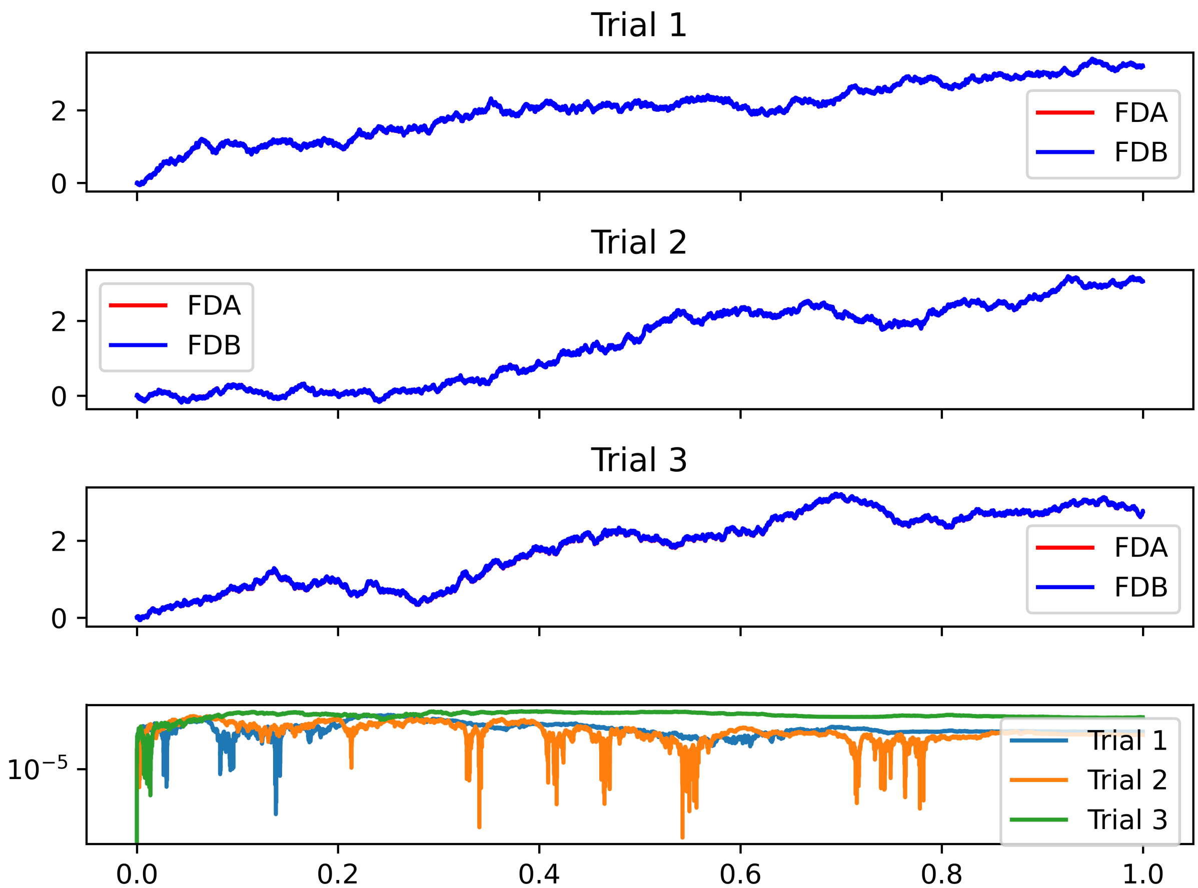

Figure 2: Numerical Results for problem 7 part c

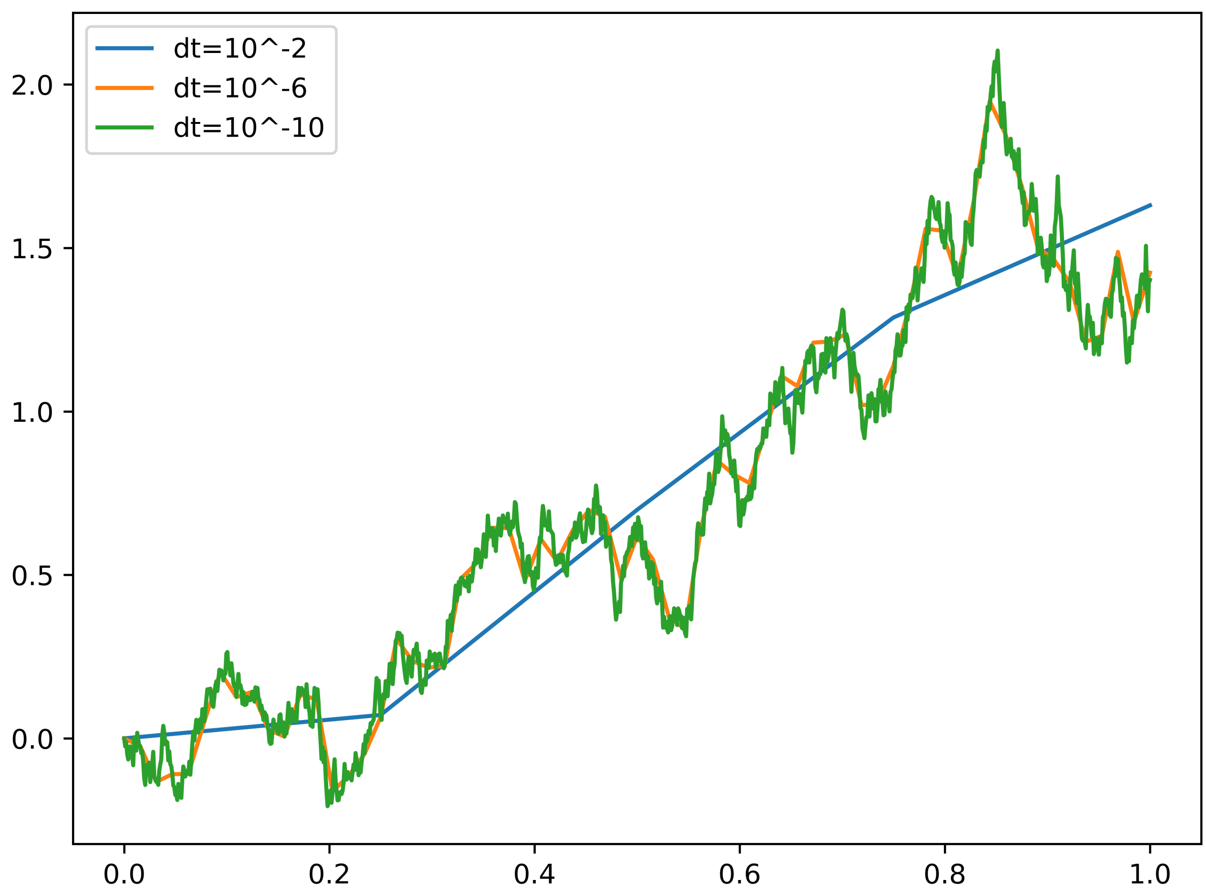

Figure 3: Numerical Results for problem 8