Formulas in Lecture 3

The characteristic function of random variable X

$$\phi_X(\xi) \equiv E\left(\exp(i\xi X)\right) = \int_{-\infty}^{+\infty} \exp(i\xi x)\rho_X(x)dx$$

Properties of characteristic functions

- $\phi_X(\xi)\Big|_{\xi=0} = E\left(\exp(i\xi X)\right)\Big|_{\xi=0} = E(1) = 1$
- $\frac{d}{d\xi}\phi_X(\xi)\Big|_{\xi=0} = iE(X)$
- $\frac{d^2}{d\xi^2}\phi_X(\xi)\Big|_{\xi=0} = -E(X^2)$
- Expansion of $\phi_X(\xi)$ around $\xi = 0$:

  $\phi_X(\xi) = 1 + iE(X)\xi - \frac{E(X^2)}{2}\xi^2 + \cdots$
- CF of the sum of two independent RVs:

  If random variables X and Y are independent, then we have

  $\phi_{(X+Y)}(\xi) = \phi_X(\xi)\cdot\phi_Y(\xi)$
- CF of a shifted RV:

  $\phi_{(X+\mu)}(\xi) = \exp(i\xi\mu)\phi_X(\xi)$
- CF of a scaled RV:

  $\phi_{(\alpha X)}(\xi) = \phi_X(\sigma\xi)$
- CF of $X \sim N(\mu, \sigma^2)$:

  $\phi_X(\xi) = \exp\left(i\mu\xi - \frac{\sigma^2\xi^2}{2}\right)$

Theorem:

Suppose X and Y are independent, and $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$.

Then $(X + Y) \sim N(\mu_1+\mu_2, \sigma_1^2 + \sigma_2^2)$.

The Wiener process, denoted by $W(t)$, satisfies

1) $W(0) = 0$
2) For $t_2 \ge t_1 \ge 0$, $W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$
3) For $t_4 \ge t_3 \ge t_2 \ge t_1 \ge 0$,

   increments $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are independent.

Formulas in Lecture 4

Properties of Wiener process:

1) $dW \sim N(0, dt)$ ==> $dW = \sqrt{dt}\, X$ where $X \sim N(0,1)$
2) $\mathrm{E}(dW) = 0$
3) $\mathrm{E}((dW)^2) = dt$
4) $dW(t_1)$ and $dW(t_2)$ are independent if the time increments are disjoint.
5) $dW = O(\sqrt{dt})$ in the statistical sense.

Ito's lemma:

Given $f(0, 0)$, at any t_f, the two SDEs below give the same $f(t_f, W(t_f))$.

$$df(t, W(t)) = f_t dt + f_w dW + \frac{1}{2} f_{ww} (dW)^2 + o(dt)$$

$$df(t, W(t)) = \left(f_t + \frac{1}{2} f_{ww} \right) dt + f_w dW + o(dt)$$

A unified view of probability and expected value

For event A, define random variable $X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise} \end{cases}$. We have

$\Pr(A) = E(X)$.

Law of total expectation: $E(X) = E(E(X|Y))$

Law of total probability: $\Pr(A) = E(\Pr(A|Y))$

The Gambler's ruin problem:

- Let $u(x) = \Pr\left(A \middle| X(0) = x\right)$, $A = \{X(t) \text{ hits } C \text{ before } 0\}$.

u(x) is the probability of breaking the bank with initial cash = x.

$u(x)$ is governed by the law of total probability.

$$u(x)=E_{dW}\Big(u(x+dW)\Big)+o(dt)$$

- Let $T(x)=E\Big(Z\Big|X(0)=x\Big)$, $Z=$ time from 0 until $X(t)=C$ or $X(t)=0$.

  $T(x)$ is the average time until the end of game with initial cash x.

  $T(x)$ is governed by the law of total expectation.

  $$T(x)=dt+E_{dW}\Big(T(x+dW)\Big)+o(dt)$$

- Let $P(x,t)=\Pr\Big(A(t)\Big|X(0)=x\Big)$, $A(t)=\Big\{X(\tau)>0 \text{ for } \tau\in[0,t]\Big\}$.

  $P(x, t)$ is the probability of surviving beyond time t with initial cash x.

  $P(x, t)$ is governed by the law of total probability.

  $$P(x,t)=E_{dW}\Big(P(x+dW,t-dt)\Big)+o(dt)$$

Solution of a general IVP of the heat equation:

$$\begin{cases} u_t = a u_{xx} \\ u(x,0)=f(x) \end{cases} \qquad \text{(E03)}$$

The solution of (E03) has the expression:

$$u(x,t)=\frac{1}{\sqrt{4\pi at}}\int_{-\infty}^{+\infty}\exp\left(\frac{-\xi^2}{4at}\right)f(x-\xi)\,d\xi$$