

AM 216 - Stochastic Differential Equations: Assignment 2

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October 6, 2025

Problem 1: MLE and the Normal Distribution

Find $\mu_{(MLE)}$ and $\sigma_{(MLE)}$.

Proof.

$$\begin{aligned}\ell(\mu, \sigma^2 | \mathbf{X}) &= -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{j=1}^n (X_j - \mu)^2 \\ \frac{\partial \ell}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu) \\ \mu_{(MLE)} &= \frac{1}{n} \sum_{i=1}^n X_i \\ \frac{\partial \ell}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (X_i - \mu)^2 \\ &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n \left(X_i - \frac{1}{n} \sum_{j=1}^n X_j \right)^2 \\ \sigma_{(MLE)} &= \sqrt{\frac{1}{n} \sum_{i=1}^n \left(X_i - \frac{1}{n} \sum_{j=1}^n X_j \right)^2}\end{aligned}$$

□

Problem 2: MLE Variance and unbiased sample variance

Show that the MLE of variance is biased.

Proof.

$$\begin{aligned}E \left(\sum_{i=1}^n \left(X_i - \frac{1}{n} \sum_{j=1}^n X_j \right)^2 \right) &= E \left(\sum_{i=1}^n \left(X_i - \frac{1}{n} \sum_{j=1}^n X_j \right)^2 \right) \\ &= E \left(\sum_{i=1}^n X_i^2 - \left(\frac{1}{n} \sum_{j=1}^n 2X_i X_j + \frac{1}{n} X_j^2 \right) \right) \\ &= \sum_{i=1}^n E \left(X_i^2 - \left(\frac{1}{n} \sum_{j=1}^n 2X_i X_j + \frac{1}{n} X_j^2 \right) \right)\end{aligned}$$

IM mad, gonna do it later

□

Problem 3: Central Limit Theorem

i)

$$\begin{aligned}\phi_X(\xi) &= \int_{-\infty}^{\infty} e^{i\xi x} f(x) dx \\ &= pe^{i\xi} + q\end{aligned}$$

ii)

$$\begin{aligned}\phi_N(\xi) &= \int_{-\infty}^{\infty} e^{i\xi x} f(x) dx \\ &= \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} e^{i\xi k}\end{aligned}$$

iii)

$$\begin{aligned}\phi_Y(\xi) &= \int_{-\infty}^{\infty} e^{i\xi y} f(y) dy \\ &= \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{i\xi k + np} f(k + np) dk \\ &= \sum_{k=-np}^{n-np} \binom{n}{k+np} p^{k+np} (1-p)^{n-k+np} e^{i\xi k + np}\end{aligned}$$

Problem 4: Sample Variance v.s. MLE variance

Problem 5: Deriving the CF of a multivariate gaussian

Problem 6: Deriving the Conditional Gaussian Distribution

Problem 7: Monty Hall's Game

Problem 8: Finite Difference Equation?