AM216 Stochastic Differential Equations	Assignment 3
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Q1. Consider the fair game in the gambler's ruin problem. Let $u(x,t)$ be the conditional probability of surviving to at least time t with $X(t) > c_0$ given starting with initial cash x.

$$u(x,t) = \Pr\Big(X(\tau) > 0, \tau \in (0,t) \text{ and } X(t) > c_0 \Big| X(0) = x\Big)$$

Derive the governing initial boundary value problem (IBVP) below for $u(x,t)$.

$$\begin{cases} u_t = \dfrac{1}{2}u_{xx}, \quad 0 < x < +\infty, \quad t > 0 \\ u(0,t) = 0 \\ u(x,0) = \begin{cases} 1, & x > c_0 \\ 0, & 0 < x < c_0 \end{cases} \end{cases} \tag{1}$$

Q2. Follow the steps below to solve IBVP (1).

i) Use odd extension to convert the IBVP to an initial value problem (IVP).

ii) Use the fundamental solution to write the solution of the IVP as an integral.

iii) Express the integral solution in terms of the error function erf( ).

Q3. Solve the boundary value problem (BVP) below.

$$\begin{cases} T_{xx} - 2mT_x = -2, \quad 0 < x < C \\ T(0) = 0, \quad T(C) = 0 \end{cases}$$

Derive the solution $T(x) = \dfrac{x}{m} - \dfrac{C}{m}\Big(\dfrac{e^{2mx}-1}{e^{2mC}-1}\Big)$.

Q4. Let $dt = \dfrac{t_f}{N}$, $t_j = j\,dt$, $dW_j = W(t_{j+1}) - W(t_j) = \sqrt{dt}X_j$, $X_j \sim N(0,1)$.

i) Show $E(X_j^4) = 3$ and $E(dW_j^4) = 3(dt)^2$.

ii) Let $Q_n = \displaystyle\sum_{j=0}^{n-1} 2t_j(dW_j)^2$. Show the following 4 items.

$$E(Q_n) = \sum_{j=0}^{n-1} 2t_j dt, \qquad \text{Var}(Q_n) = 2dt \sum_{j=0}^{n-1}(2t_j)^2 dt$$

$$\lim_{n\to\infty} E(Q_n) = \int_0^{t_f} 2t\,dt, \qquad \lim_{n\to\infty} \text{Var}(Q_n) = 0$$

Remark: This is another confirmation of Ito's lemma: $(dW)^2$ can be replaced with dt.

Q5. Let $0 < t_1 < t_2 < t_3 < t_4$. Show that

i) $\Big(W(t_3)\Big|W(t_2) = w_2,\ W(t_1) = w_1\Big) \sim \Big(W(t_3)\Big|W(t_2) = w_2\Big)$.

ii) $\Big(W(t_2)\Big|W(t_3) = w_3,\ W(t_4) = w_4\Big) \sim \Big(W(t_2)\Big|W(t_3) = w_3\Big)$.

Hint: Consider the case where random variables X_1 and X_2 are independent of Y. That is,

$$\rho_{(X_1,X_2,Y)}(x_1, x_2, y) = \rho_{(X_1,X_2)}(x_1, x_2)\rho_Y(y)$$

Use the definition to prove

$$\big(X_1\big|X_2 = x_2, Y = y\big) \sim \big(X_1\big|X_2 = x_2\big)$$

That is, a conditional probability involving (X_1, X_2) is also independent of Y. Use this result to prove items i) and ii) above.

Remark: In a conditional probability of $W(t)$, in each of the forward time and backward time directions, only the constraint at the closest time takes effect.

Q6. Consider the stationary process $Z(t) \equiv \dfrac{W(t+\Delta t) - W(t)}{\Delta t}$. For $\tau > 0$, show that

$$R(\tau) \equiv E\big(Z(s+\tau)Z(s)\big) = \begin{cases} 0, & \tau > \Delta t \\ \dfrac{\Delta t - \tau}{(\Delta t)^2}, & 0 < \tau < \Delta t \end{cases}$$

Remark: From this result, it follows $\boxed{\lim_{\Delta t \to 0} R(\tau) = \delta(\tau)}$.

Q7. Numerical demonstration of Ito's Lemma: $(dW)^2$ can be replaced with dt.
We consider two finite (stochastic) difference equations below.

$$y(0) = 0$$

$$\text{FDA:}\ \ y(t+\Delta t) = y(t) + \sqrt{\sin^2(y(t)) + 1}\,\Delta W + \big(\cos(y(t)) + 1\big)(\Delta W)^2$$

$$\text{FDB:}\ \ y(t+\Delta t) = y(t) + \sqrt{\sin^2(y(t)) + 1}\,\Delta W + \big(\cos(y(t)) + 1\big)dt$$

$$\Delta W \sim \sqrt{\Delta t}\,\varepsilon, \qquad \varepsilon \sim N(0,1)$$

Each solution contains randomness from ΔW. To compare the solutions of FDA and FDB, it is important that we use the same $\{\Delta W_0, \Delta W_1, \Delta W_2, \ldots, \Delta W_{n-1}\}$ in FDA and FDB. Numerically solve FDA and FDB to $t_f = 1$ with $\Delta t = 2^{-10}$ for 3 independent repeats.

(a) Plot the 3 sets of numerical solutions as functions of time, one set from each repeat.

(b) Let error $= \big|(\text{sol of FDB}) - (\text{sol of FDA})\big|$. Plot the 3 errors as functions of time, one from each repeat. Use log scale for error.

(c) Repeat part (b) with $\Delta t = 2^{-16}$.

Q8. Refined sampling of $W(t)$ constrained by an existing coarse sampling.

We start by sampling $W(t)$ in $t \in [0,1]$ with $(\Delta t)^{(n)} = 1/n$. Here the superscript denotes the resolution since we need to work with time grids of different resolutions. Let

$$(\Delta t)^{(n)} = 1/n, \quad W_j^{(n)} = W(j(\Delta t)^{(n)}), \quad \Delta W_j^{(n)} = W_{j+1}^{(n)} - W_j^{(n)}$$

With resolution $(\Delta t)^{(n)}$, an unconstrained $W(t)$ for $t \in [0,1]$ is obtained by

$$\begin{aligned} &\{W_k^{(n)},\ 0 \leqslant k \leqslant (n-1)\} \\ &W_k^{(n)} = \sum_{j=0}^{k-1} \Delta W_j^{(n)}, \quad \Delta W_j^{(n)} = \sqrt{(\Delta t)^{(n)}}\,\varepsilon, \quad \varepsilon \sim N(0,1) \end{aligned}$$

Each time we need ε, we draw an independent sample of $N(0,1)$. At resolution $2n$, the sampling is constrained by what already obtained at resolution n.

$$\Delta W_{2j}^{(2n)} + \Delta W_{2j+1}^{(2n)} = \Delta W_j^{(n)}$$

With this constraint, $\Delta W_{2j}^{(2n)}$ and $\Delta W_{2j+1}^{(2n)}$ are sampled as follows.

$$\begin{aligned} \Delta W_{2j}^{(2n)} &= \frac{1}{2}\Delta W_j^{(n)} + \sqrt{\frac{(\Delta t)^{(2n)}}{2}}\,\varepsilon, \quad \varepsilon \sim N(0,1) \\ \Delta W_{2j+1}^{(2n)} &= \Delta W_j^{(n)} - \Delta W_{2j}^{(2n)} \end{aligned}$$

Note that the coefficient of ε is $\sqrt{\frac{(\Delta t)^{(2n)}}{2}}$, not $\sqrt{\frac{(\Delta t)^{(n)}}{2}}$. The refined sample is

$$\{W_k^{(2n)},\ 0 \leqslant k \leqslant (2n-1)\}, \qquad W_k^{(2n)} = \sum_{j=0}^{k-1} \Delta W_j^{(2n)}$$

This constrained sample refinement can be carried out from resolution n to $2n$ to $4n$, ...

Sample $\{\Delta W\}$ with $\Delta t = 2^{-2}$ unconstrained and use the sample to solve FDB in **Q7**. Refine the sample recursively with $\Delta t = 2^{-3}, 2^{-4}, \ldots, 2^{-10}$. Use the constrained refined sample $\{\Delta W\}$ to solve FDB in **Q7**. Plot solutions at three time resolutions: $\Delta t = 2^{-2}, 2^{-6}, 2^{-10}$.

Note: If you don't have time to implement constrained refined sampling of $\{\Delta W\}$, you may start with the highest resolution $\Delta t = 2^{-10}$ and do downsampling, which is straightforward.