AM 216 - Stochastic Differential Equations: Assignment

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Problem 1: Time inversion property of the Wiener process

Proof. We begin by looking at the covariance of the two sets. We have,

$$\begin{aligned}\mathrm{Cov}(W_i, W_j) &= E(W_iW_j) - E(W_i)E(W_j), \quad j > i, \quad W_j = a + b, \quad W_i = a \\ &= E(a^2 + ab) \\ &= E(a^2) \\ &= E(W_i^2) \\ &= t_i\end{aligned}$$

This holds for $j > i$ and if $i > j$ then we have $\mathrm{Cov}(W_i, W_j) = t_j$. Thus we have, $\mathrm{Cov}(W_i, W_j) = \min(t_i, t_j)$. Next we look at

$$\begin{aligned}\mathrm{Cov}(B_i, B_j) &= t_it_j \left(E(W_{1/i}W_{1/j}) - E(W_{1/i})E(W_{1/j})\right) \\ &= t_it_jE(W^2_{1/\max(t_i,t_j)}) \\ &= t_it_j\frac{1}{\max(t_i, t_j)} \\ &= \min(t_i, t_j)\end{aligned}$$

As the problem hint states, each of these RV is clearly gaussian, and since they have the same covariance, we have that they are iid. That is,

$$\{B_i\} \sim \{W_i\}$$

□

Problem 2: Integrating the Wiener process

i) We have $W(t) = \int_0^t dW(s)$

Proof. We begin,

$$\begin{aligned}\int_0^T \int_0^s dW(s)dt &= \int_0^T \int_s^T dtdW(s) \\ &= \int_0^T (T - s)dW(s) \\ &\sim N\left(0, \int_0^T (T - s)^2dt\right) \\ &\sim N\left(0, T^3 - T^3 + \frac{T^3}{3}\right) \\ &\sim N\left(0, \frac{T^3}{3}\right)\end{aligned}$$

□

ii) We repeat but with a new integrand

Proof.

$$\begin{aligned}
\int_0^T \int_0^s t dW(s) dt &= \int_0^T \int_s^T t dt dW(s) \\
&= \frac{1}{2} \int_0^T (T^2 - s^2) dW(s) \\
&\sim N\left(0, \frac{1}{4} \int_0^T (T^2 - s^2)^2 dt\right) \\
&\sim N\left(0, \frac{1}{4} \left(T^5 - \frac{2}{3} T^5 + \frac{1}{5} T^5\right)\right) \\
&\sim N\left(0, \frac{2T^5}{15}\right)
\end{aligned}$$

□

Problem 3: Reflection principle of the Wiener process

i) Show that $\Pr(M_T \geq a) = 2\Pr(W(T) \geq a)$.

Proof. The core point in this proof is showing that the conditional probability for $\Pr(W(T) \geq a | W(\tau) = a, \tau \in [0, T]) = 0.5$. This follows because the rest of the series $t \in [\tau, T]$ is distributed symmetrically about $W(\tau) = a$. Notice that if $W(\tau) = a$, we must have $M_T \geq a$. Thus we have by bayes theorem,

$$\begin{aligned}
\Pr(W(T) \geq a | M(T) \geq a) = 0.5 &= \frac{\Pr(M(T) \geq a | W(T) \geq a) \Pr(W(T) \geq a)}{\Pr(M_T \geq a)} \\
\Pr(M_T \geq a) &= 2\Pr(W(T) \geq a)
\end{aligned}$$

□

ii) Find the PDF of M_T.

Proof. We look at the related probabilities in order to determine the pdf of M_T. We have,

$$\begin{aligned}
\Pr(M_T \geq a) &= 2\Pr(W(T) \geq a) \\
\int_a^\infty \rho_{M_T}(x) dx &= 2 \int_a^\infty \rho_{W(T)}(x) dx \\
\int_{a+dx}^\infty \rho_{M_T}(x) dx - \int_a^\infty \rho_{M_T}(x) dx &= 2 \int_{a+dx}^\infty \rho_{W(T)}(x) dx - 2 \int_a^\infty \rho_{W(T)}(x) dx \\
\int_a^{a+dx} \rho_{M_T}(x) dx &= 2 \int_a^{a+dx} \rho_{W(T)}(x) dx \\
\lim_{dx \to 0} \int_a^{a+dx} \rho_{M_T}(x) dx &= \lim_{dx \to 0} 2 \int_a^{a+dx} \rho_{W(T)}(x) dx \\
\rho_{M_T}(x) &= 2\rho_{W(T)}(x), \quad x \in [0, \infty) \\
&= \sqrt{\frac{2}{\pi T}} e^{-x^2/2T}, \quad x \geq 0
\end{aligned}$$

□

Problem 4: Lambda-chain rule for stochastic integrals

i) Stratonovich interpretation

Proof.

$$\begin{aligned}\int_0^T \cos(W(t))dW(t) &= \sin\left(W(t)\right)\Big|_0^T - \int_0^T 0dt \\ &= \sin(W(T))\end{aligned}$$

□

ii) Ito Interpretation

Proof.

$$\begin{aligned}\int_0^T e^t \sin(W(t))dW(t) &= -e^t\cos(W(t))\Big|_0^T - \int_0^T -e^t\cos(W(t)) + \frac{1}{2}e^t\cos(W(t))dt \\ &= -e^T\cos(W(T)) + 1 + \frac{1}{2}\int_0^T e^t\cos(W(t))dt\end{aligned}$$

□

Problem 5: SDE for American Stocks

i)

$$\begin{aligned}w &= E(w + w_z dW + w_s ds + w_{zz}/2dW^2 + O(ds^2)) \\ &= w + w_s ds + w_{zz}/2ds + O(ds^2) \\ w_s &= -\frac{1}{2}w_{zz}\end{aligned}$$

ii)

$$\begin{aligned}w_s &= -\frac{1}{2}w_{zz} \\ w(0,s) &= 0 \\ w(x_c,s) &= 1 \\ w(z,T) &= \begin{cases}0 & z < x_c \\ 1 & z \geq x_c\end{cases}\end{aligned}$$

iii)

$$\begin{aligned}w_\tau &= \frac{1}{2}w_{zz} \\ w(0,\tau) &= 0 \\ w(x_c,\tau) &= 1 \\ w(z,0) &= \begin{cases}0 & z < x_c \\ 1 & z \geq x_c\end{cases}\end{aligned}$$

Problem 6: SDE for European Stocks

i)

$$
\begin{aligned}
w &= E(w + w_z(bds + \sqrt{a}dW) + w_s ds + w_{zz}(b^2ds^2 + adW^2)/2 + O(ds^2)) \\
&= w + w_s ds + bw_z ds + w_{zz} ds/2 + O(ds^2) \\
w_s &= -bw_z - \frac{a}{2}w_{zz}
\end{aligned}
$$

ii)

$$
\begin{aligned}
w_s &= -bw_z - \frac{a}{2}w_{zz} \\
w(0,s) &= 0 \\
w(z,T) &= \begin{cases} 0 & z < x_c \\ 1 & z \geq x_c \end{cases}
\end{aligned}
$$

iii) No, since the probability is dependent on X at time T we have no boundary condition at $z = x_c$. That is, since $X(t)$ can cross back over x_c even if it reaches x_c before $t = T$.