Formulas in Lecture 17

Black-Scholes option pricing model

Analytical expression of option price function $C(s, t)$

$$\boxed{\begin{aligned} C(s,t) &= \frac{e^{-r\tau}K}{2}\left\{\exp\left(x+\frac{\sigma^2\tau}{2}\right)\left(1+\operatorname{erf}\left(\frac{x+\sigma^2\tau}{\sqrt{2\sigma^2\tau}}\right)\right)-\left(1+\operatorname{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}}\right)\right)\right\} \\ x &= \log\frac{s}{K}+\left(r-\frac{1}{2}\sigma^2\right)\tau, \quad \tau = T-t \end{aligned}}$$

$C(s, t)$ in terms of (η, ω)

$$\boxed{\begin{aligned} C(s,t) &= \frac{e^{-r\tau}K}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K}+r\tau, \quad \omega = \frac{1}{2}\sigma^2\tau \\ &= \frac{s}{2}e^{-\eta}\phi(\eta,\omega), \qquad e^{-\eta} = \frac{e^{-r\tau}K}{s} \end{aligned}}$$

$$\phi(\eta,\omega) = e^{\eta}\left[1+\operatorname{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\operatorname{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

Properties of function $\phi(\eta, \omega)$

- Both $\phi(\eta, \omega)$ and $e^{-\eta}\phi(\eta, \omega)$ are increasing functions of η.
- $\phi(\eta, \omega)$ is an increasing function of ω.

Expected reward for buying the option at time t_0

$$\underbrace{e^{r\tau_0}C(s_0,t_0)}_{\text{reward for putting it in savings}} = \frac{K}{2}\phi(\eta_r,\omega), \qquad \eta_r = \log\frac{s_0}{K}+r\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

$$\underbrace{E\left(\max(S(T)-K,0)\middle| S(t_0)=s_0\right)}_{\text{expected reward for buying the option}} = \frac{K}{2}\phi(\eta_\mu,\omega), \qquad \eta_\mu = \log\frac{s_0}{K}+\mu\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

The principle of risk and reward

==> $\phi(\eta_\mu,\omega) > \phi(\eta_r,\omega)$ ==> $\mu > r$

The effect of interest rate r on $C(s, t)$

$$\frac{\partial C(s,t)}{\partial r} = \frac{s}{2}\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}\cdot\frac{d\eta}{dr} = \frac{s}{2}\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}\cdot\tau > 0$$

The effect of volatility σ

$$\frac{\partial C(s,t)}{\partial \sigma} = \frac{K}{2} e^{-r\tau} \frac{\partial \phi(\eta,\omega)}{\partial \omega} \cdot \frac{d\omega}{d\sigma} = \frac{K}{2} e^{-r\tau} \frac{\partial \phi(\eta,\omega)}{\partial \omega} \cdot \sigma\tau > 0$$

Formulas in Lecture 18

Properties of $C(s, t)$ (call options)

$C(S(t), t) < 0$ is absolutely impossible.

It makes no sense to exercise before expiry T.

$C(S(t), t) < S(t) - e^{-r\tau}K$ is absolutely impossible.

$C(S(t), t) > S(t)$ is absolutely impossible.

Variations in option price

Change in the option price vs change in the stock price

$$\left| \frac{C(s+\Delta s,t) - C(s,t)}{C(s,t)} \right| > \left| \frac{\Delta s}{s} \right|$$

Change in the option price vs change in the strike price

$$\underbrace{\left| C\Big|_{K} - C\Big|_{K+\Delta K} \right|}_{\text{drop in option price}} < e^{-r\tau} \underbrace{|\Delta K|}_{\substack{\text{increase in} \\ \text{strike price}}}$$

Price of a near-the-money option

$$C(s,t) \approx e^{-r\tau} K \cdot \mathrm{erf}\left(\frac{\sqrt{\sigma^2 \tau}}{2\sqrt{2}} \right) + (s - e^{-r\tau} K) \frac{1}{2} \left(1 + \mathrm{erf}\left(\frac{\sqrt{\sigma^2 \tau}}{2\sqrt{2}} \right) \right) \quad \text{for } s \approx e^{-r\tau} K$$