AM216 Stochastic Differential Equations

Lecture 17

List of topics in this lecture

- Analytical expression of the option price function $C(s, t)$
- Expected reward at time T for buying the option at time t_0
- Nominal value at time T of the amount $C(s_0, t_0)$ at time t_0
- Effect of interest rate: option price increase with interest rate
- Effect of volatility: option price increase with volatility

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Recap

Black-Scholes option pricing model

Evolution of the underlying stock price

$$dS = \mu S\,dt + \sigma S\,dW \qquad \text{(Ito interpretation)}$$

Options associated with a stock

1 unit of call option = the right to buy 1 share of the stock at price K at time T.

1 unit of put option = the right to sell 1 share of the stock at price K at time T.

Mathematical formulation of the option price function

The option price at the current time t is a deterministic function of the current stock price $S(t)$ and the current time t:

Option price = $C(S(t), t)$ where $C(s, t)$ is a deterministic function.

The key question:

Suppose I am a market maker and I am required to set and publish $C(s, t)$.

How should I set function $C(s, t)$ to avoid a guaranteed loss?

Delta hedging portfolio

1 unit of delta hedging of time t

= owning (-1) unit of call option and $C_S(S(t), t)$ shares of stock.

Caution: the composition of delta hedging varies with t.

Net gain in $[0, T]$ for maintaining a prescribed delta hedging

Suppose that over time period [0, T], we maintain a portfolio of $F(S(t), t)$ units of delta hedging of time t, at time t.

$$G_{\text{Total}} = \int F(s,t)\left(\left(C(s,t)-C_s(s,t)s\right)r - C_t(s,t)-\frac{1}{2}C_{ss}(s,t)\sigma^2 s^2\right)\bigg|_{s=S(t)} dt$$

We set $F(s, t)$ to make $G_{\text{Total}} = \int (\quad)^2 dt$.

To avoid a guaranteed loss for the market maker, ( ) must be identically zero.

Governing FVP of $C(s, t)$

$$\begin{cases} C_t(s,t)+\frac{1}{2}\sigma^2 s^2 C_{ss}(s,t) = r\left(C(s,t)-sC_s(s,t)\right) \\ C(s,t)\big|_{t=T} = \max(s-K,0) \end{cases}$$

Analytical expression of $C(s, t)$

We use change of variables to rewrite the FVP as an IVP.

New time variable

$\tau = T - t$ time to expiration

==> $t = T - \tau$

New spatial (price) variable

$$x = \log\frac{s}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)$$

$$\Longrightarrow \quad s = K\exp\left(x - \left(r-\frac{1}{2}\sigma^2\right)\tau\right)$$

New price function

$$u(x,\tau) = e^{r(T-t)}C(s,t)$$

$$\Longrightarrow \quad C(s,t) = e^{-r\tau}u(x,\tau)$$

Derivatives of $C(s, t)$

We start with the derivatives of (τ, x) with respect to (t, s).

$$\frac{\partial\tau}{\partial t} = -1, \qquad \frac{\partial\tau}{\partial s} = 0$$

$$\frac{\partial x}{\partial t} = -\left(r-\frac{1}{2}\sigma^2\right), \qquad \frac{\partial x}{\partial s} = \frac{1}{s}$$

We express derivatives of $C(s, t)$ in terms of those of $u(x, \tau)$ using the chain rule.

$$\frac{\partial}{\partial t}C(s,t)=\frac{\partial}{\partial \tau}\left[e^{-r\tau}u(x,\tau)\right]\cdot\frac{\partial \tau}{\partial t}+\frac{\partial}{\partial x}\left[e^{-r\tau}u(x,\tau)\right]\cdot\frac{\partial x}{\partial t}$$

$$=re^{-r\tau}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial \tau}u(x,\tau)-\left(r-\frac{1}{2}\sigma^2\right)e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)$$

$$\frac{\partial}{\partial s}C(s,t)=\frac{\partial}{\partial x}\left[e^{-r\tau}u(x,\tau)\right]\cdot\frac{\partial x}{\partial s}=e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s}$$

$$\frac{\partial^2}{\partial s^2}C(s,t)=\frac{\partial}{\partial s}\left[e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s}\right]$$

$$=\frac{\partial}{\partial s}\left[e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\right]\cdot\frac{1}{s}-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s^2}$$

$$=e^{-r\tau}\frac{\partial^2}{\partial x^2}u(x,\tau)\cdot\frac{1}{s^2}-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s^2}$$

PDE for $u(x, \tau)$

Substituting these derivatives into the PDE for $C(s, t)$, we obtain the PDE for $u(x, \tau)$.

$$\underbrace{re^{-r\tau}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial \tau}u(x,\tau)-\left(r-\frac{1}{2}\sigma^2\right)e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)}_{C_t(s,t)\equiv T_1+T_2+T_3}$$

$$\underbrace{+\frac{1}{2}\sigma^2\left[e^{-r\tau}\frac{\partial^2}{\partial x^2}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\right]}_{\frac{1}{2}\sigma^2 s^2 C_{ss}(s,t)\equiv T_4+T_5}=\underbrace{r\left(e^{-r\tau}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\right)}_{r\left(C(s,t)-sC_s(s,t)\right)\equiv T_6+T_7}$$

Combining T_1 with T_6, first part of T_3 with T_7, second part of T_3 with T_5, we obtain

$$-e^{-r\tau}\frac{\partial}{\partial \tau}u(x,\tau)+\frac{1}{2}\sigma^2 e^{-r\tau}\frac{\partial^2}{\partial x^2}u(x,\tau)=0$$

which leads to a simple PDE for $u(x, \tau)$

$$\boxed{u_\tau(x,\tau)=\frac{1}{2}\sigma^2 u_{xx}(x,\tau)}$$

where $u(x, \tau)$ is related to $C(s, t)$ by

$$C(s,t)=e^{-r\tau}u(x,\tau),\qquad x=\log\frac{s}{K}+\left(r-\frac{1}{2}\sigma^2\right)\tau,\quad \tau=T-t\,.$$

Initial condition for $u(x, \tau)$

We use $s(x,\tau)\Big|_{\tau=0} = K\exp(x-(r-\frac{1}{2}\sigma^2)\tau)\Big|_{\tau=0} = Ke^x$.

$$u(x,\tau)\Big|_{\tau=0} = C(s,t)\Big|_{t=T} = \max\Big(Ke^x - K, 0\Big) = K\max\Big((e^x-1), 0\Big)$$

The initial value problem (IVP) for $u(x, \tau)$

$$\begin{cases} u_\tau(x,\tau) = \dfrac{1}{2}\sigma^2 u_{xx}(x,\tau) \\ u(x,\tau)\Big|_{\tau=0} = K\begin{cases} (e^x-1), & x>0 \\ 0, & x<0 \end{cases} \end{cases}$$

Solution of the IVP

We use the fundamental solution of the heat equation to write out $u(x, \tau)$.

$$u(x,\tau) = \int_{-\infty}^{\infty} u(y,0)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(\frac{-(y-x)^2}{2\sigma^2\tau}\right)dy$$

$$= \frac{K}{\sqrt{2\pi\sigma^2\tau}}\int_0^{\infty}(e^y-1)\exp\left(\frac{-(y-x)^2}{2\sigma^2\tau}\right)dy \equiv K(I_2 - I_1)$$

We express I_1 and I_2 in terms of the error function. Recall that

$$F_{N(0,\sigma^2\tau)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x}\exp\left(\frac{-\xi^2}{2\sigma^2\tau}\right)d\xi = \frac{1}{2}\left(1+\mathrm{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}}\right)\right)$$

We write out integral I_1 in terms of the error function.

$$I_1 = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_0^{\infty}\exp\left(\frac{-(y-x)^2}{2\sigma^2\tau}\right)dy$$

change of variables $\xi = x - y$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{x}\exp\left(\frac{-\xi^2}{2\sigma^2\tau}\right)d\xi = \frac{1}{2}\left(1+\mathrm{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}}\right)\right)$$

For integral I_2, we first complete the square in the exponent.

$$I_2 = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_0^{\infty}\exp\left(\frac{-(y-x)^2}{2\sigma^2\tau} + y\right)dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_0^{\infty}\exp\left(\frac{-\left[y^2 - 2(x+\sigma^2\tau)y + (x+\sigma^2\tau)^2\right]}{2\sigma^2\tau} + x + \frac{\sigma^2\tau}{2}\right)dy$$

$$= \exp\left(x+\frac{\sigma^2\tau}{2}\right)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_0^{\infty}\exp\left(\frac{-(y-x-\sigma^2\tau)^2}{2\sigma^2\tau}\right)dy$$

We then use change of variables $\xi = x + \sigma^2\tau - y$ to write I_2 as

$$I_2 = \exp\left(x+\frac{\sigma^2\tau}{2}\right)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{(x+\sigma^2\tau)}\exp\left(\frac{-\xi^2}{2\sigma^2\tau}\right)dy$$

$$= \exp\left(x+\frac{\sigma^2\tau}{2}\right)\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x+\sigma^2\tau}{\sqrt{2\sigma^2\tau}}\right)\right)$$

Combining I_1 and I_2, we arrive at the expression of $u(x, \tau)$

$$\boxed{u(x,\tau)=\frac{K}{2}\left\{\exp\left(x+\frac{\sigma^2\tau}{2}\right)\left(1+\operatorname{erf}\left(\frac{x+\sigma^2\tau}{\sqrt{2\sigma^2\tau}}\right)\right)-\left(1+\operatorname{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}}\right)\right)\right\}}$$

<u>Solution of $C(s,t)=e^{-r\tau}u(x,\tau)$</u>

$$\boxed{\begin{gathered}C(s,t)=\frac{e^{-r\tau}K}{2}\left\{\exp\left(x+\frac{\sigma^2\tau}{2}\right)\left(1+\operatorname{erf}\left(\frac{x+\sigma^2\tau}{\sqrt{2\sigma^2\tau}}\right)\right)-\left(1+\operatorname{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}}\right)\right)\right\}\\ x=\log\frac{s}{K}+\left(r-\frac{1}{2}\sigma^2\right)\tau, \quad \tau=T-t\end{gathered}}$$

<u>Function $\phi(\eta, \omega)$</u>

For analyzing various trends, we introduce

$$\eta \equiv x+\frac{1}{2}\sigma^2\tau=\log\frac{s}{K}+r\tau, \qquad \omega\equiv\frac{1}{2}\sigma^2\tau$$

$$\Longrightarrow \qquad x+\sigma^2\tau=\eta+\omega, \qquad 2\sigma^2\tau=4\omega$$

We write $C(s, t)$ as a function of (η, ω).

$$\boxed{C(s,t)=\frac{e^{-r\tau}K}{2}\phi(\eta,\omega), \qquad \eta=\log\frac{s}{K}+r\tau, \quad \omega=\frac{1}{2}\sigma^2\tau} \tag{C-1}$$

where function $\phi(\eta, \omega)$ is defined as

$$\phi(\eta,\omega)=e^{\eta}\left[1+\operatorname{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\operatorname{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right] \tag{F-1}$$

<u>Derivatives</u> of $\phi(\eta, \omega)$

$$\frac{d}{dz}\operatorname{erf}(z)=\frac{d}{dz}\int_0^z \frac{2}{\sqrt{\pi}}\exp(-s^2)ds=\frac{2}{\sqrt{\pi}}\exp(-z^2)$$

We use this result to calculate $\phi_\eta(\eta, \omega)$

$$e^{\eta}\frac{\partial}{\partial\eta}\operatorname{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)=\frac{2}{\sqrt{\pi}}e^{\eta}\exp\left(-\frac{(\eta^2+2\eta\omega+\omega^2)}{4\omega}\right)\frac{1}{\sqrt{4\omega}}$$

$$\frac{\partial}{\partial\eta}\operatorname{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)=\frac{2}{\sqrt{\pi}}\exp\left(-\frac{(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)\frac{1}{\sqrt{4\omega}}$$

Noticing that the two exponentials are the same.

$$e^{\eta}\exp\left(-\frac{(\eta^2+2\eta\omega+\omega^2)}{4\omega}\right)=\exp\left(-\frac{(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)$$

We obtain

$$\boxed{\frac{\partial\phi(\eta,\omega)}{\partial\eta}=e^{\eta}\left(1+\operatorname{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right)>0} \qquad \text{(DF-1)}$$

$\phi(\eta, \omega)$ is an increasing function of η.

We calculate $\phi_\omega(\eta, \omega)$ in a similar way.

$$\frac{\partial\phi(\eta,\omega)}{\partial\omega}=e^{\eta}\exp\left(\frac{-(\eta^2+2\eta\omega+\omega^2)}{4\omega}\right)\left(-\frac{\eta}{4\omega^{3/2}}+\frac{1}{4\omega^{1/2}}\right)$$
$$-\exp\left(\frac{-(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)\left(-\frac{\eta}{4\omega^{3/2}}-\frac{1}{4\omega^{1/2}}\right)$$

Again, the two exponentials are the same. We obtain

$$\boxed{\frac{\partial\phi(\eta,\omega)}{\partial\omega}=\exp\left(\frac{-(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)\frac{1}{2\omega^{1/2}}>0} \qquad \text{(DF-2)}$$

$\phi(\eta, \omega)$ is an increasing function of ω.

We combine (F-1) and (DF-1) to calculate $(e^{-\eta}\phi(\eta, \omega))_\eta$.

$$\boxed{\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}=e^{-\eta}\left(1+\operatorname{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right)>0} \qquad \text{(DF-1B)}$$

$e^{-\eta}\phi(\eta, \omega)$ is an increasing function of η.

Summary:

- Both $\phi(\eta, \omega)$ and $e^{-\eta}\phi(\eta, \omega)$ are increasing functions of η.

- $\phi(\eta, \omega)$ is an increasing function of ω.

Expression of $C_s(s, t)$

We use (DF-1) to calculate $C_s(s, t)$, which is needed in the delta hedging.

$$C(s,t) = \frac{e^{-r\tau}K}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K} + r\tau, \quad \omega = \frac{1}{2}\sigma^2\tau$$

$$\frac{\partial C(s,t)}{\partial s} = \frac{e^{-r\tau}K}{2}\frac{\partial\phi(\eta,\omega)}{\partial\eta}\cdot\frac{d\eta}{ds} = \frac{e^{-r\tau}K}{2}\mathrm{e}^{\eta}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right)\cdot\frac{1}{s}, \qquad \mathrm{e}^{\eta} = \frac{s}{K}e^{r\tau}$$

We arrive at

$$\boxed{C_s(s,t) = \frac{1}{2}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right), \qquad \eta = \log\frac{s}{K} + r\tau, \quad \omega = \frac{1}{2}\sigma^2\tau}$$

Expected reward for buying the option at time t_0

We compare the rewards of buying the option vs not buying.

Let $s_0 \equiv S(t_0)$ be the stock price at time t_0.

$C(s_0, t_0)$ is the amount needed to buy the option at time t_0.

Nominal value at time T of the amount $C(s_0, t_0)$ at time t_0

$$\left[\text{NV at } T \text{ of } C(s_0,t_0) \text{ at } t_0\right] = \mathrm{e}^{r(T-t_0)}C(s_0,t_0) = \mathrm{e}^{r\tau_0}C(s_0,t_0), \qquad \tau_0 = T - t_0$$

This is the "reward" at time T for putting $C(s_0, t_0)$ in savings at time t_0.

We write it in terms of (η, ω), using equation (C-1)

$$\mathrm{e}^{r\tau_0}C(s_0,t_0) = \frac{K}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s_0}{K} + r\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0, \quad \tau_0 = T - t_0$$

Next we calculate the expected reward at time T for buying the option at time t_0.

Evolution of $Y = \log(S)$

$$dS = \mu S\,dt + \sigma S\,dW \quad \text{(Ito)}, \quad \text{starting at } S(t_0) = s_0$$

This SDE (Ito) for S corresponds to the SDE below for $Y \equiv \log(S)$

$$dY = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma\,dW, \qquad Y = \log(S) \quad \text{starting at } Y(t_0) = \log(s_0)$$

$$\Longrightarrow \quad Y(T) = Y(t_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right)(T - t_0) + \sigma\left(W(T) - W(t_0)\right)$$

$$\Longrightarrow \quad Y(T) = \log(s_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right)\tau_0 + N\left(0, \sigma^2\tau_0\right), \qquad \tau_0 = T - t_0$$

<u>The probability density of $Y(T)$</u>.

$$\rho_Y(y) = \rho_{N\left(\log(s_0)+\mu\tau_0-\frac{1}{2}\sigma^2,\,\sigma^2\tau_0\right)}(y) = \frac{1}{\sqrt{2\pi\sigma^2\tau_0}}\exp\left(\frac{-\left(y-\log(s_0)-(\mu-\frac{1}{2}\sigma^2)\tau_0\right)^2}{2\sigma^2\tau_0}\right)$$

<u>Expected reward at time T for owning the option</u> is

$$E\left(\max(S(T)-K,0)\middle|S(t_0)=s_0\right) = E\left(\max(e^{Y(T)}-K,0)\right)$$

$$= \int_{-\infty}^{\infty}\max\left(\mathrm{e}^y-K,0\right)\rho_Y(y)dy = \int_{\log(K)}^{\infty}(\mathrm{e}^y-K)\rho_Y(y)dy \equiv J_2 - J_1$$

We work out J_1 and J_2 similar to what did previously on I_1 and I_2.

In J_1, we use change of variables: $y = -\xi$.

$$J_1 = K\int_{\log(K)}^{\infty}\rho_Y(y)dy = \frac{K}{\sqrt{2\pi\sigma^2\tau_0}}\int_{-\infty}^{-\log(K)}\exp\left(\frac{-\left(\xi+\log(s_0)+\mu\tau_0-\frac{1}{2}\sigma^2\tau_0\right)^2}{2\sigma^2\tau_0}\right)d\xi$$

$$= \frac{K}{2}\left(1+\mathrm{erf}\left(\frac{\eta_\mu-\omega}{\sqrt{4\omega}}\right)\right), \qquad \eta_\mu = \log\frac{s_0}{K}+\mu\tau_0\,, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

In J_2, we complete square and then use change of variables: $y = -\xi$.

$$J_2 = \int_{\log(K)}^{\infty} e^y\rho_Y(y)dy = \frac{1}{\sqrt{2\pi\sigma^2\tau_0}}\int_{\log(K)}^{\infty}\exp\left(\frac{2\sigma^2\tau_0 y-\left(y-\log(s_0)-\mu\tau_0+\frac{1}{2}\sigma^2\tau_0\right)^2}{2\sigma^2\tau_0}\right)dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau_0}}\exp\left(\log(s_0)+\mu\tau_0\right)\int_{\log(K)}^{\infty}\exp\left(\frac{-\left(y-\log(s_0)-\mu\tau_0-\frac{1}{2}\sigma^2\tau_0\right)^2}{2\sigma^2\tau_0}\right)dy$$

$$= \frac{K}{\sqrt{2\pi\sigma^2\tau_0}}\exp\left(\log\frac{s_0}{K}+\mu\tau_0\right)\int_{-\infty}^{-\log(K)}\exp\left(\frac{-\left(\xi+\log(s_0)+\mu\tau_0+\frac{1}{2}\sigma^2\tau_0\right)^2}{2\sigma^2\tau_0}\right)d\xi$$

$$= \frac{K}{2}\mathrm{e}^{\eta_\mu}\left(1+\mathrm{erf}\left(\frac{\eta_\mu+\omega}{\sqrt{4\omega}}\right)\right), \qquad \eta_\mu = \log\frac{s_0}{K}+\mu\tau_0\,, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

Combing J_1 and J_2, we write the expected reward at time T as

$$E\left(\max(S(T)-K,0)\middle|S(t_0)=s_0\right) = \frac{K}{2}\left\{\mathrm{e}^{\eta_\mu}\left(1+\mathrm{erf}\left(\frac{\eta_\mu+\omega}{\sqrt{4\omega}}\right)\right)-\left(1+\mathrm{erf}\left(\frac{\eta_\mu-\omega}{\sqrt{4\omega}}\right)\right)\right\}$$

$$= \frac{K}{2}\phi(\eta_\mu, \omega), \qquad \eta_\mu = \log\frac{s_0}{K} + \mu\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

We compare the two rewards.

$$\underbrace{e^{r\tau_0}C(s_0, t_0)}_{\text{reward for putting it in savings}} = \frac{K}{2}\phi(\eta_r, \omega), \qquad \eta_r = \log\frac{s_0}{K} + r\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

$$\underbrace{E\Big(\max(S(T) - K, 0) \Big| S(t_0) = s_0\Big)}_{\text{expected reward for buying the option}} = \frac{K}{2}\phi(\eta_\mu, \omega), \qquad \eta_\mu = \log\frac{s_0}{K} + \mu\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

<u>Conclusions of the comparison:</u>

- Both rewards are given by $\phi(\eta, \omega)$ with respectively η_r and η_μ.

  In (DF-1), we derived $\phi_\eta(\eta, \omega) > 0$. It follows that $\phi(\eta, \omega)$ is an increasing function of η, which, in turn, is an increasing function of r.

- The reward at time T of owning the option is a random variable. The reward may be less than the cost $C(s_0, t_0)$. In fact, it has a large probability of 0 reward.

- The principle of risk and reward:

  A higher risk demands a greater <u>expected</u> reward.

  The principle of risk and reward tells us

  $$\phi(\eta_\mu, \omega) > \phi(\eta_r, \omega)$$

  ==> $\eta_\mu > \eta_r$

  ==> $\mu > r$

- In Appendix A, we do a three-way comparison by including the expected reward at time T of buying $C(s_0, t_0)$ amount of stock at time t_0.

- The principle of risk and reward is valid in the broader sense when we use the perceived risk and the perceived expected rewards, in all forms, received from all sources.

<u>Example:</u>

The reasoning for buying a lottery ticket.

I may assign a significant value to the excitement of a possible winning (the expected reward in all forms received from all sources is higher).

Or I may (falsely) believe that my number selection scheme will significantly increase my chance of winning (the perceived expected reward is higher than the actual value; the perceived risk is lower than the actual value).

The effect of interest rate *r* on $C(s, t)$

We rewrite $C(s, t)$ given in (C-1) to transfer the dependence on r into η.

$$C(s,t)=\frac{K}{2}e^{-r\tau}\phi(\eta,\omega), \qquad \eta=\log\frac{s}{K}+r\tau, \quad \omega=\frac{1}{2}\sigma^2\tau$$

$$=\frac{s}{2}e^{-\eta}\phi(\eta,\omega), \qquad e^{-\eta}=\frac{K}{s}e^{-r\tau}$$

In (DF-1B) we derived $(e^{-\eta}\phi_\eta(\eta, \omega))_\eta > 0$. It follows that

$$\frac{\partial C(s,t)}{\partial r}=\frac{s}{2}\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}\cdot\frac{d\eta}{dr}=\frac{s}{2}\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}\cdot\tau > 0$$

<u>Conclusion:</u>

Option price $C(s, t)$ increases with interest rate r.

The effect of volatility σ

The effect of volatility σ is contained in variable ω.

$$C(s,t)=\frac{K}{2}e^{-r\tau}\phi(\eta,\omega), \qquad \eta=\log\frac{s}{K}+r\tau, \quad \omega=\frac{1}{2}\sigma^2\tau$$

In (DF-2) we derived $\phi_\omega(\eta, \omega) > 0$. It follows that

$$\frac{\partial C(s,t)}{\partial\sigma}=\frac{K}{2}e^{-r\tau}\frac{\partial\phi(\eta,\omega)}{\partial\omega}\cdot\frac{d\omega}{d\sigma}=\frac{K}{2}e^{-r\tau}\frac{\partial\phi(\eta,\omega)}{\partial\omega}\cdot\sigma\tau > 0$$

<u>Conclusion:</u>

Option price $C(s, t)$ increases with volatility σ.

The case of unknown σ

We can estimate σ from historic stock price data of the underlying stock and then use the estimated σ to predict the option price $C(s, t)$.

Conversely, we can use the current market price $C(s, t)$ of the option to estimate investors' perceived future volatility of the underlying stock.

- $C(s, t)$ increases with σ monotonically.
- For each realized sample of market price $C(s, t)$, there is a corresponding estimated value of perceived future volatility σ.

Appendix A: expected reward of buying option vs buying stock vs savings

We already calculated the two rewards.

$$\underbrace{e^{r\tau_0}C(s_0,t_0)}_{\text{reward for putting it in savings}} = \frac{K}{2}\phi(\eta_r,\omega), \qquad \eta_r = \log\frac{s_0}{K} + r\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

$$\underbrace{E\Big(\max(S(T)-K,0)\Big|S(t_0)=s_0\Big)}_{\text{expected reward for buying the option}} = \frac{K}{2}\phi(\eta_\mu,\omega), \qquad \eta_\mu = \log\frac{s_0}{K} + \mu\tau_0, \quad \omega = \frac{1}{2}\sigma^2\tau_0$$

We calculate the expected reward at time T of buying $C(s_0, t_0)$ amount of stock at time t_0.

The stock price is governed by

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \qquad S(t_0) = s_0$$

Note that in the Ito interpretation, $S(t)$ is independent of $dW(t) \equiv W(t+dt) - W(t)$.

Taking the average of both sides, we get

$$dE\big(S(t)\big) = \mu E\big(S(t)\big)dt, \qquad E\big(S(t_0)\big) = s_0$$

$$\Longrightarrow \qquad \frac{dE\big(S(t)\big)}{dt} = \mu E\big(S(t)\big)$$

$$\Longrightarrow \qquad E\big(S(t)\big) = e^{\mu(t-t_0)}s_0$$

$$E\big(S(T)\big) = e^{\mu\tau_0}s_0, \qquad \tau_0 = T - t_0$$

The expected reward at time T of buying $C(s_0, t_0)$ amount of stock at time t_0 is

$$\underbrace{e^{\mu\tau_0}C(s_0,t_0)}_{\text{expected reward for buying the stock}} = e^{\mu\tau_0 - r\tau_0} \underbrace{e^{r\tau_0}C(s_0,t_0)}_{\text{reward for putting it in savings}}$$

Using the result of $e^{r\tau_0}C(s_0,t_0)$, we write $e^{\mu\tau_0}C(s_0,t_0)$ in terms of η_r, ω and η_μ.

$$\underbrace{e^{\mu\tau_0}C(s_0,t_0)}_{\text{expected reward for buying the stock}} = \frac{K}{2}e^{\eta_\mu-\eta_r}\phi(\eta_r,\omega), \qquad \eta_r = \log\frac{s_0}{K} + r\tau_0, \quad \eta_\mu = \log\frac{s_0}{K} + \mu\tau_0$$

Recall i) $\eta_\mu > \eta_r$ and ii) $e^{-\eta}\phi(\eta, \omega)$ is an increasing function of η. We obtain

$$\phi(\eta_r,\omega) < e^{\eta_\mu-\eta_r}\phi(\eta_r,\omega) < \phi(\eta_\mu,\omega)$$

Therefore we arrive at

$$\underbrace{\mathrm{e}^{r\tau_0}C(s_0,t_0)}_{\text{reward for putting it in savings}} < \underbrace{e^{\mu\tau_0}C(s_0,t_0)}_{\substack{\text{expected reward for}\\ \text{buying the stock}}} < \underbrace{E\left(\max(S(T)-K,0)\middle| S(t_0)=s_0\right)}_{\text{expected reward for buying the option}}$$