AM216 Stochastic Differential Equations

Lecture 18

List of topics in this lecture

- The case of time-dependent drift
- Estimating volatility from market prices of options
- Scaling laws of option price
- In-the-money vs out-of-the-money options, short-term vs long-term
- Properties of option price, variations in option price

Recap

Black-Scholes option pricing model

Analytical expression of the option price function $C(s, t)$

$$C(s,t)=\frac{e^{-r\tau}K}{2}\phi(\eta,\omega)=\frac{s}{2}e^{-\eta}\phi(\eta,\omega)\,,\qquad \eta=\log\frac{s}{K}+r\tau\,,\quad \omega=\frac{1}{2}\sigma^2\tau$$

$$\phi(\eta,\omega)=\mathrm{e}^{\eta}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

- Both $\phi(\eta, \omega)$ and $\mathrm{e}^{-\eta}\phi(\eta, \omega)$ are increasing functions of η.
- $\phi(\eta, \omega)$ is an increasing function of ω.

Two expected rewards at time T

$$\underbrace{\mathrm{e}^{r\tau_0}C(s_0,t_0)}_{\text{reward for putting it in savings}}=\frac{K}{2}\phi(\eta_r,\omega)\,,\qquad \eta_r=\log\frac{s_0}{K}+r\tau_0\,,\quad \omega=\frac{1}{2}\sigma^2\tau_0$$

$$\underbrace{E\Big(\max(S(T)-K,0)\Big|S(t_0)=s_0\Big)}_{\text{expected reward for buying the option}}=\frac{K}{2}\phi(\eta_\mu,\omega)\,,\qquad \eta_\mu=\log\frac{s_0}{K}+\mu\tau_0\,,\quad \omega=\frac{1}{2}\sigma^2\tau_0$$

Principle of risk and reward:

$$\phi(\eta_\mu,\omega)>\phi(\eta_r,\omega)\quad ==>\quad \eta_\mu>\eta_r\quad ==>\quad \mu>r$$

Effect of interest rate r :

$C(s, t)$ increases with r

Effect of volatility σ :

$C(s, t)$ increases with σ

The case of time-dependent drift $\mu(t)$

$$dS = \mu(t) S dt + \sigma S dW$$

We re-visit the process of deriving the PDE for $C(s, t)$.

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At time t, buy $F(S(t), t)$ unit of delta hedging of time t .

At time $(t+dt)$, sell $F(S(t), t)$ unit of delta hedging of time t purchased at time t.

Change in cash balance at time $(t+dt)$ after selling

$$\Delta B = \underbrace{F\big(S(t),t\big)}_{\substack{\text{\# of units of}\\ \text{delta hedge}}} \Bigg[\underbrace{-1}_{\substack{\text{\# of units}\\ \text{of option}}} \times \underbrace{C\big(S(t+dt),t+dt\big)}_{\substack{\text{option price}\\ \text{at time } t+\Delta t}} + \underbrace{C_s\big(S(t),t\big)}_{\substack{\text{\# of shares}\\ \text{of stock}}} \times \underbrace{S(t+dt)}_{\substack{\text{stock price}\\ \text{at time } t+\Delta t}} \Bigg]$$

written in short notation

$$= F(S,t)\big[-C(S+dS,t+dt) + C_s(S,t)(S+dS)\big]$$

Taylor expansion in terms of dS and dt

$$= F(S,t)\left[\begin{array}{l} -C(S,t) - \underbrace{C_s(S,t)dS}_{dS \text{ term}} - C_t(S,t)dt - \frac{1}{2}C_{ss}(S,t)(dS)^2 \\ +C_s(S,t)S + \underbrace{C_s(S,t)dS}_{dS \text{ term}} + o(dt) \end{array}\right]$$

By the special design of delta hedging portfolio, the two dS terms cancel each other.

$$(dS)^2 = \big(\mu(t) S dt + \sigma S dW\big)^2 = \sigma^2 S^2 (dW)^2 + o(dt)$$

The effect of $\mu(t)$ is buried in the $o(dt)$ term, which disappears as $dt \to 0$.

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Conclusion: $\mu(t)$ does not affect $C(s, t)$.

In reality, $\mu(t)$ is unknown and unpredictable.

Estimating volatility from market prices of options

Recall that $C(s, t)$ is an increasing function of σ. Based on the observed market stock price $S(t)$ and market option price $C(S(t), t)$, we can estimate the perceived volatility.

Example:

Current time: t = the closing of May 26th, 2020.

Current stock price of Walt Disney Co: $s \equiv S(t) = \$120.95$.

Interest rate: $r = 2\%/\text{year}$; μ is not needed.

Table 1: Market prices of call options expiring June 5th, and the predicted volatility.

Expiry: T = June 5th 2020, τ = 8 trading days.

K, strike price	$C(s, t)$, market price of the option,	Predicted volatility σ^2 [1/year]
\$121	\$2.91	0.1140
\$122	\$2.47	0.1158
\$123	\$1.98	0.1092
\$124	\$1.63	0.1096
\$125	\$1.35	0.1118
\$126	\$1.05	0.1081

$E(\sigma^2) = 0.1114/\text{year}$ ==> $\sqrt{E(\sigma^2)} = 0.334/\sqrt{\text{year}}$

==> volatility = 33.4% fluctuation in a year

Table 2: Market prices of call options expiring June 19th, and the predicted volatility.

Expiry: T = June 19th 2020, τ = 18 trading days.

K, strike price	$C(s, t)$, market price of the option,	Predicted volatility σ^2 [1/year]
\$121	\$4.40	0.1138
\$122	\$3.90	0.1121
\$123	\$3.45	0.1112
\$124	\$3.05	0.1109
\$125	\$2.61	0.1067
\$126	\$2.31	0.1081

$E(\sigma^2) = 0.1105/\text{year}$ ==> $\sqrt{E(\sigma^2)} = 0.332/\sqrt{\text{year}}$

==> volatility = 33.2% fluctuation in a year

Scaling laws

Interest-rate-adjusted strike price at time t

To exercise the option at time T (buying the stock at strike price K), we need to allocate cash $e^{-r\tau}K$ at time t, which will grow to K at time T. Here $\tau = T - t$.

(Price K at time T) <==> (price $e^{-r\tau}K$ at time t)

We define the interest-rate-adjusted-strike price as $K_c \equiv e^{-r\tau}K$.

We write $C(s, t)$ and η in terms of K_c.

$$C(s,t)=\frac{e^{-r\tau}K}{2}\phi(\eta,\omega)=\frac{K_c}{2}\phi(\eta,\omega)\,,\qquad \eta=\log\frac{s}{K_c}\,,\quad \omega=\frac{1}{2}\sigma^2\tau$$

Non-dimensional ratios

We normalize the rate-adjusted strike price and the option price by the stock price.

$$q\equiv\frac{K_c}{s}\,,\qquad Q\equiv\frac{C(s,t)}{s}$$

Q is a function of q and $\omega = \sigma^2\tau/2$; Q has no explicit dependence on s.

$$Q(q,\omega)=\frac{q}{2}\phi(\eta,\omega)=\frac{e^{-\eta}}{2}\phi(\eta,\omega)\,,\qquad \eta=\log\frac{1}{q}\,,\quad \omega=\frac{1}{2}\sigma^2\tau$$

Scaling property 1: at certain conditions, $C(S(t), t) \propto S(t)$

Given t, given $K/S(t)$ and given σ, we have

$$q=\frac{e^{-r(T-t)}K}{S(t)}=\text{fixed}\,,\qquad \omega=\frac{1}{2}\sigma^2(T-t)=\text{fixed}\quad\Rightarrow\quad Q(q,\omega)=\text{fixed}$$

==> $$C(S(t),t)=S(t)\cdot Q(q,\omega)\,\propto\,S(t)$$

Conclusion:

Given t, given $K/S(t)$ and given σ, the option price is proportional to the stock price.

Example

Stock 1 is at $S_1(t) = 45$ and the call option on stock 1 with $K = 50$ is at $C_1 = 2$.

Suppose stock 2 has the same volatility and is at $S_2(t) = 90$.

The call option on stock 2 with $K = 100$ and the same T should be at $C_2 = 4$.

Scaling property 2: effect of volatility σ is only in the combination $\omega = \sigma^2\tau/2$.

Given K_c and given $S(t)$, we have $q=\frac{K_c}{S(t)}=\text{fixed}$.

$$C(S(t),t)=S(t)\cdot Q\big(q,\omega\big), \qquad \omega\equiv\frac{\sigma^2\tau}{2}$$

Conclusion:

Given K_c and given $S(t)$, when $\omega \equiv \sigma^2\tau/2$ is unchanged, the option price is unchanged.

Example:

Suppose stock 1 has volatility σ_1 and stock 2 has volatility $\sigma_2 = 2\sigma_1$. Suppose both stocks are currently at the same price $S_1(t) = S_2(t) = S(t)$.

We consider two options with times to expiry (τ_1, τ_2) and strike prices (K_1, K_2).

- the call option on stock 1 with τ_1 and $K_1 = \exp(r\tau_1)K_c$
- the call option on stock 2 with $\tau_2 = \tau_1/4$ and $K_2 = \exp(r\tau_2)K_c$

These two options should have the same option price.

Scaling property 3: effect of interest rate r is only in the combination $K_c \equiv e^{-r\tau}K$.

Given $S(t)$ and given $\omega \equiv \sigma^2\tau/2$, we have

$$C(S(t),t)=S(t)\cdot Q(q,\omega), \qquad q\equiv\frac{K_c}{S(t)}=\frac{e^{-r\tau}K}{S(t)}$$

Conclusion:

Given $S(t)$ and given $\omega \equiv \sigma^2\tau/2$, when $K_c \equiv e^{-r\tau}K$ is unchanged, the option price is unchanged even when both r and K are increased.

In-the-money vs out-of-the-money options, short-term vs long-term

In the terminology of option market, in-the-money and out-of-the-money call options are defined as follows.

Conventional definitions of in-the-money and out-of-the-money

In-the-money call option:

$K < S(t)$ (strike price < current stock price)

If the option is exercised immediately, there is a reward (not counting the premium already paid for the option)

Out-of-the-money call option:

$K > S(t)$ (strike price > current stock price)

If the option is exercised immediately, there is no reward.

Mathematically, we define in-the-money and out-of-the-money slightly differently.

Mathematical definitions of in-the-money and out-of-the-money

In-the-money call option (mathematical definition)

$K_c \equiv e^{-r\tau}K < S(t)$ (rate-adjusted strike price < current stock price)

There is a reward if the option holder does the followings

i) sells the stock at time t,

ii) set aside the proceeds (which will grow to $S(t)e^{r\tau}$ at time T), and

iii) at time T, use cash K to exercises the option to buy back the stock, which yields a cash balance of $S(t)e^{r\tau} - K > 0$.

Out-of-the-money call option (mathematical definition)

$K_c \equiv e^{-r\tau}K > s$ (rate-adjusted strike price > current stock price)

There is no reward if the option holder does the steps described above.

Remark:

The two sets of definitions are essentially the same for small $r\tau$.

Short-term in-the-money option:

$$C(s,t) = \frac{K_c}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

$$\phi(\eta,\omega) = e^{\eta}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right] - \left[1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

Mathematically, short-term means small ω and in-the-money means $\eta > 0$.

We expand erf(z) for $z > 0$ and large.

$$\mathrm{erf}(z) \approx 1 - \frac{\exp(-z^2)}{z\sqrt{\pi}} \approx 1 \qquad \text{for } z > 0 \text{ and large}$$

We expand $\phi(\eta, \omega)$ and $C(s, t)$ for $\eta > 0$ and small ω.

$$\frac{\eta+\omega}{\sqrt{4\omega}} > 0 \text{ and large}, \qquad \frac{\eta-\omega}{\sqrt{4\omega}} > 0 \text{ and large}$$

$$\phi(\eta,\omega) \approx e^{\eta} - 1$$

$$C(s,t) \approx \frac{K_c}{2}(e^{\eta}-1) = s - K_c, \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

(A more careful expansion: $C(s,t) \approx s - K_c + s\dfrac{\sigma^3\tau^{3/2}}{\eta^2\sqrt{2\pi}}\exp\left(\dfrac{-\eta^2}{2\sigma^2\tau}\right) \approx s - K_c$).

Conclusion:

For a very short-term option that is significantly in the money, the option price is essentially $(S(t) - K_c)$.

Short-term out-of-the-money option:

Mathematically, short-term means small ω and out-the-money means $\eta < 0$.

We expand $\mathrm{erf}(z)$ for $z < 0$ and large.

$$\mathrm{erf}(z) \approx -1 + \frac{\exp(-z^2)}{z\sqrt{\pi}} \approx -1 \qquad \text{for } z < 0 \text{ and large}$$

We expand $\phi(\eta, \omega)$ and $C(s, t)$ for $\eta < 0$ and small ω.

$$\frac{(\eta+\omega)}{\sqrt{4\omega}} < 0 \text{ and large}, \qquad \frac{(\eta-\omega)}{\sqrt{4\omega}} < 0 \text{ and large}$$

$$\phi(\eta,\omega) \approx 0$$

$$C(s,t) \approx 0$$

(A more careful expansion: $C(s,t) \approx s\dfrac{\sigma^3\tau^{3/2}}{\eta^2\sqrt{2\pi}}\exp\left(\dfrac{-\eta^2}{2\sigma^2\tau}\right) \approx 0$)

Conclusion:

For a very short-term option that is significantly out-of-the-money, the option price is essentially 0.

Long-term options:

Mathematically, long-term means $\omega \equiv \sigma^2\tau/2 \gg 1$, which almost does not exist in the real world of option market because $\omega \gg 1$ corresponds to a really long time.

Example:

$\sigma^2 = 0.25/\text{year}$ (50% fluctuation in a year, a significant fluctuation)

$\omega \equiv \sigma^2\tau/2 \geq 10$ requires $\tau \geq 80$ years

Even at $\omega = 10$, we have

$$\frac{\omega}{\sqrt{4\omega}} = \frac{\sigma^2\tau/2}{\sqrt{2\sigma^2\tau}} = \frac{10}{\sqrt{40}} \approx 1.58, \quad \text{not a large number.}$$

In the real world of option market, "long-term" refers to the case of $\omega \equiv \sigma^2\tau/2 \sim O(1)$, which does not have a simple asymptotic expression.

Properties of $C(s, t)$ (call options)

Option price is always positive

$C(S(t), t) \leq 0$ is absolutely impossible.

Otherwise, we can make a risk-free gain by "buying" the option and doing nothing.

It makes no sense to exercise before expiry T

Exercising at time t requires cash K at time t.

Exercising at time T requires cash $e^{-r\tau}K < K$ at time t.

American style options and European style options are essentially the same (for stocks that do not pay a dividend).

Option price is always above $S(t)-e^{-r\tau}K$

$C(S(t), t) < S(t)-e^{-r\tau}K$ is absolutely impossible.

If $C(S(t), t) < S(t)-e^{-r\tau}K$, we can make a risk-free gain by doing the steps below.

- Sell the stock at $S(t)$ at time t.
- Buy the option at $C(S(t), t)$ at time t.

  The two actions above yield a cash balance of $S(t)- C(S(t), t) > e^{-r\tau}K$, which will grow to $e^{r\tau}(S(t)-C(S(t), t)) > K$ at time T.
- At time T, use cash K to exercise the option to close the position.

  We end with a positive cash balance: $e^{r\tau}(S(t)-C(S(t), t)) - K > 0$.

Option price is always below the stock price

$C(S(t), t) > S(t)$ is absolutely impossible.

If $C(S(t), t) > S(t)$, we can make a risk-free gain by doing the steps below.

- Sell the option at $C(S(t), t)$ at time t.
- Buy the stock at $S(t)$ at time t.

  The two actions above yield a cash balance of $C(S(t), t)-S(t) > 0$.
- At time T, the stock may or may not be called away.

We end with a positive cash balance, $e^{r\tau}(C(S(t), t)-S(t)) > 0$ plus the stock (if the option is not exercised) or additional amount K (if the stock is called away).

Variations in option price

Change in the option price vs change in the stock price

We calculate $\partial C/\partial s$ from the expression of $C(s, t)$.

$$C(s,t)=\frac{e^{-r\tau}K}{2}\phi(\eta,\omega)=\frac{s}{2}e^{-\eta}\phi(\eta,\omega), \qquad \eta=\log\frac{s}{K}+r\tau, \quad \omega=\frac{1}{2}\sigma^2\tau$$

$$\phi(\eta,\omega)=e^{\eta}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

Recall that $(e^{-\eta}\phi(\eta, \omega))_\eta > 0$. We obtain

$$\frac{\partial C(s,t)}{\partial s}=\frac{1}{2}e^{-\eta}\phi(\eta,\omega)+\frac{s}{2}\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}>\frac{1}{2}e^{-\eta}\phi(\eta,\omega)=\frac{C(s,t)}{s}$$

$$\Longrightarrow \qquad \frac{C(s+\Delta s,t)-C(s,t)}{\Delta s}>\frac{C(s,t)}{s}$$

$$\Longrightarrow \qquad \left|\frac{C(s+\Delta s,t)-C(s,t)}{C(s,t)}\right| > \left|\frac{\Delta s}{s}\right|$$

Percentage-wise, the option price is more volatile than the stock price.

Change in the option price vs change in the strike price

We calculate $\partial C/\partial K$ from the expression of $C(s, t)$.

$$C(s,t)=\frac{s}{2}e^{-\eta}\phi(\eta,\omega), \qquad \eta=\log\frac{s}{K}+r\tau, \quad \omega=\frac{1}{2}\sigma^2\tau$$

Note that the effect of K is contained in η.

$$\frac{\partial C(s,t)}{\partial K}=\frac{s}{2}\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}\cdot\frac{\partial\eta}{\partial K}=\frac{-s}{2K}\frac{\partial\left(e^{-\eta}\phi(\eta,\omega)\right)}{\partial\eta}<0$$

Recall that previously we derived

$$\frac{\partial}{\partial\eta}\left(e^{-\eta}\phi(\eta,\omega)\right)=e^{-\eta}\left(1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right) < 2e^{-\eta}$$

$$\Longrightarrow \qquad 0 < -\frac{\partial C}{\partial K} < \frac{s}{K}e^{-\eta}=e^{-r\tau}$$

==> $$0 < \frac{C\big|_K - C\big|_{K+\Delta K}}{\Delta K} < \mathrm{e}^{-r\tau}$$

==> $$\left| C\big|_K - C\big|_{K+\Delta K} \right| < \mathrm{e}^{-r\tau} |\Delta K|$$

Conclusion:

- The change in option price is less than the change in strike price.
- The effect of strike price decreases as the time to expiry increases.

Price of actively traded options

Price of the at-the-money option

$$C(s,t) = K_c \frac{1}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

At-the-money means $s = K_c$ and thus $\eta = 0$.

$$\phi(\eta,\omega)\big|_{\eta=0} = \left[1+\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)\right] - \left[1-\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)\right] = 2\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)$$

When the stock price is at $s = K_c$, the option price is

$$C(s,t) = \mathrm{e}^{-r\tau}K\cdot\mathrm{erf}\left(\frac{\sqrt{\sigma^2\tau}}{2\sqrt{2}}\right) \quad \text{at } s = \mathrm{e}^{-r\tau}K$$

Price of a near-the-money option

$$C(s,t) = K_c \frac{1}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

Near-the-money means $s \approx K_c$ and thus $\eta \approx 0$.

$$\eta = \log\frac{s}{K_c} = \log\left(1+\frac{s}{K_c}-1\right) \approx \frac{s}{K_c}-1$$

We expand $C(s, t)$ around $\eta = 0$.

$$\phi(\eta,\omega)\big|_{\eta=0} = \left[\mathrm{e}^{\eta}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right) - \left(1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right)\right]_{\eta=0} = 2\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)$$

$$\frac{\partial}{\partial\eta}\phi(\eta,\omega)\Big|_{\eta=0} = e^{\eta}\left(1+\text{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right)\Bigg|_{\eta=0} = 1+\text{erf}\left(\frac{\sqrt{\omega}}{2}\right)$$

$$\phi(\eta,\omega) \approx \phi(\eta,\omega)\Big|_{\eta=0} + \eta\frac{\partial}{\partial\eta}\phi(\eta,\omega)\Big|_{\eta=0}$$

We use the expansion of $\phi(\eta, \omega)$ to approximate $C(s, t)$.

$$C(s,t) \approx \frac{K_c}{2}\left[\phi(\eta,\omega)\Big|_{\eta=0} + \eta\frac{\partial}{\partial\eta}\phi(\eta,\omega)\Big|_{\eta=0}\right]$$

$$\approx \frac{K_c}{2}\left[2\text{erf}\left(\frac{\sqrt{\omega}}{2}\right) + \left(\frac{s}{K_c}-1\right)\left(1+\text{erf}\left(\frac{\sqrt{\omega}}{2}\right)\right)\right]$$

$$\approx K_c\text{erf}\left(\frac{\sqrt{\omega}}{2}\right) + (s-K_c)\frac{1}{2}\left(1+\text{erf}\left(\frac{\sqrt{\omega}}{2}\right)\right)$$

When the stock price is near $s = K_c$, the option price is

$$C(s,t) \approx e^{-r\tau}K\cdot\text{erf}\left(\frac{\sqrt{\sigma^2\tau}}{2\sqrt{2}}\right) + (s-e^{-r\tau}K)\frac{1}{2}\left(1+\text{erf}\left(\frac{\sqrt{\sigma^2\tau}}{2\sqrt{2}}\right)\right) \quad \text{for } s \approx e^{-r\tau}K$$