

## List of topics in this lecture

- Black-Scholes option pricing model: stochastic evolution of the stock price, call option, put option, strike price, expiration date, the option price function
- Delta hedging portfolio: 1 unit of delta hedging of time  $t$ , continuously revised delta hedging, mathematical view of delta hedging, net gain from maintaining a prescribed delta hedging portfolio
- Governing equation for the option price function, the final value problem

## Recap

### Feynman-Kac formula for the forward equation

Stochastic differential equation (SDE)

$$dX = b(X, t)dt + \sqrt{a(X, t)}dW$$

$u(x, t)$  is defined using path integral

$$u(x, t) = E \left( \delta(X(t) - x) \exp \left( - \int_0^t \psi(X(s), s) ds \right) \right)$$

$u(x, t)$  satisfies the forward equation with a fatality/growth term

$$u_t = - \left( b(x, t) u \right)_x + \frac{1}{2} \left( a(x, t) u \right)_{xx} - \psi(x, t) u$$

### Meaning of $u(x, t)$

= mass density in  $x$  of the surviving cell population at time  $t$ .

Evolution of  $X(s)$  is governed solely by the SDE, independent of  $\psi(z, s)$ .

Cell population is affected by the fatality/growth effect of  $\psi(z, s)$ .

When the SDE is unknown but a large set of sample paths  $\{X_j(s), j = 1, 2, \dots, N\}$  is available, we can estimate  $u(x, t)$  from the data.

$$u(x, t) \approx \frac{1}{N \cdot \Delta x} \sum_{x < X_j(t) \leq x + \Delta x} \exp \left( - \int_0^t \psi(X_j(s), s) ds \right)$$

## Black-Scholes option pricing model

### The underlying stock:

Consider the stock of ABC company:

One share of stock = a fraction of ownership of ABC company.

### Example:

There are 1,805 million shares of Walt Disney Company stock.

Each share =  $1/(1.8 \times 10^9)$  fraction of Walt Disney Company.

### Roadmap of the discussion

1. Evolution of the stock price
2. Options associated with the stock
3. Mathematical formulation of the option price function
4. Delta hedging portfolio
5. Total gain in  $[0, T]$  for maintaining a prescribed delta hedging
6. Governing PDE for the option price function

## 1. Evolution of the stock price

Let  $S(t)$  = stock price of ABC company at time  $t$

(stock price = per share price in the open financial market).

### Assumption:

$S(t)$  is a geometric Brownian motion with a geometric drift.

Specifically, stock price  $S(t)$  is governed by the SDE

$$dS = \mu S dt + \sigma S dW \quad (\text{Ito interpretation})$$

### Parameters:

Volatility:  $\sigma$

Drift:  $\mu$

### Comment:

Here we model the stock price using the Ito interpretation of the SDE

$$dS = \mu S dt + \sigma S dW \quad (\text{S01A})$$

In some literature, the stock price is model as the SDE

$$d(\log S) = \mu_B dt + \sigma dW \quad (\text{S01B})$$

(S01A) is subject to different interpretations; (S01B) is not.

Previously, we derived that

(S01B) is equivalent to the Stratonovich interpretation of (S01A) with  $\mu = \mu_B$ ,  
which is equivalent to the Ito interpretation of the modified SDE

$$dS = \left( \mu_B + \frac{1}{2} \sigma^2 \right) S dt + \sigma S dW$$

We will use (S01A) with the Ito interpretation. When comparing the results of (S01A) and (S01B), keep in mind  $\mu = (\mu_B + \sigma^2 / 2)$ .

## 2. Options associated with the stock

An option is a contract that gives the owner (the holder) of the option the right, but not the obligation,

to buy (sell)

a specified number of shares of the underlying stock,

at a specified price

prior to or on a specified date

Terminology and notations:

- right  $\neq$  obligation  
(the option owner has no obligation to buy or sell the stock).
- call option = the right to buy
- put option = the right to sell
- 1 contract of call (put) option = the right to buy (sell) 100 shares
- strike price,  $K$  = the specified purchase (sell) price for a call (put) option
- expiry (expiration date),  $T$  = the specified data
- exercising a call (put) option: exercising the right to buy (sell)

Examples of options

Price of DIS stocks on May 4th 2022 = \$116.19/share

(DIS 220603 120 call)

= the right to buy, strike price = \$120/share, expiry = 06/03/2022

(DIS 220603 110 put)

= the right to sell, strike price = \$110/share, expiry = 06/03/2022

Caution: In the open market, option prices are listed for 1/100 contract

= price for the right to buy (sell) one share.

Price of (DIS 220603 120 call) option on May 4th 2022 = \$3.55

The cost of buying 1 contract of (DIS 220603 120 call) =  $100 \times \$3.55 = \$355$ .

Option underwriter, option buyers/sellers

Option underwriter: every option contract is underwritten by someone who sells it in the open market for a financial gain.

The option underwriter is the original option seller, and is obligated to honor the current owner of the option's right to buy (sell).

The option underwriter bets on that the future evolution of the underlying stock will make the option worth less than it is originally sold for.

Option buyer: a person who buys an option contract in the open market by paying a negotiated amount (the market price) to an option seller.

An option buyer bets on that the future evolution of the underlying stock will make the option worth more than it is bought for.

An option buyer can subsequently become an option seller by selling the acquired option in the open market.

The option underwriter can subsequently become an option buyer by buying back the option in the open market, which terminates the obligation.

Comments:

- Options have no assets of their own. Their values are solely derived from the underlying stock. For this reason, options are called financial derivatives.
- Options are not issued by the company of the underlying stock. Options are underwritten by financial gamblers. Option buyers pay money to own the right. Option underwriters receive money from buyers in exchange for granting option holders the right to buy or sell.
- Options are much more volatile than the underlying stock because the risk of options is much higher than that of the stock. The option buyer may win big if the underlying stock moves significantly in one direction; may lose all money invested in buying the options if the stock moves the other direction. For the option underwriter, the risk is even worse; the loss may be unlimited.
- Options are not limited to stocks. Options can be written on anything that has a market (and a market price). For example, oil, natural gas, wheat, corn, soy bean, gold, silver, copper, steel, lumber, ...
- In particular, options can be written on other options. They are the derivatives of derivatives. They are even more volatile (more risky!).

**3. Mathematical formulation of the option price function**

Mathematical unit of an option

For mathematical convenience, we introduce 1 “unit” of option.

1 unit of call (put) option

= 1/100 contract of call (put) option

= the right to buy (sell) 1 share of the stock at price  $K$  at time  $T$ .

Note: This is the European style option.

The American style option = the right to ... **at any time  $t \leq T$ .**

We study the European style option because it is mathematically simpler and the two are essentially equivalent.

### The option price function

Consider an option of a specified type (call or put), a specified strike price and a specified expiry.

Key assumption:

The option price at the current time  $t$  is a deterministic function of the current stock price  $S(t)$  and the current time  $t$ .

We focus on the call option. The put option can be discussed in a similar way.

Let  $C(s, t)$  denote the price per unit of the call option at time  $t$  when  $S(t) = s$ .

$C(s, t)$  is a deterministic function of two variables  $(s, t)$ .

Both the stock price and the option price are stochastic. But the two are related by a deterministic function:  $C(S(t), t)$ . The randomness in the option price is solely caused by the randomness in the stock price  $S(t)$ .

### List of variables and parameters:

$S(t)$ : stock price at time  $t$

$C(S(t), t)$ : option price at time  $t$

$\sigma$ : volatility

Volatility ( $\sigma$ ) predicts the likelihood of the stock making big moves.

$\mu$ : geometric drift in the SDE of  $S(t)$

As we will see, drift ( $\mu$ ) does not affect the option price.

$r$ : interest rate

Interest rate ( $r$ ) affects investors' appetite for risk.

$K$ : the strike price

$T$ : the expiry (expiration date)

### List of assumptions

- The stock does not pay a dividend. This is for mathematical simplicity.
- The underlying stock price  $S(t)$  is governed by
$$dS = \mu S dt + \sigma S dW \quad (\text{Ito interpretation})$$
- Volatility ( $\sigma$ ) and interest rate ( $r$ ) are known.
- The stochastic option price and the stochastic stock price are related by  $C(S(t), t)$  where  $C(s, t)$  is a deterministic function of 2 variables. Function  $C(s, t)$  also depends on parameters  $(\sigma, r, K, T)$ .
- We can buy/sell any amount of option/stock, including short selling.

Short selling = selling something we don't own.
- There is no bid-ask spread. At any time, the price we can buy is the same as the price we can sell. Again, this is for mathematical simplicity.
- There is a single interest rate. We can borrow/lend any amount at this known rate. Again, this is for mathematical simplicity.
- There is no transaction fee associated with buying/selling stock/option, no transaction fee associated with borrowing/lending.

For large financial firms, this is not completely unrealistic.  
Again, this is for mathematical simplicity.

### Key question for shaping the mathematical formulation

Suppose I am a market maker and I am required to set and publish  $C(s, t)$ , the deterministic function connecting the stock price and the option price.

How should I set function  $C(s, t)$  to avoid guaranteed loss?

Could someone design a scheme of trading stocks/options based on the published function  $C(s, t)$  to make a guaranteed gain?

### Answer:

We study the delta hedging portfolio.

### **4. Delta hedging portfolio**

We study the delta hedging involving a call option.

The delta hedging involving a put option can be discussed in a similar way.

### Composition of the delta hedging

Consider the call option with strike price  $K$  and expiry  $T$ .

1 unit of delta hedging of time  $t$

= being short one unit of call option and long  $C_s(S(t), t)$  shares of stock

= owning  $(-1)$  unit of call option and  $C_s(S(t), t)$  shares of stock.

Note:  $C_s(s, t)$  is the derivative of  $C(s, t)$ .

$$C_s(S(t), t) = \left. \frac{\partial C(s, t)}{\partial s} \right|_{s=S(t)}$$

Since we can buy/sell any amount (positive or negative, small or large), we can have  $(-1)$  unit of delta hedging, or any positive or negative fraction of delta hedging.

$(-1)$  unit of delta hedging of time  $t$

= owning  $(+1)$  unit of call option and  $-C_s(S(t), t)$  shares of stock.

Mathematical view of the delta hedging:

- The composition of 1 unit delta hedging of time  $t$  is determined by the stock price at time  $t$  and the published function  $C(s, t)$ .
- The composition of 1 unit delta hedging of time  $t$  varies with  $t$ . To maintain 1 unit delta hedging of the current time, buying/selling stocks is needed.

To update 1 unit delta hedging of time  $t$  to that of time  $(t+dt)$ , we need to

buy  $[C_s(S(t+dt), t+dt) - C_s(S(t), t)]$  shares of stock at time  $(t+dt)$ .

This is called continuously revised delta hedging.

- In the mathematical view, at time  $(t+dt)$  when updating 1 unit delta hedging of time  $t$  to that of time  $(t+dt)$ , there are two transactions:

selling 1 unit delta hedging of time  $t$  at time  $(t+dt)$ , and

buying 1 unit delta hedging of time  $(t+dt)$  at time  $(t+dt)$ .

“of time  $t$ ” refers to the composition of 1 unit delta hedging.

“at time  $(t+dt)$ ” refers to the time of buying/selling.

- We maintain a portfolio of  $F(S(t), t)$  units of delta hedging of time  $t$ , at time  $t$ , over a time period. For that purpose, we update the portfolio as follows.

selling  $F(S(t), t)$  units of delta hedging of time  $t$  at time  $(t+dt)$ , and

buying  $F(S(t+dt), t+dt)$  units of delta hedging of time  $(t+dt)$  at time  $(t+dt)$ .

These are the two transactions at each grid point over a time period.

- We view the selling as associated with time interval  $[t, t+dt]$  and view the buying as associated with time interval  $[t+dt, t+2dt]$ . In this way, we start a time interval with no position and end the time interval with no position.

The actions over time interval  $[t, t+dt]$  are

start with no position at time  $t$ ,

at time  $t$ , buy  $F(S(t), t)$  units of delta hedging of time  $t$ ;

hold it during  $[t, t+dt]$ ;

at time  $(t+dt)$ , sell  $F(S(t), t)$  units of delta hedging of time  $t$ ;

end with no position at time  $(t+dt)$ .

**Note:** This mathematical view involves a lot of “hypothetical” trading transactions. They are for the purpose of facilitating the calculation of net gain in  $[t, t+dt]$ . Many of these “hypothetical” trading transactions cancel each other. In real operation, the actual trading transactions needed are a lot less.

### 5. Total gain in $[0, T]$ for maintaining a prescribed delta hedging

Suppose that over time period  $[0, T]$ , we maintain a portfolio of  $F(S(t), t)$  units of delta hedging of time  $t$ , at time  $t$ . We do so by carrying out the actions described above in each time interval  $[t, t+dt]$ . Here  $F(s, t)$  is a function to be specified.

We first calculate the net gain in each time interval  $[t, t+dt]$ .

Net gain in  $[t, t+dt]$  for maintaining a prescribed delta hedging

- At time  $t$ , we start with 0 cash.
- At time  $t$ , after buying  $F(S(t), t)$  units of delta hedging of time  $t$ ,  
the cash balance at time  $t$  is

$$B(t) = \underbrace{F(S(t), t)}_{\substack{\text{\# of units of} \\ \text{delta hedging}}} \left[ \underbrace{1}_{\substack{\text{\# of units} \\ \text{of option}}} \times \underbrace{C(S(t), t)}_{\substack{\text{option price} \\ \text{at time } t}} - \underbrace{C_s(S(t), t)}_{\substack{\text{\# of shares} \\ \text{of stock}}} \times \underbrace{S(t)}_{\substack{\text{stock price} \\ \text{at time } t}} \right]$$

**Note:** The cash balance may be positive or negative.

- The interest earned in  $[t, t+dt]$  from the cash balance is

$$I = B(t) \times r \, dt$$

**Note:**

The interest earned may be positive or negative, depending on  $B(t)$ .

Negative earning = the interest cost of borrowing money.

- At time  $(t+dt)$ , after selling  $F(S(t), t)$  units of delta hedging of time  $t$ ,  
the change in cash balance at time  $(t+dt)$  is

$$\Delta B = \underbrace{F(S(t), t)}_{\substack{\text{\# of units of} \\ \text{delta hedging}}} \left[ \underbrace{-1}_{\substack{\text{\# of units} \\ \text{of option}}} \times \underbrace{C(S(t+dt), t+dt)}_{\substack{\text{option price} \\ \text{at time } t+\Delta t}} + \underbrace{C_s(S(t), t)}_{\substack{\text{\# of shares} \\ \text{of stock}}} \times \underbrace{S(t+dt)}_{\substack{\text{stock price} \\ \text{at time } t+\Delta t}} \right]$$

**Note:**

The composition of delta hedging of time  $t$  is unchanged in  $[t, t+dt]$ .

The prices of stock and option in the delta hedging do fluctuate in  $[t, t+dt]$ .

To facilitate the calculation of  $\Delta B$ , we introduce short notations:

$$S \equiv S(t), \quad dS \equiv S(t+dt) - S(t)$$

$$\implies S(t+dt) = S + dS$$

We write  $\Delta B$  as

$$\Delta B = F(S, t) \left[ -C(S + dS, t + dt) + C_s(S, t)(S + dS) \right]$$

Expanding the RHS in terms of  $dS$  and  $dt$ , we get

$$\Delta B = F(S, t) \left[ \begin{array}{l} -C(S, t) - \underbrace{C_s(S, t)dS}_{dS \text{ term}} - C_t(S, t)dt - \frac{1}{2}C_{ss}(S, t)(dS)^2 \\ + C_s(S, t)S + \underbrace{C_s(S, t)dS}_{dS \text{ term}} + o(dt) \end{array} \right]$$

By the special design of delta hedging, the two  $dS$  terms cancel each other.

$$\Delta B = F(S, t) \left[ -C(S, t) + C_s(S, t)S - C_t(S, t)dt - \frac{1}{2}C_{ss}(S, t)(dS)^2 + o(dt) \right]$$

- Net gain in time interval  $[t, t+dt]$  is

$$\text{net gain} = B(t) + I + \Delta B = B(t)(1 + r dt) + \Delta B$$

Let  $g(t)$  denote the rate of net gain in  $[t, t+dt]$  (net gain per time). We have

$$\begin{aligned} g(t)dt &= B(t)(1 + r dt) + \Delta B \\ &= \underbrace{F(S, t)[C(S, t) - C_s(S, t)S]}_{B(t)} (1 + r dt) \\ &\quad + \underbrace{F(S, t) \left[ -C(S, t) + C_s(S, t)S - C_t(S, t)dt - \frac{1}{2}C_{ss}(S, t)(dS)^2 \right]}_{\Delta B} \\ &= F(S, t) \left[ (C(S, t) - C_s(S, t)S)r dt - C_t(S, t)dt - \frac{1}{2}C_{ss}(S, t)(dS)^2 \right] \end{aligned}$$

We use the SDE to express  $(dS)^2$  in terms of  $dW$ .

$$dS = \mu S dt + \sigma S dW$$

$$\implies (dS)^2 = \sigma^2 S^2 (dW)^2 + o(dt)$$

We write the net gain in  $[t, t+dt]$  as

$$g(t)dt = F(s,t) \left[ \left( C(s,t) - C_s(s,t)s \right) r dt - C_t(s,t)dt - \frac{1}{2} C_{ss}(s,t) \sigma^2 s^2 (dW)^2 \right] \Big|_{s=S(t)}$$

Total gain in  $[0, T]$  for maintaining a prescribed delta hedging

$$\begin{aligned} G_{\text{Total}} &= \int_0^T g(t)dt \\ &= \int_0^T F(s,t) \left[ \left( C(s,t) - C_s(s,t)s \right) r dt - C_t(s,t)dt - \frac{1}{2} C_{ss}(s,t) \sigma^2 s^2 (dW)^2 \right] \Big|_{s=S(t)} \end{aligned}$$

Recall Ito's lemma: we can replace  $(dW)^2$  with  $dt$ . We obtain

$$G_{\text{Total}} = \int_0^T F(s,t) \left[ \left( C(s,t) - C_s(s,t)s \right) r - C_t(s,t) - \frac{1}{2} C_{ss}(s,t) \sigma^2 s^2 \right] \Big|_{s=S(t)} dt$$

## 6. Governing PDE for $C(s, t)$

Select the schedule of delta hedging, function  $F(s, t)$

Recall that we maintain  $F(S(t), t)$  units of delta hedging of time  $t$ , at time  $t$ .

We select function  $F(s, t)$  to make the total gain positive.

Based on the published function  $C(s, t)$ , we select  $F(s, t)$  as

$$F(s,t) \equiv \left( C(s,t) - C_s(s,t)s \right) r - C_t(s,t) - \frac{1}{2} C_{ss}(s,t) \sigma^2 s^2$$

With the selected  $F(s, t)$ , the total gain in  $[0, T]$  becomes

$$G_{\text{Total}} = \int_0^T \left( \left( C(s,t) - C_s(s,t)s \right) r - C_t(s,t) - \frac{1}{2} C_{ss}(s,t) \sigma^2 s^2 \right)^2 \Big|_{s=S(t)} dt \quad (\text{G01})$$

Observations:

- The integrand in (G01) is a stochastic process (a random variable at each  $t$ ). After integration, the total gain is still a random variable, a function of  $S(t, \omega)$ .
- Since the integrand is always non-negative, the total gain is always non-negative. That is, over any realized path  $S(t)$ , we will never lose money. **The delta hedging trading scheme is completely risk-free!**
- If the integrand is non-zero in a region of  $(s, t)$ , then the trading scheme gives a positive gain whenever the realized path  $S(t)$  goes through the region. For the market maker who sets  $C(s, t)$ , it is a guaranteed loss with no possibility of gain. When setting  $C(s, t)$ , the market maker must get rid of this risk.

**Therefore, the integrand in the total gain (G01) must be identically zero.**

$$\left( C(s,t) - C_s(s,t)s \right) r - C_t(s,t) - \frac{1}{2} C_{ss}(s,t) \sigma^2 s^2 = 0 \quad \text{for all } (s,t)$$

Governing equation for  $C(s, t)$

It follows that  $C(s, t)$  satisfies the PDE

$$C_t(s,t) + \frac{1}{2} \sigma^2 s^2 C_{ss}(s,t) = r \left( C(s,t) - s C_s(s,t) \right)$$

At end time  $T$  (expiry), the price of the call option is simply the amount of money the call option holder can make when

- exercising the option to buy the stock (if the strike price is lower), and
- then immediately selling the stock in the open market.

$$C(s,T) \Big|_{s=S(T)} = \begin{cases} S(T) - K, & \text{if } S(T) > K \\ 0, & \text{otherwise} \end{cases}$$

The final condition for  $C(s, t)$  is

$$C(s,t) \Big|_{t=T} = \max(s - K, 0)$$

The FVP for  $C(s, t)$  is

$$\begin{cases} C_t(s,t) + \frac{1}{2} \sigma^2 s^2 C_{ss}(s,t) = r \left( C(s,t) - s C_s(s,t) \right) \\ C(s,t) \Big|_{t=T} = \max(s - K, 0) \end{cases}$$

Notice that the drift term  $\mu$  is not in the FVP.