Formulas in Lecture 15

Feynman-Kac formula for the forward equation

Consider the Ito interpretation of SDE

$$dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$$

Definition of $u(x, t)$

$$u(x,t) \equiv E\left(\delta\left(X(t)-x\right)\exp\left(-\int_0^t \psi(X(s),s)ds\right)\right)$$

Meaning of $u(x, t)$

$\psi(z, s)$ is the fatality/growth rate at position z at time s.

$u(x, t)$ = mass density in x of the surviving cell population at time t.

$f_0(x)$ = mass density of $X(0)$.

Governing equation for $u(x, t)$

$$u_t = -\left(b(x,t)u\right)_x + \frac{1}{2}\left(a(x,t)u\right)_{xx} - \psi(x,t)u$$

The initial value problem (IVP)

$$\begin{cases} u_t = -\left(b(x,t)u\right)_x + \frac{1}{2}\left(a(x,t)u\right)_{xx} - \psi(x,t)u \\ u(x,t)\big|_{t=0} = f_0(x) \end{cases}$$

The solution is given by the Feynman-Kac path integral formula

$$u(x,t) = E\left(\delta\left(X(t)-x\right)\exp\left(-\int_0^t \psi(X(s),s)ds\right)\right)$$

Formulas in Lecture 16

Black-Scholes option pricing model

SDE for the stock price $S(t)$

$$dS = \mu S\,dt + \sigma S\,dW \qquad \text{(Ito interpretation)}$$

The option price function

The option price at the current time t is a deterministic function of the current stock price $S(t)$ and the current time t.

Option price = $C(S(t), t)$, where $C(s, t)$ is a deterministic function of (s, t).

List of variables and parameters:

$S(t)$: stock price at time t

$C(S(t), t)$: option price at time t

σ: volatility

μ: geometric drift in the SDE of $S(t)$

r: interest rate

K: the strike price

T: the expiry (expiration date)

Delta hedging portfolio

- 1 unit of delta hedging of time t

  = owning (−1) unit of call option and $C_S(S(t), t)$ shares of stock.

- In the mathematical view, when updating 1 unit delta hedging of time t to 1 unit delta hedging of time $(t+dt)$, there are two transactions:

  selling 1 unit delta hedging of time t at time $(t+dt)$, and

  buying 1 unit delta hedging of time $(t+dt)$ at time $(t+dt)$.

  "of time t" refers to the composition of 1 unit delta hedging.

  "at time $(t+dt)$" refers to the time of buying/selling.

- We maintain a portfolio of $F(S(t), t)$ units of delta hedging of time t, at time t, over a time period. For that purpose, we update the portfolio as follows.

  selling $F(S(t), t)$ units of delta hedging of time t at time $(t+dt)$, and

  buying $F(S(t+dt), t+dt)$ units of delta hedging of time $(t+dt)$ at time $(t+dt)$.

Total gain for maintaining $F(S(t), t)$ units of delta hedging of the current time

$$G_{\text{Total}} = \int_0^T F(s,t)\left[\Big(C(s,t)-C_s(s,t)s\Big)r - C_t(s,t) - \frac{1}{2}C_{ss}(s,t)\sigma^2 s^2\right]\Bigg|_{s=S(t)} dt$$

The FVP for $C(s, t)$

$$\begin{cases} C_t(s,t)+\dfrac{1}{2}\sigma^2 s^2 C_{ss}(s,t) = r\Big(C(s,t)-sC_s(s,t)\Big) \\ C(s,t)\Big|_{t=T} = \max(s-K,0) \end{cases}$$