AM216 Stochastic Differential Equations Assignment 8

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Q1. Use the λ-chain rule to solve SDEs.

Recall the λ-chain rule on $H(t, W(t))$:

$$dH = \Big(H_t + (\frac{1}{2} - \lambda)H_{ww}\Big)dt + H_w dW, \quad \lambda = 0 \text{ for Ito}$$

Consider the IVP of SDE

$$\begin{cases} dX = (2t + \frac{1}{2})X dt + X dW & \text{(Ito)} \\ X(0) = 2 \end{cases}$$

Compare the SDE and the λ-chain rule and equate the corresponding terms.

$$X(t) = H(t, W(t)), \quad (2t + \frac{1}{2})X = \Big(H_t + (\frac{1}{2} - \lambda)H_{ww}\Big), \quad X = H_w$$

Eliminate X from the equations above to derive PDEs on $H(t, w)$. Solve the PDEs with the condition $H(0, 0) = X(0) = 2$ to find $H(t, w)$ and write out the solution of SDE.

Q2. Time reversal of an SDE.

In lecture, we examined the reverse time evolution in a small time step dt, from $X(t+dt)$ back to $X(t)$. The transition density is given by Bayes' theorem.

$$\rho_{(X(t)|X(t+dt)=x_1)}(x_0) \propto \rho_{(X(t+dt)|X(t)=x_0)}(x_1)\rho_{X(t)}(x_0)$$
$$\propto \exp\Big(\frac{-1}{2\sigma^2 dt}\Big[(x_0 - x_1)^2 + 2c_1 dt(x_0 - x_1) + c_2 dt(x_0 - x_1)^2 + \cdots\Big]\Big)$$

where coefficients c_1 and c_2 are constants. Complete the square to show

$$\rho_{(X(t)|X(t+dt)=x_1)}(x_0) \propto \exp\Big(\frac{-(1 + c_2 dt)}{2\sigma^2 dt}\big(x_0 - x_1 + \frac{c_1 dt}{1 + c_2 dt}\big)^2 + \cdots\Big)$$

Remark: This result implies that $\big(X(t)\big|X(t + dt) = x_1\big)$ is approximately a Gaussian.

$$\Big(X(t)\Big|X(t + dt) = x_1\Big) \sim N\Big(\frac{-c_1 dt}{1 + c_2 dt}, \frac{\sigma^2 dt}{1 + c_2 dt}\Big)$$

Q3. Feynman-Kac formula.

Let $dX(t) = dW(t)$ and $\psi(z, s) = -z$. We define

$$u(x, t, T) \equiv E\Big(\exp(-\int_t^T \psi(X(s), s)ds)\Big|X(t)=x\Big) = E\Big(\exp(\int_t^T X(s)ds)\Big|X(t)=x\Big)$$

i) Write out the final value problem (FVP) of $u(x, t, T)$.

ii) Verify that $u(x,t,T)=\exp\big(\frac{(T-t)^3}{6}+(T-t)x\big)$ satisfies the FVP.

iii) Conclude $E\Big(\exp(\int_t^T X(s)ds)\Big|X(t){=}x\Big)=\exp\big(\frac{(T-t)^3}{6}+(T-t)x\big)$.

Q4. Use integrating factor to solve a linear SDE.

Consider the IVP of SDE below.

$$\begin{cases} dX = \dfrac{1}{1+t}X dt + dW \\ X(0)=1 \end{cases}$$

i) Use the method of integrating factor to solve the IVP. Write the solution $X(t)$ as an integral of dW, which implies that $X(t)$ is a normal random variable.

ii) Evaluate $E(X(t))$ and $\mathrm{Var}(X(t))$.

Q5. Simulations of Feynman-Kac formula.

Consider the SDE $dX=dW$ and the associated FVP for $u(x,t,T)$.

$$\begin{cases} 0=u_t+\dfrac{1}{2}u_{xx}-\psi(x,t)u, \quad \psi(x,t)\equiv\dfrac{1}{2}\Big((x-t)^2+2(x-t)-1\Big) \\ u(x,t,T)\Big|_{t=T}=f(x), \qquad f(x)\equiv\exp(\dfrac{-(x-T)^2}{2}) \end{cases}$$

where $u(x,t,T)$ is defined as

$$u(x,t,T)\equiv E\Big(\exp(-\int_t^T\psi\big(X(s),s\big)ds)\, f(X(T))\Big|X(t){=}x\Big) \tag{1}$$

i) Verify that $u(x,t,T)=\exp(\frac{-(x-t)^2}{2})$ is the solution of the FVP.

ii) Set $T=1$, $t=0$ and $x=0.7$. The analytical solution is $u(0.7,0,1)=e^{-0.7^2/2}\approx 0.7827$. Generate $N=10^5$ sample paths of $\{X(s)\colon 0\leqslant s\leqslant 1\}$ on the time grid of $\Delta s=2^{-10}$ starting with $X(0)=x$. Estimate $u(x,0,T)$ by approximating path integral in (1). Specifically, we introduce weight $w_k^{(i)}$ at each time s_k and for each sample path $\{X^{(i)}(s_j)\colon 0\leqslant j\leqslant n\}$.

$$w_k^{(i)}\equiv\exp(-\int_0^{s_k}\psi\big(X^{(i)}(s),s\big)ds)$$

We start with $w_0^{(i)}\equiv 1$ at $s_0=0$. From s_k to s_{k+1}, we update $\{w^{(i)}\}$ as follows.

$$w_{k+1}^{(i)}=w_k^{(i)}\exp(-\int_{s_k}^{s_{k+1}}\psi\big(X^{(i)}(s),s\big)ds)\approx w_k^{(i)}\exp(-\psi\big(X^{(i)}(s_k),s_k\big)\Delta s)$$

At $s_n=T$, we approximate $u(x,0,T)$ as

$$u(x,0,T)=\frac{1}{N}\sum_{i=1}^{N}w_n^{(i)}\,f\big(X^{(i)}(s_n)\big)$$

Report the numerical value of $u(0.7,0,T)$. Compare it with the analytical solution.

Q6. A problem related to Q6 of Assignment 7.

Suppose stochastic process $X(t)$ is governed by SDE

$$dX(t) = \sigma X(t) dW(t), \qquad \text{Ito interpretation}$$

Let $Y(t) \equiv \ln X(t)$. We have $dY(t) = \ln X(t+dt) - \ln X(t) = \ln \dfrac{X(t) + dX(t)}{X(t)}$

i) Expand in dX and use the SDE of $X(t)$ to derive the SDE of $Y(t)$.

ii) Derive the ODEs for $E(X(t))$ and $E(Y(t))$.

iii) For $X(0) = 1$ and $Y(0) = \ln X(0) = 0$, solve for $E(X(t))$ and $E(Y(t))$.