AM 216 - Stochastic Differential Equations: Assignment 1

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Problem 1: Two Dice

Roll a set of two fair 6-sided dice, one colored red the other white. Assume that the two dice roll independently. Record the two numbers facing up as X_1 and X_2.

i) Mathematically describe the format of outcome. Describe the sample space.

Proof. The two quantities X_1 and X_2 are samples from a given discrete probability distribution. The sample space has six equally likely outcomes $1, 2, 3, 4, 5, 6$ with an expected value of $E(X) = 3.5$. The two samples X_1 and X_2 are independent and identically distributed samples. □

ii) Let X be the absolute difference between X_1 and X_2. Is X a random variable?

Proof. Yes, the sum/difference of two random variables is also a random variable, albeit with a different sample space and expected value. We can exam the sample space to find that there are 6 possibilities of different likelihood: $S = 0, 1, 2, 3, 4, 5$, with 5 being the least likely (only two outcomes where the result is 5) and 1 being the most likely (there are ten outcomes with a result of 1). □

iii) Find the PMF of X, $E(X)$

Proof. Specifically,

k	P(X = k)
0	0.166...
1	0.277...
2	0.222...
3	0.166...
4	0.111...
5	0.055...

and, therefore,

$$E(X) = 1(0.277...) + 2(0.222...) + 3(0.166...) + 4(0.111...) + 5(0.055...) = 1.944...$$

□

iv) Let $A = \{X_2^2 \geq 2X_1\}$. Find $Pr(A|X_1 = n)$ for $n = 1, 2, \ldots, 6$. Then use the law of total probability to find $Pr(A)$.

Proof. We will construct a table to show the values of $Pr(A|X_1 = n)$ for all possible n.

n	1	2	3	4	5	6
Pr(A)	0.833...	0.833...	0.666...	0.666...	0.5	0.5

From this, we can find the probability that $Pr(X_1 = n) = 0.166\ldots \forall n$, and therefore find that $Pr(A) = 0.666\ldots$. □

Problem 2: Variance

i) Show that $\mathrm{Var}(\alpha X) = \alpha^2 \mathrm{Var}(X)$.

Proof.

$$\begin{aligned}
\mathrm{Var}(\alpha X) &= E(\alpha X)^2 - E^2(\alpha X) \\
&= E(\alpha^2 X^2) - (\alpha E(X))^2 \\
&= \alpha^2 E(X^2) - \alpha^2 E^2(X) \\
&= \alpha^2 \mathrm{Var}(X)
\end{aligned}$$

□

ii) Show that $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ if X and Y are independent.

Proof.

$$\begin{aligned}
\mathrm{Var}(X+Y) &= E(X+Y)^2 - E^2(X+Y) \\
&= E(X^2 + 2XY + Y^2) - (E^2(X) + 2E(X)E(Y) + E^2(Y)) \\
&= E(X^2) + E(2XY) + E(Y^2) - E^2(X) - 2E(X)E(Y) - E^2(Y) \\
&= E(X^2) + E(Y^2) - E^2(X) - E^2(Y) \\
&= \mathrm{Var}(X) + \mathrm{Var}(Y)
\end{aligned}$$

where the change between the third and fourth lines follows from the fact that X and Y are independent, i.e. $E(XY) = E(X)E(Y)$. □

iii) Find an example where $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)$ does not imply that X and Y are independent.

Proof. To be continued □

Problem 3: Binomial and Normal Distributions

Calculate $E(X^2 + Y^2)$ where $X \sim \mathrm{Bino}(n, p)$ and $Y \sim \mathcal{N}(\mu, \sigma)$.

Proof.

$$E(X^2 + Y^2) = E(X^2) + E(Y^2)$$

□