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| **AM216 Stochastic Differential Equations** | Assignment 4 |
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**Q1.** Time reversibility of Brownian bridge.

Consider the Brownian bridge with $w_T = 0$. Previously, we obtained

$$\boxed{\begin{aligned}&\underbrace{\Big((W(t_1), W(t_2), \ldots, W(t_{n-1}))\Big| W(T)=0\Big)}_{W(t)\text{ is constrained.}}\\ &\quad\sim (B(t_1), B(t_2), \ldots, B(t_{n-1})), \quad \underbrace{B(t) = W(t) - \frac{t}{T}W(T)}_{W(t)\text{ is unconstrained.}}\end{aligned}}$$

where $0 = t_0 \leqslant t_1 \leqslant t_2, \leqslant \cdots \leqslant t_{n-1} \leqslant t_n = T$.

i) Use this result to find $E\big(W(t_i)\big|W(T)=0\big)$ and $\text{Cov}\,\Big(W(t_i), W(t_j)\big|W(T)=0\Big)$.

ii) Using the result obtained in i) to show that

$$\begin{aligned}&E\big(W(T-t_i)\big|W(T)=0\big) = E\big(W(t_i)\big|W(T)=0\big)\\ &\text{Cov}\,\Big(W(T-t_i), W(T-t_j)\big|W(T)=0\Big) = \text{Cov}\,\Big(W(t_i), W(t_j)\big|W(T)=0\Big)\end{aligned}$$

iii) Use the result of ii) to conclude

$$\begin{aligned}&\Big((W(T-t_1), W(T-t_2), \ldots, W(T-t_{n-1}))\Big|W(T)=0\Big)\\ &\quad\sim \Big((W(t_1), W(t_2), \ldots, W(t_{n-1}))\Big|W(T)=0\Big)\end{aligned}$$

<u>Remark:</u> The Brownian bridge with $w_T = 0$ is time reversible in the sense that the two processes below are statistically indistinguishable.

$$\Big(\{W(t) : 0 \leqslant t \leqslant T\}\Big|W(T)=0\Big) \sim \Big(\{W(T-t) : 0 \leqslant t \leqslant T\}\Big|W(T)=0\Big)$$

**Q2.** Convergence in probability.

Suppose $\lim\limits_{n\to+\infty} E\big(Q_n(\omega)\big) = q$ and $\lim\limits_{n\to+\infty} \text{Var}\,\big(Q_n(\omega)\big) = 0$.

Show that $\{Q_n(\omega)\}$ converges to $q$ in probability as $n \to +\infty$.

<u>Hint:</u> First show $\lim\limits_{n\to+\infty} E\big((Q_n(\omega)-q)^2\big) = 0$. Then apply the Chebyshev-Markov inequality.

**Q3.** $I_1 \equiv \int_0^T f(s)dW(s)$ is a Gaussian.

Since it is a sum of independent Gaussians, it has a normal distribution and its distribution is completely described by $E\big(I_1\big)$ and $\text{Var}\,\big(I_1\big)$.

(a) Let $I_2 = \int_0^T \cos(n\pi \frac{t}{T}) dW(t)$. Find $E(I_2)$ and Var $(I_2)$.

(b) Let $F_n = \frac{2}{T}\int_0^T \sin(n\pi \frac{t}{T})\Big(W(t) - \frac{t}{T}W(T)\Big)dt$. Find $E(F_n)$ and Var $(F_n)$.

Hint: We rewrite $F_n$ in the form of $\int_0^t f(s) dW(s)$.

$$\begin{aligned}
F_n &= \frac{2}{T}\int_0^T \sin(n\pi \frac{t}{T})\Big(W(t) - \frac{t}{T}W(T)\Big)dt \quad \leftarrow \text{ integration by parts} \\
&= \frac{2}{n\pi}\int_0^T \cos(n\pi \frac{t}{T})\Big(dW(t) - \frac{W(T)}{T}dt\Big) \\
&= \frac{2}{n\pi}\Big(\int_0^T \cos(n\pi \frac{t}{T})dW(t) - \Big[\frac{1}{T}\int_0^T \cos(n\pi \frac{s}{T})ds\Big]\int_0^T dW(t)\Big) \\
&= \frac{2}{n\pi}\int_0^T \Big(\cos(n\pi \frac{t}{T}) - \Big[\frac{1}{T}\int_0^T \cos(n\pi \frac{s}{T})ds\Big]\Big)dW(t)
\end{aligned}$$

In this particular example, $\frac{1}{T}\int_0^T \cos(n\pi \frac{s}{T})ds = 0$. Even if it is not zero, the methodology introduced here works in the general situation.

**Q4.** Prove the theorem below, which is useful for calculating the variance of a sum.

Theorem: Let $f(\ )$ and $g(\ )$ be two functions. Suppose the set of random variables $\{(X_j, Y_j) : j = 0, 1, \ldots, (n-1)\}$ is jointly Gaussian and satisfies

(a) $E\big(f(X_j)\big) = 0$ for all $j$;

(b) $X_i$ and $X_j$ are independent for all $i \neq j$; and

(c) $X_j$ and $Y_k$ are independent for all $k \leqslant j$.

Then the statement below is true.

$$\text{Var}\Big(\sum_{j=0}^{n-1} g(Y_j)f(X_j)\Big) = \sum_{j=0}^{n-1} E\Big(g^2(Y_j)\Big)E\Big(f^2(X_j)\Big)$$

Hint: Show $E\big(g(Y_j)f(X_j)g(Y_k)f(X_k)\big) = 0$ for $j > k$.

**Q5.** Another confirmation of Ito's lemma: $dW(t)$ can be replace with $dt$.

Consider the Wiener process $W(t)$. Define quantity $Q_k$ and $f_k$ as follows.

$$\Delta t = \frac{T}{N}, \qquad t_j = j\Delta t, \qquad W_j = W(t_j), \qquad \Delta W_j = W_{j+1} - W_j$$

$$Q_k = \sum_{j=0}^{k-1} W_j^2 (\Delta W_j)^2, \qquad f_k = \sum_{j=0}^{k-1} W_j^2 \Delta t$$

$$Q_k - f_k = \sum_{j=0}^{k-1} W_j^2\big((\Delta W_j)^2 - \Delta t\big)$$

Show $E\big(Q_k - f_k\big) = 0$ and $\text{Var}(Q_k - f_k) = \sum_{j=0}^{k-1} 6t_j^2(\Delta t)^2$.

Remark: It follows that $\lim_{\Delta t \to 0} \text{Var}(Q_k - f_k) = 0$, which implies convergence in probability.

Hint: Recall that for $Z \sim N(0,1)$, we have $E(Z^4) = 3$. Use the result to show

$$E\big(W_j^4\big) = 3t_j^2, \qquad E\Big(\big((\Delta W_j)^2 - \Delta t\big)^2\Big) = 2(\Delta t)^2$$

Then combine this result and the result from Q4 ...

**Q6.** Power spectrum density (PSD) of the Ornstein-Uhlenbeck process.

Show that

$$F\big[e^{-\beta|t|}\big] \equiv \int_{-\infty}^{+\infty} e^{-i2\pi\xi t} e^{-\beta|t|} dt = \frac{2\beta}{\beta^2 + (2\pi\xi)^2}$$

**Q7.** (Optional) The Paley-Wiener representation of Wiener process.

We write the Wiener process as the sum of a shift and a Brownian bridge

$$W(t) = \frac{t}{T}W(T) + \Big(W(t) - \frac{t}{T}W(T)\Big)$$

(a) Show that $W(T)$ and $\big(W(t) - \frac{t}{T}W(T)\big)$ are independent for $0 \leqslant t \leqslant T$.

(b) We expand $\big(W(t) - \frac{t}{T}W(T)\big)$ in a Fourier sine series.

$$\Big(W(t) - \frac{t}{T}W(T)\Big) = \sum_{n=1}^{+\infty} F_n \sin(n\pi\frac{t}{T}), \quad F_n = \frac{2}{T}\int_0^T \sin(n\pi\frac{t}{T})\Big(W(t) - \frac{t}{T}W(T)\Big)dt, \;\; n > 0$$

Calculate $\text{Cov}(F_n, F_k)$ and show that $F_n$ and $F_k$ are independent Gaussians for $n \neq k$.

Hint: Both $F_n$ and $F_k$ are linear combinations of $\{dW(t)\}$. It follows that $F_n$ and $F_k$ are jointly Gaussian. To show independence, we only need $\text{Cov}(F_n, F_k) = 0$ for $n \neq k$.

**Q8.** (Optional) The Paley-Wiener representation of Wiener process (continued).

Decomposing $W(t)$ and expanding in a Fourier sine series gives

$$\begin{aligned} W(t) &= \frac{t}{T}W(T) + \Big(W(t) - \frac{t}{T}W(T)\Big) \\ &= \frac{t}{T}W(T) + \sum_{n=1}^{+\infty} F_n \sin(n\pi\frac{t}{T}), \quad \text{Cov}(F_n, F_k) = \frac{2T}{(n\pi)^2}\delta_{n,k} \end{aligned}$$

(a) Let $Z_0 = \frac{1}{\sqrt{T}}W(T)$ and $Z_n = \frac{n\pi}{\sqrt{2T}}F_n$, $n > 0$. Use the results of Q7 to show

$\{Z_0, Z_1, Z_2, \ldots, Z_n, \ldots\}$ are independent standard Gaussians.

(b) Use the result of (a) to write out the Paley-Wiener representation of $W(t)$.

$$\boxed{\begin{cases} W(t) = \dfrac{t}{\sqrt{T}}Z_0 + \sqrt{2T}\displaystyle\sum_{n=1}^{+\infty} \frac{Z_n}{n\pi}\sin(n\pi\frac{t}{T}) \\ \{Z_0, Z_1, Z_2, \ldots, Z_n, \ldots\} \overset{\text{i.i.d.}}{\sim} N(0,1) \end{cases}}$$