AM216 Stochastic Differential Equations

Lecture 04 Supplemental

Computational demonstration of Ito's lemma

The exact and two approximations of $f(t_f, W(t_f))$

Let $f(t, w)$ be a smooth function of (t, w). Replacing w with $W(t)$ yields $f(t, W(t))$, a non-smooth random function of t. We select a numerical grid on t.

$$\Delta t = \frac{t_f}{N}, \qquad t_j = j\Delta t, \qquad W_j = W(t_j), \qquad \Delta W_j = W_{j+1} - W_j$$

Consider $\Delta f_j \equiv f(t_{j+1}, W(t_{j+1})) - f(t_j, W(t_j))$, the increment of $f(t, W(t))$. We study two approximations of Δf_j.

- Approximate increment based on Taylor expansion

$$\left(\Delta f_{\text{Taylor}}\right)_j = (f_t)_j \Delta t + (f_w)_j \Delta W_j + \frac{1}{2}(f_{ww})_j (\Delta W_j)^2$$

- Approximate increment based on Ito's lemma

$$\left(\Delta f_{\text{Ito}}\right)_j = (f_t + \frac{1}{2} f_{ww})_j \Delta t + (f_w)_j \Delta W_j$$

These two approximations of Δf_j correspond to two stochastic differential equations.

Given the initial value $f(0, 0)$, using the two approximations of Δf_j, we calculate the corresponding two approximations of $f(t_f, W(t_f))$.

- Approximation based on Taylor expansion

$$\left(f_{\text{Taylor}}\right)\Big|_{t_f} = f(0,0) + \sum_{j=0}^{N-1} \left(\Delta f_{\text{Taylor}}\right)_j$$

- Approximation based on Ito's lemma

$$\left(f_{\text{Ito}}\right)\Big|_{t_f} = f(0,0) + \sum_{j=0}^{N-1} \left(\Delta f_{\text{Ito}}\right)_j$$

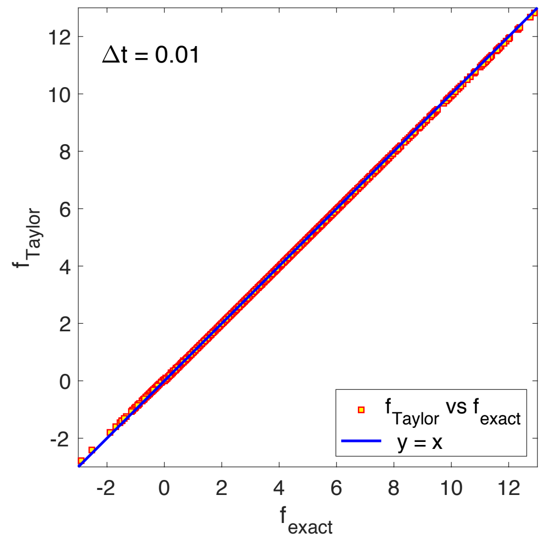
Taylor's theorem implies $\displaystyle \lim_{dt \to 0} \left(f_{\text{Taylor}}\right)\Big|_{t_f} = \left(f_{\text{exact}}\right)\Big|_{t_f}$.

Ito's lemma tells us that $\displaystyle \lim_{dt \to 0} \left(f_{\text{Ito}}\right)\Big|_{t_f} = \left(f_{\text{exact}}\right)\Big|_{t_f}$

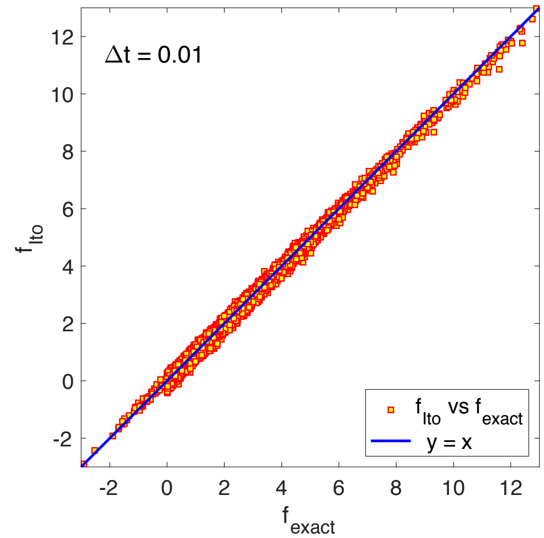
Below we use numerical simulations to confirm these two assertions.

Numerical example:

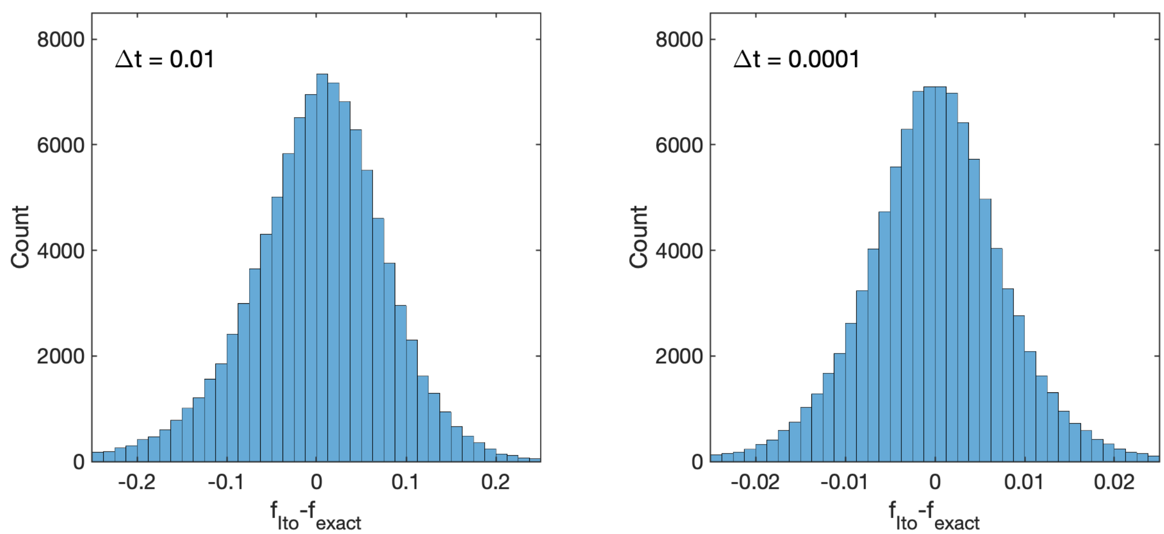
We test Ito's lemma on $f(t,w)=\dfrac{1}{2}w^2+\dfrac{t}{6}w^3$. In simulations, we use $t_f = 1$ and $\Delta t = 0.01$, and we generate 100,000 independent samples of $(f_{\text{exact}}, f_{\text{Taylor}}, f_{\text{Ito}})$.



f_{Taylor} vs f_{exact}



f_{Ito} vs f_{exact}



Histogram of $(f_{\text{Ito}} - f_{\text{exact}})$ for $\Delta t = 0.01$ (left) and $\Delta t = 0.0001$ (right).