AM 216 - Stochastic Differential Equations: Assignment 2

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Problem 1: MLE and the Normal Distribution

Find $\mu_{(MLE)}$ and $\sigma_{(MLE)}$.

Proof.

$$\begin{aligned}
\ell(\mu, \sigma^2 | \boldsymbol{X}) &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(X_j - \mu)^2 \\
\frac{\partial \ell}{\partial \mu} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu) \\
\mu_{(MLE)} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
\frac{\partial \ell}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i - \mu)^2 \\
&= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j)^2 \\
\sigma_{(MLE)} &= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)^2}
\end{aligned}$$

□

Problem 2: MLE Variance and unbiased sample variance

Show that the MLE of variance is biased.

Proof.

$$\begin{aligned}
E\left(\sum_{i=1}^{n}\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right)^2\right) &= \sum_{i=1}^{n}E\left(\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right)^2\right)\\
&= \sum_{i=1}^{n}\text{Var}\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right)+\left(E\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right)\right)^2\\
E\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right) &= E(X_i)+\frac{1}{n}\sum_{j=1}^{n}E(X_j)\\
&= \mu+\frac{1}{n}n\mu\\
&= 0\\
\text{Var}\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right) &= \text{Var}(X_i)+\text{Var}(\mu)-2\text{Cov}(X_i,\mu)\\
&= \sigma^2+\frac{1}{n}\sigma^2-\frac{2}{n}\sigma^2\\
&= \left(1-\frac{1}{n}\right)\sigma^2\\
E\left(\sum_{i=1}^{n}\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right)^2\right) &= \sum_{i=1}^{n}\left(1-\frac{1}{n}\right)\sigma^2\\
&= (n-1)\sigma^2\\
&= (n-1)\text{Var}(X)
\end{aligned}$$

□

Problem 3: Central Limit Theorem

i)

$$\begin{aligned}
\phi_X(\xi) &= \int_{-\infty}^{\infty}e^{i\xi x}f(x)dx\\
&= pe^{i\xi}+q
\end{aligned}$$

ii)

$$\begin{aligned}
\phi_N(\xi) &= \int_{-\infty}^{\infty}e^{i\xi x}f(x)dx\\
&= \sum_{k=0}^{n}\begin{pmatrix} n\\ k\end{pmatrix}p^k(1-p)^{n-k}e^{i\xi k}
\end{aligned}$$

iii)

$$\begin{aligned}
\phi_Y(\xi) &= \phi_N(\xi/\sqrt{n})\exp(i\xi\sqrt{n}p)\\
&= \sum_{k=0}^{n}\begin{pmatrix} n\\ k+np\end{pmatrix}p^k(1-p)^{n-k}e^{i\xi\left(\frac{k}{\sqrt{n}}+\sqrt{n}p\right)}
\end{aligned}$$

iv)

$$\lim_{n\to\infty} \phi_Y(\xi) = \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k+np \end{pmatrix} p^k (1-p)^{n-k} e^{i\xi\left(\frac{k}{\sqrt{n}}+\sqrt{n}p\right)}$$

My proof ends here, I have not figured out how to pull an extra ξ out of the PMF function and obtain a $e^{\xi^2(\cdots)}$. As such, I will resubmit this assignment once I have solved this particular problem.

Problem 4: Sample Variance v.s. MLE variance

See Figure 1 Generally we find that minimizing the fluctuations (using the MLE variance) decreases the

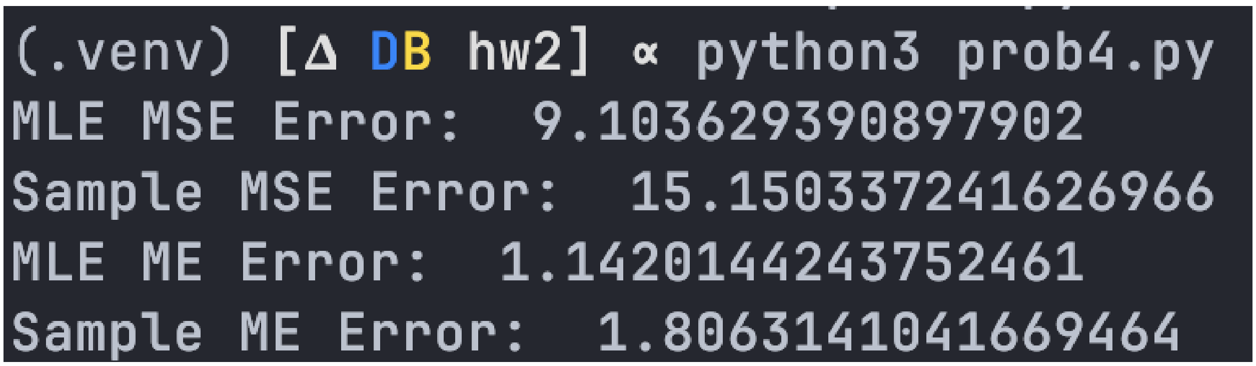

Figure 1: Results from problem 4

error in estimating the distributions variance as opposed to minimizing the bias (using the sample variance).

Problem 5: Deriving the CF of a multivariate gaussian

Proof.

$$\begin{aligned} i\xi^T x - \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) &= -\frac{1}{2}\left(x-\mu-i\Sigma\xi\right)^T\Sigma^{-1}\left(x-\mu-i\Sigma\xi\right) + \left(i\xi^T\mu - \frac{1}{2}\xi^T\Sigma\xi\right) \\ i\xi^T x &= -\frac{1}{2}\left[-i(x-\mu)^T\Sigma^{-1}\Sigma\xi + -i\xi^T\Sigma^T\Sigma^{-1}(x-\mu) - \xi^T\Sigma^T\Sigma^{-1}\Sigma\xi\right] + \left(i\xi^T\mu - \frac{1}{2}\xi^T\Sigma\xi\right) \\ &= \frac{i}{2}(x-\mu)^T\xi + \frac{i}{2}\xi^T(x-\mu) + i\xi^T\mu \\ &= i\xi^T x \end{aligned}$$

where here the symmetry of the inner product operator satisfies the movement between the third and fourth lines and its distributive properly allows the movement between the 1st and 2nd lines. □

Problem 6: Deriving the Conditional Gaussian Distribution

Proof.

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}, \qquad \Sigma^{-1} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$
$$\Sigma^{-1}\Sigma = I$$
$$\begin{bmatrix} A\Sigma_{XX} + B\Sigma_{YX} & A\Sigma_{XY} + B\Sigma_{YY} \\ B^T\Sigma_{XX} + C\Sigma_{YX} & B^T\Sigma_{XY} + C\Sigma_{YY} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$
$$A\Sigma_{XY} + B\Sigma_{YY} = 0$$
$$-A\Sigma_{XY} = B\Sigma_{YY}$$
$$-\Sigma_{XY}\Sigma_{YY}^{-1} = A^{-1}B$$
$$A\Sigma_{XX} + B\Sigma_{YX} = 0$$
$$A\left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right) = I$$
$$A^{-1} = \left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right)$$

□

Problem 7: Random Monty Hall's Game

Proof. In order to make this decision we must look at the conditional probabiliites involved with the random variant of the Monty Hall's game. In this problem, we study the conditional probability $Pr(X|Y = \text{empty})$ where X is the box we first chose and Y is the box that we first open. We have using Bayes Theorem,

$$\begin{aligned} Pr(X|Y) &= \frac{Pr(Y|X)Pr(X)}{Pr(Y)} \\ &= \frac{(1)(1/3)}{2/3} \\ &= \frac{1}{2} \end{aligned}$$

As we can see from this, since we are not guaranteed that the host reveals an empty box, we no longer have an advantage from switching. □

Problem 8: Finite Difference Equation

i) See Fig 2

ii) See Fig 3

iii) See Fig 4

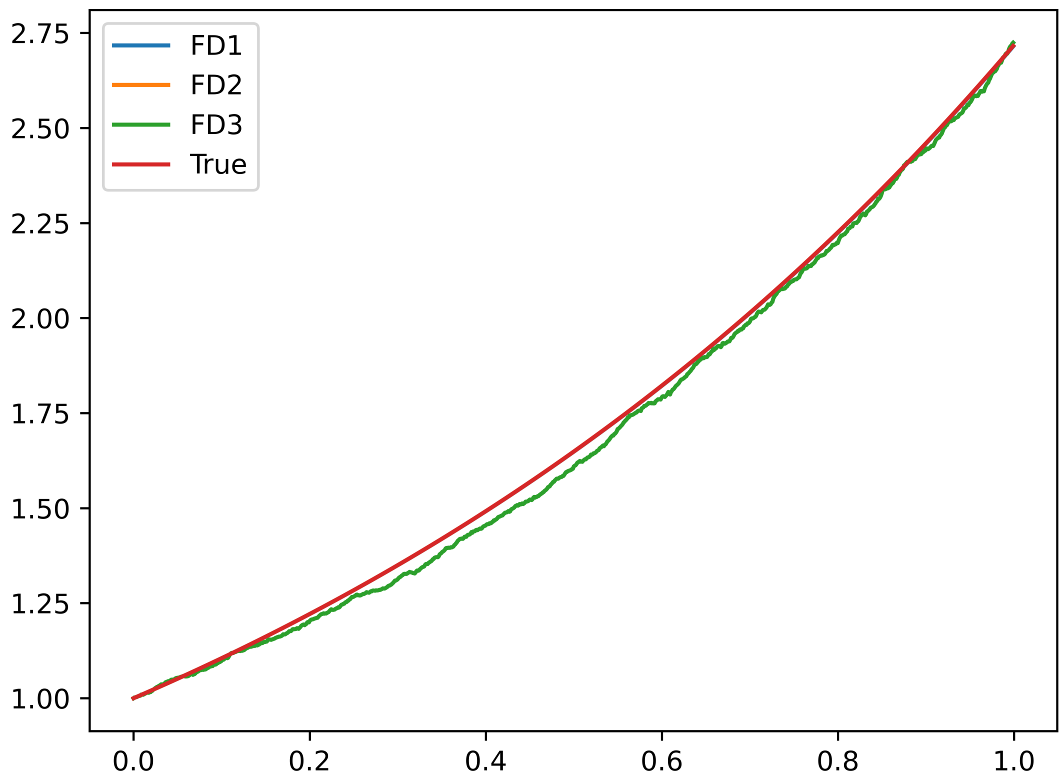

Figure 2: P8 Part a

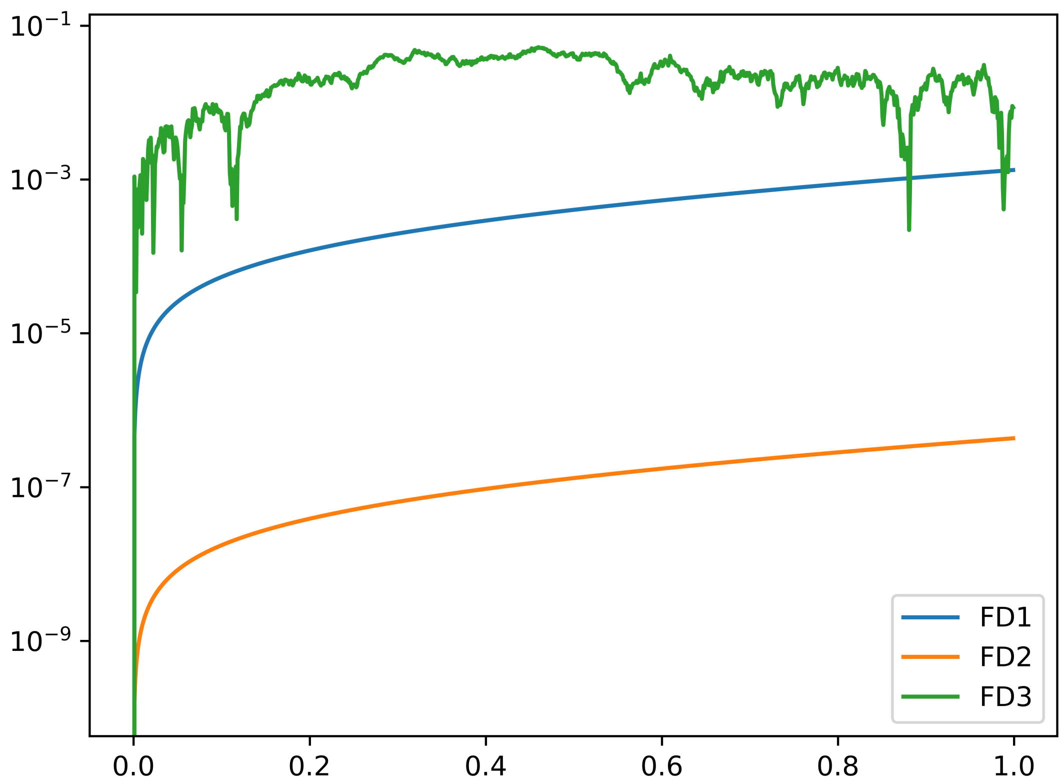

Figure 3: P8 Part b

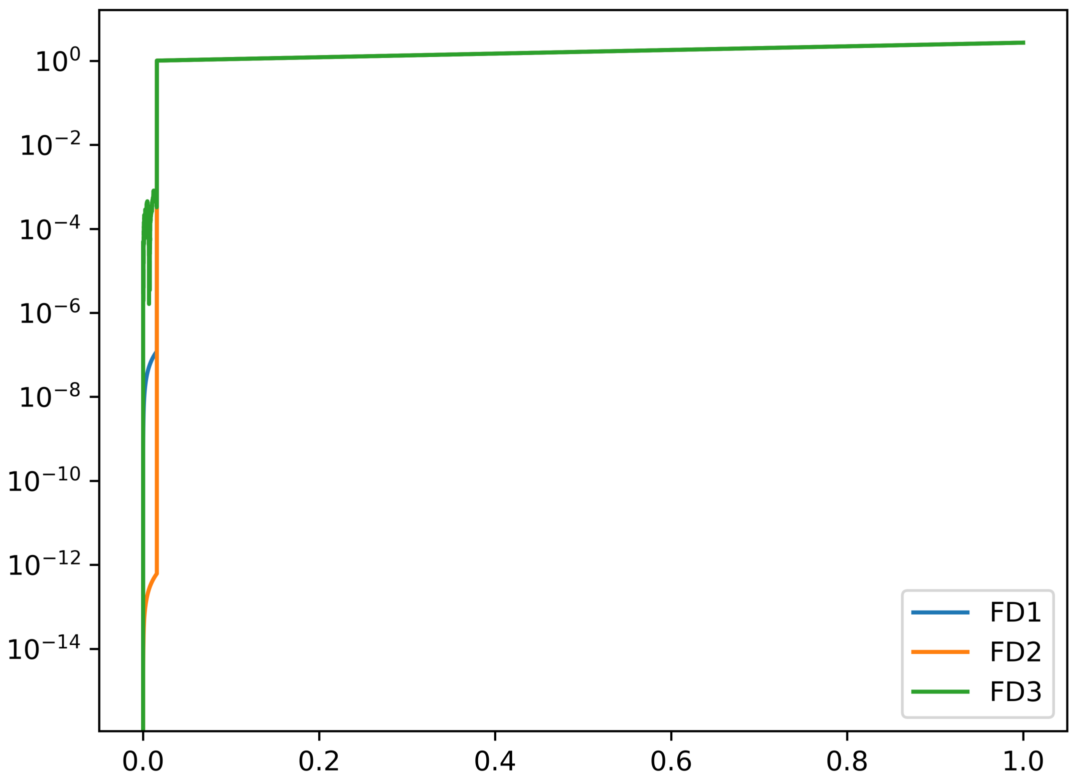

Figure 4: P8 Part c