

AM 216 - Stochastic Differential Equations: Assignment 8

Dante Buhl

November 29, 2025

Problem 1: Lambda-rule for stochastic derivatives

Proof. We begin this section by first taking $X(t) = H(t, W(t))$.

$$\begin{aligned} dX &= \left(X_t + \frac{1}{2} X_{WW} \right) dt + X_W dW \\ &= \left(2tX + \frac{1}{2} X \right) dt + X dW \end{aligned}$$

We notice that if this were true we must have several conditions which also follow with our original assumption. First and foremost,

$$\begin{aligned} X = X_W &\implies X_W = X_{WW} \\ X &\propto e^{W(t)} \\ X_t = 2tX &\implies X \propto e^{t^2} \\ X &= X_0 e^{t^2 + W(t)} \end{aligned}$$

This all follows if we take the λ -chain rule and the original SDE to be equivalent. What remains is to resolve the initial condition and this simply implies that $X_0 = 2$.

$$H(t, W(t)) = X(t) = e^{t^2 + W(t)}$$

□

Problem 2: Time Reversal of an SDE

Proof.

$$\begin{aligned} \rho_{(X(t)|X(t+dt)=x_1)}(x_0) &\propto \exp\left(\frac{-1}{2\sigma^2 dt} [(x_0 - x_1)^2 + 2c_1 dt(x_0 - x_1) + c_2 dt(x_0 - x_1)^2 + \dots]\right) \\ \hat{\sigma}^2 &= \frac{\sigma^2 dt}{1 + c_2 dt}, \quad \beta = \frac{c_1 dt}{1 + c_2 dt} \\ \rho_{(X(t)|X(t+dt)=x_1)}(x_0) &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2} [\Delta_x^2 + 2\beta\Delta_x + \dots]\right) \\ &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2} [\Delta_x^2 + 2\beta\Delta_x + \beta^2]\right) \\ &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2} [\Delta_x + \beta]^2\right) \\ &\propto \exp\left(-\frac{1 + c_2 dt}{2\sigma^2 dt} \left[x_0 - x_1 + \frac{c_1 dt}{1 + c_2 dt}\right]^2\right) \\ (X(t)|X(t+dt)=x_1) &\sim N\left(\frac{-c_1 dt}{1 + c_2 dt}, \frac{\sigma^2 dt}{1 + c_2 dt}\right) \end{aligned}$$

□

Problem 3: Feynman-Kac Formula

i) Write out the FVP

$$0 = u_t + \frac{1}{2}u_{xx} + xu$$

$$u(x, T, T) = 1$$

ii) Verify the given solution

$$u(x, t, T) = \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right)$$

$$u_t = -(x + (T-t)^2/2) \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right)$$

$$u_{xx} = (T-t)^2 \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right)$$

$$0 = \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right) \left[-\left(x + \frac{(T-t)^2}{2}\right) + \frac{(T-t)^2}{2} + x \right]$$

$$= 0$$

$$u(x, T, T) = \exp(0) = 1$$

Thus this solution satisfies the FVP.

iii) This specific problem is easy enough to directly integrate so I will do so.

Proof. We begin by expanding $X(s)$ (Note that, $\int_t^T \hat{W}(s)ds \sim N\left(0, \frac{(T-t)^3}{3}\right)$)

$$\begin{aligned} u(x, t, T) &= E\left[\exp\left(\int_t^T x + \hat{W}(s)ds\right) \middle| \hat{W}(t) = 0\right] \\ &= E\left[\exp((T-t)x) \exp\left(\int_t^T \hat{W}(s)ds\right) \middle| \hat{W}(t) = 0\right] \\ &= C \exp((T-t)x) \int_{-\infty}^{\infty} e^x e^{-x^2/2\sigma^2} dx \\ &= C \exp((T-t)x) \int_{-\infty}^{\infty} e^{-(x^2-2\sigma^2x)/2\sigma^2} dx \\ &= C \exp((T-t)x) e^{\sigma^2/2} \int_{-\infty}^{\infty} e^{-(x-\sigma^2)^2/2\sigma^2} dx \\ &= \exp((T-t)x) e^{\sigma^2/2} \\ &= \exp\left(x(T-t) + \frac{(T-t)^3}{6}\right) \end{aligned}$$

□

Problem 4: Integrating Factor to solve SDE

i)

$$\begin{aligned} dX - \frac{1}{1+t} X dt &= dW \\ d\left(\frac{X}{1+t}\right) &= \frac{dW(s)}{1+s} \\ X(t) &= (1+t) \left[c + \int_0^t \frac{dW(s)}{1+s} \right] \\ X(0) = 1 &\implies c = 1 \end{aligned}$$

ii)

$$\begin{aligned} E(X(t)) &= (1+t) \left(1 + E \left(\int_0^t \frac{dW(s)}{1+s} \right) \right) \\ &= 1 + t \\ \text{Var}(X(t)) &= (1+t)^2 \text{Var} \left(\int_0^t \frac{dW(s)}{1+s} \right) \\ &= (1+t)^2 \left(\int_t^T \frac{ds}{(1+s)^2} \right) \\ &= (1+t)^2 \left(-\frac{1}{1+s} \Big|_0^t \right) \\ &= (1+t)^2 \left(1 - \frac{1}{1+t} \right) \\ &= t + t^2 \end{aligned}$$

Problem 5: Simulating Feynman-Kac

i) We can verify this solution by simply plugging it into the original PDE and initial condition.

$$\begin{aligned} u(x, t, T) &= \exp \left(-\frac{(x-t)^2}{2} \right) \\ u_t &= (x-t)u \\ u_{xx} &= \frac{\partial}{\partial x}(-(x-t)u) \\ &= -u + (x-t)^2 u \\ 0 &= u \left[(x-t) - \frac{1}{2} + \frac{(x-t)^2}{2} - \phi \right] \\ &= 0 \end{aligned}$$

ii) The computed numerical value was $\hat{u}(0.7, 0, T) = 0.7833101185224285$ which is relatively close to the exact value of $u(0.7, 0, T) = 0.782704538242$.

Problem 6: Nonlinear Function of an RV

i)

$$\begin{aligned} dY &= \ln [X(t) + \sigma X(t)dW(t)] - \ln[X(t)] \\ &= \ln[1 + \sigma dW(t)] \end{aligned}$$

ii)

$$\begin{aligned}
E(X(t+dt)) &= E(X(t) + dX) \\
&= E(X(t)) + E(dX) \\
&= E(X(t)) + \sigma E(X(t))E(dW(t)) \\
E(X(t+dt)) - E(X(t)) &= \sigma E(X(t))E(dW(t)) \\
d(E(X(t))) &= \sigma E(X(t))E(dW(t)) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
E(Y(t+dt)) &= E(Y(t) + dY) \\
&= E(Y(t)) + E(\ln(1 + \sigma dW(t))) \\
dE(Y(t)) &= E(\ln(1 + \sigma dW(t))) \\
&\approx E\left(\ln(1) + \sigma dW(t) - \frac{\sigma^2}{2} dW^2(t) + h.o.t.\right) \\
&\approx \sigma E(dW(t)) - \frac{\sigma^2}{2} E(dW^2(t)) \\
&\approx -\frac{\sigma^2}{2}
\end{aligned}$$

iii)

$$\frac{dE_X}{dt} = 0$$

$$E_X = c$$

$$E_X(0) = 1$$

$$E_X(t) = 1$$

$$\frac{dE_Y}{dt} = -\frac{\sigma^2}{2}$$

$$E_Y(t) = c - \frac{\sigma^2}{2}t$$

$$E_Y(0) = 0$$

$$E_Y(t) = -\frac{\sigma^2}{2}t$$