AM216 Stochastic Differential Equations Assignment 5

Q1. Time inversion property of Wiener process.

Let $\{W(t)\}$ be a Wiener process. We define a new process based on $W(t)$.

$$B(t) = \begin{cases} tW(1/t) & t > 0 \\ 0 & t = 0 \end{cases}$$

Show that $\{B(t)\} \sim \{W(t)\}$. Specifically for any $0 \leqslant t_1 \leqslant t_2 \cdots \leqslant t_n$, show that

$$\Big(B(t_1), B(t_2), \ldots, B(t_{n-1})\Big) \sim \Big(W(t_1), W(t_2), \ldots, W(t_{n-1})\Big)$$

Hint: Note that $(W(1/t_1), W(1/t_2), \ldots, W(1/t_n))$ is Gaussian. It follows that on the LHS above, $(B(t_1), B(t_2), \ldots, B(t_n))$ is Gaussian. To conclude that these two Gaussians have the same distribution, we only need to show

$$\text{Cov}\left(B(t_i), B(t_j)\right) = \min(t_i, t_j) = \text{Cov}\left(W(t_i), W(t_j)\right)$$

Q2. Distribution of $\displaystyle\int f(t)W(t)dt$

i) Derive $\displaystyle\int_0^T W(t)dt \sim N(0, \frac{T^3}{3})$.

Hint: We use $W(t) = \displaystyle\int_0^t dW(s)$ to derive

$$\int_0^T W(t)dt = \int_0^T \int_0^t dW(s)dt = \int_0^T \int_s^T dt dW(s) = \int_0^T (T-s)dW(s)$$

which is a sum of independent Gaussians.

ii) Using the same method to calculate $\displaystyle\int tW(t)dt$.

Q3. Reflection principle, maximum of the Wiener process in $[0, T]$.

Let $M_T \equiv \max\limits_{t\in[0,T]} W(t)$, the maximum of $W(t)$ in $[0, T]$. M_T is a random variable.

i) For $a > 0$, derive the reflection principal:

$$\Pr(M_T \geqslant a) = 2\Pr(W(T) \geqslant a)$$

Remark: The reflection principal tells us that out of all paths crossing a given level a before time T, half end above level a at time T; the other half below level a at T.

ii) Use i) to derive the PDF of M_T.

Hint: Let $\tau_a(\omega)$ be the earliest time of $W(t,\omega)$ hitting level a, truncated at $T+1$.

$$\tau_a(\omega) = \begin{cases} \inf\{t \geqslant 0 : W(t,\omega) \geqslant a\}, & \text{if } \max\limits_{t\in[0,T+1]} W(t) \geqslant a \\ (T+1), & \text{otherwise} \end{cases}$$

The truncation is to make the random variable $\tau_a(\omega)$ well defined and bounded. We use the law of total probability to write out $\Pr(W(T) \geqslant a)$.

$$\Pr(W(T) \geqslant a) = \int_0^T \Pr\left(W(T) \geqslant a \middle| \tau_a = \tau\right) \rho_{\tau_a}(\tau) d\tau$$

$\tau_a = \tau$ means $W(t)$ hits level a for the first time at $t = \tau$.

$$= \int_0^T \Pr\left(W(T) \geqslant a \middle| W(\tau) = a, W(t) < a \text{ for } 0 \leqslant t \leqslant \tau\right) \rho_{\tau_a}(\tau) d\tau$$

Recall that with $W(\tau) = a$, any additional condition prior to time τ drops out.

$$= \int_0^T \Pr\left(W(T) - W(\tau) \geqslant 0 \middle| W(\tau) = a\right) \rho_{\tau_a}(\tau) d\tau$$

Increment $W(T) - W(\tau)$ is Gaussian and independent of $W(\tau) = a$

$$= \int_0^T \Pr(W(T) - W(\tau) \geqslant 0)\, \rho_{\tau_a}(\tau) d\tau = \cdots$$

Q4. Use the λ-chain rule to calculate the stochastic integrals below. Express the results in terms of $W(T)$ and integral(s) of the form $\int_0^T g(s, W(s)) ds$.

i) $\displaystyle\int_0^T \cos(W(t)) dW(t)$ (with Stratonovich interpretation).

ii) $\displaystyle\int_0^T e^t \sin(W(t)) dW(t)$ (with Ito interpretation).

Q5. Let $X(t)$ be the price of a company's stock at time t. You are considering betting on the performance of $X(t)$. Suppose the evolution of $X(t)$ is governed by

$$dX(t) = dW(t)$$

Let $w(z,s)$ be the probability of X reaching x_c at time $t_1 \in [s,T]$ without first hitting 0 given X starting at z at time s. Mathematically, $w(z,s)$ is

$$w(z,s) = \Pr\left(\begin{matrix} X(t_1) \geqslant x_c \text{ at some } t_1 \in [s,T] \\ X(t) > 0 \text{ for all } t \in (s,t_1) \end{matrix} \middle| X(s) = z \right)$$

i) The law of total probability gives $w(z,s) = E\big(w(z+dX, s+ds)\big) + o(ds)$. Expand in dX and ds. Use $E\big((dX)^k \big| X(s) = z\big)$ to derive the governing PDE of $w(z,s)$.

ii) Write out the final condition at $s = T$ and the boundary conditions at $z = 0$ and $z = x_c$. Write out the final boundary value problem (FBVP) for $w(z,s)$.

iii) Let $\tau \equiv T - s$. We define $u(z,\tau) \equiv w(z, T-\tau)$. Write out the initial boundary value problem (IBVP) for $u(z,\tau)$.

Remark: Probability $w(z,s)$ in this problem is related to American style options.

Q6. Let $X(t)$ be the price of a company's stock at time t. You are considering betting on the performance of $X(t)$. Suppose the evolution of $X(t)$ is governed by

$$dX(t) = b(X,t)dt + \sqrt{a(X,t)}dW(t) \quad \text{(Ito interpretation)}$$

Let $w(z,s)$ be the probability of X reaching at least x_c at time T without hitting 0 in time interval $[s,T]$ given X starting at z at time s. Mathematically, $w(z,s)$ is

$$w(z,s) = \Pr\Big(X(T) \geqslant x_c \text{ and } X(t) > 0 \text{ for all } t \in (s,T) \Big| X(s) = z\Big)$$

i) The law of total probability gives $w(z,s) = E\big(w(z+dX, s+ds)\big) + o(ds)$. Expand in dX and ds. Use $E\big((dX)^k \big| X(s) = z\big)$ to derive the governing PDE of $w(z,s)$. Observe that the particular specification of the probability does not affect the governing PDE.

ii) Write out the final condition at $s = T$ and the boundary condition at $z = 0$.

iii) Can we impose a boundary condition at $z = x_c$?

Remark: Probability $w(z,s)$ in this problem is related to European style options.