AM216 Stochastic Differential Equations Final Exam Fall 2025

Q1. Let $Y \equiv \int_0^1 \cos(\pi s) dW(s)$, which is a random variable.

i) Find $E(Y)$ and $\mathrm{Var}(Y)$.

ii) Write out $\rho_Y(y)$, the probability density of Y.

Q2. With initial condition $X(0) = 3$, solve the SDE below.

$$dX(t) = X(t)dt + 2X(t)dW(t) \quad \text{(Ito interpretation)}$$

Express $X(t)$ in terms of $W(t)$.

Q3. Let $I = \int_0^T W^2(t)dt$. Calculate $E\big(I \big| W(T) = w_1\big)$.

Hint: Consider the Brownian bridge with $W(T) = w_1$. Use it to find $E\big(W^2(t) \big| W(T) = w_1\big)$.

Q4. Let $\{W(t)\}$ be a Wiener process. Define a new process based on $W(t)$.

$$B(t) \equiv \begin{cases} t^{3/2} W(1/t^2), & t > 0 \\ 0, & t = 0 \end{cases}$$

i) Show that at any t, we have $E\big(B(t)\big) = 0$ and $\mathrm{Var}\,\big(B(t)\big) = t$.

ii) Show that, as a stochastic process, $\{B(t)\} \sim \{W(t)\}$ is NOT true. Hint: Think about the definition of $\{B(t)\} \sim \{W(t)\}$.

Q5. Consider the SDE below with $x = 1$ reflecting and $x = \varepsilon$ absorbing.

$$dX(t) = 3X^{\frac{1}{3}}(t)dt + \sqrt{9X^{\frac{4}{3}}(t)}\, dW(t) \quad \text{(Ito interpretation)}$$

Let $T(x)$ be the average time of $\{X(t)\}$ exiting the region $(\varepsilon, 1)$ given $X(0) = x$.

i) Write out the BVP for $T(x)$.

ii) Solve for $T(x)$ in the BVP.