Formulas in Lecture 5

Advantage of working with $\dfrac{dW(t)}{\sqrt{dt}}$

$$\frac{dW(t)}{\sqrt{dt}} \sim N(0,1)$$ is dimensionless and independent of dt and t.

Non-dimensionalization

Time scale: $[t_0]$ = time

Money scale: $[\sqrt{\sigma^2 t_0}] = \$$

Non-dimensional time: $t_{\text{ND}} = \dfrac{t}{t_0}$

Non-dimensional money: $X_{\text{ND}} = \dfrac{X}{\sqrt{\sigma^2 t_0}}$

Delta function:

$$\delta(x) = \lim_{d\to 0} \Pi_d(x), \qquad \Pi_d(x) = \begin{cases} \dfrac{1}{d}, & \text{for } x \in \left(\dfrac{-d}{2}, \dfrac{d}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

$$\delta(x) = \lim_{\sigma\to 0} \rho_{N(0,\sigma^2)}(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

The true meaning of the limit:

$$\int \delta(x) g(x) dx \xrightarrow{\text{Defined as}} \lim_{d\to 0} \int \Pi_d(x) g(x) dx$$

$$\int \delta(x) g(x) dx \xrightarrow{\text{Defined as}} \lim_{\sigma\to 0} \int \rho_{N(0,\sigma^2)}(x) g(x) dx$$

Fourier transform (FT):

Forward transform:

$$\underbrace{\hat{y}(\xi)}_{\text{Notation}} \equiv \underbrace{F[y(t)]}_{\text{Operator notation}} \equiv \int_{-\infty}^{+\infty} \exp(-i2\pi\xi t) y(t) dt$$

Inverse transform:

$$y(t) = F^{-1}[\hat{y}(\xi)] \equiv \int_{-\infty}^{+\infty} \exp(i2\pi\xi t)\hat{y}(\xi)d\xi$$

Properties of Fourier transform:

1) Fourier transform of a normal PDF

$$F\left[\rho_{N(0,\sigma^2)}(t)\right] = F\left[\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-t^2}{2\sigma^2}\right)\right] = \exp\left(-2\pi^2\sigma^2\xi^2\right)$$

2) Fourier transform of the delta function

$F[\delta(t)] = 1$

3) Fourier transform of $y(t) \equiv 1$

$F[1] = \delta(\xi)$

4) Parseval's theorem

$$\int \left|y(t)\right|^2 dt = \int \left|\hat{y}(\xi)\right|^2 d\xi$$

Formulas in Lecture 6

Energy spectrum density (ESD)

$$\text{ESD} \equiv E\left(\left|\hat{y}(\xi)\right|^2\right) = E\left(\left|\int_{-\infty}^{+\infty} \exp(-i2\pi\xi t)y(t)dt\right|^2\right)$$

Power spectrum density (PSD)

$$\frac{1}{2T}\int_{-T}^{T}\left|y(t)\right|^2 dt = \int_{-\infty}^{\infty}\frac{1}{2T}\left|\int_{-T}^{T}\exp(-i2\pi\xi t)y(t)dt\right|^2 d\xi \qquad \text{(Parseval's theorem)}$$

$$s(\xi) \equiv \text{PSD} \equiv \lim_{T\to\infty}\frac{1}{2T}E\left(\left|\int_{-T}^{T}\exp(-i2\pi\xi t)y(t)dt\right|^2\right)$$

Auto-correlation function (ACF) of a stationary stochastic process

$$R(\tau) \equiv E\left(y(s+\tau)\overline{y(s)}\right) = E\left(y(\tau)\overline{y(0)}\right)$$

Power spectrum density (PSD) of a stationary process

$$s(\xi) = \lim_{T\to\infty}\int_{-2T}^{2T}\exp(-i2\pi\xi\tau)R(\tau)\left(1-\frac{\tau}{2T}\right)d\tau$$

Wiener-Khinchin theorem:

For a stationary process, we have

$$s(\xi)=\int_{-\infty}^{+\infty}\exp(-i2\pi\xi t)R(t)dt$$

Definition of white noise:

A stationary process $y(t)$ is a white noise if $s(\xi) = const.$

$R(t)$ and $s(\xi)$ for $Z(t) \equiv dW/dt$

$R(t) = \delta(t)$, $s(\xi) = 1$

Bayes Theorem for events:

$$\Pr(A|B)=\frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

Bayes theorem for densities

$$\rho(X=x|Y=y) = \frac{\rho(Y=y|X=x)\cdot\rho(X=x)}{\rho(Y=y)}$$

A convenient form:

$$\rho(X=x|Y=y) \propto \rho(Y=y|X=x)\cdot\rho(X=x)$$

Conditional distribution of $(W(t_1) \mid W(t_1+t_2) = y)$

$$\rho\Big(W(t_1)=x\Big|W(t_1+t_2)=y\Big) \sim N\left(\frac{t_1 y}{t_1+t_2}, \frac{t_1 t_2}{t_1+t_2}\right)$$

The general case:

$$\rho\Big(W(a+t_1)=x\Big|W(a)=y_a \text{ and } W(a+t_1+t_2)=y_b\Big) \sim N\left(\frac{t_1 y_b+t_2 y_a}{t_1+t_2}, \frac{t_1 t_2}{t_1+t_2}\right)$$

A special case:

$$\rho\left(W(a+\frac{h}{2})=x\Big|W(a)=y_a \text{ and } W(a+h)=y_b\right) \sim N\left(\frac{y_a+y_b}{2}, \frac{h}{4}\right)$$