AM 216 - Stochastic Differential Equations: Assignment

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November 5, 2025

Problem 1: Time inversion property of the Wiener process

Problem 2: Integrating the Wiener process

i) We have $W(t) = \int_0^t dW(s)$

Proof. We begin,

$$\begin{aligned}
\int_0^T \int_0^s dW(s)dt &= \int_0^T \int_s^T dtdW(s) \\
&= \int_0^T (T-s)dW(s) \\
&\sim N\left(0, \int_0^T (T-s)^2 dt\right) \\
&\sim N\left(0, T^3 - T^3 + \frac{T^3}{3}\right) \\
&\sim N\left(0, \frac{T^3}{3}\right)
\end{aligned}$$

□

ii) We repeat but with a new integrand

Proof.

$$\begin{aligned}
\int_0^T \int_0^s tdW(s)dt &= \int_0^T \int_s^T tdtdW(s) \\
&= \frac{1}{2}\int_0^T (T^2 - s^2)dW(s) \\
&\sim N\left(0, \frac{1}{4}\int_0^T (T^2-s^2)^2 dt\right) \\
&\sim N\left(0, \frac{1}{4}\left(T^5 - \frac{2}{3}T^5 + \frac{1}{5}T^5\right)\right) \\
&\sim N\left(0, \frac{2T^5}{15}\right)
\end{aligned}$$

□

Problem 3: Reflection principle of the Wiener process

i) Show that $\Pr(M_T \geq a) = 2\Pr(W(T) \geq a)$.

Proof. The core point in this proof is showing that the conditional probability for $\Pr(W(T) \geq a | W(\tau) = a, \tau \in [0,T]) = 0.5$. This follows because the rest of the series $t \in [\tau, T]$ is distributed symmatrically about $W(\tau) = a$. Notice that if $W(\tau) = a$ and we have $M_T \geq a$. Thus we have by bayes theorem,

$$\begin{aligned}\Pr(W(T) \geq a | M(T) \geq a) = 0.5 &= \frac{\Pr(M(T) \geq a | W(T) \geq a)\Pr(W(T) \geq a)}{\Pr(M_T \geq a)} \\ \Pr(M_T \geq a) &= 2\Pr(W(T) \geq a)\end{aligned}$$

□

ii) Find the PDF of M_T.

Proof. We have that the two PDFs are related by $\rho_{M_T \geq a} = \rho_{W(T) \geq a}(x/2)$

$$\begin{aligned}\rho_{W(T)\geq a} &= \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{a}{\sqrt{2T}}\right) \\ \rho_{M_T \geq a} &= \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{a}{2\sqrt{2T}}\right)\end{aligned}$$

□

Problem 4: Lambda-chain rule for stochastic integrals

i) Stratonovich interpretation

Proof.

$$\begin{aligned}\int_0^T \cos(W(t))dW(t) &= \sin\left(W(t)\right)\Big|_0^T - \int_0^T 0dt \\ &= \sin(W(T))\end{aligned}$$

□

ii) Ito Interpretation

Proof.

$$\begin{aligned}\int_0^T e^t \sin(W(t))dW(t) &= -e^t\cos(W(t))\Big|_0^T - \int_0^T -e^t\cos(W(t)) + \frac{1}{2}e^t\cos(W(t))dt \\ &= -e^T\cos(W(T)) + 1 + \frac{1}{2}\int_0^T e^t\cos(W(t))dt\end{aligned}$$

□

Problem 5: SDE for American Stocks

Problem 6: SDE for European Stocks