AM 216 - Stochastic Differential Equations: Assignment 8

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Problem 1: Lambda-rule for stochastic derivatives

Proof. We begin this section by first taking $X(t) = H(t, W(t))$.

$$\begin{aligned} dX &= \left(X_t + \frac{1}{2}X_{WW}\right)dt + X_W dW \\ &= \left(2tX + \frac{1}{2}X\right)dt + XdW \end{aligned}$$

We notice that if this were true we must have several conditions with also follow with our original assumption. First and foremost,

$$\begin{aligned} X = X_W &\implies X_W = X_{WW} \\ X &\propto e^{W(t)} \\ X_t = 2tX &\implies X \propto e^{t^2} \\ X &= X_0 e^{t^2 + W(t)} \end{aligned}$$

This all follows if we take the λ-chain rule and the original SDE to be equivalent. What remains is to resolve the initial condition and this simply implies that $X_0 = 2$.

$$H(t, W(t)) = X(t) = e^{t^2 + W(t)}$$

□

Problem 2: Time Reversal of an SDE

Proof.

$$\begin{aligned} \rho_{(X(t)|X(t+dt)=x_1)}(x_0) &\propto \exp\left(\frac{-1}{2\sigma^2 dt}\left[(x_0 - x_1)^2 + 2c_1 dt(x_0 - x_1) + c_2 dt(x_0 - x_1)^2 + \ldots\right]\right) \\ \hat{\sigma}^2 = \frac{\sigma^2 dt}{1 + c_2 dt}&, \quad \beta = \frac{c_1 dt}{1 + c_2 dt} \\ \rho_{(X(t)|X(t+dt)=x_1)}(x_0) &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2}\left[\Delta_x^2 + 2\beta\Delta_x + \ldots\right]\right) \\ &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2}\left[\Delta_x^2 + 2\beta\Delta_x + \beta^2\right]\right) \\ &\propto \exp\left(\frac{-1}{2\hat{\sigma}^2}\left[\Delta_x + \beta\right]^2\right) \\ &\propto \exp\left(-\frac{1 + c_2 dt}{2\sigma^2 dt}\left[x_0 - x_1 + \frac{c_1 dt}{1 + c_2 dt}\right]^2\right) \\ (X(t)|X(t+dt) = x_1) &\sim N\left(\frac{-c_1 dt}{1 + c_2 dt}, \frac{\sigma^2 dt}{1 + c_2 dt}\right) \end{aligned}$$

$\square$

Problem 3: Feynman-Kac Formula

i) Write out the FVP

$$
\begin{aligned}
0 &= u_t + \frac{1}{2}u_{xx} + xu \\
u(x,T,T) &= 1
\end{aligned}
$$

ii) Verify the given solution

$$
\begin{aligned}
u(x,t,T) &= \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right) \\
u_t &= -\left(x + (T-t)^2/2\right)\exp\left(\frac{(T-t)^3}{6} + (T-t)x\right) \\
u_{xx} &= (T-t)^2\exp\left(\frac{(T-t)^3}{6} + (T-t)x\right) \\
0 &= \exp\left(\frac{(T-t)^3}{6} + (T-t)x\right)\left[-\left(x + \frac{(T-t)^2}{2}\right) + \frac{(T-t)^2}{2} + x\right] \\
&= 0 \\
u(x,T,T) &= \exp(0) = 1
\end{aligned}
$$

Thus this solution satisfies the FVP.

iii) This specific problem is easy enough to directly integrate so I will do so.

Proof. We begin by expanding $X(s)$ (Note that, $\int_t^T \hat{W}(s)ds \sim N\left(0, \frac{(T-t)^3}{3}\right)$)

$$
\begin{aligned}
u(x,t,T) &= E\left[\exp\left(\int_t^T x + \hat{W}(s)ds\right)\Big|\hat{W}(t) = 0\right] \\
&= E\left[\exp((T-t)x)\exp\left(\int_t^T \hat{W}(s)ds\right)\Big|\hat{W}(t) = 0\right] \\
&= C\exp((T-t)x)\int_{-\infty}^{\infty} e^x e^{-x^2/2\sigma^2}dx \\
&= C\exp((T-t)x)\int_{-\infty}^{\infty} e^{-\left(x^2-2\sigma^2x\right)/2\sigma^2}dx \\
&= C\exp((T-t)x)e^{\sigma^2/2}\int_{-\infty}^{\infty} e^{-\left(x-\sigma^2\right)^2/2\sigma^2}dx \\
&= \exp((T-t)x)e^{\sigma^2/2} \\
&= \exp\left(x(T-t) + \frac{(T-t)^3}{6}\right)
\end{aligned}
$$

$\square$

Problem 4: Integrating Factor to solve SDE

i)

$$\begin{aligned}
dX - \frac{1}{1+t} X dt &= dW \\
d\left(\frac{X}{1+t}\right) &= \frac{dW(s)}{1+s} \\
X(t) &= (1+t)\left[c + \int_0^t \frac{dW(s)}{1+s}\right] \\
X(0) = 1 &\implies c = 1
\end{aligned}$$

ii)

$$\begin{aligned}
E(X(t)) &= (1+t)\left(1 + E\left(\int_0^t \frac{dW(s)}{1+s}\right)\right) \\
&= 1+t \\
\text{Var}(X(t)) &= (1+t)^2 \text{Var}\left(\int_0^t \frac{dW(s)}{1+s}\right) \\
&= (1+t)^2 \left(\int_t^T \frac{ds}{(1+s)^2}\right) \\
&= (1+t)^2 \left(-\frac{1}{1+s}\Big|_0^t\right. \\
&= (1+t)^2 \left(1 - \frac{1}{1+t}\right) \\
&= t + t^2
\end{aligned}$$

Problem 5: Simulating Feynman-Kac

i) We can verify this solution by simply plugging it into the original PDE and initial condition.

$$\begin{aligned}
u(x,t,T) &= \exp\left(-\frac{(x-t)^2}{2}\right) \\
u_t &= (x-t)u \\
u_{xx} &= \frac{\partial}{\partial x}\left(-(x-t)u\right) \\
&= -u + (x-t)^2 u \\
0 &= u\left[(x-t) - \frac{1}{2} + \frac{(x-t)^2}{2} - \phi\right] \\
&= 0
\end{aligned}$$

ii) The computed numerical value was $\hat{u}(0.7, 0, T) = 0.7833101185224285$ which is relatively close to the exact value of $u(0.7, 0, T) = 0.782704538242$.

Problem 6: Nonlinear Function of an RV

i)

$$\begin{aligned}
dY &= \ln\left[X(t) + \sigma X(t) dW(t)\right] - \ln[X(t)] \\
&= \ln[1 + \sigma dW(t)]
\end{aligned}$$

ii)

$$\begin{aligned}
E(X(t+dt)) &= E(X(t)+dX) \\
&= E(X(t)) + E(dX) \\
&= E(X(t)) + \sigma E(X(t))E(dW(t)) \\
E(X(t+dt)) - E(X(t)) &= \sigma E(X(t))E(dW(t)) \\
d(E(X(t)) &= \sigma E(X(t))E(dW(t)) \\
&= 0
\end{aligned}$$

I would like to add a small bit of disccusion here. In the derivation for the ODE for $E(Y(t))$, I notice that the ODE obtained is directly determined by the order of the expansion for $\ln(1+\sigma dW)$, i.e. if we use a first order expansion we find $E(Y(t)) = Y(0)$ and if we use a second order expansion we find $E(Y(t)) = Y(0) - \frac{\sigma^2}{2}t$. It seems appropriate to use only a first order expansion as this only includes changes to $Y(t)$ which vary with respect to $dW(t)$ as $X(t)$ does. Thus, we obtain the same ODE for $E(Y(t))$ as we do for $E(X(t))$, albeit with different initial conditions.

$$\begin{aligned}
E(Y(t+dt)) &= E(Y(t)+dY) \\
&= E(Y(t)) + E(\ln(1+\sigma dW(t))) \\
dE(Y(t)) &= E(\ln(1+\sigma dW(t))) \\
&\approx E\left(\ln(1) + \sigma dW(t) + h.o.t.\right) \\
&\approx \sigma E(dW(t)) \\
&\approx 0
\end{aligned}$$

iii)

$$\begin{aligned}
\frac{dE_X}{dt} &= 0 \\
E_X &= c \\
E_X(0) &= 1 \\
E_X(t) &= 1
\end{aligned}$$

$$\begin{aligned}
\frac{dE_Y}{dt} &= 0 \\
E_Y(t) &= c \\
E_Y(0) &= 0 \\
E_Y(t) &= 0
\end{aligned}$$