AM 216 - Stochastic Differential Equations: Assignment

Dante Buhl

October 26, 2025

Problem 1: Time Reversability of Brownian Bridge

Problem 2: Convergence in Probability

We have,

$$
\begin{aligned}
\lim_{n\to\infty} \mathrm{Var}(Q_n(w)) &= 0 \\
&= \lim_{n\to\infty} \left(E(Q_n^2) - E^2(Q_n)\right) \\
\lim_{n\to\infty} E(Q_n^2) &= q^2 \\
\lim_{n\to\infty} E(|Q_n - q|^2) &\geq \varepsilon^2 \lim_{n\to\infty} Pr(|Q_n - q| \geq \varepsilon) \\
\lim_{n\to\infty} Pr(|Q_n - q| \geq \varepsilon) &\leq 0
\end{aligned}
$$

thus we can state that $\{Q_n\}$ converges to q in probability as $n \to \infty$.

Problem 3: Gaussians

i)

$$
\begin{aligned}
I_2 &= \int_0^T \cos\left(n\pi \frac{t}{T}\right) dW(t) \\
&= \int_0^T \cos\left(n\pi \frac{t}{T}\right) \sqrt{dt} N(0,1) \\
&= N\left(0, \int_0^T \cos^2\left(n\pi \frac{t}{T}\right) dt\right) \\
&= N\left(0, \frac{1}{2}\int_0^T 1 + \cos\left(2n\pi \frac{t}{T}\right) dt\right) \\
&= N\left(0, \frac{T}{2} + \frac{T}{2n\pi} \sin\left(2n\pi \frac{t}{T}\right) \Big|_0^T\right) \\
&= N\left(0, \frac{T}{2}\right)
\end{aligned}
$$

Thus we have, $E(I_2) = 0$ and $\mathrm{Var}(I_2) = T/2$

ii)

Problem 4: Variance of the sums of products of functions of independent variables lol

Problem 5: Ito's Lemma, again

Proof. We begin by determining the independence of W_j and ΔW_j. We have,

$$W_j = W_0 + \sum_{i=0}^{j-1} \Delta W_i, \quad W_i \sim N(0, dt)$$

And so ΔW_j is completely independent of W_j as it is not contained inside the sum which comprises W_j. We have next to look at $E(Q_k - f_k)$.

$$\begin{aligned}
E(Q_k - f_k) &= E\left(\sum_{j=0}^{k-1} W_j^2 \left(\Delta W_j^2 - \Delta t\right)\right) \\
&= \sum_{j=0}^{k-1} E\left(W_j^2 \left(\Delta W_j^2 - \Delta t\right)\right) \\
&= \sum_{j=0}^{k-1} E(W_j^2) E\left(\Delta W_j^2 - \Delta t\right) \\
&= \sum_{j=0}^{k-1} E(W_j^2) \left[\Delta t - \Delta t\right] \\
&= 0
\end{aligned}$$

Where the expectation of the products in line 3 is seperabla as we have shown independence. Next we look at the variance, we have

$$\begin{aligned}
\operatorname{Var}(Q_k - f_k) &= E\left(Q_k - f_k\right)^2 - E^2(Q_k - f_k) \\
&= E\left(Q_k - f_k\right)^2 \\
&= E\left(\sum_{j=0}^{k-1} W_j^4 (\Delta W_j^2 - \Delta t)^2 - 2 \sum_{i=0, i \neq j}^{k-1} W_j^2 W_i^2 (\Delta W_j^2 - \Delta t)(\Delta W_i^2 - \Delta t)\right) \\
&= \sum_{j=0}^{k-1} E\left(W_j^4 (\Delta W_j^2 - \Delta t)^2\right) - 2 \sum_{j=0}^{k-1} \sum_{i=0, i \neq j}^{k-1} E\left(W_j^2 W_i^2 (\Delta W_j^2 - \Delta t)(\Delta W_i^2 - \Delta t)\right) \\
&= (1) + (2)
\end{aligned}$$

Here, we split this calculation into two parts, (1) and (2). Let us first look at (2). Notice that while $T_j = W_j^2(\Delta W_j^2 - \Delta t)$ is comprised of independently distributed products, we do not have that T_i is independent of T_j. That is, if $j > i$ we have that W_j is conditionally dependent on W_i and that ΔW_i is in the sum which comprises W_j. We do still have, however, that ΔW_j^2 is independent of W_j and T_i. Thus, we can write,

$$\begin{aligned}
(2) &= -2 \sum_{j=0}^{k-1} \sum_{i=0, i \neq j}^{k-1} E\left((\Delta W_j^2 - \Delta t) W_j^2 T_i\right) \\
&= -2 \sum_{j=0}^{k-1} \sum_{i=0, i \neq j}^{k-1} E(\Delta W_j^2 - \Delta t) E(W_j^2 T_i) \\
&= 0
\end{aligned}$$

This also holds for the case where $i > j$ and so this is true for all terms in the sum. Finally, we look at (1). We have,

$$\begin{aligned}
(1) &= \sum_{j=0}^{k-1} E(W_j^4)E(\Delta W_j^2 - \Delta t)^2 \\
&= \sum_{j=0}^{k-1} t_j^2 E(X_j^4)E(\Delta W_j^4 + \Delta t^2 - 2\Delta t \Delta W_j^2) \\
&= \sum_{j=0}^{k-1} t_j^2 (3)\left(3\Delta t^2 + \Delta t^2 - 2\Delta t^2\right) \\
&= \sum_{j=0}^{k-1} 6 t_j^2 \Delta t^2
\end{aligned}$$

□

Problem 6: PSD of Ornstein-Uhlenbeck process

$$\begin{aligned}
F\left[e^{-\beta|t|}\right] &\equiv \int_{-\infty}^{\infty} e^{-2\pi\xi t} e^{-\beta|t|} dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi\xi t - \beta|t|} dt \\
&= \int_{-\infty}^{0} e^{(-2\pi\xi + \beta)t} dt + \int_{0}^{\infty} e^{-(2\pi\xi+\beta)t} dt \\
&= \frac{1}{-2\pi\xi + \beta} e^{(-2\pi\xi+\beta)t}\Big|_{-\infty}^{0} - \frac{1}{2\pi\xi + \beta} e^{-(2\pi\xi+\beta)t}\Big|_{0}^{\infty} \\
&= \frac{1}{-2\pi\xi + \beta} + \frac{1}{2\pi\xi + \beta} \\
&= \frac{2\beta}{\beta^2 + 4\pi^2\xi^2}
\end{aligned}$$

Problem 7: Optional: Paley Wiener represation of Wiener process

Problem 8: Optional: Paley Wiener represation of Wiener process Continued