Formulas in Lecture 1

Probability of an event:

$$\Pr(A) = \Pr(\text{outcome } \omega \in A) = \lim_{M\to\infty} \frac{\#\text{ of } \omega \in A}{M \text{ repeats}}$$

Conditional probability:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)}$$

Independent events

$$\Pr(AB) = \Pr(A)\Pr(B)$$

Law of total probability

Suppose $\{B_n, n = 1, 2, \dots\}$ is a partition of sample space Ω. Then we have

$$\Pr(A) = \sum_n \Pr(A|B_n)\Pr(B_n)$$

Probability mass function (PMF) (of a **discrete** random variable)

$$p_N(k) \equiv \Pr(N(\omega) = k)$$

Expected value in terms of PMF:

$$E(N) = \sum_k k\, p_N(k)$$

$$E(f(N)) = \sum_k f(k)\, p_N(k)$$

Probability density function (PDF) (of a **continuous** random variable)

$$\rho_X(x) = \lim_{\Delta x\to 0} \frac{\Pr\left(x < X(\omega) \le x + \Delta x\right)}{\Delta x}$$

Cumulative distribution function (CDF)

$$F_X(x) = \Pr\left(X(\omega) \le x\right)$$

Expected value in terms of PDF

$$E(X) = \int x \rho_X(x)\,dx$$

$$E(f(X)) = \int f(x)\rho_X(x)\,dx$$

Conditional probability density:

$$\rho_X\left(x\middle|Y=y\right)=\frac{\rho_{(X,Y)}(x,y)}{\rho_Y(y)}$$

Independent random variables

$$\rho_{(X,Y)}(x,y)=\rho_X(x)\rho_Y(y)$$

Conditional expectation:

$$E(X|Y=y)=\int x\rho_X(x|Y=y)dx$$

$E(X|Y)\equiv E(X|Y=y)\Big|_{y=Y}$ is a function of Y, a derived random variable.

Law of total expectation

$$E(X)=E\left(E(X|Y)\right)$$

A special case: Suppose $\{B_n, n = 1, 2, \ldots\}$ is a partition of Ω. We have

$$E(X)=\sum_n E(X|B_n)\Pr(B_n)$$

Formulas in Lecture 2

Variance:

$$\mathrm{var}(X)\equiv E\left((X-E(X))^2\right)=E(X^2)-\left(E(X)\right)^2$$

Properties of $E(X)$

- $E(aX + bY) = aE(X) + bE(Y)$ for all X and Y
- If X and Y are independent, then we have $E(X\,Y) = E(X)\,E(Y)$

Properties of var(X)

- $\mathrm{var}(\alpha X) = \alpha^2\mathrm{var}(X)$
- If X and Y are independent, then we have $\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$

Bernoulli distribution

Notation: $X \sim \mathrm{Bern}(p)$

$$X=\begin{cases} 1, & \Pr=p \\ 0, & \Pr=1-p \end{cases}$$

$E(X) = p$, $\mathrm{var}(X) = p(1-p)$

Binomial distribution

Notation: $N \sim \mathrm{Bino}(n, p)$

$$N = \sum_{i=1}^{n} X_i \,, \qquad X_i \sim \text{(iid) Bern}(p)$$

PMF: $\Pr(N = k) = C(n,k) p^k (1-p)^{n-k} \,, \qquad k = 0, 1, 2, \ldots, n$

$\mathrm{E}(X) = np, \qquad \mathrm{var}(X) = np(1-p)$

Exponential distribution

Notation: $T \sim \mathrm{Exp}(\lambda)$

PDF: $$\rho_T(t) = \begin{cases} \lambda \exp(-\lambda t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

CDF: $F_T(t) = 1 - \exp(-\lambda t) \quad$ for $t \geq 0$

$$E(T) = \frac{1}{\lambda}, \qquad \mathrm{var}(T) = \frac{1}{\lambda^2}$$

Normal distribution

Notation: $X \sim N(\mu, \sigma^2)$

PDF: $$\rho_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2} \right)$$

CDF: $$F_X(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2\sigma^2}} \right) \right) \qquad \text{where } \mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z \exp(-s^2)\, ds$$

$$E(X) = \mu, \qquad \mathrm{var}(X) = \sigma^2$$