Formulas in Lecture 7

Definition (continuity of $F(t, \omega)$ in probability)

$F(t, \omega)$ is continuous in probability if for any $\varepsilon > 0$,

$$\lim_{h \to 0} \Pr\left(\left| F(t+h, \omega) - F(t, \omega) \right| \geq \varepsilon \right) = 0$$

Chebyshev-Markov inequality:

$$\boxed{\Pr\left(|X| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^\alpha} E\left(|X|^\alpha \right)}$$ This is valid for any $\alpha > 0$.

Ornstein-Uhlenbeck Process

$$m dY = \underbrace{-bY dt}_{\text{dissipation}} + \underbrace{q dW}_{\text{fluctuation}}$$

$$dX = Y dt$$

Theorem:

$$\int_0^L f(t) dW(t) \sim N\left(0, \int_0^L f(t)^2 dt \right)$$

Solution of $Y(t)$:

$$\left(Y(t_0 + t) \middle| Y(t_0) = y_0 \right) \sim N\left(e^{-\beta t} y_0, \frac{\gamma^2}{2\beta}(1 - e^{-2\beta t}) \right) \quad \text{for } t > 0, \quad \beta = \frac{b}{m}, \quad \gamma = \frac{q}{m}$$

Equilibrium: $Y(t) \sim N\left(0, \dfrac{\gamma^2}{2\beta} \right)$ for large t

$Y(t)$ converges to a white noise as “m → 0”.

Theorem (fluctuation dissipation relation):

The fluctuation coefficient q and the drag coefficient b are related by

$$\boxed{q = \sqrt{2 k_B T b}}$$

Formulas in Lecture 8

Solution of $X(t)$

$$\left(X(t)-X(0)\right) \sim \frac{(1-e^{-\beta t})}{\beta}Y(0)+\left(\frac{\gamma}{\beta}\right)\underbrace{N\left(0,\left(t-\frac{2(1-e^{-\beta t})}{\beta}+\frac{(1-e^{-2\beta t})}{2\beta}\right)\right)}_{\text{containing } dW\text{'s in } [0,t]}$$

As "$m \to 0$", $(X(t) - X(0))$ converges to $\sqrt{2D}\,W(t)$ on any discrete time grid.

Diffusion coefficient

$$D \equiv \lim_{t\to\infty}\frac{1}{2t}\mathrm{var}\left(X(t)-X(0)\right)=2\left(\frac{\gamma}{\beta}\right)^2=\frac{k_B T}{b}$$

Backward time evolution in equilibrium OU process

Forward time evolution

$$\left(Y(t)\middle|Y(0)=y_2\right) \sim N\left(e^{-\beta t}y_2,\ \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right) \qquad \text{for } t>0$$

Backward time evolution

$$\left(Y(-t)\middle|Y(0)=y_2\right) \sim N\left(e^{-\beta t}y_2,\ \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right) \qquad \text{for } t>0$$