AM216 Stochastic Differential Equations	Assignment 2

Q1. MLE of the mean and the variance of a normal distribution.

Let $\mathbf{X} = \{X_j\}_{j=1}^{n}$ be a random sample of size n from $X \sim N(\mu, \sigma^2)$. The log-likelihood function of the sample has the expression:

$$\ell(\mu, \sigma^2 \big| \mathbf{X}) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(X_j - \mu)^2$$

The maximum likelihood estimates of mean and variance are

$$(\mu_{(\text{MLE})}, \sigma^2_{(\text{MLE})}) = \arg\max_{(\mu,\sigma^2)} \ell(\mu, \sigma^2 \big| \mathbf{X})$$

Differentiate $\ell(\mu, \sigma^2 \big| \mathbf{X})$ with respect to (μ, σ^2) to find $(\mu_{(\text{MLE})}, \sigma^2_{(\text{MLE})})$.

Hint: treat σ^2, instead of σ, as a variable.

Q2. MLE variance vs unbiased sample variance.

Let $\mathbf{X} = \{X_j\}_{j=1}^{n}$ be a random sample of size n from $X \sim N(\mu, \sigma^2)$. Derive

$$E\Big(\sum_{j=1}^{n}(X_j - \mu_{(\text{MLE})})^2\Big) = (n-1)\,\text{Var}(X), \qquad \mu_{(\text{MLE})} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

Hint: calculate $E\Big((X_j - \mu_{(\text{MLE})})^2\Big)$ for a particular j (for example, $j = 1$).

This result shows that the MLE of variance is biased.

Q3. An example of central limit theorem.

i) Let $X \sim \text{Bern}(p)$. Find $\phi_X(\xi)$, the CF of X.

ii) Let $N \sim \text{Bino}(n, p)$. Find $\phi_N(\xi)$, the CF of N.

iii) Let $Y = \dfrac{N - np}{\sqrt{n}}$ where $N \sim \text{Bino}(n, p)$. Find $\phi_Y(\xi)$, the CF of Y.

iv) At any finite ξ, show that $\lim\limits_{n\to+\infty} \phi_Y(\xi) = \exp(\dfrac{-p(1-p)\xi^2}{2})$.

Hint: use the result $\lim\limits_{n\to+\infty}(1 - \dfrac{q}{n})^n = e^{-q}$. Do you know how to derive this result?

Note that $\exp(\frac{-p(1-p)\xi^2}{2})$ is the CF of $N(0, p(1-p))$.

Q4. Evaluate the performance of sample variance vs MLE of variance.

Draw a data set of $n = 10$ independent samples of $X \sim N(\mu, \sigma^2)$ with $\mu = 0.6,\ \sigma = 1.3$.

$$(\text{A data set}) = \{X_j,\ j = 1, 2, \ldots, n\}$$

The variance can be estimated in two ways:

$$\text{MLE of variance:} \ \ \sigma^2_{\text{(MLE)}} = \frac{1}{n}\sum_{j=1}^{n}(X_j - \hat{\mu})^2, \qquad \hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

$$\text{Sample variance:} \ \ \hat{\sigma}^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \hat{\mu})^2$$

We examine the mean squared error (MSE) and mean error (ME) over $M = 500000$ repeats.

$$\text{MSE} = \frac{1}{M}\sum_{k=1}^{M}\left(\sigma^2_{\text{(est,k)}} - \sigma^2\right)^2, \qquad \text{ME} = \frac{1}{M}\sum_{k=1}^{M}\left(\sigma^2_{\text{(est,k)}} - \sigma^2\right)$$

where $\sigma^2_{\text{(est,k)}}$ denotes the estimated σ^2 from data set k. Report MSE and ME, respectively for the MLE of variance $\sigma^2_{\text{(MLE)}}$ and the sample variance $\hat{\sigma}^2$. Observe the trade-off between minimizing bias and minimizing fluctuations.

Q5. Verify a key step in deriving the CF of a multivariate Gaussian. Derive

$$\begin{aligned} & i\xi^T x - \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) \\ & \quad = -\frac{1}{2}(x-\mu-i\Sigma\xi)^T\Sigma^{-1}(x-\mu-i\Sigma\xi) + (i\xi^T\mu - \frac{1}{2}\xi^T\Sigma\xi) \end{aligned}$$

Hint: Start from the right hand side.

Q6. Verify a key step in deriving the conditional Gaussian distribution. Let

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}, \qquad \Sigma^{-1} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

Derive

$$\begin{aligned} A^{-1}B &= -\Sigma_{XY}\Sigma_{YY}^{-1} \\ A^{-1} &= (\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}) \end{aligned}$$

Hint: Expand $\Sigma^{-1}\Sigma = I$.

Q7. Another version of Monty Hall's game.

Suppose the rules of the game are set as follows.

- The host puts a cash card of $200 in one of the 3 boxes without you looking.
- After your initial selection, you are required to select and open one of the two remaining boxes (boxes you did not pick in your initial selection).
- If the box you open contains the card, then the game is ended without a winner; you and the host will start a new game from the start.

- If the box you open is empty, the host must offer you the option of paying \$5 to switch to the other remaining box.
- If the box of your final selection contains the cash card, you win \$200.

Suppose the box you open is empty and you are offered the option of paying \$5 to switch. Should you take the option to switch?

Q8. Use a finite difference equation to define a differential equation.
We consider several finite difference equations for defining theoretically the solution of a differential equation. Here the emphasis is not on the numerical accuracy; the emphasis is on the theoretical convergence as $\Delta t \to 0$. Consider a simple model ODE.

$$y' = y, \quad y(0) = 1 \quad \Longrightarrow \quad y_{\text{exact}}(t) = e^t$$

We consider three finite difference equations:

$$\begin{aligned}
\text{FD1:} \quad & y(t+\Delta t) = y(t) + y(t)\Delta t \\
\text{FD2:} \quad & y(t+\Delta t) = y(t) + y(t)\Delta t + \frac{1}{2}y(t)(\Delta t)^2 \\
\text{FDr:} \quad & y(t+\Delta t) = y(t) + y(t)(1+\varepsilon)\Delta t, \quad \varepsilon \sim N(0,1)
\end{aligned}$$

Numerically solve the 3 finite difference equations to $t_f = 1$ with $\Delta t = 2^{-10}$.

(a) Plot the 3 numerical solutions and the exact solution as functions of time.

(b) Let error $= \big|(\text{numerical sol}) - (\text{exact sol})\big|$. Plot the errors in the 3 numerical solutions as functions of time. Use log scale for error.

(c) Repeat part (b) with $\Delta t = 2^{-16}$.

The demonstrated convergence tells us that two finite difference equations differing by a Δt term can converge to the same differential equation.