AM 216 - Stochastic Differential Equations: Assignment 1

Dante Buhl

October 3, 2025

Problem 1: Two Dice

Roll a set of two fair 6-sided dice, one colored red the other white. Assume that the two dice roll independently. Record the two numbers facing up as X_1 and X_2.

i) Mathematically describe the format of outcome. Describe the sample space.

Proof. The two quantities X_1 and X_2 are samples from a given discrete probability distribution. The sample space has six equally likely outcomes $1, 2, 3, 4, 5, 6$ with an expected value of $E(X) = 3.5$. The two samples X_1 and X_2 are independent and identically distributed samples. □

ii) Let X be the absolute difference between X_1 and X_2. Is X a random variable?

Proof. Yes, the sum/difference of two random variables is also a random variable, albeit with a different sample space and expected value. We can exam the sample space to find that there are 6 possibilities of different likelihood: $S = 0, 1, 2, 3, 4, 5$, with 5 being the least likely (only two outcomes where the result is 5) and 1 being the most likely (there are ten outcomes with a result of 1). □

iii) Find the PMF of X, $E(X)$

Proof. Specifically,

k	P(X = k)
0	0.166...
1	0.277...
2	0.222...
3	0.166...
4	0.111...
5	0.055...

and, therefore,

$$E(X) = 1(0.277...) + 2(0.222...) + 3(0.166...) + 4(0.111...) + 5(0.055...) = 1.944...$$

□

iv) Let $A = \{X_2^2 \geq 2X_1\}$. Find $Pr(A|X_1 = n)$ for $n = 1, 2, \ldots, 6$. Then use the law of total probability to find $Pr(A)$.

Proof. We will construct a table to show the values of $Pr(A|X_1 = n)$ for all possible n.

n	1	2	3	4	5	6
Pr(A)	0.833...	0.833...	0.666...	0.666...	0.5	0.5

From this, we can find the probability that $Pr(X_1 = n) = 0.166\ldots \forall n$, and therefore find that $Pr(A) = 0.666\ldots$. □

Problem 2: Variance

i) Show that $\text{Var}(\alpha X) = \alpha^2 \text{Var}(X)$.

Proof.

$$\begin{aligned}\text{Var}(\alpha X) &= E(\alpha X)^2 - E^2(\alpha X) \\ &= E(\alpha^2 X^2) - (\alpha E(X))^2 \\ &= \alpha^2 E(X^2) - \alpha^2 E^2(X) \\ &= \alpha^2 \text{Var}(X)\end{aligned}$$

□

ii) Show that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ if X and Y are independent.

Proof.

$$\begin{aligned}\text{Var}(X+Y) &= E(X+Y)^2 - E^2(X+Y) \\ &= E(X^2 + 2XY + Y^2) - (E^2(X) + 2E(X)E(Y) + E^2(Y)) \\ &= E(X^2) + E(2XY) + E(Y^2) - E^2(X) - 2E(X)E(Y) - E^2(Y) \\ &= E(X^2) + E(Y^2) - E^2(X) - E^2(Y) \\ &= \text{Var}(X) + \text{Var}(Y)\end{aligned}$$

where the change between the third and fourth lines follows from the fact that X and Y are independent, i.e. $E(XY) = E(X)E(Y)$. □

iii) Find an example where $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ does not imply that X and Y are independent.

Proof. To be continued □

Problem 3: Binomial and Normal Distributions

Calculate $E(X^2 + Y^2)$ where $X \sim \text{Bino}(n, p)$ and $Y \sim \mathcal{N}(\mu, \sigma)$.

Proof.

$$\begin{aligned}E(X^2 + Y^2) &= E(X^2) + E(Y^2) \\ &=\end{aligned}$$

] □

Problem 4: Falling within some standard deviation

Proof.

$$\begin{aligned}Pr(\mu - \eta\sigma \le X \le \mu + \eta\sigma) = \cdots &= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\eta\sigma}^{\mu+\eta\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{2}{\sigma\sqrt{2\pi}} \int_{\mu}^{\mu+\eta\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\eta/\sqrt{2}} e^{-z^2} dz, \quad z = (x-\mu)/\sigma\sqrt{2} \\ &= \text{erf}\left(\frac{\eta}{\sqrt{2}}\right)\end{aligned}$$

□

Problem 5: Simple Convoluted Experiment

i) E(X)

Proof. We begin by first using the Law of Total Probability. We have,

$$Pr(X = k) = \sum_{i=0}^{n} Pr(X = k|Y = i)Pr(Y = i)$$

We notice some interesting things at first. We have as a given that $Pr(X = k|Y = m) = 0$ if $k > m$. Then applying the law of total expectation that,

$$\begin{aligned} E(X) &= \sum_{i=0}^{n} E(X|Y = i)Pr(Y = i) \\ &= \sum_{i=0}^{n} \frac{i}{2} Pr(Y = i) \\ &= \sum_{i=0}^{n} \frac{n!}{2^{n+1}(i-1)!(n-i)!} \end{aligned}$$

□

ii) E(XY)

Proof.

$$\begin{aligned} E(X) &= \sum_{i=0}^{n} E(XY|Y = i)Pr(Y = i) \\ &= \sum_{i=0}^{n} \frac{i^2}{2} Pr(Y = i) \\ &= \sum_{i=0}^{n} \frac{in!}{2^{n+1}(i-1)!(n-i)!} \end{aligned}$$

□

Problem 6: Confidence Interval

Here is a picture of my code for this problem, Figure 1.

i) 1.0

ii) 0.999858

Problem 7: Characteristic Functions

i) $E(X) = \mu$

Proof.

$$\begin{aligned} E(X) &= \frac{1}{i} \frac{\partial \phi_x}{\partial \xi} \Big|_{\xi=0} \\ &= \left(\mu - \frac{\sigma^2 \xi}{i} \right) \exp\left(i\mu\xi - \frac{\sigma^2 \xi^2}{2} \right) \Big|_{\xi=0} \\ &= \mu \end{aligned}$$

□

```python
mu = 0.6
sigma = 1.3

nr = 500000
nx = 10

act_in_95 = 0
est_in_95 = 0

for i in range(nr):
    x = np.random.normal(mu, sigma, nx)
    est_mu = x.mean()
    est_sigma = np.sqrt(np.sum((x-est_mu)**2)/(nx-1))
    if (abs(mu-est_mu) <= 2*sigma):
        act_in_95 += 1
    if (abs(mu-est_mu) <= 2*est_sigma):
        est_in_95 += 1

print('Fraction of time mean is in 95% confidence interval (actual sigma): ',\
    act_in_95/nr)
print('Fraction of time mean is in 95% confidence interval (estim. sigma): ',\
    est_in_95/nr)
```

Figure 1: Code to produce the results for this problem

ii) $\mathrm{Var}(X) = \sigma^2$

Proof.

$$
\begin{aligned}
\mathrm{Var}(X) &= (E(X))^2 - E(X^2) \\
&= \left(\frac{1}{i} \frac{\partial \phi_x}{\partial \xi}\Big|_{\xi=0} \right)^2 + \frac{\partial^2 \phi_X}{\partial \xi^2}\Big|_{\xi=0} \\
&=
\end{aligned}
$$

□

Problem 8: Making an experiment well-posed

i)