AM216 Stochastic Differential Equations	Assignment 9
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Q1. Black-Scholes option price model on a linear diffusion.

Recall the procedure we used in deriving the PDE of the option price function $C(s,t)$. In this problem, we consider the situation where the stock price $S(t)$ is governed by

$$dS = \mu dt + \sigma dW$$

Suppose we maintain a portfolio of $F(S(t),t)$ units of delta hedge of time t, at time t, over time period $[0,T]$. Write out the total gain over $[0,T]$.

Hint: In the calculation of total gain, the SDE is used only when we take care of the term $(dS)^2$. We only need to modify that part of the calculation.

Q2. Black-Scholes option price model on a linear diffusion.

We continue with the problem described in Q1.

i) Based on the total gain over $[0,T]$, write out the FVP of $C(s,t)$.

ii) Let $\tau \equiv T - t$ and $\Phi(s,\tau) \equiv C(s, T-\tau)$. Write out the IVP of $\Phi(s,\tau)$.

Q3. Boundary condition in the Black-Scholes option price model.

Recall that the SDE for the underlying stock price is

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad \text{Ito interpretation} \tag{1}$$

and the FVP of the option price function $C(s,t)$ is

$$\begin{cases} C_t(s,t) + \dfrac{1}{2}\sigma^2 s^2 C_{ss}(s,t) + r\big(sC_s(s,t) - C(s,t)\big) = 0 \\ C(s,t)\big|_{t=T} = \max(s-K, 0) \end{cases} \tag{2}$$

In Assignment 6, we showed that it takes forever for $S(t)$ to reach 0. That is, $S(t)$ naturally stays away from 0. Here we show that in IVP (2) we do not need to impose a condition on $C(s,t)$ at $s = 0$. We write the PDE in (2) in the conservation form.

$$C_t(s,t) + \frac{\partial J(s,t)}{\partial s} + a_0 C(s,t) = 0, \qquad J(s,t) = \left(\frac{1}{2}\sigma^2 s^2 C_s + a_1 s C\right)$$

$J(s,t)$ naturally satisfies the condition $J(0,t) = 0$. No additional condition needed for $J(s,t)$ at $s = 0$. Find coefficients a_0 and a_1.

Q4. Average reward of holding a call option.

Let $S(t)$ be the stock price at time t, governed by SDE (1). Consider the call option on the stock with strike price K and expiry T. Let $u(s,t,T)$ be the average reward of holding the option at time T given that the stock price at time t is $S(t) = s$.

$$u(s,t,T) \equiv E\Big(\max(S(T)-K,0)\Big|S(t)=s\Big) \tag{3}$$

The law of total expectation gives

$$\begin{aligned} u(s,t,T) &= E\Big(E\Big[\max(S(T)-K,0)\Big|S(t+dt)=s+dS\Big]\Big) \\ &= E\Big(u(s+dS,t+dt,T)\Big|S(t)=s\Big) \end{aligned}$$

Expanding $u(s+dS,t+dt,T)$ and using the moments of dS calculated from SDE (1) to derive the PDE of $u(s,t,T)$.

Q5. Another philosophical idea for deriving the Black-Scholes option price model.

Consider the hypothetical situation where we have an ensemble of independent realizations of $S(t)$ so that we can buy $E(S(t))$. In SDE (1), taking the average gives

$$\frac{dE(S(t))}{dt} = \mu E(S(t)) \quad \Longrightarrow \quad E(S(T)) = E(S(t))e^{\mu(T-t)}$$

Since the growth of $E(S(T))$ is risk-free, it must match the interest rate: $\boxed{\mu = r}$.

Consider the option on the stock with strike price K and expiry T. The cost of buying the option at time t when $S(t) = s$ is $C(s,t)$. The reward of holding the option at time T is $u(s,t,T)$ studied in Q4. Since the growth from $C(s,t)$ at time t to $u(s,t,T)$ at time T is risk-free, it must match the interest rate:

$$\boxed{C(s,t) = e^{-r(T-t)}u(s,t,T)} \tag{4}$$

Use the PDE of $u(s,t,T)$ obtained in Q4, $\mu = r$ and (9) to derive the PDE of $C(s,t)$.

Remark: This approach of deriving the governing PDE of the option price is called the risk-neutral pricing principle.

Q6. Stochastic evolution of stock price with jumps.

Consider the situation where the stock price increment $dS(t)$ has two parts: i) a geometric brownian motion (continuous in time), and ii) a Poisson jump process (discrete in time). Specifically with Ito interpretation, $S(t)$ is governed by

$$dS(t) = \underbrace{\mu S(t)dt + \sigma S(t)dW(t)}_{dS^{(1)}} + \underbrace{S(t)(J-1)dN(t)}_{dS^{(2)}} \equiv dS^{(1)} + dS^{(2)} \tag{5}$$

In (5), $dN(t)$ is the increment of the Poisson jump process.

$$dN(t) = \begin{cases} 0, & \text{prob} = 1 - \lambda dt + o(dt) \\ 1, & \text{prob} = \lambda dt + o(dt) \end{cases} \tag{6}$$

where λ is the the rate of arrival (of jumps). When $dN(t) = 1$, the stock price jumps from $S(t)$ to $S(t)J$ instantaneously where J is a random variable in $(0, +\infty)$, for example, $(\ln J) \sim N(\mu_J, \sigma_J^2)$. In model (5), $S(t)$, $dW(t)$, $dN(t)$ and J are independent. Let $f(s)$ be a smooth function. Based on (5) and (6), we have the results below.

1) $dS^{(1)}$ and $dS^{(2)}$ are independent.
2) $E\Big(dS^{(1)}\Big|S(t){=}s\Big) = \mu s dt + o(dt)$ and $E\Big((dS^{(1)})^2\Big|S(t){=}s\Big) = \sigma^2 s^2 dt + o(dt)$
3) $\Pr\big(dS^{(2)}{=}0\big) = 1 - \lambda dt + o(dt)$ and $\Pr\big(dS^{(2)}{\neq}0\big) = \lambda dt + o(dt)$
4) $\Big(s{+}dS\Big|S(t){=}s,\ dS^{(2)}{\neq}0\Big) = \Big(sJ{+}dS^{(1)}\Big|S(t){=}s\Big)$ and
5) $E\Big(f(s{+}dS)\Big|S(t){=}s,\ dS^{(2)}{\neq}0\Big) = E\Big(f(sJ{+}dS^{(1)})\Big|S(t){=}s\Big)$

$$= E\Big(f(sJ) + f'(sJ)dS^{(1)} + \frac{1}{2}f''(sJ)(dS^{(1)})^2 + o(dt)\Big|S(t){=}s\Big)$$

$$= E\big(f(sJ)\big) + E\big(f'(sJ)\big)\mu s dt + \frac{1}{2}E\big(f''(sJ)\big)\sigma^2 s^2 dt + o(dt) = E\big(f(sJ)\big) + O(dt)$$

where $E\big(f(sJ)\big) {=} \int_0^{+\infty} f(sq)\rho_J(q)dq$ depends on functions $\{f(\bullet)\}$ and $\{\rho_J(\bullet)\}$.

<u>Use these results to show</u>

i) $E\Big(f(s{+}dS)\Big|S(t){=}s,\ dS^{(2)}{=}0\Big) = f(s) + f'(s)\mu s dt + \frac{1}{2}f''(s)\sigma^2 s^2 dt + o(dt)$.

ii) $E\Big(f(s{+}dS)\Big|S(t){=}s\Big) = f(s){+}\big[f'(s)\mu s{+}\frac{1}{2}f''(s)\sigma^2 s^2\big]dt{+}\lambda\big[E\big(f(sJ)\big){-}f(s)\big]dt{+}o(dt)$.

Q7. Average reward of a call option in the case of stock price evolution with jumps.

We continue with the problem described in Q6. Let $S(t)$ be the stock price at time t, governed by SDE (5) with Poisson jump process (6). Consider the call option on the stock with strike price K and expiry T. Let $u(s, t, T)$ be the average reward of holding the option at time T given that the stock price at time t is $S(t) = s$.

$$u(s, t, T) \equiv E\Big(\max(S(T) - K, 0)\Big|S(t) = s\Big) \tag{7}$$

The law of total expectation gives

$$\begin{aligned} u(s, t, T) &= E\Big(E\Big[\max(S(T) - K, 0)\Big|S(t + dt) = s + dS\Big]\Big) \\ &= E\Big(u(s + dS, t + dt, T)\Big|S(t) = s\Big) \end{aligned} \tag{8}$$

First, apply the results of Q6 to calculate the average with respect to dS on the RHS of (8). Then expand in dt. Derive the governing PIDE (Partial Integro–Differential Equation) of $u(s, t, T)$, which is given below.

$$\boxed{u_t + \frac{1}{2}\sigma^2 s^2 u_{ss}(s, t) + \mu s u_s + \lambda\big[E\big(u(sJ, t)\big){-}u(s, t)\big] = 0}$$

where $E\big(u(sJ, t)\big){=}\int_0^{+\infty} u(sq, t)\rho_J(q)dq$.

Q8. Option price model in the case of stock price evolution with jumps.

We continue with the problem described in Q6. Let $S(t)$ be the stock price at time t, governed by SDE (5) with Poisson jump process (6). Consider the hypothetical situation where we have an ensemble of independent realizations of $S(t)$ so that we can buy $E(S(t))$. In SDE (5), taking the average and using that $S(t)$, $dW(t)$, $dN(t)$ and J are independent, we get

$$dE(S(t)) = \mu E(S(t))dt + E(S(t))(E(J)-1)E(dN(t)), \qquad E(dN(t)) = \lambda dt$$

$$\Longrightarrow \quad \frac{dE(S(t))}{dt} = \big(\mu + \lambda(E(J)-1)\big)E(S(t))$$

$$\Longrightarrow \quad E(S(T)) = E(S(t))e^{(\mu+\lambda(E(J)-1))(T-t)}$$

Since the growth of $E(S(T))$ is risk-free, it must match the interest rate:

$$\mu + \lambda(E(J)-1) = r \quad \Longrightarrow \quad \boxed{\mu = r - \lambda(E(J)-1)}$$

Consider the option on the stock with strike price K and expiry T. The cost of buying the option at time t when $S(t) = s$ is $C(s,t)$. The reward of holding the option at time T is $u(s,t,T)$ studied in Q7. Since the growth from $C(s,t)$ at time t to $u(s,t,T)$ at time T is risk-free, it must match the interest rate:

$$\boxed{C(s,t) = e^{-r(T-t)}u(s,t,T)} \tag{9}$$

Use the PIDE of $u(s,t,T)$ obtained in Q7, $\mu = r-\lambda(E(J)-1)$ and (9) to derive the governing PIDE of option price function $C(s,t)$, which is given below.

$$\boxed{C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + rsC_s - rC + \lambda\big[E\big(C(sJ,t)\big) - C(s,t) - (E(J)-1)sC_s\big] = 0}$$

<u>Remark:</u> This approach of deriving the governing PIDE of the option price is called the risk-neutral pricing principle. In the case of stock price evolution with jumps, there is more than one version of "related" risk-neutral evolution (called risk-neutral measure). For example, the two versions below are risk-neutral.

$$dS(t) = \big[r - \lambda(E(J)-1)\big]S(t)dt + \sigma S(t)dW(t) + S(t)(J-1)dN(t),$$

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) + S(t)(\tilde{J}-1)dN(t)$$

where $\tilde{J}$ is a modified version (measure) of random variable J such that $E(\tilde{J}) = 1$.