AM 216 - Stochastic Differential Equations: Assignment 2

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Problem 1: MLE and the Normal Distribution

Find $\mu_{(MLE)}$ and $\sigma_{(MLE)}$.

Proof.

$$\begin{aligned}
\ell(\mu, \sigma^2 | \boldsymbol{X}) &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(X_j - \mu)^2 \\
\frac{\partial \ell}{\partial \mu} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu) \\
\mu_{(MLE)} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
\frac{\partial \ell}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i - \mu)^2 \\
&= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j)^2 \\
\sigma_{(MLE)} &= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)^2}
\end{aligned}$$

□

Problem 2: MLE Variance and unbiased sample variance

Show that the MLE of variance is biased.

Proof.

$$\begin{aligned}
E\left(\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)^2\right) &= E\left(\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)^2\right) \\
&= E\left(\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{j=1}^{n} 2X_iX_j + \frac{1}{n}X_j^2\right)\right) \\
&= \sum_{i=1}^{n} E\left(X_i^2 - \left(\frac{1}{n}\sum_{j=1}^{n} 2X_iX_j + \frac{1}{n}X_j^2\right)\right)
\end{aligned}$$

IM mad, gonna do it later □

Problem 3: Central Limit Theorem

i)

$$
\begin{aligned}
\phi_X(\xi) &= \int_{-\infty}^{\infty} e^{i\xi x} f(x) dx \\
&= pe^{i\xi} + q
\end{aligned}
$$

ii)

$$
\begin{aligned}
\phi_N(\xi) &= \int_{-\infty}^{\infty} e^{i\xi x} f(x) dx \\
&= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k} e^{i\xi k}
\end{aligned}
$$

iii)

$$
\begin{aligned}
\phi_Y(\xi) &= \int_{-\infty}^{\infty} e^{i\xi y} f(y) dy \\
&= \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{i\xi k+np} f(k+np) dk \\
&= \sum_{k=-np}^{n-np} \left(\begin{array}{c} n \\ k+np \end{array} \right) p^{k+np} (1-p)^{n-k+np} e^{i\xi k+np}
\end{aligned}
$$

Problem 4: Sample Variance v.s. MLE variance

Problem 5: Deriving the CF of a multivariate gaussian

Problem 6: Deriving the Conditional Gaussian Distribution

Problem 7: Monty Hall's Game

Problem 8: Finite Difference Equation?