Formulas in Lecture 9

Convergence in probability

Definition (convergence in probability)

Let $\{Q_N(\omega)\}$ be a sequence of random variables. We say that $\{Q_N(\omega)\}$ converges to q in probability as $N \to +\infty$, if for any $\varepsilon > 0$,

$$\lim_{N\to\infty} \Pr\left(|Q_N(\omega) - q| > \varepsilon\right) = 0$$

Theorem (a sufficient condition for convergence in probability)

Suppose $\lim_{N\to\infty} E\left(Q_N(\omega)\right) = q$ and $\lim_{N\to\infty} \mathrm{var}\left(Q_N(\omega)\right) = 0$.

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Then, $\{Q_N(\omega)\}$ converges to q in probability as $N \to +\infty$.

Proof: homework problem.

Theorem (a useful formula for calculating $\mathrm{var}\left(\sum_{j=0}^{N-1} Y_j X_j\right)$)

Suppose random variables $\{X_j, j = 0, 1, \ldots, N-1\}$ and $\{Y_k, k = 0, 1, \ldots, N-1\}$ satisfy

1. $E(X_j) = 0$ for all j,
2. X_j is independent of X_i for all $i \neq j$, and
3. X_j is independent of Y_k for all $k \leq j$.

Then we have
$$\mathrm{var}\left(\sum_{j=0}^{N-1} Y_j X_j\right) = \sum_{j=0}^{N-1} E(Y_j^2) E(X_j^2).$$

Different interpretations of $\int_0^t f(s, W(s)) dW(s)$

Discretization:

$$\Delta s = \frac{t}{N}, \quad s_j = j\Delta s, \quad f_j = f(s_j, W(s_j)), \quad \Delta W_j = W_{j+1} - W_j$$

Ito interpretation:

$$I_{\text{Ito}} = \lim_{N\to\infty} \sum_{j=0}^{N-1} f_j \Delta W_j$$

Stratonovich interpretation:

$$I_{\text{Stratonovich}} = \lim_{N\to\infty} \sum_{j=0}^{N-1} \frac{1}{2}\left(f_j + f_{j+1}\right)\Delta W_j$$

Theorem (Ito's lemma)

$$\lim_{N\to\infty}\left(\sum_{j=0}^{N-1} g(s_j, W_j)(\Delta W_j)^2 - \sum_{j=0}^{N-1} g(s_j, W_j)\Delta s\right) = 0$$

Theorem:

Ito interpretation and Stratonovich interpretation of $\int_0^t f(s, W(s))dW(s)$ are related by

$$I_{\text{Stratonovich}} = I_{\text{Ito}} + \frac{1}{2}\int_0^t f_w(s, W(s))ds$$

Stochastic integrals based on axioms

1) Fundamental theorem of calculus: $\int_a^b dH(t, W(t)) = H(t, W(t))\Big|_a^b$

2) λ–chain rule $dH(t, W(t)) = H_t dt + H_w dW(t) + \left(\tfrac{1}{2} - \lambda\right) H_{ww} dt$

   ==> $\int H_w dW = \int dH - \int \left(H_t + (\tfrac{1}{2} - \lambda) H_{ww}\right) dt$

   Different interpretations of $\int H_w dW$ are reflected in different values of λ.

   Ito: λ = 0; Stratonovich: λ = 0.5

Formulas in Lecture 10

Different interpretations of $dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$

Ito interpretation:

$$\Delta X = b(X(t), t)\Delta t + \sqrt{a(X(t), t)}\,\Delta W + o(\Delta t)$$

Stratonovich interpretation:

$$\Delta X = \frac{1}{2}\Big(b(X(t), t) + b(X(t+\Delta t), t+\Delta t)\Big)\Delta t + \frac{1}{2}\left(\sqrt{a(X(t), t)} + \sqrt{a(X(t+\Delta t), t+\Delta t)}\right)\Delta W + o(\Delta t)$$

Theorem: The Stratonovich of $dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$ **is equivalent to** the Ito of the modified SDE $dX = \left(b(X,t) + \frac{1}{4} a_x(X,t) \right) dt + \sqrt{a(X,t)}\, dW$.

Backward equation and forward equation of $dX = b(X,t)dt + \sqrt{a(X,t)}\, dW$

The moments of dX (Ito interpretation)

$$E\left(dX \middle| X(s) = z\right) = b(z,s)ds + o(ds)$$

$$E\left((dX)^2 \middle| X(s) = z\right) = a(z,s)ds + \left(b(z,s)ds\right)^2 = a(z,s)ds + o(ds)$$

$$E\left((dX)^n \middle| X(s) = z\right) = o(ds), \qquad \text{for } n \ge 3$$

Transition probability density (a 4-variable function)

$$q(x, \underbrace{t}_{\substack{\uparrow \\ \text{end time}}} \,|\, z, \underbrace{s}_{\substack{\uparrow \\ \text{starting time}}}) = \frac{1}{dx} \Pr\left(x < X(t) \le x + dx \middle| X(s) = z \right), \qquad t > s$$

The moments in terms of transition PD $q(x, t \mid z, s)$

$q(z+y, s+ds|z, s)$, as a function of y, is the probability density of $(dX \mid X(s) = z)$.

0) $\int q(z+y, s+ds | z, s) dy = 1$

Equivalently $\int q(x, s+ds | z, s) dx = 1$

1) $\int q(z+y, s+ds | z, s)\, y\, dy = E\left(dX \middle| X(s) = z\right) = b(z,s)ds + o(ds)$

Equivalently $\int q(x, s+ds | z, s)(x-z) dx = b(z,s)ds + o(ds)$

2) $\int q(z+y, s+ds | z, s)\, y^2 dy = E\left((dX)^2 \middle| X(s) = z\right) = a(z,s)ds + o(ds)$

Equivalently $\int q(x, s+ds | z, s)(x-z)^2 dx = a(z,s)ds + o(ds)$

3) $\int q(z+y, s+ds | z, s)\, y^n dy = E\left((dX)^n \middle| X(s) = z\right) = o(ds)$, for $n \ge 3$

Equivalently $\int q(x, s+ds | z, s)(x-z)^n dx = o(ds)$, for $n \ge 3$

Backward view (the law of total probability)

We fix (x, t) and view q as a function of (z, s): $\quad q(z, s) \equiv q(x, t|z, s)$

$$\underbrace{q(x,t\,|\,z,s)}_{\substack{q(\bullet,s)\\ [s\to t]}} = \int \underbrace{q(x,t\,|\,z+y,s+ds)}_{\substack{q(\bullet,s+ds)\\ [s+ds\to t]}}\ \underbrace{q(z+y,s+ds\,|\,z,s)}_{\substack{\text{density of } dX\\ [s\to s+ds]}}\,dy$$

$$q(\cdot\,,s+ds) \quad \longrightarrow \quad q(\cdot\,,s)$$

We move the starting time backward from $(s+ds)$ to s.

Forward view (the law of total probability)

We fix (z, s) and view q as a function of (x, t): $\quad q(x, t) \equiv q(x, t|z, s)$

$$\underbrace{q(x,t+dt\,|\,z,s)}_{\substack{q(\bullet,t+dt)\\ [s\to t+dt]}} = \int \underbrace{q(x,t+dt\,|\,y,t)}_{\substack{\text{density of } X(t+dt)|X(t)=y\\ [t\to t+dt]}}\ \underbrace{q(y,t\,|\,z,s)}_{\substack{q(\bullet,t)\\ [s\to t]}}\,dy$$

$$q(\cdot\,,t) \quad \longrightarrow \quad q(\cdot\,,t+dt)$$

We move the end time forward from t to $(t+dt)$.

Kolmogorov backward equation for $q(z, s) \equiv q(x, t \mid z, s)$

$$0 = q_s + b(z,s)q_z + \frac{1}{2}a(z,s)q_{zz}$$

Final value problem (FVP) of the backward equation

$$\begin{cases} u_s = -b(z,s)u_z - \dfrac{1}{2}a(z,s)u_{zz} \\ u(z,s)\big|_{s=T} = u_T(z) \end{cases}$$

Converting it to an IVP

Let $\tau \equiv T-s$, the time until the specified end time T. We define

$$\phi(z,\tau) \equiv u(z,T-\tau), \quad \alpha(z,\tau) \equiv a(z,T-\tau), \quad \beta(z,\tau) \equiv b(z,T-\tau)$$

$\phi(z, \tau)$ is governed by an initial value problem:

$$\begin{cases} \phi_\tau = \beta(z,\tau)\phi_z + \tfrac{1}{2}\alpha(z,\tau)\phi_{zz} \\ \phi(z,0) = u_T(z) \end{cases}$$