**AM216 Stochastic Differential Equations**

**List of topics in this lecture**

- A mathematical derivation of Smoluchowski-Kramers approximation
- Scaling of $X$, $Y$ and $F(X, t)$
- The method of solvability condition

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**Recap**

**Smoluchowski-Kramers approximation**

<u>Langevin equation</u> for a small particle in water

$$dX = Y\,dt$$

$$mdY = -bY\,dt + F(X,t)dt + \sqrt{2k_B Tb}\,dW$$

where    $X$ : position;        $Y$ : velocity;

$a$ : radius of the particle;

$m = 4\pi/3\ a^3 = O(a^3)$ :      mass of the particle

$b = 6\pi\,\eta\,a = O(a)$ :          drag coefficient of the particle

<u>Time scale of inertia</u> and <u>thermal excitation</u>:    $t_0 = m/b = O(a^2)$

<u>S-K approximation:</u>

For $t_{\text{scale}} \gg t \gg t_0$, $X(t)$ satisfies

$$X(t) - X(0) = \underbrace{\frac{F_0}{b}t}_{\text{Term III}} + \underbrace{t_0\left(Y(0) - \frac{F_0}{b}\right)}_{\text{Term I}} + \underbrace{N\left(0,\ (\gamma t_0)^2 t_0\left(\frac{t}{t_0}\right)\right)}_{\text{Term II}} + \text{T.S.T.} \qquad \text{(SK0)}$$

where $t_{\text{scale}}$ is the time scale of physical evolution. We neglect Term I, and keep Term II and Term III. On a "coarse" grid, $X(t)$ is governed by

$$dX = \frac{F(X,t)}{b}dt + \sqrt{2D}\,dW$$

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**A mathematical derivation of Smoluchowski-Kramers approximation**

Let $\varepsilon \equiv \sqrt{t_0 / t_{\text{scale}}} = O(a/R)$ be a small parameter.

Here $t_{\text{scale}}$ is the time scale of physical evolution and $R$ is a size scale whose time scale of inertia is $t_{\text{scale}}$. We assume that ==the time is measured in units of $t_{\text{scale}}$==, and the particle position/size is measured in units of $R$. We write $O(a/R)$ simply as $O(a)$.

<u>Scaling of position $X$</u>

The over-damped Langevin equation for $X$ is

$$dX = \frac{F(X,t)}{b} dt + \sqrt{2D}\, dW, \qquad D = O(a^{-1}) = O(\varepsilon^{-1})$$

This equation is not free of $\varepsilon$. When $F = 0$ and $t = O(1)$, we have

$$X(1) - X(0) \sim \sqrt{2D} = O(\varepsilon^{-1/2})$$

We select a scaling for $X$ to make it $O(1)$.

$$\hat{X} = \sqrt{\varepsilon} \quad \Rightarrow \quad \hat{X}(1) - \hat{X}(0) = O(1)$$

<u>The over-damped Langevin equation after scaling $X$</u>

We consider the case where for $t = O(1)$, the effect of external driving force is comparable to that of the diffusion

$$\frac{F(X,t)}{b} = O(\sqrt{D}) = O(\varepsilon^{-1/2}), \qquad b = O(a) = O(\varepsilon)$$

==> $\quad F(X,t) = O(\sqrt{\varepsilon})$

After scaling $X$, we have

$$\sqrt{\varepsilon}\, dX = \sqrt{\varepsilon}\, \frac{F(X,t)}{b} dt + \sqrt{\varepsilon}\, \sqrt{2D}\, dW$$

==> $\quad \boxed{d\hat{X} = f(\hat{X},t)dt + \sqrt{2D_0}\, dW}$

where

$$D_0 \equiv \varepsilon D = O(1), \qquad f(\hat{X},t) \equiv \sqrt{\varepsilon}\, \frac{F(X,t)}{b} = O(1)$$

==Note:   after scaling $X$, the over-damped Langevin equation is free of $\varepsilon$.==

<u>Scaling of velocity $Y$</u>

We scale $Y$ separately from the scaling of $X$. Here $Y$ is the instantaneous velocity, dominated by thermal excitations, much larger than $X(1)$–$X(0)$.

When $F = 0$, the equipartition of energy gives us

$$\sqrt{E(Y^2)} = \sqrt{\frac{k_B T}{m}} = O(\varepsilon^{-3/2}), \qquad m = O(a^3) = O(\varepsilon^3)$$

We select a scaling for $Y$ to make it $O(1)$

$$\hat{Y} = \varepsilon^{3/2} Y \qquad \Rightarrow \qquad \sqrt{E(\hat{Y}^2)} = O(1)$$

The FULL Langevin equation after scaling of $Y$

$$m\,dY = -bY\,dt + F(X,t)dt + b\sqrt{2D}\,dW$$

==> $\quad \dfrac{m}{b}dY = -Y\,dt + \dfrac{F(X,t)}{b}dt + \sqrt{2D}\,dW$

==> $\quad \varepsilon^2(\varepsilon^{-3/2}d\hat{Y}) = -(\varepsilon^{-3/2}\hat{Y})dt + \varepsilon^{-1/2}f(\hat{X},t)dt + \sqrt{2(\varepsilon^{-1}D_0)}\,dW$

==> $\quad \varepsilon^{1/2}d\hat{Y} = -\varepsilon^{-3/2}\hat{Y}\,dt + \varepsilon^{-1/2}f(\hat{X},t)dt + \varepsilon^{-1/2}\sqrt{2D_0}\,dW$

==> $\quad \boxed{d\hat{Y} = \dfrac{-1}{\varepsilon^2}\hat{Y}\,dt + \dfrac{1}{\varepsilon}f(\hat{X},t)dt + \dfrac{1}{\varepsilon}\sqrt{2D_0}\,dW}$

Since $X$ and $Y$ are scaled <u>separately</u>, after scaling they are related by

$$dX = Y\,dt \quad \Rightarrow \quad d(\varepsilon^{1/2}X) = \varepsilon^{-1}(\varepsilon^{3/2}Y)dt \quad \Rightarrow \quad d\hat{X} = \varepsilon^{-1}\hat{Y}\,dt$$

<u>Notation:</u>

After scaling, we recycle the simple notation of ($X$, $Y$, $t$) to write out the starting SDE and the end SDE of the S-K approximation.

The starting SDE of S-K approximation

$$dX = \frac{1}{\varepsilon}Y\,dt$$

$$dY = \frac{-1}{\varepsilon^2}Y\,dt + \frac{1}{\varepsilon}f(X,t)dt + \frac{1}{\varepsilon}\sqrt{2D_0}\,dW \qquad\qquad \text{(S-1)}$$

The end SDE of S-K approximation

$$dX = f(X,t)dt + \sqrt{2D_0}\,dW \qquad\qquad \text{(S-2)}$$

<u>Backward equation</u> for the starting SDE

For the starting SDE (S-1), we consider function

$$u(x,y,t) \equiv \Pr\Big\{ X \text{ having crossed } x_B \text{ by time } t \,\Big|\, X(0)=x, Y(0)=y \Big\}$$

We derive the governing equation of $u(x, y, t)$. For $dt$ small enough, we have

$$u(x,y,t)=E\big(u(x+dX,y+dY,t-dt)\big)+o(dt) \qquad\qquad \text{(E-1)}$$

<u>Note</u> that in the SDE, $\varepsilon$ = fixed. As $dt \to 0$, we have $dX=O(dt)$ and $dY=O(\sqrt{dt})$.

Specifically we have the moments of $dX$ and $dY$ from the SDE.

$$\big(dX\,|\,X(0)=x,Y(0)=y\big)=\frac{y}{\varepsilon}dt \quad \text{not a random variable}$$

$$E\big(dY\,|\,X(0)=x,Y(0)=y\big)=\left(\frac{-1}{\varepsilon^2}y+\frac{1}{\varepsilon}f(x,t)\right)dt$$

$$E\big((dY)^2\,|\,X(0)=x,Y(0)=y\big)=\frac{1}{\varepsilon^2}2D_0 dt+o(dt)$$

$$E\big((dY)^n\,|\,X(0)=x,Y(0)=y\big)=o(dt) \quad \text{for } n\geq 3$$

We expand (E-1)

$$u(x,y,t)=E\big(u(x+dX,y+dY,t-dt)\big)+o(dt)$$

$$=E\left(u(x,y,t)+u_x dX+u_y dY+\frac{1}{2}u_{yy}(dY)^2-u_t dt\right)+o(dt)$$

$$=u+u_x\frac{y}{\varepsilon}dt+u_y\left(\frac{-1}{\varepsilon^2}y+\frac{1}{\varepsilon}f(x,t)\right)dt+\frac{1}{2}u_{yy}\frac{1}{\varepsilon^2}2D_0 dt-u_t dt+o(dt)$$

…

The governing equation for $u(x,y,t)$ is the backward equation of (S-1).

$$\varepsilon^2 u_t=-yu_y+D_0 u_{yy}+\varepsilon\big(yu_x+f(x,t)u_y\big) \qquad\qquad \text{(BE-1)}$$

<u>Backward equation</u> for the end SDE

For the end SDE (S-2), we consider function

$$w(x,t)\equiv\Pr\big\{X \text{ having crossed } x_B \text{ by time } t\,|\,X(0)=x\big\}$$

The governing equation for $w(x,t)$ is the backward equation of (S-2).

$$\frac{\partial w}{\partial t}=D_0\frac{\partial^2 w}{\partial x}+f(x,t)\frac{\partial w}{\partial x} \qquad\qquad \text{(BE-2)}$$

<u>Convergence</u> of $u(x,y,t)$ to $w(x,t)$

For backward equation (BE-1), we seek solutions of the form

$$u(x,y,t) = u_0(x,y,t) + \varepsilon u_1(x,y,t) + \varepsilon^2 u_2(x,y,t) + \cdots$$

We substitute the expansion into (BE-1) and keep terms up to $O(\varepsilon^2)$.

$$\varepsilon^2 \frac{\partial u_0}{\partial t} = -y\left( \frac{\partial u_0}{\partial y} + \varepsilon \frac{\partial u_1}{\partial y} + \varepsilon^2 \frac{\partial u_2}{\partial y} \right) + D_0 \left( \frac{\partial^2 u_0}{\partial y^2} + \varepsilon \frac{\partial^2 u_1}{\partial y^2} + \varepsilon^2 \frac{\partial^2 u_2}{\partial y^2} \right)$$

$$+ \varepsilon \left( y\left( \frac{\partial u_0}{\partial x} + \varepsilon \frac{\partial u_1}{\partial x} \right) + f(x,t)\left( \frac{\partial u_0}{\partial y} + \varepsilon \frac{\partial u_1}{\partial y} \right) \right) + o(\varepsilon^2)$$

$$==> \quad 0 = \underbrace{\left( -y \frac{\partial u_0}{\partial y} + D_0 \frac{\partial^2 u_0}{\partial y^2} \right)}_{O(1) \text{ terms}} + \varepsilon \underbrace{\left( -y \frac{\partial u_1}{\partial y} + D_0 \frac{\partial^2 u_1}{\partial y^2} + y \frac{\partial u_0}{\partial x} + f(x,t) \frac{\partial u_0}{\partial y} \right)}_{O(\varepsilon) \text{ terms}}$$

$$+ \varepsilon^2 \underbrace{\left( -y \frac{\partial u_2}{\partial y} + D_0 \frac{\partial^2 u_2}{\partial y^2} + y \frac{\partial u_1}{\partial x} + f(x,t) \frac{\partial u_1}{\partial y} - \frac{\partial u_0}{\partial t} \right)}_{O(\varepsilon^2) \text{ terms}} + o(\varepsilon^2)$$

Below, we show that

- $u_0(x, y, t) = u_0(x, t)$, independent of $y$.
- $u_0(x, t)$ satisfies backward equation (BE-2).

<u>Step A:</u>  $u_0(x, y, t)$ is independent of $y$.

The balance of $O(1)$ terms gives the equation

$$-y \frac{\partial u_0}{\partial y} + D_0 \frac{\partial^2 u_0}{\partial y^2} = 0$$

Multiplying by the integrating factor yields

$$\exp\left( \frac{-y^2}{2D_0} \right)\left( \frac{-y}{D_0} \right)\frac{\partial u_0}{\partial y} + \exp\left( \frac{-y^2}{2D_0} \right)\frac{\partial^2 u_0}{\partial y^2} = 0$$

$$==> \quad \frac{\partial}{\partial y}\left[ \exp\left( \frac{-y^2}{2D_0} \right)\frac{\partial u_0}{\partial y} \right] = 0$$

$$==> \quad \exp\left( \frac{-y^2}{2D_0} \right)\frac{\partial u_0}{\partial y} = c_1(x,t) \quad \text{independent of } y$$

==> $\dfrac{\partial u_0}{\partial y} = c_1(x,t)\exp\left(\dfrac{y^2}{2D_0}\right)$

==> $u_0 = c_0(x,t) + c_1(x,t)\displaystyle\int_0^y \exp\left(\dfrac{z^2}{2D_0}\right)dz$

Recall that $u(x,y,t)$ = probability.

==> $u(x,y,t)$ is bounded as $y \to \infty$.

==> $u_0(x,y,t)$, as the leading term, is bounded as $y \to \infty$

==> $c_1(x,t) = 0$

==> $u_0 = c_0(x,t)$    independent of $y$

<u>Step B:</u>    $u_0(x,t)$ satisfies backward equation (BE-2)

<u>Step B1:</u>   The balance of $O(\varepsilon)$ terms gives

$$-y\frac{\partial u_1}{\partial y} + D_0\frac{\partial^2 u_1}{\partial y^2} + y\frac{\partial u_0}{\partial x} + f(x,t)\frac{\partial u_0}{\partial y} = 0$$

We write it as an equation for $u_1$ and multiply by the integrating factor

$$-y\frac{\partial u_1}{\partial y} + D_0\frac{\partial^2 u_1}{\partial y^2} = -y\frac{\partial u_0}{\partial x}$$

==> $\exp\left(\dfrac{-y^2}{2D_0}\right)\left(\dfrac{-y}{D_0}\right)\dfrac{\partial u_1}{\partial y} + \exp\left(\dfrac{-y^2}{2D_0}\right)\dfrac{\partial^2 u_1}{\partial y^2} = \exp\left(\dfrac{-y^2}{2D_0}\right)\left(\dfrac{-y}{D_0}\right)\dfrac{\partial u_0}{\partial x}$

==> $\dfrac{\partial}{\partial y}\left(\exp\left(\dfrac{-y^2}{2D_0}\right)\dfrac{\partial u_1}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\exp\left(\dfrac{-y^2}{2D_0}\right)\right)\dfrac{\partial u_0}{\partial x}$

Noticing that $\partial u_0/\partial x$ is independent of $y$ and integrating in $y$, we get

$$\exp\left(\dfrac{-y^2}{2D_0}\right)\frac{\partial u_1}{\partial y} = \exp\left(\dfrac{-y^2}{2D_0}\right)\frac{\partial u_0}{\partial x} + c_3(x,t)$$

==> $\dfrac{\partial u_1}{\partial y} = \dfrac{\partial u_0}{\partial x} + c_3(x,t)\exp\left(\dfrac{y^2}{2D_0}\right)$

==> $u_1 = y\dfrac{\partial u_0}{\partial x} + c_2(x,t) + c_3(x,t)\displaystyle\int_0^y \exp\left(\dfrac{z^2}{2D_0}\right)dz$

$u_1$ is solvable without imposing any constraint on $u_0$.

To find a constraint on $u_0$, we need to examine $O(\varepsilon^2)$ terms.

Step B2:   The balance of $O(\varepsilon^2)$ terms gives

$$-y\frac{\partial u_2}{\partial y}+D_0\frac{\partial^2 u_2}{\partial y^2}+y\frac{\partial u_1}{\partial x}+f(x,t)\frac{\partial u_1}{\partial y}-\frac{\partial u_0}{\partial t}=0$$

==>   $$-y\frac{\partial u_2}{\partial y}+D_0\frac{\partial^2 u_2}{\partial y^2}=\frac{\partial u_0}{\partial t}-y\frac{\partial u_1}{\partial x}-f(x,t)\frac{\partial u_1}{\partial y}$$

==>   $$L_1[u_2]=\frac{\partial u_0}{\partial t}+L_2[u_1]$$

where operators $L_1$ and $L_2$ are defined as

$$L_1[\cdot]\equiv -y\frac{\partial\cdot}{\partial y}+D_0\frac{\partial^2\cdot}{\partial y^2}$$

$$L_2[\cdot]\equiv -y\frac{\partial\cdot}{\partial x}-f(x,t)\frac{\partial\cdot}{\partial y}$$

Theorem (solvability condition)

Equation $L[u] = g$ is solvable if and only if $\langle g, v\rangle = 0$ for all $v$ satisfying $L^*[v] = 0$
==(with $v(y) \to 0$ rapidly as $|y| \to \infty$)==

A simple demonstration of the theorem:

Let $A$ be an $m{\times}n$ matrix.

$A\,u = b$ is solvable if and only if $b \in \mathrm{Col}(A)$

if and only if $\langle b, v\rangle$ for all $v \in$ ==$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$==.

if and only if $\langle b, v\rangle$ for all $v$ satisfying $A^Tv = 0$.

Important note:

When operator $L_1$ is in variable $y$, we view $v(y)$ as a function of $y$ and the inner-product $\langle g, v\rangle$ is an integral with respect to $y$.

Step B3:   Solvability of $L_1[u_2]=\dfrac{\partial u_0}{\partial t}+L_2[u_1]$.

$u_2$ is solvable if and only if $\left\langle \dfrac{\partial u_0}{\partial t}+L_2[u_1], v \right\rangle=0$ for all $v$ satisfying $L_1^*[v] = 0$.

<u>Step B4:</u>   Solution of $L_1^*[v] = 0$

$$L_1[\cdot] = -y\frac{\partial \cdot}{\partial y} + D_0\frac{\partial^2 \cdot}{\partial y^2} \quad ==> \quad L_1^*[\cdot] = \frac{\partial(y\cdot)}{\partial y} + D_0\frac{\partial^2 \cdot}{\partial y^2}$$

$L_1^*[v] = 0$ yields

$$\frac{\partial(y\,v)}{\partial y} + D_0\frac{\partial^2 v}{\partial y^2} = 0$$

$$==> \quad \frac{y}{D_0}v + \frac{\partial v}{\partial y} = d_1$$

$$==> \quad \exp\left(\frac{y^2}{2D_0}\right)\frac{y}{D_0}v + \exp\left(\frac{y^2}{2D_0}\right)\frac{\partial v}{\partial y} = d_1\exp\left(\frac{y^2}{2D_0}\right)$$

$$==> \quad \frac{\partial}{\partial y}\left[\exp\left(\frac{y^2}{2D_0}\right)v\right] = d_1\exp\left(\frac{y^2}{2D_0}\right)$$

$$==> \quad \exp\left(\frac{y^2}{2D_0}\right)v(y) = d_1\int_0^y\exp\left(\frac{z^2}{2D_0}\right)dz + d_0$$

$$==> \quad v(y) = d_1\underbrace{\exp\left(\frac{-y^2}{2D_0}\right)\int_0^y\exp\left(\frac{z^2}{2D_0}\right)dz}_{\text{Not decaying to zero rapidly as } y\to\infty} + d_0\exp\left(\frac{-y^2}{2D_0}\right)$$

Anyway, we select    $v(y) = d_0\exp\left(\dfrac{-y^2}{2D_0}\right)$

<u>Step B5:</u>   Back to solvability of $L_1[u_2] = \dfrac{\partial u_0}{\partial t} + L_2[u_1]$.

We set $d_0$ to make $v(y)$ a normal density.

$$v(y) = \frac{1}{\sqrt{2\pi D_0}}\exp\left(\frac{-y^2}{2D_0}\right) = \rho_{N(0,D_0)}(y)$$

$u_2$ is solvable if and only if $\displaystyle\int_{-\infty}^{+\infty}\left(\frac{\partial u_0}{\partial t} + L_2[u_1]\right)\rho_{N(0,D_0)}(y)dy = 0$.

Notice that $\partial u_0/\partial t$ is independent of $y$. The solvability condition gives us

$$\frac{\partial u_0}{\partial t} + \int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy = 0 \qquad\qquad\qquad \text{(Cond-1)}$$

<u>Step B6:</u>  Calculation of $\displaystyle\int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy$

Recall that $u_1 = y\dfrac{\partial u_0}{\partial x} + c_2(x,t) + c_3(x,t)\displaystyle\int_0^y \exp\left(\dfrac{z^2}{2D_0}\right)dz$ . We calculate $L_2[u_1]$.

$$L_2[u_1] = -y\frac{\partial u_1}{\partial x} - f(x,t)\frac{\partial u_1}{\partial y} = \underbrace{-y^2\frac{\partial^2 u_0}{\partial x^2} - f(x,t)\frac{\partial u_0}{\partial x}}_{\text{effect of }u_0} \underbrace{-y\frac{\partial c_2}{\partial x}}_{\text{effect of }c_2}$$

$$\underbrace{-y\frac{\partial c_3}{\partial x}\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz - f(x,t)c_3\exp\left(\frac{y^2}{2D_0}\right)}_{\text{effect of }c_3}$$

We examine the inner product of each part with $\rho_{N(0, D)}(y)$.

$u_0(x, t)$ is independent of $y$.

$$\int_{-\infty}^{+\infty}\underbrace{\left[-y^2\frac{\partial^2 u_0}{\partial x^2} - f(x,t)\frac{\partial u_0}{\partial x}\right]}_{\text{effect of }u_0}\rho_{N(0,D_0)}(y)dy = -D_0\frac{\partial^2 u_0}{\partial x^2} - f(x,t)\frac{\partial u_0}{\partial x}$$

$c_2(x, t)$ is independent of $y$.

$$\int_{-\infty}^{+\infty}\underbrace{-y\frac{\partial c_2}{\partial x}}_{\text{effect of }c_2}\rho_{N(0,D_0)}(y)dy = 0$$

$c_3(x, t)$ is independent of $y$.

$$\int_{-\infty}^{+\infty}\underbrace{\left[-y\frac{\partial c_3}{\partial x}\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz - f(x,t)c_3\exp\left(\frac{y^2}{2D_0}\right)\right]}_{\text{effect of }c_3}\rho_{N(0,D_0)}(y)dy = \infty \quad \text{if } c_3 \neq 0.$$

To make $\displaystyle\int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy$ finite, we must have $c_3(x, t) \equiv 0$.

Thus, we obtain the expression

$$\int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy = -D_0\frac{\partial^2 u_0}{\partial x^2} - f(x,t)\frac{\partial u_0}{\partial x} \qquad\qquad \text{(Res-1)}$$

<u>Step B7:</u>   Governing equation for $u_0$

Substitute result (Res-1) into solvability condition (Cond-1), we conclude

$$\frac{\partial u_0}{\partial t} = D_0 \frac{\partial^2 u_0}{\partial x^2} + f(x,t) \frac{\partial u_0}{\partial x}$$

which is backward equation (BE-2).

<u>Remark:</u>

In the derivation above, the key is the method of solvability condition.

Below we illustrate the method of solvability condition in two simple examples.

<u>Example:</u>   (the method of solvability condition)

Consider solving a 2×2 linear system $Ax = b$ where

$$A = \begin{pmatrix} 1 & 1+\varepsilon \\ 1+\varepsilon & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1-\varepsilon \\ 1+\varepsilon \end{pmatrix}$$

The exact solution is

$$x_{exa} = \begin{pmatrix} \dfrac{3+\varepsilon}{2+\varepsilon} \\ \dfrac{-(1+\varepsilon)}{2+\varepsilon} \end{pmatrix} \approx \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$$

Here we use asymptotic analysis to solve this example.

The goal is to see the application of solvability condition in a simple setting.

We seek solutions of the form

$$x = x^{(0)} + \varepsilon x^{(1)} + \cdots$$

We write matrix $A$ and vector $b$ as

$$A = A^{(0)} + \varepsilon A^{(1)}, \qquad A^{(0)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$b = b^{(0)} + \varepsilon b^{(1)}, \qquad b^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad b^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We expand linear system $Ax - b = 0$ into $O(1)$ terms, $O(\varepsilon)$ terms, ...

$$\left( A^{(0)} + \varepsilon A^{(1)} \right)\left( x^{(0)} + \varepsilon x^{(1)} + \cdots \right) - \left( b^{(0)} + \varepsilon b^{(1)} \right) = 0$$

==> $\left(A^{(0)}x^{(0)}-b^{(0)}\right)+\varepsilon\left(A^{(0)}x^{(1)}+A^{(1)}x^{(0)}-b^{(1)}\right)+\cdots=0$

First we look at the O(1) terms.

$$A^{(0)}x^{(0)}=b^{(0)}$$

This linear system is <u>underdetermined.</u>

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

==> $\quad x_1^{(0)}+x_2^{(0)}=1$ $\hspace{5cm}$ (CNS-1)

From the O(1) terms, we obtain <u>only one constraint</u> for two unknowns.

Next we look at the O($\varepsilon$) terms.

$$A^{(0)}x^{(1)}=-A^{(1)}x^{(0)}+b^{(1)} \hspace{3cm} \text{(L-1)}$$

Matrix $A^{(0)}$ is symmetric. The null space of $A^{(0)}$ is

$$\mathrm{Nul}\left(A^{(0)}\right)=\{v_0\}, \quad v_0=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Linear system (L-1) is solvable if and only if

$$\left\langle -A^{(1)}x^{(0)}+b^{(1)},\ v_0\right\rangle=0$$

==> $\quad x_1^{(0)}-x_2^{(0)}-2=0$ $\hspace{4cm}$ (CNS-2)

Combining constraints (CNS-1) and (CNS-2), we conclude

$$x^{(0)}=\begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$$

<u>Example:</u> (the method of solvability condition)

Consider the BVP of 2nd order linear ODE

$$\begin{cases} L[u]+\varepsilon xu=3\sin(2x), & L[u]\equiv u''+u \\ u(0)=0, & u(\pi)=0 \end{cases}$$

Here we use asymptotic analysis to solve this example.

We seek solutions of the form

$$u(x)=u^{(0)}(x)+\varepsilon u^{(1)}(x)+\cdots$$

We expand the ODE into O(1) terms, O($\varepsilon$) terms, ...

$$L\left[u^{(0)''}(x)+\varepsilon u^{(1)''}(x)\right]+\varepsilon x u^{(0)}(x)-3\sin(2x)=0$$

$$\Longrightarrow \quad \left(L\left[u^{(0)}(x)\right]-3\sin(2x)\right)+\varepsilon\left(L\left[u^{(1)}(x)\right]+x u^{(0)}(x)\right)+\cdots=0$$

First we look at the O(1) terms.

$$\begin{cases} L\left[u^{(0)}\right]=3\sin(2x) \\ u^{(0)}(0)=0, \quad u^{(0)}(\pi)=0 \end{cases}$$

This BVP is <u>underdetermined</u>.

$$u^{(0)}=-\sin(2x)+c\sin(x) \quad \text{is a solution for any } c. \tag{CNS-3}$$

Next we look at the O($\varepsilon$) terms.

$$\begin{cases} L\left[u^{(1)}\right]=-x u^{(0)}(x) \\ u^{(1)}(0)=0, \quad u^{(1)}(\pi)=0 \end{cases} \tag{BVP-1}$$

Operator $L$ with zero-BCs is self-adjoint (symmetric).

The null space of $L$ with zero-BCs is

$$\text{Nul}(L)=\{v_0(x)\}, \quad v_0(x)=\sin(x)$$

(BVP-1) is solvable if and only if

$$\left\langle -x u^{(0)}(x), \ v_0\right\rangle=0$$

$$\Longrightarrow \quad \int_0^\pi \left(x\sin(2x)-c x\sin(x)\right)\sin(x)dx=0$$

$$\Longrightarrow \quad -\frac{8}{9}-c\frac{\pi^2}{4}=0 \qquad \Longrightarrow \quad c=\frac{-32}{9\pi^2} \tag{CNS-4}$$

Combining constraints (CNS-3) and (CNS-4), we conclude

$$u^{(0)}=-\sin(2x)-\frac{32}{9\pi^2}\sin(x)$$