AM 216 - Stochastic Differential Equations: Assignment 7

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Problem 1: Average Quadratic Variation of Brownian Bridge

Proof. We begin this proof by writing the distribution of the brownian bridge. We have,

$$(B(t)|W(T) = w_T) \sim \left(W(t) + \frac{t}{T}(w_T - W(T)\right)$$

Thus we may begin,

$$\begin{aligned}
\lim_{N\to\infty} E\left(\sum_{j=0}^{N-1}(\Delta W_j)^2|W(T) = 0\right) &= \lim_{N\to\infty}\sum_{j=0}^{N-1} E(\Delta W_j)^2|W(T) = 0) \\
&= \lim_{N\to\infty}\sum_{j=0}^{N-1} E\left(\Delta\tilde{W}_j^2 + \frac{\Delta t_j^2}{T^2}\tilde{W}^2(T) - 2\frac{\Delta t_j \Delta\tilde{W}_j\tilde{W}(T)}{T}\right) \\
&= \lim_{N\to\infty}\sum_{j=0}^{N-1} \Delta t_j + \frac{\Delta t_j^2}{T^2}T - 2\frac{\Delta t_j^2}{T} \\
&= \lim_{N\to\infty}\sum_{j=0}^{N-1} \Delta t_j - \frac{\Delta t_j^2}{T} \\
&= \lim_{N\to\infty} T - \frac{NT^2}{N^2T} \\
&= T
\end{aligned}$$

□

Problem 2: Microscopic Behavior at Reflecting Boundary

i) *Proof.*

$$\begin{aligned}
E(\max(0,\alpha)) &= \int_{-\infty}^{\infty}\max(\alpha, z)\rho_Z(z)dz \\
&= \int_{-\infty}^{\alpha}\alpha\rho_Z(z)dz + \int_{\alpha}^{\infty} z\rho_Z(z)dz \\
&= \alpha F_Z(\alpha) - \rho_Z(z)\Big|_{\alpha}^{\infty} \\
&= \alpha F_Z(\alpha) + \rho_Z(\alpha)
\end{aligned}$$

□

ii) *Proof.*

$$\begin{aligned} E(\max(0,Y)) &= E\left(\max\left(0, \sigma Z + \mu\right)\right) \\ &= E\left(\max\left(-\frac{\mu}{\sigma}, Z\right)\sigma + \mu\right) \\ &= \mu + \sigma\left(-\frac{\mu}{\sigma}F_Z\left(-\frac{\mu}{\sigma}\right) + \rho_Z\left(-\frac{\mu}{\sigma}\right)\right) \end{aligned}$$

□

iii) *Proof.*

$$\begin{aligned} E(dX(t)|X(t)=0) &= E(\max(0, b(0)dt + \sqrt{a(0)}dW)) \\ &= E(\max(0, \mu + \sigma Z)), \quad \mu = b(0)dt, \quad \sigma = \sqrt{a(0)dt} \\ &= b(0)dt + \sqrt{a(0)dt}\left(b(0)\sqrt{\frac{dt}{a(0)}}F_Z\left(b(0)\sqrt{\frac{dt}{a(0)}}\right) + \rho_Z\left(b(0)\sqrt{\frac{dt}{a(0)}}\right)\right) \\ &= \sqrt{a(0)dt}\rho_Z\left(b(0)\sqrt{\frac{dt}{a(0)}}\right) + O(dt) \approx \sqrt{a(0)dt}\rho_Z(0) \end{aligned}$$

□

Problem 3: The Issue of X(t) = 0: Part 1

Proof. We begin with the method of integrating factor. We have,

$$\begin{aligned} \mu(x) &= e^{\int \frac{1}{2x}dx} = \sqrt{x} \\ \frac{\partial}{\partial x}\left(T_x\mu(x)\right) &= -\frac{\mu(x)}{2x} \\ T_x &= -\frac{1}{\mu(x)}\int \frac{\mu(x)}{2x}dx \\ &= -1 + \frac{C}{\sqrt{x}} \\ T_x(1) = 0 &\implies C = 1 \\ T(x) &= -x + 2\sqrt{x} + C_2 \\ T(\varepsilon) = 0 &\implies C = \varepsilon - 2\sqrt{\varepsilon} \\ T(x) &= -x + 2\sqrt{x} + \varepsilon - 2\sqrt{\varepsilon} \end{aligned}$$

□

Problem 4: The Issue of X(t) = 0: Part 2

i) *Proof.*

$$\begin{aligned} dU &= Y^2 + dY^2 + 2Y\,dY - Y^2 \\ &= dt + \sqrt{4U}dU \end{aligned}$$

This is the exact SDE we were given for X in the last problem. □

ii) *Proof.*

$$\begin{aligned} T_{yy} &= -2 \\ T_y &= -2y + C \\ T &= -y^2 + C_1 y + C_2 \\ T'(1) = 0 \implies C_1 = 2, \quad T(\sqrt{\varepsilon}) = 0 &\implies C_2 = \epsilon - 2\sqrt{\varepsilon} \\ T(y) &= -y^2 + 2y + \varepsilon - 2\sqrt{\varepsilon} \end{aligned}$$

This is the exact same solution if we take $X = Y^2$. □

Problem 5: The Issue of X(t) = 0: Part 3

i)

$$\begin{aligned} T_{xx} - \frac{2}{x}T_x &= -\frac{2}{x^2} \\ \frac{\partial}{\partial x}\left(T_x x^{-2}\right) &= -\frac{2}{x^4} \\ T_x &= x^2\left(\frac{2}{3}x^{-3} + C\right) \\ T_x &= \frac{2}{3x} + Cx^2 \\ T &= \frac{2}{3}\ln(x) + \frac{C_1}{3}x^3 + C_2 \\ T'(1) = 0 \implies C_1 = -\frac{2}{3}, \quad T(\varepsilon) = 0 &\implies C_2 = -\frac{2}{3}\ln(\varepsilon) + \frac{2}{9}\varepsilon^3 \\ T(x) &= \frac{2}{9}(\varepsilon^3 - x^3) + \frac{2}{3}\ln\left|\frac{x}{\varepsilon}\right| \end{aligned}$$

ii) *Proof.*

$$\begin{aligned} \lim_{\varepsilon \to 0^+} T(x) &= -\frac{2}{3}x^3 + \frac{2}{3}\ln\left|\frac{x}{0^+}\right| \\ &= \infty \end{aligned}$$

□

Problem 6: Solution of dX = RHS

i)

$$\begin{aligned} dX &= X(e^{\alpha dW + \beta dt} - 1) \\ &= X\left(0 + \alpha dW(1) + \beta dt(1) + \frac{\alpha^2}{2}dW^2(1) + h.o.t.\right) \\ &= \left(\beta + \frac{\alpha^2}{2}\right)X dt + \alpha X dW \end{aligned}$$

ii) They are very similar and dX can be recovered using:

$$\begin{aligned} \sigma = \alpha, &\quad b = \beta + \frac{\sigma^2}{2} \\ \alpha = \sigma, &\quad \beta = b - \frac{\sigma^2}{2} \end{aligned}$$