AM216 Stochastic Differential Equations	Assignment 7

Q1. Average quadratic variation of Brownian bridge $\{W(t) : 0 \leqslant t \leqslant T \big| W(T) = 0\}$.

We start with the unconstrained Wiener process $\{W(t) : 0 \leqslant t \leqslant T\}$ on a grid:

$$\Delta t = \frac{T}{N}, \quad t_j = j\Delta t, \quad W_j = W(t_j), \quad \Delta W_j = W_{j+1} - W_j$$

For the unconstrained $\{W(t)\}$, the average quadratic variation is

$$E\Big((\Delta W_j)^2\Big) = \Delta t, \qquad E\Big(\sum_{j=0}^{N-1} (\Delta W_j)^2\Big) = N\Delta t = T$$

Calculate the average quadratic variation for the Brownian bridge and show that

$$\lim_{N\to\infty} E\Big(\sum_{j=0}^{N-1} (\Delta W_j)^2 \Big| W(T) = 0\Big) = T$$

<u>Hint:</u> Previously showed that the Brownian bridge has the same distribution as a tilted unconstrained $\{W(t)\}$. Use that to calculate $E\big((\Delta W_j)^2 \big| W(T) = 0\big)$.

Q2. Effect of microscopic behavior at a reflecting boundary.

Consider $dX = b(X)dt + \sqrt{a(X)}\,dW$ with a reflecting boundary at $x = 0$. The boundary condition on $u(x,t) = E\big(f(X(T))\big|X(T-t){=}x\big)$ at $x = 0$ is derived based on

$$u(x,t) = E\big(u(x+dX, t-dt)\big), \qquad E\big(dX(t)\big|X(t){=}0\big) = O(\sqrt{dt})$$

We demonstrate that the detailed microscopic behavior at $x = 0$ does not change the property $E\big(dX(t)\big|X(t){=}0\big) = O(\sqrt{dt})$. Previously, we set $\big(dX(t)\big|X(t){=}0\big) = \Big|b(0)dt + \sqrt{a(0)}\,dW\Big|$. Now we consider a different version of $\big(dX(t)\big|X(t){=}0\big)$.

$$\big(dX(t)\big|X(t){=}0\big) = \max\big(0,\ (b(0)dt + \sqrt{a(0)}\,dW)\big)$$

To show $E\big(dX(t)\big|X(t){=}0\big) = O(\sqrt{dt})$, we derive the three items below.

i) Let $Z \sim N(0,1)$. Derive

$$E\Big(\max(\alpha, Z)\Big) = \alpha F_Z(\alpha) + \rho_Z(\alpha)$$

where $\rho_Z(z) \equiv \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$ is the PDF and $F_Z(z) \equiv \frac{1}{2} + \frac{1}{2}\text{erf}(\frac{z}{\sqrt{2}})$ is the CDF of Z.

ii) Let $Y = \mu + \sigma Z \sim N(\mu, \sigma^2)$. Derive

$$E\Big(\max(0, Y)\Big) = \mu + \sigma\Big(\frac{-\mu}{\sigma} F_Z(\frac{-\mu}{\sigma}) + \rho_Z(\frac{-\mu}{\sigma})\Big)$$

iii) Use $(b(0)dt + \sqrt{a(0)}\,dW) \sim N\big(b(0)dt,\, (\sqrt{a(0)dt})^2\big)$ to derive

$$E\big(dX(t)\big|X(t)=0\big) = (\sqrt{a(0)dt})\rho_Z(0) + O(dt)$$

Q3. The issue of $X(t)$ hitting 0.

Suppose the stochastic process $X(t)$ is governed by the SDE

$$dX = dt + \sqrt{4X}dW \quad \text{(Ito interpretation)} \tag{1}$$

Let $T(x)$ be the average time until $X(t)$ exiting $(\varepsilon, 1)$ given $X(0) = x$. At the two endpoints, $x = \varepsilon$ is absorbing and $x = 1$ is reflecting. $T(x)$ is governed by the BVP below.

$$\begin{cases} 2x\,T_{xx} + T_x = -1 \\ T(\varepsilon) = 0, \quad T'(1) = 0 \end{cases}$$

Use the method of integrating factor to solve the BVP to derive

$$T(x) = (2\sqrt{x} - x) - (2\sqrt{\varepsilon} - \varepsilon)$$

<u>Remark:</u> From the result above, it follows that $\lim_{\varepsilon\to 0} T(x) = (2\sqrt{x} - x) < \infty$. That is, with a reflecting boundary at $x = 1$ and given sufficient time, $X(t)$ will hit 0.

Q4. The issue of $X(t)$ hitting 0.

Suppose the stochastic process $Y(t)$ is governed by the SDE $dY = dW$. Let $T(y)$ be the average time until $Y(t)$ exiting $(\sqrt{\varepsilon}, 1)$ given $Y(0) = y$. At the two endpoints, $y = \sqrt{\varepsilon}$ is absorbing and $y = 1$ is reflecting. $T(y)$ is governed by the BVP below.

$$\begin{cases} \frac{1}{2}T_{yy} = -1 \\ T(\sqrt{\varepsilon}) = 0, \quad T'(1) = 0 \end{cases}$$

i) Let $U(t) \equiv Y^2(t)$. Expand $dU = (Y + dY)^2 - Y^2$ and replace $(dW)^2$ with dt to derive the SDE for $U(t)$. Compare with the SDE in Q3.

ii) Solve the BVP above to find $T(y)$. Compare with $T(x)$ in Q3.

<u>Remark:</u> Based on the SDEs of $X(t)$ and $Y^2(t)$, we identify $X(t) = Y^2(t)$. The SDE of $Y(t)$ gives $Y(t) = Y(0) + W(t)$. With a reflecting boundary at $y = 1$ and given sufficient time, $Y(t)$ will hit 0. $Y(t)$ hitting 0 is much more intuitive than $X(t)$ hitting 0 (see Q5).

Q5. The issue of $X(t)$ hitting 0.

Suppose the stochastic process $X(t)$ is governed by the SDE

$$dX = -X\,dt + \sqrt{X^2}\,dW \quad \text{(Ito interpretation)} \tag{2}$$

Let $T(x)$ be the average time until $X(t)$ exiting $(\varepsilon, 1)$ given $X(0) = x$. At the two endpoints, $x = \varepsilon$ is absorbing and $x = 1$ is reflecting. $T(x)$ is governed by the BVP below.

$$\begin{cases} \dfrac{1}{2}x^2 T_{xx} - x\, T_x = -1 \\ T(\varepsilon) = 0, \quad T'(1) = 0 \end{cases}$$

i) Use the method of integrating factor to solve the BVP to derive

$$T(x) = (\frac{2}{3}\ln x - \frac{2}{9}x^3) - (\frac{2}{3}\ln \varepsilon - \frac{2}{9}\varepsilon^3)$$

ii) At $x > 0$, take the limit as $\varepsilon \to 0_+$ to conclude $\lim\limits_{\varepsilon \to 0_+} T(x) = +\infty$.

Remark: Note that in SDE (2), the deterministic drifting is negative, driving $X(t)$ toward 0. Even with this negative deterministic drifting, it takes longer and longer to go below $x = \varepsilon$ as ε is reduced. On average it takes infinitely large time for $X(t)$ to reach 0.

Q6. Solution of $dX = bX(t)dt + \sigma X(t)dW$.

Let $X(t) \equiv C\exp(\alpha W(t) + \beta t)$. We expand $dX = X(t + dt) - X(t)$ in dt and dW.

$$\begin{aligned} dX &= C\exp\big(\alpha W(t) + \beta t + \alpha dW + \beta dt\big) - C\exp\big(\alpha W(t) + \beta t\big) \\ &= C\exp\big(\alpha W(t) + \beta t\big)\Big(\exp\big(\alpha dW + \beta dt\big) - 1\Big) = \cdots \end{aligned}$$

i) Replace $(dW)^2$ with dt to derive the SDE of $X(t)$.

ii) Compare the SDE obtained in i) with the SDE

$$dX = bX(t)dt + \sigma X(t)dW \tag{3}$$

Adjust coefficients (α, β) to write out the solution of (3) in terms $(W(t), b, \sigma)$.

Q7. (Optional) Time reversal of an SDE.

Consider the discrete time version (i.e., finite dt) of the SDE below.

$$X(t + dt) = X(t) + f(X(t))dt + \sigma dW(t), \qquad dW(t) \equiv W(t + dt) - W(t)$$

Show that

$$\Big(X(t)\Big|X(t + dt) = x_1,\ X(t + 2dt) = x_2\Big) \sim \Big(X(t)\Big|X(t + dt) = x_1\Big)$$

Remark: The result shows that the time reversal of $X(t)$ is also a Markov process.