AM216 Stochastic Differential Equations

Lecture 05
Copyright by Hongyun Wang, UCSC

List of topics in this lecture

- Non-dimensionalization, advantage of working with $\dfrac{dW(t)}{\sqrt{dt}}$
- Gambler's ruin, Case 2: a biased game
- White noise dW/dt is not a regular function in the conventional sense
- Interpretation of the delta function: taking the limit after operations
- Fourier transform (FT), properties of FT

Recap

Properties of $W(t)$:

- $dW = O(\sqrt{dt})$,
- Arclength of $W(t)$ over finite time is infinity
- Ito's lemma: $(dW)^2$ can be replaced with dt.

The Gambler's ruin problem (Continued)

Case 2: a biased game

$$dX = -m\,dt + dW, \qquad m < 0$$

It is a biased game: $E_{dW}(dX) = -m\,dt < 0$; on average, you are losing money.

Before we solve case 2, let us study scaling and non-dimensionalization, which is a key component in modeling, analysis and simulations.

Scaling and non-dimensionalization

We look at the dimensions of various physical quantities

$[X]$ = \$, $\quad X$ and dX have the dimension of money

$[t]$ = time, $\quad t$ and dt have the dimension of time

$E(W^2) = t$

==> $[W]=[\sqrt{t}]=\sqrt{\text{time}}$, W and dW have the dimension of $\sqrt{\text{time}}$

The original physical equation of case 1 (before non-dimensionalization) should be

$$dX = \sqrt{\sigma^2}\, dW$$

where the dimension of σ is

$$[\sigma] = \frac{[dX]}{[dW]} = \frac{\$}{\sqrt{\text{time}}}.$$

The original physical equation of case 2 is

$$dX = -m dt + \sqrt{\sigma^2}\, dW \qquad \text{(E01)}$$

where

$$[X] = \$, \quad [dt] = \text{time}, \quad [m] = \frac{\$}{\text{time}}, \quad [\sigma] = \frac{\$}{\sqrt{\text{time}}}, \quad [dW] = \sqrt{\text{time}}$$

We re-write the physical equation as

$$dX = -m dt + \sqrt{\sigma^2 dt}\, \frac{dW(t)}{\sqrt{dt}} \qquad \text{(E02)}$$

Advantage of working with $\frac{dW(t)}{\sqrt{dt}}$

$$dW(t) \sim N(0, dt) = \sqrt{dt}\, N(0,1)$$

==> $\frac{dW(t)}{\sqrt{dt}} \sim N(0,1)$ is dimensionless and independent of dt and t.

This property is especially useful in non-dimensionalization!

Caution: $\frac{dW(t)}{\sqrt{dt}}$ is not $\frac{dW(t)}{dt}$, which we will discuss later.

Notation for dimensionless

$$\left[\frac{X}{\$}\right] = \text{one}, \quad \left[\frac{dW}{\sqrt{dt}}\right] = \text{one}$$

Caution: "one" means "dimensionless". It does not mean numerical value 1.

$$\frac{\$8}{\$} = 8, \quad \left[\frac{\$8}{\$}\right] = \text{one}$$

Objectives of non-dimensionalizing (E02)

- A dimensionless equation
- Getting rid of parameter σ.

Scales for various physical quantities

Time scale: $[t_0]$ = time

In this problem, we select the time scale t_0, for example, t_0 = 1 minute.

Money scale: $\left[\sqrt{\sigma^2 t_0}\right] = \$$

The money scale is derived from the given σ and the selected time scale t_0.

Non-dimensional quantities

Non-dimensional time: $t_{\text{ND}} = \dfrac{t}{t_0}$

Non-dimensional money: $X_{\text{ND}} = \dfrac{X}{\sqrt{\sigma^2 t_0}}$

Non-dimensional equation

Start with the physical equation: $dX = -m\,dt + \sqrt{\sigma^2 dt}\,\dfrac{dW(t)}{\sqrt{dt}}$

We write all physical quantities in terms of non-dimensional quantities and then substitute into the physical equation

$$t = t_0\, t_{\text{ND}}\,, \qquad X = \sqrt{\sigma^2 t_0}\, X_{\text{ND}}$$

$$dt = t_0\, dt_{\text{ND}}\,, \qquad dX = \sqrt{\sigma^2 t_0}\, dX_{\text{ND}}\,, \qquad \frac{dW(t)}{\sqrt{dt}} = \frac{dW(t_{\text{ND}})}{\sqrt{dt_{\text{ND}}}}$$

Recall $\boxed{\dfrac{dW(t)}{\sqrt{dt}} \sim N(0,1) \text{ is dimensionless and independent of } dt \text{ and } t.}$

==> $$\sqrt{\sigma^2 t_0}\, dX_{\text{ND}} = -m t_0 dt_{\text{ND}} + \sqrt{\sigma^2 t_0 dt_{\text{ND}}}\,\frac{dW(t_{\text{ND}})}{\sqrt{dt_{\text{ND}}}}$$

Divide the equation by $\sqrt{\sigma^2 t_0}$, we obtain

$$dX_{\text{ND}} = -m\frac{t_0}{\sqrt{\sigma^2 t_0}}dt_{\text{ND}} + \sqrt{dt_{\text{ND}}}\,\frac{dW(t_{\text{ND}})}{\sqrt{dt_{\text{ND}}}}$$

Re-writing it in terms of dW, we arrive at the dimensionless equation

$$dX_{\mathrm{ND}} = -m_{\mathrm{ND}} dt_{\mathrm{ND}} + dW(t_{\mathrm{ND}}), \qquad m_{\mathrm{ND}} \equiv m\sqrt{\frac{t_0}{\sigma^2}}$$

Once we have the dimensionless equation, we can drop the subscript "ND" and revert back to the simple notation $(X,\ t,\ C,\ m)$.

$$dX = -mdt + dW$$

Summary

When we work with the dimensionless equation, $dX = -mdt + dW$,

we need to keep in mind that

$$X = \frac{X_{\mathrm{phy}}}{\sqrt{\sigma^2 t_0}}, \qquad C = \frac{C_{\mathrm{phy}}}{\sqrt{\sigma^2 t_0}}, \qquad t = \frac{t_{\mathrm{phy}}}{t_0}, \qquad m = m_{\mathrm{phy}}\sqrt{\frac{t_0}{\sigma^2}}$$

where the subscript $_{\mathrm{phy}}$ denotes the physical quantity before non-dimensionalization.

Solutions of case 2

For the biased game, we again study the two questions.

Question #1: How long can you play?

Question #2: What is the chance that you break the bank?

Answer to Question #2

The strategy we use is the same as that in case 1.

Let $\quad u(x) = \Pr\left(A \middle| X(0) = x\right), \quad A \equiv \{X(t) \text{ hits C before } 0\}$

Strategy:

Find a boundary value problem (BVP) governing $u(x)$.

Boundary condition:

$u(C) = 1$ and $u(0) = 0$.

Differential equation:

Start with $X(0) = x \in (0, C)$. After a small time step, $X(dt)$ has the expression

$$X(dt) = x + dX, \qquad dX = -m\,dt + dW$$

We need to calculate moments of dX.

$$E_{dW}(dX) = E_{dW}(-mdt + dW) = -mdt$$

$$E_{dW}\left((dX)^2\right) = E_{dW}\left((-mdt)^2 + 2(-mdt)dW + (dW)^2\right) = dt + o(dt)$$

For a fixed $x \in (0, C)$, when dt is small enough (depending on x), we have

$$u(x) = E_{dW}\Big(u(x+dX)\Big) + o(dt) \qquad \text{(The law of total probability)}$$

$$= E_{dW}\left(u(x) + u_x dX + \frac{1}{2}u_{xx}(dX)^2 \right) + o\Big(dt\Big)$$

(This is where we need moments of dX.)

$$= u(x) - u_x m dt + \frac{1}{2}u_{xx}dt + o\Big(dt\Big)$$

Divide by dt and then take the limit as $dt \to 0$, we obtain a 2nd order ODE for $u(x)$

$$u_{xx} - 2mu_x = 0$$

Function $u(x)$ satisfies the boundary value problem (BVP)

$$\begin{cases} u_{xx} - 2mu_x = 0 & \text{differential equation} \\ u(0) = 0, \quad u(C) = 1 & \text{boundary conditions} \end{cases}$$

Solution of the BVP:

$$u(x) = \frac{e^{-2mC}(e^{2mx} - 1)}{1 - e^{-2mC}} \qquad \text{(homework problem)}$$

When mC is moderately large (for example, $mC \geq 5$), we have

$$u(x) = \frac{e^{-2mC}(e^{2mx} - 1)}{1 - e^{-2mC}} \approx e^{-2mC}(e^{2mx} - 1)$$

which decays exponentially with the factor e^{-2mC}.

Comparison of fair game vs biased game

We look at $u(C/2)$, the probability of breaking the bank when you and the casino start with the same amount of cash, $(C/2)$.

Fair game: $u(x) = x/C$ ==> $u\left(\frac{C}{2}\right) = \frac{1}{2}$

Biased game: $u\left(\frac{C}{2}\right) \approx e^{-2mC}(e^{2mC/2} - 1) \approx e^{-mC}$

which decays exponentially with mC.

For example, $mC = 5$ ==> $e^{-mC} = e^{-5} = 0.0067$.

Answer to Question #1

Let $\quad T(x) = E\left(Z \,\middle|\, X(0) = x\right), \quad Z \equiv \left(\text{time from 0 until } X(t) = C \text{ or } X(t) = 0\right).$

Strategy:

Find a boundary value problem (BVP) governing $T(x)$.

Boundary condition:

$T(C) = 0$ and $T(0) = 0$.

Differential equation:

Start with $X(0) = x \in (0, C)$. After a small time step, $X(dt)$ has the expression

$X(dt) = x + dX, \qquad dX = -m\, dt + dW$

The moments of dX are

$$E_{dW}(dX) = -mdt\,, \qquad E_{dW}\left((dX)^2\right) = dt + o(dt)$$

For a fixed $x \in (0, \mathrm{C})$, when dt is small enough (depending on x), we have

$$T(x) = dt + E_{dW}\left(T(x+dX)\right) + o(dt) \qquad \text{(The law of total expectation)}$$

$$= dt + E_{dW}\left(T(x) + T_x dX + \frac{1}{2}T_{xx}(dX)^2\right) + o(dt)$$

(This is where we need moments of dX.)

$$= dt + T(x) - T_x m dt + \frac{1}{2}T_{xx} dt + o(dt)$$

Divide by dt and then take the limit as $dt \to 0$, we obtain an ODE for $T(x)$

$$T_{xx} - 2mT_x = -2$$

Function $T(x)$ satisfies the boundary value problem (BVP)

$$\begin{cases} T_{xx} - 2mT_x = -2 & \text{differential equation} \\ T(0) = 0, \quad T(C) = 0 & \text{boundary conditions} \end{cases}$$

The solution of the BVP:

$$T(x) = \frac{x}{m} - \frac{C}{m}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right) \qquad \text{(homework problem)}$$

When mC is moderately large (for example, $mC \geq 5$), we have

$$T(x) = \frac{x}{m}\cdot\left(1 - \frac{C}{x}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right)\right) \approx \frac{x}{m} \qquad \text{for } x \leq \frac{C}{2}$$

Here we have used $\dfrac{C}{x}\left(\dfrac{e^{2mx}-1}{e^{2mC}-1}\right) << 1$, which is derived in Appendix A.

The result, $T(x) \approx x/m$, is consistent with the intuitive picture that if your cash decreases with speed m, then your initial cash x will last a time period of (x/m).

Meaning of $mC \geq 5$ in terms of physical quantities:

$$m = m_{\text{phy}}\sqrt{\frac{t_0}{\sigma^2}}, \qquad C = \frac{C_{\text{phy}}}{\sqrt{\sigma^2 t_0}}, \qquad x = \frac{x_{\text{phy}}}{\sqrt{\sigma^2 t_0}}$$

$$\Longrightarrow \qquad mC = \frac{m_{\text{phy}} C_{\text{phy}}}{\sigma^2}$$

$mC \geq 5$ corresponds to $\dfrac{m_{\text{phy}} C_{\text{phy}}}{\sigma^2} \geq 5$.

An example (with physical parameters)

Consider a biased game with physical parameters below.

$$\sigma = 5\frac{\$}{\sqrt{\text{min}}}, \qquad m_{\text{phy}} = 0.25\frac{\$}{\text{min}}$$

$$C_{\text{phy}} = 1000\$, \qquad x_{\text{phy}} = 500\$$$

The scales are

t_0 = 1 min (we select t_0), $\qquad \sqrt{\sigma^2 t_0} = 5\$$

The dimensionless quantities are

$$m = m_{\text{phy}}\sqrt{\frac{t_0}{\sigma^2}} = 0.05, \qquad C = \frac{C_{\text{phy}}}{\sqrt{\sigma^2 t_0}} = 200, \qquad mC = 10$$

$$x = \frac{x_{\text{phy}}}{\sqrt{\sigma^2 t_0}} = 100, \qquad mx = 5$$

Since mC = 10, the approximate expressions for $u(x)$ and $T(x)$ are valid.

Probability of breaking the bank:

$$u(x) \approx e^{-2mC}(e^{2mx}-1) = e^{-20}(e^{10}-1) \approx e^{-10} = 4.54\times 10^{-5}$$

The chance of breaking the bank is virtually zero even though you and the casino start with the same amount \$500.

Average time until the end of game:

$$T(x) \approx \frac{x}{m} = \frac{100}{0.05} = 2000$$

The physical time until the end of game is

$T_{\text{phy}} = T\, t_0 = 2000$ minutes.

White noise $\dfrac{dW}{dt}$

Consider the stochastic differential equation (SDE)

$$dX = -m\,dt + dW$$

We write the "formal" derivative of X as

$$\frac{dX}{dt} = -m + \frac{dW}{dt}$$

Recall that in SDE, dt is finite until we take the limit as $dt \to 0$.

Here $\lim\limits_{dt \to 0} \dfrac{dW}{dt}$ does not exist in the conventional sense.

Key strategy: We take the limit AFTER its interactions with other entities.

The short story of white noise

1) $Z(t) \equiv \dfrac{dW}{dt} = \dfrac{1}{\sqrt{dt}} \cdot \dfrac{dW}{\sqrt{dt}}, \qquad \dfrac{dW}{\sqrt{dt}} \sim N(0,1)$

   $Z(t)$ diverges to $\pm\infty$ as $dt \to 0$. $Z(t)$ is not a regular function.

2) $E\big(Z(t)Z(s)\big) = \delta(t-s)$

3) $\int e^{-i2\pi\xi t} E\big(Z(t)Z(0)\big)\, dt = 1$

4) $Z(t)$ is a white noise (we will clarify what that means).

Before we discuss the details in the long story of white noise, we review some of the mathematical tools/methods we will use.

Mathematical preparations

Delta function (Dirac's delta function):

Definition 1:

Consider the limit of a boxcar function.

$$\lim_{d \to 0} \Pi_d(x)$$

where $\Pi_d(x) = \begin{cases} \dfrac{1}{d}, & \text{for } x \in \left(\dfrac{-d}{2}, \dfrac{d}{2}\right) \\ 0, & \text{otherwise} \end{cases}$

This limit does not exist in the conventional sense.

However, for any smooth function $g(x)$, we have

$$\lim_{d\to 0} \int \Pi_d(x) g(x) dx = g(0)$$

We "formally" denote $\lim_{d\to 0} \Pi_d(x)$ by $\delta(t)$.

$$\delta(x) = \lim_{d\to 0} \Pi_d(x)$$

$\delta(x)$ "formally" satisfies

$$\int \delta(x) g(x) dx = g(0) \qquad \text{for all smooth functions } g(x).$$

The true meaning of the LHS is $\int \delta(x) g(x) dx \xrightarrow{\text{Defined as}} \lim_{d\to 0} \int \Pi_d(x) g(x) dx$

The <u>key strategy</u> in making sense of $\lim_{d\to 0} \Pi_d(x)$ is that we take the limit AFTER integrating $\Pi_d(x)$ with smooth function $g(x)$.

<u>Definition 2:</u>

In a similar way, we can define δ(x) as the limit of a normal distribution

$$\delta(x) = \lim_{\sigma\to 0} \rho_{N(0,\sigma^2)}(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

These two definitions are equivalent. We use whichever is convenient.

<u>Remark:</u>

Both definitions are based on the probability density of a scaled random variable with the multiplier converging to zero.

<u>Definition 1:</u> $\delta(x) = \lim_{\sigma\to 0} \rho_{\sigma X}(x)$, $\quad X \sim \text{uniform}\left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$

<u>Definition 2:</u> $\delta(x) = \lim_{\sigma\to 0} \rho_{\sigma X}(x)$, $\quad X \sim N(0,1)$

<u>Fourier transform</u> (FT):

Forward transform:

$$\underbrace{\hat{y}(\xi)}_{\text{Notation}} \equiv \underbrace{F[y(t)]}_{\text{Operator notation}} \equiv \int_{-\infty}^{+\infty} \exp(-i2\pi\xi t)\, y(t)\, dt$$

Inverse transform:

$$y(t) = F^{-1}[\hat{y}(\xi)] \equiv \int_{-\infty}^{+\infty} \exp(i2\pi\xi t)\, \hat{y}(\xi)\, d\xi$$

Remark: There are several versions of FT. They are all equivalent by a scaling.

Alternative FT 1: $F[y(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-i\omega t)\, y(t)\, dt$

Alternative FT 2: $F[y(t)] = \int_{-\infty}^{+\infty} \exp(-i\omega t)\, y(t)\, dt$

Properties of Fourier transform:

1) Fourier transform of a normal PDF

$$F\left[\rho_{N(0,\sigma^2)}(t)\right] = F\left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-t^2}{2\sigma^2}\right)\right] = \exp\left(-2\pi^2\sigma^2\xi^2\right)$$

Proof:

Use the characteristic function (CF) of a normal RV we derived in Lecture 3. Then use the connection between the CF and the Fourier transform of PDF.

2) Fourier transform of the delta function

$F[\delta(t)] = 1$

Proof: We view the delta function as the limit of normal distribution

$$\delta(t) = \lim_{\sigma \to 0} \rho_{N(0,\sigma^2)}(t)$$

We apply the Fourier transform and then take the limit as $\sigma \to 0$.

$$F\left[\delta(t)\right] = \lim_{\sigma \to 0} F\left[\rho_{N(0,\sigma^2)}(t)\right] = \lim_{\sigma \to 0} \exp\left(-2\pi^2\sigma^2\xi^2\right) = 1$$

3) Fourier transform of $y(t) \equiv 1$

$F[1] = \delta(\xi)$

Proof: $F[1] = \int_{-\infty}^{+\infty} \exp(-i2\pi\xi t)\, dt$ does not converge in the conventional sense!

We view 1 as the limit of

$$1 = \lim_{\sigma \to \infty} \exp\left(\frac{-t^2}{2\sigma^2}\right)$$

We apply the Fourier transform and then take the limit as $\sigma \to \infty$.

$$F[1] = \lim_{\sigma\to\infty} F\left[\exp\left(\frac{-t^2}{2\sigma^2}\right)\right], \qquad \rho_{N(0,\sigma^2)}(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-t^2}{2\sigma^2}\right)$$

$$= \lim_{\sigma\to\infty}\sqrt{2\pi\sigma^2}\,F\left[\rho_{N(0,\sigma^2)}(t)\right] = \lim_{\sigma\to\infty}\sqrt{2\pi\sigma^2}\exp\left(-2\pi^2\sigma^2\xi^2\right)$$

$$= \lim_{\sigma\to\infty}\frac{1}{\sqrt{2\pi\left(\frac{1}{4\pi^2\sigma^2}\right)}}\exp\left(\frac{-\xi^2}{2\left(\frac{1}{4\pi^2\sigma^2}\right)}\right) = \lim_{\sigma\to\infty}\rho_{N\left(0,\frac{1}{4\pi^2\sigma^2}\right)}(\xi)$$

$$= \lim_{s\to 0}\rho_{N(0,s^2)}(\xi) = \delta(\xi)$$

<u>Key observation:</u>

If an operator acting on the limit of a function is invalid in the conventional sense, we can try to <u>make sense</u> of it by <u>delaying taking the limit</u>. That is, we first apply the operator and then we take the limit afterwards. That is why in the discussion of stochastic differential equations, *dt* is finite until we take the limit at the end.

<u>Example:</u>

"Formally" we can conveniently write

$$\int \underbrace{\lim_{\sigma\to 0}\rho_{N(0,\sigma^2)}(t)}_{\text{Not a regular function}}\, g(t)\,dt = \int \delta(t)g(t)\,dt = g(0)$$

The true mathematical meaning is

$$\lim_{\sigma\to 0}\int \rho_{N(0,\sigma^2)}(t)g(t)\,dt = g(0)$$

which makes sense and is mathematically rigorous.

Appendix A

Theorem: When mC is moderately large and $x \le C/2$, we have

$$\frac{C}{x}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right) << 1$$

Proof: We first introduce a lemma.

Lemma: Function $f(s) \equiv \dfrac{e^{s}-1}{s}$ increases monotonically for $s > 0$.

Proof:

$$\frac{e^{s}-1}{s} = \frac{1}{s}\left(\sum_{n=0}\frac{1}{n!}s^{n}-1\right) = \sum_{n=1}\frac{1}{n!}s^{n-1}$$

Each term in the summation is positive, and increases monotonically for $s > 0$.

End of proof

Apply the lemma to $\dfrac{e^{2mx}-1}{x}$ for $x \le C/2$, we get

$$\frac{e^{2mx}-1}{x} = 2m\cdot\underbrace{\frac{(e^{2mx}-1)}{2mx}}_{f(2mx)} \le 2m\cdot\underbrace{\frac{(e^{2mC/2}-1)}{2mC/2}}_{f(2mC/2)} = \frac{2(e^{mC}-1)}{C}$$

Using this inequality, we write $\dfrac{C}{x}\left(\dfrac{e^{2mx}-1}{e^{2mC}-1}\right)$ as

$$\frac{C}{x}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right) = \frac{C}{(e^{2mC}-1)}\left(\frac{e^{2mx}-1}{x}\right) \le \frac{C}{(e^{2mC}-1)}\frac{2(e^{mC}-1)}{C}$$

$$= \frac{2(e^{mC}-1)}{(e^{2mC}-1)} = \frac{2}{e^{mC}+1} \approx 2e^{-mC} << 1 \qquad \text{for moderately large } mC$$

Therefore, we conclude $\dfrac{C}{x}\left(\dfrac{e^{2mx}-1}{e^{2mC}-1}\right) << 1$ when mC is moderately large and $x \le C/2$.