AM216 Stochastic Differential Equations	Assignment 1

Q1. Basic concepts of probability.

Roll a set of two fair 6-sided dice, one colored red the other white. Assume that the two dice roll independently. Record the two numbers facing up as X_1 and X_2.

i) Mathematically describe the format of outcome. Describe the sample space.

ii) Let X be the absolute difference between X_1 and X_2. Is X a random variable?

iii) Find the PMF of X. Find $E(X)$.

iv) Let $A = \{X_2^2 \geqslant 2X_1\}$. Find $\Pr(A|X_1 = n)$ for $n = 1, 2, \ldots, 6$. Use the law of total probability to find $\Pr(A)$.

Q2. Properties of Var( ).

i) Use the definition, $\text{Var}(Y) \equiv E(Y^2)–E(Y)^2$, to derive $\text{Var}(\alpha X) = \alpha^2 \,\text{Var}(X)$.

ii) Finish the proof that if X and Y are independent, then we have $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

iii) Find an example demonstrating that $\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y)$ does not imply X and Y are independent.

Q3. Properties of $E($ $)$ and Var( ).

Let $X \sim \text{Bino}(n, p)$ and $Y \sim N(\mu, \sigma^2)$. We know that

$$E(X) = np, \qquad \text{Var}(X) = np(1–p), \qquad E(Y) = \mu, \qquad \text{Var}(Y) = \sigma^2$$

Note that X and Y are not given as independent. Calculate $E(X^2 + Y^2)$.

Q4. Probability of a normal random variable falling in a given interval.

Let $X \sim N(\mu, \sigma^2)$. Finish the derivation of the inequality below.

$$\Pr\Big(\mu - \eta\sigma \leqslant X \leqslant \mu + \eta\sigma\Big) = F_X(\mu + \eta\sigma) - F_X(\mu - \eta\sigma) = \cdots = \text{erf}\left(\frac{\eta}{\sqrt{2}}\right)$$

The results above gives a numerical formula for calculating the probability on the left since the error function erf( ) is available in all packages.

Q5. A simple convoluted experiment.

Consider a coin with Pr(“head”) $= p$. In the first part of an experiment, we first flip the coin n times where n is a prescribed fixed number. Let

$$Y = \text{number of heads in the sequence of } n \text{ flips in the first part}$$

Then in the second part of the experiment, we flip the coin Y times. Let

$$X = \text{number of heads in the sequence of } Y \text{ flips in the second part}$$

Use the law of total expectation to calculate

i) $E(X)$, and

ii) $E(XY)$

Hint: $Y \sim \text{Bino}(n, p), \quad (X \big| Y = m) \sim \text{Bino}(m, p)$.

Q6. Verify the concept of confidence interval.

Write a code to draw numerically a data set of $n = 10$ independent samples of $X \sim N(\mu, \sigma^2)$ with $\mu = 0.6,\ \sigma = 1.3$.

$$(\text{A data set}) = \{X_j,\ j = 1, 2, \ldots, n\}$$

In real applications, one data set is measured while the true values of μ and σ are unknown. We estimate the mean μ as the sample mean of the data set. To illustrate the statistical behavior of the process, we repeat this experiment $M = 500,000$ times.

i) Consider the idealized case where $\sigma = 1.3$ is given. For each data set, calculate the exact 95% confidence interval for estimating μ. Out of $M = 500,000$ repeats, calculate the fraction in which the confidence interval actually contains the true μ. Report the fraction.

ii) Consider the realistic case where σ is unknown. For each data set, calculate the approximate 95% confidence interval for estimating μ.

$$\hat{\sigma} = \sqrt{\frac{1}{(n-1)} \sum_{j=1}^{n} (X_j - \hat{\mu})^2}, \qquad \hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} X_j$$

Out of $M = 500,000$ repeats, calculate the fraction in which the confidence interval actually contains the true μ. Report the fraction.

Q7. Analytical utility of characteristic functions.

A normal distribution is defined by its probability density

$$\rho_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

The corresponding characteristic function is

$$\phi_X(\xi) = \exp\left(i\mu\xi - \frac{\sigma^2\xi^2}{2}\right)$$

Use the properties of characteristic function $\phi_X(\xi)$ to show

i) $E(X) = \mu$, and

ii) $\text{Var}(X) = \sigma^2$

Hint: It is more convenient to differentiate

$$E\Big(\exp(i\xi(X-\mu))\Big) = \phi_X(\xi)\exp(-i\xi\mu) = \exp\left(-\frac{\sigma^2\xi^2}{2}\right)$$

The results above justify that we name parameters μ and σ^2 in the probability density, respectively, as the mean and the variance. This approach does not involve integration by parts. This approach will be very useful in the discussion of multivariate Gaussian.

Q8. An example illustrating that the well-posedness of a probability problem depends on whether or not we have enough information to repeat the random experiments.
Consider a bin containing two types of coins in unknown proportions.

$$\text{Coin type 1:}\ \ \Pr(\text{“head”}) = 0.5$$
$$\text{Coin type 2:}\ \ \Pr(\text{“head”}) = 0.7$$

Take a coin randomly from the bin and flip it. We study the probability that the coin is type 2 when the flip outcome is “head”. Mathematically, we study the conditional probability

$$\Pr(\text{“type 2”} \big| \text{“head”})$$

i) Is this probability problem completely specified (i.e, we have enough information to repeat the experiment in a lab setting)?

ii) If not, What information we might impose to make it well-posed?