AM216 Stochastic Differential Equations

Lecture 08
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List of topics in this lecture

- OU process (continued), solution of particle position $X(t)$
- Behavior of $X(t)$, diffusion coefficient, converging to $W(t)$
- Going backward in time using Bayes theorem
- Time reversibility of an equilibrium system
- Different interpretations of stochastic integrals

Recap

Ornstein-Uhlenbeck process (OU):

$$m\,dY = \underbrace{-bY\,dt}_{\text{dissipation}} + \underbrace{q\,dW}_{\text{fluctuation}}\ , \qquad q=\sqrt{2k_B T b}$$

Four goals of the discussion

Goal 1: Solve for $Y(t)$, the particle velocity

$$\Big(Y(t_0+t)\Big|Y(t_0)=y_0\Big) \sim N\left(e^{-\beta t}y_0,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\right) \qquad \text{for } t>0$$

$$\text{Equilibrium:}\quad Y(t) \sim N\left(0,\ \frac{\gamma^2}{2\beta}\right) \qquad \text{for large } t>0$$

Goal 2A: $Y(t)$ is a colored noise

Goal 2B: $Y(t)$ converges to a white noise as "m converges to zero"

Goal 3: Fluctuation-dissipation theorem: $q=\sqrt{2k_B T b}$.

Goal 4: Study the behavior of $X(t)$, the particle position

$$Y(t)=e^{-\beta t}Y(0)+e^{-\beta t}G(t)\,, \qquad G(t)\equiv\int_0^t \gamma e^{\beta s}\,dW(s)$$

$$X(t)-X(0)=\int_0^t Y(s)ds=\frac{1}{\beta}(1-e^{-\beta t})Y(0)+\frac{\gamma}{\beta}G_2(t)$$

where $G_2(t) \equiv \int_0^t (1-e^{-\beta(t-s)}) dW(s) \sim$ normal.

Goal 4: (continued): We calculate the mean and variance of $G_2(t)$.

$$E\big(G_2(t)\big) = \int_0^t (1-e^{-\beta(t-s)}) E\big(dW(s)\big) = 0$$

$$\mathrm{var}\big(G_2(t)\big) = \int_0^t (1-e^{-\beta(t-s)})^2 ds = t - \frac{2}{\beta}(1-e^{-\beta t}) + \frac{1}{2\beta}(1-e^{-2\beta t})$$

We write out the distribution of $(X(t) - X(0))$.

$$\big(X(t) - X(0)\big) \sim \frac{(1-e^{-\beta t})}{\beta} Y(0) + \left(\frac{\gamma}{\beta}\right) \underbrace{N\left(0, \left(t - \frac{2(1-e^{-\beta t})}{\beta} + \frac{(1-e^{-2\beta t})}{2\beta}\right)\right)}_{\text{containing } dW\text{'s in } [0,t]} \qquad \text{(E01)}$$

Remark:

We cannot integrate $G(t)$ directly because $G(t_1)$ and $G(t_2)$ are not independent. We need to write the integral as a sum of dW's.

In Goal 4, we discuss two cases for $X(t)$.

Goal 4A: finite m

We show that over long time, $(X(t) - X(0))$ demonstrates a diffusion behavior.

The diffusion coefficient is defined as

$$D \equiv \lim_{t\to\infty} \frac{1}{2t} \mathrm{var}\big(X(t) - X(0)\big)$$

We use (E01) to show the limit exists and to calculate the limit.

$$D \equiv \lim_{t\to\infty} \frac{1}{2t} \mathrm{var}\big(X(t) - X(0)\big) = \frac{1}{2}\left(\frac{\gamma}{\beta}\right)^2$$

Substituting $\beta = \frac{b}{m}$, $\gamma = \frac{q}{m}$, and $q = \sqrt{2k_B T b}$, we have

$$\left(\frac{\gamma}{\beta}\right)^2 = \frac{q^2}{b^2} = \frac{2k_B T b}{b^2} = \frac{2k_B T}{b} \qquad \text{(E02)}$$

Thus, we arrive at $\boxed{D = \frac{k_B T}{b}}$.

This is called the Einstein-Smoluchowski relation.

It relates the drag coefficient to the diffusion coefficient.

Remark: The diffusion coefficient is independent of the mass density of the particle. It is affected by the particle size via the drag coefficient b.

Goal 4B: $m \to 0$ (while b and q stay unchanged)

We show that $(X(t) - X(0))$ converges to $\sqrt{2D}\,W(t)$ on any discrete time grid.

Specifically, we show that for $t_2 > t_1 > 0$, as $m \to 0$, we have

- $X(t_1) - X(0) \to \sqrt{2D}\,N(0, t_1)$
- $X(t_1 + t_2) - X(t_1) \to \sqrt{2D}\,N(0, t_2)$
- $(X(t_1) - X(0))$ and $(X(t_1+t_2) - X(t_1))$ are independent.

Using (E01), we write $(X(t_1) - X(0))$ as

$$\big(X(t_1) - X(0)\big) \sim (1 - e^{-\beta t_1})\frac{Y(0)}{\beta} + \sqrt{2D}\,\underbrace{N\left(0, \left(t_1 - \frac{2(1-e^{-\beta t_1})}{\beta} + \frac{(1-e^{-2\beta t_1})}{2\beta}\right)\right)}_{\text{containing } dW\text{'s in } [0, t_1]}$$

As $m \to 0$, we have

$$\beta = \frac{b}{m} = O(m^{-1}), \qquad \gamma = \frac{q}{m} = O(m^{-1}), \qquad \frac{\gamma}{\beta} = O(1)$$

$$2D \equiv \left(\frac{\gamma}{\beta}\right)^2 = O(1) \quad \text{and} \quad \frac{1}{\beta}(1 - e^{-\beta t_1}) = O(m) \to 0$$

Caution: $\lim_{m\to 0} |Y(0)| = \infty$. The Maxwell-Boltzmann distribution gives

$$Y(0) \sim N\left(0, \frac{\gamma^2}{\beta}\right) = O\left(\sqrt{\frac{\gamma^2}{\beta}}\right) = O(m^{-0.5})$$

$$\Longrightarrow \qquad \frac{Y(0)}{\beta} = O(m^{0.5}) \to 0$$

Taking the limit as $m \to 0$, we obtain

- $\big(X(t_1) - X(0)\big) \xrightarrow{\text{as } m\to 0} \sqrt{2D}\ \underbrace{N(0, t_1)}_{\substack{\text{containing } dW\text{'s} \\ \text{in } [0, t_1]}}$

Similarly, we have

$$\left(X(t_1+t_2)-X(t_1)\right) \sim (1-e^{-\beta t_2})\frac{Y(t_1)}{\beta}+\sqrt{2D}\,\underbrace{N\left(0,\left(t_2-\frac{2(1-e^{-\beta t_2})}{\beta}+\frac{(1-e^{-2\beta t_2})}{2\beta}\right)\right)}_{\text{containing } dW\text{'s in } [t_1,t_1+t_2]}$$

- $$\left(X(t_1+t_2)-X(t_1)\right) \xrightarrow{\text{as } m\to 0} \sqrt{2D}\,\underbrace{N(0,t_2)}_{\substack{\text{containing } dW\text{'s} \\ \text{in}[t_1,t_1+t_2]}}$$

Notice that $\left(X(t_1+t_2)-X(t_1)\right)-(1-e^{-\beta t_2})\dfrac{Y(t_1)}{\beta}$ contains dW's in $[t_1, t_1+t_2]$.

Since $(1-e^{-\beta t_2})\dfrac{Y(t_1)}{\beta}=O(m^{0.5})\to 0$ as $m \to 0$, we arrive at

- $(X(t_1)-X(0))$ and $(X(t_1+t_2)-X(t_1))$ are independent in the limit of $m \to 0$.

Therefore, as $m \to 0$, $(X(t) - X(0))$ converges to $\sqrt{2D}\,W(t)$ on any discrete time grid.

Remarks:

1. The diffusion coefficient of the standard Wiener process is 1/2 (not 1).

$$D_{\text{Wiener}} \equiv \frac{1}{2t}\text{var}\left(W(t)\right)=\frac{1}{2}$$

2. In the limit of $m \to 0$, $(X(t) - X(0))$ exhibits the behavior of a scaled Wiener process, called the Brownian motion, named after Scottish botanist Robert Brown.
3. The derivation above is for the "simplified story". The real story where radius $a \to 0$ while ρ_{mass} is fixed, is presented in Appendix A.

Going backward in time in an equilibrium OU process

In the discussion of Goals #1–4 above, we focused on going forward in time.

$$E\left(Y(t)\middle|Y(0)\right)=e^{-\beta t}Y(0) \qquad \text{for } t > 0$$

Question:

What happens for $(-t) < 0$? Do we have

$$E\left(Y(-t)\middle|Y(0)\right)=e^{+\beta t}Y(0) \ ?$$

which diverges to infinity as $t \to +\infty$. That seems unreasonable.

Answer: $t_{\text{new}} = -t_{\text{old}}$ does not work in stochastic differential equations.

Recall that when we scale dW, it is best to work with $\dfrac{dW}{\sqrt{dt}}$

$$dW(t)=\sqrt{dt}\cdot\frac{dW(t)}{\sqrt{dt}}, \qquad \frac{dW(t)}{\sqrt{dt}}\sim N(0,1) \quad \text{independent of } t \text{ and } dt$$

It is clear that this works only for $dt > 0$, not for $t_{\text{new}} = -t_{\text{old}}$.

Key point:

In stochastic differential equations, scaling $t_{\text{new}} = -t_{\text{old}}$ does not work!

Bayes theorem describes Pr(A | B) in terms of Pr(B | A). We use Bayes theorem to calculate the backward time evolution based on the forward time evolution.

Bayes theorem for densities:

$$\rho\Big(Y(-t)=y_1\Big|Y(0)=y_2\Big) \propto \rho\Big(Y(0)=y_2\Big|Y(-t)=y_1\Big)\cdot\rho\Big(Y(-t)=y_1\Big)$$

Backward time evolution in an equilibrium OU process

We assume that the equilibrium has been reached long time ago (at $t = -\infty$) and $Y(t)$ is already a stationary process for all t (including negative t). In particular, the unconstrained $Y(t)$ has the equilibrium distribution for all t.

$$Y(-t) \sim N\left(0,\frac{\gamma^2}{2\beta}\right)$$

$$\Longrightarrow \quad \rho\Big(Y(-t)=y_1\Big) \propto \exp\left(\frac{-y_1^{\,2}}{2\gamma^2/(2\beta)}\right)$$

For the forward time evolution, we already derived

$$\Big(Y(t_1+t)\Big|Y(t_1)=y_1\Big) \sim N\left(e^{-\beta t}y_1,\ \frac{\gamma^2}{2\beta}\Big(1-e^{-2\beta t}\Big)\right) \qquad \text{for } t>0 \text{ and any } t_1$$

$$\Longrightarrow \quad \rho\Big(Y(0)=y_2\Big|Y(-t)=y_1\Big)\propto\exp\left(\frac{-(y_2-e^{-\beta t}y_1)^2}{2(1-e^{-2\beta t})\gamma^2/(2\beta)}\right)$$

Substituting into Bayes theorem, we obtain

$$\rho\Big(Y(-t)=y_1\Big|Y(0)=y_2\Big) \propto \rho\Big(Y(0)=y_2\Big|Y(-t)=y_1\Big)\cdot\rho\Big(Y(-t)=y_1\Big)$$

$$\propto\exp\left(\frac{-(y_2-e^{-\beta t}y_1)^2}{2(1-e^{-2\beta t})\gamma^2/(2\beta)}\right)\cdot\exp\left(\frac{-y_1^{\,2}}{2\gamma^2/(2\beta)}\right)$$

Note that here y_1 is the independent variable of the PDF and we only need to keep track factors that depend on y_1.

$$\rho\Big(Y(-t)=y_1\Big|Y(0)=y_2\Big)\propto\exp\left(\frac{-\Big[e^{-2\beta t}y_1^{\,2}-2e^{-\beta t}y_2\cdot y_1+(1-e^{-2\beta t})y_1^{\,2}\Big]}{2(1-e^{-2\beta t})\gamma^2/(2\beta)}\right)$$

$$\propto \exp\left(\frac{-\left[y_1^{\,2}-2e^{-\beta t}y_2\cdot y_1\right]}{2(1-e^{-2\beta t})\gamma^2/(2\beta)}\right)\propto \exp\left(\frac{-(y_1-e^{-\beta t}y_2)^2}{2(1-e^{-2\beta t})\gamma^2/(2\beta)}\right)$$

We recognize that this is a normal distribution.

It follows that in an equilibrium system, the backward time evolution is described by

$$\left(Y(-t)\middle|Y(0)=y_2\right)\sim N\left(e^{-\beta t}y_2,\ \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right)\qquad \text{for } t>0$$

We compare it with the forward time evolution

$$\left(Y(t)\middle|Y(0)=y_2\right)\sim N\left(e^{-\beta t}y_2,\ \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right)\qquad \text{for } t>0$$

<u>Conclusions/remarks:</u>

- At equilibrium, the evolution of going backward in time is statistically the same as the evolution of going forward in time. This is called the <u>time reversibility of equilibrium</u>.
- The time reversibility of equilibrium is a universal law applicable to all thermodynamic systems.
- The intuitive meaning of time reversibility is that if we are given a time series of a system in equilibrium, we won't be able to tell the direction of the time no matter how long and how detailed the time series is.
- Bayes theorem is very powerful in expressing the backward time evolution in terms of the forward time evolution.

Going backward in time in non-equilibrium OU process (optional)

Suppose the system starts with $Y(0) = 0$.

For $t_1 > 0$ and $t_2 > 0$, we use Bayes theorem to calculate $\rho\left(Y(t_1)=y_1\middle|Y(t_1+t_2)=y_2\right)$.

Bayes theorem for densities:

$$\rho\left(Y(t_1)=y_1\middle|Y(t_1+t_2)=y_2\right)\propto \rho\left(Y(t_1+t_2)=y_2\middle|Y(t_1)=y_1\right)\cdot\rho\left(Y(t_1)=y_1\right)$$

We already derived

- $\left(Y(t_1)\middle|Y(0)=0\right)\sim N\left(0,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t_1})\right)\qquad \text{for } t_1>0$

==> $\rho\left(Y(t_1)=y_1\right)\propto \exp\left(\frac{-y_1^{\,2}}{2(1-e^{-2\beta t_1})\gamma^2/(2\beta)}\right)$

- $\left(Y(t_1+t_2)\middle|Y(t_1)=y_1\right) \sim N\left(e^{-\beta t_2}y_1,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t_2})\right)$ for $t_1>0,\ t_2>0$

$$==> \quad \rho\left(Y(t_1+t_2)=y_2\middle|Y(t_1)=y_1\right) \propto \exp\left(\frac{-(y_2-e^{-\beta t_2}y_1)^2}{2(1-e^{-2\beta t_2})\gamma^2/(2\beta)}\right)$$

Substituting into Bayes theorem, we obtain

$$\rho\left(Y(t_1)=y_1\middle|Y(t_1+t_2)=y_2\right) \propto \rho\left(Y(t_1+t_2)=y_2\middle|Y(t_1)=y_1\right)\cdot\rho\left(Y(t_1)=y_1\right)$$

$$\propto \exp\left(\frac{-(y_2-e^{-\beta t_2}y_1)^2}{2(1-e^{-2\beta t_2})\gamma^2/(2\beta)}\right)\cdot\exp\left(\frac{-y_1{}^2}{2(1-e^{-2\beta t_1})\gamma^2/(2\beta)}\right)$$

(we only need to keep track factors that depend on y_1).

$$\propto \exp\left(\frac{-\left[(1-e^{-2\beta(t_1+t_2)})y_1{}^2-2(1-e^{-2\beta t_1})e^{-\beta t_2}y_2\cdot y_1\right]}{2(1-e^{-2\beta t_1})(1-e^{-2\beta t_2})\gamma^2/(2\beta)}\right)$$

$$\propto \exp\left(\frac{-\left(y_1-\frac{(1-e^{-2\beta t_1})}{(1-e^{-2\beta(t_1+t_2)})}e^{-\beta t_2}y_2\right)^2}{2\frac{(1-e^{-2\beta t_1})(1-e^{-2\beta t_2})}{(1-e^{-2\beta(t_1+t_2)})}\gamma^2/(2\beta)}\right)$$

It follows that

$$\left(Y(t_1)\middle|Y(t_1+t_2)=y_2\right)\sim N\left(\frac{(1-e^{-2\beta t_1})}{(1-e^{-2\beta(t_1+t_2)})}e^{-\beta t_2}y_2,\ \frac{(1-e^{-2\beta t_1})}{(1-e^{-2\beta(t_1+t_2)})}\frac{\gamma^2}{2\beta}(1-e^{-2\beta t_2})\right)$$

We discuss two special cases for t_1 and t_2

<u>Case i)</u> $\quad t_1 \to +\infty$ while t_2 = fixed

$$\frac{(1-e^{-2\beta t_1})}{(1-e^{-2\beta(t_1+t_2)})}e^{-\beta t_2}y_2 \to e^{-\beta t_2}y_2 \qquad \text{for large } t_1$$

$$\frac{(1-e^{-2\beta t_1})}{(1-e^{-2\beta(t_1+t_2)})}\frac{\gamma^2}{2\beta}(1-e^{-2\beta t_2}) \to \frac{\gamma^2}{2\beta}(1-e^{-2\beta t_2}) \qquad \text{for large } t_1$$

$$==> \quad \left(Y(t_1)\middle|Y(t_1+t_2)=y_2\right) \sim N\left(e^{-\beta t_2}y_2,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t_2})\right) \quad \text{for large } t_1$$

This is the same as the equilibrium case, not a surprise at all.

Case ii) $t_1 = t_2 = h$

$$\frac{(1-e^{-2\beta t_1})}{(1-e^{-2\beta(t_1+t_2)})}e^{-\beta t_2}y_2 = \frac{e^{-\beta h}y_2}{1+e^{-2\beta h}}$$

$$\frac{(1-e^{-2\beta t_1})}{(1-e^{-2\beta(t_1+t_2)})}\frac{\gamma^2}{2\beta}(1-e^{-2\beta t_2}) = \frac{\gamma^2}{2\beta}\left(\frac{1-e^{-2\beta h}}{1+e^{-2\beta h}}\right)$$

$$\left(Y(h)\middle|Y(2h)=y_2\right) \sim N\left(\frac{e^{-\beta h}y_2}{1+e^{-2\beta h}},\ \frac{\gamma^2}{2\beta}\left(\frac{1-e^{-2\beta h}}{1+e^{-2\beta h}}\right)\right)$$

We compare it with the forward time evolution

$$\rho\left(Y(2h)\middle|Y(h)=y_1\right) \sim N\left(e^{-\beta h}y_1,\ \frac{\gamma^2}{2\beta}\left(1-e^{-2\beta h}\right)\right)$$

When βh is not large, this case clearly demonstrates the difference between forward time evolution and backward time evolution in a non-equilibrium system.

Different interpretations of stochastic integrals

Beauty of the deterministic calculus

Consider the integral of a deterministic function $f(s)$.

$$\int_0^t f(s)ds = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j)\Delta s$$

where $\Delta s = \frac{t}{N}$, $s_j = j\,\Delta s$, $\tilde{s}_j \in [s_j\ s_{j+1}]$

Note: When $f(s)$ is piecewise continuous, the choice of $\tilde{s}_j \in [s_j, s_{j+1}]$ does not affect the limit. We can use any $\tilde{s}_j \in [s_j, s_{j+1}]$. In particular,

$$\lim_{N\to\infty}\sum_{j=0}^{N-1} f(s_j)\Delta s = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(s_{j+1})\Delta s = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(s_{j+1/2})\Delta s$$

A simple stochastic integral

$$\int_0^t f(s)dW(s) = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j)\Delta W_j$$

where $\tilde{s}_j \in [s_j\ s_{j+1}]$, $\Delta W_j = W(s_{j+1}) - W(s_j)$

The Riemann sum, $\lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j)\Delta W_j$, is a normal RV with mean = 0 and

$$\text{variance} = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j)^2 \Delta s = \int_0^t f(s)^2\, ds$$

When $f(s)$ is piecewise continuous, the choice of $\tilde{s}_j \in [s_j, s_{j+1}]$ does not affect the limit. We can use any $\tilde{s}_j \in [s_j, s_{j+1}]$.

Another simple stochastic integral

$$\int_0^t f(s, W(s))ds = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j, W(\tilde{s}_j))\Delta s$$

When $f(s, w)$ is smooth, the choice of $\tilde{s}_j \in [s_j, s_{j+1}]$ does not affect the limit

(homework problem).

A more complicated stochastic integral:

$$\int_0^t f(s, W(s))dW(s) = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j, W(\tilde{s}_j))\Delta W_j$$

where $\quad \tilde{s}_j \in [s_j\ s_{j+1}], \quad \Delta W_j = W(s_{j+1}) - W(s_j)$

Note that

- $f(s, W(s))$ is not a deterministic function of s.
- $f(\tilde{s}_j, W(\tilde{s}_j))$ is a random variable, potentially correlated with ΔW_j depending on the choice of $\tilde{s}_j \in [s_j, s_{j+1}]$.
- As a result, different choices of $\tilde{s}_j \in [s_j, s_{j+1}]$ lead to different results.
- Thus, integral $\int_0^t f(s, W(s))dW(s)$ is subject to different interpretations.

Appendix A The limit of $X(t)$ as radius $a \to 0$ while ρ_{mass} is fixed.

Recall that in the "simplified story", as $m \to 0$ while b and q are fixed, we have

$$2D = O(1) \qquad \text{and} \qquad (X(t) - X(0)) \text{ converges to } \sqrt{2D}\,W(t)$$

Now we consider the real story. As $a \to 0$ while ρ_{mass} is fixed, we have

$$m = O(a^3), \qquad b = O(a), \qquad q = \sqrt{2k_B T b} = O(\sqrt{a})$$

$$\beta = \frac{b}{m} = O(a^{-2}), \qquad \gamma = \frac{q}{m} = O(a^{-2.5})$$

$$\frac{\gamma}{\beta} = O(a^{-0.5}), \qquad D = \frac{1}{2}\left(\frac{\gamma}{\beta}\right)^2 = O(a^{-1}) \to \infty$$

The behavior of diffusion coefficient D suggests scaling the displacement by $\sqrt{a}$.

We show that $\sqrt{a}\left(X(t_1) - X(0)\right)$ converges to $cW(t)$ on any discrete time grid where coefficient $c \equiv \sqrt{a}\sqrt{2D} = O(1)$. Specifically, we show that for $t_2 > t_1 > 0$, as $a \to 0$,

- $\sqrt{a}\left(X(t_1) - X(0)\right) \to c\,N(0, t_1)$
- $\sqrt{a}\left(X(t_1 + t_2) - X(t_1)\right) \to c\,N(0, t_2)$
- $(X(t_1) - X(0))$ and $(X(t_1+t_2) - X(t_1))$ are independent.

Using (E01), we write $\sqrt{a}\,(X(t_1) - X(0))$ as

$$\sqrt{a}\left(X(t_1) - X(0)\right) \sim (1 - e^{-\beta t_1})\frac{\sqrt{a}\,Y(0)}{\beta} + \underbrace{c\,N\left(0, \left(t_1 - \frac{2(1 - e^{-\beta t_1})}{\beta} + \frac{(1 - e^{-2\beta t_1})}{2\beta}\right)\right)}_{\text{containing } dW\text{'s in } [0, t_1]}$$

The Maxwell-Boltzmann distribution gives

$$Y(t) \sim N\left(0, \frac{\gamma^2}{\beta}\right) = O\left(\sqrt{\frac{\gamma^2}{\beta}}\right) = O\left(\sqrt{\frac{a^{-5}}{a^{-2}}}\right) = O(a^{-1.5})$$

$$\Longrightarrow \qquad \frac{\sqrt{a}\,Y(t)}{\beta} = \frac{\sqrt{a}\,O(a^{-1.5})}{O(a^{-2})} = O(a) \to 0$$

Taking the limit as $a \to 0$ and using $\frac{1}{\beta}(1 - e^{-\beta t_1}) \to 0$, we obtain

- $\sqrt{a}\left(X(t_1) - X(0)\right) \xrightarrow{\text{as } a \to 0} \underbrace{c\,N(0, t_1)}_{\substack{\text{containing } dW\text{'s} \\ \text{in } [0, t_1]}}$

Similarly, we have

$$\sqrt{a}\Big(X(t_1+t_2)-X(t_1)\Big)\sim(1-e^{-\beta t_2})\frac{\sqrt{a}\,Y(t_1)}{\beta}+c\underbrace{N\left(0,\left(t_2-\frac{2(1-e^{-\beta t_2})}{\beta}+\frac{(1-e^{-2\beta t_2})}{2\beta}\right)\right)}_{\text{containing } dW\text{'s in } [t_1,t_1+t_2]}$$

- $\sqrt{a}\Big(X(t_1+t_2)-X(t_1)\Big)\xrightarrow{\text{as } a\to 0}\underbrace{c^2N(0,t_2)}_{\substack{\text{containing } dW\text{'s} \\ \text{in}[t_1,t_1+t_2]}}$

Again, $\sqrt{a}\Big(X(t_1+t_2)-X(t_1)\Big)-(1-e^{-\beta t_2})\dfrac{\sqrt{a}\,Y(t_1)}{\beta}$ contains dW's in $[t_1, t_1+t_2]$.

Since $(1-e^{-\beta t_2})\dfrac{\sqrt{a}\,Y(t_1)}{\beta}=O(a)\to 0$ as $a\to 0$, we arrive at

- $(X(t_1)-X(0))$ and $(X(t_1+t_2)-X(t_1))$ are independent in the limit of $a\to 0$.

Therefore, we conclude that $\sqrt{a}\Big(X(t)-X(0)\Big)$ converges to $cW(t)$ as $a\to 0$.

In other words, for a particle of small radius a, the displacement $(X(t)-X(0))$ is approximately $\dfrac{c}{\sqrt{a}}W(t)$ with the magnitude diverging to ∞ as $a\to 0$.