DNS of Stratified Turbuluence with Rotation and Stochastic Forcing

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1 Previous Work

2 Current Work

3 Gaussian Processes

My current job is to design a stochastic forcing structure using the Gaussian random process. Gaussian Processes are a way of generating a regression from current data, fitting a line almost if you will. We are using gaussian processes to use the current data to inform a new point going forward in the code.

The concept of the Gaussian Process is not a novel idea. Its purpose is to generate new points which fit onto an informed window of uncertainty around a given set of initial data. Ultimately, the process samples a gaussian distribution whose mean and covariance matrices are created through the use of precise linear algebra and a kernel chosen to optimize on the desired properties of the gaussian regression.

The purpose of the Gaussian Process in the context of this work is to create a statistically stationary stochastic forcing in which to perturb and drive eddies in a stable manner as done in (Waite 2004) **SOURCE**. In our Spectral Code, the Gaussian Forcing was enforced on low horizontal wavenumbers as to affect the mean background flow, without directly interacting with the turbulence structures.

$$\begin{aligned} \boldsymbol{G}(k,t) &= \langle G_x(k,t), G_y(k,t) \rangle \\ \boldsymbol{k_h} \cdot \boldsymbol{G}(k,t) &= 0 \end{aligned}$$

3.1 Dealing with Finite Precision

The procedure in which a Gaussian Process is generated is usually not a very complex Linear Algebra structure. The formulation is as below.

$$\begin{aligned} \boldsymbol{f}_* &\sim \mathcal{N}(\boldsymbol{\mu}_*, \boldsymbol{\Sigma}_*) & \boldsymbol{K} &= \mathcal{K}(\boldsymbol{x}, \boldsymbol{x}) \\ \boldsymbol{\mu}_* &= \boldsymbol{K}_*^T \times \boldsymbol{K}^{-1} \times \boldsymbol{f} & \boldsymbol{K}_* &= \mathcal{K}(\boldsymbol{x}, \boldsymbol{x}_*) \\ \boldsymbol{\Sigma}_* &= \boldsymbol{K}_{**} - \boldsymbol{K}_*^T \times \boldsymbol{K}^{-1} \times \boldsymbol{K}_* & \boldsymbol{K}_{**} &= \mathcal{K}(\boldsymbol{x}_*, \boldsymbol{x}_*) \end{aligned}$$

It should be noted that the covariance matrices, $\boldsymbol{K}$, are generated through the use of the Exponential Squared Kernal as defined below where, $\boldsymbol{a}$ and $\boldsymbol{b}$, are vectors with all real values (i.e. $\alpha_i, \beta_j \in \mathbb{R}$), and the kernel function, f, depends on the Gaussian Scale Parameter, σ.

$$\begin{aligned} f(x_1, x_2) &= \exp\left(\frac{-(x_1 - x_2)^2}{2\sigma^2}\right) \\ \boldsymbol{a} &= [\alpha_1, \alpha_2, \cdots, \alpha_n] \\ \boldsymbol{b} &= [\beta_1, \beta_2, \cdots, \beta_m] \end{aligned} \qquad \mathcal{K}(\boldsymbol{a}, \boldsymbol{b}) = \begin{bmatrix} f(\boldsymbol{a}[1], \boldsymbol{b}[1]) & f(\boldsymbol{a}[1], \boldsymbol{b}[2]) & \cdots & f(\boldsymbol{a}[1], \boldsymbol{b}[m]) \\ f(\boldsymbol{a}[2], \boldsymbol{b}[1]) & f(\boldsymbol{a}[2], \boldsymbol{b}[2]) & \cdots & f(\boldsymbol{a}[2], \boldsymbol{b}[m]) \\ \vdots & \vdots & \ddots & \vdots \\ f(\boldsymbol{a}[n], \boldsymbol{b}[1]) & f(\boldsymbol{a}[n], \boldsymbol{b}[2]) & \cdots & f(\boldsymbol{a}[n], \boldsymbol{b}[m]) \end{bmatrix}$$

4 Code Design and Algorithm Structure