

Dante Buhl  
 Jason Johnstone  
 Arstan Tulekeyev  
 Nathan Van Duker

## Appendix

### 1.1 Derivation of $\mathbf{u}_{\text{fgm}}/\mathbf{u}_b$

Numerical computations of the fastest growing mode yield fourier modes for  $\mathbf{u}_{\text{fgm}}$ . We have the following eigenvector to describe the coefficients of each component. The first three components are the coefficients  $u_{-1}, u_0, u_1$ , the middle three are  $v_{-1}, v_0, v_1$ , and the last three are  $w_{-1}, w_0, w_1$ .

$$\begin{bmatrix} u_{\text{fgm}} \\ v_{\text{fgm}} \\ w_{\text{fgm}} \\ t_{\text{fgm}} \\ p_{\text{fgm}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0.5 & -0.91 & -0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.36 & 0 & 0.36 \end{bmatrix} \begin{bmatrix} e^{-ik_y y} \\ 1 \\ e^{ik_y y} \end{bmatrix} e^{\lambda t + ik_x x}$$

We notice that all of the coefficients for  $w$  are zero, thus we find that  $w_{\text{fgm}} = 0$ . Now we simply must find the real parts of the sum of the three fourier modes for  $u_{\text{fgm}}$  and  $v_{\text{fgm}}$ .

$$\begin{aligned} u_{\text{fgm}} &= e^{\lambda t + i(k_x x - k_y y)} + e^{\lambda t + i(k_x x + k_y y)} \\ v_{\text{fgm}} &= 0.5 \cdot e^{\lambda t + i(k_x x - k_y y)} - 0.91 \cdot e^{\lambda t + ik_x x} - 0.5 \cdot e^{\lambda t + i(k_x x + k_y y)} \end{aligned}$$

We will of course be taking the "frozen-in-time" approximations for the background flow, so we will proceed without writing  $\lambda t$ .

$$\begin{aligned} u_{\text{fgm}} &= e^{ik_x x} (e^{-ik_y y} + e^{ik_y y}) \\ v_{\text{fgm}} &= -0.5 \cdot e^{ik_x x} (e^{ik_y y} - e^{-ik_y y}) - 0.91 \cdot e^{ik_x x} \end{aligned}$$

We notice that the forms of these complex exponentials yield complex and real components and so we identify which will contribute to the real part.

$$\begin{aligned} u_{\text{fgm}} &= 2 \cos(k_x x) \cdot \cos(k_y y) \\ v_{\text{fgm}} &= \sin(k_x x) \cdot \sin(k_y y) - 0.91 \cos(k_x x) \end{aligned}$$

Finally we add the shear term in order to complete  $\mathbf{u}_b$ .

$$\begin{aligned} \mathbf{u}_b &= (\bar{u} + u_{\text{fgm}}) \hat{e}_x + v_{\text{fgm}} \hat{e}_y \\ \bar{u} &= \sin(k_y y) \\ u_{\text{fgm}} &= 2 \cos(k_x x) \cos(k_y y) \\ v_{\text{fgm}} &= \sin(k_x x) \sin(k_y y) - 0.91 \cos(k_x x) \end{aligned}$$

$$\begin{aligned} \bar{u} &= \frac{1}{2i} (e^{ik_y y} - e^{-ik_y y}) \\ u_{\text{fgm}} &= \frac{1}{2} (e^{ik_x x} + e^{-ik_x x}) (e^{ik_y y} + e^{-ik_y y}) \\ v_{\text{fgm}} &= -\frac{1}{4} (e^{ik_x x} - e^{-ik_x x}) (e^{ik_y y} - e^{-ik_y y}) - \frac{0.91}{2} (e^{ik_x x} + e^{-ik_x x}) \end{aligned}$$

$$u_{\text{fgm}} = \frac{1}{2} (e^{ik_x x + ik_y y} + e^{ik_x x - ik_y y} + e^{-ik_x x + ik_y y} + e^{-ik_x x - ik_y y}) \quad (1)$$

$$v_{\text{fgm}} = -\frac{1}{4} (e^{ik_x x + ik_y y} - e^{ik_x x - ik_y y} - e^{-ik_x x + ik_y y} + e^{-ik_x x - ik_y y}) - \frac{0.91}{2} (e^{ik_x x} + e^{-ik_x x}) \quad (2)$$

## 1.2 Derivation of non-linear terms

This section has two essential components to investigate. In Navier-Stokes, the infamous non-linear advection terms are often the at the center of simplification. In this problem, we consider simplified non-linear terms in order to reduce the algebraic complexity.

$$\begin{aligned}\mathbf{u} \cdot \nabla \mathbf{u} &= (\mathbf{u}_b + \tilde{\mathbf{u}}) \cdot \nabla (\mathbf{u}_b + \tilde{\mathbf{u}}) \\ &\simeq \mathbf{u}_b \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_b\end{aligned}$$

From this simplification, we realize there are a few essential outcomes of these terms. We will investigate multiplication of  $\mathbf{u}_b$  and its derivatives, by a function  $q$ , and its derivatives,  $\frac{\partial q}{\partial x}$ ,  $\frac{\partial q}{\partial y}$ . Remember, we have that the ansatz  $q$  is given by the following form,

$$q = \sum_n \sum_m \hat{q} e^\theta, \quad \theta = \lambda t + i(nk_x x + mk_y y + k_z z) \quad (3)$$

$$\frac{\partial q}{\partial x} = \sum_n \sum_m i n k_x \hat{q} e^\theta \quad (4)$$

$$\frac{\partial q}{\partial y} = \sum_n \sum_m i m k_y \hat{q} e^\theta \quad (5)$$

## 1.3 Derivation of continuity equation

Finally, we consider the continuity equation:

$$\begin{aligned}\nabla \cdot \tilde{\mathbf{u}} &= \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \\ \Rightarrow n k_x \hat{u}_{m,n} + m k_y \hat{v}_{m,n} + k_z \hat{w}_{m,n} &= 0\end{aligned} \quad (6)$$