Turbulent dissipation scales (03/30/2023)

3D turbulence

In 3D turbulence, forced at large scales, assume there is a given energy flux $\epsilon > 0$ from large to small scales. We can define a Reynolds number for each scale l,

$$Re_l = \frac{(\epsilon l)^{1/3} l}{\nu} = \frac{\epsilon^{1/3} l^{4/3}}{\nu}.$$

The Kolmogorov or dissipation length scale l_K, where $Re_{l_K} = 1$ is where viscosity starts to play a dominant role in the dynamics. Rearranging the equation, we find

$$l_K = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}.$$

Note that we can also obtain this result from dimensional analysis, assuming that $l_K = \nu^\alpha \epsilon^\beta$. Since dimensionally $[\epsilon] = L^2/T^3$ and $[\nu] = L^2/T$, to get dimensions of length one has to solve the algebraic system of equations

$$2\alpha + 2\beta = 1, \qquad -\alpha - 3\beta = 0,$$

which also gives $\alpha = 3/4$, $\beta = -1/4$.

2D turbulence

In 2D turbulence, we can simply redo the dimensional analysis, but have to take into account that it is enstrophy, not energy, which cascades to small scales. The enstrophy flux η has dimensions $[\eta] = 1/T^3$ and $[\nu] = L^2/T$ as before. Letting the dissipation scale $l_d = \nu^\alpha \eta^\beta$, we get

$$2\alpha = 1, \qquad -\alpha - 3\beta = 0,$$

we find $\alpha = 1/2$, and $\beta = -1/6$, such that the 2D turbulence dissipation scale is given by

$$l_d = \left(\frac{\nu^3}{\eta}\right)^{1/6}. \tag{1}$$

Note that if the forcing is spectrally localized at a scale $\ell_f = \frac{2\pi}{k_f}$, then

$$\eta = k_f^2 \epsilon.$$

Energy injection by random forcing

We consider the Navier-Stokes equation in d dimensions

$$\partial_t \mathbf{u} = \underbrace{-\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p}_{=\mathcal{N}} + \mathbf{f} + \underbrace{\nu \nabla^2 \mathbf{u}}_{=\mathcal{D}}, \qquad \nabla \cdot \mathbf{u} = 0 \tag{2}$$

Denote the domain volume by $V = \int 1 d^n x$. For a given time step size dt, we choose $\mathbf{f} = \sqrt{\frac{2\epsilon}{dt}}\boldsymbol{\eta}(\mathbf{x},t)$, where η has the following properties

$$\int \boldsymbol{\eta}(\mathbf{x},t) d^{\mathrm{d}}x = \mathbf{0}, \qquad \frac{1}{V}\int \boldsymbol{\eta}\cdot\boldsymbol{\eta} d^{\mathrm{d}}x = 1.$$

Further we assume that

$$\int \boldsymbol{\eta}(\mathbf{x},t)\cdot\boldsymbol{\eta}(\mathbf{x},t') d^{\mathrm{d}}x \approx 0,$$

if $t \neq t'$ and that the forcing is drawn independently of the velocity field at the same time

$$\int \boldsymbol{\eta}(\mathbf{x},t)\cdot\boldsymbol{u}(\mathbf{x},t) d^{\mathrm{d}}x \approx \int \boldsymbol{\eta}(\mathbf{x},t)\cdot\mathcal{N}(\mathbf{x},t) d^{\mathrm{d}}x \approx \int \boldsymbol{\eta}(\mathbf{x},t)\cdot\mathcal{D}(\mathbf{x},t) d^{\mathrm{d}}x \approx 0.$$

The energy density is given by $\mathcal{E} = \frac{1}{2V}\int \mathbf{u}\cdot\mathbf{u} d^n x$. The rate of change between two consecutive time steps is then given by

$$\frac{d\mathcal{E}}{dt} \approx \frac{\mathcal{E}_{n+1}-\mathcal{E}_n}{dt} = \frac{1}{2V dt}\int (\boldsymbol{u}_{n+1}^2 - \boldsymbol{u}_n^2) d^{\mathrm{d}}x \tag{3}$$

Similarly discretizing the NSE using a direct 1st order Euler scheme, we get

$$\mathbf{u}_{n+1} \approx \mathbf{u}_n + \sqrt{2\epsilon dt}\boldsymbol{\eta} + dt\left(\mathcal{N}_n + \mathcal{D}_n\right). \tag{4}$$

Hence

$$\frac{d\mathcal{E}}{dt} \approx \frac{1}{2V dt}\int \left[\left(\mathbf{u}_n + \sqrt{2\epsilon dt}\boldsymbol{\eta} + dt\mathcal{N}_n + dt\mathcal{D}_n\right)^2 - \boldsymbol{u}_n^2\right] d^{\mathrm{d}}x \tag{5}$$

$$= \frac{1}{2V dt}\int \left[2\epsilon dt\boldsymbol{\eta}^2 + dt^2(\mathcal{N}_n^2 + \mathcal{D}_n^2) + 2\boldsymbol{u}_n[\sqrt{2\epsilon dt}\boldsymbol{\eta} + dt\left(\mathcal{N}_n + \mathcal{D}_n\right)] + 2dt^{3/2}\sqrt{2\epsilon}\boldsymbol{\eta}\cdot(\mathcal{N}_n + \mathcal{D}_n) + dt^2\mathcal{N}_n\cdot\mathcal{D}_n\right] d^{\mathrm{d}}x \tag{6}$$

$$= \frac{1}{V}\int \left[\epsilon\boldsymbol{\eta}^2 + \boldsymbol{u}_n\cdot(\mathcal{N}_n + \mathcal{D}_n) + \sqrt{2\epsilon dt}\boldsymbol{\eta}\cdot(\mathcal{N}_n + \mathcal{D}_n) + \mathcal{O}(dt)\right] d^{\mathrm{d}}x \tag{7}$$

$$= \underbrace{\frac{\epsilon}{V}\int \boldsymbol{\eta}^2 d^{\mathrm{d}}x}_{=\epsilon} + \underbrace{\frac{1}{V}\int \boldsymbol{u}_n\cdot\mathcal{N} d^{\mathrm{d}}x}_{=0\text{ (periodic BCs)}} + \underbrace{\frac{1}{V}\int \boldsymbol{u}_n\cdot\mathcal{D}_n d^{\mathrm{d}}x}_{<0\text{ (viscous dissipation of energy)}} + \underbrace{\frac{\sqrt{2\epsilon dt}}{V}\int \boldsymbol{\eta}\cdot(\mathcal{N}_n + \mathcal{D}_d) d^{\mathrm{d}}x}_{\approx 0} + O(dt). \tag{8}$$