REGIMES OF STRATIFIED TURBULENCE ACROSS PARAMETER SPACE: FROM ASYMPTOTIC ANALYSIS TO DNS

Pascale Garaud (Applied Mathematics, UCSC)

Collaborators :

Kasturi Shah (Cambridge)

Laura Cope (Leeds)

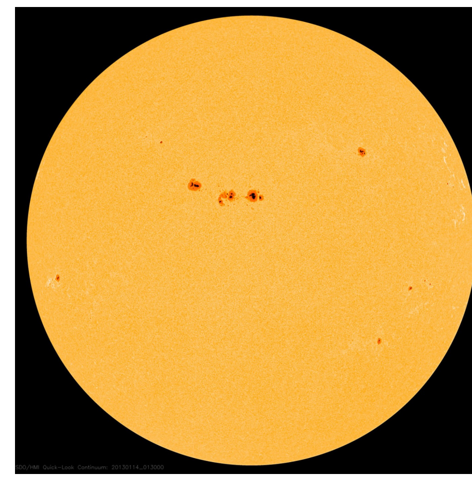
Greg Chini (UNH)

Colm-cille Caulfield (Cambridge)

MOTIVATION

Goal

To understand turbulence in stably stratified regions of the Earth, and of giant planets and stars.

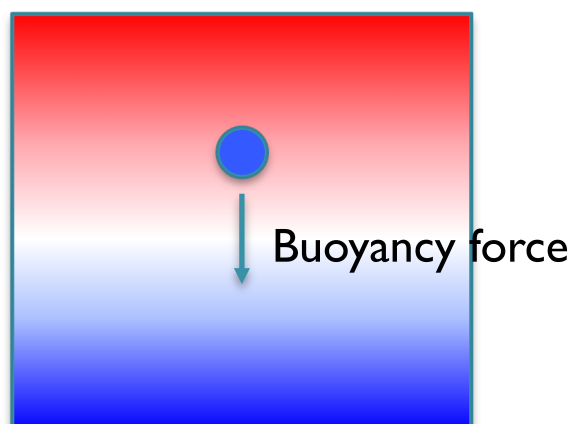


Stratified turbulence

Stratified turbulence is characterized by

- Three-dimensional <u>chaotic</u> fluid flow
- With vertical motion strongly inhibited by buoyancy (gravity)

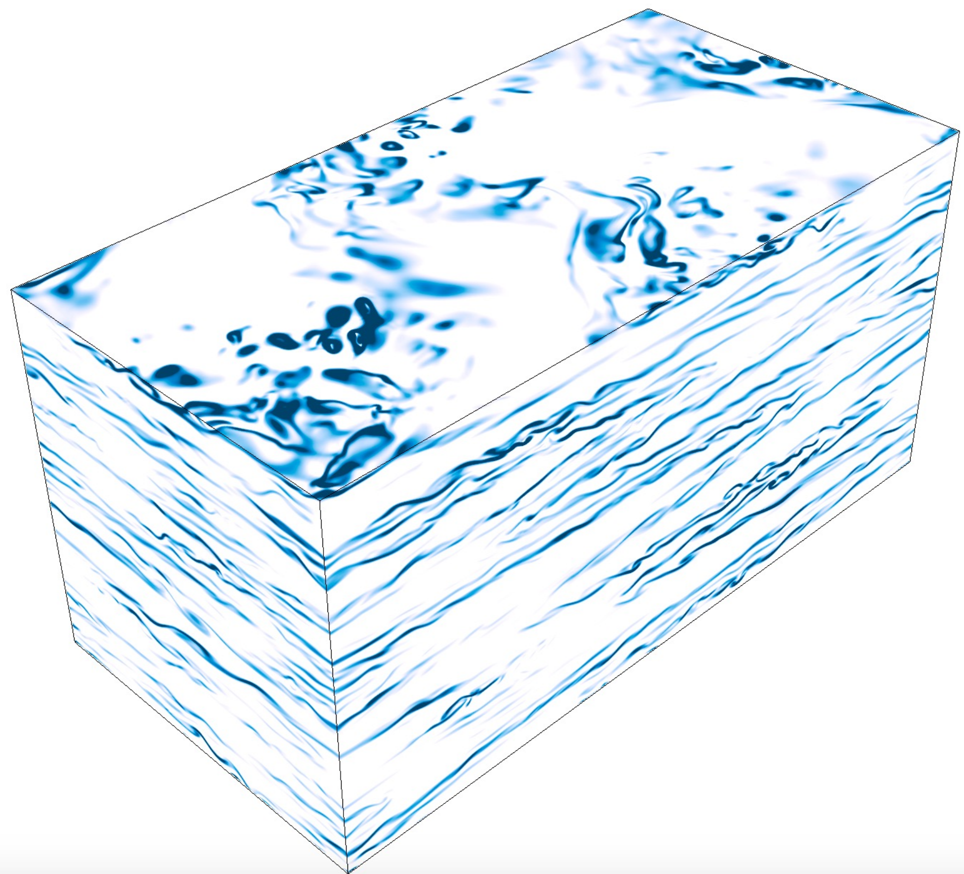
$$\frac{d\bar{\rho}}{dz} < 0$$



Stratified turbulence

→ Flow is anisotropic (larger horizontal than vertical scales)

→ Flow can be intermittent



Stratified turbulence

- Usually generated by instabilities of the large-scale flows (winds, ocean currents, planetary or stellar circulation).
- Plays an important role in :
  1. Vertical mixing of chemical species, momentum and heat
  2. Vertical mixing of buoyancy, which is crucial in the maintenance of global meridional overturning circulation
- All these processes are important to long-term evolution of climate, planet, star.

Stratified turbulence

AGU ADVANCING EARTH AND SPACE SCIENCES

About | Membership | Publications | Meetings | Data Services | Careers | Honors | Science Poli

Home / Hazardous clear-air turbulence up 55% from 1979, study shows

HAZARDOUS CLEAR-AIR TURBULENCE UP 55% FROM 1979, STUDY SHOWS

THE INCREASE, WHICH HAS LONG BEEN SUSPECTED, IS CONSISTENT WITH WARMER AIR TEMPERATURES. SKIES ARE PREDICTED TO GET BUMPIER AS CLIMATE CHANGE PROGRESSES

8 June 2023

A ‘TOY’ PROBLEM

Basic model

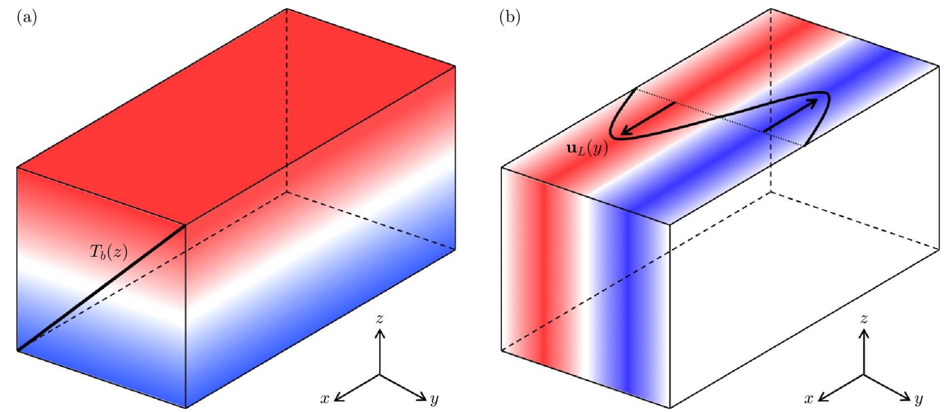
We ignore all ‘superfluous’ physics

- Rotation
- Magnetic fields
- Multiple scalars contributing to buoyancy

We drive flow only on the largest scales, and let it become naturally turbulent.

Basic model

Horizontal "Kolmogorov flow" model with vertical density stratification



$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\frac{1}{\rho_m} \boldsymbol{\nabla} p + b \boldsymbol{e}_z + \nu \nabla^2 \boldsymbol{u} + \frac{F_0}{\rho_m} \sin(y) \boldsymbol{e}_x$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\frac{\partial b}{\partial t} + \mathbf{u} \cdot \nabla b + N^2 u_z = \kappa_T \nabla^2 b$$

with triply-periodic boundary conditions on $\boldsymbol{u}, p, b$

Basic model

Non-dimensionalize with

$$[\text{length}] = L = \text{horizontal shear scale},$$

$$[\text{velocity}] = U = \sqrt{\frac{LF_0}{\rho_m}} = \text{horizontal flow velocity}$$

$$[\text{buoyancy}] = LN^2$$

$$[\text{pressure}] = \rho_m U^2$$

so

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \frac{b}{Fr^2}\boldsymbol{e}_z + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \sin(y)\boldsymbol{e}_x$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\frac{\partial b}{\partial t} + \mathbf{u} \cdot \nabla b + u_z = \frac{1}{Pe}\nabla^2 b$$

with

$$Re = \frac{UL}{\nu}, Pe = \frac{UL}{\kappa_T}, Fr = \frac{U}{NL}$$

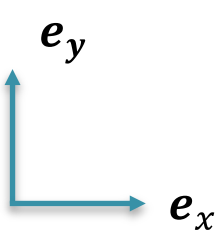
$$(viscosity)^{-1} \quad (diffusion)^{-1} \quad (stratification)^{-1}$$

Nonlinear evolution

Transient nonlinear evolution is complex, and depends on input parameters

Example: from moderately stratified case

View from top, u_x

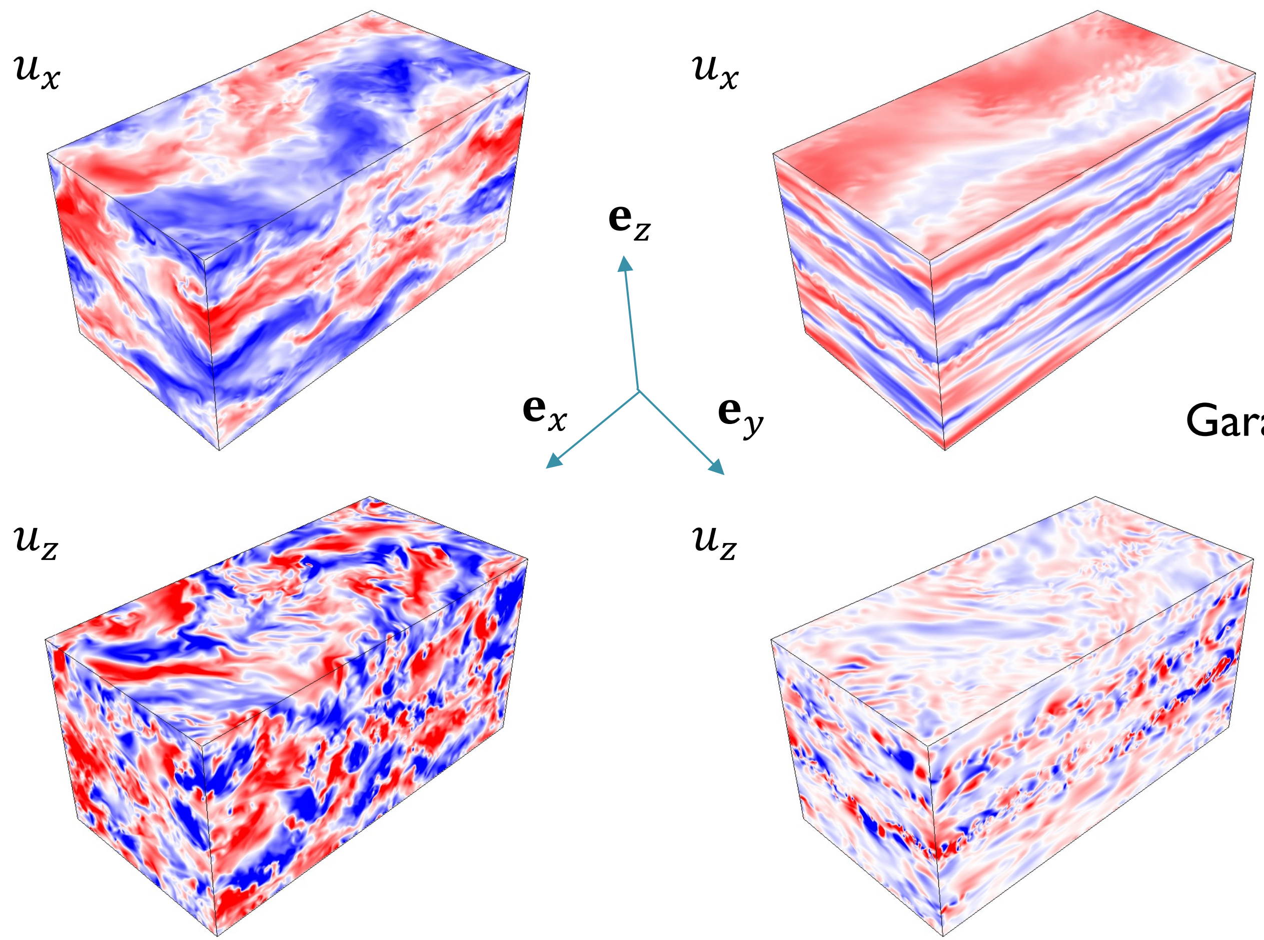


$$Re = 1000, Pe = 10, Fr = 0.33$$

Nonlinear evolution ($Re = 600, Pe = 60$)

$Fr = 0.33$, moderate stratification

$Fr = 0.1$, stronger stratification



Garaud 2020

Vertical length scale & vertical velocity decrease as stratification increases

Rationale for asymptotic analysis

This is a 3-parameter problem (ignoring possible effects of domain aspect ratio):

$$Re, Pe, Fr$$

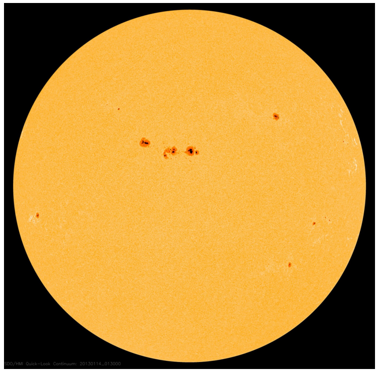
or equivalently

$$Re, Pr, Fr \text{ where } Pr = \frac{\nu}{\kappa_T} = \frac{Pe}{Re}$$

Note: Pr is a property of the fluid, not of the flow.

$Pr = O(1 - 10)$

$Pr = O(0.01 - 0.1)$



$Pr = O(10^{-9} - 10^{-4})$

Rationale for asymptotic analysis

<u>This is a hard problem!</u>

- Too many parameters to vary systematically
- Parameters are asymptotically large (Re, Pe), or asymptotically small (Fr, Pr),.

  DNS can only probe moderate Re, Pe, Fr
- On Earth, we can use observations, lab experiments, to test models. But not for stellar fluids (no fluid on Earth has $Pr \ll 0.01$)

Rationale for asymptotic analysis

Since DNS only get us that far …

Since experiments and observations can only probe $\mathrm{Pr} = O(1)$ ...

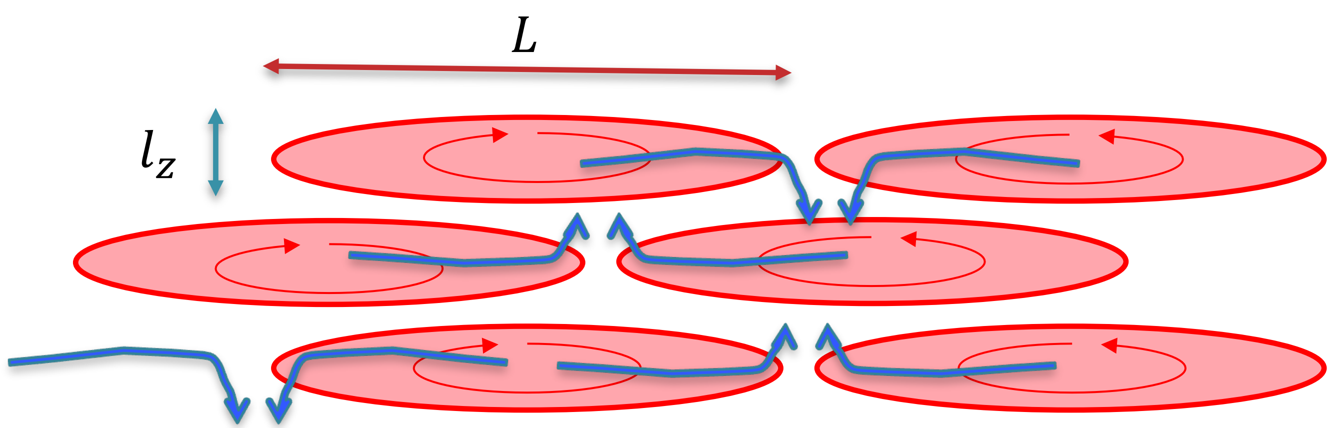
We will use asymptotic analysis, validated by DNS, to study the properties of turbulence in strongly stratified horizontally sheared flows at both moderate and low Pr

'TRADITIONAL' REDUCED MODELS OF STRATIFIED MIXING

Traditional approach

Observations, lab experiments at $Pr = O(1)$, suggests strong stratification ($Fr \ll 1$) causes strong flow anisotropy with

Vertical scale, velocity << Horizontal scales, velocities



Traditional approach

Billant & Chomaz 2001, Brethouwer et al. 2007 → Rescale governing equations with the anticipation of <u>anisotropy</u> in the limit $Fr \ll 1$:

$$\zeta = \frac{z}{\alpha} \text{ with } \alpha \ll 1$$

where α (aspect ratio l_z/L) is presumably related to Fr.

Traditional approach

Anisotropically scaled equations:

$$\nabla_h \cdot \boldsymbol{u}_h + \frac{1}{\alpha}\frac{\partial u_z}{\partial \zeta} = 0$$

$$\frac{\partial \boldsymbol{u}_h}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h \boldsymbol{u}_h + \frac{u_z}{\alpha}\frac{\partial \boldsymbol{u}_h}{\partial \zeta} = -\nabla_h p + \frac{1}{\alpha^2 Re}\left(\alpha^2 \nabla_h^2 \boldsymbol{u}_h + \frac{\partial^2 \boldsymbol{u}_h}{\partial \zeta^2}\right) + \sin(y)\boldsymbol{e}_x$$

$$\frac{\partial u_z}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h u_z + \frac{u_z}{\alpha}\frac{\partial u_z}{\partial \zeta} = -\frac{1}{\alpha}\frac{\partial p}{\partial \zeta} + \frac{b}{Fr^2} + \frac{1}{\alpha^2 Re}\left(\alpha^2 \nabla_h^2 u_z + \frac{\partial^2 u_z}{\partial \zeta^2}\right)$$

$$\frac{\partial b}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h b + \frac{u_z}{\alpha}\frac{\partial b}{\partial \zeta} + u_z = \frac{1}{\alpha^2 Pe}\left(\alpha^2 \nabla_h^2 b + \frac{\partial^2 b}{\partial \zeta^2}\right)$$

where

- $u_h = O(1)$ by construction
- $Pe, Re \gg 1$, but $\alpha, Fr \ll 1$

Traditional approach

Immediate implications for stratified turbulence:

Mass conservation (assuming flow does not become 2D):

$$\nabla_h \cdot \boldsymbol{u}_h + \frac{1}{\alpha}\frac{\partial u_z}{\partial \zeta} = 0 \rightarrow u_z = O(\alpha)$$

Horizontal momentum equation:

$$\frac{\partial \boldsymbol{u}_h}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h \boldsymbol{u}_h + \frac{u_z}{\alpha}\frac{\partial \boldsymbol{u}_h}{\partial \zeta} = -\nabla_h p + \frac{1}{\alpha^2 Re}\left(\alpha^2 \nabla_h^2 \boldsymbol{u}_h + \frac{\partial^2 \boldsymbol{u}_h}{\partial \zeta^2}\right) + sin(y)\boldsymbol{e}_x$$

$\rightarrow p = O(1)$

$\rightarrow$ Note emergent "buoyancy" Reynolds number $\alpha^2 Re \equiv Re_b \ll Re$

$\rightarrow$ turbulence requires Re_b to not be too small (Bartello & Tobias 2013), i.e.

$$Re_b \geq O(1)$$

Traditional approach for $Pr = O(1)$

Buoyancy equation (with $u_z = O(\alpha)$)

$$\underbrace{\frac{\partial b}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h b + \frac{u_z}{\alpha}\frac{\partial b}{\partial \zeta}}_{O(b)} + \underbrace{u_z}_{O(\alpha)} = \underbrace{\frac{1}{\alpha^2 Pe}\left(\alpha^2 \nabla_h^2 b + \frac{\partial^2 b}{\partial \zeta^2}\right)}_{O\left(\frac{b}{\alpha^2 Pe}\right)}$$

→ Dominant balance assuming stratification is relevant requires $b = O(\alpha)$

Define $\alpha^2 Pe = Pe_b$ as the "buoyancy Peclet number"

Note: $Pe_b = Pr Re_b$

If $Pr = O(1), Re_b \geq O(1)$

$Pe_b = Pr Re_b \geq O(1)$

so diffusion term is small.

Traditional approach for $Pr = O(1)$

Vertical momentum equation wit $u_z = O(\alpha)$, $b = O(\alpha)$

$$\underbrace{\frac{\partial u_z}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h u_z + \frac{u_z}{\alpha}\frac{\partial u_z}{\partial \zeta}}_{\substack{O(\alpha) \\ \text{(small)}}} = \underbrace{\underbrace{-\frac{1}{\alpha}\frac{\partial p}{\partial \zeta}}_{\substack{O(\alpha^{-1}) \\ \text{(large)}}} + \underbrace{\frac{b}{Fr^2}}_{\substack{O(\alpha/Fr^2) \\ \text{(large)}}}}_{\substack{\text{Dominant balance} \\ \text{requires } \alpha = Fr}} + \underbrace{\frac{1}{Re_b}\left(\alpha^2 \nabla_h^2 u_z + \frac{\partial^2 u_z}{\partial \zeta^2}\right)}_{\substack{O(\alpha/Re_b) \\ \text{(small)}}}$$

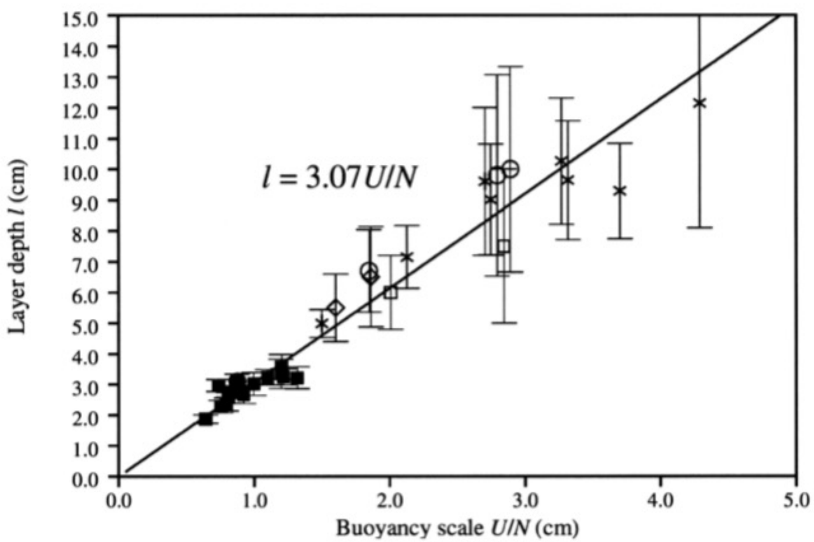
Conclusions (e.g. Brethouwer et al. 2007):

- Vertical scale is $l_z = O(\alpha L) = O(Fr L) \propto \frac{U}{N}$
- Vertical velocity is $u_z = O(\alpha U) = O(Fr U) \propto \frac{U^2}{NL}$

Validation

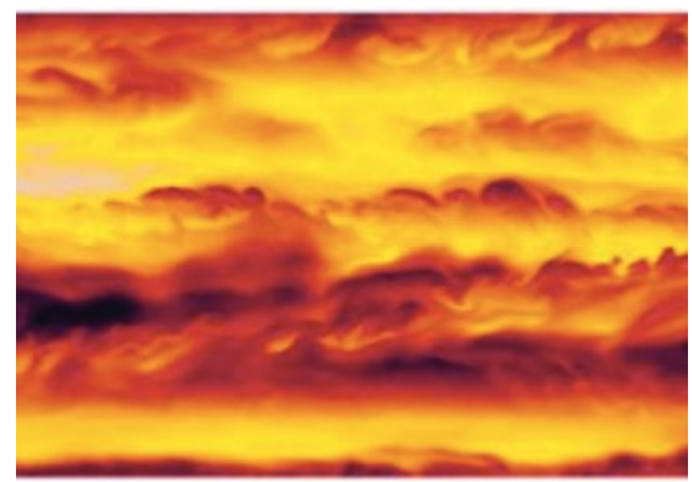
Emergence of $l_z = U/N$ scale has been validated in both laboratory and numerical experiments at $Pr \geq O(1)$.

Lab experiments, Holford & Linden 1999



Salt-stratified ($Pr \gg 1$)

Numerical experiments



Brethouwer et al. 2007

Air (?) ($Pr = 1?$)

WHAT ABOUT LOW PR?

Traditional approach for $Pr \ll 1$

Shah et al. 2024

<u>Buoyancy equation</u> (with $u_z = O(\alpha)$)

$$\underbrace{\frac{\partial b}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h b + \frac{u_z}{\alpha}\frac{\partial b}{\partial \zeta}}_{O(b)} + \underbrace{u_z}_{O(\alpha)} = \underbrace{\frac{1}{Pe_b}\left(\alpha^2 \nabla_h^2 b + \frac{\partial^2 b}{\partial \zeta^2}\right)}_{O\left(\frac{b}{Pe_b}\right)}$$

$$\underbrace{\text{If } Pr \ll 1 \rightarrow PrRe_b = Pe_b \ll 1}_{}$$

Dominant balance requires
$b = O(\alpha Pe_b) = O(\alpha^3 Pe)$

$\rightarrow$ Buoyancy equation becomes $u_z = \dfrac{1}{Pe_b}\dfrac{\partial^2 b}{\partial \zeta^2}$

Low Peclet Number balance
(Lignieres 1999)

Traditional approach for $Pr \ll 1$

Vertical momentum equation with $u_z = O(\alpha)$, and $b = O(\alpha Pe_b)$

$$\frac{\partial u_z}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h u_z + \frac{u_z}{\alpha}\frac{\partial u_z}{\partial \zeta} = -\underbrace{\frac{1}{\alpha}\frac{\partial p}{\partial \zeta}}_{\substack{O(\alpha^{-1}) \\ \text{(large)}}} + \underbrace{\frac{b}{Fr^2}}_{\substack{O(\alpha Pe_b/Fr^2) \\ \text{(large)}}} + \frac{1}{Re_b}\left(\alpha^2 \nabla_h^2 u_z + \frac{\partial^2 u_z}{\partial \zeta^2}\right)$$

Dominant balance requires $\alpha = \dfrac{Fr}{\sqrt{Pe_b}} \leftrightarrow \alpha = \left(\dfrac{Fr^2}{Pe}\right)^{1/4}$

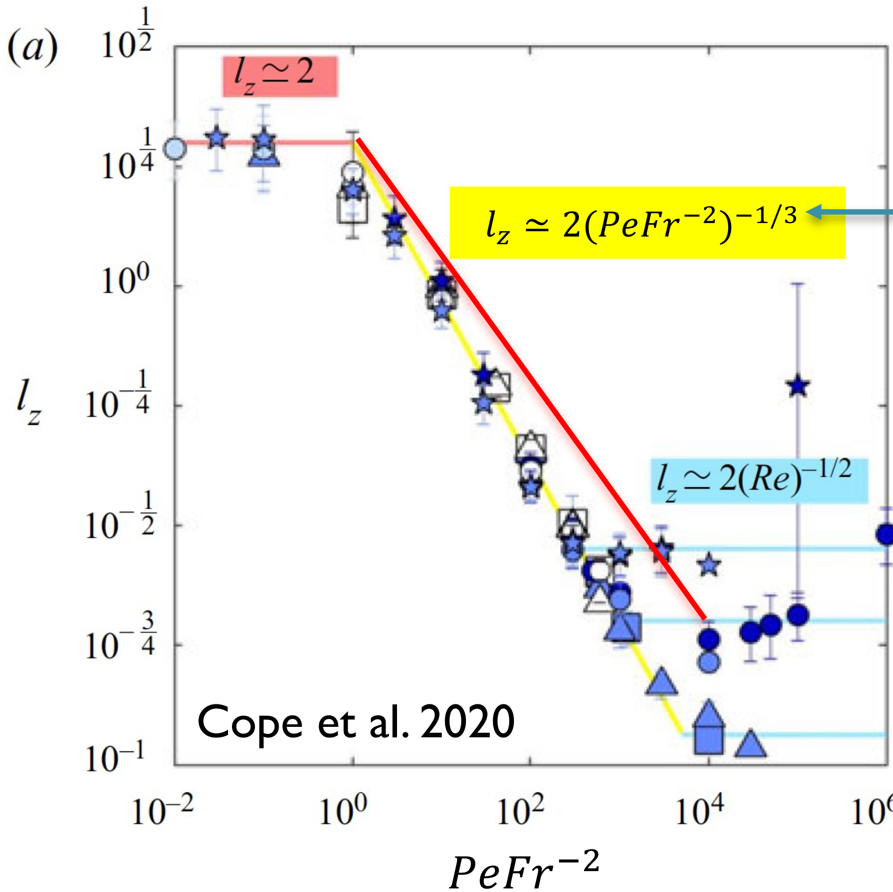
Conclusions (Shah et al. 2024, see also Lignieres, 2021, Skoutnev 2023): stratified turbulence in low Pr fluids has:

$$l_z \propto \left(\frac{Fr^2}{Pe}\right)^{\frac{1}{4}} L, u_z \propto \left(\frac{Fr^2}{Pe}\right)^{\frac{1}{4}} U$$

Validation ? (or not…)

DNS data of Cope et al. 2020 in $Pr \to 0$ limit does not agree with this theoretical result!

Cope et al. 2020

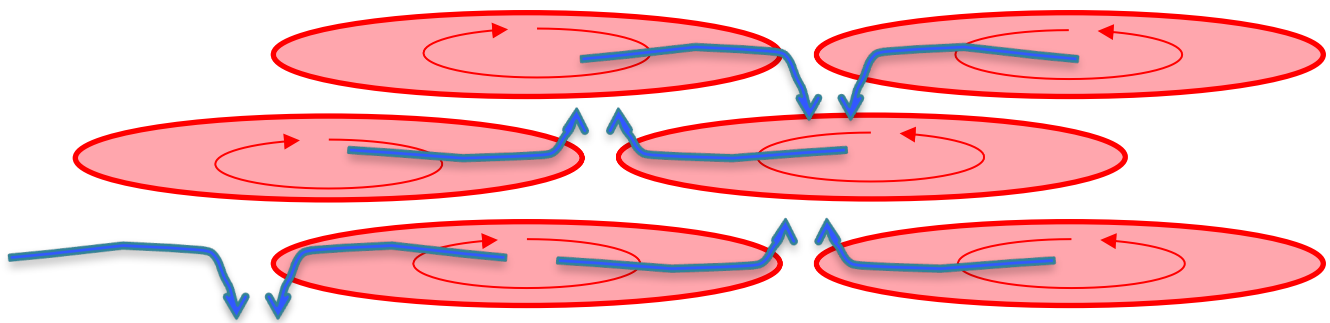


Power index not -1/4

Validation ? (or not…)

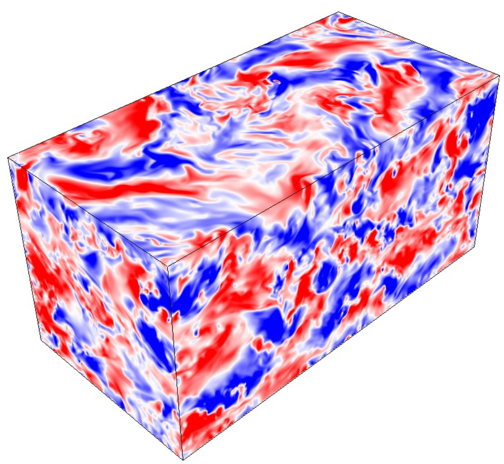
Why the difference ?

Traditional approach only describes the dynamics of <u>almost</u> laminar flows where horizontal scales are <u>large by construction</u>

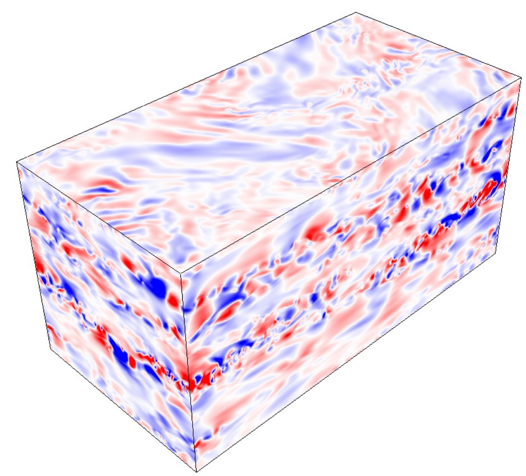


Validation ? (or not…)

This is inconsistent with the data, which shows small scales are important, especially at more moderate stratification but also in stronger stratification



$Fr = 0.33$, moderate stratification

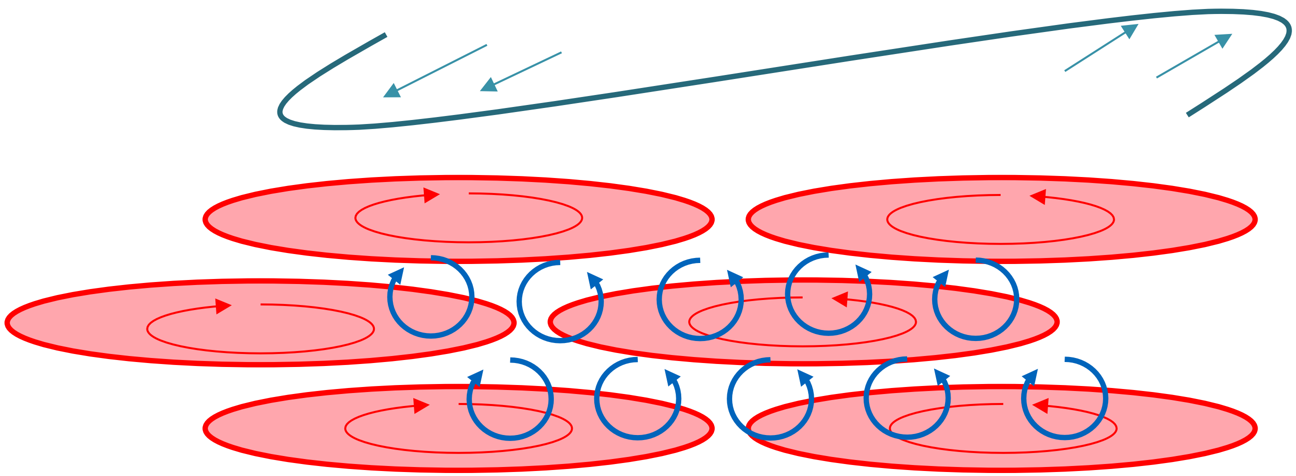


$Fr = 0.1$, stronger stratification

MULTISCALE MODELS OF STRATIFIED TURBULENCE

The importance of emergent vertical shear (cf. Zahn 1992)

Horizontal shear → Horizontal turbulence U, L



↓

Vertically decorrelated

↓

Emergent vertical shear → Vertical shear instability → Small-scale vertical Mixing

Multiscale modeling approach

Chini et al. 2022

To model this mathematically,

we need a multiscale model!

Chini et al. 2022 (at $Pr = O(1)$) created a model for stratified turbulence where

1. Vertical scale << large horizontal scales
2. Turbulence is multiscale with fast/slow time and small/large spatial scales

Multiscale modeling approach

Step 1: As before, rescale governing equations with the anticipation of small vertical scales in the limit of strong stratification ($Fr \ll 1$) :

$$\zeta = \frac{z}{\alpha} \text{ with } \alpha \ll 1$$

where the aspect ratio $\alpha = l_z/L$ is presumably related to Fr.

Multiscale modeling approach

Step 2: Assume flow has both large ("slow") and small ("fast") horizontal scales and fast and slow time scales,

$$q(\boldsymbol{x}_h, \zeta, t) = \bar{q}(\boldsymbol{x}_s, \zeta, t_s) + q'(\boldsymbol{x}_f, \boldsymbol{x}_s, \zeta, t_f, t_s)$$

$$\frac{\partial}{\partial t} = \frac{1}{\alpha}\frac{\partial}{\partial t_f} + \frac{\partial}{\partial t_s}$$

$$\nabla_h = \frac{1}{\alpha}\nabla_f + \nabla_s$$

<u>Note:</u>

- $\bar{u}_h$, slow horizontal scale and slow time are $O(1)$ by chosen non-dimensionalization
- $\overline{q'} = 0$ by assumption
- Fast horizontal scales ~ vertical scale (<u>isotropy on small scales</u>)
- Fast time ~ vertical shearing timescale l_z/U

Multiscale modeling approach

Immediate implications:

Mass conservation:

$$\nabla_s \cdot \bar{\boldsymbol{u}}_h + \frac{1}{\alpha}\frac{\partial \bar{u}_z}{\partial \zeta} + \frac{1}{\alpha}\nabla_f \cdot \boldsymbol{u'}_h + \frac{1}{\alpha}\frac{\partial u'_z}{\partial \zeta} + h.o.t = 0$$

Average over small scales:

$$\nabla_s \cdot \bar{\boldsymbol{u}}_h + \frac{1}{\alpha}\frac{\partial \bar{u}_z}{\partial \zeta} = 0 \rightarrow \bar{u}_z = O(\alpha)$$

Subtract from total equation to get perturbation equation:

$$\frac{1}{\alpha}\nabla_f \cdot \boldsymbol{u'}_h + \frac{1}{\alpha}\frac{\partial u'_z}{\partial \zeta} + h.o.t = 0 \rightarrow \boldsymbol{u'}_h = O(u'_z)$$

isotropy on small scales

Multiscale modeling approach

Horizontal momentum equation: averaged over small scales + use continuity and $\bar{u}_z = O(\alpha)$. At leading order

$$\frac{\partial \overline{\boldsymbol{u}}_h}{\partial t_s} + \overline{\boldsymbol{u}}_h \cdot \nabla_s \overline{\boldsymbol{u}}_h + \frac{\bar{u}_z}{\alpha}\frac{\partial \overline{\boldsymbol{u}}_h}{\partial \zeta} + \frac{1}{\alpha}\frac{\partial}{\partial \zeta}\overline{u'_z \boldsymbol{u}'_h}$$
$$= -\nabla_s \bar{p} + \frac{1}{Re_b}\left(\alpha^2 \nabla_s^2 \overline{\boldsymbol{u}}_h + \frac{\partial^2 \overline{\boldsymbol{u}}_h}{\partial \zeta^2}\right) + \sin(y)\boldsymbol{e}_x$$

$$\boldsymbol{u}'_h = O(u'_z) = O\left(\alpha^{1/2}\right)$$

Multiscale modeling approach

<u>Implications</u>:

→ **all quantities should be expanded in asymptotic series in $\boldsymbol{O}(\alpha^{1/2})$**

$$q(\boldsymbol{x}_h, \zeta, t) = \bar{q} + q' \text{ where}$$

$$\bar{q} = \bar{q}_o + \alpha^{1/2}\bar{q}_1 + \alpha\bar{q}_2 + \cdots,$$

$$q' = q'_0 + \alpha^{1/2}\, q'_1 + \alpha q'_2 + \cdots$$

Note: this result is completely agnostic about the size of Pr.

We then proceed as usual, substituting ansatz into equations, and looking at them order by order to obtain dominant scalings for each quantity, as functions of input parameters.

Multiscale modeling approach:

$\mathrm{Pr} = O(1)\ vs\ \mathrm{Pr} \ll 1$

Shah et al. 2024

Inspection of the mean and fluctuating buoyancy equations reveal the role of Pr (or, more practically, $Pe_b = PrRe_b$)

$$\frac{\partial \bar{b}}{\partial t_s} + \bar{\boldsymbol{u}}_h \cdot \nabla_s \bar{b} + \frac{\bar{u}_z}{\alpha}\frac{\partial \bar{b}}{\partial \zeta} + \frac{1}{\alpha}\frac{\partial}{\partial \zeta}\overline{u'_z b'} + \bar{u}_z = \frac{1}{Pe_b}\left(\alpha^2 \nabla_s^2 \bar{b} + \frac{\partial^2 \bar{b}}{\partial \zeta^2}\right)$$

$$\frac{\partial b'}{\partial t_f} + \bar{\boldsymbol{u}}_h \cdot \nabla_f b' + \alpha^{\frac{1}{2}} u'_{z1}\frac{\partial \bar{b}}{\partial \zeta} + \alpha^{\frac{3}{2}} u'_{z1} = \frac{\alpha}{Pe_b}\left({\nabla_f}^2 b' + \frac{\partial^2 b'}{\partial \zeta^2}\right) + h.o.t$$

Non-diffusive regime (Chini et al. 2022)

$Pe_b \geq O(1)$

Intermediate regime (Shah et al. 2024)

$\alpha \leq Pe_b \ll 1$

Diffusive regime (Shah et al. 2024)

$Pe_b \ll \alpha$

Case $Pe_b = \alpha^2 Pe \geq 1$ (non-diffusive regime)

Recovers exactly results of Chini et al. 2022

- Asymptotic analysis reveals $\alpha = Fr$ (as in single scale analysis)
- Scalings:

$$\bar{u}_h, \bar{p} = O(1)$$
$$\bar{u}_z, \bar{b} = O(\alpha)$$
$$u'_h, u'_z, p' = O\left(\alpha^{1/2}\right)$$
$$b' = O\left(\alpha^{3/2}\right)$$

cf. Riley & Lindborg 2013, Maffioli & Davidson 2016

- Mean and fluctuation equations separate into <u>closed quasilinear system at leading order</u>

Case $\alpha \ll Pe_b \ll 1$ (intermediate regime)

- Asymptotic series in $\alpha^{1/2}$ now also needs to be expanded in small parameter Pe_b
- Asymptotic analysis reveals $\alpha = Fr$ (as in single scale analysis and non-diffusive regime)
- Scalings:

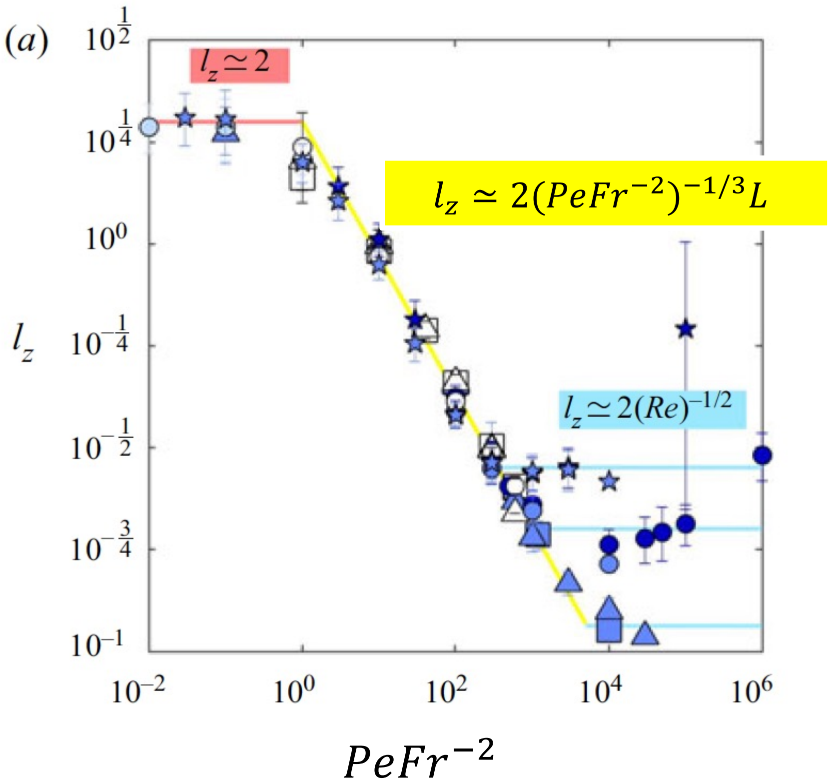
$$\bar{u}_h, \bar{p} = O(1)$$
$$\bar{u}_z = O(\alpha)$$
$$\bar{b} = O(\alpha Pe_b)$$
$$u'_h, u'_z, p' = O\left(\alpha^{1/2}\right)$$
$$b' = O\left(\alpha^{3/2}\right)$$

- Mean and fluctuation equations separate into closed quasilinear system at leading order

Case $Pe_b \ll \alpha$ (diffusive regime)

Cope et al. 2020

- Asymptotic analysis reveals $\alpha = \left(\frac{Fr^2}{Pe}\right)^{1/3}$
- This recovers Cope et al. 2020 empirical scalings!



Case $Pe_b \ll \alpha$ (diffusive regime)

Cope et al. 2020

- Asymptotic analysis reveals $\alpha = \left(\frac{Fr^2}{Pe}\right)^{1/3}$
- Scalings:

$$\bar{u}_h, \bar{p} = O(1)$$
$$\bar{u}_z = O(\alpha)$$
$$\bar{b} = O(\alpha Pe_b)$$
$$u'_h, u'_z, p' = O\left(\alpha^{1/2}\right) = \left(\frac{Fr^2}{Pe}\right)^{1/6}$$
$$b' = O\left(\alpha^{1/2} Pe_b\right) = Pe\left(\frac{Fr^2}{Pe}\right)^{5/6}$$

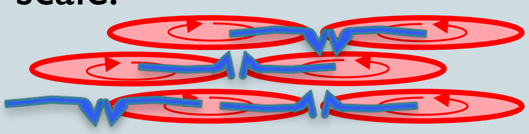
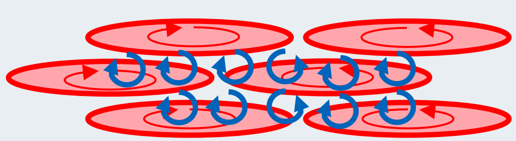
cf. Cope et al. 2020

- Mean and fluctuation equations separate into <u>closed quasilinear system at leading order</u>

SUMMARY (SO FAR)

Summary of traditional vs. multiscale approach

Traditional vs. multiscale approach make quite different predictions for vertical length scale and velocity of stratified turbulence

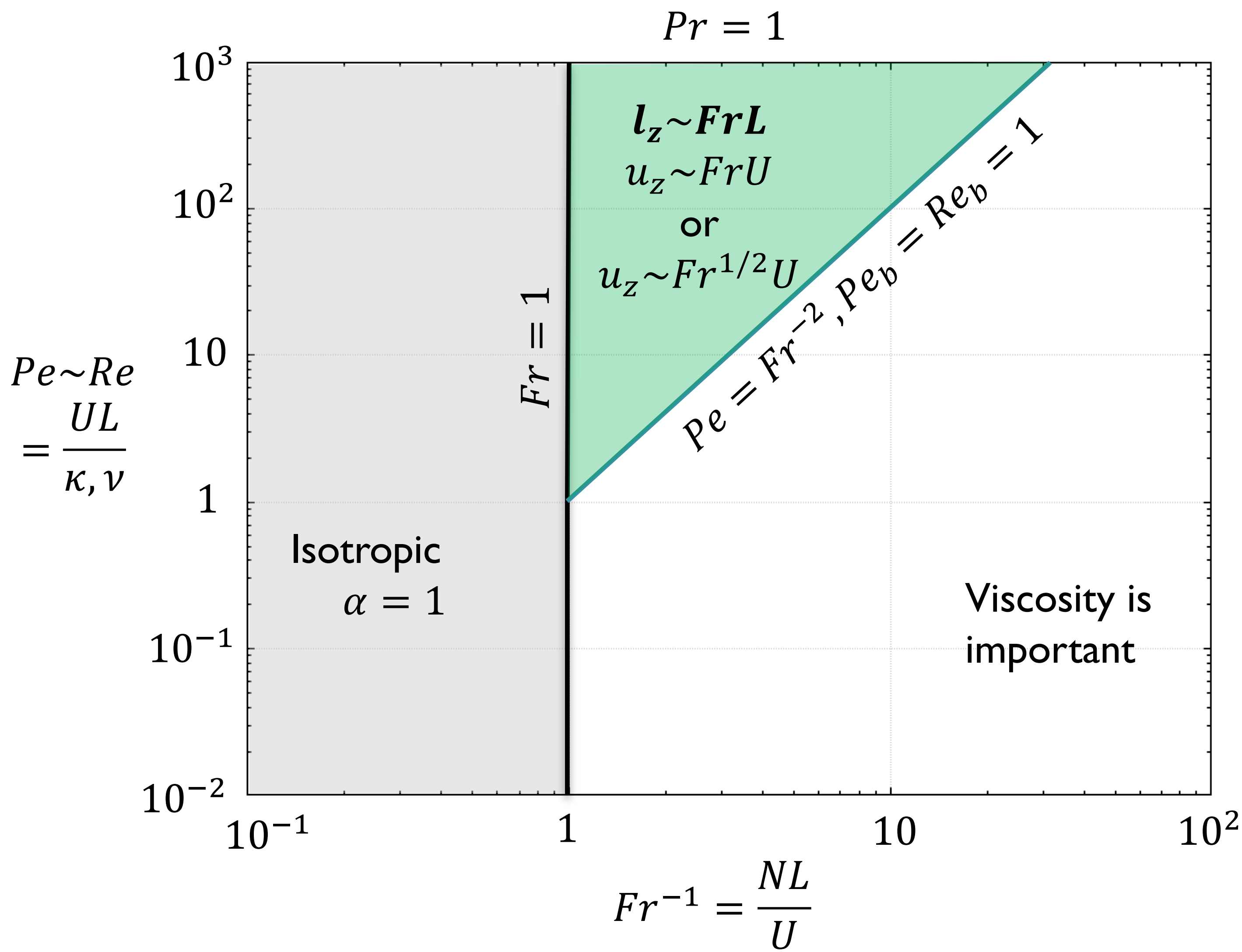
	Non-diffusive (and intermediate) regime	Diffusive regime
Single large horizontal scale: 	$l_z \sim FrL$ $\boldsymbol{u_z \sim FrU}$ Billant & Chomaz 2001 Brethouwer et al. 2007	$l_z \sim (Fr^2/Pe)^{1/4} L$ $\boldsymbol{u_z \sim (Fr^2/Pe)^{1/4} U}$ Shah et al. 2024, as in Lignieres 2021 Skoutnev 2023
Two horizontal scales: 	$l_z \sim FrL$ $\boldsymbol{u_z \sim Fr^{1/2} U}$ Chini et al. 2022 as in Riley & Lindborg 2013, Davidson & Maffioli 2016	$l_z \sim (Fr^2/Pe)^{1/3} L$ $\boldsymbol{u_z \sim (Fr^2/Pe)^{1/6} U}$ Shah et al. 2024, as in Cope et al. 2020

Regime diagrams

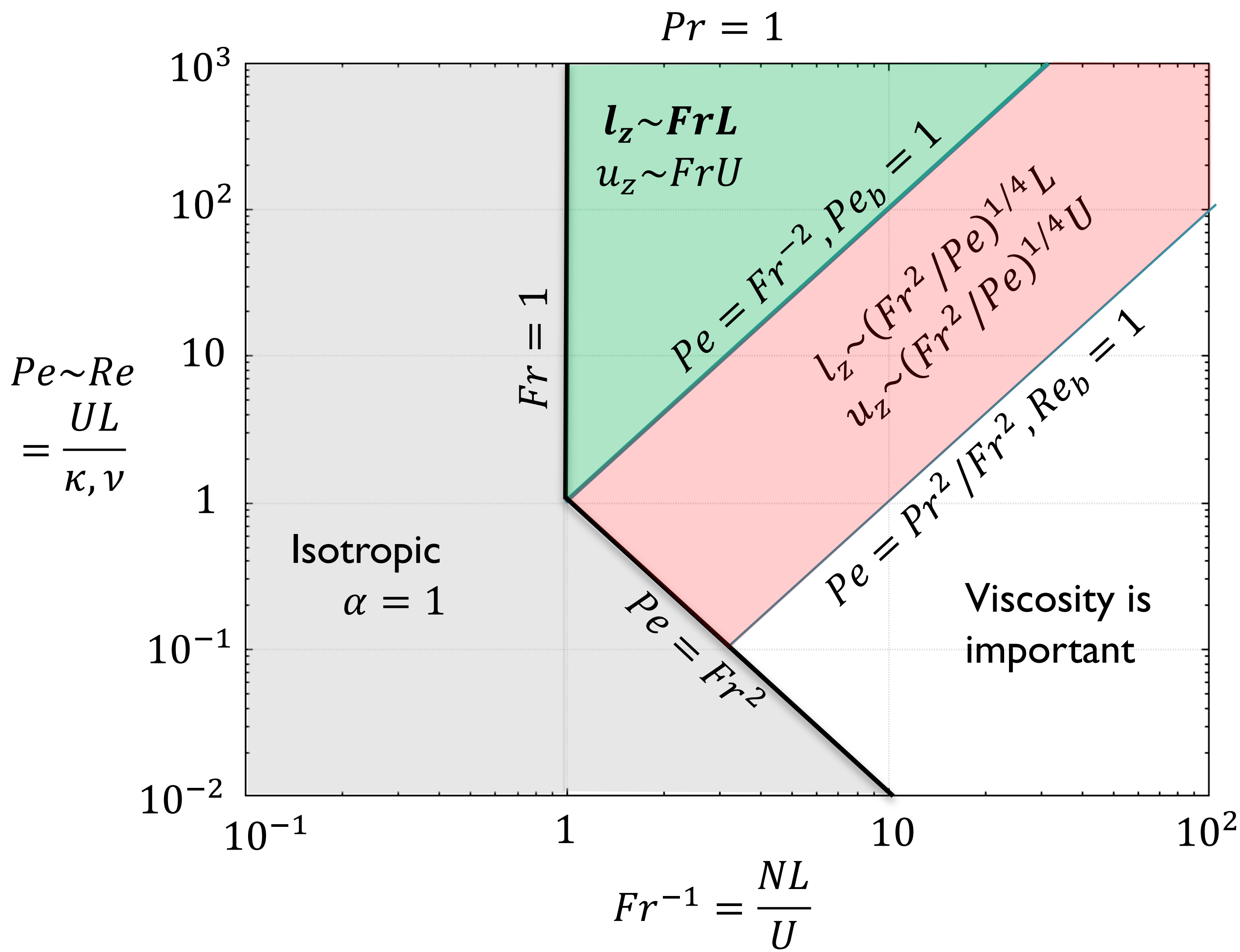
In both traditional (single horizontal scale), and multiscale models, asymptotic analysis also reveals regime of validity of the scalings derived.

These usually depend on size of $Re_b = \alpha^2 Re$ and $Pe_b = \alpha^2 Pe$

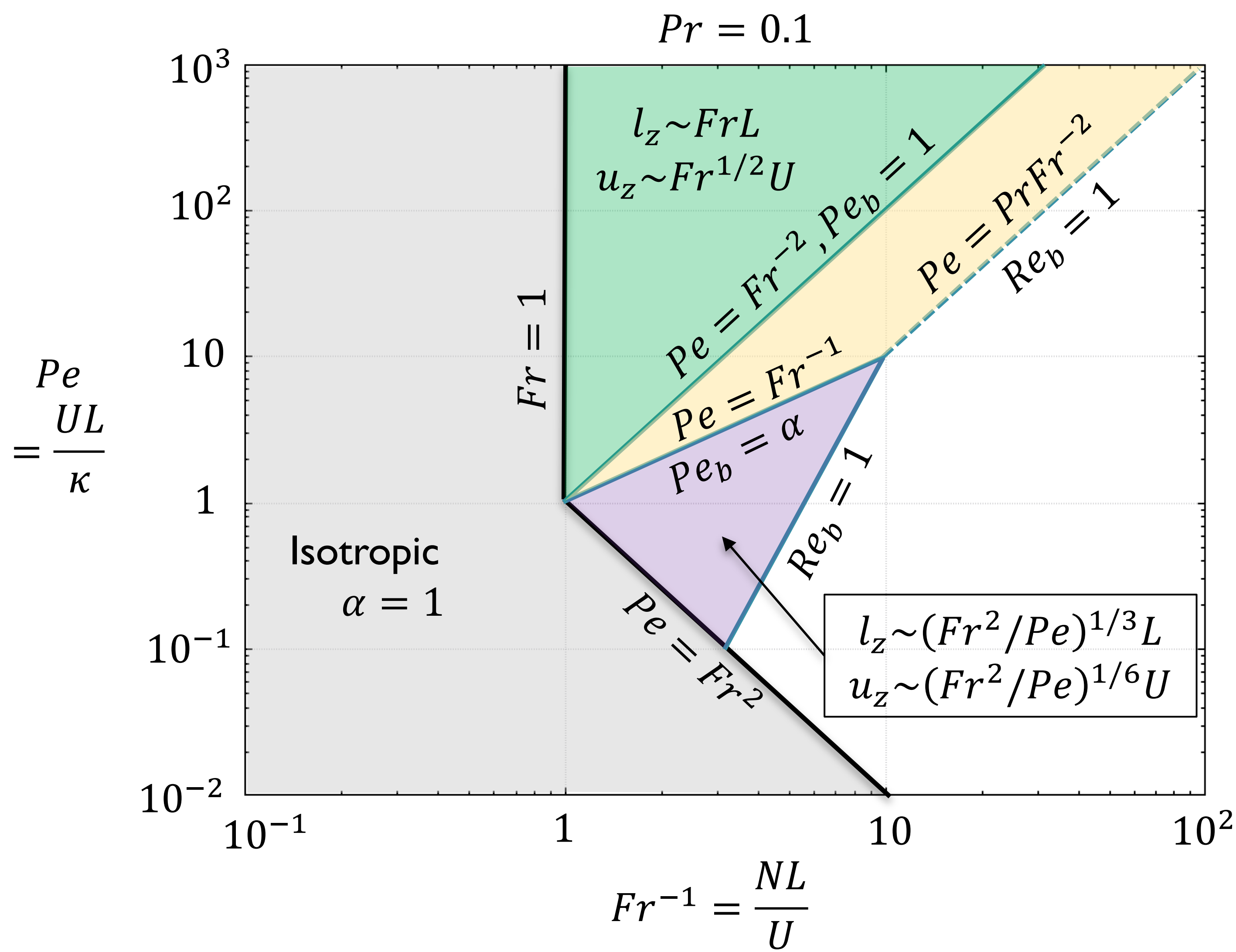
Regime diagram at Pr = 1 (both models)



Regime diagram at $Pr = 0.1$, traditional



Regime diagram at Pr = 0.1, multiscale



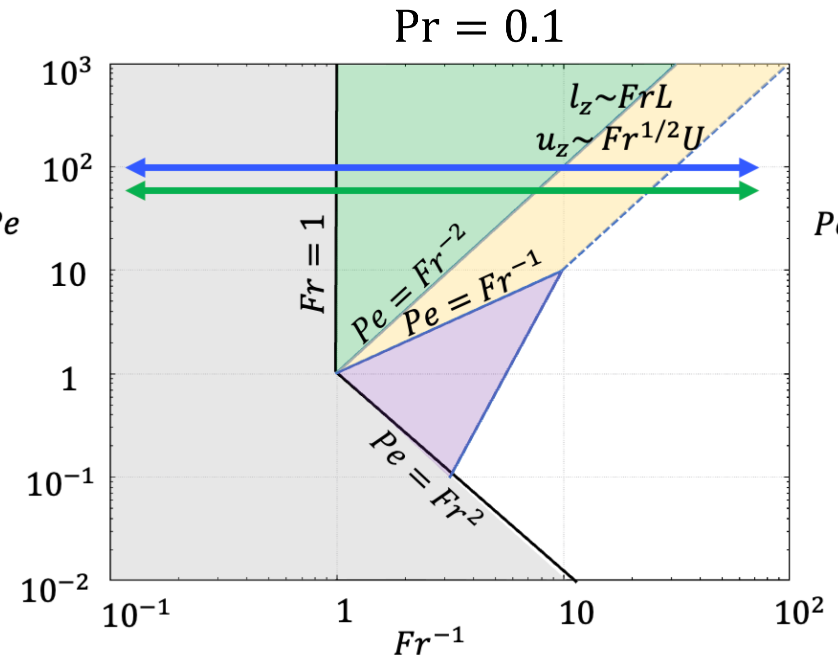
Which model?

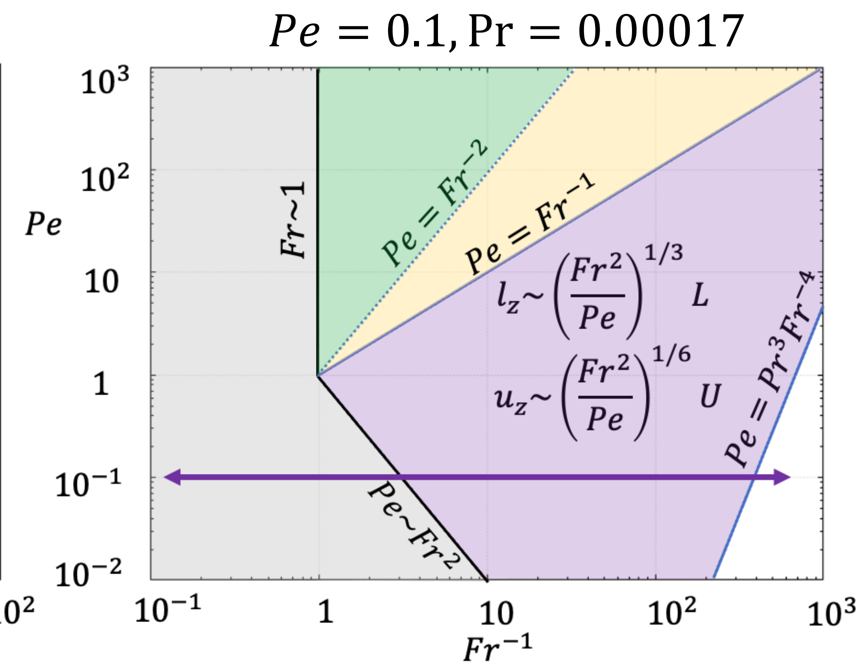
Which of these models (traditional single horizontal scale, or multiscale) best fits the data?

COMPARISON WITH DNS

Tests

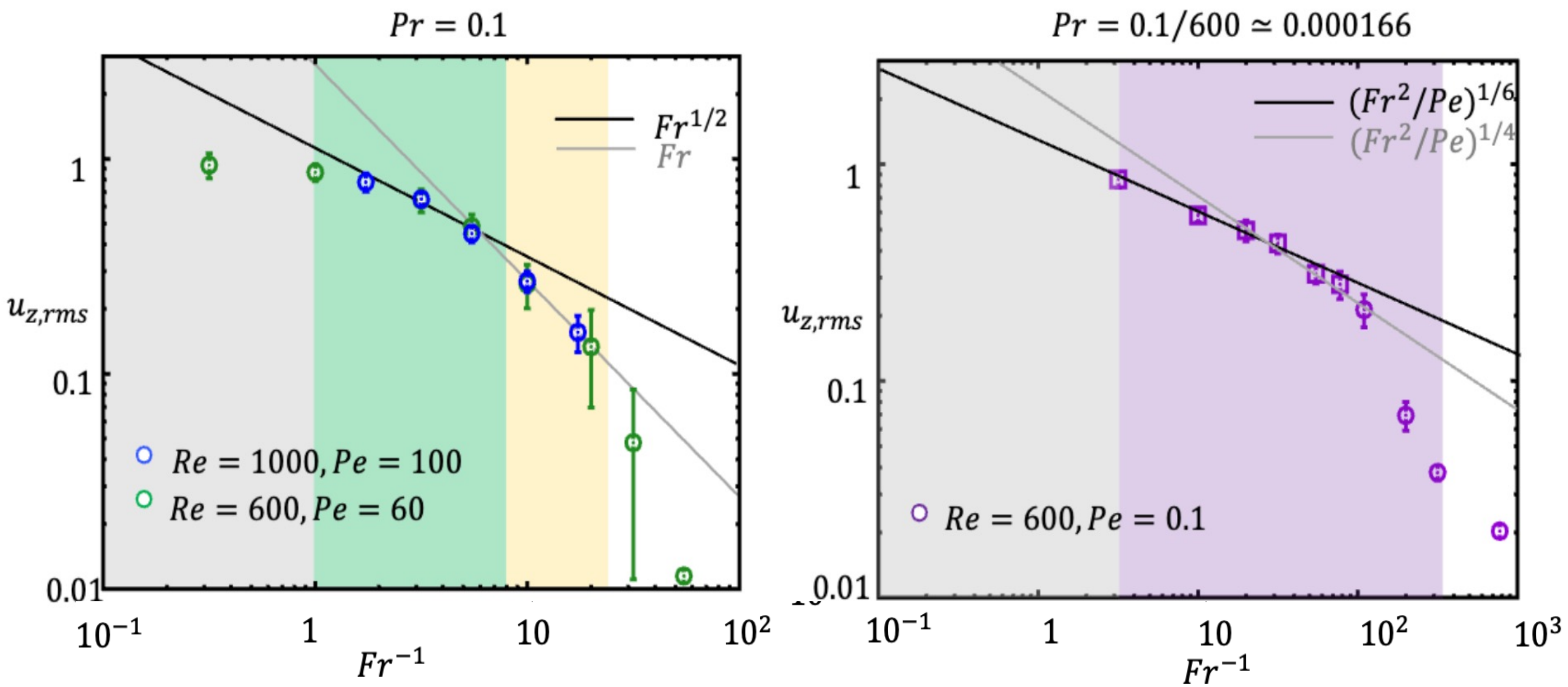
In order to test the quality of both models, we ran two series of numerical experiments





Tests

We measured the rms vertical velocity:

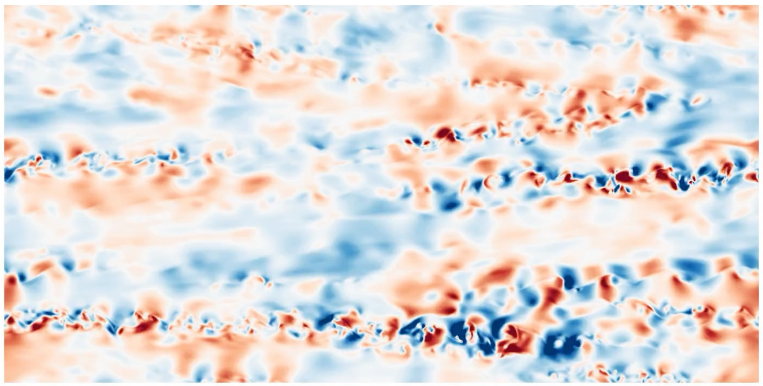


The multiscale model performs better at weaker / intermediate stratification, and the traditional model performs better at stronger stratification

Tests

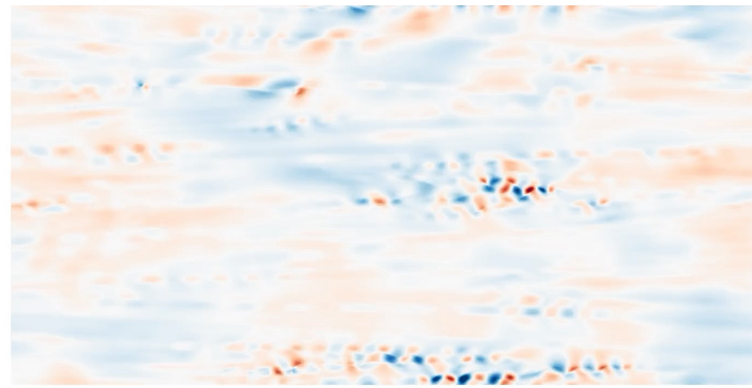
Looking at the data, this is not surprising:

$Re = 1000, Pe = 100, Fr = 0.1$



u_z

$Re = 1000, Pe = 100, Fr = 0.06$

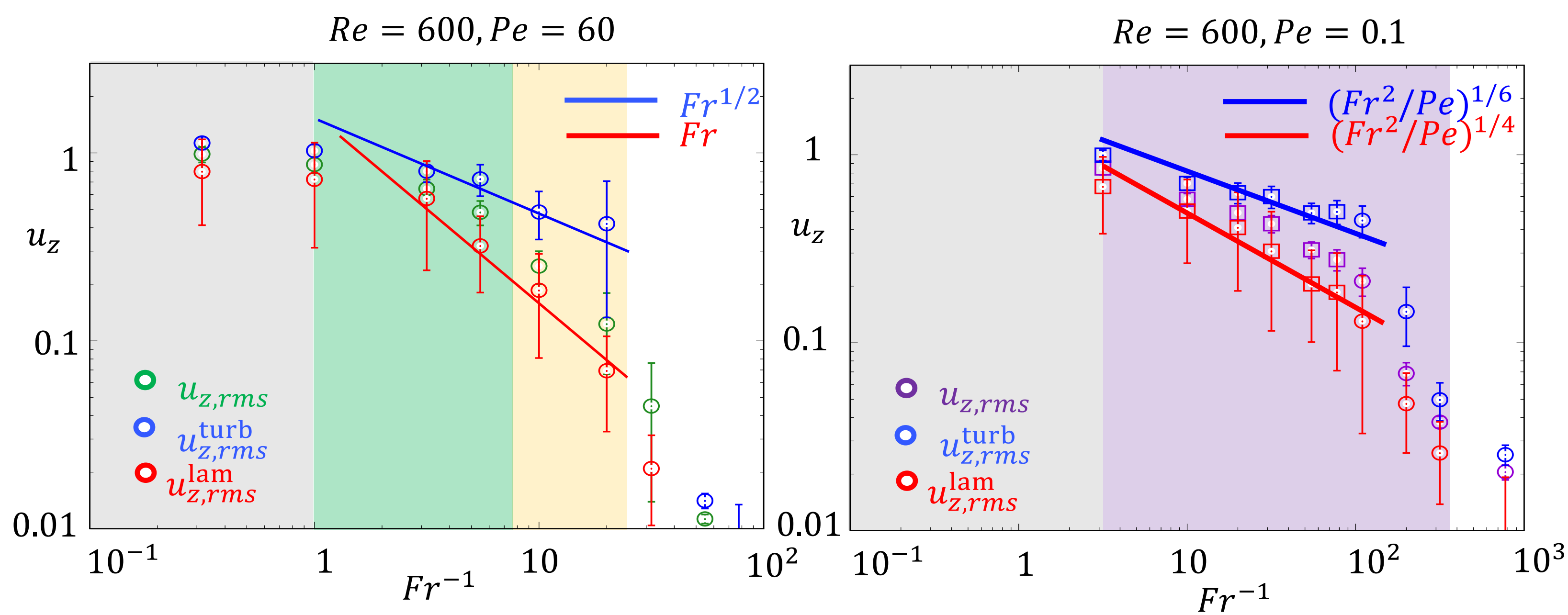


u_z

- The stronger the stratification, the more intermittent the turbulence
- Multiscale model is only expected to hold in turbulent patches.
- Traditional model only expected to hold outside of turbulent patches.

Tests

We measure u_z separately in and out of the turbulent patches (ask me how):



- Multiscale model <u>holds</u> in turbulent patches!
- Traditional model <u>holds</u> outside of turbulent patches!

→ Turbulent patch volume fraction decreases as stratification increases

TAKE-HOME MESSAGES

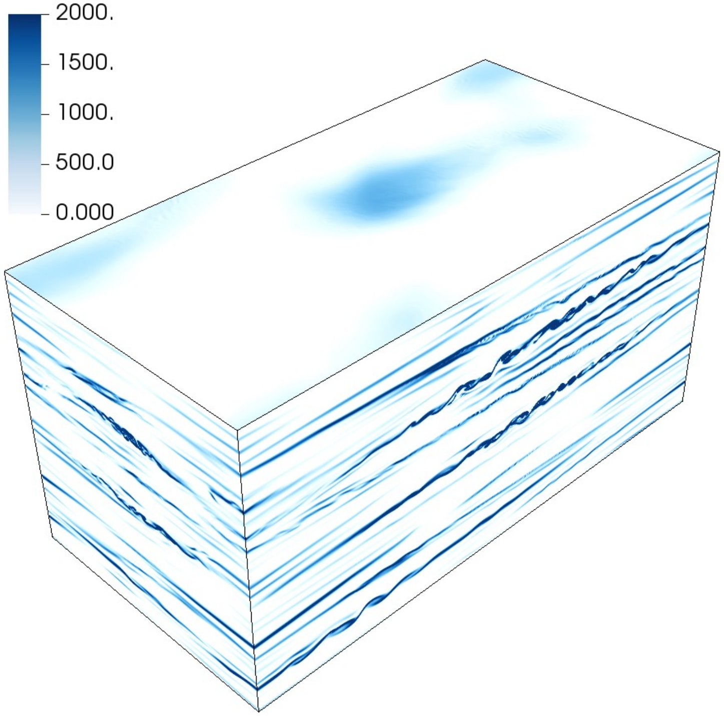
Take-home messages

- Various models have been proposed for stratified turbulence, including single horizontal scale and multiscale models.
- Models predict regime diagram for stratified turbulence, as well as scaling laws for various quantities of interest (length scale, vertical velocity, buoyancy fluctuations)
- **DNS have validated all models in their regions of validity.**
- At low / intermediate stratification, multiscale model is best; at stronger stratification, single scale model is best (until viscosity takes over).

EXTRAS

Spatio-temporal patchiness

$|\boldsymbol{\omega}|^2$



$Re = 600, Pe = 60, Fr = 0.05$

Commonly-used procedure to identify turbulent regions is to use viscous dissipation rate (eq. enstrophy).

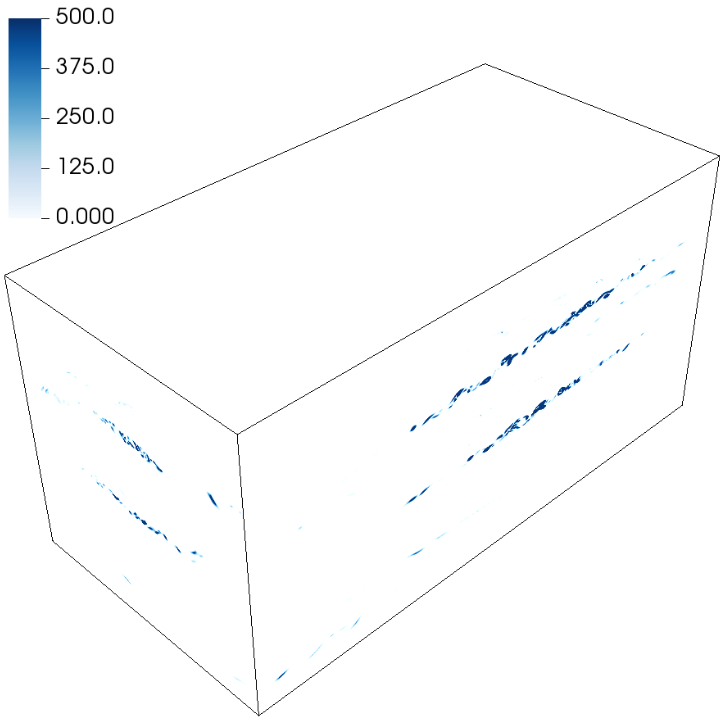
But: this picks up turbulent patches **and** regions of strong laminar dissipation.

$$\omega_y = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}$$

$O(Fr^{-1})$ $O(Fr^{-1/2})$

Spatio-temporal patchiness

ω_z^2



Instead, use vertical vorticity:

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$$

$O(1)$ in laminar regions

$O(Fr^{-1/2})$ in turbulent regions.

Diagnostic is independent of vertical velocity.

$Re = 600, Pe = 60, Fr = 0.05$

Spatio-temporal patchiness

Define:

- regular rms (whole domain average):

$$u_{z,rms} = \langle u_z^2 \rangle^{1/2}$$

- vorticity-weighted rms (emphasizes turbulent regions):

$$u_{z,turb} = \frac{\langle u_z^2 \omega_z^2 \rangle^{1/2}}{\langle \omega_z^2 \rangle^{1/2}}$$

- inverse vorticity-weighted rms (emphasizes 'laminar' regions):

$$u_{z,lam} = \frac{\langle u_z^2 \omega_z^{-2} \rangle^{1/2}}{\langle \omega_z^{-2} \rangle^{1/2}}$$

MULTISCALE MODELS OF STRATIFIED TURBULENCE

1. At $\mathrm{Pr} = O(1)$

Case $Pr = O(1)$ (Chini et al. 2022)

$$\underbrace{\frac{\partial \bar{b}}{\partial t_s} + \bar{\boldsymbol{u}}_h \cdot \nabla_s \bar{b} + \frac{\bar{u}_z}{\alpha}\frac{\partial \bar{b}}{\partial \zeta}}_{O(\bar{b})} + \underbrace{\frac{1}{\alpha}\frac{\partial}{\partial \zeta}\overline{u'_z b'}}_{O(\alpha^{-1/2} b')} + \underbrace{\bar{u}_z}_{O(\alpha)} = \underbrace{\frac{1}{Pe_b}\left(\alpha^2 \nabla_s^2 \bar{b} + \frac{\partial^2 \bar{b}}{\partial \zeta^2}\right)}_{O\left(\frac{\bar{b}}{Pe_b}\right)\ \text{(small)}}$$

→ Stratification relevant only if $\bar{b} = O(\alpha)$

→ Turbulent flux affects mean if $b' = O(\alpha^{3/2})$

This suggests that

$$\bar{b} = \alpha \bar{b}_2 + \cdots$$
$$b' = \alpha^{3/2} b'_3 + \cdots$$

Case $Pr = O(1)$ (Chini et al. 2022)

Substituting asymptotic series in each equation reveals:

- From vertical momentum equation we find, as before, $\alpha = Fr$

  → has already been validated experimentally

- Mean and fluctuation equations naturally separate into closed quasilinear system at leading order (Chini et al. 2022)

Case $Pr = O(1)$ (Chini et al. 2022)

Mean equations (non-diffusive)

$$\nabla_s \cdot \bar{\boldsymbol{u}}_{h0} + \frac{\partial \bar{u}_{z2}}{\partial \zeta} = 0$$

$$\frac{\partial \bar{\boldsymbol{u}}_{h0}}{\partial t_s} + \bar{\boldsymbol{u}}_{h0} \cdot \nabla_s \bar{\boldsymbol{u}}_{h0} + \bar{u}_{z2} \frac{\partial \bar{\boldsymbol{u}}_{h0}}{\partial \zeta} + \frac{\partial}{\partial \zeta} \overline{u'_{z1} \boldsymbol{u}'_{h1}} = -\nabla_s \bar{p}_0 + \frac{1}{Re_b} \frac{\partial^2 \bar{\boldsymbol{u}}_{h0}}{\partial \zeta^2} + \sin(y) \boldsymbol{e}_x$$

$$\frac{\partial \bar{b}_2}{\partial t_s} + \bar{\boldsymbol{u}}_{h0} \cdot \nabla_s \bar{b}_2 + \frac{\partial}{\partial \zeta} \overline{u'_{z1} b'_3} + \bar{u}_{z2} \left(1 + \frac{\partial \bar{b}_2}{\partial \zeta}\right) = \frac{1}{Pe_b} \frac{\partial^2 \bar{b}_2}{\partial \zeta^2}$$

$$-\frac{\partial \bar{p}_0}{\partial \zeta} + \bar{b}_2 = 0$$

Fluctuation equations (non-diffusive)

$$\nabla_s \cdot \boldsymbol{u}'_{h1} + \frac{\partial u'_{z1}}{\partial \zeta} = 0$$

$$\frac{\partial \boldsymbol{u}'_{h1}}{\partial t_f} + \bar{\boldsymbol{u}}_{h0} \cdot \nabla_s \boldsymbol{u}'_{h1} + u'_{z1} \frac{\partial \bar{\boldsymbol{u}}_{h0}}{\partial \zeta} = -\nabla_s p'_1 + \frac{\alpha}{Re_b} \left(\nabla_f^2 \boldsymbol{u}'_{h1} + \frac{\partial^2 \boldsymbol{u}'_{h1}}{\partial \zeta^2} \right)$$

$$\frac{\partial b'_3}{\partial t_f} + \bar{\boldsymbol{u}}_{h0} \cdot \nabla_f b'_3 + u'_{z1} \left(1 + \frac{\partial \bar{b}_2}{\partial \zeta}\right) = \frac{\alpha}{Pe_b} \left(\nabla_f^2 b'_3 + \frac{\partial^2 b'_3}{\partial \zeta^2} \right)$$

$$\frac{\partial u'_{z1}}{\partial t_f} + \bar{\boldsymbol{u}}_{h0} \cdot \nabla_f u'_{z1} = -\frac{\partial p'_1}{\partial \zeta} + b'_3 + \frac{\alpha}{Re_b} \left({\nabla_f}^2 u'_{z1} + \frac{\partial^2 u'_{z1}}{\partial \zeta^2} \right)$$

MULTISCALE MODELS OF STRATIFIED TURBULENCE

2. At $Pr \ll 1$

Case Pr $\ll$ 1 (Shah et al., in prep)

Mean buoyancy equation:

$$\underbrace{\frac{\partial \bar{b}}{\partial t_s} + \bar{\boldsymbol{u}}_h \cdot \nabla_s \bar{b} + \frac{\bar{u}_z}{\alpha}\frac{\partial \bar{b}}{\partial \zeta}}_{O(\bar{b})} + \underbrace{\frac{1}{\alpha}\frac{\partial}{\partial \zeta}\overline{u'_z b'}}_{O(\alpha^{-1/2} b')} + \underbrace{\bar{u}_z}_{O(\alpha)} = \underbrace{\frac{1}{Pe_b}\left(\alpha^2 \nabla_s^2 \bar{b} + \frac{\partial^2 \bar{b}}{\partial \zeta^2}\right)}_{O\left(\frac{\bar{b}}{Pe_b}\right)}$$

- If $Pe_b = PrRe_b > O(1)$ despite $Pr \ll 1$ then we recover the Chini et al. 2022 results "as is".
- But if $Pe_b \ll 1$ diffusive terms are important: we can show

$$\underbrace{\bar{u}_z = \frac{1}{Pe_b}\frac{\partial^2 \bar{b}}{\partial \zeta^2}}_{Pe_b \ll 1 \text{ dominant balance}}$$

$\bar{b} = O(\alpha Pe_b)$ as before

Case Pr ≪ 1 (Shah et al., in prep)

Implications:

We need a **2-parameter asymptotic expansion** in $\alpha^{1/2}$ and Pe_b

$$\bar{q} = \bar{q}_{00} + \alpha^{1/2}\bar{q}_{01} + \alpha\bar{q}_{02} + \cdots + Pe_b\left(\bar{q}_{10} + \alpha^{1/2}\bar{q}_{11} + \cdots\right) + \cdots$$

$$q' = q'_{00} + \alpha^{1/2}\, q'_{01} + \alpha\, q'_{02} + \cdots + Pe_b\left(q'_{10} + \alpha^{1/2}\, q'_{11} + \cdots\right) + \cdots$$

So far we have found that at leading order

$$\bar{u}_h = \bar{u}_{h00} + \cdots, \qquad \bar{p} = \bar{p}_{00} + \cdots$$

$$\bar{w} = \alpha\bar{w}_{02} + \cdots, \bar{b} = \alpha Pe_b \bar{b}_{12} + \cdots$$

$$u'_h = \alpha^{1/2} u'_{h01} + \cdots$$

$$w' = \alpha^{1/2} w'_{01} + \cdots$$

$$p' = \alpha^{1/2} p'_{01} + \cdots$$

What is b'?

Case Pr ≪ 1 (Shah et al., in prep)

Perturbation buoyancy equation:

dominant balance depends on size of $\frac{\alpha}{Pe_b}$

$$\frac{\partial b'}{\partial t_f} + \bar{\boldsymbol{u}}_h \cdot \nabla_f b' + \alpha^{\frac{3}{2}} Pe_b u'_{z1} \frac{\partial \bar{b}_{12}}{\partial \zeta} + \alpha^{\frac{3}{2}} u'_{z1} = \frac{\alpha}{Pe_b}\left(\nabla_f{}^2 b' + \frac{\partial^2 b'}{\partial \zeta^2}\right) + h.o.t$$

If $\frac{\alpha}{Pe_b} \leq O(1)$

(perturbations non-diffusive)

$\rightarrow b' = O\left(\alpha^{3/2}\right) = \alpha^{3/2} b'_3 + \cdots$

as before

If $\frac{\alpha}{Pe_b} \gg 1$

(perturbations are diffusive)

$\rightarrow b' = O\left(\alpha^{1/2} Pe_b\right) = \alpha^{1/2} Pe_b b'_{11} + \cdots$

Case $Pe_b \ll \alpha$ (fully diffusive regime)

Mean equations (diffusive)

$$\nabla_s \cdot \bar{\boldsymbol{u}}_{h00} + \frac{\partial \bar{u}_{z02}}{\partial \zeta} = 0$$

$$\frac{\partial \bar{\boldsymbol{u}}_{h00}}{\partial t_s} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_s \bar{\boldsymbol{u}}_{h00} + \bar{u}_{z02} \frac{\partial \bar{\boldsymbol{u}}_{h00}}{\partial \zeta} + \frac{\partial}{\partial \zeta} \overline{u'_{z01} \boldsymbol{u}'_{h01}} = -\nabla_s \bar{p}_{00} + \frac{1}{Re_b} \frac{\partial^2 \bar{\boldsymbol{u}}_{h00}}{\partial \zeta^2} + \sin(y) \boldsymbol{e}_x$$

$$\frac{\partial \bar{b}_{12}}{\partial t_s} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_s \bar{b}_{12} + \frac{\partial}{\partial \zeta} \overline{u'_{z01} b'_{03}} + \bar{u}_{z02} \left(1 + \frac{\partial \bar{b}_{12}}{\partial \zeta}\right) = \frac{1}{Pe_b} \frac{\partial^2 \bar{b}_{12}}{\partial \zeta^2}$$

$$-\frac{\partial \bar{p}_{00}}{\partial \zeta} + \bar{b}_{12} = 0$$

Fluctuation equations (diffusive)

$$\nabla_s \cdot \boldsymbol{u}'_{h01} + \frac{\partial u'_{z01}}{\partial \zeta} = 0$$

$$\frac{\partial \boldsymbol{u}'_{h01}}{\partial t_f} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_s \boldsymbol{u}'_{h01} + u'_{z01} \frac{\partial \bar{\boldsymbol{u}}_{h00}}{\partial \zeta} = -\nabla_s p'_{01} + \frac{\alpha}{Re_b} \left(\nabla_f^2 \boldsymbol{u}'_{h01} + \frac{\partial^2 \boldsymbol{u}'_{h01}}{\partial \zeta^2} \right)$$

$$\frac{\partial b'_{11}}{\partial t_f} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_f b'_{11} + u'_{z01} \left(1 + \frac{\partial \bar{b}_{12}}{\partial \zeta}\right) = \frac{\alpha}{Pe_b} \left(\nabla_f^2 b'_{11} + \frac{\partial^2 b'_{11}}{\partial \zeta^2} \right)$$

$$\frac{\partial u'_{z01}}{\partial t_f} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_f u'_{z01} = -\frac{\partial p'_{01}}{\partial \zeta} + b'_{11} + \frac{\alpha}{Re_b} \left({\nabla_f}^2 u'_{z01} + \frac{\partial^2 u'_{z01}}{\partial \zeta^2} \right)$$

Case $\alpha \ll Pe_b \ll 1$ (intermediate regime)

Mean equations (diffusive)

$$\nabla_s \cdot \bar{\boldsymbol{u}}_{h00} + \frac{\partial \bar{u}_{z02}}{\partial \zeta} = 0$$

$$\frac{\partial \bar{\boldsymbol{u}}_{h00}}{\partial t_s} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_s \bar{\boldsymbol{u}}_{h00} + \bar{u}_{z02} \frac{\partial \bar{\boldsymbol{u}}_{h00}}{\partial \zeta} + \frac{\partial}{\partial \zeta} \overline{u'_{z01} \boldsymbol{u}'_{h01}} = -\nabla_s \bar{p}_{00} + \frac{1}{Re_b} \frac{\partial^2 \bar{\boldsymbol{u}}_{h00}}{\partial \zeta^2} + \sin(y) \boldsymbol{e}_x$$

$$\frac{\partial \bar{b}_{12}}{\partial t_s} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_s \bar{b}_{12} + \frac{\partial}{\partial \zeta} \overline{u'_{z01} b'_{03}} + \bar{u}_{z02} \left(1 + \frac{\partial \bar{b}_{12}}{\partial \zeta}\right) = \frac{1}{Pe_b} \frac{\partial^2 \bar{b}_{12}}{\partial \zeta^2}$$

$$-\frac{\partial \bar{p}_{00}}{\partial \zeta} + \bar{b}_{12} = 0$$

Fluctuation equations (non-diffusive)

$$\nabla_s \cdot \boldsymbol{u}'_{h01} + \frac{\partial u'_{z01}}{\partial \zeta} = 0$$

$$\frac{\partial \boldsymbol{u}'_{h01}}{\partial t_f} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_s \boldsymbol{u}'_{h01} + u'_{z01} \frac{\partial \bar{\boldsymbol{u}}_{h00}}{\partial \zeta} = -\nabla_s p'_{01} + \frac{\alpha}{Re_b} \left(\nabla_f^2 \boldsymbol{u}'_{h01} + \frac{\partial^2 \boldsymbol{u}'_{h01}}{\partial \zeta^2} \right)$$

$$\frac{\partial b'_{03}}{\partial t_f} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_f b'_{03} + u'_{z01} \left(1 + \frac{\partial \bar{b}_{12}}{\partial \zeta}\right) = \frac{\alpha}{Pe_b} \left(\nabla_f^2 b'_{03} + \frac{\partial^2 b'_{03}}{\partial \zeta^2} \right)$$

$$\frac{\partial u'_{z01}}{\partial t_f} + \bar{\boldsymbol{u}}_{h00} \cdot \nabla_f u'_{z01} = -\frac{\partial p'_{01}}{\partial \zeta} + b'_{03} + \frac{\alpha}{Re_b} \left({\nabla_f}^2 u'_{z01} + \frac{\partial^2 u'_{z01}}{\partial \zeta^2} \right)$$

Properties of the quasilinear system

In this quasilinear system

- In each regime, perturbations grow exponentially due to shear instability of the mean flow…

$$\frac{\partial \boldsymbol{u}'_{h01}}{\partial t_f} + \overline{\boldsymbol{u}}_{h00} \cdot \nabla_s \boldsymbol{u}'_{h01} + u'_{z01} \frac{\partial \overline{\boldsymbol{u}}_{h00}}{\partial \zeta} = -\nabla_s p'_{01} + \frac{\alpha}{Re_b} \left(\nabla_f^2 \boldsymbol{u}'_{h01} + \frac{\partial^2 \boldsymbol{u}'_{h01}}{\partial \zeta^2} \right)$$

provided the buoyancy restoring force is not too strong, e.g.

$$\frac{\partial u'_{z01}}{\partial t_f} + \overline{\boldsymbol{u}}_{h00} \cdot \nabla_f u'_{z01} = -\frac{\partial p'_{01}}{\partial \zeta} + b'_{11} + \frac{\alpha}{Re_b} \left({\nabla_f}^2 u'_{z01} + \frac{\partial^2 u'_{z01}}{\partial \zeta^2} \right)$$

- Buoyancy force controlled by buoyancy evolution equation (regime dependent)

Properties of the quasilinear system

- If fluctuations are non-diffusive, criterion for marginal stability is well-known Richardson criterion

$$\frac{N^2}{S^2} = O(1) \leftrightarrow \frac{N^2}{(U/l_z)^2} = O(1) \leftrightarrow l_z = \frac{U}{N}$$

…. which is exactly what we found.

- If fluctuations are diffusive, criterion for marginal stability is (less) well-known Dudis/Jones criterion

$$\frac{N^2}{S^2} Pe_l = O(1) \leftrightarrow \frac{N^2}{(U/l_z)^2}\frac{Ul_z}{\kappa} = O(1) \leftrightarrow l_z^3 = \frac{\kappa U}{N^2}$$

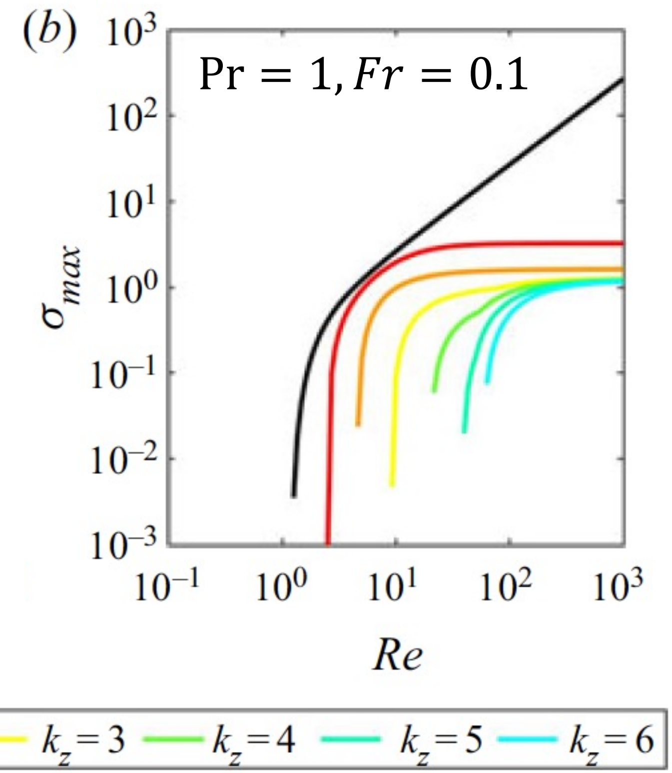
…. which is also exactly what we found.

The quasilinear system adjusts itself to be in marginal equilibrium with respect to shear instability of the mean flow!

Linear stability

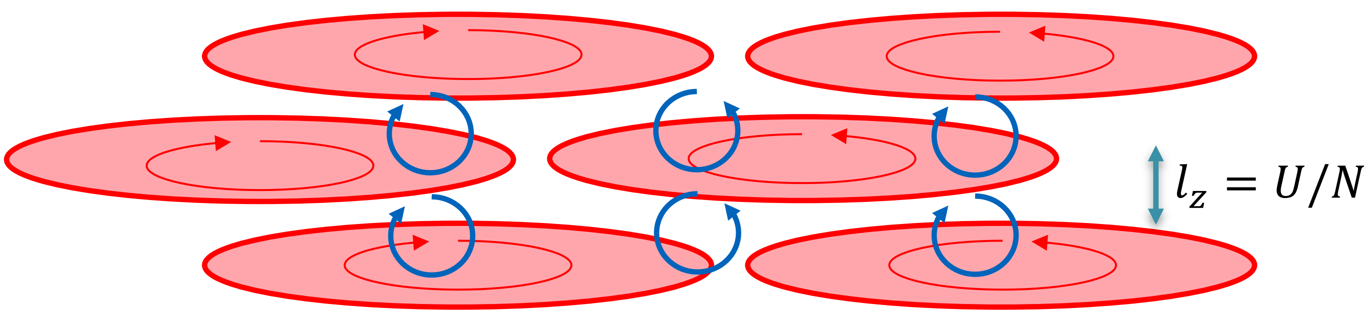
Linear stability analysis (Balmforth & Young, 2002, Arobone & Sarkar 2012, Cope et al. 2020) shows

- Fastest-growing mode is vertically invariant with no vertical motion ($\frac{\partial}{\partial z} \equiv 0, u_z \equiv 0$).
- Other modes exist that have vertical structure & flow motion (but grow much more slowly)



Physical consequences

(1) "Self-organized criticality"



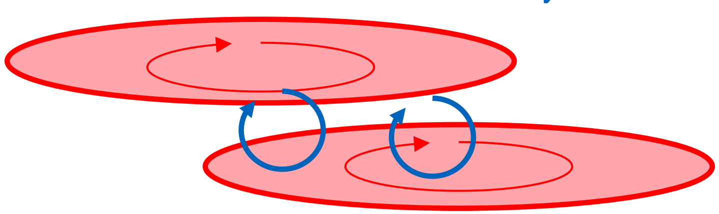
$$S \sim \frac{U}{l_z} = \frac{U}{U/N} = N \;\rightarrow\; J = O(1) \quad \text{(marginal stability)}$$

Physical consequences

(2) Critical Reynolds number for stratified mixing:

→ Vertical shear only becomes turbulent if emergent Reynolds number on small scales l_z is “large”:

$$\frac{u_z l_z}{\nu} = Fr^2 \frac{UL}{\nu} = Re_b > O(1)$$



Buoyancy Reynolds number
must be large to be in turbulent regime

If $Re_b \ll 1$ instead, viscously dominated regime with different scaling laws

The importance of the Péclet number

Pr is not important on its own: it matters because of the Péclet number,

$$Pe = \frac{"u \cdot \nabla T"}{"\kappa_T \nabla^2 T"} = \frac{UL}{\kappa_T}$$

- Thermal diffusion is negligible when $Pe \gg 1$
- Thermal diffusion is important when $Pe = O(1)$ or $Pe \ll 1$ (small scales and/or slow flows)

The importance of the Péclet number

Note how $Pe = \frac{UL}{\kappa_T} = \frac{UL}{\nu}\frac{\nu}{\kappa_T} \rightarrow Pe = Pr\,Re$

- When $Pr = O(1)$ (oceans)

$$Re \gg 1 \quad \leftrightarrow \quad Pe \gg 1$$

so turbulent flows ($Re \gg 1$) are always non-diffusive ($Pe \gg 1$) .

- When $Pr \ll 1$ (stars) it is possible to have

$$Re \gg 1 \;\; and \;\; Pe \ll 1$$

so turbulent flows ($Re \gg 1$) can be diffusive ($Pe \ll 1$).

This opens up completely new regimes of stratified turbulence!