AM 227
UCSC, Spring 2024, Final Project
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Appendix

1.1 Derivation of $\boldsymbol{u}_{\mathrm{fgm}}/\boldsymbol{u}_b$

Numerical computations of the fastest growing mode yield fourier modes for $\boldsymbol{u}_{\mathrm{fgm}}$. We have the following eigenvector to describe the coefficients of each component. The first three components are the coefficients u_{-1}, u_0, u_1, the middle three are v_{-1}, v_0, v_1, and the last three are w_{-1}, w_0, w_1.

$$\begin{bmatrix} u_{\mathrm{fgm}} \\ v_{\mathrm{fgm}} \\ w_{\mathrm{fgm}} \\ t_{\mathrm{fgm}} \\ p_{\mathrm{fgm}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0.5 & -0.91 & -0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.36 & 0 & 0.36 \end{bmatrix} \begin{bmatrix} e^{-ik_y y} \\ 1 \\ e^{ik_y y} \end{bmatrix} e^{\lambda t + ik_x x}$$

We notice that all of the coefficients for w are zero, thus we find that $w_{\mathrm{fgm}} = 0$. Now we simply must find the real parts of the sum of the three fourier modes for u_{fgm} and v_{fgm}.

$$\begin{aligned} u_{\mathrm{fgm}} &= e^{\lambda t + i(k_x x - k_y y)} + e^{\lambda t + i(k_x x + k_y y)} \\ v_{\mathrm{fgm}} &= 0.5 \cdot e^{\lambda t + i(k_x x - k_y y)} - 0.91 \cdot e^{\lambda t + ik_x x} - 0.5 \cdot e^{\lambda t + i(k_x x + k_y y)} \end{aligned}$$

We will of course be taking the "frozen-in-time" approximations for the background flow, so we will proceed without writing λt.

$$\begin{aligned} u_{\mathrm{fgm}} &= e^{ik_x x}\left(e^{-ik_y y} + e^{ik_y y}\right) \\ v_{\mathrm{fgm}} &= -0.5 \cdot e^{ik_x x}\left(e^{ik_y y} - e^{-ik_y y}\right) - 0.91 \cdot e^{ik_x x} \end{aligned}$$

We notice that the forms of these complex exponentials yield complex and real components and so we identify which will contribute to the real part.

$$\begin{aligned} u_{\mathrm{fgm}} &= 2\cos(k_x x) \cdot \cos(k_y y) \\ v_{\mathrm{fgm}} &= \sin(k_x x) \cdot \sin(k_y y) - 0.91\cos(k_x x) \end{aligned}$$

Finally we add the shear term in order to complete $\boldsymbol{u}_b$.

$$\begin{aligned} \boldsymbol{u}_b &= (\overline{u} + u_{\mathrm{fgm}})\hat{e}_x + v_{\mathrm{fgm}}\hat{e}_y \\ \overline{u} &= \sin(k_y y) \\ u_{\mathrm{fgm}} &= 2\cos(k_x x)\cos(k_y y) \\ v_{\mathrm{fgm}} &= \sin(k_x x)\sin(k_y y) - 0.91\cos(k_x x) \end{aligned}$$

$$\begin{aligned} \overline{u} &= \frac{1}{2i}\left(e^{ik_y y} - e^{-ik_y y}\right) \\ u_{\mathrm{fgm}} &= \frac{1}{2}\left(e^{ik_x x} + e^{-ik_x x}\right)\left(e^{ik_y y} + e^{-ik_y y}\right) \\ v_{\mathrm{fgm}} &= -\frac{1}{4}\left(e^{ik_x x} - e^{-ik_x x}\right)\left(e^{ik_y y} - e^{-ik_y y}\right) - \frac{0.91}{2}\left(e^{ik_x x} + e^{-ik_x x}\right) \end{aligned}$$

$$u_{\mathrm{fgm}} = \frac{1}{2}\left(e^{ik_x x + ik_y y} + e^{ik_x x + ik_y y} + e^{ik_x x + ik_y y} + e^{ik_x x + ik_y y}\right) \tag{1}$$

$$v_{\mathrm{fgm}} = -\frac{1}{4}\left(e^{ik_x x + ik_y y} - e^{ik_x x - ik_y y} - e^{-ik_x x + ik_y y} + e^{-ik_x x - ik_y y}\right) - \frac{0.91}{2}\left(e^{ik_x x} + e^{-ik_x x}\right) \tag{2}$$

1.2 Derivation of non-linear terms

This section has two essential components to investigate. In Navier-Stokes, the infamous non-linear advection terms are often the at the center of simplicifation. In this problem, we consider simplified non-linear terms in order to reduce the algebraic complexity.

$$\begin{aligned} \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= (\boldsymbol{u}_b + \tilde{\boldsymbol{u}}) \cdot \nabla (\boldsymbol{u}_b + \tilde{\boldsymbol{u}}) && (3) \\ &\simeq \boldsymbol{u}_b \cdot \nabla \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}_b && (4) \end{aligned}$$

From this simplification, we realize there are a few essential outcomes of these terms.

1.3 Derivation of continuity equation

Finally, we consider the continuity equation:

$$\nabla \cdot \tilde{\boldsymbol{u}} = \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z}$$

$$\Rightarrow n k_x \hat{u}_{m,n} + m k_v \hat{v}_{m,n} + k_z \hat{w}_{m,n} = 0 \qquad (5)$$