Lecture 2

The definite integral, I

- Indefinite integrals, continued.
- Review of summation
- Motivational example.

1/

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = ?$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$? \longrightarrow \int x^{-1}\,dx = \ln x + C$$

Problem:

x^{-1} is defined for all $x \neq 0$, but $\ln x$ is defined only for $x > 0$ . . .

2/ <u>Idea</u>: try $\ln|x|$.

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad ✓$$

check:

$$\frac{d}{dx}\ln|x| = \dots \frac{d}{dx}\begin{cases} \ln x & : x>0 \\ \ln(-x) & : x<0 \end{cases}$$

$$x>0: \dots = (\ln x)' = \frac{1}{x}$$

$$x<0: \dots (\ln(-x))' = \underbrace{\frac{1}{-x}\cdot(-1)}_{\text{chain rule}} = \frac{1}{x}$$

👏

3/ Example

$$\int \frac{x^2 - 3\sqrt{x}}{x^3}\,dx = \int \frac{x^2}{x^3} - \frac{3x^{1/2}}{x^3}\,dx$$

$$= \int x^{-1}\,dx - 3\int x^{-5/2}\,dx$$

$$= \ln|x| - 3\cdot\frac{x^{-3/2}}{-3/2} + C$$

$$= \ln|x| + 2x^{-3/2} + C$$

4/ **So far** :

"rules"

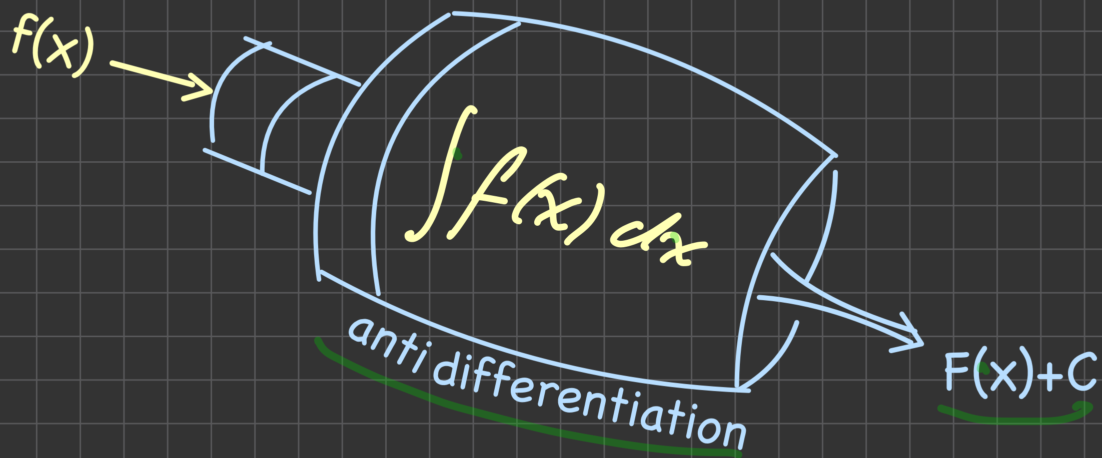
1. $\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
2. $a \neq 0 \longrightarrow \int a\cdot f(x)\,dx = a\int f(x)\,dx$

formulas

3. $k \neq -1 \longrightarrow \int x^k\,dx = \frac{1}{k+1}x^{k+1}+C$
4. $\int x^{-1}\,dx = \ln|x| + C$
5. $\int e^x\,dx = e^x + C$

5/

The **Indefinite** Integral

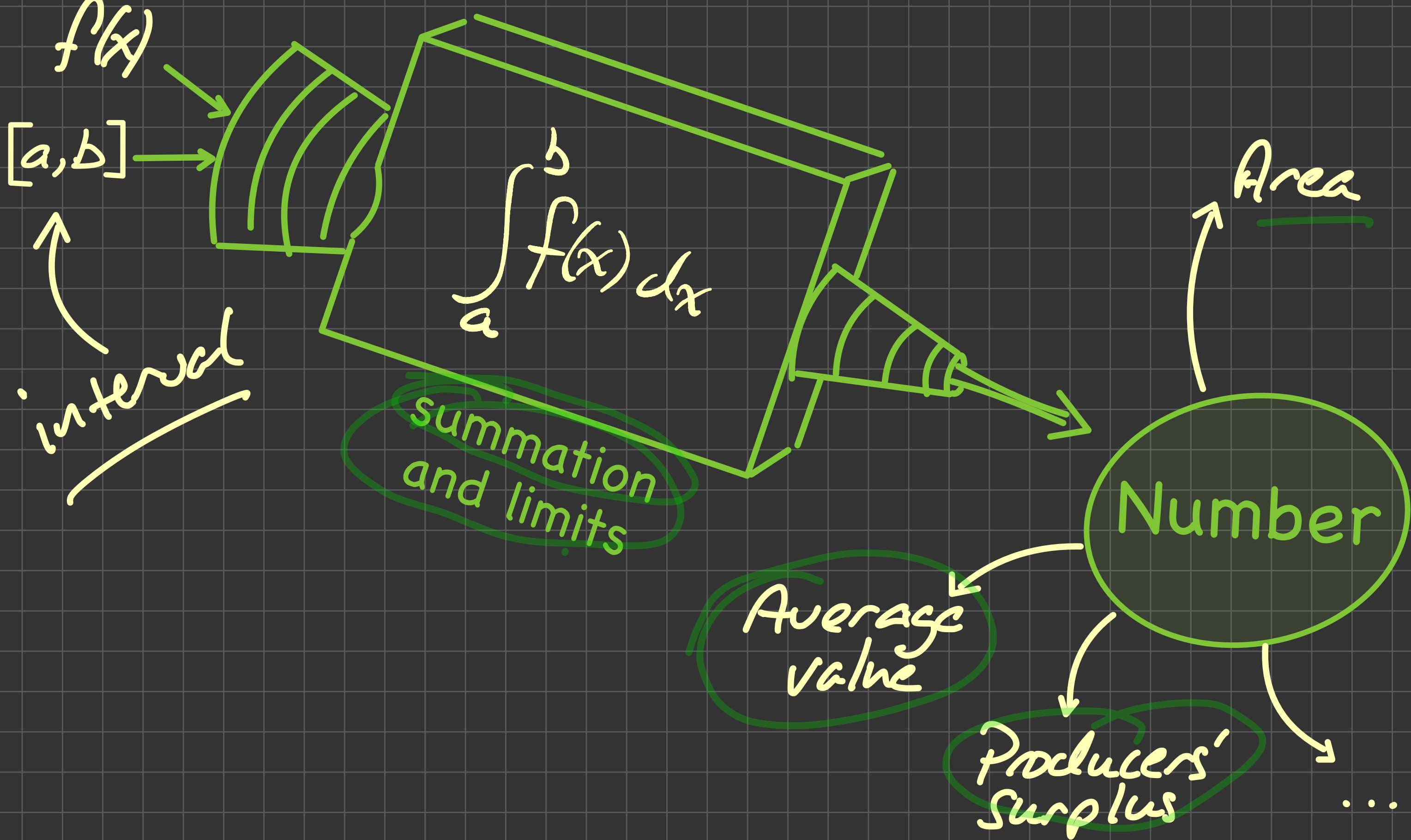


$$\begin{cases} F'(x) = f(x) \\ \text{and } C \text{ any} \\ \text{real number} \end{cases}$$

an infinite set of functions

6/

The **Definite** Integral



7/ "Summation Notation" : $\boxed{\boxed{\text{shorthand}}}$

⭐ Notation:

"Sigma"

$$A_1 + A_2 + \cdots + A_n = \sum_{i=1}^{n} A_i$$

In $\mathbb{B}$:

1. $A_i = f(x_i^*)(x_i - x_{i-1}) \quad (x_{i-1} \le x_i^* \le x_i)$
2. x_i depends on i, n and $[a, b]$.

8/ Example:

$$\sum_{i=1}^{10} \frac{i^2}{10} = \frac{1}{10} + \frac{4}{10} + \frac{9}{10} + \cdots + \frac{100}{10}$$

$$= 38.5$$

Basic Properties:

(i) Distributive Property:

$$cA_1 + cA_2 + \cdots + cA_n = c(A_1 + \cdots + A_n)$$

$$\Longleftrightarrow \sum_{i=1}^{n} c f(x_i) = c \cdot \sum_{i=1}^{n} f(x_i)$$

9/

(ii) <u>Commutative</u> and <u>associative</u> properties — we can rearrange and group the terms, as we want:

$$\sum_{i=1}^{n} \left(f(x_i) + g(x_i)\right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} g(x_i)$$

10/ Example

$$\sum_{j=1}^{10}\left(2j^2+3j+1\right) = 2\sum_{j=1}^{10} j^2 + 3\sum_{j=1}^{10} j + \sum_{j=1}^{10} 1$$

$3\cdot\frac{10\cdot 11}{2} = 165$

$= 10$

$$2\cdot\frac{\cancel{10}^{5}(11)\cdot\cancel{21}^{7}}{\cancel{6}\,\cancel{2}} = 770$$

$770+165+10 = 945$

11/

Some formulas

0. $$\sum_{k=1}^{n} 1 = \overbrace{1+1+\cdots+1}^{n} = n$$

k^0

1. $$S_1(n) = \sum_{k=1}^{n} k^1 = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$2\cdot S_1(n) = \begin{cases} \ 1+2+3+\cdots+n \\ +\ n+(n-1)+(n-2)+\cdots+1 \end{cases}$$

$$= (n+1)+(n+1)+(n+1)+\cdots+(n+1)$$

$$= n\cdot(n+1)$$

12/

Or...

💡 $(k+1)^2 - k^2 = k^2+2k+1-k^2 = 2k+1$

$$\Longrightarrow \sum_{k=1}^{n}\left((k+1)^2 - k^2\right) = \sum_{k=1}^{n}(2k+1)$$

$$\text{(i)} \quad \sum_{k=1}^{n}(2k+1) = 2\cdot\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$\| \qquad = 2\cdot S_1(n) + n$$

?

$$\text{(ii)} \quad \sum_{k=1}^{n}\left((k+1)^2 - k^2\right) = \ldots ?$$

13/

$$\ldots = \overset{k=1}{\cancel{2^2} - 1^2} + \overset{k=2}{\cancel{3^2} - \cancel{2^2}} + \overset{k=3}{\cancel{4^2} - \cancel{3^2}} + \cancel{\ldots}$$

$$\cancel{\ldots +} \underset{k=n-1}{\cancel{n^2} - \cancel{(n-1)^2}} + \underset{k=n}{(n+1)^2 - \cancel{n^2}}$$

$$= (n+1)^2 - 1^2 = n^2 + 2n + 1 - 1 = n^2 + 2n$$

$$\Longrightarrow \quad 2 \cdot S_1(n) + \cancel{n} = n^2 + \cancel{2}n$$

$$2 \cdot S_1(n) = n^2 + n$$

$$\Longrightarrow \quad S_1(n) = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

14/

0. $$\sum_{k=1}^{n} k^0 = \sum_{k=1}^{n} 1 = n$$

1. $$\sum_{k=1}^{n} k^1 = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

2. $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{3n^2+n}{6}$$

3. $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{2n^3+n^2}{4}$$

15/

In general:

$$S_m = \sum_{k=1}^{n} k^m = \frac{n^{m+1}}{m+1} + P_m(n)$$

polynomial of degree $\underline{\underline{m}}$

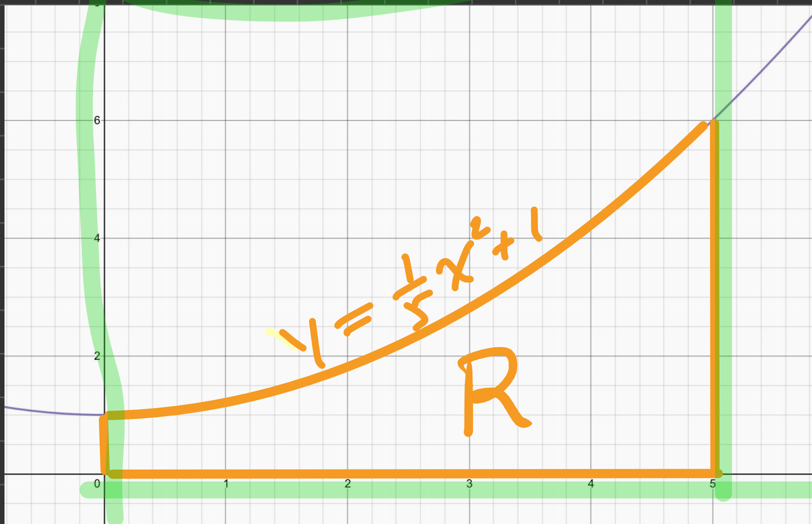
🤯

$$\int x^m \, dx = \frac{x^{m+1}}{m+1} + C$$

16/

Motivational Example

Find the area of the region R bounded by: $x = 0$, $y = 0$, $x = 5$ and $y = \frac{1}{5}x^2 + 1$



<u>Problem</u>: Area is defined in terms of squares (or rectangles), and more generally, for regions whose boundaries are straight lines ...

... $y = \frac{1}{5}x^2+1$ is <u>not</u> a straight line.

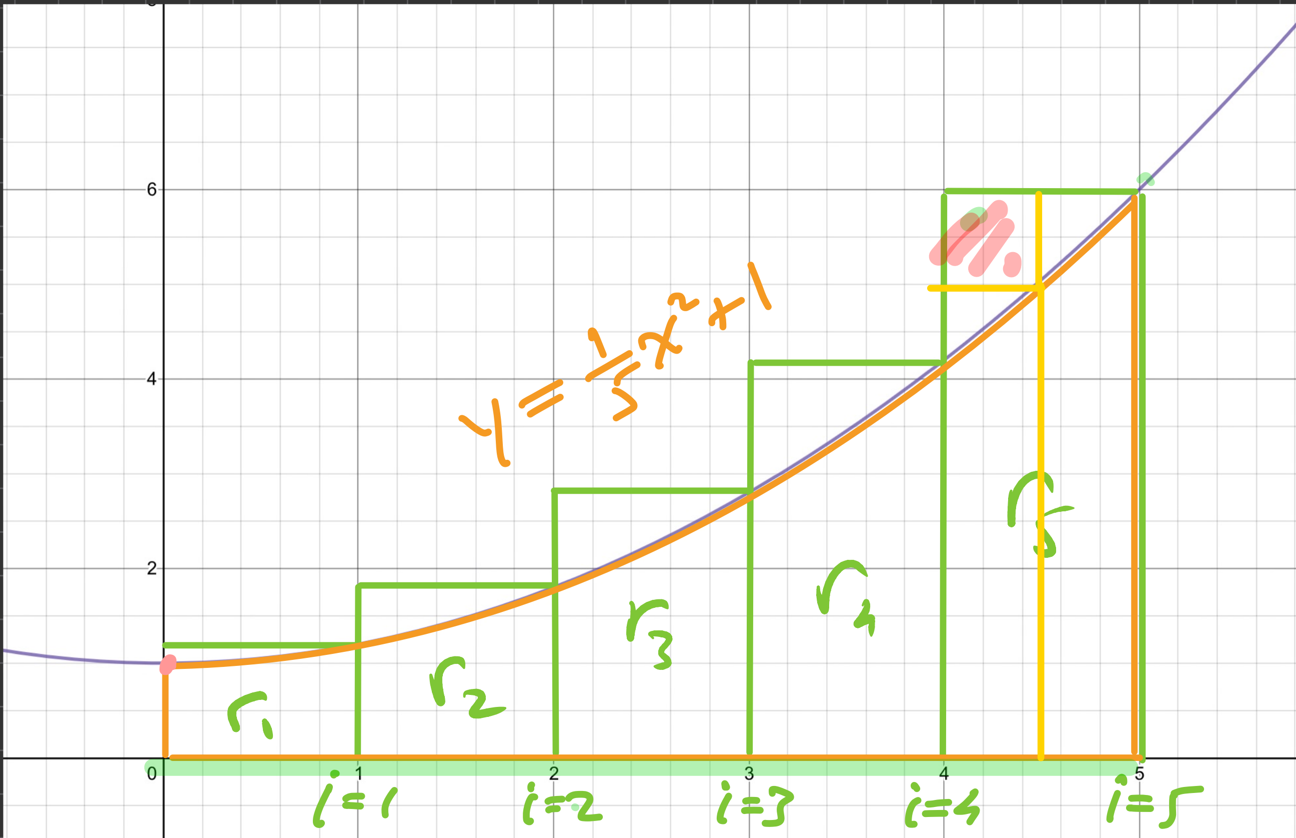
18/

Idea:

Approximate R with rectangles and use their areas to estimate $\text{area}(R)$.

19/

$$\text{Area}(R) \approx \sum_{i=1}^{5} \text{area}(r_i)$$



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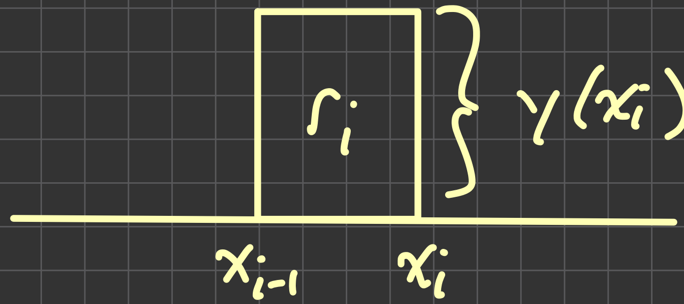
$$\text{area}(R) \approx \sum_{i=1}^{5} \text{area}(r_i) = \ldots = $$



$$y(x_i) = \frac{1}{5}x_i^2 + 1$$

$$\text{area}(r_i) = (x_i - x_{i-1}) \cdot \left(\frac{1}{5}x_i^2 + 1\right)$$

In this case: $x_i = i$

$$\longrightarrow \text{area}(r_i) = 1 \cdot \left(\frac{1}{5} \cdot i^2 + 1\right)$$

21/

$$\Longrightarrow \quad \operatorname{area}(R) \approx \sum_{i=1}^{5} \operatorname{area}(r_i)$$

$$= \sum_{i=1}^{5} \left(\frac{1}{5} i^2 + 1\right)$$

$$= \frac{1}{5} \cdot \sum_{i=1}^{5} i^2 + \sum_{i=1}^{5} 1$$

$$= \frac{1}{\cancel{5}} \cdot \frac{\cancel{5} \cdot \cancel{6} \cdot 11}{\cancel{6}} + 5 = 16$$

22/ Now what..? 🤔

🤓 Repeat: with more, <u>thinner</u> rectangles...

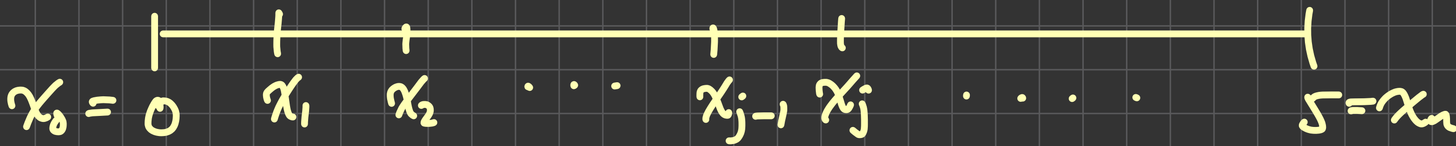
... to make the approximation <u>more precise</u>.

23)

How many rectangles? 🤔

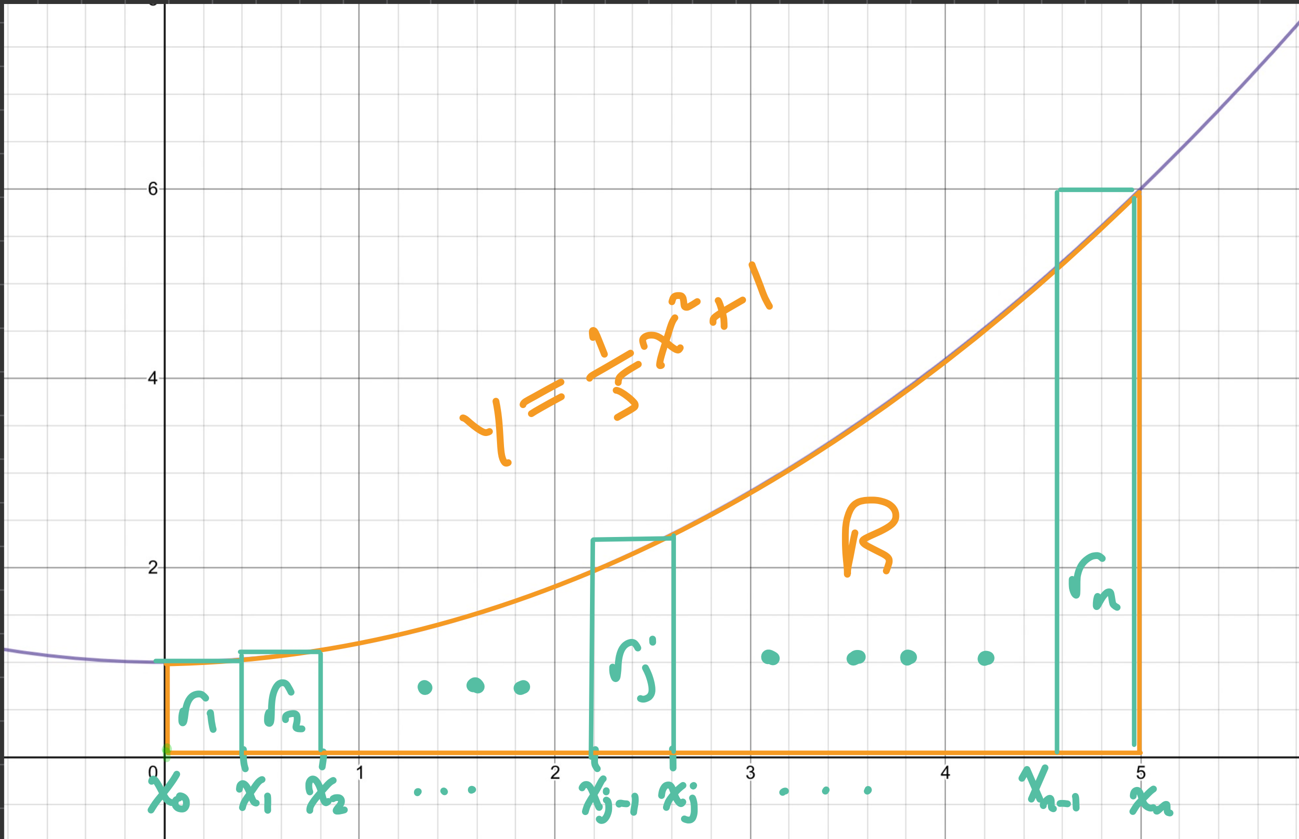
🤓 ... n.

(i) divide $[0,5]$ into n equal subintervals:



$$\Longrightarrow x_j - x_{j-1} = \frac{5}{n}$$

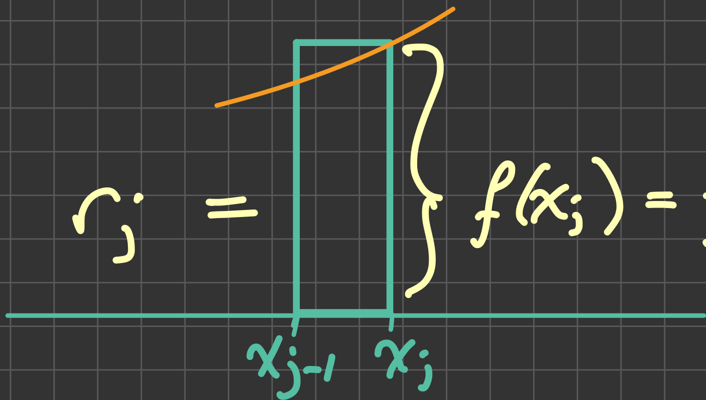
24/ Approximate R with rectangles:



25/

$$\text{Area}(R) \approx \sum_{j=1}^{n} \text{area}(r_j)$$

$r_j =$



$f(x_j) = \frac{1}{5}x_j^2 + 1$

To be continued...