Numerical Methods for the Solution of Differential Equations (AMS 213B)
Homework 2 - Due Sunday April 28

Instructions

Please submit to CANVAS one PDF file (your solution to the assignment), and one .zip file that includes any computer code you develop for the assignment. The PDF file must be a document compiled from Latex source code (mandatory for PhD students), or a PDF created using any other other word processor (MS and SciCAM students). No handwritten work should be submitted.

Question	points
1	30
2	20
3	30
4	20

Question 1 (30 points). Consider the three-dimensional linear dynamical system

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 10 & 10 & 10 \end{bmatrix}, \tag{1}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 10 & -10 \\ -100 & -1 & 0 \\ 0 & 10 & -100 \end{bmatrix}. \tag{2}$$

a) (10 points) Determine the largest value of Δt, say Δt^*, for which the three-step Adams-Bashforth method (AB3)

$$\boldsymbol{u}_{k+3} = \boldsymbol{u}_{k+2} + \frac{\Delta t}{12}\left(23\boldsymbol{f}_{k+2} - 16\boldsymbol{f}_{k+1} + 5\boldsymbol{f}_k\right) \tag{3}$$

is absolutely stable when applied to (1). You are allowed to use a numerical solver to compute the eigenvalues of $\boldsymbol{A}$.

b) (10 points) Plot the region of absolute stability of AB3.

c) (10 points) Plot the numerical solution of (1) (all three components in one figure) you obtain with AB3 in the time interval $[0, 10]$ for $\Delta t = 10^{-4}$, $\Delta t = \Delta t^* + 0.0001$ and $\Delta t = \Delta t^* - 0.0001$ (three different figures), where Δt^* is the critical value of Δt you determined in part a). Comment on your numerical results.

Solution:

(a) (b)

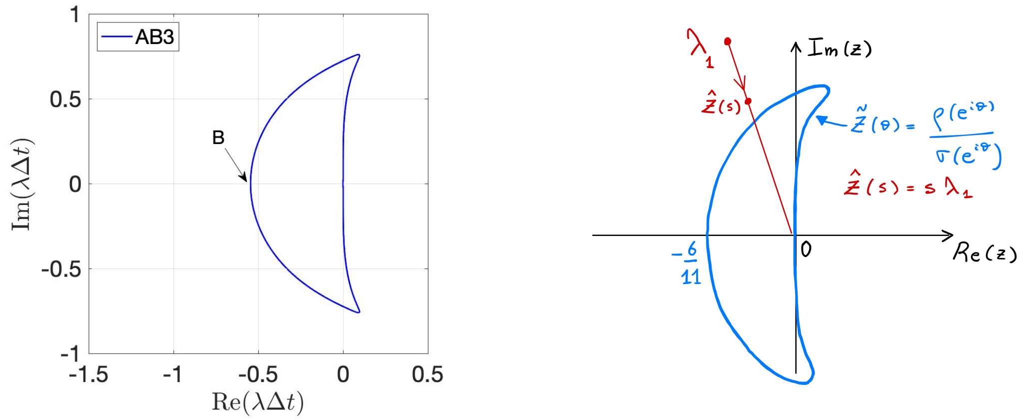

Figure 1: (a) Region of absolute stability of AB3. (b) Intersection of the line connecting λ_1 to the origin with the boundary of the region of absolute stability.

b) The boundary of the absolute stability stability region of AB3 satisfies the equation

$$\lambda \Delta t = \frac{\rho\left(e^{i\vartheta}\right)}{\sigma\left(e^{i\vartheta}\right)} \qquad \vartheta \in [0, 2\pi], \tag{4}$$

where

$$\rho(z) = z^3 - z^2, \qquad \sigma(z) = \frac{23}{12}z^2 - \frac{4}{3}z + \frac{5}{12}. \tag{5}$$

In Figure 1(a) we plot the region of absolute stability. AB3 is absolutely stable inside the closed curve.

a) The eigenvalues of the matrix $\boldsymbol{A}$ are obtained numerically as

$$\lambda_{1,2} = -\,0.966628909233147 \pm 30.125472190148354i, \tag{6}$$
$$\lambda_3 = -\,99.066742181533712. \tag{7}$$

The numerical solution of (1) we obtain with AB3 is absolutely stable (i.e., it decays to zero) for all $\Delta t < \Delta t^*$ where Δt^* is the critical value of Δt that rescales all eigenvalues of $\boldsymbol{A}$ within the region of absolute stability of AB3, with some eigenvalues possibly sitting at the boundary (critical condition). The coordinates of the point B in Figure 1(a) are

$$\mathrm{Re}(B) = \frac{\rho(-1)}{\sigma(-1)} = -\frac{6}{11} = -0.545454545454\cdots \qquad \mathrm{Im}(B) = 0. \tag{8}$$

Hence, $\Delta t \lambda_3$ is on the boundary of the absolute stability region of AB3 if and only if

$$\Delta t^* = \frac{1}{\lambda_3}\frac{6}{11} \simeq 0.005505929976531. \tag{9}$$

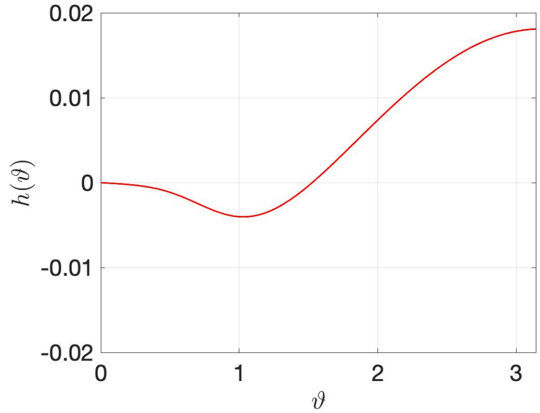

Figure 2: Graph of the function (11). The function has a zero at $\vartheta^* = 1.527216388016579$.

On the other hand, the intersection of the line connecting λ_1 to the origin (see Figure 1(b)) with the boundary of the region of absolute stability can be determined by computing the real solutions the following equation

$$\Delta t = \frac{1}{\lambda_1} \frac{\rho\left(e^{i\vartheta}\right)}{\sigma\left(e^{i\vartheta}\right)} \qquad \vartheta \in [0, \pi]. \tag{10}$$

To find such real solutions, in Figure 2 we plot the imaginary part of (10), i.e.,

$$h(\vartheta) = \operatorname{Im}\left[\frac{1}{\lambda_1} \frac{\rho\left(e^{i\vartheta}\right)}{\sigma\left(e^{i\vartheta}\right)}\right] \qquad \vartheta \in [0, \pi] \tag{11}$$

The imaginary part is zero at $\vartheta^* = 0$ (trivial solution) and at

$$\vartheta^* = 1.527216388016579. \tag{12}$$

This value is computed by using Matlab solver `fsolve()`, using $\vartheta = 2$ as initial guess (see the attached Matlab function `plot_h.m`). A substitution of (12) into (10) yields[1]

$$\Delta t = 0.023655478822220. \tag{15}$$

[1] An alternative method to compute Δt is to take the modulus of equation (10), i.e.,

$$\Delta t = \left|\frac{1}{\lambda_1}\right| \left|\frac{\rho\left(e^{i\vartheta}\right)}{\sigma\left(e^{i\vartheta}\right)}\right| \qquad \vartheta \in [0, \pi]. \tag{13}$$

This equation must be supplemented with the correct value of ϑ, i.e., the angle ϑ that solves the nonlinear equation

$$\arg(\lambda_1) = \arg\left(\frac{\rho\left(e^{i\vartheta}\right)}{\sigma\left(e^{i\vartheta}\right)}\right), \tag{14}$$

where $\arg(\lambda_1)$ denotes the argument of the complex number λ_1. Note that ϑ is *not* the argument of the complex number λ_1, but rather the solution of the nonlinear equation (14). In other words, ϑ is the argument on the unit disk $e^{i\vartheta}$ that corresponds to $\arg(\lambda_1)$ via the nonlinear mapping $\arg\left[\rho\left(e^{i\vartheta}\right)/\sigma\left(e^{i\vartheta}\right)\right]$, as detailed in equation (14). The solution to such equation is $\vartheta^* = 1.527216388016579$.

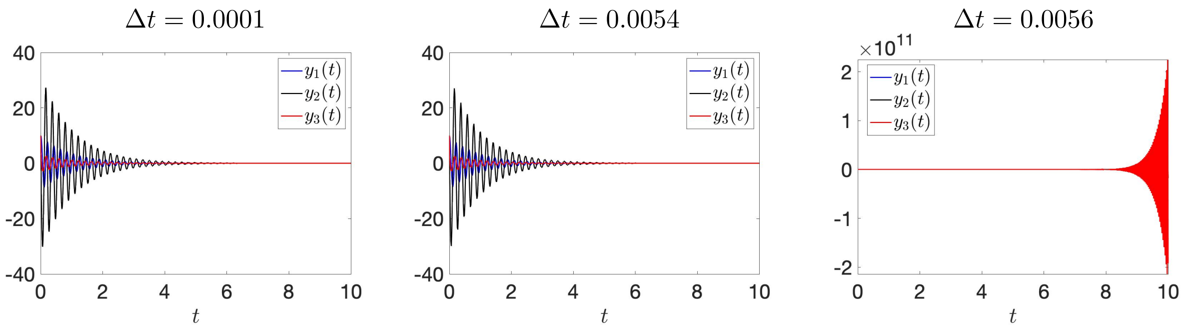

Figure 3: Solution of the system (1) with the AB3 method for different Δt. It is seen that for Δt slightly larger than the critical one defined in equation (9) the numerical solution does not decay to zero anymore.

This value of Δt is larger than the one in equation (9). Hence the critical Δt that guarantees absolute stability of AB3 for the problem (3) is the one defined in equation (9).

c) In Figure 3 we plot the solution we obtain with AB3 for different Δt.

Question 2 (20 points). Consider the BDF3 scheme

$$\boldsymbol{u}_{k+3} - \frac{18}{11}\boldsymbol{u}_{k+2} + \frac{9}{11}\boldsymbol{u}_{k+1} - \frac{2}{11}\boldsymbol{u}_k = \frac{6}{11}\Delta t \boldsymbol{f}_{k+3} \tag{16}$$

a) (10 points) Show that scheme is convergent with order 3.

b) (10 points) Plot the boundary of the absolute stability region for BDF3. Is BDF3 A-stable? Justify your answer.

Solution:

a) To show that BDF3 is convergent we need to show that the method is consistent and zero stable. The first characteristic polynomial of BDF3 is

$$\rho(z) = z^3 - \frac{18}{11}z^2 + \frac{9}{11}z - \frac{2}{11}. \tag{17}$$

For consistency we just need to check that

$$\rho(1) = 0 \qquad \rho'(1) = \frac{6}{11}. \tag{18}$$

Clearly,

$$\rho(1) = \frac{11}{11} - \frac{18}{11} + \frac{9}{11} - \frac{2}{11} = 0 \tag{19}$$

$$\rho'(z) = 3z^2 - 2\frac{18}{11}z + \frac{9}{11} \quad \Rightarrow \quad \rho'(1) = \frac{33}{11} - \frac{36}{11} + \frac{9}{11} = \frac{6}{11}, \tag{20}$$

and therefore BDF3 is consistent. Regarding the order of consistency, let us compute the coefficients

$$C_s = \frac{1}{s!}\sum_{j=0}^{q}\left(j^s \alpha_j - s j^{s-1}\beta_j\right) \qquad s = 2,3 \tag{21}$$

discussed in the course note 3. For BDF3 we have

$$\alpha_3 = 1, \qquad \alpha_2 = -\frac{18}{11}, \quad \alpha_1 = \frac{9}{11} \qquad \alpha_0 = -\frac{2}{11} \tag{22}$$

$$\alpha_3 = \frac{6}{11}, \quad \beta_2 = 0, \qquad \beta_1 = 0, \qquad \beta_0 = 0. \tag{23}$$

Therefore,

$$C_0 = \rho(1) = 0 \tag{24}$$

$$C_1 = \rho'(1) - \beta_3 = 0, \tag{25}$$

$$C_2 = \frac{1}{2}\left[\frac{9}{11} - 4\frac{18}{11} + 9 - 6\frac{6}{11}\right] = 0, \tag{26}$$

$$C_3 = \frac{1}{6}\left[\frac{9}{11} - 8\frac{18}{11} + 27 - 27\frac{6}{11}\right] = 0, \tag{27}$$

$$C_4 = \frac{1}{24}\left[\frac{9}{11} - 16\frac{18}{11} + 81 - 108\frac{6}{11}\right] = \frac{1}{26}\left[81 - \frac{927}{11}\right] \neq 0. \tag{28}$$

Therefore BDF3 is consistent with order 3. For zero-stability we simply notice that the roots of (17) are (see Figure 4(a))

$$z_1 = 1, \tag{29}$$

$$z_{2,3} = 0.318181818181819 \pm 0.283863545381745i. \tag{30}$$

All roots are within the unit disk (see Figure 4(a)). Therefore the root condition is satisfied, i.e., BDF3 is zero stable. In summary, BDF3 is consistent and zero-stable, which implies that it is convergent. The convergence order coincides with the order of consistency, i.e., BDF3 is an order 3 method.

b) For A-stability we notice that the boundary of the region of absolute stability of BDF3 intersects the imaginary axis, and therefore the method is not A-stable. This result is consistent with the second Dahlquist barrier theorem stating that there is no implicit multistep method of order greater than two that is A-stable.

Question 3 (30 points). Consider the following explicit linear multistep scheme

$$\boldsymbol{u}_{k+2} - 4\boldsymbol{u}_{k+1} + 3\boldsymbol{u}_k = -2\Delta t \boldsymbol{f}(\boldsymbol{u}_k, t_k). \tag{31}$$

a) (10 points) Show that (31) is consistent and compute the order of consistency.

b) (10 points) Show that (31) is not zero-stable. Is the scheme convergent? Justify your answer.

c) (10 points) Show that (31) is unconditionally absolutely unstable.

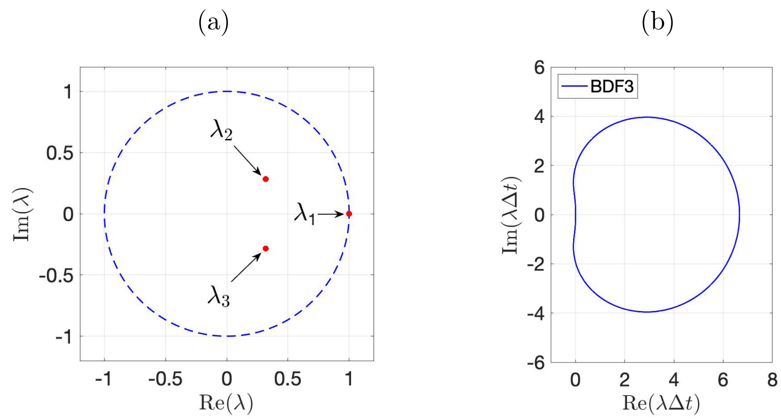

Figure 4: (a) Roots of the first characteristic polynomial associated with the BDF3 method. (b) Boundary of the region of absolute stability for BDF3. BDF3 is absolutely stable outside such region. Note that BDF3 is not A-stable since the boundary of the stability region intersects the imaginary axis and therefore there are points in $\mathbb{C}^-$ such that BDF3 is not absolutely stable.

Solution:

a) The local truncation error of the scheme is

$$\boldsymbol{\tau}_{k+2} = \frac{\boldsymbol{y}_{k+2} - 4\boldsymbol{y}_{k+1} + 3\boldsymbol{y}_k}{\Delta t} + 2\boldsymbol{f}_k, \tag{32}$$

where $\boldsymbol{y}_k$ denotes the analytical solution of the ODE evaluated at t_k. By using Taylor series we obtain

$$\boldsymbol{y}_{k+2} = \boldsymbol{y}_k + 2\Delta t \boldsymbol{f}_k + 2\Delta t^2 \frac{d^2\boldsymbol{y}_k}{dt^2} + \frac{8\Delta t^3}{6}\frac{d^3\boldsymbol{y}_k}{dt^3} + \cdots \tag{33}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + \Delta t \boldsymbol{f}_k + \frac{\Delta t^2}{2}\frac{d^2\boldsymbol{y}_k}{dt^2} + \frac{\Delta t^3}{6}\frac{d^3\boldsymbol{y}_k}{dt^3} + \cdots \tag{34}$$

A substitution of (33)-(34) into (32) yields

$$\boldsymbol{\tau}_{k+2} = \frac{2\Delta t^2}{3}\frac{d^3\boldsymbol{y}_k}{dt^3} + \cdots . \tag{35}$$

Hence, the scheme is consistent with order 2. Alternatively, one can use the general consistency theory for linear multistep methods in the course note 3 (see the answer to question 2).

b) Regarding zero-stability, we notice that the first characteristic polynomial associated with (31) is

$$\rho(z) = z^2 - 4z + 3 \tag{36}$$

The roots of $\rho(z)$ are

$$z = 2 \pm 1. \tag{37}$$

Hence the method is not zero-stable. Consistent methods that are not zero-stable are unconditionally absolutely unstable. The proof of this statement is provided in the course note 5,

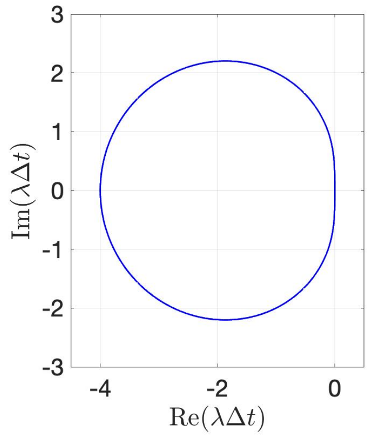

Figure 5: Set of points for which at least one root of the stability polynomial $\pi(z) = \rho(z) - \lambda\Delta t\sigma(z)$ associated with the scheme (31) has modulus equal to one. The other root has modulus always larger than one, and therefore the scheme (31) is unconditionally absolutely unstable.

"absolute stability of numerical methods for ODEs", at the top of page 10. In Figure 5 we plot the curve

$$\lambda\Delta t = \frac{\rho(e^{i\vartheta})}{\sigma(e^{i\vartheta})}, \tag{38}$$

representing the set of points for which at least one root of the stability polynomial

$$\pi(z) = \rho(z) - \lambda\Delta t\sigma(z) \tag{39}$$

has modulus equal to one. By computing the roots of $\pi(z)$ at arbitrary points within the curve and outside the curve numerically, it can be verified that the scheme (31) is unconditionally absolutely unstable. In fact, there exist at least one root of $\pi(z)$ with modulus larger than one for $\Delta t\lambda$ inside or outside the curve shown in Figure 5.

Question 4 (20 points) Consider the RK method corresponding to the following Butcher array

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 \\ \hline & 2/9 & 1/3 & 4/9 \end{array}$$

a) (10 points) Write down the given RK method and show that the method is convergent.

b) (10 points) Plot the region of absolute stability of RK method defined above. Is the method A-stable? Justify your answer.

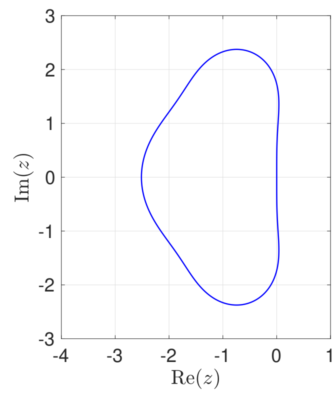

Figure 6: Region of absolute stability of the explicit RK3 method given in Question 4.

Solution:

a) The given three-stage Runge-Kutta method can be written as

$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + \Delta t \left(\frac{2}{9}\boldsymbol{K}_1 + \frac{1}{3}\boldsymbol{K}_2 + \frac{4}{9}\boldsymbol{K}_3 \right), \tag{40}$$

where

$$\boldsymbol{K}_1 = \boldsymbol{f}(\boldsymbol{u}_k, t_k), \tag{41}$$

$$\boldsymbol{K}_2 = \boldsymbol{f}\left(\boldsymbol{u}_k + \frac{\Delta t}{2}\boldsymbol{K}_1, t_k + \frac{\Delta t}{2}\right), \tag{42}$$

$$\boldsymbol{K}_3 = \boldsymbol{f}\left(\boldsymbol{u}_k + \frac{3\Delta t}{4}\boldsymbol{K}_2, t_k + \frac{3}{4}\Delta t\right). \tag{43}$$

The first characteristic polynomial is

$$\rho(z) = z - 1 \tag{44}$$

and it has only one simple root at $z = 1$ ($\rho(1) = 0$). This implies that (40), like any other RK method, is always zero-stable. We have seen in class that consistency of RK is granted if the sum of the weights b_i equals one. For the given three-stage RK method we have

$$\sum_{i=1}^{3} b_i = \frac{2}{9} + \frac{1}{3} + \frac{4}{9} = 1. \tag{45}$$

Therefore, the method is consistent and zero-stable, which implies that the method is convergent. It can be shown that the order of convergence is three in this case.

b) Regarding absolute stability, in Figure 6 we plot the boundary of the absolute stability region. The given RK3 method is stable within the region enclosed by the curve, and unstable outside. Hence the method is conditionally absolutely stable, i.e., it is not A-stable.