Homework 2: Report

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Problem 1: Absolute Stability for AB3

a) Determine the largest value of Δt, for which the three-step Adams-Bashforth method (AB3)

Proof. We use the condition for absolute stability:

$$\lim_{k\to\infty} ||\boldsymbol{u}_k|| = 0 \tag{1}$$

For this specific numerical method, we have the following characteristic polynomial for the numerical method. (Note that since the columns of B are linearly independent we have that B is diagonalizable).

$$\boldsymbol{u}_{k+3} = \boldsymbol{u}_{k+2} + \frac{\Delta t}{12}\left(23\boldsymbol{f}_{k+2} - 16\boldsymbol{f}_{k+1} + 5\boldsymbol{f}_k\right) \tag{2}$$

$$\boldsymbol{u}_{k+3} - \boldsymbol{u}_{k+2} = \frac{\Delta t}{12}\left(23\boldsymbol{A}\boldsymbol{u}_{k+2} - 16\boldsymbol{A}\boldsymbol{u}_{k+1} + 5\boldsymbol{A}\boldsymbol{u}_k\right) \tag{3}$$

$$\boldsymbol{w}_{k+3} - \boldsymbol{w}_{k+2} = \frac{\Delta t}{12}\Lambda\left(23\boldsymbol{w}_{k+2} - 16\boldsymbol{w}_{k+1} + 5\boldsymbol{w}_k\right) \tag{4}$$

From this form of the Adams Bashforth method, we have that the coefficients α_i and β_i are as follows,

$$\boldsymbol{\alpha} = [0, 0, -1, 1], \quad \boldsymbol{\beta} = \left[\frac{5}{12}, -\frac{16}{12}, \frac{23}{12}, 0\right] \tag{5}$$

$$\sum_{i=0}^{3}(\alpha_i - \Delta t\lambda_m\beta_i)\boldsymbol{w}_{k+i}^m = 0 \tag{6}$$

At this point, bother to find the eigenvalues of the matrix A which form Λ. Using a matlab eigenvalue solver, we find the eigenvalues of A to be,

$$\lambda \approx [-0.9667 \pm i30.1255, -99.0667]$$
$$\mathbb{R}(\lambda) \approx [-0.9667, -99.0667]$$

We also only consider the real part of λ as this is what will contribute to the convergence/stability. At this point we have 2 equations to solve in order to find the requirement on Δt for the absolute convergence. The two equations are related to the characteristic polynomial for the iteration process.

$$\pi(z) = \rho(z) - \Delta t\lambda_i\sigma(z) = 0$$
$$\rho(z) = \sum_{j=0}^{q}\alpha_j z^j, \quad \sigma(z) = \sum_{j=0}^{q}\beta_j z^j$$
$$\rho(z) = z^3 - z^2, \quad \sigma(z) = \frac{23z^2 - 16z + 5}{12}$$

$$\pi(z) = 0, \implies \Delta t = \frac{\rho(z)}{\lambda_i\sigma(z)} \tag{7}$$

This equation now becomes a constrained optimization problem. We consider two constraints. First, the value of Δt must be a real value. We notice that the equation given to solve for Δt is composed of three complex quantities, so there is no guarantee that the obtained value for Δt is real. Second, we notice that the eigenvalues of the system that we wish to be a contraction must be of absolute value less than 1. That is, $|z| < 1$. Therefore, we have two constraints, on a 3 variable optimization problem. We obtain three equations.

$$z = x + iy, \quad |z| < 1 \implies x^2 + y^2 < 1 \tag{8}$$

$$\mathbb{I}(\Delta t) = 0 \implies \mathbb{I}\left(\frac{\rho(x,y)}{\sigma(x,y)}\right) = 0 \tag{9}$$

$$\max \mathbb{R}(\Delta t) \implies \nabla_h \mathbb{R}\left(\frac{\rho(x,y)}{\sigma(x,y)}\right) = \vec{0} \tag{10}$$

The actual function being considered here is quite unpleasant to write out explicitly. So the rest of this problem will proceed by the result of numerical work. The easiest way to solve this problem is as it is put in the notes. We consider the boundary of the domain for the eigenvalue, z, that will yield a contraction. This is of course the unit circle, $z = e^{i\theta}$. Then, we find the region of absolute stability from $\frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$ (**Fig. 1**). This yields a plot on the complex plane with eigenvalues that satisfy the problem

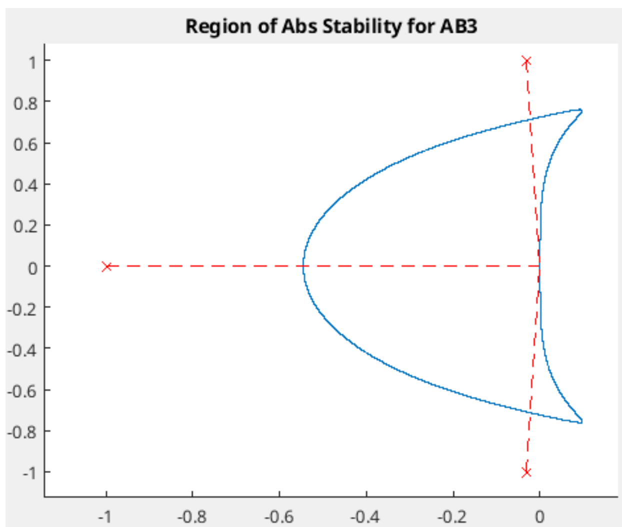

Figure 1: Region of Abs. Stab. for AB3 (Blue) (inside boundary)

we wish to solve. Then we overplot the normalized eigenvalues from our original matrix A, and draw lines originating from the origin to the eigenvalues of A in the complex plane. In order to ensure that Δt is a real-valued, we must pick points on the region of absolute stability which are colinear with the eigenvalues of A. Then we determine the maximum Δt by the ratio between the distance from the eigenvalue to the origin and the distance from the point colinear on the boundary of the region of absolute stability to the origin. This is then our Δt. We find for this problem, that the eigenvalue, $\lambda = -99.0667$ is furthest from the region of absolute stability, and appropriate Δt for this eigenvalue is very close to $\Delta t = 0.00550593$. $\square$

b) See **Fig. 1**.

c) We can plot the results of AB3 with three values of Δt. We have $\Delta t \in [10^{-4}, \Delta t^* - 10^{-4}, \Delta t^* + 10^{-4}]$, where $\Delta t^* = 0.0055$ (we only consider up to 4 decimal digits since we perturb it by 10^{-4}). The results can be seen in figure 2. Evidently, above the critical Δt the numerical method diverges and doesn't resemble that of the other two plots. This demonstrates the concept of absolute stability and we can see its import on numerical methods for ODEs and PDEs.

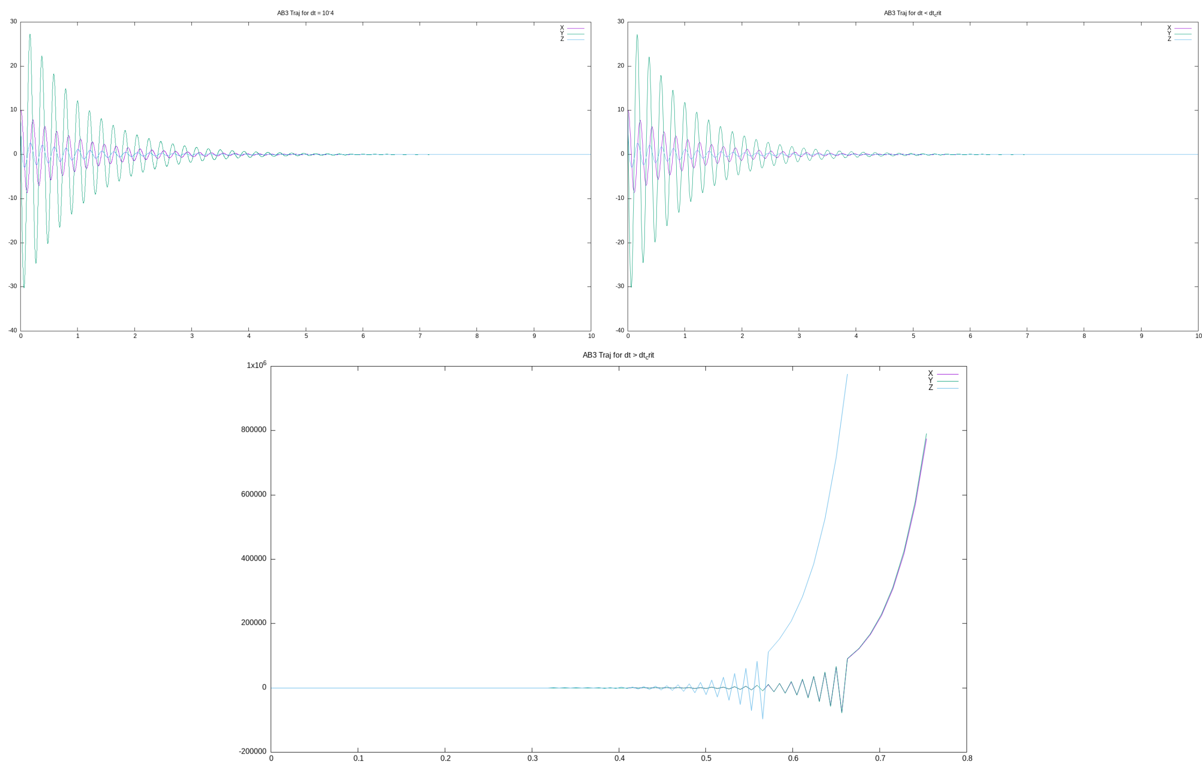

Figure 2: Plots of AB3 with distinct Δt. (Top-left: $\Delta t = 10^{-4}$, Top-right: $\Delta t = \Delta t^* - 10^{-4}$, Bottom: $\Delta t = \Delta t^* + 10^{-4}$)

Question 2: Convergence and Asbolute Stability for the BDF3 Method

a) *Proof.* To show that BDF3 is convergent with order 3, we have to show two things. First that BDF3 is convergent, and then that its order of consistency is of order 2. To show it is convergent, we need to demonstrate consistency and zero-stability. Zero-stability is given by looking at the roots of the first characteristic polynomial. We know it has a root at $z = 1$ by looking at the coefficients. The two other complex roots are solved using the quadratic formula.

$$\begin{aligned}\rho(z) &= z^3 - \frac{18}{11}z^2 + \frac{9}{11}z - \frac{2}{11} \\ &= (z-1)\left(z^2 - \frac{7}{11}z + \frac{2}{11}\right) \\ &= (z-1)\left(z - \frac{7 - i\sqrt{39}}{22}\right)\left(z - \frac{7 + i\sqrt{39}}{22}\right)\end{aligned}$$

We can compute the absolute value of the two complex roots and it is shown that they are both within the unit disk

$$\left|\frac{7 \pm i\sqrt{39}}{22}\right| = \frac{1}{22}\sqrt{49 + 39} = 0.4264\ldots$$

Therefore the method is zero stable because all of its roots are either within the unit disk or on the unit disk and a simple root.

Next we show consistency with order three using Taylor Expansions.

$$
\begin{aligned}
\tau_{k+3} &= \frac{\boldsymbol{y}_{k+3} - \frac{18}{11}\boldsymbol{y}_{k+2} + \frac{9}{11}\boldsymbol{y}_{k+1} - \frac{2}{11}\boldsymbol{y}_k}{\Delta t} - \frac{6}{11}\dot{\boldsymbol{y}}_{k+3} \\
\boldsymbol{y}_k &= \boldsymbol{y}_{k+3} - 3\Delta t\dot{\boldsymbol{y}}_{k+3} + \frac{9}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+3} - \frac{27}{6}\Delta t^3\dddot{\boldsymbol{y}}_{k+3} + \frac{81}{24}\Delta t^4\ddddot{\boldsymbol{y}}_{k+3} + h.o.t. \\
\boldsymbol{y}_{k+1} &= \boldsymbol{y}_{k+3} - 2\Delta t\dot{\boldsymbol{y}}_{k+3} + \frac{4}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+3} - \frac{8}{6}\Delta t^3\dddot{\boldsymbol{y}}_{k+3} + \frac{16}{24}\Delta t^4\ddddot{\boldsymbol{y}}_{k+3} + h.o.t. \\
\boldsymbol{y}_{k+2} &= \boldsymbol{y}_{k+3} - \Delta t\dot{\boldsymbol{y}}_{k+3} + \frac{1}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+3} - \frac{1}{6}\Delta t^3\dddot{\boldsymbol{y}}_{k+3} + \frac{1}{24}\Delta t^4\ddddot{\boldsymbol{y}}_{k+3} + h.o.t. \\
\tau_{k+3} &= \frac{\boldsymbol{y}_{k+3}}{\Delta t}\left(1 + \frac{-18+9-2}{11}\right) + \dot{\boldsymbol{y}}_{k+3}\left(\frac{18-18+6-6}{11}\right) \\
&\quad + \frac{\Delta t}{2}\ddot{\boldsymbol{y}}\left(\frac{-18+36-18}{11}\right) + \frac{\Delta t^2}{6}\dddot{\boldsymbol{y}}_k + 3\left(\frac{18-72+54}{11}\right) \\
&\quad + \frac{\Delta t^3}{24}\ddddot{\boldsymbol{y}}_{k+3}\left(\frac{-18+144-161}{11}\right) + h.o.t. \\
|\tau_{k+3}| &= \frac{-35\Delta t^3}{24}|\ddddot{\boldsymbol{y}}_{k+3}|
\end{aligned}
$$

Therefore BDF3 is consistent with order three, and since it is zero-stable, it is convergent with order three as well!

□

b) No, BDF3 is not A-stable. This is clear from the plot since the region of absolute stability intersects the "imaginary" (y) axis in the complex plane. Because of this, there is some part of $\mathbb{C}^-$ which is not in the region of absolute stability. Thus, BDF3 is not A-Stable.

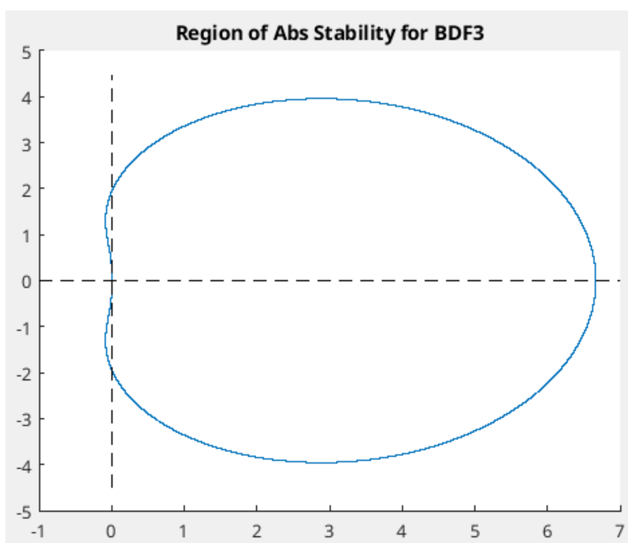

Figure 3: Region of Abs. Stab. (outside blue boundary)

Question 3: Consistency, Convergence, and Stability for an LMM

a) *Proof.* Proving consistency with order two is a mere matter of condidering the Taylor Expansion for this LMM. We have by the definition of local truncation error,

$$\boldsymbol{y}_{k+2} - 4\boldsymbol{y}_{k+1} + 3\boldsymbol{y}_k = -2\Delta t\boldsymbol{f}(\boldsymbol{u}_k, t_k) + \Delta t\tau_{k+2} \tag{11}$$

$$
\begin{aligned}
\boldsymbol{y}_{k+2} &= \boldsymbol{y}_k + 2\Delta t\dot{\boldsymbol{y}}_k + \frac{4\Delta t^2}{2}\ddot{\boldsymbol{y}}_k + \frac{8\Delta t^3}{6}\dddot{\boldsymbol{y}}_k + h.o.t. \\
\boldsymbol{y}_{k+1} &= \boldsymbol{y}_k + \Delta t\dot{\boldsymbol{y}}_k + \frac{\Delta t^2}{2}\ddot{\boldsymbol{y}}_k + \frac{\Delta t^3}{6}\dddot{\boldsymbol{y}}_k + h.o.t. \\
\Delta t\tau_{k+2} &= \boldsymbol{y}_k + 2\Delta t\dot{\boldsymbol{y}}_k + \frac{4\Delta t^2}{2}\ddot{\boldsymbol{y}}_k + \frac{8\Delta t^3}{6}\dddot{\boldsymbol{y}}_k + h.o.t \\
&\quad - 4\left(\boldsymbol{y}_k + \Delta t\dot{\boldsymbol{y}}_k + \frac{\Delta t^2}{2}\ddot{\boldsymbol{y}}_k + \frac{\Delta t^3}{6}\dddot{\boldsymbol{y}}_k + h.o.t.\right) \\
&\quad + 3\boldsymbol{y}_k + 2\Delta t\boldsymbol{f}(\boldsymbol{u}_k, t_k) \\
\tau_{k+2} &= \frac{4\Delta t^2}{6}\dddot{\boldsymbol{y}}_k, \qquad \Longrightarrow \qquad |\tau_{k+2}| = \frac{4\Delta t^2}{6}|\dddot{\boldsymbol{y}}_k|
\end{aligned}
$$

Thus we obtain order 2 consistency for this LMM. □

b) *Proof.* Next in order to demonstrate the zero-stability for this LMM we consider the root condition for the first characteristic polynomial. We have from this LMM, a characteristic polynomial,

$$\rho(z) = z^2 - 4z + 3 \tag{12}$$

This polynomial has roots $z = 1$ and $z = 3$, the second of which lies outside of the unit disk. Therefore this LMM fails the root condition and is necessarily zero-unstable. This also implies that this LMM is not convergent since congergence is dependent on consistency and zero-stability. □

c) *Proof.* Finally, we can cite the lemma from the notes which states that all zero-unstable, consistent LMMs are necessarily unconditionally absolutely unstable. We can also look at the region of absolute stability in the figure 3. This region of absolute stability has no intersection with $\mathbb{C}^-$ whatsoever. Thus, there is no condition which this LMM is absolutely stable. Hence, this LMM is A-unstable.

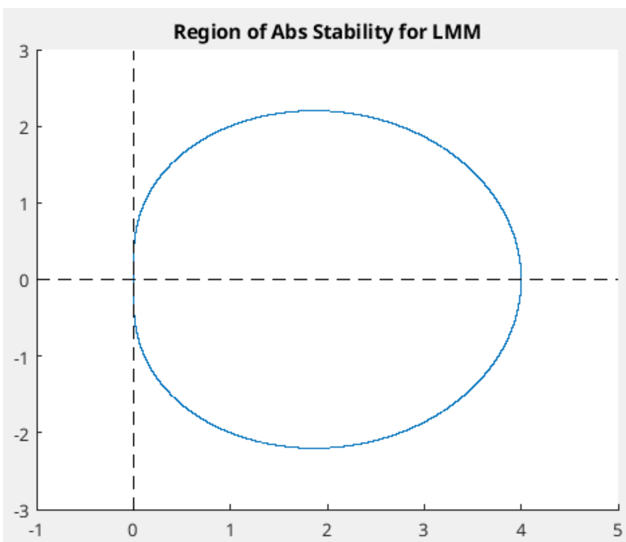

Figure 4: Region of Abs. Stab. (inside blue boundary)

Note that the region of absolute stability is obtained by plotting,

$$\lambda\Delta t = \frac{e^{i2\theta} - 4e^{i\theta} + 3}{2}$$

□

Question 4: Convergence and Stability for an RK Method

a) *Proof.* Proving an explcit RK3 method is convergent requires two things. First, we must show that it is consistent, and then that it is zero-stable. Zero-stability is implied since all explicit RK methods are necessarily one-step methods, i.e. $\rho(z) = z - 1$ which has one simple root $\rightarrow$ satisfies the root condition. Consistency is much more difficult. Referencing the notes, we see that there is a well defined criterion for explicit RK3 methods to have consistency. We will employ that here. There are 4 conditions,

$$
\begin{aligned}
b_1 + b_2 + b_3 &= 1, & \frac{2+4}{9} + \frac{1}{3} &= 1 \\
b_2 c_2 + b_3 c_3 &= \frac{1}{2}, & \frac{1}{3}\cdot\frac{1}{2} + \frac{4}{9}\cdot\frac{3}{4} &= \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \\
b_2 c_2^2 + b_3 c_3^2 &= \frac{1}{3}, & \frac{1}{3}\cdot\frac{1}{4} + \frac{4}{9}\cdot\frac{9}{16} &= \frac{1}{12} + \frac{1}{4} = \frac{1}{3} \\
b_3 a_{32} c_2 &= \frac{1}{6}, & \frac{4}{9}\cdot\frac{3}{4}\cdot\frac{1}{2} &= \frac{1}{6}
\end{aligned}
$$

They are, so we conclude that since this RK3 method is zero-stable and convergent that it is a convergent method. □

b) *Proof.* Finally with this RK3 method we will use the standard method for determining absolute stability. We look for $|S(z)| < 1$, and for this method to be A-stable, we require that this is satisfied for all $z \in \mathbb{C}^-$. To compute this, we use the defintion of $S(z)$ from the notes, where A and b are parts of the coefficient matrix for this RK3 method, and h is a column vector of ones. After plotting the contours of this function on the complex plane, we see that the region of absolute stability is small and does not contain $\mathbb{C}^-$. This is doubly confirmed by the fact that this method is an explicit method, implying that it couldn't be A-stable.

$$S(z) = \det(\mathrm{I} - zA + zhb^T) \tag{13}$$

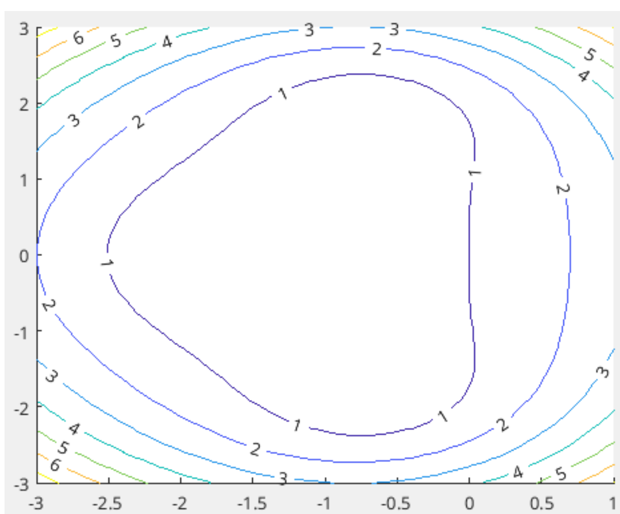

Figure 5: Contour Plot of $|S(z)|$ (Region of Abs. Stab. (inside $|S(z)|$=1 contour)

□