Numerical Methods for the Solution of Differential Equations (AMS 213B)
Final Exam - Due Wednesday June 12th

Instructions

Please submit to CANVAS one PDF file (your solution to the exam), and one .zip file that includes any computer code you develop for the exam. The PDF file must be a document compiled from Latex source code (mandatory for PhD students), or a PDF created using any other other word processor (MS and SciCAM students). No handwritten work should be submitted.

You are allowed to consult any materials you wish during the exam, including books, lecture notes, in-class notes, internet resources, etc. You are not allowed to communicate with regard to any aspect of the exam with any individual: this includes sharing or requesting materials such as computer codes or solutions, or discussing methods/algorithms related to the final exam during the exam time through email, text messages, or any other form of communication.

Question	Points
1	20
2	70
3	10

Consider the following initial/boundary value problem (IBVP) for the heat equation in cylindrical coordinates

$$\begin{cases} \dfrac{\partial U}{\partial t} = \nu \left(\dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} \right) & r \in [r_1, r_2] \qquad t \geq 0 \\ U(r,0) = U_0(r) \\ U(r_1,t) = 0 \\ U(r_2,t) = 0 \end{cases} \tag{1}$$

where

$$r_1 = 1, \qquad r_2 = 3, \qquad \nu = \frac{1}{4}, \tag{2}$$

and

$$U_0(r) = 15(r-r_1)^2(r_2-r)^2 e^{-\sin(2r)-r} \qquad r \in [r_1, r_2]. \tag{3}$$

The IBVP (1) governs the propagation of heat within an annulus with radii r_1 (inner radius) and r_2 (outer radius) under the assumption that the initial condition and the boundary conditions are independent of the angular coordinate ϑ. By using the method of separation of variables it can be shown that the analytical solution of (1) is

$$U(r,t) = \sum_{k=1}^{\infty} e^{-\beta_k^2 \nu t} \widehat{R}_k(r) \int_{r_1}^{r_2} r' \widehat{R}_k(r') U_0(r') dr', \tag{4}$$

where

$$\widehat{R}_k(r) = \frac{R_k(r)}{\|R_k\|_{L^2_w}}, \tag{5}$$

and

$$R_k(r) = J_0(\beta_k r)Y_0(\beta_k r_2) - J_0(\beta_k r_2)Y_0(\beta_k r), \qquad \|R_k(r)\|^2_{L^2_w} = \frac{2}{\pi^2\beta_k^2}\frac{J_0^2(\beta_k r_1) - J_0^2(\beta_k r_2)}{J_0^2(\beta_k r_1)}. \tag{6}$$

In equation (6), J_0 denotes the Bessel function of the first kind of order zero, while Y_0 is the Bessel function of the second kind of order zero. The numbers β_j appearing in (4) and (6) are solutions to the nonlinear equation[1]

$$H(\beta) = J_0(\beta r_1)Y_0(\beta r_2) - J_0(\beta r_2)Y_0(\beta r_1) = 0 \qquad \text{(eigenvalue equation).} \tag{8}$$

Question 1 (20 points)

a) (15 points) Plot the solution you obtain when keeping 60 modes in (4) (i.e., truncate the series expansion in (4) to 60 terms) at times $t = \{0, 0.5, 1, 2\}$ versus r. To this end, you will need to compute the coefficients β_k using (8) and the integrals appearing in (4) numerically. For β_k you can solve (8) using the Newton's method or any other nonlinear solver such as the Matlab function `fzero()`. In both cases you will need need an appropriate set of initial guesses that allow you to identify the zeros[2] of $H(\beta)$ in (8). The first three β_k are

$$\beta_1 = 1.548458778289446, \qquad \beta_2 = 3.129084015718067, \qquad \beta_3 = 4.703797206969750.$$

For the integrals in (4) you can compute them to machine accuracy using the Matlab function `quadl()` with tolerance set to 10^{-15}, or any quadrature rule of your choice. For instance you can use the Gauss-Legendre quadrature rule `lgwt.m` provided in CANVAS (150 quadrature points are sufficient to hit machine accuracy).

b) (5 points) Plot the functions $\{R_1(r), R_2(r), R_4(r), , R_6(r)\}$ versus r for $r \in [r_1, r_2]$ and the squared norms $\|R_k(r)\|^2_{L^2_w}$ defined in (6) versus k for $k = 1, 2, \ldots, 30$.

Question 2 (70 points)

Compute the numerical solution of the IBVP (1) using second-order centered finite differences in the radial direction and the three-step Adams-Bashforth (AB3) time integration scheme.

a) (10 points) Consider the following evenly-spaced grid with $n+2$ nodes (n specified below)

$$r^{(j)} = r_1 + j\Delta r \qquad \Delta r = \frac{r_2 - r_1}{n+1} \qquad j = 0, ..., n+1, \tag{9}$$

[1] Note that $\widehat{R}_j(r)$ are normalized eigenfunctions of the following eigenvalue problem

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} + \beta^2 R = 0 \\ R(r_1) = 0 \\ R(r_2) = 0 \end{cases} \tag{7}$$

involving a Bessel equation of order zero.

[2] The solutions of (8) can be visualized by plotting $H(\beta)$ versus β. In fact, β_k $k = 1, 2, \ldots$ are the zeros of the function $H(\beta)$.

and set $u_j(t) \simeq U(r^{(j)}, t)$. Write the semi-discrete form of the IBVP (1), as

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{M}\boldsymbol{u}(t), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{10}$$

where $\boldsymbol{u}(t) = [u_1(t) \cdots u_n(t)]^T$ (solution at the inner nodes) and $\boldsymbol{M}$ represents the second-order centered finite-difference discretization of the right-hand side of (1). Write the $n \times n$ (negative-definite) matrix $\boldsymbol{M}$ explicitly. Hint: discretize $\partial^2 U/\partial r^2$ using the classical centered second-order finite difference formula, and $r^{-1}\partial U/\partial r$ using the formula:

$$\left.\frac{1}{r}\frac{\partial U}{\partial r}\right|_{r=r^{(j)}} \simeq \frac{u_{j+1}(t) - u_{j-1}(t)}{2\Delta r(r_1 + j\Delta r)} \qquad j = 1, \ldots n. \tag{11}$$

b) **(10 points)** Compute the spectral radius $\rho(\boldsymbol{M})$ of the matrix $\boldsymbol{M}$ using a Matlab function of your choice. Plot $\rho(\boldsymbol{M})$ versus the number of inner nodes n for all integers n such that

$$n = 10 + 10(k-1) \qquad k = 1, \ldots, 20. \tag{12}$$

c) **(10 points)** Discretize (10) in time with the three-step Adams-Bashforth (AB3) method. Show that the largest time step Δt^* for absolute stability is related to the spectral radius $\rho(\boldsymbol{M})$ of the matrix $\boldsymbol{M}$ by

$$\Delta t^* = \frac{6}{11}\frac{1}{\rho(\boldsymbol{M})}. \tag{13}$$

Plot $\log(\Delta t^*)$ versus $\log(n)$ for all integers n defined in (12). What is the decay rate of Δt^* versus n? What is the largest time step Δt^* that grants us absolute stability for $n = 200$? Comment on your results.

d) **(30 points)** Set $\Delta t = 5 \times 10^{-5}$. Integrate (10) numerically with the AB3 scheme and compute the time-dependent maximum point-wise error between the exact solution and the numerical solution versus time within the time interval $[0, 2]$, i.e.,

$$e_n(t) = \max_{j=1,\ldots,n} |U(t, r_j) - u_j(t)|, \qquad t \in [0, 2], \tag{14}$$

Here, $U(t, r)$ is the exact solution[3] (4), while and $u_j(t)$ $(j = 1, \ldots, n)$ is the numerical solution of (10) with n interior nodes, respectively. Plot $\log(e_n(t))$ versus t for $n = \{50, 100, 150, 200\}$ on a temporal grid with sufficient resolution, e.g., 200 time instants in $[0, 2]$.

e) **(10 points)** Plot $e_n(2)$ (error at final time) as a function of n in a log-log plot for all n defined in (12). Does the error decay as expected for the spatial discretization you considered? Justify your answer.

Question 3 (10 points)

Show that the following finite-difference discretization of right hand side of (1)

$$\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right)_{r=r^{(j)}} \simeq \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{\Delta r^2} + \frac{u_{j+1}(t) - u_{j-1}(t)}{2\Delta r(r_1 + j\Delta r)} \tag{15}$$

is second-order accurate in Δr.

[3] Truncate the series expansion in (4) to 60 terms.