Midterm: Report

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Question 1: BVP using Shooting Method and RK4

a) The system can be solved analytically by simply integrating 4 times and then solving for the boundary conditions.

$$\int\int\int\int \frac{d^4y}{dx^4}dx^4 = \int\int\int\int x^2 dx^4$$
$$y(x) = \frac{x^6}{360} + c_1x^3 + c_2x^2 + c_3x + c_4$$

We can see very clearly from the boundary conditions at $x = 0$ that $c_4 = c_3 = 0$. We finish solving for c_1 and c_2.

$$0 = \frac{1}{360} + c_1 + c_2, \quad 0 = \frac{1}{60} + 3c_1 + 2c_2$$
$$c_1 = -\frac{1}{90}, \quad c_2 = \frac{1}{120}$$

$$y(x) = \left(\frac{x^6}{360} - \frac{x^3}{90} + \frac{x^2}{120}\right)$$

b) The numerical solution obtained by the shooting method takes the following form,

$$\frac{dy_1}{dx} = y_2, \quad y_1(0) = 0$$
$$\frac{dy_2}{dx} = y_3, \quad y_2(0) = 0$$
$$\frac{dy_3}{dx} = y_4, \quad y_3(0) = v_2$$
$$\frac{dy_4}{dx} = x^2, \quad y_4(0) = v_1$$

Thus we proceed by integrating this ODE with a standard numerical method and at each iteration adjust v_1, v_2 so that we get closer and closer to the bouncary condition at $x = 1$. To this end, I will implement something similar to the second method. We will update v_1 to minimize the error of y_1. Then we I will update v_2 to minimize the error of y_2. After iterating several times, we obtain accuracy with error at the boundaries less then 10^{-10}. The corresponding initial conditions are $v_1 = -0.066666\ldots$, $v_2 = 0.016666\ldots$.

c) See Fig. 1

d) See Fig. 2

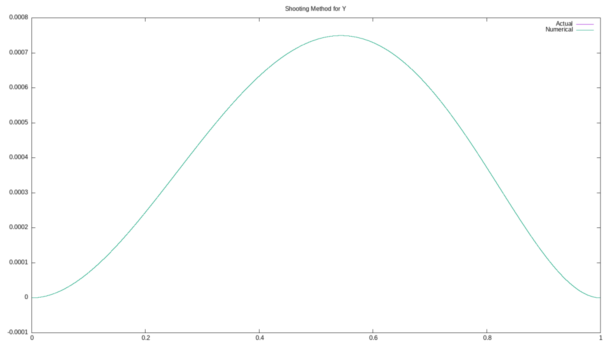


Figure 1: Numerical Solution for Shooting Method

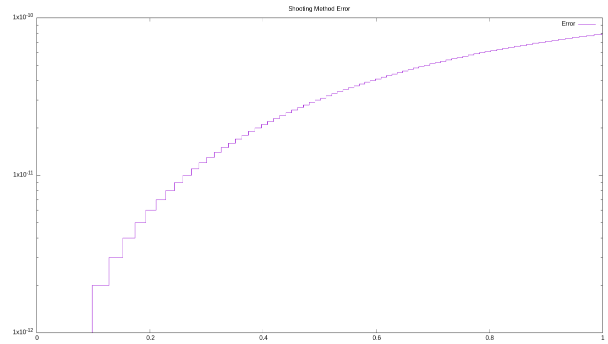


Figure 2: Numerical Error for Shooting Method

Question 2: Convergence and Absolute Stability for an Implicit RK3 Method

a) Prove convergence for Implicit RK3.

Proof. Showing convergence is a matter of showing zero-stability and consistency. This is a one-step method, so we have the first characteristic polynomial, $\rho(z) = z - 1$. This of course satisfies the root condition and therefore is zero-stable. Showing consistency is a matter of using a taylor expansion.

$$
\begin{aligned}
\boldsymbol{y}_{k+1} &= \boldsymbol{y}_k + \frac{\Delta t}{6}\left(\boldsymbol{K}_1 + 4\boldsymbol{K}_2 + \boldsymbol{K}_3\right) + \Delta t \tau_{k+1} \\
\tau_{k+1} &= \frac{\boldsymbol{y}_{k+1} - \boldsymbol{y}_k}{\Delta t} - \frac{1}{6}\left(B\boldsymbol{y}_k + 4B\left(\boldsymbol{y}_k + \frac{\Delta t}{4}\left(\boldsymbol{K}_1 + \boldsymbol{K}_2\right)\right) + B\left(\boldsymbol{y}_k + \Delta t\boldsymbol{K}_2\right)\right) \\
\tau_{k+1} &= \frac{\boldsymbol{y}_{k+1} - \boldsymbol{y}_k}{\Delta t} - \dot{\boldsymbol{y}}_k - \frac{\Delta t}{6}B\left(\boldsymbol{K}_1 + 2\boldsymbol{K}_2\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{y}_{k+1} &= \boldsymbol{y}_k + \Delta t\dot{\boldsymbol{y}}_k + \Delta t^2(h.o.t.) \\
\frac{\boldsymbol{y}_{k+1} - \boldsymbol{y}_k}{\Delta t} &= \dot{\boldsymbol{y}}_k + \Delta t(h.o.t.) \\
\tau_{k+1} &= -\frac{\Delta t}{6}B\left(\boldsymbol{K}_1 + 2\boldsymbol{K}_2\right) + \Delta t(h.o.t.) \\
|\tau_{k+1}| &= \frac{\Delta t}{6}\left|B\left(\boldsymbol{K}_1 + 2\boldsymbol{K}_2\right) + \Delta t(h.o.t.)\right|
\end{aligned}
$$

Thus we have shown consistency with at least order 1, so we must have that this Implicit RK3 method is convergent with at least order 1 since it is both zero-stable and consistent. □

b) Plot Region of Absolute Statbility (See Fig. 3)

c) This problem is solved by plotting the region of absolute stability and finding the eignevalues of the matrix B. We notice that since B is an upper triangular matrix that its eigenvalues are found on its diagonal. So we have that B has all real eigenvalues, $\lambda = \{-1, -2, -4, -16\}$. Next we compare to see which eigenvalue is furthest from the region of absolute stability. It is evidently the eigenvalue, $\lambda = -16$. So we look at the closest point in our region of absolute stability. From the plot we can see that the closest point is $\lambda\Delta t \approx -5.4199 + 0i$. Therefore we have that the largest Δt must be, $\Delta t \approx 0.3387\ldots$.

d) As seen in Fig. 4, when $\Delta t > \Delta t^*$ we can see that the oscillations in the system grow with time while the oscillations for $\Delta t < \Delta t^*$ decay with time. Ultimately the stability of this method is questionable

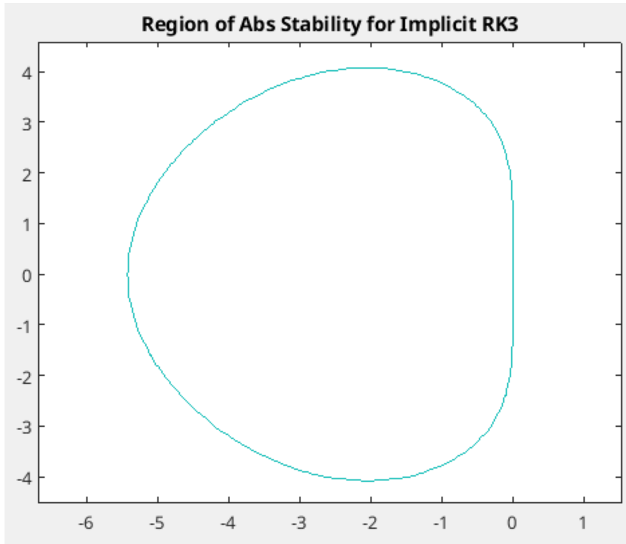


Figure 3: Region of Absolute Stability for 2.b.

near Δt^*, as we can see in the plot of the true solution, however this does coincide with the result of absolute stability for this value of Δt. To remind the reader, the only requirement is for $\boldsymbol{u}_k \to 0$ as $k \to \infty$

Question 3: Convergence and Absolute Stability for an LMM

$$\boldsymbol{u}_{k+3} - \frac{1}{3}\left(\boldsymbol{u}_{k+2} + \boldsymbol{u}_{k+1} + \boldsymbol{u}_k\right) = \frac{\Delta t}{12}\left[23\boldsymbol{f}_{k+2} - 2\boldsymbol{f}_{k+1} + 3\boldsymbol{f}_k\right] \tag{1}$$

a) *Proof.* (Zero-Stability)

We begin this proof first by showing that this LMM is zero-stable.

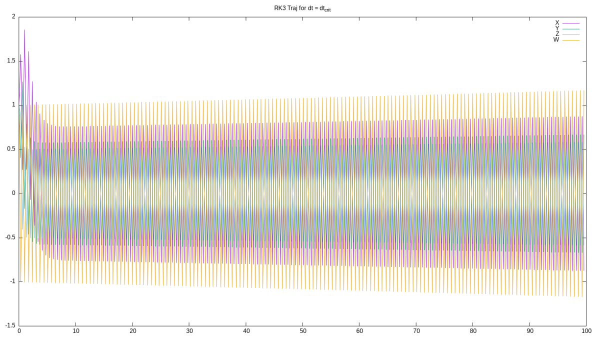
$$\rho(z) = z^3 - \frac{1}{3}z^2 - \frac{1}{3}z - \frac{1}{3}$$
$$\rho(z) = (z-1)\left(z^2 + \frac{2}{3}z + \frac{1}{3}\right)$$
$$\rho(z) = (z-1)\left(z - \frac{-\frac{2}{3} \pm \sqrt{\frac{4}{9} - \frac{4}{3}}}{2}\right)$$
$$\rho(z) = (z-1)\left(z + \frac{1 \pm i\sqrt{2}}{3}\right)$$

These are the three complex roots of the characteristic polynomial. We now simply must check if all three are within or on the boundary of the unit disk.

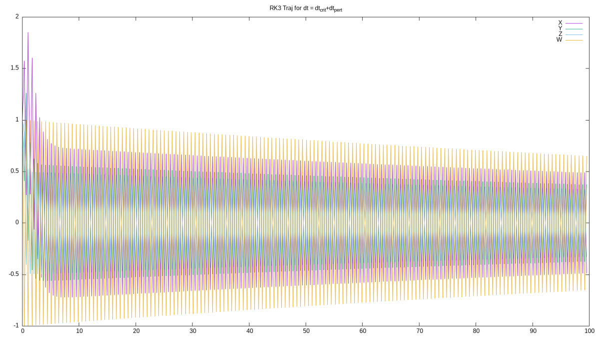
$$|1| = 1 \leq 1, \qquad \left|\frac{1 \pm i\sqrt{2}}{3}\right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3} \leq 1$$

As I have just shown, in fact all three roots of the first characteristic polynomial fall within the unit disk or on the boundary, thus we have that this LMM is zero-stable.

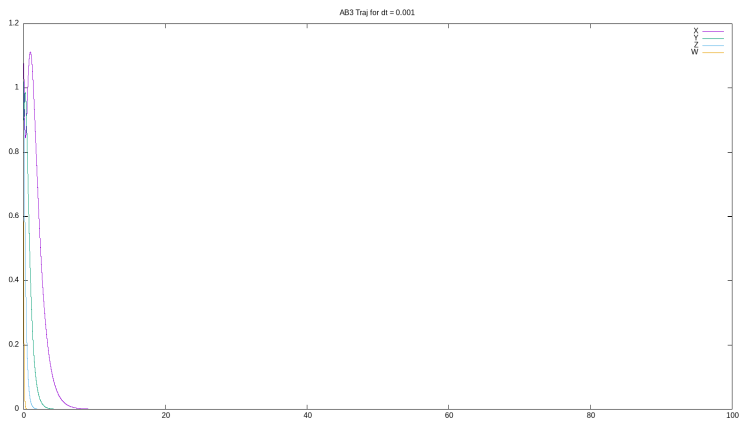
(Consistency)



Δt above threshold



Δt below threshold



True Solution

Figure 4: Evolution of the IC with Δt above and below the calculated threshold.

We we will look at the consistency of this LMM. We look at the definition of truncation error in our system.

$$\boldsymbol{y}_{k+3} - \frac{1}{3}(\boldsymbol{y}_{k+2} + \boldsymbol{y}_{k+1} + \boldsymbol{y}_k) = \frac{\Delta t}{12}[\dot{\boldsymbol{y}}_{k+2} - 2\dot{\boldsymbol{y}}_{k+1} + 3\dot{\boldsymbol{y}}_k] + \Delta t \tau_{k+3}$$

In order to simplify this system an demonstrate convergence we will convert this problem into terms of $\boldsymbol{q}_{k+2}$ using Taylor Expansions.

$$\boldsymbol{y}_{k+3} = \boldsymbol{y}_{k+2} + \Delta t \dot{\boldsymbol{y}}_{k+2} + \frac{\Delta t^2}{2}\ddot{\boldsymbol{y}}_{k+2} + \frac{\Delta t^3}{6}\dddot{\boldsymbol{y}}_{k+2} + \frac{\Delta t^4}{24}\ddddot{\boldsymbol{y}}_{k+2} + h.o.t. \tag{2}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_{k+2} - \Delta t \dot{\boldsymbol{y}}_{k+2} + \frac{\Delta t^2}{2}\ddot{\boldsymbol{y}}_{k+2} - \frac{\Delta t^3}{6}\dddot{\boldsymbol{y}}_{k+2} + \frac{\Delta t^4}{24}\ddddot{\boldsymbol{y}}_{k+2} + h.o.t. \tag{3}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_{k+2} - 2\Delta t \dot{\boldsymbol{y}}_{k+2} + \frac{4\Delta t^2}{2}\ddot{\boldsymbol{y}}_{k+2} - \frac{8\Delta t^3}{6}\dddot{\boldsymbol{y}}_{k+2} + \frac{16\Delta t^4}{24}\ddddot{\boldsymbol{y}}_{k+2} + h.o.t. \tag{4}$$

$$\dot{\boldsymbol{y}}_{k+1} = \dot{\boldsymbol{y}}_{k+2} - \Delta t \ddot{\boldsymbol{y}}_{k+2} + \frac{\Delta t^2}{2}\dddot{\boldsymbol{y}}_{k+2} - \frac{\Delta t^3}{6}\ddddot{\boldsymbol{y}}_{k+2} + h.o.t \tag{5}$$

$$\dot{\boldsymbol{y}}_{k} = \dot{\boldsymbol{y}}_{k+2} - 2\Delta t \ddot{\boldsymbol{y}}_{k+2} + \frac{4\Delta t^2}{2}\dddot{\boldsymbol{y}}_{k+2} - \frac{8\Delta t^3}{6}\ddddot{\boldsymbol{y}}_{k+2} + h.o.t \tag{6}$$

Using this we can build a linear system to satisfy order 1 consistency, then we can check to see if any of the other orders happen to be satisfied by the solution. If the following system has a solution then we have,

$$\begin{bmatrix} 1 & -1/3 & -1/3 \\ 1 & 1/3 & 2/3 \\ 1/2 & -1/6 & -2/3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2 \\ -1/3 \end{bmatrix} \implies \tau_{k+3} = C\Delta t^2$$

Furthermore, if we can show that these coefficients satisfy higher order equations then we can demonstrate higher order convergence. As it happens, the solution to this system is, $A = B = C = 1$. Thus we have that this system is at least consistent with order 2, and thus at least convergent with order 2 since zero-stability was already implied. So we look at the higher orders of convergence/consistency.

$$\begin{bmatrix} 1/6 & 1/18 & 8/18 \\ 1/24 & -1/72 & -16/72 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} i \stackrel{?}{=} \begin{bmatrix} 5/12 \\ -2/9 \end{bmatrix}$$

$$\begin{bmatrix} 1/6 & 1/18 & 8/18 \\ 1/24 & -1/72 & -16/72 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} - \begin{bmatrix} 5/12 \\ -2/9 \end{bmatrix} = \begin{bmatrix} 0.2499\ldots \\ -5.40277\ldots \end{bmatrix}$$

Thus no higher order of convergence is obtained. Therefore this system is convergence with order 2. □

b) Absolute Stability and A-Stability

We can plot the region of absolute stability for this LMM. We have the first and second characterstic polynomials are the following.

$$\rho(z) = z^3 - \frac{1}{3}\left(z^2 + z + 1\right), \quad \sigma(z) = \frac{1}{12}\left(23z^2 - 2z + 3\right)$$
$$\frac{\rho(z)}{\sigma(z)} = \lambda\Delta t$$

We can then plot this by evaluating $\frac{\rho(z)}{\sigma(z)}$ with $z = e^{i\theta}$ and plotting in the complex plane. We can see

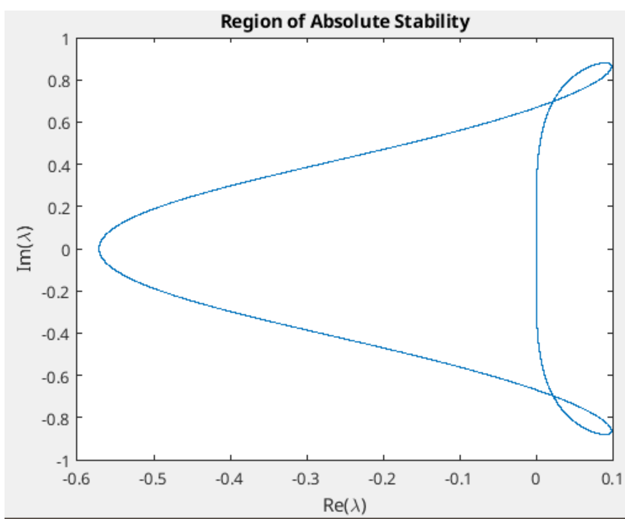


Figure 5: Region of Absolute Stability for 3.b

from this plot that this LMM is certainly not A-Stable. The reason being that the region of absolute stability is only conditionally absolutely stable. This is seen in the plot which clearly illurstrates the region of absolute stability including only a small subset of $\mathbb{C}^-$.