Homework 2: Report

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Problem 1: Absolute Stability for AB3

a) Determine the largest value of Δt, for which the three-step Adams-Bashforth method (AB3)

Proof. We use the condition for absolute stability:

$$\lim_{k\to\infty} ||\boldsymbol{u}_k|| = 0 \tag{1}$$

For this specific numerical method, we have the following characteristic polynomial for the numerical method. (Note that since the columns of B are linearly independent we have that B is diagonalizable).

$$\boldsymbol{u}_{k+3} = \boldsymbol{u}_{k+2} + \frac{\Delta t}{12}\left(23\boldsymbol{f}_{k+2} - 16\boldsymbol{f}_{k+1} + 5\boldsymbol{f}_k\right) \tag{2}$$

$$\boldsymbol{u}_{k+3} - \boldsymbol{u}_{k+2} = \frac{\Delta t}{12}\left(23\boldsymbol{A}\boldsymbol{u}_{k+2} - 16\boldsymbol{A}\boldsymbol{u}_{k+1} + 5\boldsymbol{A}\boldsymbol{u}_k\right) \tag{3}$$

$$\boldsymbol{w}_{k+3} - \boldsymbol{w}_{k+2} = \frac{\Delta t}{12}\Lambda\left(23\boldsymbol{w}_{k+2} - 16\boldsymbol{w}_{k+1} + 5\boldsymbol{w}_k\right) \tag{4}$$

From this form of the Adams Bashforth method, we have that the coefficients α_i and β_i are as follows,

$$\boldsymbol{\alpha} = [0, 0, -1, 1], \quad \boldsymbol{\beta} = \left[\frac{5}{12}, -\frac{16}{12}, \frac{23}{12}, 0\right] \tag{5}$$

$$\sum_{i=0}^{3}(\alpha_i - \Delta t\lambda_m\beta_i)\boldsymbol{w}_{k+i}^m = 0 \tag{6}$$

At this point, bother to find the eigenvalues of the matrix A which form Λ. Using a matlab eigenvalue solver, we find the eigenvalues of A to be,

$$\lambda \approx [-0.9667 \pm i30.1255, -99.0667] \tag{7}$$

$$\mathbb{R}(\lambda) \approx [-0.9667, -99.0667] \tag{8}$$

We also only consider the real part of λ as this is what will contribute to the convergence/stability. At this point we have 2 equations to solve in order to find the requirement on Δt for the absolute convergence. The two equations are related to the characteristic polynomial for the iteration process.

$$\pi(z) = p(z) - \Delta t\lambda_i\sigma(z) = 0 \tag{9}$$

$$p(z) = \sum_{j=0}^{q}\alpha_j z^j, \quad \sigma(z) = \sum_{j=0}^{q}\beta_j z^j \tag{10}$$

We solve (9) twice in order, once for each eigenvalue of our original linear transformation, A. The exact polynomial becomes,

$$z^3 - z^2 - \frac{dt\lambda}{12}\left(\frac{5}{12} - \frac{16}{12}z + \frac{23}{12}z^2\right) = 0 \tag{11}$$

$$z^3 - z^2\left(1 + \frac{23\Delta t\lambda}{12}\right) + \frac{16\Delta t\lambda}{12}z - \frac{5\Delta t\lambda}{12} = 0 \tag{12}$$

$$
\begin{aligned}
z^3 - z^2\left(1 + \frac{23\Delta t(-0.9667)}{12}\right) + \frac{16\Delta t(-0.9667)}{12} z - \frac{5\Delta t(-0.9667)}{12} &= 0 \\
z^3 - z^2\left(1 + \frac{23\Delta t(-99.0667)}{12}\right) + \frac{16\Delta t(-99.0667)}{12} z - \frac{5\Delta t(-99.0667)}{12} &= 0
\end{aligned} \tag{13}
$$

□

b)

Question 2: Convergence and Asbolute Stability for the BDF3 Method

Criterion for Consistency for a lineaer multistep method

$$\rho(1) = 0 \tag{14}$$

$$\rho(1) - \sigma(1) = 0 \tag{15}$$

a)

Question 3: Consistency, Convergence, and Stability for an LMM

Question 4: Convergence and Stability for an RK Method