Numerical Methods for the Solution of Differential Equations (AMS 213B)
Homework 4 Solution

Question 1 (20 points). Consider the following initial-boundary value problem for the linear advection equation

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial U}{\partial x} = 0 \qquad t \geq 0 \qquad x \in [0, 2\pi] \\ U(x,0) = \sin^2(x) \\ \text{Periodic B.C.} \end{cases} \tag{1}$$

Discretize (1) with the following scheme

$$u_j^{k+1} = u_j^k - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^{k+1} - u_{j-1}^{k+1}\right) \tag{2}$$

on an evenly-spaced grid with N spatial points in $[0, 2\pi]$

$$x_j = j\Delta x, \qquad \Delta x = \frac{2\pi}{N}, \qquad j = 0, \ldots, N. \tag{3}$$

In (2) u_j^k represents an approximation of $U(x_j, t_k)$.

(a) (10 points) Compute the local truncation error of the scheme (2). Is the scheme consistent? To which order in Δt and Δx?

(b) (10 points) Use Von-Neumann analysis to study stability of the scheme (2). Is the scheme convergent? Justify your answer.

Answers:

(a) The local truncation error for this method is

$$\tau(\Delta t, \Delta x) = \frac{u_j^{k+1} - u_j^k}{\Delta t} + \frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{2\Delta x}. \tag{4}$$

Now, we assume the existence of the analytical solution $u(x,t)$ and using the Taylor expansion around Δt yields

$$u_j^{k+1} = u_j^k + \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_k).$$

Similarly, using the Taylor series expansion around both Δx and Δx gives

$$\begin{aligned} u_{j+1}^{k+1} &= u_j^k + \Delta t \frac{\partial u}{\partial t}(x_j, t_k) + \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) \\ &\quad + \frac{\Delta t \Delta x}{2}\frac{\partial^2 u}{\partial x \partial t}(x_j, t_k) + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k) \\ u_{j-1}^{k+1} &= u_j^k + \Delta t \frac{\partial u}{\partial t}(x_j, t_k) - \Delta x \frac{\partial u}{\partial x}(x_j, t_k) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_k) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_k) \\ &\quad - \frac{\Delta t \Delta x}{2}\frac{\partial^2 u}{\partial x \partial t}(x_j, t_k) - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_k). \end{aligned}$$

Substituting the expansions above into the local truncation error, we get

$$
\begin{aligned}
\tau(\Delta t, \Delta x) =& \frac{1}{\Delta t}\left(\Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}\right) \\
&- \frac{1}{2\Delta x}\left[\left(u_j^k + \Delta t\frac{\partial u}{\partial t} + \Delta x\frac{\partial u}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t \Delta x}{2}\frac{\partial^2 u}{\partial x \partial t} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)\right. \\
&- \left.\left(u_j^k + \Delta t\frac{\partial u}{\partial t} - \Delta x\frac{\partial u}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta t \Delta x}{2}\frac{\partial^2 u}{\partial x \partial t} - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)\right] + \cdots \\
=& \quad \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x \partial t} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \cdots \\
=& \quad \underbrace{\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}}_{=0} + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)
\end{aligned}
$$

where the first and second term equal zero since u satisfies the PDE. Hence, this method is consistent with order 1 in time and order 2 in space.

(b) Using the discrete Fourier transform, we have

$$
u_j^k = \frac{1}{N}\sum_{p=0}^{N-1} c_p^k e^{ipj\xi}, \qquad \xi = \Delta x
$$

and substituting into the scheme give

$$
\begin{aligned}
&\sum_{p=0}^{N-1} c_p^{k+1} e^{ipj\xi} = \sum_{p=0}^{N-1} c_p^k e^{ipj\xi} - \frac{\Delta t}{2\Delta x}\sum_{p=0}^{N-1} c_p^{k+1}\left(e^{ip(j+1)\xi} - e^{ip(j-1)\xi}\right) \\
&\Rightarrow \sum_{p=0}^{N-1} c_p^{k+1} e^{ipj\xi}\left(1 + \frac{\Delta t}{2\Delta x}\left(e^{ip\xi} - e^{-ip\xi}\right)\right) = \sum_{p=0}^{N-1} c_p^k e^{ipj\xi} \\
&\Rightarrow c_p^{k+1}\left(1 + \frac{i\Delta t}{\Delta x}\sin(p\Delta x)\right) = c_p^k \\
&\Rightarrow c_p^{k+1} = \left[\frac{1}{1 + \frac{i\Delta t}{\Delta x}\sin(p\Delta x)}\right] c_p^k
\end{aligned}
$$

which means that we have a diagonal amplification matrix $\boldsymbol{G}$ with diagonal entries given by

$$
G_p(\Delta t, \Delta x) = \frac{1}{1 + \frac{i\Delta t}{\Delta x}\sin(p\Delta x)}.
$$

Since $\boldsymbol{G}$ is diagonal, the Von-Neumann condition

$$
\rho(G) \leq 1 + \gamma\Delta t
$$

for some $\gamma \in \mathbb{R}$ is necessary and sufficient. We have

$$
\begin{aligned}
\rho(\boldsymbol{G}) &= \max_{i=0,1,\dots,p} |G_p(\Delta t, \Delta x)| \\
&= \max_{i=0,1,\dots,p} \frac{1}{\sqrt{1 + \dfrac{i\Delta t}{\Delta x}\sin(p\Delta x)}} \leq 1.
\end{aligned}
$$

Hence, the scheme is stable and therefore convergent.

Question 2 (30 points). Consider the following initial value problem

$$
\begin{cases}
\dfrac{\partial U(x,y,t)}{\partial t} + \dfrac{\partial}{\partial x}\left(f_1(x,y)U(x,y,t)\right) + \dfrac{\partial}{\partial y}\left(f_2(x,y)U(x,y,t)\right) = 0 \\
U(x,y,0) = \dfrac{1}{2\pi^2}\sin(x+y)^2
\end{cases}
\tag{5}
$$

where

$$
f_1(x,y) = \sin(x)\sin(y), \qquad f_2(x,y) = 1 - e^{\sin(x+y)}. \tag{6}
$$

Given the periodicity of f_1, f_2 and the initial condition $U(x,y,0)$, it is convenient to solve (5) in the spatial domain

$$
\mathbb{T} = \left\{(x,y) \in \mathbb{R}^2 : \quad 0 \leq x \leq 2\pi, \quad 0 \leq y \leq 2\pi\right\} \tag{7}
$$

and set periodic boundary conditions along the four edges of $\mathbb{T}$, i.e.,

$$
\begin{cases}
U(0,y,t) = U(2\pi,y,t), & 0 \leq y \leq 2\pi \\
U(x,0,t) = U(x,2\pi,t), & 0 \leq x \leq 2\pi
\end{cases}
\tag{8}
$$

(a) (10 points) By using the Gauss theorem, prove that the integral of the solution to (5)-(8) in $\mathbb{T}$ is constant in time and equal to one, i.e.,

$$
\int_{\mathbb{T}} U(x,y,t)dxdy = 1, \qquad \forall t \geq 0. \tag{9}
$$

(b) (20 points) Compute the numerical solution of the initial value problem (5) in the spatial domain $\mathbb{T}$ defined in (7) by using the method of characteristics. To this end, discretize such domain with an evenly-spaced grid with N points in both x and y variables (N^2 grid points total):

$$
x_i = \frac{2\pi i}{N} \qquad y_j = \frac{2\pi j}{N} \qquad i,j = 1,\dots,N. \tag{10}
$$

Follow the steps below to compute the solution of (5) on the grid (10) at time $t = t^*$ using the methods of characteristics.

i) The characteristic curve $(x(t), y(t))$ starting at $(x^{(0)}, y^{(0)})$ at $t = 0$ is governed by the ODE system

$$
\begin{cases}
\dfrac{dx}{dt} = \sin(x)\sin(y) \\
\dfrac{dy}{dt} = 1 - \exp[\sin(y+x)] \\
x(0) = x^{(0)} \\
y(0) = y^{(0)}
\end{cases}
\tag{11}
$$

We trace the characteristic curve from each grid point (x_i, y_j) at $t = t^*$ backward in time to $t = 0$. To this end, let $\tilde{x}(t) = x(t^* - t)$ and $\tilde{y}(t) = y(t^* - t)$. Clearly, $(\tilde{x}(t), \tilde{y}(t))$ is governed by ODE system

$$\begin{cases} \dfrac{d\tilde{x}}{dt} = -\sin(\tilde{x})\sin(\tilde{y}) \\ \dfrac{d\tilde{y}}{dt} = -(1 - \exp[\sin(\tilde{y} + \tilde{x})]) \\ \tilde{x}(0) = x_i \\ \tilde{y}(0) = y_j \end{cases} \tag{12}$$

Note that $(\tilde{x}(t^*), \tilde{y}(t^*))$ gives us the starting point (at $t = 0$) of the characteristic curve that reaches the grid point (x_i, y_j) exactly at time $t = t^*$. Use RK4 to solve the ODE systems (12) (from $t = 0$ to $t = t^*$) for each grid point (x_i, y_j).

ii) Once the starting point $(x_i^{(0)}, y_j^{(0)})$ is found, we integrate u forward in time along the characteristic curve with the ODE system

$$\begin{cases} \dfrac{dx}{dt} = \sin(x)\sin(y) \\ \dfrac{dy}{dt} = 1 - \exp[\sin(y + x)] \\ \dfrac{du}{dt} = -\Big(\cos(x)\sin(y) - \cos(x+y)\exp[\sin(y+x)]\Big)\, u \\ x(0) = x_i^{(0)} \\ y(0) = y_j^{(0)} \\ u(0) = U\left(x_i^{(0)}, y_j^{(0)}, 0\right) \end{cases} \tag{13}$$

where the initial condition $U(x, y, 0)$ is defined for arbitrary (x, y) in (5). The desired solution at (x_i, y_j) at $t = t^*$ is $U(x_i, y_j, t^*) \simeq u(t^*)$.

Set $N = 80$ and solve (5) using the method of characteristics, and generate a 2D contour plot with 20 contour levels[1] of the solution in the domain (7) at each of $t = [0,\ 0.25,\ 0.5,\ 0.75,\ 1,\ 1.25]$.

Answers:

(a) Let $\boldsymbol{f} = (f_1, f_2)$. By integrating (5) with respect to x and y over domain $\mathbb{T}$ we obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{T}} u(x, y, t)dxdy = -\int_{\mathbb{T}} \nabla \cdot (\boldsymbol{f}(x, y)u(x, y, t))\, dxdy. \tag{14}$$

By using the Gauss theorem, we transform the integral of the divergence into an a flux integral through the boundary of $\mathbb{T}$, i.e., a square. We have

$$\int_{\mathbb{T}} \nabla \cdot (\boldsymbol{f}(x, y)u(x, y, t))\, dxdy = \int_{\partial\mathbb{T}} \widehat{\boldsymbol{n}} \cdot (\boldsymbol{f}(x, y)u(x, y, t))\, dxdy, \tag{15}$$

[1] Use the Matlab command `contourf(X,Y,U,20)`.

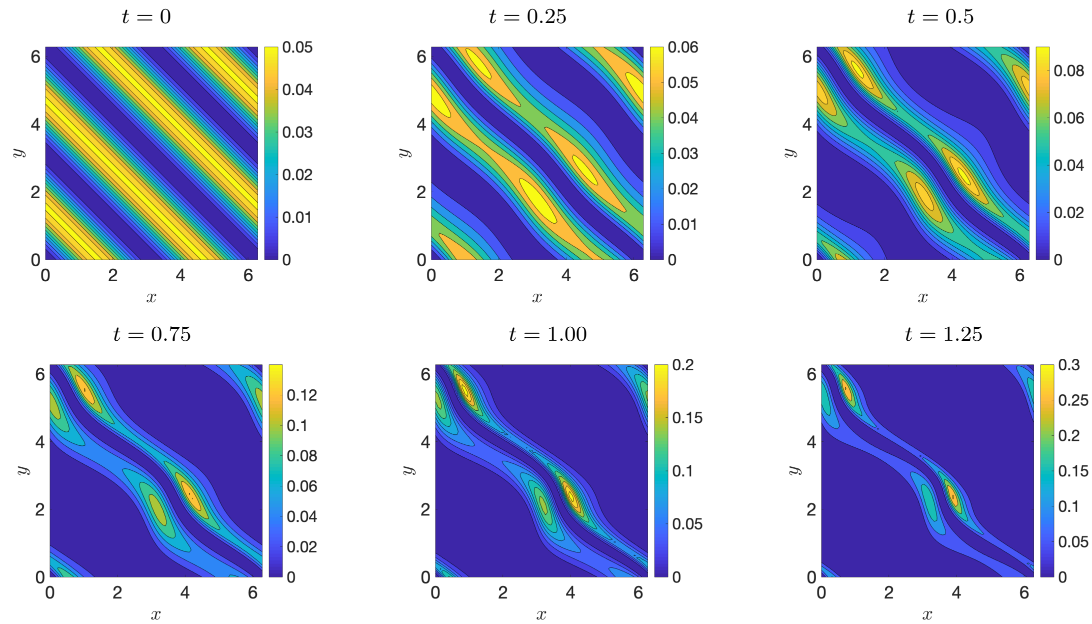

Figure 1: Solution of the initial value problem (5) computed using the method of characteristics. on a grid with $N = 80$ points in each variable x and y.

where $\widehat{\boldsymbol{n}}$ is the outward unit vector. Thanks to the periodicity of the boundary conditions (both $\boldsymbol{f}$ and u), we have that the overall flux in (15) is zero for all $t \geq 0$. Therefore,

$$\int_{\mathbb{T}} u(x, y, t) dx dy = \int_{\mathbb{T}} u(x, y, 0) dx dy. \tag{16}$$

The last integral can be computed analytically as

$$\begin{aligned} \int_{\mathbb{T}} u(x, y, 0) dx dy = & \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sin(x+y)^2 dx dy \\ = & \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left(\sin(x)\sin(y) + \cos(x)\cos(y)\right)^2 dx dy \\ = & 1 \end{aligned} \tag{17}$$

This proves mass conservation of (9).

(b) In Figure 1 we plot the solution to (5) computed with the method of characteristics on a grid with $N = 80$ points in each variable x and y.

Question 3 (50 points). Compute the numerical solution of the initial-boundary value problem (5)-(8) using second-order centered finite differences in x and y. To this end, consider the evenly spaced grid:

$$x_i = \frac{2\pi i}{N} \qquad y_j = \frac{2\pi j}{N} \qquad i, j = 0, ..., N+1. \tag{18}$$

The semi-discrete form of (5) on the grid (18) is

$$\frac{du_{i,j}}{dt} = -\frac{g^{(1)}_{i+1,j} - g^{(1)}_{i-1,j}}{2h} - \frac{g^{(2)}_{i,j+1} - g^{(2)}_{i,j-1}}{2h}, \qquad i, j = 1, ..., N, \tag{19}$$

where $u_{i,j}(t)$ is an approximation of $U(x_i, y_j, t)$, $h = \Delta x = \Delta y = 2\pi/N$ is the grid spacing in both x and y directions, and

$$g^{(1)}_{i,j}(t) = f_1(x_i, y_j)u(x_i, y_j, t), \qquad g^{(2)}_{i,j}(t) = f_2(x_i, y_j)u(x_i, y_j, t). \tag{20}$$

The finite difference scheme (19) requires $u_{i,0}$, $u_{i,N+1}$, $u_{0,j}$, and $u_{N+1,j}$. These quantities can be obtained by enforcing the periodic boundary conditions (8) as

$$\begin{aligned} u_{i,0} &= u_{i,N}, \qquad u_{i,N+1} = u_{i,1} \qquad \text{for all } i, \\ u_{0,j} &= u_{N,j}, \qquad u_{N+1,j} = u_{1,j} \qquad \text{for all } j. \end{aligned}$$

1. (20 points) Compute the numerical solution of (19) by using the two-step Adams-Bashforth method (use one step of Heun's method to start-up the scheme). To this end, set $N = 80$ and $\Delta t = 0.0005$. Provide a 2D contour plot with 20 contour levels of the solution at times $t = [0,\ 0.25,\ 0.5,\ 0.75,\ 1,\ 1.25]$.

2. (10 points) Set $N = 80$ and $\Delta t = 0.0005$. Compute the integral of the finite-difference numerical solution versus time. To this end, use the following quadrature rule

$$\int_{\mathbb{T}} U(x, y, t_k)dxdy \simeq \frac{4\pi^2}{N^2} \sum_{i,j=1}^{N} u_{i,j}(t_k). \tag{21}$$

   Plot your results on the evenly-spaced temporal grid $t = [0, 0.5, 1, 1.5, 2]$. Is the integral (21) constant in time as you would expect from (9)?

3. (10 points) Compute the maximum pointwise error between the numerical solution you obtained with the method of characteristics and the solution you obtained with the finite difference scheme on the grid (10) with $N = 80$ points in each direction (set $\Delta t = 0.0005$). Plot the error in a log-linear scale vs. time at $t = [0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2]$.

4. (10 points) Compute the mean squared error between the numerical solution you obtained with finite-differences and the solution you obtained with the method of characteristics (benchmark solution). Specifically, use the quadrature rule (21) to compute

$$\begin{aligned} e_2(t) &= \int_{\mathbb{T}} \left(u_{FD}(x, y, t) - u_{CH}(x, y, t)\right)^2 dxdy \\ &\simeq \frac{4\pi^2}{N^2} \sum_{i,j=1}^{N} \left(u_{\mathrm{FD}}(x_i, y_j, t) - u_{\mathrm{CH}}(x_i, y_j, t)\right)^2 \end{aligned} \tag{22}$$

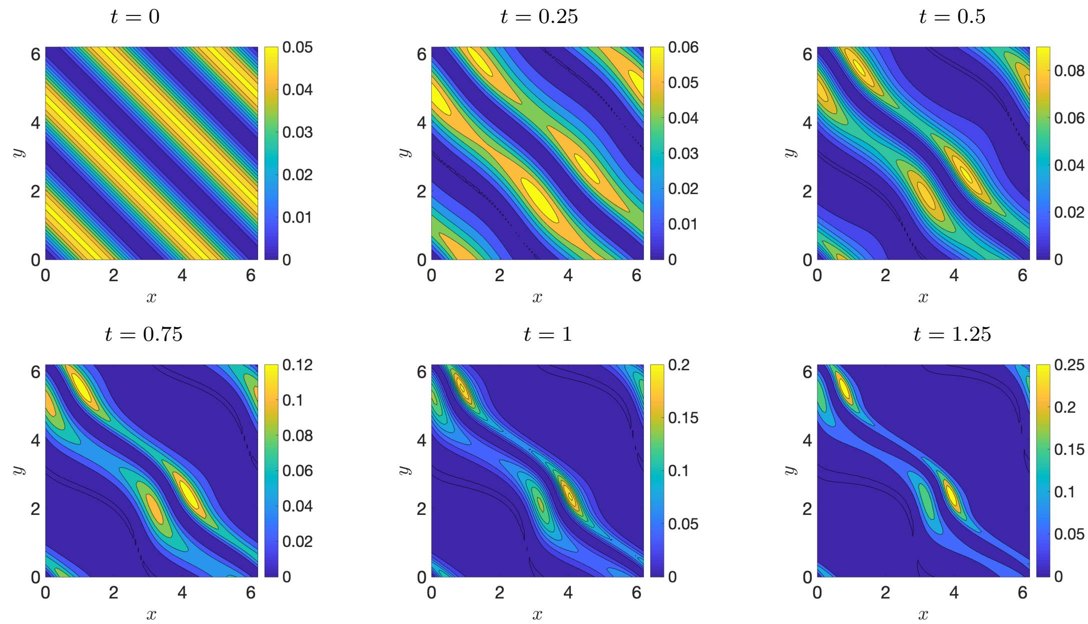

Figure 2: Solution of the initial value problem (5) computed using the finite-difference scheme (19) on the grid (10) with $N = 80$ points, and two-step Adams-Bashorth integration with $\Delta t = 0.0005$.

at $t = 1$ for $N = [40, 60, 100, 140]$. In equation (22), $u_{\mathrm{FD}}(x_i, y_j, t)$ represents the solution to (19) (finite-difference method), while $u_{\mathrm{CH}}(x_i, y_j, t)$ is the solution you obtained via the method of characteristics, evaluated on the grid (10). Comment on your results. In particular, does the error decrease as N increases? At which rate? Comment on your results.

Answers:

1. In Figure 2 we plot the numerical solution we obtain with the finite-difference scheme (19) on the grid (10) with $N = 80$ points, and two-step Adams-Bashorth temporal integration with $\Delta t = 0.0005$.

2. In Figure 3 we plot the integral of the numerical solution we obtained with the method of characteristics and the finite-difference method on the grid (10) with 80 points ($\Delta t = 0.0005$). Such integrals are computed with the Fourier pseudo-spectral quadrature rule (21). It is seen that the scheme (19) preserves mass quite well for $N = 80$ and within the time interval $[0, 2]$.

3. In Figure 4 we plot the maximum pointwise error and the $L^2(\mathbb{T})$ error (22) of the the finite-difference numerical solution relative to the benchmark solution obtained with the method of characteristics. Specifically, we consider the grid (10) with $N = 80$, $\Delta t = 0.0005$.

4. In Figure 5 we plot the $L^2(\mathbb{T})$ errors of the finite-difference solution at $t = 1$ versus the number of grid points N in each variable.

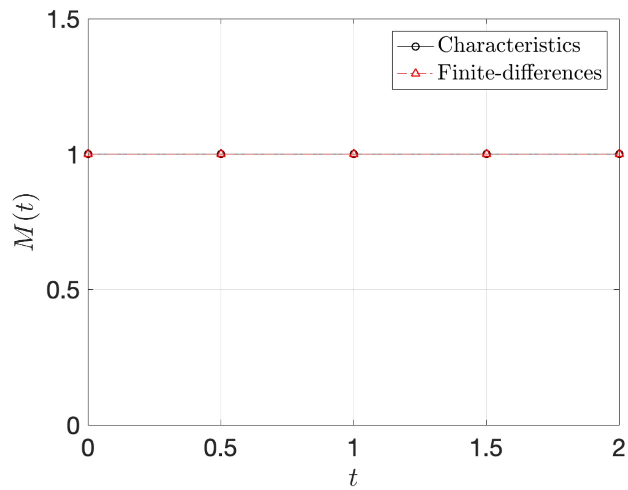

Figure 3: Integrals of the solution (21) computed by solving (5) with the method of characteristics and with the finite-difference method on the grid (10) with 80 points. It is seen that both the method of characteristics and the second-order finite difference method accurately preserve mass (in the given simulation setting).

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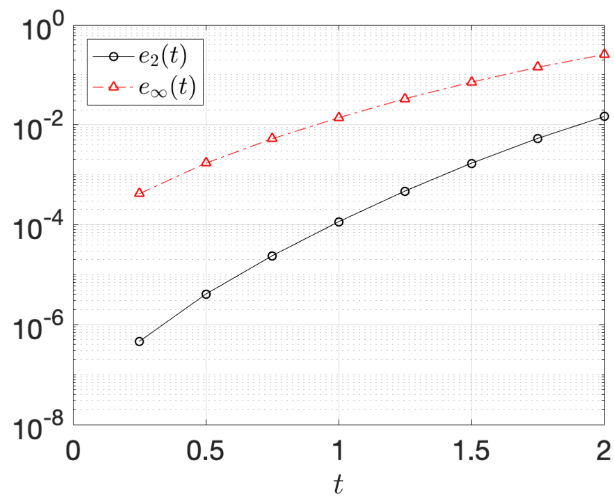

Figure 4: Maximum point-wise error and $L^2(\mathbb{T})$ error (22) versus time of the the finite-difference numerical solution relative to the benchmark solution obtained with the method of characteristics. These errors are obtained on the grid (10) with $N = 80$, and $\Delta t = 0.0005$.

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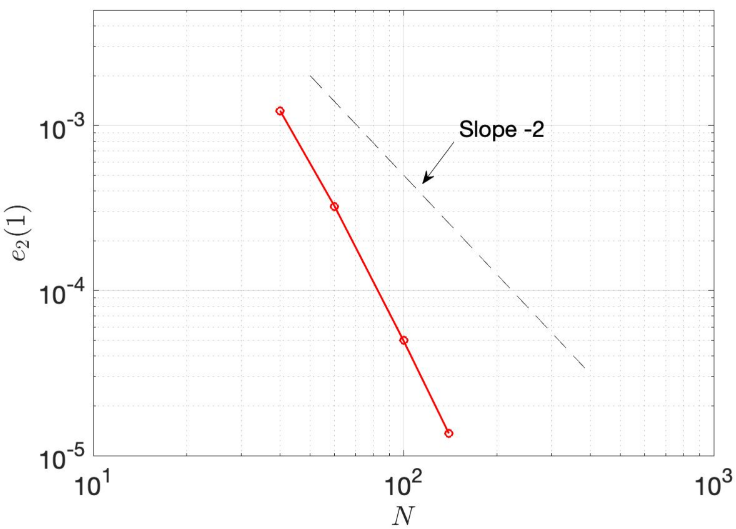

Figure 5: Mean squared errors of the finite-difference solution at $t = 1$ versus the number of grid points N in each variable, and versus N^2. The error is relative to the benchmark characteristic solution. It is seen that the error scales approximately quadratically with N, suggesting that the scheme (19) is second-order in N.