Numerical Methods for the Solution of Differential Equations (AMS 213B)
Homework 3 - Due Monday May 20

Instructions

Please submit to CANVAS one PDF file (your solution to the assignment), and one .zip file that includes any computer code you develop for the assignment. The PDF file must be a document compiled from Latex source code (mandatory for PhD students), or a PDF created using any other other word processor (MS and SciCAM students). No handwritten work should be submitted.

Question	points
1	40
2	60
Extra credit	30

Question 1 (40 points). Consider the following boundary value problem (BVP) for the Poisson's equation in two dimensions

$$\begin{cases} \dfrac{\partial^2 U(x,y)}{\partial x^2} + \dfrac{\partial^2 U(x,y)}{\partial y^2} = f(x,y) \qquad (x,y) \in \Omega \\ U(x,y) = g(x,y) \qquad (x,y) \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = [0,2] \times [0,1]$, $\partial\Omega$ is the boundary of Ω,

$$g(x,y) = 2 - x^2 - 2\sin(\pi y^2), \qquad \text{and} \qquad f(x,y) = -20 + 3x^2 + 4y^2. \tag{2}$$

(a) (30 points) Write a computer code to compute the numerical solution of the BVP (1)-(2) using second-order centered finite differences. To this end, discretize the domain using a tensor product grid with N points in x (including endpoints) and M points in y (including endpoints).

(b) (5 points) Plot the numerical solution you obtain for $(N, M) = (81, 51)$ as a surface plot.

(c) (5 points) Plot the numerical solution at $y = 0.2$ and $y = 0.5$ versus x (two graphs in the same figure).

Hint: To solve Poisson's equation (1)-(2) you can first transform it into an equivalent problem with zero boundary conditions via the linear transformation

$$\eta(x,y) = U(x,y) - g(x,y).$$

The resulting boundary value problem for $\eta(x,y)$ can be solved by inverting the matrix system described in the course note 7 at page 14. Alternatively, you can use the least squares method applied to the finite-difference discretization of (1)-(2). To this end, you can use any optimizer of your choice.

Question 2 (60 points). Consider the following initial-boundary value problem for the heat equation

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2} \qquad t \geq 0 \qquad x \in [-1,1] \\ U(x,0) = (3+x) + 5(1-x^2)^2 \qquad \text{(initial condition)} \\ U(-1,t) = 2, \qquad U(1,t) = 4 \qquad \text{(boundary conditions)} \end{cases} \tag{3}$$

(a) (15 points) Determine the analytical solution of (3). (Hint: the solution of (3) can be decomposed as

$$U(x,t) = (3+x) + \eta(x,t) \tag{4}$$

where $\eta(x,t)$ satisfies the PDE

$$\frac{\partial \eta}{\partial t} = \frac{\partial^2 \eta}{\partial x^2} \tag{5}$$

with initial condition $\eta(x,0) = 5(1-x^2)^2$ and zero Dirichlet boundary conditions at $x = -1$ and $x = 1$).

(b) (5 points) Plot the analytical solution as a surface plot on a grid with 100×100 evenly-spaced points in $(x,t) \in [-1,1] \times [0,2]$.

(c) (15 points) Write a code to compute the numerical solution of (3) using second-order finite differences in space on an evenly-spaced grid with N points (N to be chosen later) in $[-1,1]$, including the endpoints. Integrate the semi-discrete form of the PDE in time using the Crank-Nicolson method. Set $\Delta t = 10^{-4}$.

(d) (15 points) Write a code to compute the numerical solution of (3) using the Gauss-Chebyshev-Lobatto collocation method on a Gauss-Chebyshev-Lobatto grid with N points (N to be chosen later) in $[-1,1]$, including the endpoints. As before, integrate the semi-discrete form in time using the Crank-Nicolson method. Set $\Delta t = 10^{-4}$.

(e) (10 points) Plot maximum pointwise error

$$e(T) = \max_{i=1,\ldots,N} |U(x_i,T) - u_i(T)| \tag{6}$$

in a log scale between the analytical solution $U(x,t)$ you obtained in (a) and the numerical solutions $\{u_i(t)\}$ you obtained in (b) and (c) at time $T = 2$ versus the number of grid points N for $N = \{5, 10, 15, 20, 25, 30, 50, 100, 150, 200\}$ (two "semilogy" error plots in the same Figure, each with 10 points). Which method converges faster? Based on the analysis of the error plots, what can you say about the convergence order of each method?

Extra Credit (30 points).

(a) (20 points) Write a code to compute the numerical solution of the Kuramoto-Sivashinsky initial-boundary value problem

$$\begin{cases} \dfrac{\partial U}{\partial t} + u\dfrac{\partial U}{\partial x} + \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^4 U}{\partial x^4} = 0 \qquad t \geq 0 \qquad x \in [-25,25] \\ U(x,0) = \sin(x)e^{-(x-10)^2/2} \\ \text{Periodic boundary conditions} \end{cases} \tag{7}$$

using second-order centered finite-differences in space and the two-step Adams-Bashforth method in time. Run your simulations with $N = 200$ evenly-spaced spatial grid points in $[-25, 25]$, including endpoints, up $T = 100$ time units using time step $\Delta t = 10^{-4}$.

(b) (5 points) Plot the numerical solution as a surface on a 200×1001 space-time grid[1] (200 spatial points in $[-25, 25]$ and 1001 time instants in $[0, 100]$.

(c) (5 points) Plot the numerical solution at time $t = 62$ as a function of x.

[1] To this end, you can use the Matlab commands:

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surf(X,T,U);    shading interp;    view(0,90).
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