Homework 3: Report

Dante Buhl

June 3, 2024

Question 1: Consistency and Stability

$$\begin{cases} U_t + U_x = 0 \qquad x \in [0, 2\pi], \quad t \geq 0 \\ U(x,0) = \sin^2(x) \\ \text{Periodic B.C.} \end{cases} \tag{1}$$

$$u_j^{k+1} = u_j^k - \frac{\Delta t}{2\Delta x}(u_{j+1}^{k+1} - u_{j-1}^{k+1}) \tag{2}$$

a) Compute the Local Truncation Error of the scheme in (2). Is the scheme consistent? If so, to which orders in Δt and Δx

Proof.

$$\begin{aligned}
\Delta t \tau_j^{k+1} &= \boldsymbol{y}_j^{k+1} - \boldsymbol{y}_j^k + \frac{\Delta t}{2\Delta x}(\boldsymbol{y}_{j+1}^{k+1} - \boldsymbol{y}_{j-1}^{k+1}) \\
\tau_j^{k+1} &= \dot{\boldsymbol{y}}_j^{k+1} + \frac{\Delta t}{2}\ddot{\boldsymbol{y}}_j^{k+1} + O(\Delta t^2) + \frac{1}{2\Delta x}(2\Delta x \boldsymbol{y}_{xj}^{k+1} + \frac{2\Delta x^3}{6}\boldsymbol{y}_{xj}^{k+1} + O(\Delta x^5)) \\
\tau_j^{k+1} &= \frac{\Delta t}{2}\ddot{\boldsymbol{y}}_j^{k+1} + \frac{\Delta x^2}{6}\boldsymbol{y}_{xxxj}^{k+1} + O(\Delta t^2) + O(\Delta x^4)
\end{aligned}$$

Thus we have shown that the local truncation error, τ_j^{k+1} scales with Δt with order 1, and with Δx with order 2. Thus, the scheme is consistent with order 1 in time and order 2 in space. □

b) Compute the Von-Neumann Stability Analysis of (2). Is the scheme convergent?

Proof. There are two methods to show that this method is stable. We can show this wih Lax-Richtmyer stability or with Von Neumann stability theory. First, we can show rather easily,

$$\begin{aligned}
u^{k+1}(\mathrm{I} + \Delta t D1_{PFD2}) &= u^k \\
u^{k+1} &= (\mathrm{I} + \Delta t D1_{PFD2})^{-1} u^k \\
B &= (\mathrm{I} + \Delta t D1_{PFD2})^{-1}
\end{aligned}$$

We can consider the eigenvalues of B^{-1} which are not difficult to compute and show rather easily that the spectral radius of B has an upper bound of 1. Note that $D1_{PFD2}$ is the periodic first differentiation matrix for second ordered finite differences. This looks like the following logic,

$$\begin{aligned}
\det(\mathrm{I}(1-\lambda) + \Delta t D1_{PFD2}) &= (1-\lambda)^N + \frac{\Delta t}{2\Delta x}((-1)^N + 1) \\
\lambda &= 1 \mp \sqrt[N]{-\frac{\Delta t}{2\Delta x}((-1)^N + 1)}
\end{aligned}$$

Obviously this produces eigenvalues all of which are larger than 1 in magnitude. Thus we can use some logic from linear algebra finding that the eigenvalues of an inverse matrices are the multiplicative inverses of the eigenvalues of the original matrix. That is, we have bounded the eigenvalues of the transformation with a spectral radius of less than 1 and have proved stability. However, the von neumann method is analogous.

We use the substitution $u_j^k = c_p^k e^{ijp\xi}$.

$$
\begin{aligned}
c_p^{k+1}(1 + \frac{\Delta t}{2\Delta x}(e^{ip\xi} - e^{-ip\xi})) &= c_p^k \\
c_p^{k+1}(1 + i\frac{\Delta t}{\Delta x}\sin(p\xi)) &= c_p^k \\
c_p^{k+1} &= \frac{1}{1 + i\frac{\Delta t}{\Delta x}\sin(p\xi)} c_p^k \\
\frac{1}{1 + i\frac{\Delta t}{\Delta x}\sin(p\xi)} &= \frac{1 - i\frac{\Delta t}{\Delta x}\sin(p\xi)}{1 + \frac{\Delta t^2}{\Delta x^2}\sin^2(p\xi)} \\
\left|\frac{1}{1 + i\frac{\Delta t}{\Delta x}\sin(p\xi)}\right| &= \frac{\sqrt{1 + \frac{\Delta t^2}{\Delta x^2}\sin^2(p\xi)}}{1 + \frac{\Delta t^2}{\Delta x^2}\sin^2(p\xi)} \leq 1
\end{aligned}
$$

Therefore we have shown using Von Neumann stability theory that the scheme is stable. Since we have coupled stability and consistency we have convergence with order 1 in Δt and order 2 in Δx. □

Question 2: Method of Characteristics for Advection

$$
\begin{cases} U_t + (fU)_x + (gU)_y = 0 \\ U(x, y, 0) = \frac{1}{2\pi^2}\sin^2(x + y) \end{cases} \tag{3}
$$

$$
f(x, y) = \sin(x)\sin(y), \qquad g(x, y) = 1 - e^{\sin(x+y)} \tag{4}
$$

$$
\mathbb{T} = \left\{(x, y) \in \mathbb{R}^2 : \quad 0 \leq x \leq 2\pi, \quad 0 \leq y \leq 2\pi\right\} \tag{5}
$$

$$
\begin{cases} U(0, y, t) = U(2\pi, y, t), & 0 \leq y \leq 2\pi \\ U(x, 0, t) = U(x, 2\pi, t), & 0 \leq x \leq 2\pi \end{cases} \tag{6}
$$

a) Show that the following integral evaluates to 1.

$$
\int_{\mathbb{T}} U(x, y, t)dxdy = 1, \quad \forall t \geq 0
$$

Proof.

$$
\begin{aligned}
U_t &= -(fU)_x - (gU)_y \\
\int_{\mathbb{T}} U_t dA &= \int_{\mathbb{T}} -(fU)_x - (gU)_y dA \\
\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA &= \int_{\mathbb{T}} -(fU)_x - (gU)_y dA \\
\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA &= \int_{\partial\mathbb{T}} -\langle fU, gU\rangle \cdot \eta ds \\
\eta &= \begin{cases} \langle 0, 1\rangle, y = 2\pi \\ \langle 0, -1\rangle, y = 0 \\ \langle 1, 0\rangle, x = 2\pi \\ \langle -1, 0\rangle, x = 0 \end{cases}
\end{aligned}
$$

$$\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA = \int_0^{2\pi} fU|_{x=0}dy - \int_0^{2\pi} gU|_{y=2\pi}dx - \int_0^{2\pi} fU|_{x=2\pi}dy + \int_0^{2\pi} gU|_{y=0}dx$$

$$\begin{aligned}
\int_0^{2\pi} gU|_{y=0}dx &= \int_0^{2\pi} gU|_{y=2\pi}dx \\
\int_0^{2\pi} fU|_{x=0}dy &= 0 \\
\int_0^{2\pi} fU|_{x=2\pi}dy &= 0 \\
\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA &= 0
\end{aligned}$$

Therefore we have that the value of the integral is constant. Now we must simply show that it is at one timestep equal to one.

$$\begin{aligned}
\int_{\mathbb{T}} U(x,y,0)dA &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \sin^2(x+y)dxdy \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} 1-\cos(2x+2y)dxdy \\
&= \frac{1}{4\pi^2}\left(4\pi^2 - \int_0^{2\pi}\int_0^{2\pi} \cos(2x+2y)dxdy\right) \\
&= 1 - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \cos(2x+2y)dxdy \\
&= 1 - \frac{1}{4\pi^2}\int_0^{2\pi} 0dxdy \\
&= 1
\end{aligned}$$

Thus we have that U will satisfy this integral at every timestep. □

b) See figure 1.

Question 3: Finite Differences for Advection

a) See figure 2.

b) Yes! The discrete integral is constant up to numerical precision as we would expect (this is hard to see in the plot but if you look at fdint.dat after running the code it is clear that this is true).

c) See figure 4.

d) See figure 5. The MSE descreases with N. The order of decay seems to be quadratic after comparing the slope to that of a quadratic decay. This seems good! It is also important to consider that this plot is only as accurate as the combined accuracy of the method of Characteristics and finite differences combined. After fitting a line to the plot, a line of best fit seems to yield and order 2 decay!

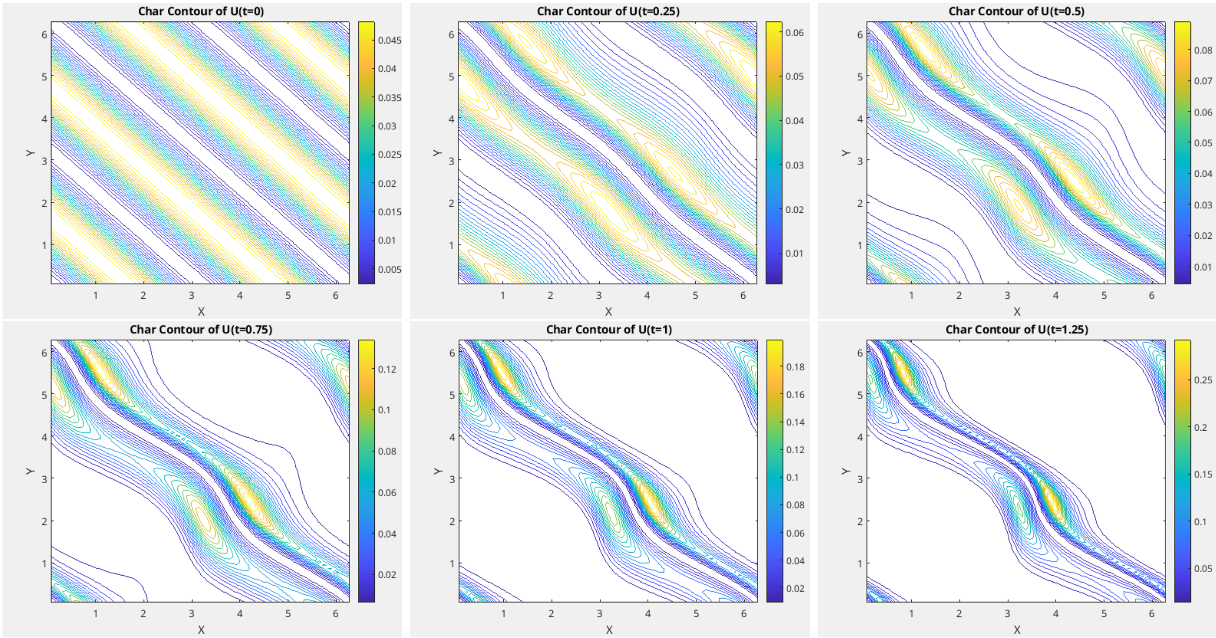

Figure 1: Contour Plots of Solution using Method of Characteristics

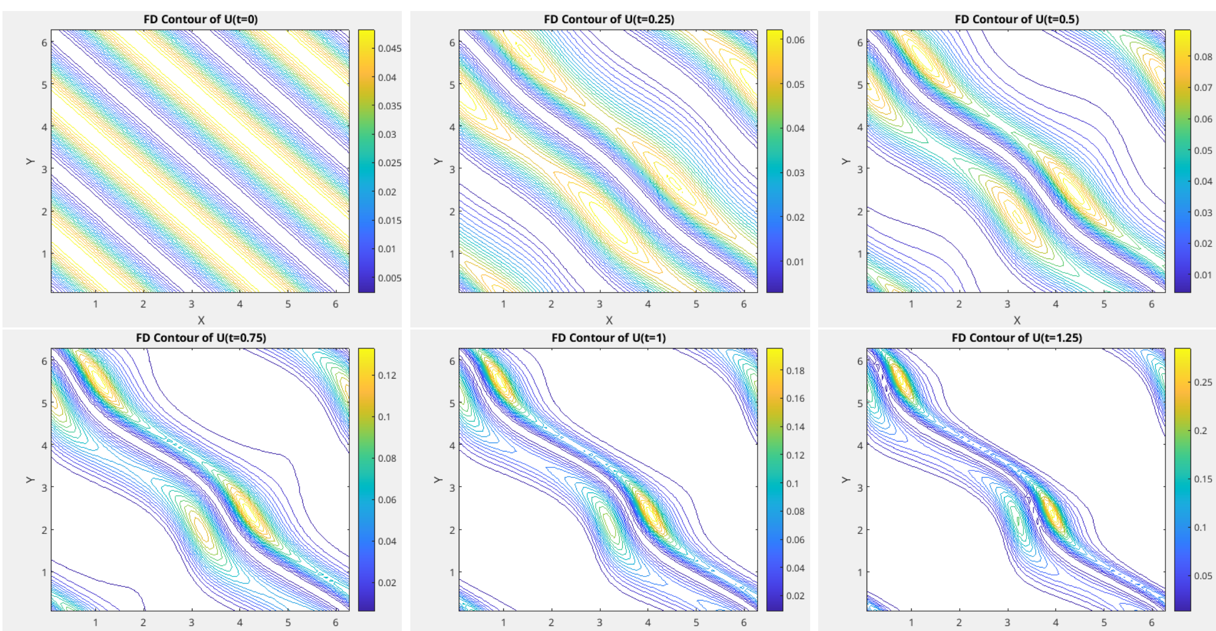

Figure 2: Contour Plots of Solution using Finite Differences

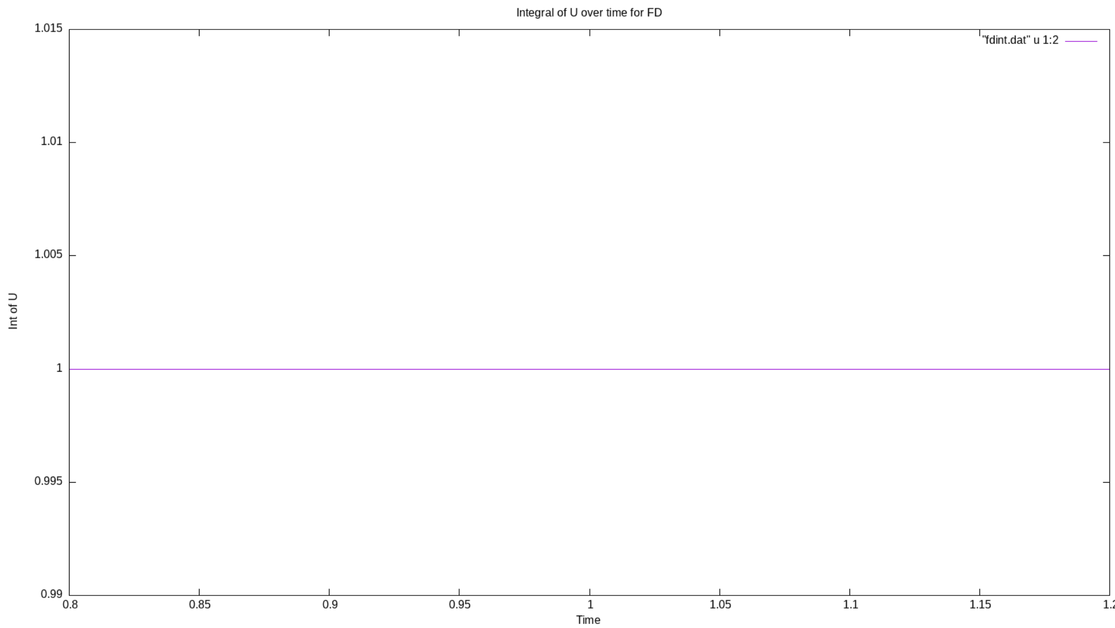

Figure 3: Area Integral of Solution over Time

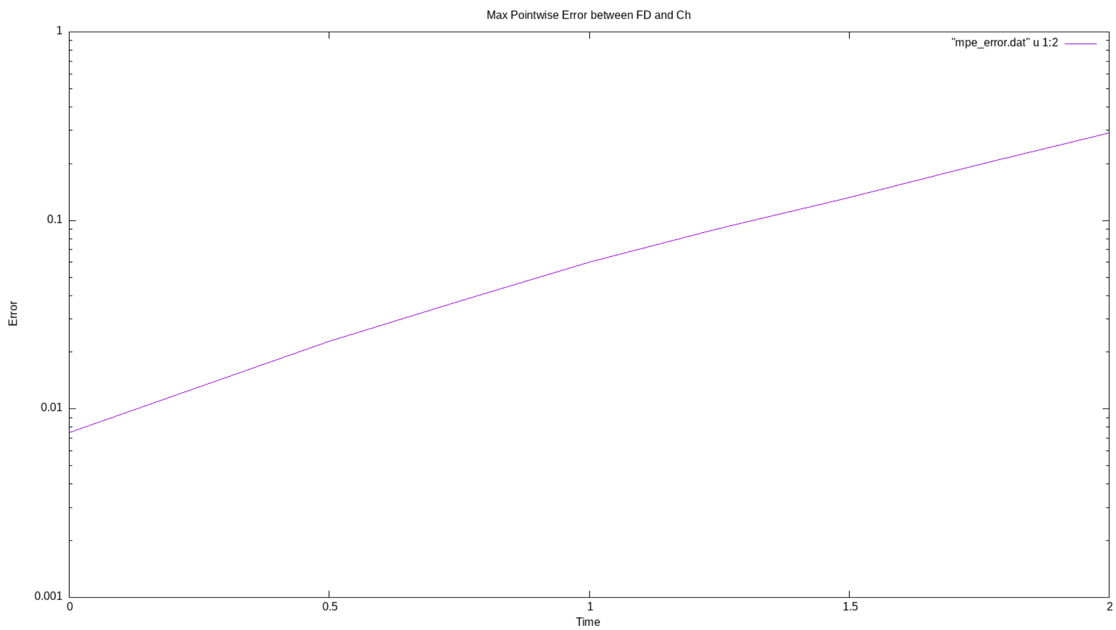

Figure 4: Plot of Max Pointwise Error as a function of time

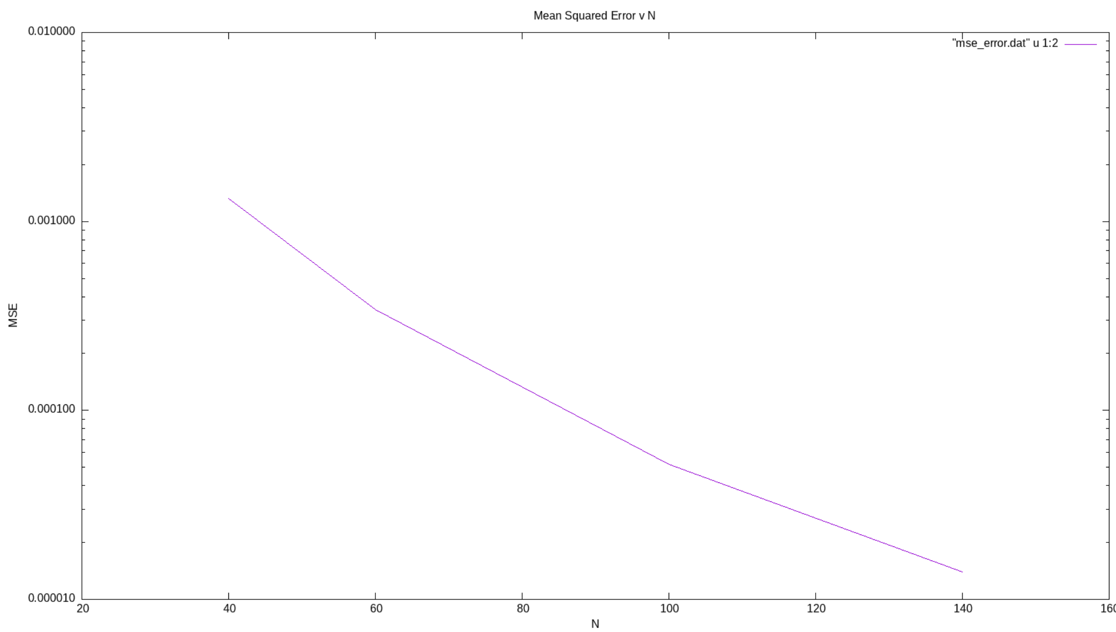

Figure 5: Plot of Mean Square Error as a function of gridsize