Numerical Methods for the Solution of Differential Equations (AM 213B)
Homework 1 - Due Monday April 15

Instructions

Submit to CANVAS one PDF file (your solution to the assignment), and one .zip file that includes any computer code you develop for the assignment. The PDF file must be a document compiled from Latex source code (mandatory for PhD students), or a PDF created using any other other word processor (MS and SciCAM students). No handwritten work should be submitted.

Question	points
1	30
2	70

Question 1. Let $f(x)$ be a continuously differentiable function defined over the interval $[0, 1]$.

a) **(10 points)** Derive the four-point finite-difference backward differentiation formula (BDF3) approximation of the first derivative of f.

$$f'(x_j) \simeq \frac{11f(x_j) - 18f(x_{j-1}) + 9f(x_{j-2}) - 2f(x_{j-3})}{6\Delta x} \tag{1}$$

b) **(10 points)** Prove that (1) converges with order 3 in Δx to the analytical derivative $f'(x_j)$.

c) **(10 points)** Show numerically that (1) converges with order 3 by applying it to the periodic function

$$f(x) = \log\left(2 + \sin(2\pi x)\right) \qquad x \in [0, 1]. \tag{2}$$

To this end, plot the derivative of the function (2) and the finite difference approximation (1) you obtain on the evenly-spaced grid

$$x_j = \frac{j}{n} \qquad j = 0, \ldots, n \tag{3}$$

for $n = 20$ and $n = 60$ (two different Figures). In another Figure, plot the maximum pointwise error between the analytical and numerical derivatives evaluated on the grid (3), i.e.,

$$e(n) = \max_{j=0,\ldots,n} \left| f'(x_j) - \frac{1}{\Delta x}\left(\frac{11}{6}f(x_j) - 3f(x_{j-1}) + \frac{3}{2}f(x_{j-2}) - \frac{1}{3}f(x_{j-3})\right)\right|, \tag{4}$$

as a function of n, for n up to 10^4 (see the log-log plot in Figure 9 of the course note 1). What is the convergence order you obtain numerically for BDF3 as $\Delta x = 1/n$ goes to zero? Justify your answer.

Question 2. Consider the two-dimensional linear system of ODEs

$$\begin{cases} \dfrac{d\boldsymbol{y}}{dt} = \boldsymbol{A}\boldsymbol{y} \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases} \tag{5}$$

where

$$\boldsymbol{y}_0 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{A} = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}. \tag{6}$$

a) **(15 points)** Compute the analytical solution of (5)-(6) and plot $y_1(t)$ versus t, $y_2(t)$ versus t, and $y_2(t)$ versus $y_1(t)$ for $t \in [0, 10]$.

b) **(20 points)** Write a computer code to compute the numerical solution of the initial value problem (5) using the three-stage Runge-Kutta method (RK3) defined by the Butcher array

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

and the three-step Adams-Moulton method[1] (AM3)

$$\boldsymbol{u}_{k+3} = \boldsymbol{u}_{k+2} + \frac{\Delta t}{24}\left[9\boldsymbol{f}\left(\boldsymbol{u}_{k+3}, t_{k+3}\right) + 19\boldsymbol{f}\left(\boldsymbol{u}_{k+2}, t_{k+2}\right) - 5\boldsymbol{f}\left(\boldsymbol{u}_{k+1}, t_{k+1}\right) + \boldsymbol{f}\left(\boldsymbol{u}_k, t_k\right)\right]. \tag{7}$$

To this end,

c) **(5 points)** Provide the explicit formulations of RK3 and AM3 tailored for the linear dynamical system (5)-(6), i.e., for $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{y}$.

d) **(20 points)** Study convergence of the numerical solution you obtain with RK3 and AM3 as a function of Δt. To this end, run simulations for different values of Δt, i.e., $\Delta t = \{0.1, 0.05, 0.005, 0.0005\}$, fixed final time $T = 10$, and plot the error

$$e_2(t_k) = \|\boldsymbol{u}_k - \boldsymbol{y}(t_k)\|_2 \qquad k = 0, 1, \ldots \tag{8}$$

in logarithmic scale versus time for each case. In (8), $\boldsymbol{y}(t_k)$ denotes the analytical solution of (5)-(6) evaluated at time t_k while $\boldsymbol{u}_k$ is the numerical solution you obtain with AM3 or RK3.

e) **(10 points)** Plot the error (8) at final time in logarithmic scale versus Δt for both AM3 or RK3. What is the order of convergence you observe numerically for AM3 and RK3?

[1] Note that since the system is linear, i.e., $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{y}$, the implementation of the implicit Adams-Moulton method does not require a nonlinear solver, but only one linear solve at each time step. To start-up the AM3 method, i.e., to compute $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, use the RK3 method.