Numerical Methods for the Solution of Differential Equations (AMS 213B)

Homework 4 - Due MOnday June 3

Instructions

Please submit to CANVAS one PDF file (your solution to the assignment), and one .zip file that includes any computer code you develop for the assignment. The PDF file must be a document compiled from Latex source code (mandatory for PhD students), or a PDF created using any other other word processor (MS and SciCAM students). No handwritten work should be submitted.

Question	points
1	20
2	30
3	50

Question 1 (20 points). Consider the following initial-boundary value problem for the linear advection equation

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial U}{\partial x} = 0 \qquad t \geq 0 \qquad x \in [0, 2\pi] \\ U(x,0) = \sin^2(x) \\ \text{Periodic B.C.} \end{cases} \tag{1}$$

Discretize (1) with the following scheme

$$u_j^{k+1} = u_j^k - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^{k+1} - u_{j-1}^{k+1}\right) \tag{2}$$

on an evenly-spaced grid with N spatial points in $[0, 2\pi]$

$$x_j = j\Delta x, \qquad \Delta x = \frac{2\pi}{N}, \qquad j = 0, \ldots, N. \tag{3}$$

In (2) u_j^k represents an approximation of $U(x_j, t_k)$.

(a) (10 points) Compute the local truncation error of the scheme (2). Is the scheme consistent? To which order in Δt and Δx?

(b) (10 points) Use Von-Neumann analysis to study stability of the scheme (2). Is the scheme convergent? Justify your answer.

Question 2 (30 points). Consider the following initial value problem

$$\begin{cases} \dfrac{\partial U(x,y,t)}{\partial t} + \dfrac{\partial}{\partial x}\left(f_1(x,y)U(x,y,t)\right) + \dfrac{\partial}{\partial y}\left(f_2(x,y)U(x,y,t)\right) = 0 \\ U(x,y,0) = \dfrac{1}{2\pi^2}\sin(x+y)^2 \end{cases} \tag{4}$$

where

$$f_1(x,y) = \sin(x)\sin(y), \qquad\qquad f_2(x,y) = 1 - e^{\sin(x+y)}. \tag{5}$$

Given the periodicity of f_1, f_2 and the initial condition $U(x,y,0)$, it is convenient to solve (4) in the spatial domain

$$\mathbb{T} = \left\{(x,y) \in \mathbb{R}^2 : \quad 0 \leq x \leq 2\pi, \quad 0 \leq y \leq 2\pi\right\} \tag{6}$$

and set periodic boundary conditions along the four edges of $\mathbb{T}$, i.e.,

$$\begin{cases} U(0,y,t) = U(2\pi,y,t), & 0 \leq y \leq 2\pi \\ U(x,0,t) = U(x,2\pi,t), & 0 \leq x \leq 2\pi \end{cases} \tag{7}$$

(a) (10 points) By using the Gauss theorem, prove that the integral of the solution to (4)-(7) in $\mathbb{T}$ is constant in time and equal to one, i.e.,

$$\int_{\mathbb{T}} U(x,y,t)dxdy = 1, \qquad \forall t \geq 0. \tag{8}$$

(b) (20 points) Compute the numerical solution of the initial value problem (4) in the spatial domain $\mathbb{T}$ defined in (6) by using the method of characteristics. To this end, discretize such domain with an evenly-spaced grid with N points in both x and y variables (N^2 grid points total):

$$x_i = \frac{2\pi i}{N} \qquad y_j = \frac{2\pi j}{N} \qquad i,j = 1,\dots,N. \tag{9}$$

Follow the steps below to compute the solution of (4) on the grid (9) at time $t = t^*$ using the methods of characteristics.

i) The characteristic curve $(x(t), y(t))$ starting at $(x^{(0)}, y^{(0)})$ at $t = 0$ is governed by the ODE system

$$\begin{cases} \dfrac{dx}{dt} = \sin(x)\sin(y) \\ \dfrac{dy}{dt} = 1 - \exp[\sin(y+x)] \\ x(0) = x^{(0)} \\ y(0) = y^{(0)} \end{cases} \tag{10}$$

We trace the characteristic curve from each grid point (x_i, y_j) at $t = t^*$ backward in time to $t = 0$. To this end, let $\tilde{x}(t) = x(t^* - t)$ and $\tilde{y}(t) = y(t^* - t)$. Clearly, $(\tilde{x}(t), \tilde{y}(t))$ is governed by ODE system

$$\begin{cases} \dfrac{d\tilde{x}}{dt} = -\sin(\tilde{x})\sin(\tilde{y}) \\ \dfrac{d\tilde{y}}{dt} = -(1 - \exp[\sin(\tilde{y}+\tilde{x})]) \\ \tilde{x}(0) = x_i \\ \tilde{y}(0) = y_j \end{cases} \tag{11}$$

Note that $(\tilde{x}(t^*), \tilde{y}(t^*))$ gives us the starting point (at $t = 0$) of the characteristic curve that reaches the grid point (x_i, y_j) exactly at time $t = t^*$. Use RK4 to solve the ODE systems (11) (from $t = 0$ to $t = t^*$) for each grid point (x_i, y_j).

ii) Once the starting point $(x_i^{(0)}, y_j^{(0)})$ is found, we integrate u forward in time along the characteristic curve with the ODE system

$$\begin{cases} \dfrac{dx}{dt} = \sin(x)\sin(y) \\ \dfrac{dy}{dt} = 1 - \exp[\sin(y+x)] \\ \dfrac{du}{dt} = -\Big(\cos(x)\sin(y) - \cos(x+y)\exp[\sin(y+x)]\Big)\,u \\ x(0) = x_i^{(0)} \\ y(0) = y_j^{(0)} \\ u(0) = U\left(x_i^{(0)}, y_j^{(0)}, 0\right) \end{cases} \tag{12}$$

where the initial condition $U(x, y, 0)$ is defined for arbitrary (x, y) in (4). The desired solution at (x_i, y_j) at $t = t^*$ is $U(x_i, y_j, t^*) \simeq u(t^*)$.

Set $N = 80$ and solve (4) using the method of characteristics, and generate a 2D contour plot with 20 contour levels[1] of the solution in the domain (6) at each of $t = [0,\ 0.25,\ 0.5,\ 0.75,\ 1,\ 1.25]$.

Question 3 (50 points). Compute the numerical solution of the initial-boundary value problem (4)-(7) using second-order centered finite differences in x and y. To this end, consider the evenly spaced grid:

$$x_i = \frac{2\pi i}{N} \qquad y_j = \frac{2\pi j}{N} \qquad i, j = 0, ..., N+1. \tag{13}$$

The semi-discrete form of (4) on the grid (13) is

$$\frac{du_{i,j}}{dt} = -\frac{g^{(1)}_{i+1,j} - g^{(1)}_{i-1,j}}{2h} - \frac{g^{(2)}_{i,j+1} - g^{(2)}_{i,j-1}}{2h}, \qquad i, j = 1, ..., N, \tag{14}$$

where $u_{i,j}(t)$ is an approximation of $U(x_i, y_j, t)$, $h = \Delta x = \Delta y = 2\pi/N$ is the grid spacing in both x and y directions, and

$$g^{(1)}_{i,j}(t) = f_1(x_i, y_j)u(x_i, y_j, t), \qquad g^{(2)}_{i,j}(t) = f_2(x_i, y_j)u(x_i, y_j, t). \tag{15}$$

The finite difference scheme (14) requires $u_{i,0}$, $u_{i,N+1}$, $u_{0,j}$, and $u_{N+1,j}$. These quantities can be obtained by enforcing the periodic boundary conditions (7) as

$$\begin{aligned} u_{i,0} &= u_{i,N}, \qquad u_{i,N+1} = u_{i,1} \qquad \text{for all } i, \\ u_{0,j} &= u_{N,j}, \qquad u_{N+1,j} = u_{1,j} \qquad \text{for all } j. \end{aligned}$$

1. (20 points) Compute the numerical solution of (14) by using the two-step Adams-Bashforth method (use one step of Heun's method to start-up the scheme). To this end, set $N = 80$ and $\Delta t = 0.0005$. Provide a 2D contour plot with 20 contour levels of the solution at times $t = [0,\ 0.25,\ 0.5,\ 0.75,\ 1,\ 1.25]$.

[1] Use the Matlab command `contourf(X,Y,U,20)`.

2. (10 points) Set $N = 80$ and $\Delta t = 0.0005$. Compute the integral of the finite-difference numerical solution versus time. To this end, use the following quadrature rule

$$\int_{\mathbb{T}} U(x, y, t_k) dx dy \simeq \frac{4\pi^2}{N^2} \sum_{i,j=1}^{N} u_{i,j}(t_k). \tag{16}$$

Plot your results on the evenly-spaced temporal grid $t = [0, 0.5, 1, 1.5, 2]$. Is the integral (16) constant in time as you would expect from (8)?

3. (10 points) Compute the maximum point-wise error between the numerical solution you obtained with the method of characteristics and the solution you obtained with the finite difference scheme on the grid (9) with $N = 80$ points in each direction (set $\Delta t = 0.0005$). Plot the error in a log-linear scale vs. time at $t = [0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2]$.

4. (10 points) Compute the mean squared error between the numerical solution you obtained with finite-differences and the solution you obtained with the method of characteristics (benchmark solution). Specifically, use the quadrature rule (16) to compute

$$\begin{aligned} e_2(t) &= \int_{\mathbb{T}} \left(u_{FD}(x, y, t) - u_{CH}(x, y, t)\right)^2 dx dy \\ &\simeq \frac{4\pi^2}{N^2} \sum_{i,j=1}^{N} \left(u_{\mathrm{FD}}(x_i, y_j, t) - u_{\mathrm{CH}}(x_i, y_j, t)\right)^2 \end{aligned} \tag{17}$$

at $t = 1$ for $N = [40, 60, 100, 140]$. In equation (17), $u_{\mathrm{FD}}(x_i, y_j, t)$ represents the solution to (14) (finite-difference method), while $u_{\mathrm{CH}}(x_i, y_j, t)$ is the solution you obtained via the method of characteristics, evaluated on the grid (9). Comment on your results. In particular, does the error decrease as N increases? At which rate? Comment on your results.