Homework 2: Report

Dante Buhl

April. 29^{th} 2024

Problem 1: Absolute Stability for AB3

a) Determine the largest value of Δt, for which the three-step Adams-Bashforth method (AB3)

Proof. We use the condition for absolute stability:

$$\lim_{k\to\infty} ||\boldsymbol{u}_k|| = 0 \tag{1}$$

For this specific numerical method, we have the following characteristic polynomial for the numerical method. (Note that since the columns of B are linearly independent we have that B is diagonalizable).

$$\boldsymbol{u}_{k+3} = \boldsymbol{u}_{k+2} + \frac{\Delta t}{12}\left(23\boldsymbol{f}_{k+2} - 16\boldsymbol{f}_{k+1} + 5\boldsymbol{f}_k\right) \tag{2}$$

$$\boldsymbol{u}_{k+3} - \boldsymbol{u}_{k+2} = \frac{\Delta t}{12}\left(23\boldsymbol{A}\boldsymbol{u}_{k+2} - 16\boldsymbol{A}\boldsymbol{u}_{k+1} + 5\boldsymbol{A}\boldsymbol{u}_k\right) \tag{3}$$

$$\boldsymbol{w}_{k+3} - \boldsymbol{w}_{k+2} = \frac{\Delta t}{12}\Lambda\left(23\boldsymbol{w}_{k+2} - 16\boldsymbol{w}_{k+1} + 5\boldsymbol{w}_k\right) \tag{4}$$

From this form of the Adams Bashforth method, we have that the coefficients α_i and β_i are as follows,

$$\boldsymbol{\alpha} = [0, 0, -1, 1], \quad \boldsymbol{\beta} = \left[\frac{5}{12}, -\frac{16}{12}, \frac{23}{12}, 0\right] \tag{5}$$

$$\sum_{i=0}^{3}(\alpha_i - \Delta t\lambda_m\beta_i)\boldsymbol{w}^m_{k+i} = 0 \tag{6}$$

At this point, bother to find the eigenvalues of the matrix A which form Λ. Using a matlab eigenvalue solver, we find the eigenvalues of A to be,

$$\lambda \approx [-0.9667 \pm i30.1255, -99.0667]$$
$$\mathbb{R}(\lambda) \approx [-0.9667, -99.0667]$$

We also only consider the real part of λ as this is what will contribute to the convergence/stability. At this point we have 2 equations to solve in order to find the requirement on Δt for the absolute convergence. The two equations are related to the characteristic polynomial for the iteration process.

$$\pi(z) = \rho(z) - \Delta t\lambda_i\sigma(z) = 0$$
$$\rho(z) = \sum_{j=0}^{q}\alpha_j z^j, \quad \sigma(z) = \sum_{j=0}^{q}\beta_j z^j$$
$$\rho(z) = z^3 - z^2, \quad \sigma(z) = \frac{23z^2 - 16z + 5}{12}$$

$$\pi(z) = 0, \implies \Delta t = \frac{\rho(z)}{\lambda_i\sigma(z)} \tag{7}$$

This equation now becomes a constrained optimization problem. We consider two constraints. First, the value of Δt must be a real value. We notice that the equation given to solve for Δt is composed of three complex quantities, so there is no guarantee that the obtained value for Δt is real. Second, we notice that the eigenvalues of the system that we wish to be a contraction must be of absolute value less than 1. That is, $|z| < 1$. Therefore, we have two constraints, on a 3 variable optimization problem. We obtain three equations.

$$z = x + iy, \quad |z| < 1 \implies x^2 + y^2 < 1 \tag{8}$$

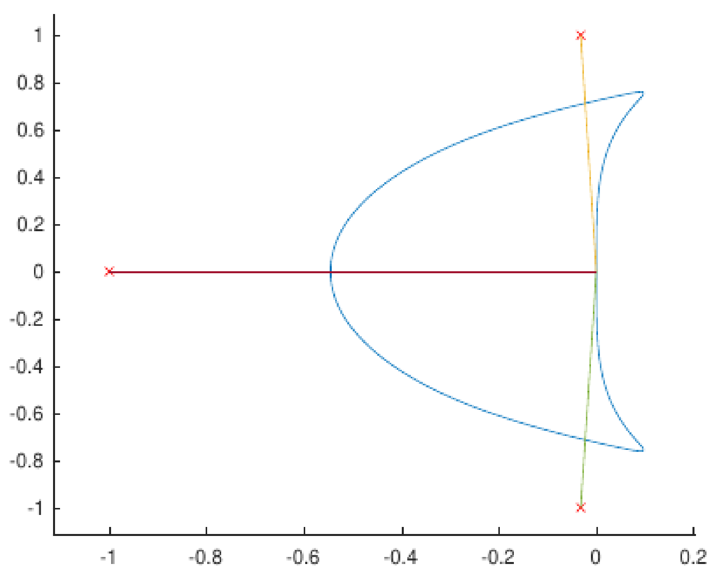
$$\mathbb{I}(\Delta t) = 0 \implies \mathbb{I}\left(\frac{\rho(x,y)}{\sigma(x,y)}\right) = 0 \tag{9}$$

$$\max \mathbb{R}(\Delta t) \implies \nabla_h \mathbb{R}\left(\frac{\rho(x,y)}{\sigma(x,y)}\right) = \vec{0} \tag{10}$$

$$\begin{aligned}
\rho(x,y) &= x^3 - 3xy^2 - x^2 + y^2 + i\left(3x^2y - y^3 - 2xy\right) \\
\sigma(x,y) &= \frac{23x^2 - 23y^2 - 16x + 5 + i\left(46xy - 16y\right)}{12}
\end{aligned}$$

$$\begin{aligned}
\frac{\rho(x,y)}{\sigma(x,y)} &= \frac{\rho(x,y)\overline{\sigma}(x,y)}{|\sigma(x,y)|^2} \\
\mathbb{R}\left(\frac{\rho(x,y)}{\sigma(x,y)}\right) &= \frac{\mathbb{R}\left(\rho(x,y)\overline{\sigma}(x,y)\right)}{|\sigma(x,y)|^2} \\
\mathbb{I}\left(\frac{\rho(x,y)}{\sigma(x,y)}\right) &= \frac{\mathbb{I}\left(\rho(x,y)\overline{\sigma}(x,y)\right)}{|\sigma(x,y)|^2}
\end{aligned}$$

The actual function beign considered here is quite unpleasant to write out explicitly. So the rest of this problem will proceed by the result of numerical work. The easiest way to solve this problem is as it is put in the notes. We consider the boundary of the domain for the eigenvalue, z, that will yield a contraction. This is of course the unit circle, $z = e^{i\theta}$. Then, we find the region of absolute stability from $\frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$.



This yields a plot on the complex plane with eigenvalues that satisfy the problem we wish to solve. Then we overplot the normalized eigenvalues from our original matrix A, and draw lines originating from the origin to the eigenvalues of A in the complex plane. In order to ensure that Δt is a real-valued, we must pick points on the region of absolute stability which are colinear with the eigenvalues of A. Then we determine the maximum Δt by the ratio between the distance from the eigenvalue to the origin and the distance from the point colinear on the boundary of the region of absolute stability to the origin. This is then our Δt. We find for this problem, that the eigenvalue, $\lambda = -99.0667$ is furthest from the region of absolute stability, and appropriate Δt for this eigenvalue is very close to $\Delta t = 0.00550593$. □

b)

Question 2: Convergence and Asbolute Stability for the BDF3 Method

Criterion for Consistency for a linear multistep method

$$\rho(1) = 0 \tag{11}$$

$$\rho(1) - \sigma(1) = 0 \tag{12}$$

a)

Question 3: Consistency, Convergence, and Stability for an LMM

Question 4: Convergence and Stability for an RK Method