Homework 3: Report

Dante Buhl

June 2, 2024

Question 1: Consistency and Stability

$$\begin{cases} U_t + U_x = 0 \qquad x \in [0, 2\pi], \quad t \geq 0 \\ U(x,0) = \sin^2(x) \\ \text{Periodic B.C.} \end{cases} \tag{1}$$

$$u_j^{k+1} = u_j^k - \frac{\Delta t}{2\Delta x}(u_{j+1}^{k+1} - u_{j-1}^{k+1}) \tag{2}$$

a) Compute the Local Truncation Error of the scheme in (2). Is the scheme consistent? If so, to which orders in Δt and Δx

Proof.

$$\begin{aligned} \Delta t \tau_j^{k+1} &= \boldsymbol{y}_j^{k+1} - \boldsymbol{y}_j^k + \frac{\Delta t}{2\Delta x}(\boldsymbol{y}_{j+1}^{k+1} - \boldsymbol{y}_{j-1}^{k+1}) \\ \tau_j^{k+1} &= \dot{\boldsymbol{y}}_j^{k+1} + \frac{\Delta t}{2}\ddot{\boldsymbol{y}}_j^{k+1} + O(\Delta t^2) + \frac{1}{2\Delta x}(2\Delta x \boldsymbol{y}_{xj}^{k+1} + \frac{2\Delta x^3}{6}\boldsymbol{y}_{xj}^{k+1} + O(\Delta x^5)) \\ \tau_j^{k+1} &= \frac{\Delta t}{2}\ddot{\boldsymbol{y}}_j^{k+1} + \frac{\Delta x^2}{6}\boldsymbol{y}_{xxxj}^{k+1} + O(\Delta t^2) + O(\Delta x^4) \end{aligned}$$

Thus we have shown that the local truncation error, τ_j^{k+1} scales with Δt with order 1, and with Δx with order 2. Thus, the scheme is consistent with order 1 in time and order 2 in space. □

b) Compute the Von-Neumann Stability Analysis of (2). Is the scheme convergent?

Question 2: Method of Characteristics for Advection

$$\begin{cases} U_t + (fU)_x + (gU)_y = 0 \\ U(x,y,0) = \frac{1}{2\pi^2}\sin^2(x+y) \end{cases} \tag{3}$$

$$f(x,y) = \sin(x)\sin(y), \qquad g(x,y) = 1 - e^{\sin(x+y)} \tag{4}$$

$$\mathbb{T} = \{(x,y) \in \mathbb{R}^2 : \quad 0 \leq x \leq 2\pi, \quad 0 \leq y \leq 2\pi\} \tag{5}$$

$$\begin{cases} U(0,y,t) = U(2\pi,y,t), \quad 0 \leq y \leq 2\pi \\ U(x,0,t) = U(x,2\pi,t), \quad 0 \leq x \leq 2\pi \end{cases} \tag{6}$$

a) Show that the following integral evaluates to 1.

$$\int_{\mathbb{T}} U(x,y,t)dxdy = 1, \quad \forall t \geq 0$$

Proof.

$$\begin{aligned}
U_t &= -(fU)_x - (gU)_y \\
\int_{\mathbb{T}} U_t dA &= \int_{\mathbb{T}} -(fU)_x - (gU)_y dA \\
\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA &= \int_{\mathbb{T}} -(fU)_x - (gU)_y dA \\
\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA &= \int_{\partial\mathbb{T}} -\langle fU, gU\rangle \cdot \eta ds \\
\eta &= \begin{cases} \langle 0,1\rangle, y = 2\pi \\ \langle 0,-1\rangle, y = 0 \\ \langle 1,0\rangle, x = 2\pi \\ \langle -1,0\rangle, x = 0 \end{cases}
\end{aligned}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA = \int_0^{2\pi} fU|_{x=0}dy - \int_0^{2\pi} gU|_{y=2\pi}dx - \int_0^{2\pi} fU|_{x=2\pi}dy + \int_{2\pi}^{0} gU|_{y=0}dx$$

$$\begin{aligned}
\int_0^{2\pi} gU|_{y=0}dx &= \int_0^{2\pi} gU|_{y=2\pi}dx \\
\int_0^{2\pi} fU|_{x=0}dy &= 0 \\
\int_0^{2\pi} fU|_{x=2\pi}dy &= 0 \\
\frac{\partial}{\partial t}\int_{\mathbb{T}} U dA &= 0
\end{aligned}$$

Therefore we have that the value of the integral is constant. Now we must simply show that it is at one timestep equal to one.

$$\begin{aligned}
\int_{\mathbb{T}} U(x,y,0)dA &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \sin^2(x+y)dxdy \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} 1 - \cos(2x+2y)dxdy \\
&= \frac{1}{4\pi^2}\left(4\pi^2 - \int_0^{2\pi}\int_0^{2\pi} \cos(2x+2y)dxdy\right) \\
&= 1 - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \cos(2x+2y)dxdy \\
&= 1 - \frac{1}{4\pi^2}\int_0^{2\pi} 0dxdy \\
&= 1
\end{aligned}$$

Thus we have that U will satisfy this integral at every timestep. ☐

b) Contour Plots

Question 3: Finite Differences for Advection

a) Contour Plots

b) Integral of FD Scheme

c) Max Pointwise Error

d) Mean Squared Error