Numerical Methods for the Solution of Differential Equations (AMS 213B)
Midterm Exam

Instructions

Please submit to CANVAS one PDF file (your solution to the exam), and one .zip file that includes any computer code you develop for the exam. The PDF file must be a document compiled from Latex source code (mandatory for PhD students), or a PDF created using any other other word processor (MS and SciCAM students). No handwritten work should be submitted.

Question	Points
1	45
2	35
3	20

Question 1 (45 points). Consider the fully clamped Euler-Bernoulli beam sketched in Figure 1

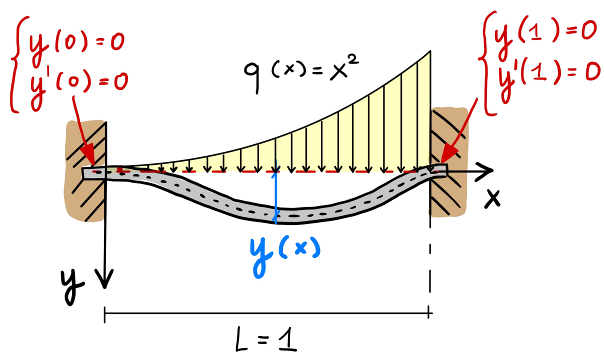

Figure 1: Euler-Bernoulli beam modeled by the two-point boundary value problem (1).

The vertical displacement $y(x)$ satisfies the following two-point boundary value problem

$$EI\frac{d^4y}{dx^4} = q(x), \qquad y(0) = 0, \qquad y(1) = 0, \qquad \frac{dy(0)}{dx} = 0, \qquad \frac{dy(1)}{dx} = 0, \tag{1}$$

where $EI = 1$ and[1]

$$q(x) = x^2 \quad \text{(load)}.$$

[1]This corresponds to a beam made of steel (modulus of elasticity $E = 200 \times 10^9$ N/m^2) with rectangular section 0.75 cm and width 2 mm.

a) (10 points) Determine the analytical solution $y(x)$ of the boundary value problem (1).

b) (25 points) Determine the numerical solution to the problem by using the shooting method with Newton's iterations. To this end, use the explicit RK4 scheme defined by the following Butcher array

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

to solve the initial value corresponding to the shooting method. In particular, set

$$\Delta x = \frac{1}{N}, \quad \text{and} \quad N = 60000 \tag{2}$$

in the RK4 method.

c) (5 points) Plot the numerical solution $u(x_k)$ you obtain with the shooting method versus x_k on the on the grid

$$x_k = k\Delta x \qquad k = 0, \ldots, N \tag{3}$$

where Δx and N are defined in equation (2).

d) (5 points) Plot the error

$$e_k = |y(x_k) - u(x_k)| \tag{4}$$

between the analytical solution $y(x_k)$ you obtained at point a) and the numerical solution you obtained at point c) on the grid (3). To this end, use a logarithmic scale, i.e., plot $\log(e_k)$ versus x_k.

Question 2 (35 points). Consider the implicit RK3 method defined by the Butcher array

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1 & 0 & 1 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

a) (5 points) Prove that method is convergent.

b) (10 points) Plot the region of absolute stability of the method.

c) (10 points) Determine the largest Δt for which the implict RK3 defined in Question 2 applied to the initial value problem

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{B}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{5}$$

where

$$\boldsymbol{B} = \begin{bmatrix} -1 & 3 & -5 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -16 \end{bmatrix} \qquad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{6}$$

is absolutely stable.

d) (10 points) Verify your predictions numerically, i.e., verify that for Δt slightly smaller (or slightly larger) than the one you computed at point c) the solution converges to zero (or diverges to infinity).

Question 3 (20 points). Consider the linear multi-step method

$$\boldsymbol{u}_{k+3} - \frac{1}{3}\left(\boldsymbol{u}_{k+2} + \boldsymbol{u}_{k+1} + \boldsymbol{u}_k\right) = \frac{\Delta t}{12}\left[23\boldsymbol{f}(\boldsymbol{u}_{k+2}, t_{k+2}) - 2\boldsymbol{f}(\boldsymbol{u}_{k+1}, t_{k+1}) + 3\boldsymbol{f}(\boldsymbol{u}_k, t_k)\right]. \tag{7}$$

a) (10 points) Show that the method is convergent and determine the convergence order.

b) (10 points) Plot the region of absolute stability. Is the linear multistep method (7) A-stable? Justify your answer.