Final Report

Dante Buhl

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Question 1: Bessels Function Solution

$$\begin{cases} U_t = \nu\left(U_{rr} + \frac{1}{r}U_r\right) & r \in [1,3], \quad t \geq 0 \\ U(r,0) = 15(r-r_1)^2(r_2-r)^2 e^{-\sin(2r)-r} \\ U(1,t) = 0 \\ U(3,t) = 0 \end{cases} \tag{1}$$

$$r_1 = 1, \quad r_2 = 3, \quad \nu = \frac{1}{4} \tag{2}$$

a) Plot the solution keeping 60 modes of the bessels function expansion at times 0, 0.5, 1, 2. As seen in figure 1, the solution decays in time as expected from a heat equation in cylindrical coordinates since the heat seems to dissipate faster on the outer rim of the cylinder, which is typical since there is more surface area on the outer rim.

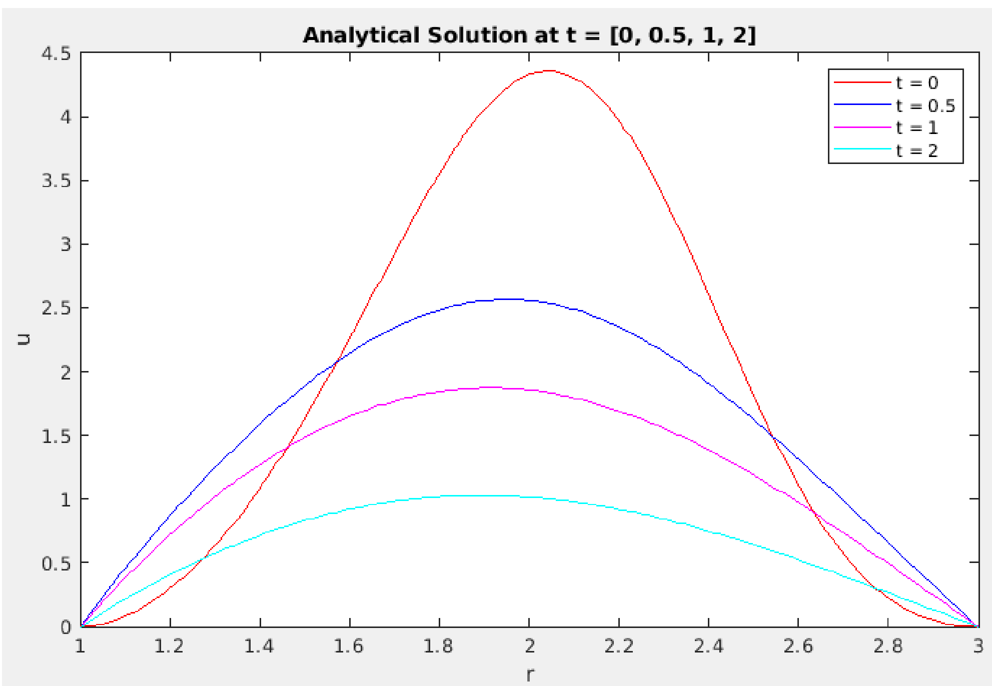

Figure 1: Plot of Truncated Bessels Solution at $t = [0, 0.5, 1, 2]$

b) In figure 2, the basis functions $\hat{R_n}$ can be seen for values $n = [1, 2, 4, 6]$. As we can see the number of nodes for each function is equal to $n + 1$ as it should be. We can also see that all functions have zeros at the endpoints of the domain which is lovely. In figure 3, a plot of the normalizing factor can be seen to increase with mode number.

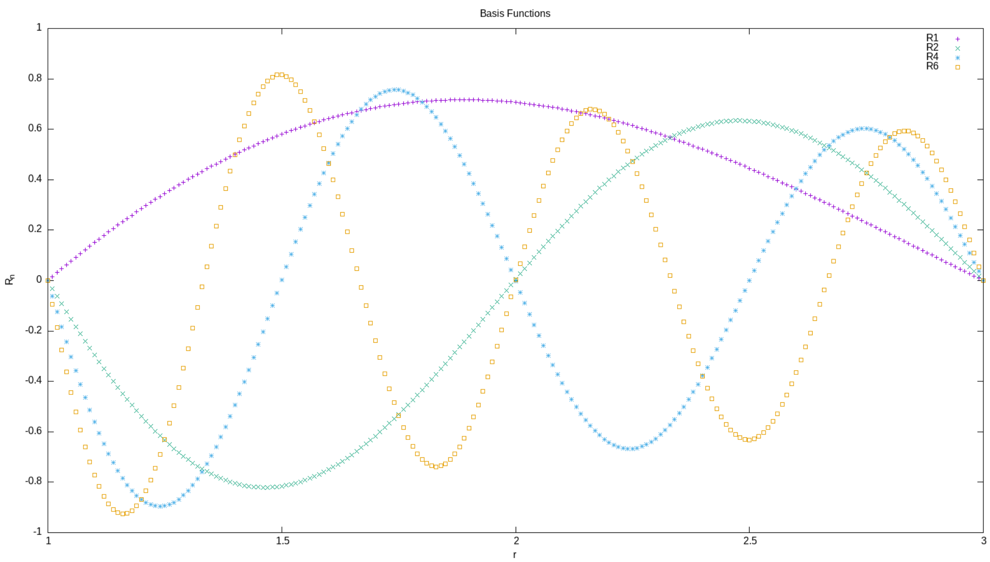

Figure 2: Plot of 4 of the basis functions, R_1, R_2, R_4, R_6.

Question 2: Finite Differences Solution

a) Write the matrix M explicitly

$$M = \nu D_2 + \nu \frac{1}{r} D_1$$

Where D_1 and D_2 are the standard second order finite difference differentiation matrices (non-periodic).

$$M = \nu \begin{bmatrix} \frac{-2}{\Delta r^2} & \frac{1}{\Delta r^2} + \frac{1}{2r\Delta r} & 0 & 0 & \cdots \\ \frac{1}{\Delta r^2} - \frac{1}{2r\Delta r} & \frac{-2}{\Delta r^2} & \frac{1}{\Delta r^2} + \frac{1}{2r\Delta r} & 0 & \cdots \\ 0 & \frac{1}{\Delta r^2} - \frac{1}{2r\Delta r} & \frac{-2}{\Delta r^2} & \frac{1}{\Delta r^2} + \frac{1}{2r\Delta r} & \\ 0 & 0 & \frac{1}{\Delta r^2} - \frac{1}{2r\Delta r} & \frac{-2}{\Delta r^2} & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{bmatrix}$$

b) Compute the spectral radius of M. This is done using an eigenvalue solver, I use LAPACK's DGEEV routine for general, non-symmetric matrices. I find that the spectral radius increases with n/k.

c) Find the critical value of dt according to the problem. It can be seen in figure 5 that Δt^* decreases in order 2 with respect to the number of gridpoints. Seen in the plot along side the data in purple is an order 2 decay line in green and an order 3 decay line in blue. It is very clear that the purple line is nearly parallel with the green one indicating order 2 decay in Δt^*.

Proof. First we must show that the conditional absolute stability of this scheme. We begin by looking at the characteristic polynomial of this method. We have,

$$u^{k+1} = u^k + \frac{\Delta t}{12} \boldsymbol{M} \left(23u^k - 16u^{k-1} + 5u^{k-2}\right) \tag{AB3}$$

$$z^3 - z^2 = \lambda \Delta t \left(\frac{23z^2 - 16z + 5}{12}\right)$$

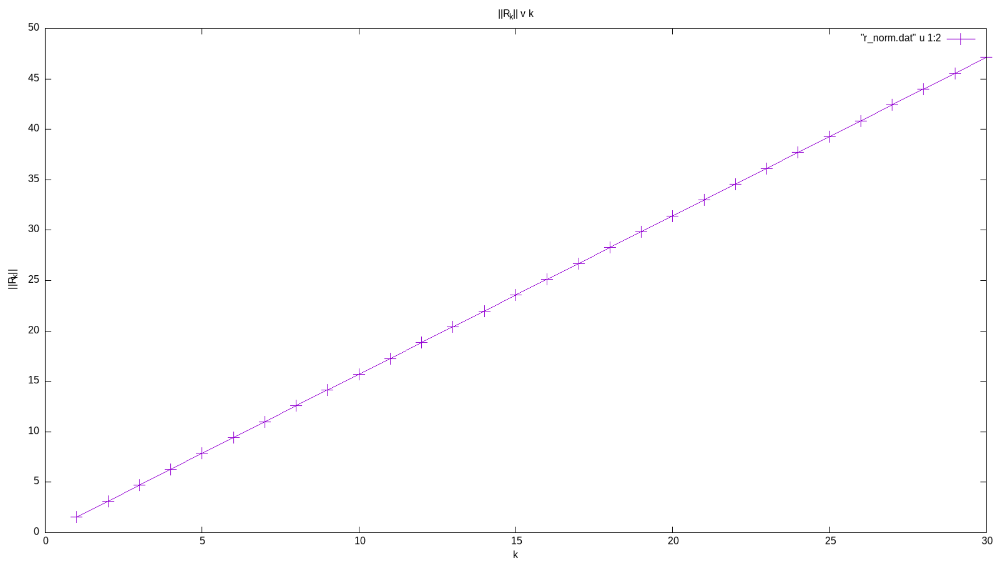

Figure 3: Normalizing Factors $||\hat{R}_k||_{L^2_w}$ versus k

$$\Delta t = \frac{12}{\lambda}\left(\frac{z^3 - z^2}{23z^2 - 16z + 5}\right)$$
$$\Delta t = \frac{12}{\lambda}\left(\frac{e^{3i\theta} - e^{2i\theta}}{23e^{2i\theta} - 16e^{i\theta} + 5}\right)$$

Now in order to show that this fraction when colinear with the largest eigenvalue of $\boldsymbol{M}$ has a maxinum of $\frac{6}{11}$ we must consider two things, first the largest eigenvalue of $\boldsymbol{M}$ which we will show analytically to be real valued, and a constrianed optimization problem for $\frac{\rho(z)}{\sigma(z)}$. First we consider the eigenvalues of $\boldsymbol{M}$. One will first notice the Tridiagonal form of $\boldsymbol{M}$. This is a special form of a matrix, and furthermore the matrix is also known to have a "Toeplitz" form. Theese two properties combined have been shown to produce an analytical form for the eigenvalues of that matrix. As seen in a paper by Kulkarni, Schmidt, and Tsui, the eigenvalaues of Tridiagonal (pseudo-)Toeplitz matrices have the following form,

$$\lambda = a - 2\sqrt{bc}\cos\left(\frac{i\pi}{n+1}\right), \quad i = 1, \ldots, n$$
$$a = \frac{-2}{\Delta r^2}, \quad b = \frac{1}{\Delta r^2} + \frac{1}{2r\Delta r}, \quad c = \frac{1}{\Delta r^2} - \frac{1}{2r\Delta r}$$
$$bc = \frac{1}{\Delta r^4} - \frac{1}{2r\Delta r^3} + \frac{1}{2r\Delta r^3} - \frac{1}{4r^2\Delta r^2}$$
$$bc = \frac{1}{\Delta r^4} - \frac{1}{4r^2\Delta r^2}$$

Now we notice that if $\Delta r << 1$ then $\frac{1}{\Delta r^4} >> \frac{1}{\Delta r^2}$, which implies that $\sqrt{bc}$ is a real valued quantity, and therefore so is each value of λ for this matrix $\boldsymbol{M}$.

Next we look for all of the real-valued minimums (minimums because λ is bounded to be negative by its given form). In order to do this we will consider only $\mathrm{Re}\left(\frac{\rho(z)}{\sigma(z)}\right)$.

$$\rho(x+iy) = x^3 - xy^2 - x^2 + y^2 + i(x^2y - y^3 - xy)$$
$$\sigma(x+iy) = 23x^2 - 23y^2 - 16x + 5 + i\,(46xy - 16y)$$
$$\mathrm{Re}\left(\frac{\rho(z)}{\sigma(z)}\right) = \mathrm{Re}\left(\frac{\rho(z)\bar{\sigma}(z)}{\sigma(z)\bar{\sigma}(z)}\right)$$

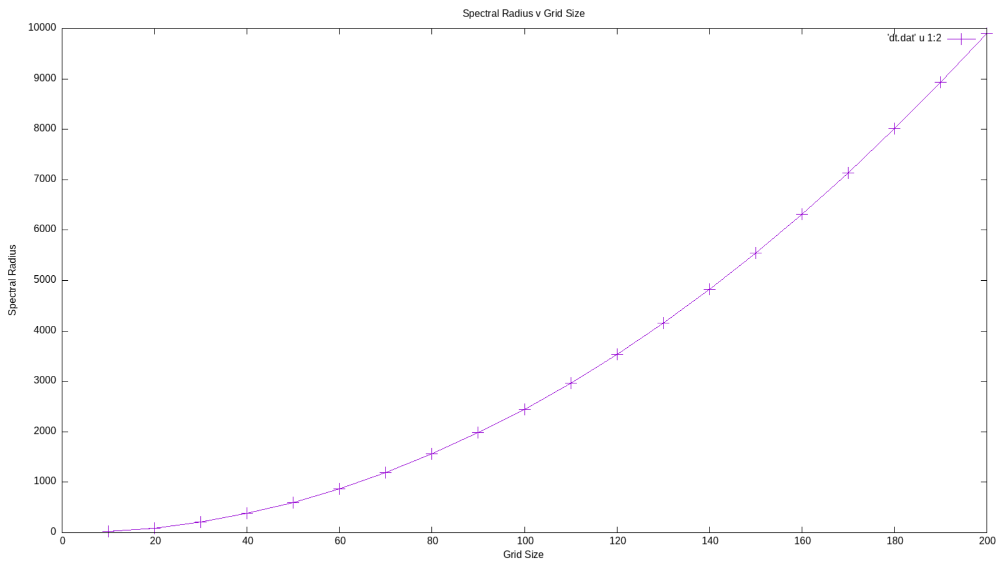

Figure 4: Plot of $\rho(M)$ versis gridsize

$$f(x,y) = \text{Re}\left(\frac{\rho(z)}{\sigma(z)}\right) = \frac{(23x^2 - 23y^2 - 16x + 5)(x^3 - xy^2 - x^2 + y^2) - (46xy - 16y)\left(x^2y - y^3 - xy\right)}{(23x^2 - 23y^2 - 16x + 5)^2 + (46xy - 16y)^2}$$

$$L(x,y,\mu) = f(x,y) - \mu(x^2 + y^2 - 1)$$

□

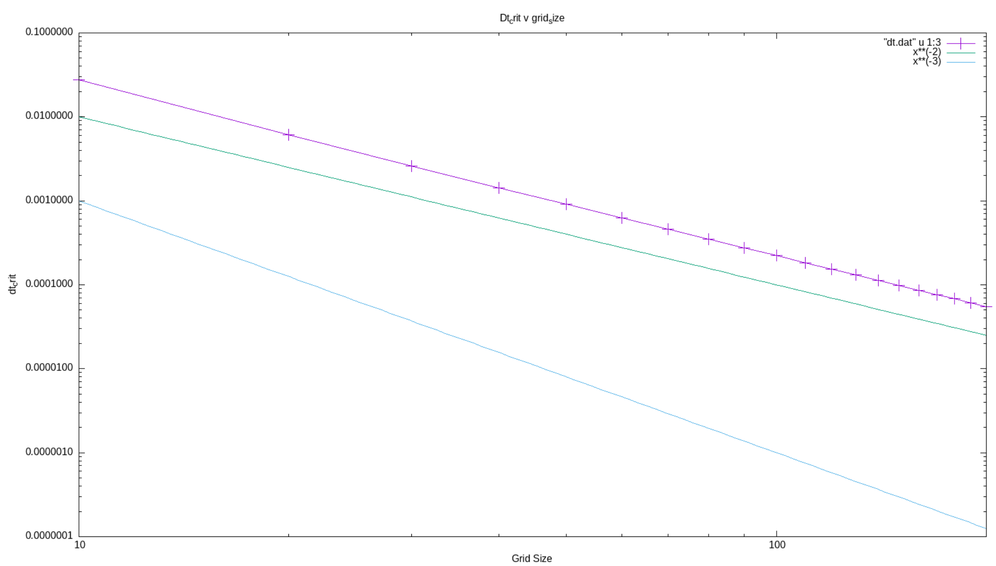

Figure 5: Plot of Δt^* versis gridsize

d) Integrate using AB3 and plot the maximum pointwise error. As seen in figure 6, the Maximum pointwise error seems to decline as the gridsize increases. Note that all 4 lines plotted start at essentially the same value of error and quickly diverge into different orders of accuracy. It should also be noted that the inherent error in the system at $t = 0$ is around the order of 10^{-3} or 10^{-4} with only 60 bessels modes.

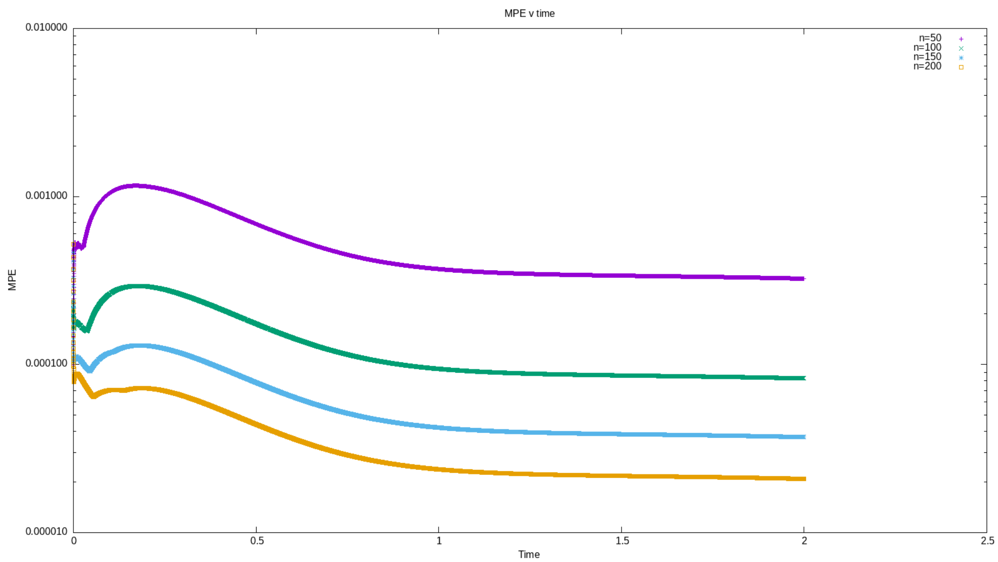

Figure 6: Maximum Pointwise Error between Truncated Solution and Numerical Solution

e) Plot the final error as a function of times. It can be seen in figure 7, that the maximum pointwise error at $t = 2$ is dependent on the gridsize. It can be seen in the plot that compared with order 2 decay (the green line), that this error decays with order 2 in the gridsize.

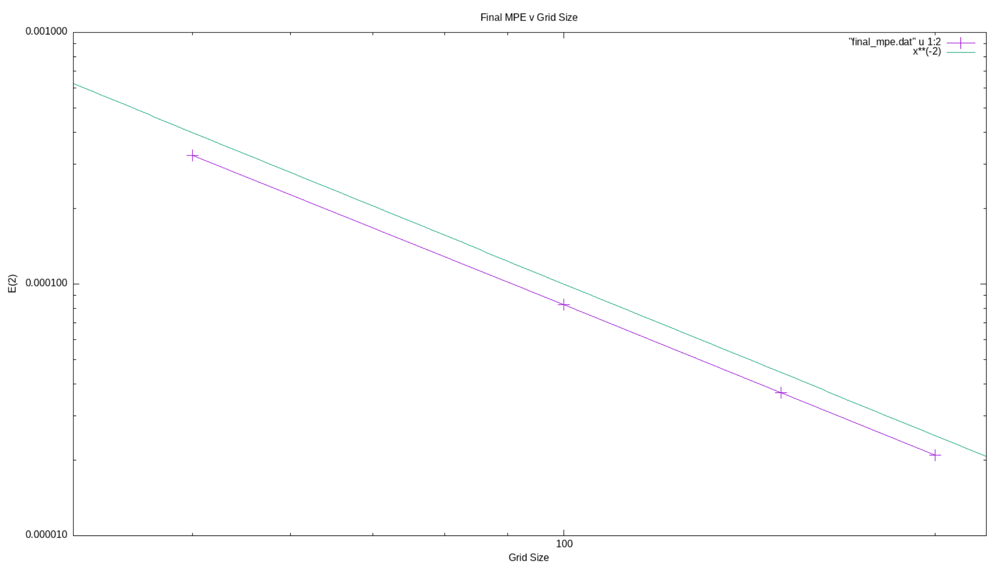

Figure 7: Plot of $e_n(2)$ versus n alongside n^{-2}

Question 3: Prove Second Order Accuracy in Δr

a) We prove that this finite differnce discretization is second order accurate in Δr. To do this we look at the taylor expansions of U.

Proof. As said already. we beign with the Taylor Expansions.

$$y_{j-1} = y_j - \Delta r \left(\frac{\partial y}{\partial r}\right)_j + \frac{\Delta r^2}{2}\left(\frac{\partial^2 y}{\partial r^2}\right)_j - \frac{\Delta r^3}{6}\left(\frac{\partial^3 y}{\partial r^3}\right)_j + \frac{\Delta r^4}{24}\left(\frac{\partial^4 y}{\partial r^4}\right)_j$$
$$y_{j+1} = y_j + \Delta r \left(\frac{\partial y}{\partial r}\right)_j + \frac{\Delta r^2}{2}\left(\frac{\partial^2 y}{\partial r^2}\right)_j + \frac{\Delta r^3}{6}\left(\frac{\partial^3 y}{\partial r^3}\right)_j + \frac{\Delta r^4}{24}\left(\frac{\partial^4 y}{\partial r^4}\right)_j$$

We then substitute these taylor expansions into the finite difference scheme and solve for the local truncation error. The result is,

$$\tau_j + \left(\frac{\partial^2 y}{\partial r^2}\right)_j + \frac{1}{r_j}\left(\frac{\partial y}{\partial r}\right)_j = \frac{\Delta r^2 \left(\frac{\partial^2 y}{\partial r^2}\right)_j + \frac{\Delta r^4}{12}\left(\frac{\partial^4 y}{\partial r^4}\right)_j + O(\Delta r^6)}{\Delta r^2} + \frac{2\Delta r \left(\frac{\partial y}{\partial r}\right)_j + \frac{\Delta r^3}{3}\left(\frac{\partial^3 y}{\partial r^3}\right)_j + O(\Delta r^5)}{2 r_j \Delta r}$$
$$\tau_j = \Delta r^2 \left(\frac{1}{12}\left(\frac{\partial^4 y}{\partial r^4}\right)_j + \frac{1}{6 r_j}\left(\frac{\partial^3 y}{\partial r^3}\right)_j + h.o.t\right)$$

Thus we find that this scheme as expected (it being a second order finite difference scheme) is in fact a second ordered finite difference scheme. □