Homework 3: Report

Dante Buhl

May 22, 2024

Question 1: BVP for 2D Poisson's Equation

a) Write Code to solve (1)

$$\begin{cases} \nabla^2 U(x,y) = f(x,y) & (x,y) \in \Omega \\ U(x,y) = g(x,y) & (x,y) \in \partial\Omega, \\ f(x,y) = -20 + 3x^2 + 4y^2 \\ g(x,y) = 2 - x^2 + 2\sin(\pi y^2) \end{cases} \tag{1}$$

b) See figure 1

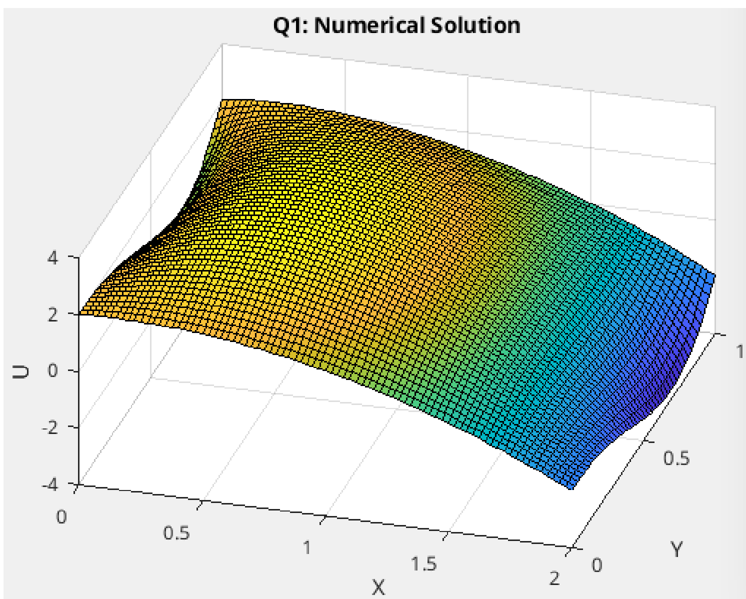

Figure 1: Numerical Solution to (1) with $N = 81, M = 51$

c) See figure 2

Question 2: IBVP for 1D Heat Equation

a) Determine the analytical solution for (2)

$$\begin{cases} U_t = U_{xx} \quad x \in [-1,1], \quad t \geq 0 \\ U(x,0) = (3+x) + 5(1-x^2)^2 \\ U(-1,t) = 2, \quad U(1,t) = 4 \end{cases} \tag{2}$$

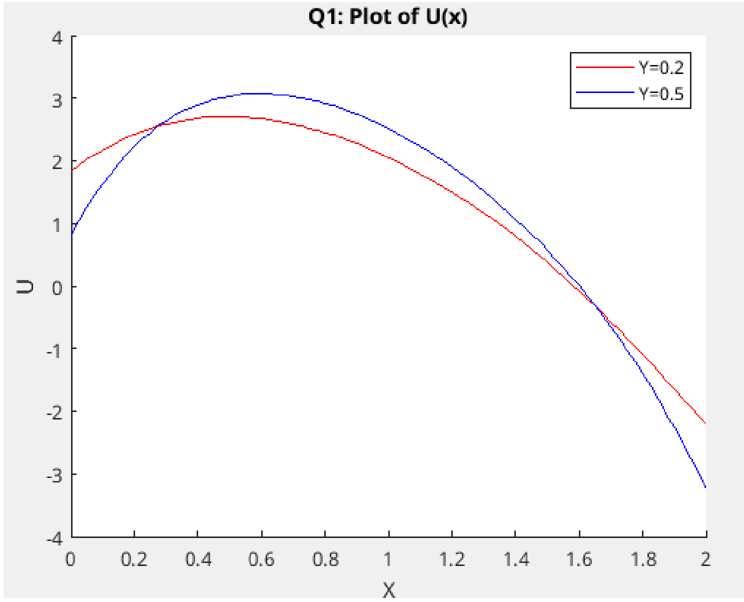

Figure 2: Numerical Solution to (1) at $Y = 0.2, 0.5$

$$g(x) = 3 + x + 5 - 10x^2 + 5x^4$$

Proof. To begin we look at the general solution for the Heat Equation with Homogeneous BC.

$$\eta = U - 3 - x$$

$$\begin{cases} \eta_t = \eta_{xx} \qquad x \in [-1, 1], \quad t \geq 0 \\ \eta(x, 0) = 5(1 - x^2)^2 \\ \eta(-1, t) = 0, \quad \eta(1, t) = 0 \end{cases}$$

This system can be solved with the general form.

$$\eta = \sum_n c_n \sin\left(\frac{n\pi}{2}(x+1)\right) e^{-\frac{n^2\pi^2}{4}t}$$

$$\sum_n c_n \sin\left(\frac{n\pi}{2}(x+1)\right) = 5(1-x^2)^2$$

$$c_n = \frac{2}{2}\int_{-1}^{1} 5(1-x^2)^2 \sin\left(\frac{n\pi}{2}(x+1)\right) dx$$

$$c_n = 5\int_{-1}^{1} (1 - 2x^2 + x^4) \sin\left(\frac{n\pi}{2}(x+1)\right) dx$$

We will solve this term by term.

$$\int_{-1}^{1} \sin\left(\frac{n\pi}{2}(x+1)\right) dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right)$$

$$
\begin{aligned}
\int_{-1}^{1} x^2 \sin\left(\frac{n\pi}{2}(x+1)\right) dx &= -\frac{2x^2}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{4}{n\pi}\int_{-1}^{1} x\cos\left(\frac{n\pi}{2}(x+1)\right) dx \\
&= -\frac{2x^2}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{8x}{n^2\pi^2}\sin\left(\frac{n\pi}{2}(x+1)\right) - \frac{8}{n^2\pi^2}\int_{-1}^{1}\sin\left(\frac{n\pi}{2}(x+1)\right) dx \\
&= -\frac{2x^2}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{8x}{n^2\pi^2}\sin\left(\frac{n\pi}{2}(x+1)\right) + \frac{16}{n^3\pi^3}\cos\left(\frac{n\pi}{2}(x+1)\right)
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} x^4 \sin\left(\frac{n\pi}{2}(x+1)\right) dx &= -\frac{2x^4}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{8}{n\pi}\int_{-1}^{1} x^3\cos\left(\frac{n\pi}{2}(x+1)\right) dx \\
&= -\frac{2x^4}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{16}{n^2\pi^2}x^3\sin\left(\frac{n\pi}{2}(x+1)\right) - \frac{48}{n^3\pi^3}\int_{-1}^{1} x^2\sin\left(\frac{n\pi}{2}(x+1)\right) dx \\
&= -\frac{2x^4}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{16}{n^2\pi^2}x^3\sin\left(\frac{n\pi}{2}(x+1)\right) \\
&- \frac{48}{n^2\pi^2}\left(-\frac{2x^2}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{8x}{n^2\pi^2}\sin\left(\frac{n\pi}{2}(x+1)\right) + \frac{16}{n^3\pi^3}\cos\left(\frac{n\pi}{2}(x+1)\right)\right) \\
&= -\frac{2x^4}{n\pi}\cos\left(\frac{n\pi}{2}(x+1)\right) + \frac{16}{n^2\pi^2}x^3\sin\left(\frac{n\pi}{2}(x+1)\right) \\
&+ \frac{96x^2}{n^3\pi^3}\cos\left(\frac{n\pi}{2}(x+1)\right) - \frac{384x}{n^4\pi^4}\sin\left(\frac{n\pi}{2}(x+1)\right) + \frac{768}{n^5\pi^5}\cos\left(\frac{n\pi}{2}(x+1)\right)
\end{aligned}
$$

Obviously, terms attached to $\sin(\theta)$ will evaluate to zero, and terms with $\cos(\theta)$ which are multiplied by x^k will be proportional to $((-1)^n - 1)$. We have then,

$$
\begin{aligned}
c_n &= 5\int_{-1}^{1}(1-2x^2+x^4)\sin\left(\frac{n\pi}{2}(x+1)\right) dx \\
&= 5\left((-1)^n-1\right)\left(-\frac{2}{n\pi} - 2\left(-\frac{2}{n\pi} + \frac{16}{n^3\pi^3}\right) - \frac{2}{n\pi} + \frac{96}{n^3\pi^3} - \frac{768}{n^5\pi^5}\right) \\
&= 5\left((-1)^n-1\right)\left(\frac{64}{n^3\pi^3} - \frac{768}{n^5\pi^5}\right)
\end{aligned}
$$

Thus we have found the solution to our PDE.

$$
U(x,t) = 3 + x + \sum_n 5\left((-1)^n-1\right)\left(\frac{64}{n^3\pi^3} - \frac{768}{n^5\pi^5}\right)\sin\left(\frac{n\pi}{2}(x+1)\right)e^{-\frac{n^2\pi^2}{4}t}
$$

□

b) See figure 3

c) See figure 4

d) See figure 5

e) See figure 6. From the figure shown. One can deduce that initially the spectral method converges much more quickly. However whether due to a bug in my code (or seemingly every student's code) the error does not decrease nicely with N. In fact for larger N the error seems to rise. According to other students, a similar phenomenon is shown except with different orders of accuracy. In my figure, it seems that the Finite Diffence method converges with order 2, while the spectral method doesn't seem to follow any convergence scaling order.

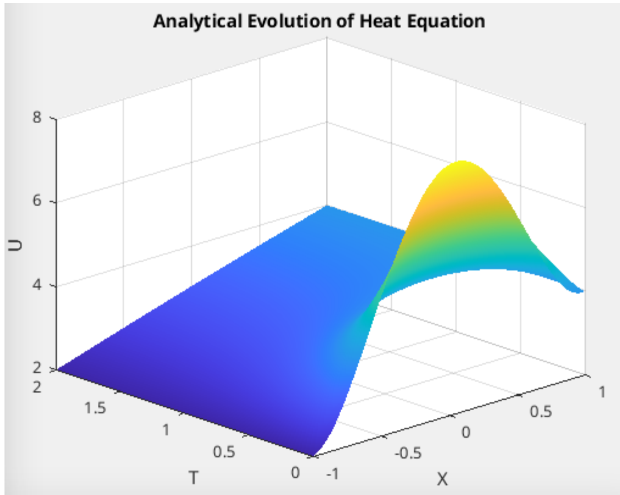

Figure 3: Analytical Solution to (2)

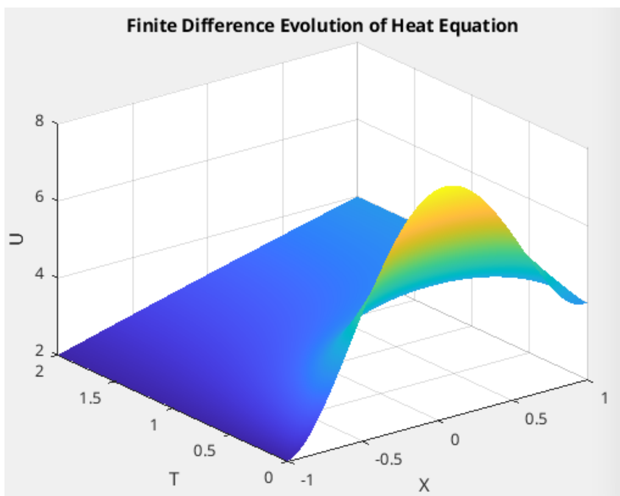

Figure 4: Finite Solution to (2)

Question 3: Extra Credit

a) I tried, didn't make anything that looked pretty. But if you look at my fortran code a lot of the routines it needs are there.

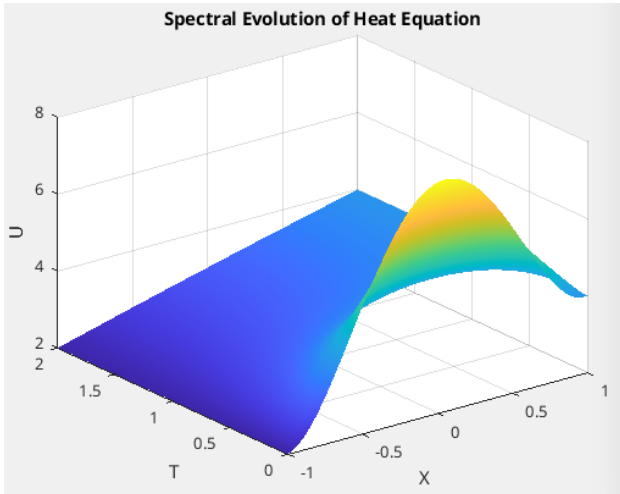

Figure 5: Spectral Solution to (2)

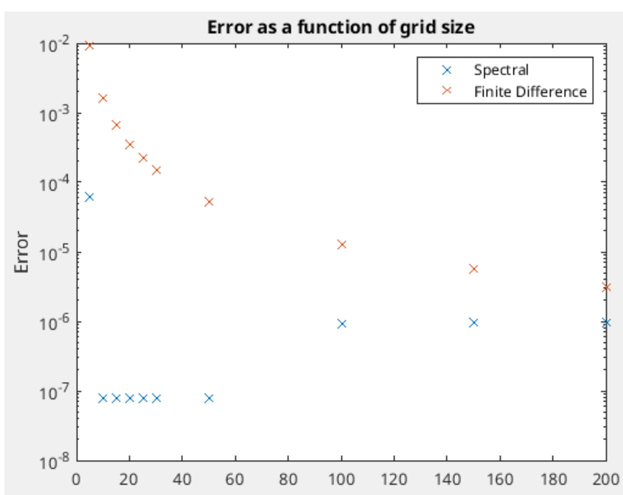

Figure 6: Numerical Error for the FD2 and Spectral methods