Numerical Methods for the Solution of Differential Equations (AMS 213B)
Midterm Exam - Solution

Question 1 (45 points). Consider the fully clamped Euler-Bernoulli beam sketched in Figure 1

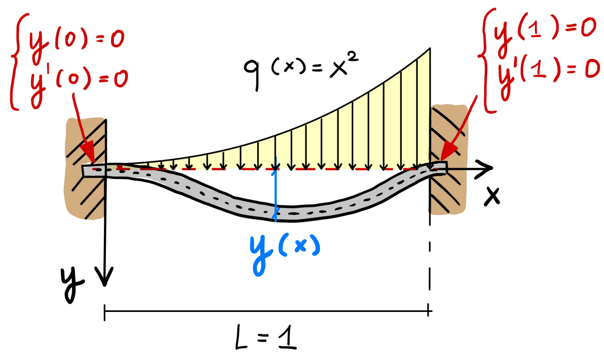

Figure 1: Euler-Bernoulli beam modeled by the two-point boundary value problem (1).

The vertical displacement $y(x)$ satisfies the following two-point boundary value problem

$$EI\frac{d^4y}{dx^4} = q(x), \qquad y(0) = 0, \qquad y(1) = 0, \qquad \frac{dy(0)}{dx} = 0, \qquad \frac{dy(1)}{dx} = 0, \tag{1}$$

where $EI = 1$ and[1]

$$q(x) = x^2 \quad \text{(load)},$$

a) (10 points) Determine the analytical solution $y(x)$ of the problem (1);

b) (25 points) Determine the numerical solution to the problem by using the shooting method. To this end, use the explicit RK4 scheme defined by the following Butcher array

0	0	0	0	0
1/2	1/2	0	0	0
1/2	0	1/2	0	0
1	0	0	1	0
	1/6	1/3	1/3	1/6

to solve the initial value corresponding to the shooting method. In particular, set

$$\Delta x = \frac{1}{N}, \quad \text{and} \quad N = 60000 \tag{2}$$

in the RK4 method.

c) (5 points) Plot the numerical solution $u(x_k)$ you obtain with the shooting method versus x_k on the on the grid

$$x_k = k\Delta x \qquad k = 0, \dots, N \tag{3}$$

where Δx and N are defined in (2).

[1]This corresponds to a beam made of steel (modulus of elasticity $E = 200 \times 10^9$ N/m^2) with rectangular section 0.75 cm and width 2 mm.

d) (5 points) Compute the error

$$e_k = |y(x_k) - u(x_k)| \tag{4}$$

between the analytical solution $y(x_k)$ you obtained at point a) and the numerical solution you obtained at point c) on the grid (3). Plot $\log(e_k)$ versus x_k.

Solution:

a) Setting $EI = 1$ and integrating the system four times we get

$$\begin{aligned}
\frac{d^3y(x)}{dx^3} =& c_1 + \frac{x^3}{3}, \\
\frac{d^2y(x)}{dx^2} =& c_2 + c_1 x + \frac{x^4}{12}, \\
\frac{dy(x)}{dx} =& c_3 + c_2 x + c_1\frac{x^2}{2} + \frac{x^5}{60}, \\
y(x) =& c_4 + c_3 x + c_2\frac{x^2}{2} + c_1\frac{x^3}{6} + \frac{x^6}{360}.
\end{aligned}$$

Enforcing the boundary conditions yields

$$\begin{aligned}
y(0) =& c_4 = 0 \\
\frac{dy(0)}{dx} =& c_3 = 0, \\
y(1) =& \frac{c_2}{2} + \frac{c_1}{6} + \frac{1}{360} = 0, \\
\frac{dy(1)}{dx} =& c_2 + \frac{c_1}{2} + \frac{1}{60} = 0,
\end{aligned}$$

which reduces to solving the linear system

$$\begin{bmatrix} 1/6 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{360}\begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

This gives the constants $c_1 = -1/15, c_2 = 1/60$ and consequently the analytical solution is

$$y(x) = \left(\frac{x^2}{120} - \frac{x^3}{90} + \frac{x^6}{360}\right)$$

b) We transform the boundary value problem (1) to the following initial value problem

$$\begin{cases}
\frac{dz_0}{dx} = z_1, \qquad \frac{dz_1}{dx} = z_2, \qquad \frac{dz_2}{dx} = z_3, \qquad \frac{dz_3}{dx} = x^2 \\
z_0(0) = 0 \\
z_1(0) = 0 \\
z_2(0) = v_1 \\
z_3(0) = v_2
\end{cases} \tag{5}$$

where we have two unknown parameters v_1 and v_2. To determine these unknown values we define the vector-valued error function

$$\boldsymbol{E}(\boldsymbol{v}) = \begin{bmatrix} z_0(L;\boldsymbol{v}) - 0 \\ z_1(L;\boldsymbol{v}) - 0 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

and solve the system of equations $\boldsymbol{E}(\boldsymbol{v}) = \mathbf{0}$. To this end, we can use Newton's iterations. We have seen in class that since the dependence $z_0(L; \boldsymbol{v})$ and $z_1(L; \boldsymbol{v})$ is linear in $\boldsymbol{v}$ (the system (5) is linear), then the Jacobian of $\boldsymbol{E}(\boldsymbol{v})$

$$\boldsymbol{J}_{\boldsymbol{E}}(\boldsymbol{v}) = \begin{bmatrix} \dfrac{\partial z_0}{\partial v_1}(L; \boldsymbol{v}) & \dfrac{\partial z_0}{\partial v_2}(L; \boldsymbol{v}) \\ \dfrac{\partial z_1}{\partial v_1}(L; \boldsymbol{v}) & \dfrac{\partial z_1}{\partial v_2}(L; \boldsymbol{v}) \end{bmatrix}. \tag{6}$$

does *not* depend on $\boldsymbol{v}$. Therefore one Newton's iteration is sufficient to compute the correct $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ which, if substituted into (5) yields the numerical solution to the BVP (1). For every initial condition $\boldsymbol{v} = \begin{bmatrix} v_1^{(0)} & v_2^{(0)} \end{bmatrix}^T$, we have

$$\boldsymbol{v} = \boldsymbol{v}^{(0)} - \boldsymbol{J}_{\boldsymbol{E}}^{-1} \boldsymbol{E}\left(\boldsymbol{v}^{(0)}\right). \tag{7}$$

Here $\boldsymbol{v}$ is the solution to the shooting problem. To compute $\boldsymbol{v}$ from any initial guess $\boldsymbol{v}^{(0)}$ we need the matrix $\boldsymbol{J}_{\boldsymbol{E}}$ As derived in the course notes (Chapter 7), the evolution equation for the partial derivatives of $z_i(x; \boldsymbol{v})$ $(i = 1, \ldots, 4)$ with respect v_1 and v_2 is given by the linear dynamical system

$$\frac{d}{dx}\boldsymbol{Q}(x; \boldsymbol{v}) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{C}} \boldsymbol{Q}(x; \boldsymbol{v}) \tag{8}$$

where

$$\boldsymbol{Q}(x; \boldsymbol{v}) = \begin{bmatrix} \dfrac{\partial z_0}{\partial v_1}, & \dfrac{\partial z_1}{\partial v_1}, & \dfrac{\partial z_2}{\partial v_1}, & \dfrac{\partial z_3}{\partial v_1}, & \dfrac{\partial z_0}{\partial v_2}, & \dfrac{\partial z_1}{\partial v_2}, & \dfrac{\partial z_2}{\partial v_2}, & \dfrac{\partial z_3}{\partial v_2} \end{bmatrix}^T. \tag{9}$$

The initial condition for (8) is obtained by differentiating the initial condition in (5) with respect to v_1 and v_2. This yields

$$\boldsymbol{Q}(0, \boldsymbol{v}) = \begin{bmatrix} 0, & 0, & 1, & 0, & 0, & 0, & 0, & 1 \end{bmatrix}^T. \tag{10}$$

The components of $\boldsymbol{J}_{\boldsymbol{E}}$ coincide with the first, second, fifth and sixth components of $\boldsymbol{Q}(L, \boldsymbol{v})$ obtained by integrating (8)-(10) from $x = 0$ to $x = L$, i.e.,

$$\boldsymbol{J}_{\boldsymbol{E}} = \begin{bmatrix} \dfrac{\partial z_0}{\partial v_1}(L; \boldsymbol{v}) & \dfrac{\partial z_0}{\partial v_2}(L; \boldsymbol{v}) \\ \dfrac{\partial z_1}{\partial v_1}(L; \boldsymbol{v}) & \dfrac{\partial z_1}{\partial v_2}(L; \boldsymbol{v}) \end{bmatrix} = \begin{bmatrix} Q_1(L; \boldsymbol{v}) & Q_5(L; \boldsymbol{v}) \\ Q_2(L; \boldsymbol{v}) & Q_6(L; \boldsymbol{v}) \end{bmatrix} \tag{11}$$

The attached MATLAB script `Euler_Bernoulli_Beam_Shooting_Method.m` provides the implementation of the shooting method for this problem.

c) In Figure 2 we compare the analytical solution to numerical solution we obtained using the shooting method on the given grid.

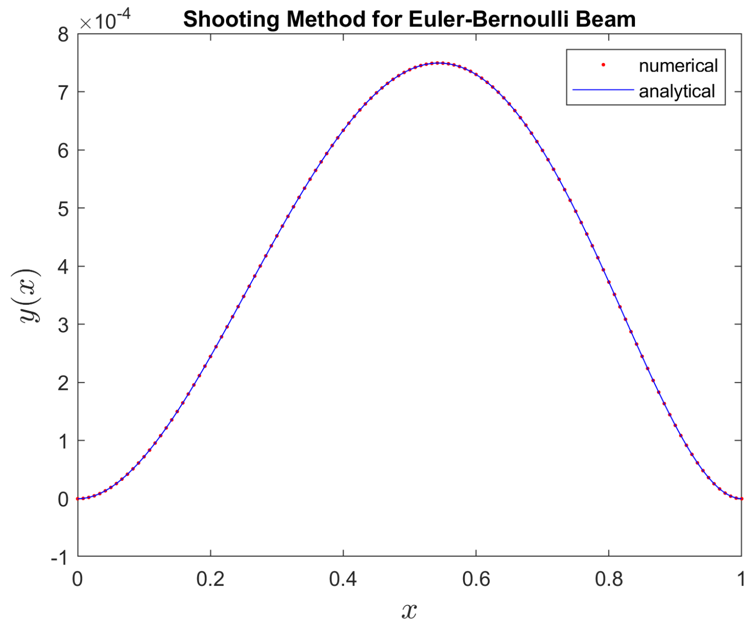

Figure 2: Plot of the numerical solution (blue) obtained using the Shooting Method and RK4 integration scheme and the analytical solution (red) of the Euler-Bernoulli beam modeled by the two-point boundary value problem (1).

d) In Figure 3 we plot the error between the analytical solution and the numerical solution, as requested by the assignment.

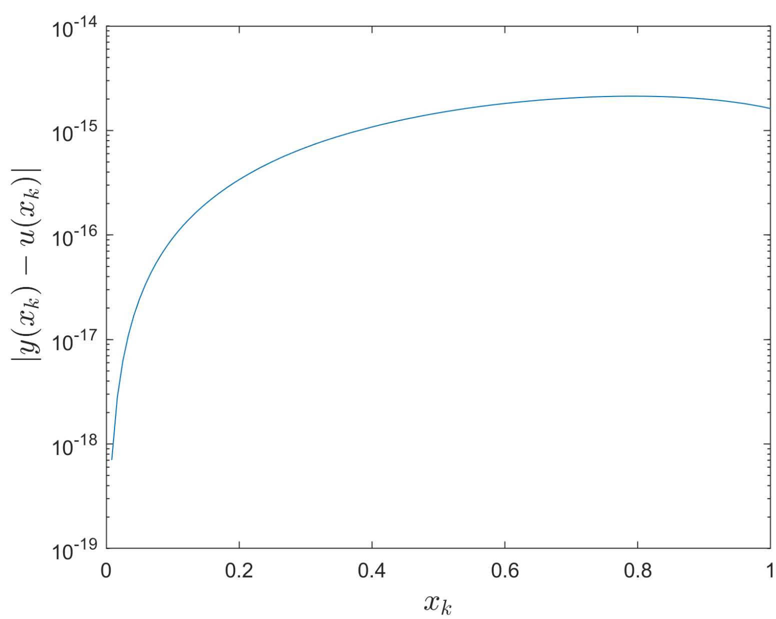

Figure 3: Time dependent errors obtained by solving the two-point boundary value problem (1) using the Shooting Method with RK4 integration scheme.

Question 2 (35 points). Consider the implicit RK3 method defined by the Butcher array

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1 & 0 & 1 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

a) (5 points) Prove that method is convergent.

b) (10 points) Plot the region of absolute stability of the method.

c) (10 points) Determine the largest Δt for which the implict RK3 defined in Question 2 applied to the initial value problem

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{B}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{12}$$

where

$$\boldsymbol{B} = \begin{bmatrix} -1 & 3 & -5 & 7 \\ 0 & -2 & 4 & -6 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -16 \end{bmatrix} \qquad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{13}$$

is absolutely stable.

d) (5 points) Verify your predictions numerically, i.e., verify that for Δt slightly smaller (or slightly larger) than the one you computed at point c) the solution converges to zero (or diverges to infinity).

Solution:

a) To prove convergence, we need to show that he numerical method consistent and zero-stable. Since the implicit RK3 method is a one-step method, it is necessarily zero-stable. Consistency directly follows from

$$\sum_{i=1}^{3} b_i = \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1.$$

b) Let

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 1/4 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{bmatrix}, \quad \boldsymbol{h} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The region of absolute stability for implicit RK3 is obtained as the zero level set of the stability function $|S(z)| - 1$ where

$$S(z) = \frac{\det\left(\boldsymbol{I} - z\boldsymbol{A} + z\boldsymbol{h}\boldsymbol{b}^T\right)}{\det\left(\boldsymbol{I} - z\boldsymbol{A}\right)}, \qquad s \in \mathbb{C}. \tag{14}$$

The region of absolute stability is shown in Figure 4. The region inside the curve is absolutely stable, while the region outside is absolutely unstable. This implies that the given RK3 method is not A-stable.

c) To determine the largest Δt for which the RK3 method applied to (12)-(13) is absolutely stable we first determine the real zeros of the function $|S(z)| - 1$, i.e., we solve

$$|S(z)| - 1 = 0 \qquad z \in \mathbb{R}, \tag{15}$$

using any root finding algorithm. In particular, the Matlab function `fsolve()` yields

$$z^* = -5.419951893353394. \tag{16}$$

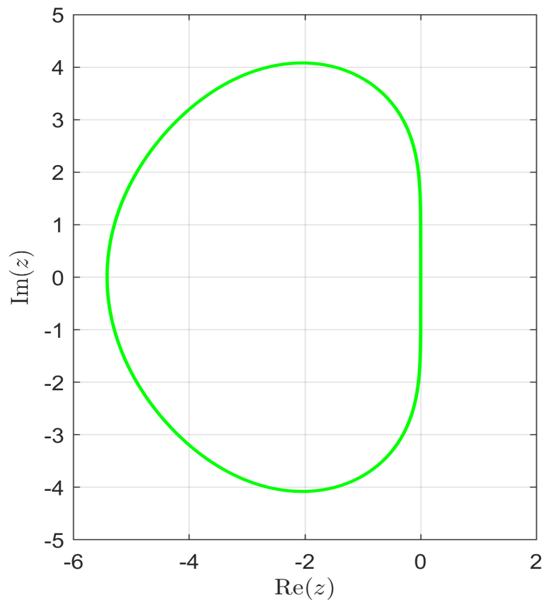

Figure 4: Region of absolute stability for the given implicit RK3 method.

At this point, we set up the inequality

$$\Delta t \lambda_{\min} \leq z^* \tag{17}$$

where $\lambda_{\min} = -16$ is the smallest eigenvalue of the matrix $\boldsymbol{B}$ defined in (13). This yields the following critical value of Δt

$$\Delta t^* = \frac{z^*}{\lambda_{\min}} = 0.338746993334587. \tag{18}$$

For $\Delta t \geq \Delta t^*$ the RK3 integrator is absolutely unstable, while for $\Delta t < \Delta t^*$ it is absolutely stable.

d) In Figure 5 we plot the numerical solution of (12) we obtain with the implict RK3 using $\Delta t = 0.03$. In Figure 6 we show that for Δt slightly smaller or slightly larger than the critical Δt^* defined in (18), the numerical solution we obtain with the implicit RK3 converges to zero, or diverges to infinity, respectively.

Question 3 (20 points). Consider the linear multistep method

$$\boldsymbol{u}_{k+3} - \frac{1}{3}\left(\boldsymbol{u}_{k+2} + \boldsymbol{u}_{k+1} + \boldsymbol{u}_k\right) = \frac{\Delta t}{12}\left(23\boldsymbol{f}(\boldsymbol{u}_{k+2}, t_{k+2}) - 2\boldsymbol{f}(\boldsymbol{u}_{k+1}, t_{k+1}) + 3\boldsymbol{f}(\boldsymbol{u}_k, t_k)\right). \tag{19}$$

a) (10 points) Show that the method is convergent and determine the convergence order.

b) (10 points) Plot the region of absolute stability. Is the midpoint method A-stable? Justify your answer.

Solution:

a) To show that the LMM method (19) is convergent we need to prove that the method is consistent and zero-stable. The first and second characteristic polynomials associated with (19) are

$$\rho(z) = z^3 - \frac{1}{3}\left(z^2 + z + 1\right), \qquad \sigma(z) = \frac{1}{12}\left(23z^2 - 2z + 3\right) \tag{20}$$

The coefficients of ρ and σ are, respectively

$$\alpha_3 = 1 \qquad \alpha_2 = -\frac{1}{3}, \qquad \alpha_1 = -\frac{1}{3}, \qquad \alpha_0 = -\frac{1}{3}, \tag{21}$$

$$\beta_3 = 0 \qquad \beta_2 = \frac{23}{12}, \qquad \beta_1 = -\frac{2}{12}, \qquad \beta_0 = \frac{3}{12}. \tag{22}$$

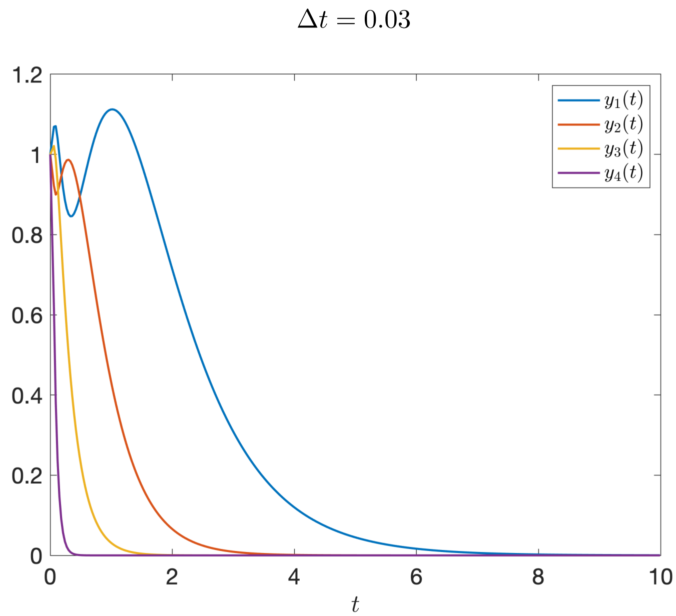

Figure 5: Solution of the system (12)-(13) with the implicit RK3 method for $\Delta t = 0.03$.

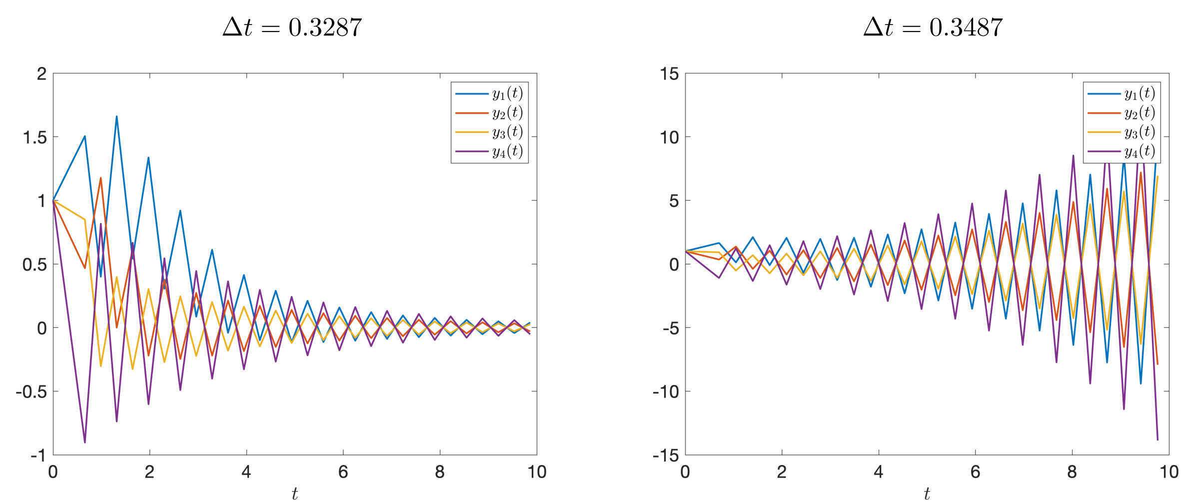

Figure 6: Solution of the system (12)-(13) with the implicit RK3 method for different Δt. It is seen that for Δt slightly larger than the critical value $\Delta t^* = 0.338746993334587$, the numerical solution does not decay to zero.

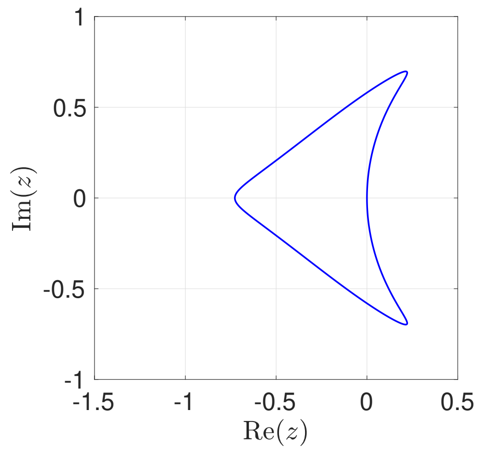

Figure 7: Region of absolute stability of the LMM method (19).

The roots of $\rho(z)$ are easily obtained as

$$z_1 = 1 \qquad z_{2,3} = -\frac{1}{3} \pm i\frac{\sqrt{2}}{3}. \tag{23}$$

The modulus of $z_{2,3}$ is

$$|z_{2,3}| = \frac{\sqrt{3}}{3} < 1. \tag{24}$$

Therefore $z_{2,3}$ are within the unit disk, implying that the method (19) is zero stable. For consistency, we notice that

$$C_0 = \rho(1) = 0 \qquad C_1 = \rho'(1) - \sigma(1) = 2 - 2 = 0. \tag{25}$$

Therefore the LMM method (19) is consistent. Regarding the order of consistency, recall that if

$$C_s = \frac{1}{s!}\sum_{j=1}^{3}\left(j^s - sj^{s-1}\beta_j\right) = 0 \qquad \text{for } s = 2, \dots, p \tag{26}$$

then the method has order p. Substituting the coefficients (21)-(22) into (26) yields

$$\begin{aligned} C_2 =& \frac{1}{2}\left[(\alpha_1 - 2\beta_1) + (4\alpha_2 - 4\beta_2) + 9\alpha_3\right] \\ =& \frac{1}{2}\left[(\frac{1}{3} - \frac{1}{3} - \frac{4}{3} - \frac{23}{3} + \frac{27}{3}\right] \\ =& 0. \end{aligned} \tag{27}$$

and

$$\begin{aligned} C_3 =& \frac{1}{6}\left[(\alpha_1 - 3\beta_1) + (8\alpha_2 - 12\beta_2) + 27\alpha_3\right] \\ =& \frac{1}{6}\left[(-\frac{1}{3} - \frac{9}{12} - \frac{8}{3} + \frac{24}{12} + 27\right] \\ =& \frac{74}{18} \neq 0. \end{aligned} \tag{28}$$

Hence, (19) is a three step explicit linear multistep method converging with order two.

b) In figure 7 we plot the region of absolute stability of the LMM method (19). The method is absolutely stable in the region within the closed curve.