Midterm: Report

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Question 1: BVP using Shooting Method and RK4

a) The system can be solved analytically by simply integrating 4 times and then solving for the boundary conditions.

$$EI \int\int\int\int \frac{d^4y}{dx^4} dx^4 = \int\int\int\int x^2 dx^4$$
$$EIy(x) = \frac{x^6}{360} + c_1x^3 + c_2x^2 + c_3x + c_4$$

We can see very clearly from the boundary conditions at $x = 0$ that $c_4 = c_3 = 0$. We finish solving for c_1 and c_2.

$$0 = \frac{1}{360} + c_1 + c_2, \quad 0 = \frac{1}{60} + 3c_1 + 2c_2$$
$$c_1 = -\frac{1}{90}, \quad c_2 = \frac{1}{120}$$

$$y(x) = \frac{1}{EI}\left(\frac{x^6}{360} - \frac{x^3}{90} + \frac{x^2}{120}\right)$$

b) Determine the Numerical Solution using the shooting method. The numerical solution obtained by the shooting method takes the following form,

$$\frac{dy_1}{dx} = y_2, \quad y_1(0) = 0$$
$$\frac{dy_2}{dx} = y_3, \quad y_2(0) = 0$$
$$\frac{dy_3}{dx} = y_4, \quad y_3(0) = v_1$$
$$\frac{dy_4}{dx} = x^2, \quad y_4(0) = v_2$$

Thus we proceed by integrating this ODE with a standard numerical method and at each iteration adjust v_1, v_2 so that we get closer and closer to the bouncary condition at $x = 1$. To this end, I will implement something similar to the second method. We will update v_1 to minimize the error of y_1. Then we I will update v_2 to minimize the error of y_2. After iterating several times, we obtain accuracy of 10^{-10}, $v_1 =$, $v_2 =$.

c) Plot the numerical solution obtained with the shooting method

d) Plot the error

Question 2: Convergence and Absolute Stability for an Implicit RK3 Method

a) Prove convergence for Implicit RK3.

Proof. Showing convergence is a matter of sending zero-stability and consistency. This is a one-step method, so we have the first characteristic polynomial, $\rho(z) = z - 1$. This of course satisfies the root condition and therefore is zero-stable. Showing consistency ☐

b) Plot Region of Absolute Statbility (See Fig. 1)

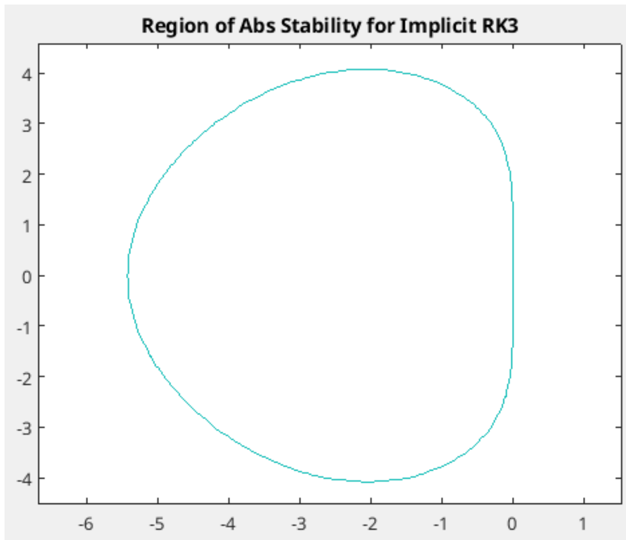

Figure 1: Region of Absolute Stability for 2.b.

c) This problem is solved by plotting the region of absolute stability and finding the eignevalues of the matrix B. We notice that since B is an upper triangular matrix that its eigenvalues are found on its diagonal. So we have that B has all real eigenvalues, $\lambda = \{-1, -2, -4, -16\}$. Next we compare to see which eigenvalue is furthest from the region of absolute stability. It is evidently the eigenvalue, $\lambda = -16$. So we look at the closest point in our region of absolute stability. From the plot we can see that the closest point is $\lambda\Delta t = -5.4199 + 0i$. Therefore we have that the largest Δt must be, $\Delta t \approx 0.338746\ldots$.

d) Show that this is validated numerically.

Question 3: Convergence and Absolute Stability for an LMM

$$\boldsymbol{u}_{k+3} - \frac{1}{3}(\boldsymbol{u}_{k+2} + \boldsymbol{u}_{k+1} + \boldsymbol{u}_k) = \frac{\Delta t}{12}\left[23\boldsymbol{f}_{k+2} - 2\boldsymbol{f}_{k+1} + 3\boldsymbol{f}_k\right] \tag{1}$$

a) *Proof.* (Zero-Stability)

We begin this proof first by showing that this LMM is zero-stable.

$$\rho(z) = z^3 - \frac{1}{3}z^2 - \frac{1}{3}z - \frac{1}{3}$$
$$\rho(z) = (z-1)\left(z^2 + \frac{2}{3}z + \frac{1}{3}\right)$$
$$\rho(z) = (z-1)\left(z - \frac{-\frac{2}{3} \pm \sqrt{\frac{4}{9} - \frac{4}{3}}}{2}\right)$$
$$\rho(z) = (z-1)\left(z + \frac{1 \pm i\sqrt{2}}{3}\right)$$

These are the three complex roots of the characteristic polynomial. We now simply must check if all three are within or on the boundary of the unit disk.

$$|1| = 1 \leq 1, \quad \left|\frac{1 \pm i\sqrt{2}}{3}\right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3} \leq 1$$

As I have just shown, in fact all three roots of the first characteristic polynomial fall within the unit disk or on the boundary, thus we have that this LMM is zero-stable.

(Consistency)

We we will look at the consistency of this LMM. We look at the definition of truncation error in our system.

$$\boldsymbol{y}_{k+3} - \frac{1}{3}(\boldsymbol{y}_{k+2} + \boldsymbol{y}_{k+1} + \boldsymbol{y}_k) = \frac{\Delta t}{12}[\dot{\boldsymbol{y}}_{k+2} - 2\dot{\boldsymbol{y}}_{k+1} + 3\dot{\boldsymbol{y}}_k] + \Delta t \tau_{k+3}$$

We look at the taylor expansions for several points in the LMM.

$$\boldsymbol{y}_k = \boldsymbol{y}_{k+3} - 3\Delta t\dot{\boldsymbol{y}}_k - \frac{9}{2}\Delta t^2\ddot{\boldsymbol{y}}_k - \frac{27}{6}\Delta t^3 \dddot{\boldsymbol{y}}_k - \frac{81}{24}\Delta t^4 \ddddot{\boldsymbol{y}}_k + h.o.t.$$
$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_{k+3} - 2\Delta t\dot{\boldsymbol{y}}_{k+1} - \frac{4}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+1} - \frac{8}{6}\Delta t^3 \dddot{\boldsymbol{y}}_{k+1} - \frac{16}{24}\Delta t^4 \ddddot{\boldsymbol{y}}_{k+1} + h.o.t.$$
$$\boldsymbol{y}_{k+2} = \boldsymbol{y}_{k+3} - \Delta t\dot{\boldsymbol{y}}_{k+2} - \frac{1}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+2} - \frac{1}{6}\Delta t^3 \dddot{\boldsymbol{y}}_{k+2} - \frac{1}{24}\Delta t^4 \ddddot{\boldsymbol{y}}_{k+2} + h.o.t.$$

$$\tau_{k+3} = \frac{\boldsymbol{y}_{k+3} - \frac{1}{3}(\boldsymbol{y}_{k+2} + \boldsymbol{y}_{k+1} + \boldsymbol{y}_k)}{\Delta t} - \frac{1}{12}[\dot{\boldsymbol{y}}_{k+2} - 2\dot{\boldsymbol{y}}_{k+1} + 3\dot{\boldsymbol{y}}_k]$$

$$\tau_{k+3} = -\frac{\boldsymbol{y}_{k+3} - 3\Delta t\dot{\boldsymbol{y}}_k - \frac{9}{2}\Delta t^2\ddot{\boldsymbol{y}}_k - \frac{27}{6}\Delta t^3 \dddot{\boldsymbol{y}}_k - \frac{81}{24}\Delta t^4 \ddddot{\boldsymbol{y}}_k + h.o.t.}{3\Delta t}$$
$$- \frac{\boldsymbol{y}_{k+3} - 2\Delta t\dot{\boldsymbol{y}}_{k+1} - \frac{4}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+1} - \frac{8}{6}\Delta t^3 \dddot{\boldsymbol{y}}_{k+1} - \frac{16}{24}\Delta t^4 \ddddot{\boldsymbol{y}}_{k+1} + h.o.t.}{3\Delta t}$$
$$- \frac{\boldsymbol{y}_{k+3} - \Delta t\dot{\boldsymbol{y}}_{k+2} - \frac{1}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+2} - \frac{1}{6}\Delta t^3 \dddot{\boldsymbol{y}}_{k+2} - \frac{1}{24}\Delta t^4 \ddddot{\boldsymbol{y}}_{k+2} + h.o.t.}{3\Delta t}$$
$$+ \frac{\boldsymbol{y}_{k+3}}{\Delta t} - \frac{1}{12}[23\dot{\boldsymbol{y}}_{k+2} - 2\dot{\boldsymbol{y}}_{k+1} + 3\dot{\boldsymbol{y}}_k]$$

$$\tau_{k+3} = \frac{9\dot{\boldsymbol{y}}_k + 10\dot{\boldsymbol{y}}_{k+1} - 19\dot{\boldsymbol{y}}_{k+2}}{12} + h.o.t$$

$\square$

b) Absolute Stability and A-Stability

We can plot the region of absolute stability for this LMM. We have the first and second characterstic polynomials are the following.

$$\rho(z) = z^3 - \frac{1}{3}\left(z^2 + z + 1\right), \quad \sigma(z) = \frac{1}{12}\left(23z^2 - 2z + 3\right)$$
$$\frac{\rho(z)}{\sigma(z)} = \lambda \Delta t$$

We can then plot this by evaluating $\frac{\rho(z)}{\sigma(z)}$ with $z = e^{i\theta}$ and plotting in the complex plane. We can see

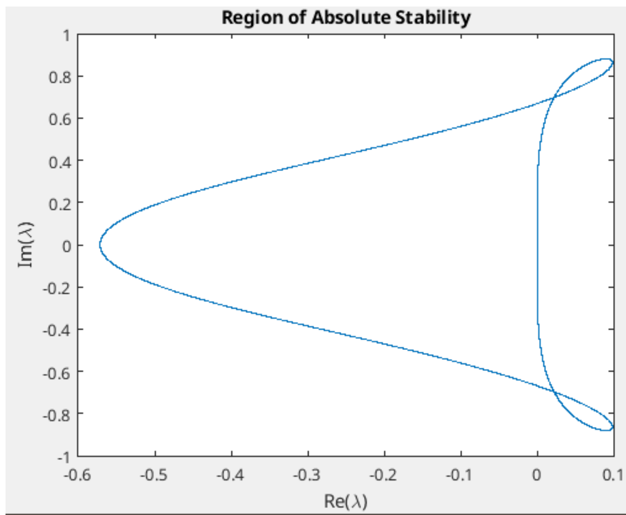

Figure 2: Region of Absolute Stability for 3.b

from this plot that this LMM is certainly not A-Stable. The reason being that the region of absolute stability is only conditionally absolutely stable. This is seen in the plot which clearly illurstrates the region of absolute stability including only a small subset of $\mathbb{C}^-$.