Lecture 3

The definite integral, II

- Motivational example, continued.
- Riemann sums and the Riemann integral.

Warm-up:

$$\int (4x+2)(3-\sqrt{x})\,dx$$

$$= \int 12x - 4x^{3/2} + 6 - 2x^{1/2}\,dx$$

$$= 12\cdot\frac{x^2}{2} - 4\cdot\frac{x^{5/2}}{5/2} + 6x - 2\cdot\frac{x^{3/2}}{3/2} + C$$

$$= -\frac{8}{5}x^{5/2} + 6x^2 - \frac{4}{3}x^{3/2} + 6x + C$$

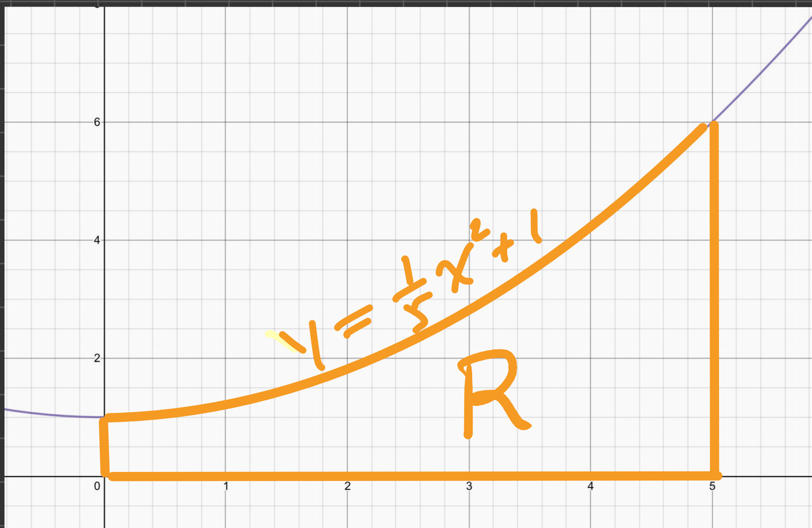
1/

Last time:

Find the area of the region R

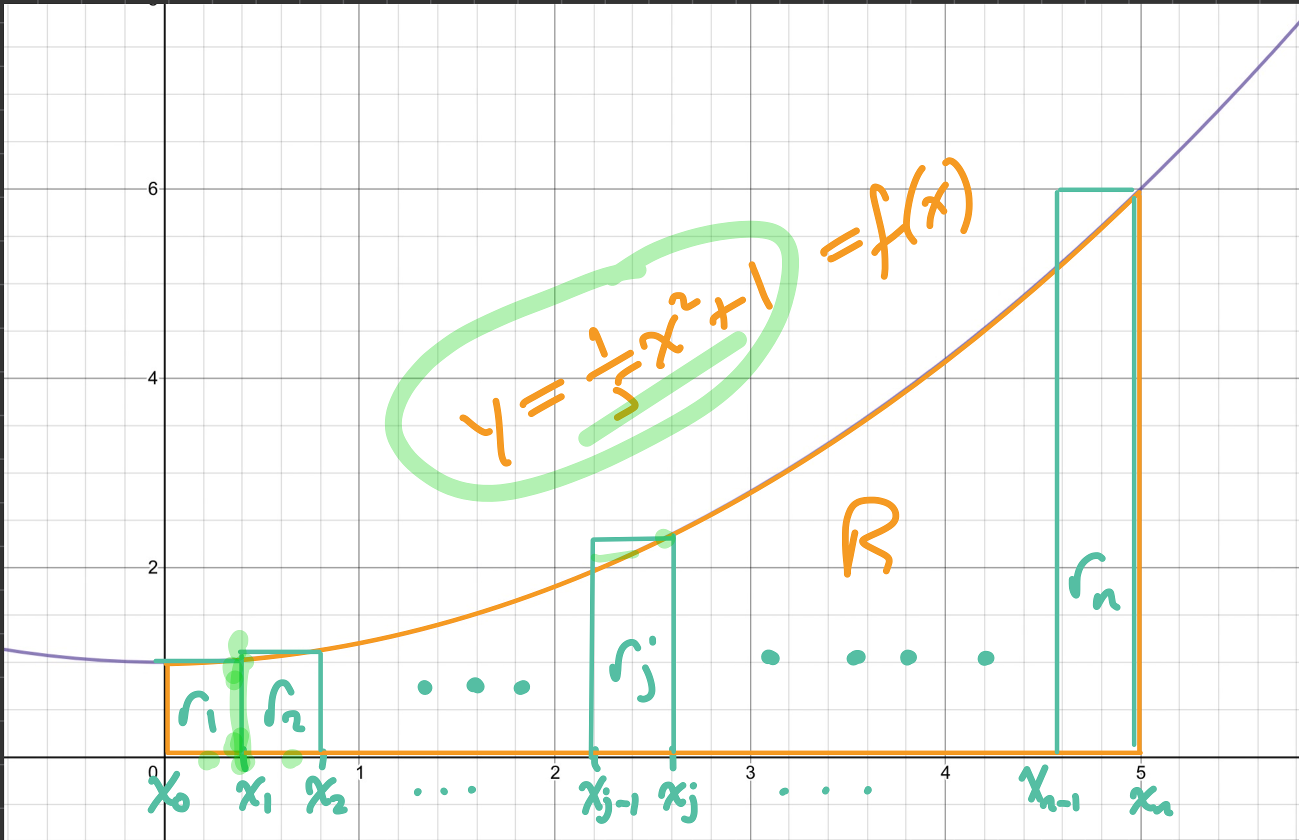
bounded by: $x = 0$, $y = 0$, $x = 5$

and $y = \frac{1}{5}x^2 + 1$



$\longrightarrow$

2/ Approximate R with rectangles:



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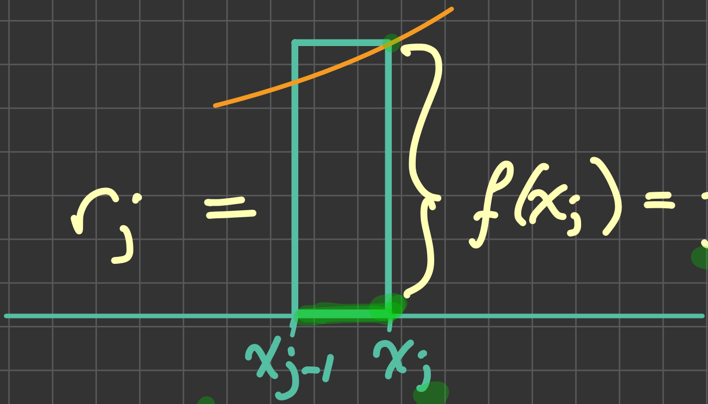
$$\text{Area}(R) \approx \sum_{j=1}^{n} \text{area}(r_j)$$

$r_j =$



$f(x_j) = \frac{1}{5}x_j^2 + 1$

$$\text{Area}(R) \approx \sum_{j=1}^{n} (x_j - x_{j-1}) \cdot \left(\frac{1}{5}x_j^2 + 1\right) = \ldots$$

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To evaluate $\sum_{j=1}^{n} (x_j - x_{j-1}) \cdot \left(\frac{1}{5} x_j^2 + 1\right)$

we need to know x_j

<u>Recall:</u> we divided $[0,5]$ into

n equal subintervals

(i) $\text{length}([x_{j-1}, x_j]) = x_j - x_{j-1} = \frac{5}{n}$

<u>Notation:</u> $\Delta x_j = x_j - x_{j-1}$

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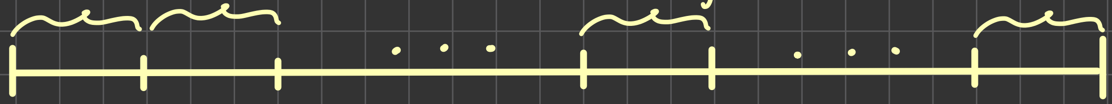
$$\Delta x_1 = \Delta x_2 = \dots = \Delta x_j = \dots = \Delta x_n = \frac{5}{n}$$



$$x_0 = 0 \quad x_1 \quad x_2 \quad \cdot \quad \cdot \quad x_{j-1} \quad x_j \quad \cdots \quad x_{n-1} \quad 5 = x_n$$

(ii) $x_1 = x_0 + \Delta x_1 = 0 + \frac{5}{n} = \frac{5}{n}$

$x_2 = x_1 + \Delta x_2 = \frac{5}{n} + \frac{5}{n} = 2 \cdot \frac{5}{n}$

$x_3 = x_2 + \Delta x_3 = 2 \cdot \frac{5}{n} + \frac{5}{n} = 3 \cdot \frac{5}{n}$

etc.

$$\Longrightarrow \boxed{x_j = j \cdot \frac{5}{n}} \qquad 0 \leq j \leq n$$

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$$\text{Area}(R) \approx \sum_{j=1}^{n} \text{area}(r_j)$$

$$= \sum_{j=1}^{n} \left(\frac{1}{5} \cdot x_j^2 + 1\right) \cdot \Delta x_j$$

$$= \sum_{j=1}^{n} \left(\frac{1}{5} \cdot \left(j \cdot \frac{5}{n}\right)^2 + 1\right) \cdot \frac{5}{n}$$

$$= \frac{5}{n} \cdot \left[\sum_{j=1}^{n} \left(\frac{1}{5} \cdot \left(j \cdot \frac{5}{n}\right)^2 + 1\right)\right]$$

7/ Arithmetic...

$$\text{Area} \approx \frac{5}{n} \cdot \sum_{j=1}^{n} \left(\frac{1}{5} \cdot \left(j \cdot \frac{5}{n}\right)^2 + 1 \right)$$

$$= \frac{5}{n} \cdot \sum_{j=1}^{n} \underbrace{\frac{1}{5} \cdot \left(j \cdot \frac{5}{n}\right)^2}_{?} + \frac{5}{n} \cdot \sum_{j=1}^{n} 1$$

$$= \frac{5}{n^2} \cdot \frac{5}{n} \cdot \sum_{j=1}^{n} j^2 + \frac{5}{\cancel{n}} \cdot \cancel{n} \to 5$$

...

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$$\ldots \text{Area}(R) \approx 5 + \frac{25}{n^3} \boxed{\sum_{j=1}^{n} j^2}$$

$$= 5 + \frac{25}{n^3} \cdot \boxed{\left(\frac{n^3}{3} + \frac{3n^2+n}{6} \right)}$$

$$= \frac{40}{3} + \frac{25}{6} \cdot \left(\frac{3}{n} + \frac{1}{n^2} \right)$$

Intuition:

As the number of subintervals grows, the approximation ⭐ becomes increasingly accurate...

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$$\text{Area}(R) = \lim_{n\to\infty} \sum_{j=1}^{n} \text{area}(r_j)$$

$$= \lim_{n\to\infty} \left(\frac{40}{3} + \frac{25}{6} \cdot \left(\frac{3}{n} + \frac{1}{n^2} \right) \right)$$

$$= \frac{40}{3} + \frac{25}{6} \cdot \overbrace{\lim_{n\to\infty} \left(\frac{3}{n} + \frac{1}{n^2} \right)}^{0}$$

$$= \frac{40}{3}$$

👏

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Observation

1. $\int \frac{1}{5}x^2+1 \, dx = \frac{1}{15}x^3+x+C$

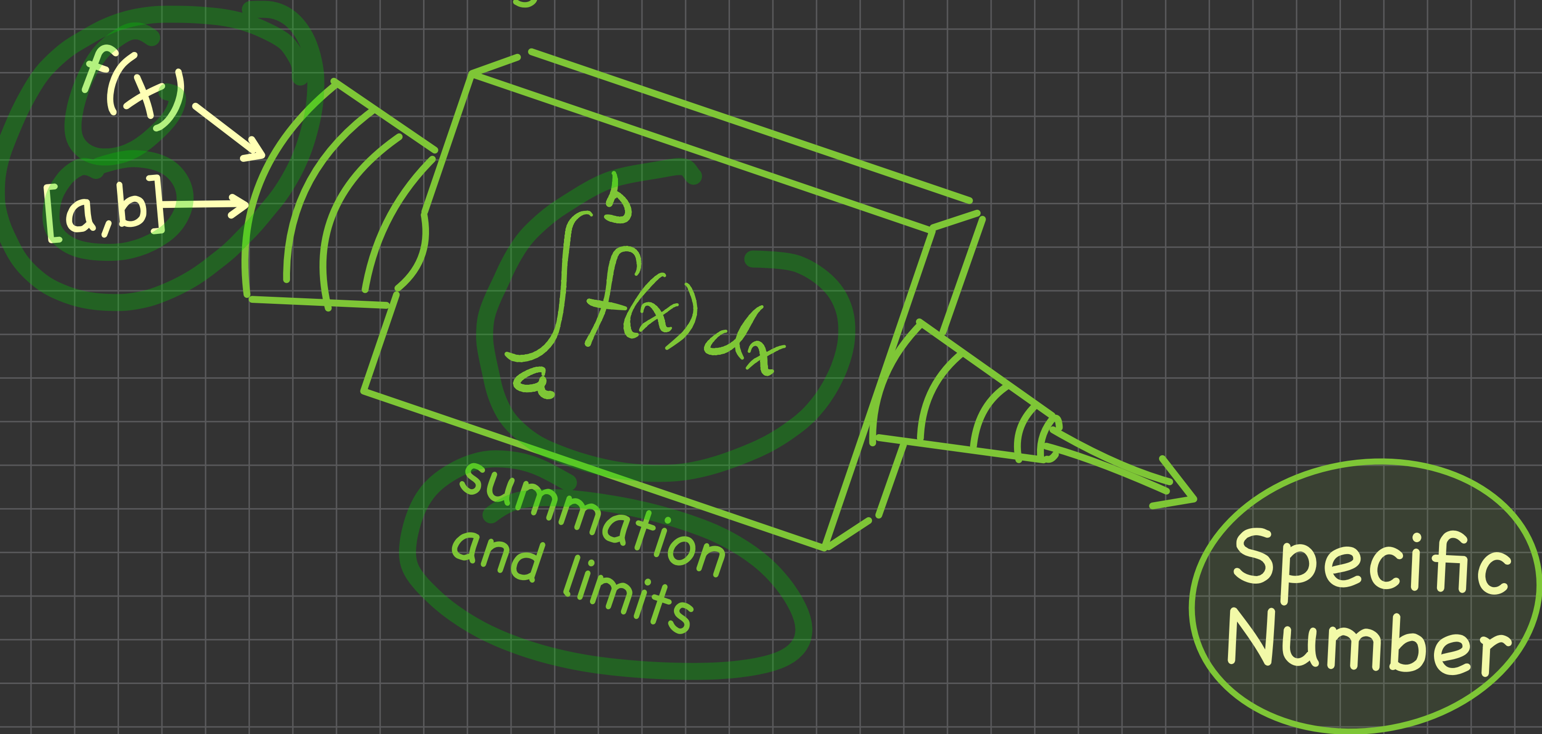
$$= A(x) + C$$

2. $A(5) - A(0) = \left(\frac{125}{15} + 5\right) - 0$

$$= \frac{40}{3}$$

11/ Recall:

The **Definite** Integral



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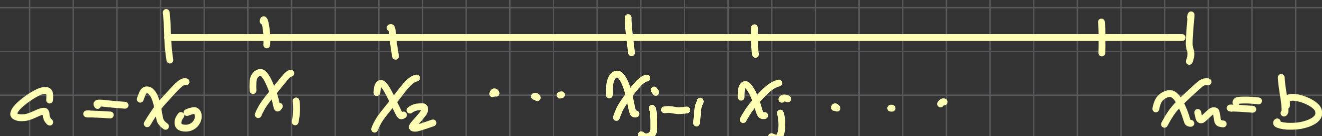
The definite integral of $f(x)$ on the interval $[a,b]$

1. Divide $[a,b]$ into n subintervals:

$$a = x_0 < x_1 < \ldots < x_{j-1} < x_j < \ldots < x_n = b$$



$[x_{j-1}, x_j]$ = j^{th} subinterval

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2. Choose x_j^* in $[x_{j-1}, x_j]$

$$\longrightarrow x_{j-1} \leq x_j^* \leq x_j$$

$$\Delta x_j = x_j - x_{j-1}$$

a **Riemann Sum**

for $f(x)$ in $[a,b]$

$$= \sum_{j=1}^{n} f(x_j^*) \cdot \Delta x_j$$

Bernhard Riemann

Riemann c. 1863

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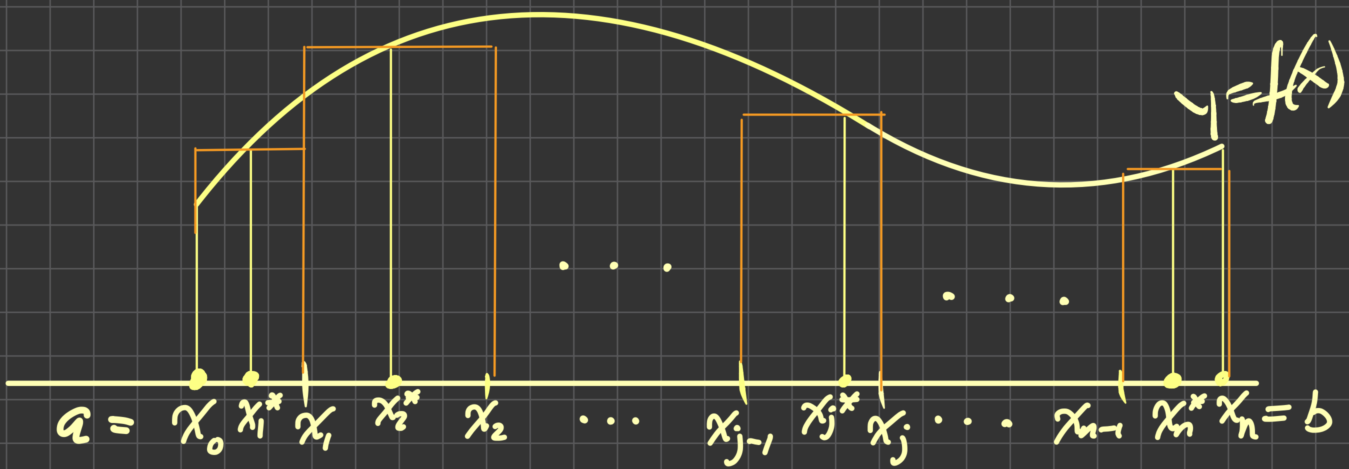
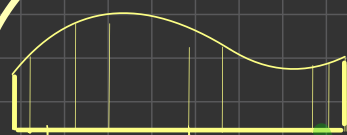
👀 If $f(x) \geq 0$ in $[a,b]$ ...



if all $\Delta x_j \approx 0$.

$$\sum_{j=1}^{n} f(x_j^*)\cdot\Delta x_j \approx \text{area}\Big($$  $$\Big)$$

15/ $D_n = \max\limits_{1\le j\le n} \Delta x_j$ = biggest length

$\Longrightarrow$ If $D_n \approx 0$, then $\Delta x_j \approx 0$ for all j.

<u>Definition:</u>

The definite integral of $f(x)$ on the interval $[a,b]$:

$$\int_a^b f(x)\,dx = \lim_{D_n \to 0} \left(\sum_{j=1}^{n} f(x_j^*) \cdot \Delta x_j \right)$$

...

16/ ... if the limit exists

$\longrightarrow$ All possible Riemann sums are approaching the same value as D_n approaches 0.

"All possible Riemann sums"

$\longrightarrow$ any partition of $[a, b]$ into subintervals, and any choices of x_j^*.

Indefinite Integral

$$\int f(x)\,dx = F(x) + C$$

(family of functions)

Definite Integral

$$\int_a^b f(x)\,dx = \underline{\text{number}}$$

limits of integration

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If $f(x)$ is continuous 👀

in $[a, b]$, the limit

$$\lim_{D_n \to 0} \sum_{j=1}^{n} f(x_j^*) \Delta x_j = \int_a^b f(x)\,dx$$

always exists

So: we can choose partitions and x_j^* however we want, as long as $D_n \longrightarrow 0$.

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Common Choices:

1. Divide $[a,b]$ into n equal subintervals ...

(i) $\Delta x_j = \frac{b-a}{n} = D_n \xrightarrow[n\to\infty]{} 0$

(ii) $x_j = a + j \cdot \Delta x_j = a + j \cdot \frac{b-a}{n}$

2. $x_j^* = x_j$ (righthand point)

or

$x_j^* = x_{j-1}$ (lefthand point)

20/ Righthand sums:

$$\underline{RHS} = \sum_{j=1}^{n} f\Big(\overbrace{a+j\cdot\frac{b-a}{n}}^{x_j}\Big)\cdot\overbrace{\frac{b-a}{n}}^{\Delta x_j}$$

Lefthand sums:

$$\underline{LHS} = \sum_{j=1}^{n} f\Big(\overbrace{a+(j-1)\cdot\frac{b-a}{n}}^{x_{j-1}}\Big)\cdot\overbrace{\frac{b-a}{n}}^{\Delta x_j}$$

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Example:

$$\int_0^5 \frac{1}{5}x^2+1\,dx$$

R.H.S.

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \underbrace{\left(\frac{1}{5}\cdot\left(\frac{5i}{n}\right)^2+1\right)}_{f(x_i)} \cdot \underbrace{\frac{5}{n}}_{\Delta x_i}\right)$$

$$\ldots = \lim_{n\to\infty}\left(\frac{40}{3}+\frac{25}{6}\left(\frac{3}{n}+\frac{1}{n^2}\right)\right)$$

$$= \frac{40}{3}$$

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Example

$$\int_0^2 x^3 - x \, dx = \dots$$

interval: $[0,2]$, R.H.S.: $\begin{cases} \Delta x_k = \frac{2}{n} \\ x_k = k \cdot \frac{2}{n} \end{cases}$

$$\dots = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \underbrace{\left(\left(\frac{2k}{n}\right)^3 - \frac{2k}{n} \right)}_{f(x_k)} \cdot \underbrace{\frac{2}{n}}_{\Delta x_k} \right)$$

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Arithmetic :

$$\sum_{k=1}^{n}\left(\left(\frac{2k}{n}\right)^3-\frac{2k}{n}\right)\cdot\frac{2}{n}$$

$$=\frac{2}{n}\cdot\sum_{k=1}^{n}\frac{8k^3}{n^3}-\frac{2}{n}\cdot\sum_{k=1}^{n}\frac{2k}{n}$$

$$=\frac{16}{n^4}\sum_{k=1}^{n}k^3-\frac{4}{n^2}\sum_{k=1}^{n}k=\ \ldots$$

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Recall :

1. $$\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}$$

2. $$\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

So . . .

25/

So :

$$\frac{16}{n^4}\sum_{k=1}^{n} k^3 - \frac{4}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{16}{n^4}\left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}\right) - \frac{4}{n^2}\cdot\left(\frac{n^2}{2}+\frac{n}{2}\right)$$

$$= 4 + \frac{8}{n} + \frac{4}{n^2} - 2 - \frac{2}{n}$$

$$= \boxed{2 + \frac{6}{n} + \frac{4}{n^2}}$$

26/ $$\int_0^2 x^3 - x \, dx = \ldots$$

$\underline{\text{RHS}} + \underline{\text{Arithmetic}}$

$$\ldots = \lim_{n\to\infty}\left(2 + \cancel{\frac{6}{n}}^{\,0} + \cancel{\frac{4}{n^2}}^{\,0}\right)$$

$$= \underline{2}$$

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👀 $$\int x^3 - x \, dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$$

$$= F(x) + C$$

👀 $$F(2) - F(0) = \frac{1}{4}\cdot 16 - \frac{1}{2}\cdot 4$$

$$= 2$$

$$= \int_0^2 x^3 - x \, dx$$

Next time:

The fundamental theorem of calculus (FTC)