Midterm: Report

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Question 1: BVP using Shooting Method and RK4

a) The system can be solved analytically by simply integrating 4 times and then solving for the boundary conditions.

$$EI\int\int\int\int \frac{d^4y}{dx^4}dx^4 = \int\int\int\int x^2 dx^4$$
$$EIy(x) = \frac{x^6}{360} + c_1x^3 + c_2x^2 + c_3x + c_4$$

We can see very clearly from the boundary conditions at $x = 0$ that $c_4 = c_3 = 0$. We finish solving for c_1 and c_2.

$$0 = \frac{1}{360} + c_1 + c_2, \quad 0 = \frac{1}{60} + 3c_1 + 2c_2$$
$$c_1 = -\frac{1}{90}, \quad c_2 = \frac{1}{120}$$

$$y(x) = \frac{1}{EI}\left(\frac{x^6}{360} - \frac{x^3}{90} + \frac{x^2}{120}\right)$$

b) Determine the Numerical Solution using the shooting method.

c) Plot the numerical solution obtained with the shooting method

d) Plot the error

Question 2: Convergence and Absolute Stability for an Implicit RK3 Method

a) Prove convergence for Implicit RK3.

Proof. Showing convergence is a matter of sending zero-stability and consistency. This is a one-step method, so we have the first characteristic polynomial, $\rho(z) = z - 1$. This of course satisfies the root condition and therefore is zero-stable. Showing consistency □

b) Plot Region of Absolute Statbility (See Fig. 1)

c) Find the largest Δt for the IVP defined in problem 2. This problem is solved by plotting the region of absolute stability and finding the eignevalues of the matrix B. We notice that since B is an upper triangular matrix that its eigenvalues are found on its diagonal. So we have that B has all real eigenvalues, $\lambda = \{-1, -2, -4, -16\}$. Next we compare to see which eigenvalue is furthest from the region of absolute stability. It is evidently the eigenvalue, $\lambda = -16$. So we look at the closest point in our region of absolute stability. From the plot we can see that the closest point is $\lambda\Delta t = -5.4199 + 0i$. Therefore we have that the largest Δt must be, $\Delta t \approx 0.338746\ldots$.

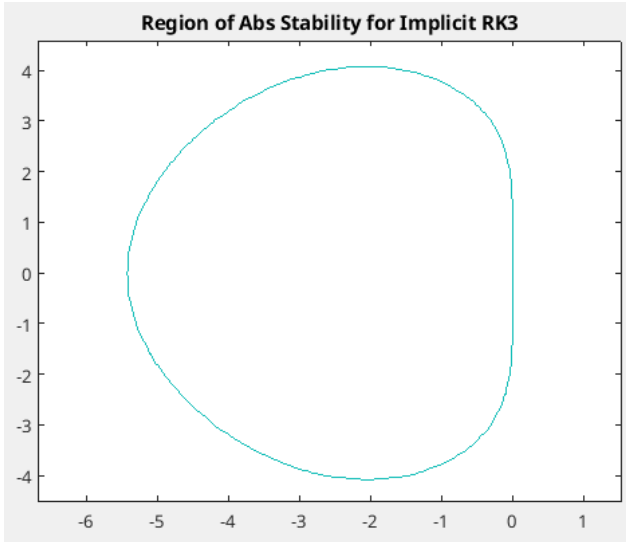

Figure 1: Region of Absolute Stability for 2.b.

d) Show that this is validated numerically.

Question 3: Convergence and Absolute Stability for an LMM

$$\boldsymbol{u}_{k+3} - \frac{1}{3}\left(\boldsymbol{u}_{k+2} + \boldsymbol{u}_{k+1} + \boldsymbol{u}_k\right) = \frac{\Delta t}{12}\left[23\boldsymbol{f}_{k+2} - 2\boldsymbol{f}_{k+1} + 3\boldsymbol{f}_k\right] \tag{1}$$

a) *Proof.* (Zero-Stability)

We begin this proof first by showing that this LMM is zero-stable.

$$\rho(z) = z^3 - \frac{1}{3}z^2 - \frac{1}{3}z - \frac{1}{3}$$

$$\rho(z) = (z-1)\left(z^2 + \frac{2}{3}z + \frac{1}{3}\right)$$

$$\rho(z) = (z-1)\left(z - \frac{-\frac{2}{3} \pm \sqrt{\frac{4}{9} - \frac{4}{3}}}{2}\right)$$

$$\rho(z) = (z-1)\left(z + \frac{1 \pm i\sqrt{2}}{3}\right)$$

These are the three complex roots of the characteristic polynomial. We now simply must check if all three are within or on the boundary of the unit disk.

$$|1| = 1 \leq 1, \quad \left|\frac{1 \pm i\sqrt{2}}{3}\right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3} \leq 1$$

As I have just shown, in fact all three roots of the first characteristic polynomial fall within the unit disk or on the boundary, thus we have that this LMM is zero-stable.

(Consistency)

We we will look at the consistency of this LMM. We look at the definition of truncation error in our system.

$$\boldsymbol{y}_{k+3} - \frac{1}{3}(\boldsymbol{y}_{k+2} + \boldsymbol{y}_{k+1} + \boldsymbol{y}_k) = \frac{\Delta t}{12}[\dot{\boldsymbol{y}}_{k+2} - 2\dot{\boldsymbol{y}}_{k+1} + 3\dot{\boldsymbol{y}}_k] + \Delta t \tau_{k+3}$$

We look at the taylor expansions for several points in the LMM.

$$\begin{aligned}
\boldsymbol{y}_k &= \boldsymbol{y}_{k+3} - 3\Delta t\dot{\boldsymbol{y}}_k - \frac{9}{2}\Delta t^2\ddot{\boldsymbol{y}}_k - \frac{27}{6}\Delta t^3\dddot{\boldsymbol{y}}_k - \frac{81}{24}\Delta t^4\ddddot{\boldsymbol{y}}_k + h.o.t. \\
\boldsymbol{y}_{k+1} &= \boldsymbol{y}_{k+3} - 2\Delta t\dot{\boldsymbol{y}}_{k+1} - \frac{4}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+1} - \frac{8}{6}\Delta t^3\dddot{\boldsymbol{y}}_{k+1} - \frac{16}{24}\Delta t^4\ddddot{\boldsymbol{y}}_{k+1} + h.o.t. \\
\boldsymbol{y}_{k+2} &= \boldsymbol{y}_{k+3} - \Delta t\dot{\boldsymbol{y}}_{k+2} - \frac{1}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+2} - \frac{1}{6}\Delta t^3\dddot{\boldsymbol{y}}_{k+2} - \frac{1}{24}\Delta t^4\ddddot{\boldsymbol{y}}_{k+2} + h.o.t.
\end{aligned}$$

$$\tau_{k+3} = \frac{\boldsymbol{y}_{k+3} - \frac{1}{3}(\boldsymbol{y}_{k+2} + \boldsymbol{y}_{k+1} + \boldsymbol{y}_k)}{\Delta t} - \frac{1}{12}[\dot{\boldsymbol{y}}_{k+2} - 2\dot{\boldsymbol{y}}_{k+1} + 3\dot{\boldsymbol{y}}_k]$$

$$\begin{aligned}
\tau_{k+3} = &-\frac{\boldsymbol{y}_{k+3} - 3\Delta t\dot{\boldsymbol{y}}_k - \frac{9}{2}\Delta t^2\ddot{\boldsymbol{y}}_k - \frac{27}{6}\Delta t^3\dddot{\boldsymbol{y}}_k - \frac{81}{24}\Delta t^4\ddddot{\boldsymbol{y}}_k + h.o.t.}{3\Delta t} \\
&- \frac{\boldsymbol{y}_{k+3} - 2\Delta t\dot{\boldsymbol{y}}_{k+1} - \frac{4}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+1} - \frac{8}{6}\Delta t^3\dddot{\boldsymbol{y}}_{k+1} - \frac{16}{24}\Delta t^4\ddddot{\boldsymbol{y}}_{k+1} + h.o.t.}{3\Delta t} \\
&- \frac{\boldsymbol{y}_{k+3} - \Delta t\dot{\boldsymbol{y}}_{k+2} - \frac{1}{2}\Delta t^2\ddot{\boldsymbol{y}}_{k+2} - \frac{1}{6}\Delta t^3\dddot{\boldsymbol{y}}_{k+2} - \frac{1}{24}\Delta t^4\ddddot{\boldsymbol{y}}_{k+2} + h.o.t.}{3\Delta t} \\
&+ \frac{\boldsymbol{y}_{k+3}}{\Delta t} - \frac{1}{12}[23\dot{\boldsymbol{y}}_{k+2} - 2\dot{\boldsymbol{y}}_{k+1} + 3\dot{\boldsymbol{y}}_k]
\end{aligned}$$

$$\tau_{k+3} = \frac{9\dot{\boldsymbol{y}}_k + 10\dot{\boldsymbol{y}}_{k+1} - 19\dot{\boldsymbol{y}}_{k+2}}{12} + h.o.t$$

□

b) Absolute Stability and A-Stability

We can plot the region of absolute stability for this LMM. We have the first and second characterstic polynomials are the following.

$$\rho(z) = z^3 - \frac{1}{3}\left(z^2 + z + 1\right), \quad \sigma(z) = \frac{1}{12}\left(23z^2 - 2z + 3\right)$$
$$\frac{\rho(z)}{\sigma(z)} = \lambda\Delta t$$

We can then plot this by evaluating $\frac{\rho(z)}{\sigma(z)}$ with $z = e^{i\theta}$ and plotting in the complex plane. We can see from this plot that this LMM is certainly not A-Stable. The reason being that the region of absolute stability is only conditionally absolutely stable. This is seen in the plot which clearly illurstrates the region of absolute stability including only a small subset of $\mathbb{C}^-$.

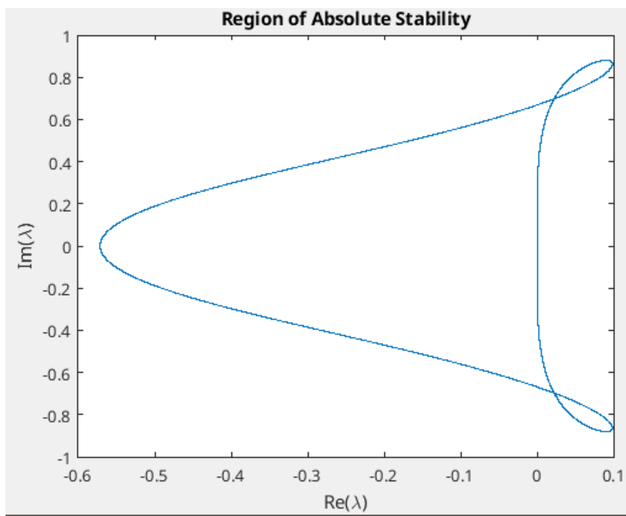

Figure 2: Region of Absolute Stability for 3.b