AM 260 - Computational Fluid Dynamis: Homework 3

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Problem 1: Lax-Friedrichs Method

Problem 2: Lax-Wendroff Method

(a) Show that the LW method is convergent if $|C_a| \leq 1$.

In order to demonstrate consistency and stability, we perform taylor expansions to demonstrate consistency (and at which order it is consistent), and then von Neumann stability analysis in order to prove stability.

Consistency

$$\begin{aligned}
U_j^{n+1} &= U_j^n + \Delta t U_{t,j}^n + \frac{\Delta t^2}{2} U_{tt,j}^n + \frac{\Delta t^3}{6} U_{ttt,j}^n + \frac{\Delta t^4}{24} U_{tttt,j}^n + O(\Delta t^5) \\
U_{j+1}^n &= U_j^n + \Delta x U_{x,j}^n + \frac{\Delta x^2}{2} U_{xx,j}^n + \frac{\Delta x^3}{6} U_{xxx,j}^n + \frac{\Delta x^4}{24} U_{xxxx,j}^n + O(\Delta x^5) \\
U_{j-1}^n &= U_j^n - \Delta x U_{x,j}^n + \frac{\Delta x^2}{2} U_{xx,j}^n - \frac{\Delta x^3}{6} U_{xxx,j}^n + \frac{\Delta x^4}{24} U_{xxxx,j}^n + O(\Delta x^5)
\end{aligned}$$

$$\begin{aligned}
\lim_{\Delta t, \Delta x \to 0} E_{LT} &= \lim_{\Delta t, \Delta x \to 0} \frac{1}{\Delta t} U_j^{n+1} - U_j^n + \frac{1}{2} C_a \left(U_{j+1}^n - U_{j-1}^n \right) - \frac{1}{2} C_a^2 \left(U_{j+1}^n - 2U_j^n + U_{j-1}^n \right) \\
&= \lim_{\Delta t, \Delta x \to 0} U_{t,j}^n + \frac{\Delta t}{2} U_{tt,j}^n + \frac{\Delta t^2}{6} U_{ttt,j}^n + O(\Delta t^3) + a U_{x,j}^n + a \frac{\Delta x^2}{6} U_{xxx,j}^n + O(\Delta x^4) \\
&\quad - a C_a \left(\frac{\Delta x}{2} U_{xx,j}^n + \frac{\Delta x^3}{24} U_{xxxx,j}^n + O(\Delta x^5) \right) \\
&= \lim_{\Delta t, \Delta x \to 0} \frac{\Delta t^2}{6} U_{ttt,j}^n + a \frac{\Delta x^2}{6} U_{xxx,j}^n + O(\Delta^3)
\end{aligned}$$

Therefore, we have that this method is consistent with $O(\Delta t^2 + \Delta x^2)$.

Stability

$$\begin{aligned}
G &= (1 - C_a^2) + \frac{1}{2} \left(C_a^2 - C_a \right) e^{i k_x \Delta x} + \frac{1}{2} \left(C_a^2 + C_a \right) e^{-i k_x \Delta x} \\
G &= (1 - C_a^2) + C_a^2 \cos(k_x \Delta x) - i C_a \sin(k_x \Delta x) \\
|G| &= (1 - C_a^2)^2 + C_a^4 \cos^2(k_x \Delta x) + 2(1 - C_a^2) C_a^2 \cos(k_x \Delta x) + C_a^2 \sin^2(k_x \Delta x) \\
&= 1 - 2C_a^2 + C_a^4 + C_a^4 \cos^2() + 2C_a^2 \cos() - 2C_a^4 \cos() + C_a^2 \sin^2() \\
&= 1 + C_a^2 \left(2\cos + \sin^2 - 2 \right) + C_a^4 \left(1 + \cos^2 - 2\cos \right)
\end{aligned}$$

We proceed from here casewise. Take, $|C_a| = 1$. We have,

$$|G| = 1 + 2\cos - 2 + 1 - 2\cos + 1 = 1$$

in which case, the method is stable. We next consider $|C_a| \leq 1$.

(b) Show that the LW method is $O(\Delta t^2 + \Delta x^2)$.

Problem 3: von Neumann Stability Analysis

We can show that this method is unconditionally unstable with only a few lines of algebra.

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right), \quad U_j^n = G^n e^{ijk_x\Delta x}$$

$$G = 1 - \frac{a\Delta t}{2\Delta x}\left(e^{ik_x\Delta x} - e^{-ik_x\Delta x}\right)$$

$$G = 1 - \frac{a\Delta t}{2\Delta x} i\sin(k_x\Delta x)$$

$$|G| = 1 + \left(\frac{a\Delta t}{2\Delta x}\right)^2 \sin(k_x\Delta x)^2 > 1$$

Therefore, we have that this method is unconditionally unstable, i.e. there is no condition on which $|G| \leq 1$.

Problem 4: Modified Lax-Friedrichs Coefficient

Problem 5: von Neumann Analysis of the Heat Equation

Problem 6: Sinusoidal Adv. with LF

Problem 7: Discontinuous IC with LF

Problem 8: Sinusoidal Adv. with LW

Problem 9: Discontinuous IC with LW