AM 260, Spring 2025

Note on the basic discretization setup

1. Overview

We are interested in solving a linear advection PDE given as

$$u_t + au_x = 0, \tag{1}$$

where a is a constant advection velocity.

2. Initial and boundary conditions

We impose an initial condition at $t = 0$:

$$\text{IC:}\;\; u(x, 0) = u_0(x) \tag{2}$$

and a boundary condition on a bounded domain $x_a \le x \le x_b$:

$$\text{BC:}\;\; u(x_a, t) = g_a(t) \text{ and } u(x_b, t) = g_b(t),\;\; \text{for } t > 0, \tag{3}$$

where g_a and g_b are known functions.

3. Discretization in space and time

Let us take the discretization of interior cells with which we have a spatial resolution of N and a temporal resolution of M:

$$x_i = x_a + (i - \frac{1}{2})\Delta x,\;\; i = 1, \dots, N + 4, \tag{4}$$

$$t^n = n\Delta t,\;\; n = 0, \dots, M. \tag{5}$$

The N interior points are defined with $i = 3, \dots, N + 2$, whereas the four points with $i = 1, 2$ and $i = N + 3, N + 4$ are the guardcell points (see below). Notice that the cell interface-centered grid points are written using the 'half-integer' indices:

$$x_{i+\frac{1}{2}} = x_i + \frac{\Delta x}{2}. \tag{6}$$

4. Imposing Boundary Conditions via guard-cells (or ghost-cells)

We introduce the so-called 'guard-cells' or 'ghost-cells' (simply GCs) on each end, having two extra layers of GC points on each side of boundaries,

$$x_1 = x_a - \frac{3\Delta x}{2}, \tag{7}$$

$$x_2 = x_a - \frac{\Delta x}{2}, \tag{8}$$

$$x_{N+3} = x_b + \frac{\Delta x}{2}, \tag{9}$$

$$x_{N+4} = x_b + \frac{3\Delta x}{2}. \tag{10}$$

With these two extra layers of GC points (i.e., two GC points on each end), the differential equation is discretized with numerical approximations $U_i^n \approx u(x_i, t^n)$, which are spatially differenced and temporally evolved on the N interior points. The boundary conditions, on the other hand, are explicitly imposed throught the four GC points, simply

$$\begin{aligned} U_1^n &= g_a(t^n), & (11)\\ U_2^n &= g_a(t^n), & (12)\\ U_{N+3}^n &= g_b(t^n), & (13)\\ U_{N+4}^n &= g_b(t^n). & (14) \end{aligned}$$

Of particularly useful boundary conditions are periodic condition and outflow condition, which respectively are given as

$$\begin{aligned} U_1^n &= U_{N+1}^n, & (15)\\ U_2^n &= U_{N+2}^n, & (16)\\ U_{N+3}^n &= U_3^n, & (17)\\ U_{N+4}^n &= U_4^n, & (18) \end{aligned}$$

for periodic condition, and

$$\begin{aligned} U_1^n &= U_3^n, & (19)\\ U_2^n &= U_3^n, & (20)\\ U_{N+3}^n &= U_{N+2}^n, & (21)\\ U_{N+4}^n &= U_{N+2}^n, & (22) \end{aligned}$$

for outflow condition.

5. The CFL condition

As studied, the CFL condition provides a necessary condition for choosing the length of Δt depending on the PDE under consideration. The CFL condition amounts to say that, if we let C_a to be the CFL number that satisfies $0 < C_a \le 1$, a timestep Δt needs to satisfy

$$0 < \Delta t \le C_a \frac{\Delta x}{|a|} \tag{23}$$

for a numerical method for advection to be stable.

It is important to note that the CFL condition is only a *necessary* condition for stability (and hence convergence). It is not always *sufficient* to guarantee stability, which means that a numerical method satisfying the CFL condition could still become unstable.