Multiscale Expansion for Rotating Stratified Turbulence

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1. Equations, Component Expansion, and Scales

1.1 Governing Equations for Rotating Stratified Flows

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro}\left(\boldsymbol{e}_z \times \boldsymbol{u}\right) = -\nabla p + \frac{1}{Fr^2} b \boldsymbol{e}_z + \boldsymbol{F} + \frac{1}{Re}\nabla^2 \boldsymbol{u} \tag{1}$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + w = \frac{1}{Pe}\nabla^2 b \tag{2}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{3}$$

Here we define our nondimensionalized equations along with the nondimensional numbers: the Rossby Ro, Reynolds Re, Froude Fr and Péclet Pe numbers defined according to the following characteristic scales.

$$\boldsymbol{u} = U\boldsymbol{u}', \quad t = \frac{L}{U}t', \quad \boldsymbol{x} = L\boldsymbol{x}', \quad p = \frac{U^2\rho}{L}p', \quad b = N^2 L$$
$$Re = \frac{UL}{\nu}, \quad Pe = \frac{UL}{\kappa}, \quad Ro = \frac{U}{2\Omega L}, \quad Fr = \frac{U}{NL}$$

1.2 Choice of Fast and Slow Spatial/Time Scales

To create a multiscale model for the statistically steady state of rotating stratified turbulence, we define the crucial length scales that specific dynamics occur on. Specifically, we expect both large/slow and small/fast scales in the horizontal, vertical, and temporal coordinates of the flow. We take all of our unit coordinates (x, y, z) to be defined with respect to the characteristic horizontal length scale of the flow taken from the stochastic forcing process ($L_h = 2\pi/\sqrt{2} = \sqrt{2}\pi$). Furthermore, we take our unit time to be scaled according to the advection timescale of the flow L_h/U. We then define our small/fast coordinates to be rescaled by the aspect ratio $\alpha = l_z/L_h$, which compares the small vertical length scale typical of layered anistropic stratified turbulence to the horizontal forcing length scale L_h.

$$\chi = \frac{x}{\alpha}, \quad \xi = \frac{y}{\alpha}, \quad \eta = \frac{z}{\alpha}, \quad \tau = \frac{t}{\alpha}$$

Derivative operators are also expanded in order to resolve dynamics over the various spatio-temporal scales. We have specifically,

$$\nabla_\perp = \nabla_{\perp S} + \frac{1}{\alpha}\nabla_{\perp F} = (\partial_x, \partial_y) + \frac{1}{\alpha}(\partial_\chi, \partial_\xi)$$
$$\partial_z = \partial_z + \frac{1}{\alpha}\partial_\eta$$
$$\partial_t = \partial_t + \frac{1}{\alpha}\partial_\tau$$

1.3 Component Expansion of the Primary Variables

We will continue with a decomposition of our scalar (and each component of our vector) fields as defined below,

$$q = \langle q \rangle (x, y, z, t) + \bar{q}(x, y, z, \eta, t) + q'(x, \chi, y, \xi, z, \eta, t, \tau) \tag{4}$$

This decomposition introduces a "bracket" component which is invariant under the following isotropic fast-averaging method defined using the same notation.

$$\langle (\cdot) \rangle \equiv \int_\tau \int_\eta \int_\xi \int_\chi (\cdot) d\chi d\xi d\eta d\tau \tag{5}$$

The "bar" component, which varies on the slow scales and only the fast vertical scale, is invariant under the horizontal fast-averaging method defined similarly. By contrast, it is not invariant under the "bracket" average, as it averages over the small vertical scales to zero. The "bracket" component of the flow is also invariant under the "bar" average as it only varies on the slow scales.

$$\bar{(\cdot)} \equiv \int_\tau \int_\xi \int_\chi (\cdot) d\chi d\xi d\tau \tag{6}$$

Finally, the "prime" or perturbation component of the decomposition varies on all of the fast and slow scale variables. This particular component of the decomposition averages to zero under both of the fast-averaging operators.

2. Multi-scale expansion of the Governing Equations

We begin the multi-scale expansion of the governing equations and proceed to isolate the governing dynamics of each component of our flow by utilizing the averaging operators. We begin with the expansion of the divergence-free condition.

$$\nabla \cdot \boldsymbol{u} = 0 \tag{7}$$

$$\left(\nabla_{\perp S} + \frac{1}{\alpha} \nabla_{\perp F} \right) \cdot \left(\langle \boldsymbol{u} \rangle_\perp + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp \right) + \left(\partial_z + \frac{1}{\alpha} \partial_\eta \right) \left(\langle w \rangle + \bar{w} + w' \right) = 0 \tag{8}$$

$$\nabla_{\perp S} \cdot \langle \boldsymbol{u} \rangle_\perp + \nabla_{\perp S} \cdot \bar{\boldsymbol{u}}_\perp + \nabla_{\perp S} \cdot \boldsymbol{u}'_\perp + \frac{1}{\alpha} \nabla_{\perp F} \cdot \boldsymbol{u}'_\perp + \partial_z \langle w \rangle + \partial_z \bar{w} + \partial_z w' + \frac{1}{\alpha} \partial_\eta \bar{w} + \frac{1}{\alpha} \partial_\eta w' = 0 \tag{9}$$

$$\langle (9) \rangle \to \nabla_{\perp S} \cdot \langle \boldsymbol{u} \rangle_\perp + \partial_z \langle w \rangle = 0 \tag{10}$$

$$\overline{(9) - \langle (9) \rangle} \to \nabla_{\perp S} \cdot \bar{\boldsymbol{u}}_\perp + \partial_z \bar{w} + \frac{1}{\alpha} \partial_\eta \bar{w} = 0 \tag{11}$$

$$(9) - \langle (9) \rangle - \overline{(9) - \langle (9) \rangle} \to \nabla_{\perp S} \cdot \boldsymbol{u}'_\perp + \frac{1}{\alpha} \nabla_{\perp F} \cdot \boldsymbol{u}'_\perp + \partial_z w' + \frac{1}{\alpha} \partial_\eta w' = 0 \tag{12}$$

2.1 The Buoyancy Equation

$$\begin{aligned} \partial_t \left(\langle b \rangle + \bar{b} + b' \right) + \frac{1}{\alpha} \partial_\tau b' + \left(\langle \boldsymbol{u}_\perp \rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp \right) \cdot \left(\nabla_{\perp S} + \frac{1}{\alpha} \nabla_{\perp F} \right) \left(\langle b \rangle + \bar{b} + b' \right) \\ + \left(\langle w \rangle + \bar{w} + w' \right) \left(\partial_z + \frac{1}{\alpha} \partial_\eta \right) \left(\langle b \rangle + \bar{b} + b' \right) + \left(\langle w \rangle + \bar{w} + w' \right) \\ = \frac{1}{Pe} \left[\left(\nabla_{\perp S} + \frac{1}{\alpha} \nabla_{\perp F} \right)^2 \left(\langle b \rangle + \bar{b} + b' \right) + \left(\partial_z + \frac{1}{\alpha} \partial_\eta \right)^2 \left(\langle b \rangle + \bar{b} + b' \right) \right] \end{aligned} \tag{13}$$

$$\langle(13)\rangle \to \partial_t \langle b\rangle + \langle \boldsymbol{u}_\perp\rangle \cdot \nabla_{\perp S}\langle b\rangle + \langle w\rangle \partial_z \langle b\rangle + \langle w\rangle$$
$$= -\nabla_{\perp S}\cdot\left\langle \bar{\boldsymbol{u}}_\perp \bar{b}\right\rangle - \nabla_{\perp S}\cdot\langle \boldsymbol{u}'_\perp b'\rangle - \partial_z\left\langle \bar{w}\bar{b}\right\rangle - \partial_z\langle w'b'\rangle + \frac{1}{Pe}\left(\nabla^2_{\perp S}\langle b\rangle + \partial_z^2\langle b\rangle\right) \quad (14)$$

$$\overline{(13) - \langle(13)\rangle} \to \partial_t\bar{b} + (\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp)\cdot\nabla_{\perp S}\bar{b} + \bar{\boldsymbol{u}}_\perp\cdot\nabla_{\perp S}\left(\langle b\rangle + \bar{b}\right) + (\langle w\rangle + \bar{w})\left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)\bar{b}$$
$$+ \bar{w}\partial_z\langle b\rangle + \bar{w} = \nabla_{\perp S}\cdot\left\langle \bar{\boldsymbol{u}}_\perp\bar{b}\right\rangle + \nabla_{\perp S}\cdot\langle \boldsymbol{u}'_\perp b'\rangle + \partial_z\left\langle \bar{w}\bar{b}\right\rangle + \partial_z\langle w'b'\rangle$$
$$- \nabla_{\perp S}\cdot\overline{\boldsymbol{u}'_\perp b'} - \partial_z\overline{w'b'} - \frac{1}{\alpha}\partial_\eta\overline{w'b'} + \frac{1}{Pe}\left(\nabla^2_{\perp S}\bar{b} + \partial_z^2\bar{b} + \frac{1}{\alpha^2}\partial_\eta^2\bar{b} + \frac{2}{\alpha}\partial_z\partial_\eta\bar{b}\right) \quad (15)$$

$$(13) - \langle(13)\rangle - \overline{(13) - \langle(13)\rangle} \to \partial_t b' + \frac{1}{\alpha}\partial_\tau b' + (\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp)\cdot\nabla_{\perp S}b'$$
$$+ \frac{1}{\alpha}(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp)\cdot\nabla_{\perp F}b' + \boldsymbol{u}'_\perp\cdot\nabla_{\perp S}\left(\langle b\rangle + \bar{b}\right) + (\langle w\rangle + \bar{w} + w')\partial_z b'$$
$$+ \frac{1}{\alpha}(\langle w\rangle + \bar{w} + w')\partial_\eta b' + w'\partial_z\left(\langle b\rangle + \bar{b}\right) + \frac{1}{\alpha}w'\partial_\eta\bar{b} + w' = \nabla_{\perp S}\cdot\overline{\boldsymbol{u}'_\perp b'} + \partial_z\overline{w'b'} + \frac{1}{\alpha}\partial_\eta\overline{w'b'}$$
$$+ \frac{1}{Pe}\left(\nabla^2_{\perp S}b' + \frac{2}{\alpha}\nabla_{\perp S}\nabla_{\perp F}b' + \frac{1}{\alpha^2}\nabla_{\perp F}b' + \partial_z^2 b' + \frac{2}{\alpha}\partial_z\partial_\eta b' + \frac{1}{\alpha^2}\partial_\eta^2 b'\right) \quad (16)$$

2.2 The Vertical Momentum Equation

$$\partial_t(\langle w\rangle + \bar{w} + w') + \frac{1}{\alpha}\partial_\tau w' + (\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp)\cdot\left(\nabla_{\perp S} + \frac{1}{\alpha}\nabla_{\perp F}\right)(\langle w\rangle + \bar{w} + w')$$
$$+ (\langle w\rangle + \bar{w} + w')\left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)(\langle w\rangle + \bar{w} + w') = -\left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)(\langle p\rangle + \bar{p} + p') + \frac{1}{Fr^2}\left(\langle b\rangle + \bar{b} + b'\right)$$
$$+ \frac{1}{Re}\left[\left(\nabla_{\perp S} + \frac{1}{\alpha}\nabla_{\perp F}\right)^2(\langle w\rangle + \bar{w} + w') + \left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)(\langle w\rangle + \bar{w} + w')\right] \quad (17)$$

$$\langle(17)\rangle \to \partial_t\langle w\rangle + \langle \boldsymbol{u}_\perp\rangle\cdot\nabla_{\perp S}\langle w\rangle + \langle w\rangle\partial_z\langle w\rangle = -\partial_z\langle p\rangle + \frac{1}{Fr^2}\langle b\rangle$$
$$= -\nabla_{\perp S}\cdot\langle \bar{\boldsymbol{u}}_\perp\bar{w}\rangle - \nabla_{\perp S}\cdot\langle \boldsymbol{u}'_\perp w'\rangle - \partial_z\langle \bar{w}\bar{w}\rangle - \partial_z\langle w'w'\rangle + \frac{1}{Re}\left(\nabla^2_{\perp S}\langle w\rangle + \partial_z^2\langle w\rangle\right) \quad (18)$$

$$\overline{(17) - \langle(17)\rangle} \to \partial_t\bar{w} + (\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp)\cdot\nabla_{\perp S}\bar{w} + \bar{\boldsymbol{u}}_\perp\cdot\nabla_{\perp S}(\langle w\rangle + \bar{w}) + (\langle w\rangle + \bar{w})\left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)\bar{w}$$
$$+ \bar{w}\partial_z\langle w\rangle = -\partial_z\bar{p} - \frac{1}{\alpha}\partial_\eta\bar{p} + \frac{1}{Fr^2}\bar{b} + \nabla_{\perp S}\cdot\langle \bar{\boldsymbol{u}}_\perp\bar{w}\rangle + \nabla_{\perp S}\cdot\langle \boldsymbol{u}'_\perp w'\rangle + \partial_z\langle \bar{w}\bar{w}\rangle + \partial_z\langle w'w'\rangle$$
$$- \nabla_{\perp S}\cdot\overline{\boldsymbol{u}'_\perp w'} - \partial_z\overline{w'w'} - \frac{1}{\alpha}\partial_\eta\overline{w'w'} + \frac{1}{Re}\left(\nabla^2_{\perp S}\bar{w} + \partial_z^2\bar{w} + \frac{1}{\alpha^2}\partial_\eta^2\bar{w} + \frac{2}{\alpha}\partial_z\partial_\eta\bar{w}\right) \quad (19)$$

$$(17) - \langle(17)\rangle - \overline{(17) - \langle(17)\rangle} \to \partial_t w' + \frac{1}{\alpha}\partial_\tau w' + (\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp)\cdot\nabla_{\perp S}w'$$
$$+ \frac{1}{\alpha}(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp)\cdot\nabla_{\perp F}w' + \boldsymbol{u}'_\perp\cdot\nabla_{\perp S}(\langle w\rangle + \bar{w}) + (\langle w\rangle + \bar{w} + w')\partial_z w' + \frac{1}{\alpha}(\langle w\rangle + \bar{w} + w')\partial_\eta w'$$
$$+ w'\partial_z(\langle w\rangle + \bar{w}) + \frac{1}{\alpha}w'\partial_\eta\bar{w} = -\partial_z p' - \frac{1}{\alpha}\partial_\eta p' + \frac{1}{Fr^2}b' + \nabla_{\perp S}\cdot\overline{\boldsymbol{u}'_\perp w'} + \partial_z\overline{w'w'} + \frac{1}{\alpha}\partial_\eta\overline{w'w'}$$
$$+ \frac{1}{Re}\left(\nabla^2_{\perp S}w' + \frac{2}{\alpha}\nabla_{\perp S}\nabla_{\perp F}w' + \frac{1}{\alpha^2}\nabla_{\perp F}w' + \partial_z^2 w' + \frac{2}{\alpha}\partial_z\partial_\eta w' + \frac{1}{\alpha^2}\partial_\eta^2 w'\right) \quad (20)$$

2.3 The Horizontal Momentum Equation

$$\begin{aligned}\partial_t\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right) + \frac{1}{\alpha}\partial_\eta \boldsymbol{u}'_\perp + \left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right)\cdot\left(\nabla_{\perp S} + \frac{1}{\alpha}\nabla_{\perp F}\right)\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right)\\ + \left(\langle w\rangle + \bar{w} + w'\right)\left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right) + \frac{1}{Ro}\left(\boldsymbol{e}_z \times \left(\langle \boldsymbol{u}\rangle + \bar{\boldsymbol{u}} + \boldsymbol{u}'\right)\right)\\ = -\left(\nabla_{\perp S} + \frac{1}{\alpha}\nabla_{\perp F}\right)\left(\langle p\rangle + \bar{p} + p'\right) + \boldsymbol{F}\\ + \frac{1}{Re}\left[\left(\nabla_{\perp S} + \frac{1}{\alpha}\nabla_{\perp F}\right)^2\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right) + \left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)^2\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right)\right] \quad (21)\end{aligned}$$

$$\begin{aligned}\langle(21)\rangle \to \partial_t\langle \boldsymbol{u}_\perp\rangle + \langle \boldsymbol{u}_\perp\rangle\cdot\nabla_{\perp S}\langle \boldsymbol{u}_\perp\rangle + \langle w\rangle\partial_z\langle \boldsymbol{u}_\perp\rangle + \frac{1}{Ro}\left(\boldsymbol{e}_z\times\langle \boldsymbol{u}\rangle\right) = -\nabla_{\perp S}\langle p\rangle + \langle \boldsymbol{F}\rangle\\ - \nabla_{\perp S}\cdot\langle \bar{\boldsymbol{u}}_\perp\bar{\boldsymbol{u}}_\perp\rangle - \nabla_{\perp S}\cdot\langle \boldsymbol{u}'_\perp\boldsymbol{u}'_\perp\rangle - \partial_z\langle \bar{w}\bar{\boldsymbol{u}}_\perp\rangle - \partial_z\langle w'\boldsymbol{u}'_\perp\rangle + \frac{1}{Re}\left(\nabla^2_{\perp S}\langle \boldsymbol{u}_\perp\rangle + \partial_z^2\langle \boldsymbol{u}_\perp\rangle\right) \quad (22)\end{aligned}$$

$$\begin{aligned}\overline{(21) - \langle(21)\rangle} \to \partial_t\bar{\boldsymbol{u}}_\perp + \left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp\right)\cdot\nabla_{\perp S}\bar{\boldsymbol{u}}_\perp + \bar{\boldsymbol{u}}_\perp\cdot\nabla_{\perp S}\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp\right) + \left(\langle w\rangle + \bar{w}\right)\left(\partial_z + \frac{1}{\alpha}\partial_\eta\right)\bar{\boldsymbol{u}}_\perp\\ + \bar{w}\partial_z\langle \boldsymbol{u}_\perp\rangle = -\nabla_{\perp S}\bar{p} + \bar{\boldsymbol{F}} + \nabla_{\perp S}\cdot\langle \bar{\boldsymbol{u}}_\perp\bar{\boldsymbol{u}}_\perp\rangle + \nabla_{\perp S}\cdot\langle \boldsymbol{u}'_\perp\boldsymbol{u}'_\perp\rangle + \partial_z\langle \bar{w}\bar{\boldsymbol{u}}_\perp\rangle + \partial_z\langle w'\boldsymbol{u}'_\perp\rangle\\ - \nabla_{\perp S}\cdot\overline{\boldsymbol{u}'_\perp\boldsymbol{u}'_\perp} - \partial_z\overline{w'\boldsymbol{u}'_\perp} - \frac{1}{\alpha}\partial_\eta\overline{w'\boldsymbol{u}'_\perp} + \frac{1}{Re}\left(\nabla^2_{\perp S}\bar{\boldsymbol{u}}_\perp + \partial_z^2\bar{\boldsymbol{u}}_\perp + \frac{1}{\alpha^2}\partial_\eta^2\bar{\boldsymbol{u}}_\perp + \frac{2}{\alpha}\partial_z\partial_\eta\bar{\boldsymbol{u}}_\perp\right) \quad (23)\end{aligned}$$

$$\begin{aligned}(21) - \langle(21)\rangle - \overline{(21) - \langle(21)\rangle} \to \partial_t\boldsymbol{u}'_\perp + \frac{1}{\alpha}\partial_\tau\boldsymbol{u}'_\perp + \left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right)\cdot\nabla_{\perp S}\boldsymbol{u}'_\perp +\\ \frac{1}{\alpha}\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp + \boldsymbol{u}'_\perp\right)\cdot\nabla_{\perp F}\boldsymbol{u}'_\perp + \boldsymbol{u}'_\perp\cdot\nabla_{\perp S}\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp\right) + \left(\langle w\rangle + \bar{w} + w'\right)\partial_z\boldsymbol{u}'_\perp + \frac{1}{\alpha}\left(\langle w\rangle + \bar{w} + w'\right)\partial_\eta\boldsymbol{u}'_\perp\\ + w'\partial_z\left(\langle \boldsymbol{u}_\perp\rangle + \bar{\boldsymbol{u}}_\perp\right) + \frac{1}{\alpha}w'\partial_\eta\bar{\boldsymbol{u}}_\perp + \frac{1}{Ro}\left(\boldsymbol{e}_z\times\boldsymbol{u}'\right) = -\nabla_{\perp S}p' - \frac{1}{\alpha}\nabla_{\perp F}p' + \boldsymbol{F}' + \nabla_{\perp S}\cdot\overline{\boldsymbol{u}'_\perp\boldsymbol{u}'_\perp} + \partial_z\overline{w'\boldsymbol{u}'_\perp}\\ + \frac{1}{\alpha}\partial_\eta\overline{w'\boldsymbol{u}'_\perp} + \frac{1}{Re}\left(\nabla^2_{\perp S}\boldsymbol{u}'_\perp + \frac{2}{\alpha}\nabla_{\perp S}\nabla_{\perp F}\boldsymbol{u}'_\perp + \frac{1}{\alpha^2}\nabla_{\perp F}\boldsymbol{u}'_\perp + \partial_z^2\boldsymbol{u}'_\perp + \frac{2}{\alpha}\partial_z\partial_\eta\boldsymbol{u}'_\perp + \frac{1}{\alpha^2}\partial_\eta^2\boldsymbol{u}'_\perp\right) \quad (24)\end{aligned}$$

2.4 Scaling Arguments for multiscale expansion

3. Adding fast time dependence to Bracket Component

Here we introduce a fast-time dependence to the "bracket" component of the flow variables.

$$\langle q\rangle = \langle q\rangle(x, y, z, t, \tau)$$

This will affect the leading order of the "bracket" equations, i.e. the fast-time derivative will dominate the dynamics of "bracket" components. We see that equations (14), (18), and (22) are modified.

$$\begin{aligned}(14) \to \partial_t\langle b\rangle + \frac{1}{\alpha}\partial_\tau\langle b\rangle + \langle \boldsymbol{u}_\perp\rangle\cdot\nabla_{\perp S}\langle b\rangle + \langle w\rangle\partial_z\langle b\rangle + \langle w\rangle\\ = -\nabla_{\perp S}\cdot\left\langle \bar{\boldsymbol{u}}_\perp\bar{b}\right\rangle - \nabla_{\perp S}\cdot\langle \boldsymbol{u}'_\perp b'\rangle - \partial_z\left\langle \bar{w}\bar{b}\right\rangle - \partial_z\langle w'b'\rangle + \frac{1}{Pe}\left(\nabla^2_{\perp S}\langle b\rangle + \partial_z^2\langle b\rangle\right) \quad (25)\end{aligned}$$

$$
\begin{aligned}
(18) \rightarrow \partial_t \langle w \rangle + \frac{1}{\alpha}\partial_\tau \langle w \rangle + \langle \boldsymbol{u}_\perp \rangle \cdot \nabla_{\perp S} \langle w \rangle + \langle w \rangle \partial_z \langle w \rangle = -\partial_z \langle p \rangle + \frac{1}{Fr^2} \langle b \rangle \\
= -\nabla_{\perp S} \cdot \langle \bar{\boldsymbol{u}}_\perp \bar{w} \rangle - \nabla_{\perp S} \cdot \langle \boldsymbol{u}'_\perp w' \rangle - \partial_z \langle \bar{w}\bar{w} \rangle - \partial_z \langle w' w' \rangle + \frac{1}{Re} \left(\nabla^2_{\perp S} \langle w \rangle + \partial_z^2 \langle w \rangle \right)
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
(22) \rightarrow \partial_t \langle \boldsymbol{u}_\perp \rangle + \frac{1}{\alpha}\partial_\tau \langle \boldsymbol{u}_\perp \rangle + \langle \boldsymbol{u}_\perp \rangle \cdot \nabla_{\perp S} \langle \boldsymbol{u}_\perp \rangle + \langle w \rangle \partial_z \langle \boldsymbol{u}_\perp \rangle + \frac{1}{Ro} \left(\boldsymbol{e}_z \times \langle \boldsymbol{u} \rangle \right) = -\nabla_{\perp S} \langle p \rangle + \langle \boldsymbol{F} \rangle \\
- \nabla_{\perp S} \cdot \langle \bar{\boldsymbol{u}}_\perp \bar{\boldsymbol{u}}_\perp \rangle - \nabla_{\perp S} \cdot \langle \boldsymbol{u}'_\perp \boldsymbol{u}'_\perp \rangle - \partial_z \langle \bar{w} \bar{\boldsymbol{u}}_\perp \rangle - \partial_z \langle w' \boldsymbol{u}'_\perp \rangle + \frac{1}{Re} \left(\nabla^2_{\perp S} \langle \boldsymbol{u}_\perp \rangle + \partial_z^2 \langle \boldsymbol{u}_\perp \rangle \right)
\end{aligned} \tag{27}
$$