AM 260 - Computational Fluid Dynamis: Homework 3

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Problem 1: Lax-Friedrichs Method

(a) Show that the LW method is convergent if $|C_a| \leq 1$.

Consistency Here are the taylor expansions which will be used to show consistency both for 1.a. and 2.a.

$$\begin{aligned}
U_j^{n+1} &= U_j^n + \Delta t U_{t,j}^n + \frac{\Delta t^2}{2} U_{tt,j}^n + \frac{\Delta t^3}{6} U_{ttt,j}^n + \frac{\Delta t^4}{24} U_{tttt,j}^n + O(\Delta t^5) \\
U_{j+1}^n &= U_j^n + \Delta x U_{x,j}^n + \frac{\Delta x^2}{2} U_{xx,j}^n + \frac{\Delta x^3}{6} U_{xxx,j}^n + \frac{\Delta x^4}{24} U_{xxxx,j}^n + O(\Delta x^5) \\
U_{j-1}^n &= U_j^n - \Delta x U_{x,j}^n + \frac{\Delta x^2}{2} U_{xx,j}^n - \frac{\Delta x^3}{6} U_{xxx,j}^n + \frac{\Delta x^4}{24} U_{xxxx,j}^n + O(\Delta x^5)
\end{aligned}$$

$$\begin{aligned}
\Delta t E_{LT} = &\, U_j^n + \Delta t U_{j,t}^n + \frac{\Delta t^2}{2} U_{j,tt}^n + \dots \\
&- \frac{1}{2}\left(U_j^n + \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots \right) \\
&- \frac{1}{2}\left(U_j^n - \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots \right) \\
&+ \frac{a\Delta t}{2\Delta x}\left(U_j^n + \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots \right) \\
&- \frac{a\Delta t}{2\Delta x}\left(U_j^n - \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots \right)
\end{aligned}$$

$$\begin{aligned}
\lim_{\Delta t, \Delta x \to 0} E_{LT} &= \lim_{\Delta t, \Delta x \to 0} \frac{1}{\Delta t}\left(U_j^{n+1} - \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) + \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)\right) \\
&= \lim_{\Delta t, \Delta x \to 0} U_{t,j}^n + \frac{\Delta t}{2} U_{tt,j}^n + O(\Delta t^2) + \frac{\Delta x^2}{2\Delta t} U_{xx,j}^n + O(\Delta x^3) + U_{x,j}^n + \frac{\Delta x^2}{6} U_{xxx,j}^n \\
&= \lim_{\Delta t, \Delta x \to 0} \frac{\Delta t}{2} U_{tt,j}^n + \frac{\Delta x}{2a} U_{xx,j}^n + O(\Delta^2)
\end{aligned}$$

Therefore, we have shown that the local truncation error is bounded by $\Delta t + \Delta x$ with order 1.

Stability Next to show stability we look at the von Neumann stability analysis. We have,

$$\begin{aligned}
G &= \frac{1}{2}\left(e^{ik_x\Delta x} + e^{-ik+x\Delta x}\right) - \frac{C_a}{2}\left(e^{ik_x\Delta x} - e^{-ik+x\Delta x}\right) \\
&= \cos(k_x\Delta x) - iC_a \sin(k_x\Delta x) \\
|G| &= \cos^2(k_x\Delta x) + C_a^2 \sin^2(k_x\Delta x) \\
&= \cos^2(k_x\Delta x) + \sin^2(k_x\Delta x) + (C_a^2 - 1)\sin^2(k_x\Delta x) = 1 - (1 - C_a^2)\sin^2(k_x\Delta x) \leq 1
\end{aligned}$$

Where here, since $|C_a| \leq 1$ we must have that $C_a^2 \leq 1$ and the right most term is negative semi-definite, thereby bounding $|G|$ to ensure stability.

(b) Show that the LF method is $O(\Delta t + \Delta x)$.

This has been shown in the proof for consistency in 1.a.

(c) Rewrite the LF method in the conservative form,

Problem 2: Lax-Wendroff Method

(a) Show that the LW method is convergent if $|C_a| \leq 1$.

In order to demonstrate consistency and stability, we perform taylor expansions to demonstrate consistency (and at which order it is consistent), and then von Neumann stability analysis in order to prove stability.

Consistency

$$\begin{aligned}
\lim_{\Delta t, \Delta x \to 0} E_{LT} &= \lim_{\Delta t, \Delta x \to 0} \frac{1}{\Delta t} U_j^{n+1} - U_j^n + \frac{1}{2} C_a \left(U_{j+1}^n - U_{j-1}^n \right) - \frac{1}{2} C_a^2 \left(U_{j+1}^n - 2U_j^n + U_{j-1}^n \right) \\
&= \lim_{\Delta t, \Delta x \to 0} U_{t,j}^n + \frac{\Delta t}{2} U_{tt,j}^n + \frac{\Delta t^2}{6} U_{ttt,j}^n + O(\Delta t^3) + aU_{x,j}^n + a\frac{\Delta x^2}{6} U_{xxx,j}^n + O(\Delta x^4) \\
&\quad -aC_a \left(\frac{\Delta x}{2} U_{xx,j}^n + \frac{\Delta x^3}{24} U_{xxxx,j}^n + O(\Delta x^5) \right) \\
&= \lim_{\Delta t, \Delta x \to 0} \frac{\Delta t^2}{6} U_{ttt,j}^n + a\frac{\Delta x^2}{6} U_{xxx,j}^n + O(\Delta^3)
\end{aligned}$$

Therefore, we have that this method is consistent with $O(\Delta t^2 + \Delta x^2)$.

Stability

$$\begin{aligned}
G &= (1 - C_a^2) + \frac{1}{2} \left(C_a^2 - C_a \right) e^{ik_x \Delta x} + \frac{1}{2} \left(C_a^2 + C_a \right) e^{-ik_x \Delta x} \\
G &= (1 - C_a^2) + C_a^2 \cos(k_x \Delta x) - iC_a \sin(k_x \Delta x) \\
|G| &= (1 - C_a^2)^2 + C_a^4 \cos^2(k_x \Delta x) + 2(1 - C_a^2) C_a^2 \cos(k_x \Delta x) + C_a^2 \sin^2(k_x \Delta x) \\
&= 1 - 2C_a^2 + C_a^4 + C_a^4 \cos^2() + 2C_a^2 \cos() - 2C_a^4 \cos() + C_a^2 \sin^2() \\
&= 1 + C_a^2 \left(2\cos + \sin^2 - 2 \right) + C_a^4 \left(1 + \cos^2 - 2\cos \right)
\end{aligned}$$

We proceed from here casewise. Take, $|C_a| = 1$. We have,

$$|G| = 1 + 2\cos - 2 + 1 - 2\cos + 1 = 1$$

in which case, the method is stable. We next consider $|C_a| \leq 1$, for this case, it is hard to simplify the RHS (due to the Sinusoidal terms) in order to show that $|G| - 1$ is negative semi-definite. This can however easily be verified using any plotting routine. Here is a link to a desmos graph which shows an animation of $|G| - 1$ and demonstrates the fact that it is a negative semi-definite term .

(b) Show that the LW method is $O(\Delta t^2 + \Delta x^2)$.

This has already been shown in the proof for consistency of the Lax-Wendroff method, whereby the Local Truncation Error is shown to be bounded by Δt^2 and Δx^2.

Problem 3: von Neumann Stability Analysis

We can show that this method is unconditionally unstable with only a few lines of algebra.

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right), \quad U_j^n = G^n e^{ijk_x\Delta x}$$
$$G = 1 - \frac{a\Delta t}{2\Delta x}\left(e^{ik_x\Delta x} - e^{-ik_x\Delta x}\right)$$
$$G = 1 - \frac{a\Delta t}{2\Delta x} i\sin(k_x\Delta x)$$
$$|G| = 1 + \left(\frac{a\Delta t}{2\Delta x}\right)^2 \sin(k_x\Delta x)^2 > 1$$

Therefore, we have that this method is unconditionally unstable, i.e. there is no condition on which $|G| \leq 1$.

Problem 4: Modified Lax-Friedrichs Coefficient

We show the diffusion coefficient for the Lax-Friedrichs method by taylor expanding the original PDE.

$$\Delta t u_t(x,t) + \frac{\Delta t^2}{2}u_{tt}(x,t) = \frac{1}{2}\left(\Delta x^2 u_{xx}(x,t) + \frac{\Delta x^4}{12}\right) - \frac{C_a}{2}\left(2\Delta x u_x + \frac{\Delta x^3}{3}u_{xxx}(x,t)\right)$$
$$u_t + au_x = -\frac{\Delta t}{2}u_{tt} + \frac{\Delta x^2}{2\Delta t}u_{xx} - \frac{a\Delta x^2}{6\Delta t}u_{xxx} + O(\Delta x^4, \Delta t^2)$$
$$u_t + au_x = \frac{\Delta x^2}{2\Delta t}\left(-\frac{a^2\Delta t^2}{\Delta x^2} + 1\right)u_{xx} + O(\Delta x^2, \Delta t^2)$$
$$\kappa = \frac{\Delta x^2}{2\Delta t}\left(1 - \frac{a^2\Delta t^2}{\Delta x^2}\right)$$

Problem 5: von Neumann Analysis of the Heat Equation

In order to show this, we substitute the von Neumann ansatz into the update function.

$$G = 1 + C_k\left(e^{ik_x\Delta x} - 2 + e^{-ik_x\Delta x}\right)$$
$$G = 1 - 2C_k + 2C_k\cos(k_x\Delta x)$$
$$G = 1 + 2C_k(\cos(k_x\Delta x) - 1))$$
$$1 - 4C_k \leq G \leq 1, \quad |G| \leq 1 \implies C_k \leq 0.5$$

Problem 6: Sinusoidal Adv. with LF

Problem 7: Discontinuous IC with LF

Problem 8: Sinusoidal Adv. with LW

Problem 9: Discontinuous IC with LW

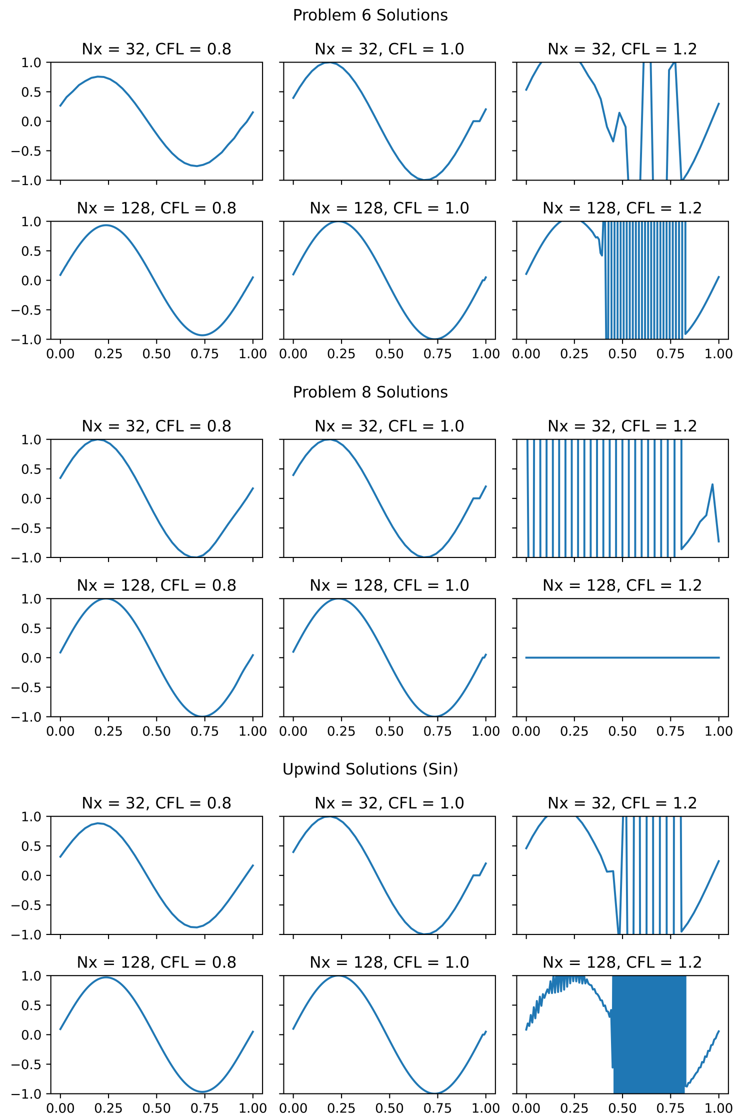

Figure 1: Sharp Discontinuous IC advected by Lax-Friedrichs Method (a), Lax-Wendroff Method (b), and Upwind Method (c) for various grid sizes and CFL numbers.

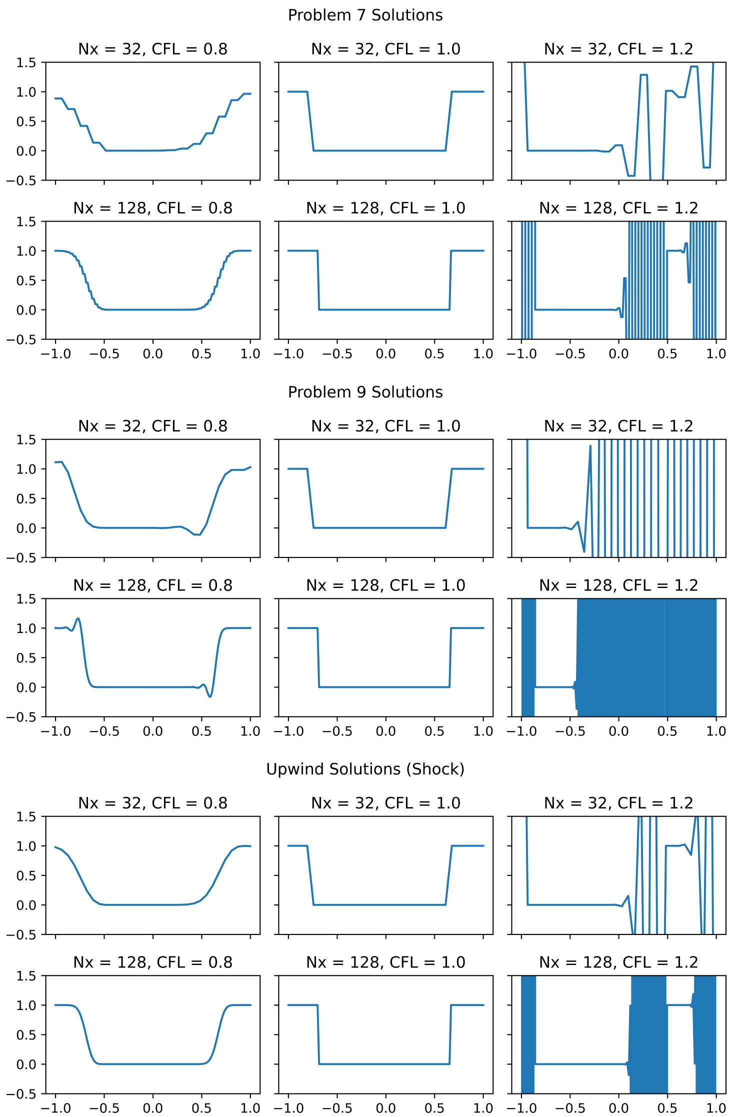

Figure 2: Sharp Discontinuous IC advected by Lax-Friedrichs Method (a), Lax-Wendroff Method (b), and Upwind Method (c) for various grid sizes and CFL numbers.