AM 260 - Computational Fluid Dynamis: Homework 2

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Problem 1: Show equivalency between derivations F1-F4

Problem 2: Solve the Burgers' equation for the following IC

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$
$$u(x,0) = \begin{cases} 2, & |x| < 1/2 \\ -1, & |x| > 1/2 \end{cases}$$

We solve this using the method of characteristics.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau} = 0 \tag{1}$$
$$\frac{\partial t}{\partial \tau} = 1, \quad \frac{\partial x}{\partial \tau} = u \tag{2}$$
$$t = \tau, \quad x = u\tau + s \tag{3}$$

We find that u is constant in τ or rather time and that the slope of each characteristic also does not change in time. We now implement the initial condition upon the characteristic solution. We find,

$$x(s,\tau) = \begin{cases} 2\tau + s, & |s| < 1/2 \\ -\tau + s, & |s| > 1/2 \end{cases} \tag{4}$$

The characteristics can be sketched as shown in Figure 1

Problem 3: Solve the scalar conservation law with subsequent IC

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{e^u}{2}\right) = 0$$
$$u(x,0) = \begin{cases} 2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

We begin to solve this problem by determining which type of discontinuity we have present in the initial condition. We notice that the IC produces values of u such that at $x = 1$ we have a shock wave, and at $x = -1$ we have a rarefraction wave. Thus we determine the solution by identifying the shock speed s and filling in the rarefraction wave. In order to do so we first resolve the shock.

$$F(u) = F'(u) = \frac{1}{2}e^u$$
$$s = \frac{F(2) - F(0)}{2 - 0} \approx \frac{3.19}{2} = 1.595$$

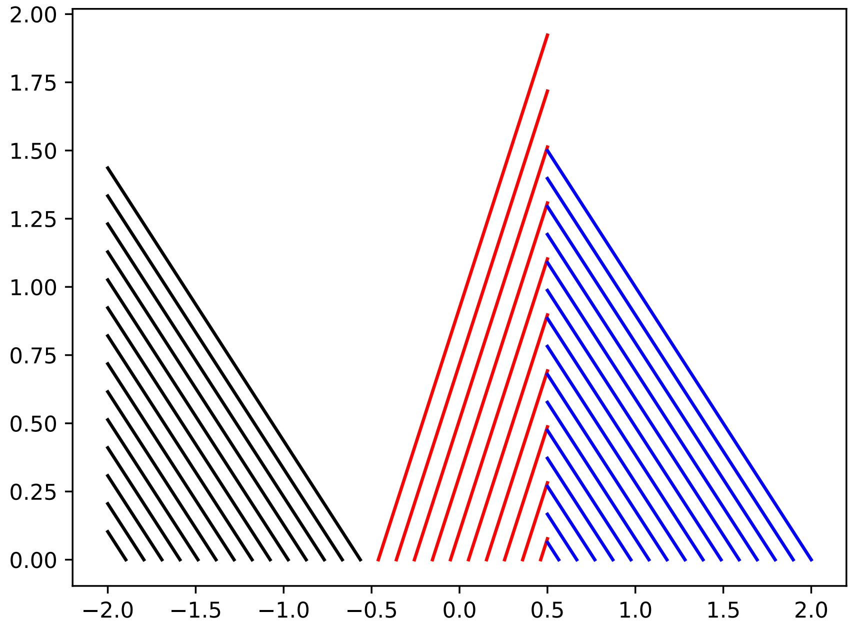

Figure 1: Plot of characteristics ($t \leq t_b$)

We find that the shock propagates with a speed in the x-t plane of 1.595. Note that with this information we can find the time t_b in which the shock front intersects with the tail of the rarefraction wave. We have that the shock front and the right most tail of the rarefraction wave are separated by $\Delta x = 2$. The right tail of the shock has a speed of $c \approx 3.69$. Therefore, we have,

$$3.69t_b = 1.595t_b + 2$$
$$t_b = \frac{2}{2.095} \approx 0.954$$

In order to fill in the rarefraction wave, we adjust the initial condition to be continuous near the discontinuity at $x = -1$. We have,

$$x = \begin{cases} \frac{t}{2} + s & \text{if } s + 1 \leq -\epsilon \\ \frac{te^{s/\epsilon+1}}{2} - 1 & \text{if } -\epsilon < s + 1 < \epsilon \\ \frac{te^2}{2} + s & \text{if } s + 1 \geq \epsilon \end{cases}$$

We can then complete the inverse mapping,

$$s = \begin{cases} x - \frac{t}{2} & \text{if } x \leq \frac{t}{2} - 1 \\ \epsilon\left(\ln\left(\frac{2(x-1)}{t}\right) + 1\right) & \text{if } \frac{t}{2} - 1 < x < \frac{te^2}{2} - 1 \\ x - \frac{te^2}{2} & \text{if } x \geq \frac{te^2}{2} - 1 \end{cases}$$

Problem 4: Weak solutions of the conservation laws

Problem 5: Review on WENO and Numerical Methodology (no response required)