AM 260 - Computational Fluid Dynamics: Homework 3

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Problem 1: Lax-Friedrichs Method

(a) Show that the LW method is convergent if $|C_a| \leq 1$.

Consistency Here are the taylor expansions which will be used to show consistency both for 1.a. and 2.a.

$$U_j^{n+1} = U_j^n + \Delta t U_{t,j}^n + \frac{\Delta t^2}{2} U_{tt,j}^n + \frac{\Delta t^3}{6} U_{ttt,j}^n + \frac{\Delta t^4}{24} U_{tttt,j}^n + O(\Delta t^5)$$
$$U_{j+1}^n = U_j^n + \Delta x U_{x,j}^n + \frac{\Delta x^2}{2} U_{xx,j}^n + \frac{\Delta x^3}{6} U_{xxx,j}^n + \frac{\Delta x^4}{24} U_{xxxx,j}^n + O(\Delta x^5)$$
$$U_{j-1}^n = U_j^n - \Delta x U_{x,j}^n + \frac{\Delta x^2}{2} U_{xx,j}^n - \frac{\Delta x^3}{6} U_{xxx,j}^n + \frac{\Delta x^4}{24} U_{xxxx,j}^n + O(\Delta x^5)$$

$$\begin{aligned}\Delta t E_{LT} = &\, U_j^n + \Delta t U_{j,t}^n + \frac{\Delta t^2}{2} U_{j,tt}^n + \dots \\ &- \frac{1}{2}\left(U_j^n + \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots\right) \\ &- \frac{1}{2}\left(U_j^n - \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots\right) \\ &+ \frac{a\Delta t}{2\Delta x}\left(U_j^n + \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots\right) \\ &- \frac{a\Delta t}{2\Delta x}\left(U_j^n - \Delta x U_{j,x}^n + \frac{\Delta x^2}{2} U_{j,xx}^n + \dots\right)\end{aligned}$$

$$\begin{aligned}\lim_{\Delta t, \Delta x \to 0} E_{LT} &= \lim_{\Delta t, \Delta x \to 0} \frac{1}{\Delta t}\left(U_j^{n+1} - \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) + \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)\right) \\ &= \lim_{\Delta t, \Delta x \to 0} U_{t,j}^n + \frac{\Delta t}{2} U_{tt,j}^n + O(\Delta t^2) + \frac{\Delta x^2}{2\Delta t} U_{xx,j}^n + O(\Delta x^3) + U_{x,j}^n + \frac{\Delta x^2}{6} U_{xxx,j}^n \\ &= \lim_{\Delta t, \Delta x \to 0} \frac{\Delta t}{2} U_{tt,j}^n + \frac{\Delta x}{2a} U_{xx,j}^n + O(\Delta^2)\end{aligned}$$

Therefore, we have shown that the local truncation error is bounded by $\Delta t + \Delta x$ with order 1.

Stability Next to show stability we look at the von Neumann stability analysis. We have,

$$\begin{aligned}G &= \frac{1}{2}\left(e^{ik_x\Delta x} + e^{-ik+x\Delta x}\right) - \frac{C_a}{2}\left(e^{ik_x\Delta x} - e^{-ik+x\Delta x}\right) \\ &= \cos(k_x\Delta x) - iC_a \sin(k_x \Delta x) \\ |G| &= \cos^2(k_x\Delta x) + C_a^2 \sin^2(k_x\Delta x) \\ &= \cos^2(k_x\Delta x) + \sin^2(k_x\Delta x) + (C_a^2 - 1)\sin^2(k_x\Delta x) = 1 - (1 - C_a^2)\sin^2(k_x\Delta x) \leq 1\end{aligned}$$

Where here, since $|C_a| \leq 1$ we must have that $C_a^2 \leq 1$ and the right most term is negative semi-definite, thereby bounding $|G|$ to ensure stability.

(b) Show that the LF method is $O(\Delta t + \Delta x)$.

This has been shown in the proof for consistency in 1.a.

(c) Rewrite the LF method in the conservative form,

In order to solve for the fluxes $\hat{f}^n_{i+1/2}$ we will first solve for $\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2}$ and then isolate each term using addition.

$$\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2} = -\frac{\Delta x}{\Delta t}\left(\frac{1}{2}U^n_{i+1} - U^n_i + \frac{1}{2}U^n_{i-1}\right) + \frac{1}{2}\left(f(U^n_{i+1} - f(U^n_{i-1})\right)$$

Since this holds for any arbitrary i we can add additional terms to this sum in order to increase the distance between the terms on the LHS.

$$\begin{aligned}
\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2} &= -\frac{\Delta x}{\Delta t}\left(\frac{1}{2}U^n_{i+1} - U^n_i + \frac{1}{2}U^n_{i-1}\right) + \frac{1}{2}\left(f(U^n_{i+1}) - f(U^n_{i-1})\right) \\
\hat{f}^n_{i+3/2} - \hat{f}^n_{i+1/2} &= -\frac{\Delta x}{\Delta t}\left(\frac{1}{2}U^n_{i+2} - U^n_{i+1} + \frac{1}{2}U^n_{i}\right) + \frac{1}{2}\left(f(U^n_{i+2}) - f(U^n_{i})\right) \\
\hat{f}^n_{i-1/2} - \hat{f}^n_{i-3/2} &= -\frac{\Delta x}{\Delta t}\left(\frac{1}{2}U^n_{i} - U^n_{i-1} + \frac{1}{2}U^n_{i-2}\right) + \frac{1}{2}\left(f(U^n_{i}) - f(U^n_{i-2})\right)
\end{aligned}$$

Adding these three equations together, we find some terms cancel. We have remaining,

$$\hat{f}^n_{i+3/2} - \hat{f}^n_{i-3/2} = -\frac{\Delta x}{2\Delta t}\left(U^n_{i+2} - U^n_{i+1} + U^n_{i-2} - U^n_{i-1}\right) + \frac{1}{2}\left(f(U^n_{i+2}) + f(U^n_{i+1}) - f(U^n_{i-2}) - f(U^n_{i-1})\right)$$

We are able to deduce the value for each of the LHS terms separately using symmetry and by the fact that each flux should only have information from the two indices adjacent ($i+3/2$ should only be influenced by $i+1, i+2$ and similar argument for $i-3/2$). Thus we have,

$$\begin{aligned}
\hat{f}^n_{i+3/2} &= -\frac{\Delta x}{2\Delta t}\left(U^n_{i+2} - U^n_{i+1}\right) + \frac{1}{2}\left(f(U^n_{i+2}) + f(U^n_{i+1})\right) \\
\hat{f}^n_{i+1/2} &= -\frac{\Delta x}{2\Delta t}\left(U^n_{i+1} - U^n_{i}\right) + \frac{1}{2}\left(f(U^n_{i+1}) + f(U^n_{i})\right)
\end{aligned}$$

Problem 2: Lax-Wendroff Method

(a) Show that the LW method is convergent if $|C_a| \leq 1$.

In order to demonstrate consistency and stability, we perform taylor expansions to demonstrate consistency (and at which order it is consistent), and then von Neumann stability analysis in order to prove stability.

Consistency

$$\begin{aligned}
\lim_{\Delta t, \Delta x \to 0} E_{LT} &= \lim_{\Delta t, \Delta x \to 0} \frac{1}{\Delta t}U^{n+1}_j - U^n_j + \frac{1}{2}C_a\left(U^n_{j+1} - U^n_{j-1}\right) - \frac{1}{2}C_a^2\left(U^n_{j+1} - 2U^n_j + U^n_{j-1}\right) \\
&= \lim_{\Delta t, \Delta x \to 0} U^n_{t,j} + \frac{\Delta t}{2}U^n_{tt,j} + \frac{\Delta t^2}{6}U^n_{ttt,j} + O(\Delta t^3) + aU^n_{x,j} + a\frac{\Delta x^2}{6}U^n_{xxx,j} + O(\Delta x^4) \\
&\quad -aC_a\left(\frac{\Delta x}{2}U^n_{xx,j} + \frac{\Delta x^3}{24}U^n_{xxxx,j} + O(\Delta x^5)\right) \\
&= \lim_{\Delta t, \Delta x \to 0} \frac{\Delta t^2}{6}U^n_{ttt,j} + a\frac{\Delta x^2}{6}U^n_{xxx,j} + O(\Delta^3)
\end{aligned}$$

Therefore, we have that this method is consistent with $O(\Delta t^2 + \Delta x^2)$.

Stability

$$G = (1 - C_a^2) + \frac{1}{2}\left(C_a^2 - C_a\right) e^{ik_x\Delta x} + \frac{1}{2}\left(C_a^2 + C_a\right) e^{-ik_x\Delta x}$$
$$G = (1 - C_a^2) + C_a^2 \cos(k_x\Delta x) - iC_a \sin(k_x\Delta x)$$
$$|G| = (1 - C_a^2)^2 + C_a^4 \cos^2(k_x\Delta x) + 2(1 - C_a^2)C_a^2 \cos(k_x\Delta x) + C_a^2 \sin^2(k_x\Delta x)$$
$$= 1 - 2C_a^2 + C_a^4 + C_a^4 \cos^2() + 2C_a^2 \cos() - 2C_a^4 \cos() + C_a^2 \sin^2()$$
$$= 1 + C_a^2\left(2\cos + \sin^2 - 2\right) + C_a^4\left(1 + \cos^2 - 2\cos\right)$$

We proceed from here casewise. Take, $|C_a| = 1$. We have,

$$|G| = 1 + 2\cos - 2 + 1 - 2\cos + 1 = 1$$

in which case, the method is stable. We next consider $|C_a| \leq 1$, for this case, it is hard to simplify the RHS (due to the Sinusoidal terms) in order to show that $|G| - 1$ is negative semi-definite. This can however easily be verified using any plotting routine. Here is a link to a desmos graph which shows an animation of $|G| - 1$ and demonstrates the fact that it is a negative semi-definite term .

(b) Show that the LW method is $O(\Delta t^2 + \Delta x^2)$.

This has already been shown in the proof for consistency of the Lax-Wendroff method, whereby the Local Truncation Error is shown to be bounded by Δt^2 and Δx^2.

Problem 3: von Neumann Stability Analysis

We can show that this method is unconditionally unstable with only a few lines of algebra.

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right), \quad U_j^n = G^n e^{ijk_x\Delta x}$$
$$G = 1 - \frac{a\Delta t}{2\Delta x}\left(e^{ik_x\Delta x} - e^{-ik_x\Delta x}\right)$$
$$G = 1 - \frac{a\Delta t}{2\Delta x} i \sin(k_x\Delta x)$$
$$|G| = 1 + \left(\frac{a\Delta t}{2\Delta x}\right)^2 \sin(k_x\Delta x)^2 > 1$$

Therefore, we have that this method is unconditionally unstable, i.e. there is no condition on which $|G| \leq 1$.

Problem 4: Modified Lax-Friedrichs Coefficient

We show the diffusion coefficient for the Lax-Friedrichs method by taylor expanding the original PDE.

$$\Delta t u_t(x,t) + \frac{\Delta t^2}{2} u_{tt}(x,t) = \frac{1}{2}\left(\Delta x^2 u_{xx}(x,t) + \frac{\Delta x^4}{12}\right) - \frac{C_a}{2}\left(2\Delta x u_x + \frac{\Delta x^3}{3} u_{xxx}(x,t)\right)$$
$$u_t + au_x = -\frac{\Delta t}{2} u_{tt} + \frac{\Delta x^2}{2\Delta t} u_{xx} - \frac{a\Delta x^2}{6\Delta t} u_{xxx} + O(\Delta x^4, \Delta t^2)$$
$$u_t + au_x = \frac{\Delta x^2}{2\Delta t}\left(-\frac{a^2\Delta t^2}{\Delta x^2} + 1\right) u_{xx} + O(\Delta x^2, \Delta t^2)$$
$$\kappa = \frac{\Delta x^2}{2\Delta t}\left(1 - \frac{a^2\Delta t^2}{\Delta x^2}\right)$$

This diffusion coefficient is interesting because it is inversely proportional to the temporal discretization, i.e. smaller timesteps make the method more diffuse. This will cause the velocity field to dampen more

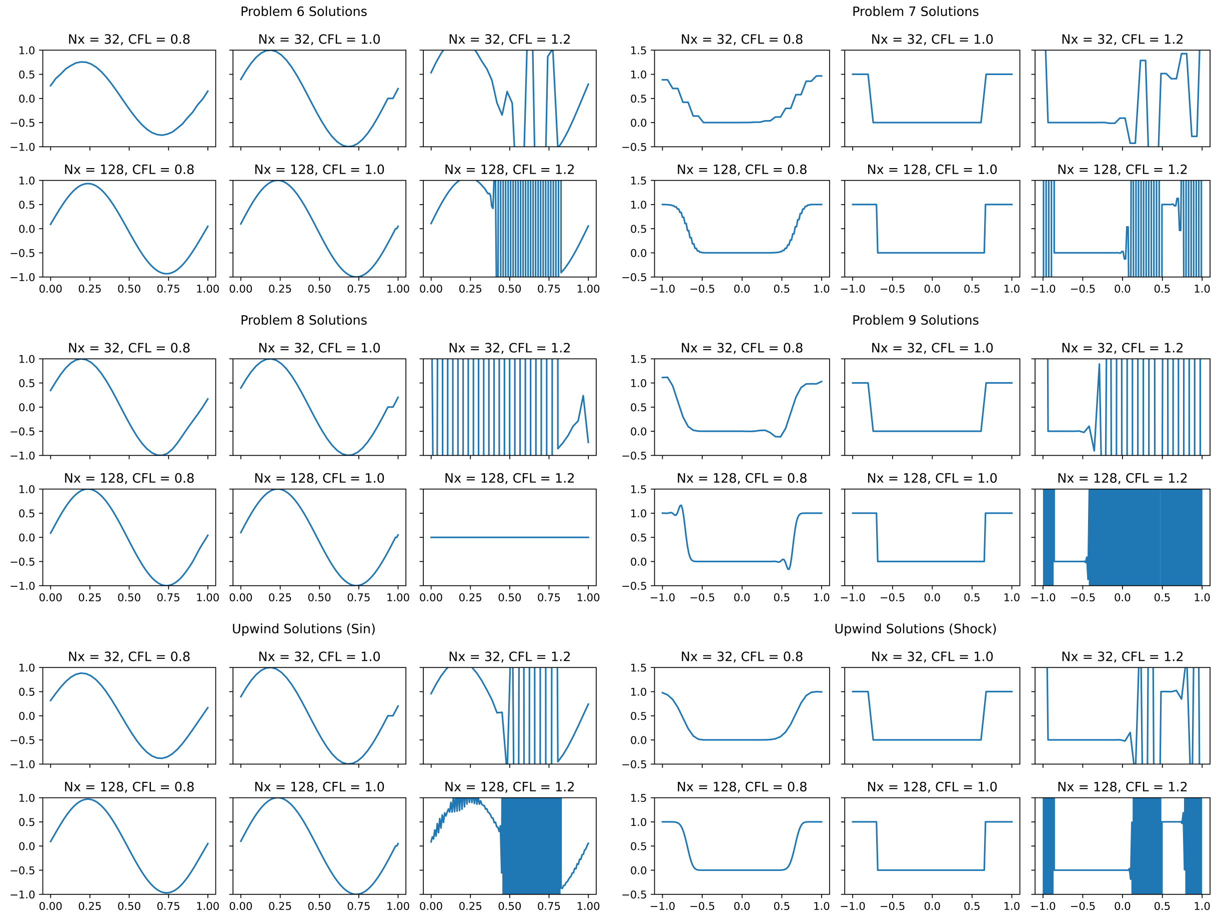

Figure 1: Various initial conditions advected by Lax-Friedrichs Method (top), Lax-Wendroff Method (middle), and Upwind Method (bottom) for various grid sizes and CFL numbers. The left column shows the "Sin Wave" initial condition and the right column shows the "Shock/Rarefraction Wave" initial condition.

quickly as the timestep becomes smaller. At the same time it is stabilized by the gridsize, i.e. smaller spatial discretizations will cause the diffusion coefficient to become smaller (as it varies proportional to the square of Δx).

Problem 5: von Neumann Analysis of the Heat Equation

In order to show this, we substitute the von Neumann ansatz into the update function.

$$G = 1 + C_k \left(e^{ik_x\Delta x} - 2 + e^{-ik_x\Delta x}\right)$$
$$G = 1 - 2C_k + 2C_k\cos(k_x\Delta x)$$
$$G = 1 + 2C_k(\cos(k_x\Delta x) - 1))$$
$$1 - 4C_k \leq G \leq 1, \quad |G| \leq 1 \implies C_k \leq 0.5$$

Problem 6: Sinusoidal Adv. with LF

My numerical implementation of the Lax-Friedrichs method for the sin wave initial condition was done in Fortran and then visualized using Python. The results can be seen in Figure 1 alongside comparisions of the same solution using the Lax-Wendroff method and the Upwind method. These are all plots of the solution at $t = t_{cycle}$ which is computed by considering the advection speed compared to the natural length scale of the system, i.e.

$$t_{cycle} = \frac{L_x}{c} = \frac{1}{1} = 1$$

Since the Lax-Friedrichs method is diffusive (as shown in an earlier homework problem), we expect the amplitude of the sin wave to decrease with time, which is indeed what we see for the lowest resolution result. As expected, once the CFL number approaches and then increases past one, the method is no longer stable. This is visualized by the oscillations seen in the CFL = 1.2 plots.
Compared to the upwind method, we see that the LF method appears to have slighly worse errors for the same grid sizes. This is most apparent in the $CFL = 1.2$ cases where the oscillations away from the true solution have a much larger length scale compared to the upwind solutions. Otherwise, the method seems to be very similar to the upwind method.

Problem 7: Discontinuous IC with LF

The same methodology and style of results for problem 6 can be seen in Figure 1. Here is LF method is used in the same way as before, only with a different initial condition. Unlike before, where the upwind and LF method appears to be very similar, we see that the LF is much worse at resolving shocks and rarefraction waves. In the $Nx = 32$ and $CFL = 0.8$ we see very jagged edges on the contour of the shock front and rarefraction fan. This appears to be partially caused by the resolution of the system, but also speaks to the inability of the method. The upwind method does not preserve the shock front well either, but its solution is at least smooth. Both fail quite spectacularly (as expected) at $CFL = 1.2$.

Problem 8: Sinusoidal Adv. with LW

See Figure 1 for the results. Unlike the LF method, this methods reaction to a large CFL number appears to to be quite erratic. The $Nx = 128$ and $CFL = 1.2$ appears to have diffused completely, while the $Nx = 32$ and $CFL = 1.2$ appears to have blown up completely. Besides this, the $CFL = 0.8, 1$ are very reasonable solutions and appear to be quite stable.

Problem 9: Discontinuous IC with LW

See Figure 1 for the results. Perhaps the most interesting aspect about the solutions to this method is that (because the method is not diffusive), we notice rather than a degredation of the shock front / rarefraction wave, we see oscillations (similar to Gibbs phenomenon) around the discontinuity points. This is especially visible in the $Nx = 128$ and $CFL = 0.8$ plot. Again, the method appears to be highly sensitive to the CFL number (the 1.2 plots particularly). If I've learned anything its that I should never have a CFL number above 0.8.