AM 260 - Computational Fluid Dynamis: Homework 2

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Problem 1: Show equivalency between derivations F1-F4

We will proceed by showing the equivalencies in the following order as suggested: $(F1) \to (F2) \to (F4) \to (F3) \to (F1)$:
We begin with the first conversion.

$$
\begin{gathered}
(F1): \frac{\partial}{\partial t}\int_V \rho dV = -\int_{\delta V} \rho \boldsymbol{u} \cdot dS \\
\frac{\partial}{\partial t}\int_V \rho dV + \int_V \nabla \cdot \rho \boldsymbol{u} dV = 0 \\
\frac{D}{Dt}\int_V \rho dV + \int_V \rho \nabla \cdot \boldsymbol{u} dV = 0 \\
\frac{D}{Dt}\int_V \rho dV = 0
\end{gathered}
$$

Where here the last term is dropped due to the conversion to the FCV moving with the flow and no longer remaining stationary.
The transition from $(F2) \to (F4)$ is much more straightforward. We simply consider the integral formulation at a very specific control volume such that the equality must be considered pointwise.

$$
\frac{D}{Dt}\int_{\delta V} \rho dV = \frac{D}{Dt}(\rho \delta V) = \frac{D\rho}{Dt}\delta V + \rho \frac{D\delta V}{Dt} = \frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0
$$

Next we consider the transformation from $(F4) \to (F3)$. We simply have to simplify the equation in a different way. We have,

$$
\begin{gathered}
\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = 0 \\
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} = 0
\end{gathered}
$$

Thus the conservative and non-conservative forms are exactly the same, simply expressed algebraically distinctly. Finally, we recover the first expression by integrating this quantity over some arbitrary control volume. We have,

$$
\begin{aligned}
\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} dV &= \int_V \frac{\partial \rho}{\partial t} dV + \int_{\delta V} \rho \boldsymbol{u} \cdot dS \\
&= \frac{\partial}{\partial t}\int_V \rho dV + \int_{\delta V} \rho \boldsymbol{u} \cdot dS
\end{aligned}
$$

where the last step is possible because the FCV does not change in time, only the density of that FCV.

Problem 2: Solve the Burgers' equation for the following IC

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$
$$u(x,0) = \begin{cases} 2, & |x| < 1/2 \\ -1, & |x| > 1/2 \end{cases}$$

We solve this using the method of characteristics.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau} = 0 \tag{1}$$
$$\frac{\partial t}{\partial \tau} = 1, \quad \frac{\partial x}{\partial \tau} = u \tag{2}$$
$$t = \tau, \quad x = u\tau + s \tag{3}$$

We find that u is constant in τ or rather time and that the slope of each characteristic also does not change in time. We notice at there are two discontinuities in the initial condition leading to a shock wave at $x = 1/2$ and a rarefraction wave originating from $x = -1/2$. We attempt to find the piecewise solution in the $x - t$ plane for these two discontinuities. For the shock wave, we look to the Rankine-Hugonoit condition in order to determine the propagation speed of the shock front. We have,

$$s = \frac{F(u_R) - F(u_L)}{u_R - u_L} = \frac{1/2 - 2}{-1 - 2} = \frac{1}{2}$$

Thus we have that the shock front will have the characteristic defined by $x = t/2 + \frac{1}{2}$. Next we look to fill in the rarefraction wave using the notion of self-similarity. That is we center the wave at $x = -1/2$ find the solution u to be self-similar with respect to $(x + 1/2)/t$. We have,

$$u(x,t) = \begin{cases} -1 & \text{if } x < -t - \frac{1}{2} \\ \frac{x+1/2}{t} & \text{if } -t < x + \frac{1}{2} < 2t \\ 2 & \text{if } 2t - \frac{1}{2} < x < \frac{t}{2} + \frac{1}{2} \\ -1 & \text{if } x > \frac{t}{2} + \frac{1}{2} \end{cases}$$

Notice that the third case for the solution in the $x - t$ plane only exists for a finite time t_b. We solve for this time where the rarefraction wave and shock front intersect. Specifically we have,

$$2t_b - \frac{1}{2} = \frac{t_b}{2} + \frac{1}{2}$$
$$\frac{3}{2}t_b = 1 \implies t_b = \frac{2}{3}$$

A drawing of the characteristics in the $x - t$ plane can be seen in Figure 1

Problem 3: Solve the scalar conservation law with subsequent IC

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{e^u}{2}\right) = 0$$
$$u(x,0) = \begin{cases} 2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

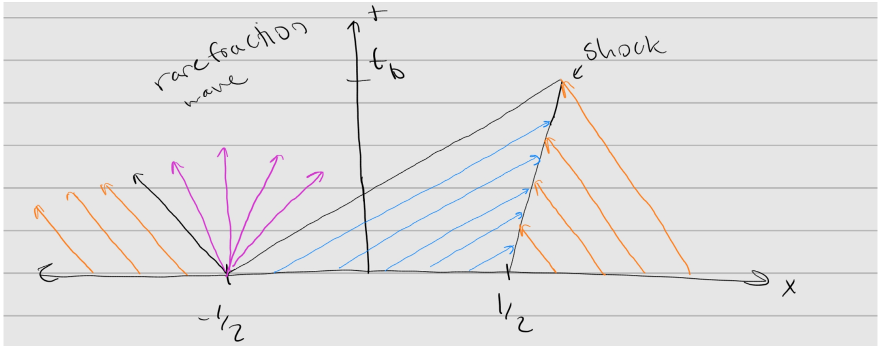

Figure 1: Drawing of the characteristics from problem 2.

We begin to solve this problem by determining which type of discontinuity we have present in the initial condition. We notice that the IC produces values of u such that at $x = 1$ we have a shock wave, and at $x = -1$ we have a rarefraction wave. Thus we determine the solution by identifying the shock speed s and filling in the rarefraction wave. In order to do so we first resolve the shock.

$$F(u) = F'(u) = \frac{1}{2}e^u$$
$$s = \frac{F(2) - F(0)}{2 - 0} \approx \frac{3.19}{2} = 1.595$$

We find that the shock propagates with a speed in the x-t plane of 1.595. Note that with this information we can find the time t_b in which the shock front intersects with the tail of the rarefraction wave. We have that the shock front and the right most tail of the rarefraction wave are separated by $\Delta x = 2$. The right tail of the shock has a speed of $c \approx 3.69$. Therefore, we have,

$$3.69t_b = 1.595t_b + 2$$
$$t_b = \frac{2}{2.095} \approx 0.954$$

In order to fill in the rarefraction wave, we impose self-similarity with respect to $(x + 1)/t$.

$$\frac{e^u}{2} = \frac{x+1}{t}$$
$$u = \ln\left(\frac{2(x+1)}{t}\right)$$
$$u(x,t) = \begin{cases} 0 & \text{if } x < \frac{t}{2} - 1 \\ \ln\left(\frac{2(x+1)}{t}\right) & \text{if } \frac{t}{2} < x + 1 < \frac{e^2 t}{2} \\ 2 & \text{if } \frac{e^2 t}{2} - 1 < x < \frac{(e^2-1)t}{4} + 1 \\ 0 & \text{if } x > \frac{(e^2-1)t}{4} + 1 \end{cases}$$

A drawing of the characteristics can be seen in Figure 2

Problem 4: Weak solutions of the conservation laws

Part A:
For this section we must show that the the solution,

$$u(x,t) = \begin{cases} 1 & \text{for } x < t/2 \\ 0 & \text{for } x > t/2 \end{cases}$$

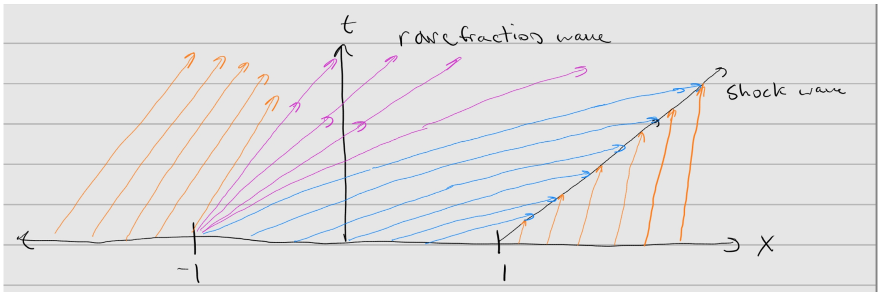

Figure 2: Drawing of the characteristics from problem 3.

is a weak solution of the burgers equation. In order to do so, we look at the definition of a weak solution, i.e.

$$\int_0^\infty \int_{-\infty}^\infty \frac{\partial \phi}{\partial t} u + \frac{\partial \phi}{\partial x} f(u) dx dt = -\int_\infty^\infty \phi(x,0)u(x,0)dx$$

We begin by evaluating the first portion of the integral with integration by parts. We have,

$$\int \int \phi_t u dx dt = \int \phi u dx \Big|_0^\infty - \int \int \phi u_t dx dt$$

After subsituting into the original equation we find,

$$\int \phi u dx \Big|_0^\infty + \int \int -\phi u_t + \phi_x f(u) dx dt$$

$$\int \phi u dx \Big|_0^\infty + \int \int \phi f_x + \phi_x f dx dt$$

$$\int \phi u dx \Big|_0^\infty + \int \int \frac{\partial}{\partial x} (\phi f) dx dt$$

$$\int \phi u dx \Big|_0^\infty + \int \phi f \Big|_{-\infty}^\infty dt$$

$$\int \phi u dx \Big|_0^\infty = -\int_{\mathbb{R}} \phi(x,0)u(x,0)dx$$

Part B:
We begin this problem by examining the integral a bit closer. We have,

$$L = \int_a^b u dx = \begin{cases} 0 & \text{if } \frac{t}{2} < a < b \\ b-a & \text{if } a < b < \frac{t}{2} \\ \frac{t}{2} - a & \text{if } a < \frac{t}{2} < b \end{cases}$$

Notice that in only one of these cases does the integral have a non-zerp derivative in time. We look at this case specifically and evaluate the value of that time derivative. We have,

$$\frac{\partial L}{\partial t} = \begin{cases} 0 & \text{if } \frac{t}{2} < a < b \\ 0 & \text{if } a < b < \frac{t}{2} \\ \frac{1}{2} & \text{if } a < \frac{t}{2} < b \end{cases}$$

We look then at the specific values for $F(a,t)$ and $F(b,t)$. We have,

$$\left(u(a,t)^2 - u(b,t)^2\right)/2 = \begin{cases} 0 & \text{if } \frac{t}{2} < a < b \\ 0 & \text{if } a < b < \frac{t}{2} \\ \frac{1}{2} & \text{if } a < \frac{t}{2} < b \end{cases}$$

Therefore we have that the given u satisfies the integral form of the conservation law.

Problem 5: Review on WENO and Numerical Methodology (no response required)