AM 260, Spring 2025
Homework 2

Posted on Thu, Apr 17, 2025
Due 11:59 pm, Sat, Apr 26, 2025

- You are encouraged to use LaTex or MS-words like text editors for homework. A scanned copy of a handwritten solutions will still be accepted on condition that your handwriting is clean and well-organized, and your scanned copy is fully readable.

Problem 1 In Chapter 1, we derived four different equations assuming four different models. Show that all four approaches discussed in (F1)-(F4) for the continuity equation are in fact all equivalent mathematically. That is, one of them can be obtained from any of the others. (Hint: You can show that there are equivalent relationships in a loop: (F1) $\Rightarrow$ (F2) $\Rightarrow$ (F4) $\Rightarrow$ (F3) $\Rightarrow$ (F1)).

Problem 2 Solve Burgers' equation on $\mathbb{R}$ for small enough $t \le t_b$ that allows the exact piecewise-linear weak solution with the following initial conditions:

$$u(x,0) = \begin{cases} 2 & \text{if } |x| < 1/2 \\ -1 & \text{if } |x| > 1/2 \end{cases} \tag{1}$$

Find the time t_b when the tail of the rarefaction and the shock wave first intersect each other. Draw a wave diagram for the weak solution in the x-t plane.

Problem 3 Consider the scalar conservation law $u_t + (\frac{e^u}{2})_x = 0$ with initial data $u(x,0) = u_0(x)$:

$$u_0(x) = \begin{cases} 2 \text{ if } -1 < x < 1, \\ \\ 0 \text{ otherwise }. \end{cases} \tag{2}$$

(a) Sketch the characteristics and shock paths in the x-t plane. Please clearly identify the exact solution in each compression and rarefaction region in the x-t plane. Use $e^2 \approx 7.38$.

(b) Find $t = t_b$ at which the shock and the expansion fan begin to cross.

Problem 4 Let $u(x,t)$ be defined for $(x,t) \in \mathbb{R}^2$ by

$$u(x,t) = \begin{cases} 1 \text{ for } x < t/2 \\ \\ 0 \text{ for } x > t/2. \end{cases} \tag{3}$$

(a) By using the definition of a weak solution, show that u is a weak solution of $u_t + uu_x = 0$. Please assume your test functions $\phi(x,t)$ is continuously differentiable with compact support, i.e., $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$.

(b) Show that u satisfies the integral form

$$\frac{d}{dt}\int_a^b u(x,t)dx = F(a,t) - F(b,t) \tag{4}$$

of the conservation law when $F(u) = \frac{u^2}{2}$. (Hint: Consider three cases: (i) $t/2 < a < b$, (ii) $a < t/2 < b$, and (iii) $a < b < t/2$.)

Problem 5 On the course Canvas page, go to "Other reading materials" and find the first review paper "Review 1 on WENO" under the section, "WENO methods in FDM, FVM, and DG". This paper is by Prof. Chi-Wang Shu, "High Order Weighted Essentially Non-Oscillatory Schemes for Convection Dominated Problems." Read and study Sections 3.1 – 3.4 therein, focusing particularly on the mathematical differences between finite volume and finite difference methods. Nothing needs to be submitted for this assignment, but make sure you study it as the paper covers a couple of basic mathematical concepts that are important in CFD.

For further study, you can also read Chapter 3 of the lecture note, which is another journal article by Joaquim Peiró and Spencer Sherwin.