AM 260, Spring 2025
Homework 3

Posted on Tue, Apr 29, 2025
Due 11:59 pm, Mon, May 12, 2025

- You are recommended to use LaTex or MS-words like text editors for homework. A scanned copy of a handwritten solutions will still be accepted on condition that your handwriting is clean and well-organized, and your scanned copy is fully readable.

Part 1: Theory Problems

Problem 1. Consider the Lax-Friedrichs (LF) method for solving the scalar advection $u_t + f(u)_x = 0$ with $f(u) = au$, where $a > 0$ or $a < 0$,

$$U_i^{n+1} = \frac{1}{2}\Big(U_{i+1}^n + U_{i-1}^n\Big) - \frac{\Delta t}{2\Delta x}\Big(f(U_{i+1}^n) - f(U_{i-1}^n)\Big). \tag{1}$$

(a) Show that the LF method is consistent and stable for $|C_a| \leq 1$, where $C_a = \frac{a\Delta t}{\Delta x}$.

(b) Show that the LF method is $\mathcal{O}(\Delta t + \Delta x)$.

(c) Rewrite the LF method in the conservative form,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\Big(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n\Big), \tag{2}$$

that is to say, find the expressions for $\hat{f}_{i\pm 1/2}^n$ as functions of U_k^n and the original flux $f(U_k^n)$, $k = -1, 0, 1$.

Problem 2. Consider the Lax-Wendroff (LW) method for solving the scalar advection $u_t + au_x = 0$ with $f(u) = au$, where $a > 0$ or $a < 0$, and $C_a = \frac{a\Delta t}{\Delta x}$,

$$U_i^{n+1} = U_i^n - \frac{C_a}{2}\Big(U_{i+1}^n - U_{i-1}^n\Big) + \frac{C_a^2}{2}\Big(U_{i+1}^n - 2U_i^n + U_{i-1}^n\Big). \tag{3}$$

(a) Show that the LW method is consistent and stable if $|C_a| \leq 1$.

(b) Show that the LW method is $\mathcal{O}(\Delta t^2 + \Delta x^2)$.

Problem 3. Use the von Neumann analysis to show that the forward in time centered in space scheme (FTCS) for the advection $u_t + au_x = 0$ with $a > 0$ or $a < 0$,

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) \tag{4}$$

is unconditionally unstable (i.e., unstable for any choices of $\Delta t > 0$).

Problem 4. Find the modified equation of the Lax-Friedrichs method and show the diffusion coefficient is given as

$$\kappa = \frac{a\Delta x}{2C_a}\left(1 - C_a^2\right), \quad \text{where } C_a = \frac{a\Delta t}{\Delta x}. \tag{5}$$

Treat C_a as a constant instead of treating its components, $a, \Delta t$, and Δx, separately. Discuss your findings.

Problem 5. Use the von Neumann stability analysis to show that the CFL condition for the 1D heat equation

$$U_j^{n+1} = U_j^n + C_k\Big(U_{j+1}^n - 2U_j^n + U_{j-1}^n\Big), \tag{6}$$

with $C_k = \kappa\frac{2\Delta t}{\Delta x^2}$, becomes $C_a \le 1$.

Part 2: Coding Problems

For graduate students, use Fortran 90 or C to implement the following schemes. If you are an undergraduate student, you can use MATLAB or Python instead. A template MATLAB code for the upwind method to solve the linear scalar advection $u_t + au_x = 0$ is available as an example. Study this MATLAB code first. To learn the basic discretization strategies, take a look at the separate document, "Note on the basic discretization setup."

Problem 6. Implement the LF method in Eqn. (1) to numerically solve the sinusoidal advection problem

$$u_t + au_x = 0, \quad a = 1, \tag{7}$$

with an IC: $u(x, 0) = \sin(2\pi x)$, on $x \in [0, 1]$. Use the periodic boundary condition on both ends at $x = 0$ and $x = 1$.

Run your code on two different grid resolutions of $N = 32, 128$ with CFL numbers of 0.8, 1.0, and 1.2. Show your plots at $t = t_{cycle1}$ at all two grid resolutions, where t_{cycle1} is the time the sinusoidal wave returns to the initial position (Hint: You can easily find t_{cycle1} analytically first. How?). Describe your findings and compare the LF results with the first-order upwind method provided in the MATLAB code. You should be able to implement the upwind method in your code following the MATLAB code example.

Problem 7. Repeat the comparison study in Problem 6 on $[-1, 1]$ using a discontinuous initial condition,

$$u(x, 0) = \begin{cases} 1 \text{ for } |x| < 1/3, \\ \\ 0 \text{ for } 1/3 < |x| \leq 1. \end{cases} \tag{8}$$

As before, use the periodic boundary condition on both ends at $x = -1$ and $x = 1$. Use the same sets of grid resolutions, the CFL numbers, and $t = t_{cycle1}$ as in Problem 6.

Problem 8. Repeat Problem 6 using the LW method in Eqn. (3).

Problem 9. Repeat Problem 7 using the LW method in Eqn. (3).