Homework 1

Gomez - Math 19B

Due: Jan 19th, 2024

All exercises are taken from Section 5.1 in the textbook

1. Exercise 8: Let $f(x) = x^2 + x - 2$.

   - Calculate R_3 and L_3 over [2, 5]. NOTE: R_3 and L_3 denote the right and left endpoint approximations of the Area under $f(x)$, each made of three rectangles/intervals.

     $$R_3 = \Delta x(h_1 + h_2 + h_3) \qquad L_3 = \Delta x(h_1 + h_2 + h_3)$$

     $$R_3 = (1)(f(3) + f(4) + f(5)) \qquad L_3 = (1)(f(2) + f(3) + f(4))$$

     $$R_3 = 10 + 18 + 28 = 56 \qquad L_3 = 4 + 10 + 18 = 32$$

     $R_3 = 56$, $L_3 = 32$

   - Sketch the graph of f and the rectangles that make up each approximation. Is the area under the graph larger or smaller than R_3? The area under f(x) is smaller than R_3

     Than L_3 ? The area under f(x) is larger than L_3

2. Derive the Midpoint Approximation Formula, M_n, for a function, $f(x)$, on the interval $[a, b]$ through the process of this problem.

   - Find the base width for each rectangle, Δx, in terms of n.

     $$\Delta x = \frac{\text{length of interval}}{\text{number of rectangles}} = \frac{b-a}{n}$$

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   - Find the height of the i-th rectangle, h_i, in terms of f and x_i, where x_i denotes the midpoint of each interval.

     $$h_i = \text{height of function at midpoint} = f(x_i)$$

     $h_i = f(x_i)$

- Find the midpoint of the i-th subinterval, x_i, in terms of i, Δx.

$$x_i = \text{midpoint of i-th rectangle} = a + \Delta x \cdot \frac{i}{2}$$

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- Find the area of the i-th rectangle, A_i, in terms of i, f, and Δx using the equation for the area of a rectangle.

$$A_i = b \cdot h_i = \Delta x \cdot f(x_i) = \Delta x \cdot f(a + \Delta x \cdot \frac{i}{2})$$

$A_i = \Delta x \cdot f(a + \Delta x \cdot \frac{i}{2})$

- Express the midpoint approximation, M_n, as the sum of the areas of all rectangles in the partition in terms of i, Δx, and f.

$$M_n = \sum_{i=1}^{n} \Delta x \cdot f(a + \Delta x \cdot \frac{i}{2}) = \Delta x \sum_{i=1}^{n} f(a + \Delta x \cdot \frac{i}{2})$$

$M_n = \Delta x \sum_{i=1}^{n} f(a + \Delta x \cdot \frac{i}{2}) = \frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{i(b-a)}{2n})$ Either answer is fine, since I forgot to specify to write the answer in terms of n, f, and i.

Formulas (3)-(5)

$$\sum_{i=1}^{N} i = 1 + 2 + \cdots + N = \frac{N(N+1)}{2} = \frac{N^2}{2} + \frac{N}{2} \tag{3}$$

$$\sum_{i=1}^{N} i^2 = 1 + 4 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \tag{4}$$

$$\sum_{i=1}^{N} i^3 = 1 + 8 + \cdots + N^3 = \frac{N^2(N+1)^2}{4} = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4} \tag{5}$$

3. Exercise 44: Compute the sum using linearity and the formulas (3)-(5) for power sums covered in the textbook (pg. 304)

$$\sum_{i=1}^{10} (i^3 - 2i^2)$$

$$\sum_{i=1}^{10} (i^3 - 2i^2) = \sum_{i=1}^{10} i^3 - 2\sum_{i=1}^{10} i^2$$

$$= \frac{10^4}{4} + \frac{10^3}{2} + \frac{10^2}{4} - 2\left[\frac{10^3}{3} + \frac{10^2}{2} + \frac{10}{6}\right]$$

$$= 3025 - 770 = 2255$$

$\sum_{i=1}^{10}(i^3 - 2i^2) = 2255$

4. Exercise 52: use linearity and formulas (3)–(5) to evaluate the limit.

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N^2}$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N^2} = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} i$$

$$= \lim_{N\to\infty} \frac{1}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right)$$

$$= \lim_{N\to\infty} \frac{1}{2} + \frac{1}{2N} = \frac{1}{2} + 0 = \frac{1}{2}$$

$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N^2} = \frac{1}{2}$