AM112/212A,   Assignment #1

1. Solve $y''(t)+\varepsilon y'(t)+y(t)=0$ for a general solution for small ε > 0.

   Observe that the solution oscillates and the amplitude decays exponentially.

2. Show that $\int_0^L \sin(\frac{(n-\frac{1}{2})\pi}{L}x)\sin(\frac{(m-\frac{1}{2})\pi}{L}x)\,dx=\begin{cases}0, & n\neq m, n>0, m>0\\ L/2, & n=m>0\end{cases}$.

<u>Remark:</u> Eigenfunctions $\left\{\sin(\frac{(n-\frac{1}{2})\pi}{L}x),\ n=1,2,\ldots\right\}$ are orthogonal to each other.

<u>Hint:</u> Recall the trignometric formulas: $\sin\alpha\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$

3. We expand *f*(*x*) in eigenfunctions $\left\{\sin(\frac{(n-\frac{1}{2})\pi}{L}x),\ n=1,2,\ldots\right\}$.

   Suppose *f*(*x*) has the expansion: $f(x)=\sum_{n=1}^{\infty}c_n\sin(\frac{(n-\frac{1}{2})\pi}{L}x)$.

(a) Multiply both sides by $\sin(\frac{(m-\frac{1}{2})\pi}{L}x)$ and integrate over [0, *L*] to derive

$$c_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{(n-\frac{1}{2})\pi}{L}x)\,dx$$

(b) Use this formula to expand $f(x) = x/L$ in $\left\{\sin(\frac{(n-\frac{1}{2})\pi}{L}x),\ n=1,2,\ldots\right\}$.

4. Use separation of variables to solve (IBVP-P4) with $f_2(x)=(1-\frac{x}{L}),\ x\in[0,L]$.

$$\begin{cases} u_t = k u_{xx}, & \text{differential equation (DE)} \\ u(x,0) = f_2(x), & \text{initial condition (IC)} \\ u(0,t) = 0,\ u(L,t) = 0, & \text{homogeneous BCs (HBCs)} \end{cases} \qquad \text{(IBVP-P4)}$$

5. Solve (IBVP-P5) with $f_3(x) \equiv 1$, $x \in [0, L]$.

$$\begin{cases} u_t = k u_{xx}, & \text{differential equation (DE)} \\ u(x,0) = f_3(x), & \text{initial condition (IC)} \\ u(0,t) = -1,\ u(L,t) = 1, & \text{boundary conditions (BCs)} \end{cases} \qquad \text{(IBVP-P5)}$$

Hint: First convert it to an IBVP with homogeneous BCs. Then use the result of Problem 4.

6. Solve the ODE eigenvalue problem

$$X''(x) = -\lambda X(x), \quad X'(0) = 0, \quad X(L) = 0 \qquad \text{(EIG-P6)}$$

7. (a) Show the orthogonality of eigenfunctions $\{X_n(x)\}$ obtained in Problem 6.

   (b) Derive the formula for coefficients in the expansion of $f(x)$ in $\{X_n(x)\}$.

   (c) Use the formula to expand $f(x) = x/L$, $x \in [0, L]$ in $\{X_n(x)\}$.

8. Use separation of variables to solve (IBVP-P8) below with $f(x) = x/L$, $x \in [0, L]$.

$$\begin{cases} u_t = k u_{xx}, & \text{differential equation (DE)} \\ u(x,0) = f(x), & \text{initial condition (IC)} \\ u_x(0,t) = 0,\ u(L,t) = 0, & \text{homogeneous BCs (HBCs)} \end{cases} \qquad \text{(IBVP-P8)}$$

Hint: Use the results of Problems 6 and 7.