AM 227
UCSC, Spring 2024, Final Project
Prof. Garaud
Due March 22,

Dante Buhl
Jason Johnstone
Arstan Tulekeyev
Nathan Van Duker

We consider a stratified fluid, where the equations of motion are described as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + BT\hat{e}_z + \frac{1}{Re}\nabla^2 \boldsymbol{u} \tag{1}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T + u_z = \frac{1}{Pe}\nabla^2 T \tag{2}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{3}$$

Suppose the flow is split into three components $\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u'} + \tilde{\boldsymbol{u}} \equiv \boldsymbol{u_b} + \tilde{\boldsymbol{u}}$. And the pressure similarly splits into $p = \overline{p} + p' + \tilde{p}$. Where

$$\overline{\boldsymbol{u}} = \sin(y)e_x$$

$$\boldsymbol{u'} = A\boldsymbol{u}_{fgm}(x, y)$$

4. We substitute $\boldsymbol{u} = \boldsymbol{u}_b + \tilde{\boldsymbol{u}}$ into the governing equations, simplify and linearize around $\boldsymbol{u}_b$. Begining with equation (1) above,

$$\frac{\partial \tilde{u}}{\partial t} + (\boldsymbol{u}_b + \tilde{\boldsymbol{u}}) \cdot \nabla(\boldsymbol{u}_b + \tilde{\boldsymbol{u}}) = -\nabla(p_b + \tilde{p}) + Ri(T_b + \tilde{T})\hat{e}_z + \frac{1}{Re}\nabla^2(\boldsymbol{u}_b + \tilde{\boldsymbol{u}})$$

$$\frac{\partial \tilde{u}}{\partial t} + (\boldsymbol{u}_b \cdot \nabla u_b) + (\tilde{u} \cdot \nabla u_b) + (u_b \cdot \nabla \tilde{u}) = -\nabla p_b - \nabla \tilde{p} + Ri(T_b + \tilde{T})\hat{e}_z + \frac{1}{Re}(\nabla^2 u_b + \nabla^2 \tilde{u})$$

We have $\dfrac{\partial u_b}{\partial t} + u_b \cdot \nabla u_b = -\nabla p_b + B\tilde{T} + \frac{1}{Re}\nabla^2 \tilde{u}$. Therefore,

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla u_b) + (u_b \cdot \nabla \tilde{u}) = -\nabla \tilde{p} + Ri(\tilde{T})\hat{e}_z + \frac{1}{Re}(\nabla^2 \tilde{u}) \tag{4}$$

Performing a similar calculation with equation (2) finds,

$$\frac{\partial \tilde{T}}{\partial t} + (\tilde{u} \cdot \nabla T_b) + (u_b \cdot \nabla \tilde{T}) + \tilde{w} = \frac{1}{Pe}(\nabla^2 \tilde{T}) \tag{5}$$

Finally, for equation (3), we have

$$\nabla \cdot u = \nabla \cdot \overline{u} + \nabla \cdot u' + \nabla \cdot \tilde{u} = 0$$

$$\Rightarrow 0 + 0 + \nabla \cdot \tilde{u} = 0 \tag{6}$$

5. We will construct a Fourier expansion of $\tilde{u}$ that takes into account the fact that the equations now explicitly depend on both x and y. We will choose the Ansatz

$$\tilde{q} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} q_{n,m} e^{\lambda t + i(nk_x x + mk_y y + k_z z)} \equiv \sum_{n,m} q_{n,m} e^{\gamma}$$

6. Next, we split $\boldsymbol{u}_b, \tilde{u}$ and the momentum equation into, x, y, and z components, to produce a total of 5 equations. We take the -1,0,1 terms of the Fourier expansion for u_{fgm}. As shown previously, this captures the majority of the corresponding eigenvalue.

$$\boldsymbol{u}_b = u_b\hat{e}_x + v_b\hat{e}_y + w_b\hat{e}_z$$

$$\begin{aligned}
u_b &= \sin(k_y y) + A(2\cos(k_x x)\cos(k_y y)) && (7)\\
v_b &= A(-0.91\cos(k_x x) + \sin(k_x x)\sin(k_y y)) && (8)\\
w_b &= 0 = T_b && (9)\\
p_b &= 0.366[e^{i(k_x x - k_y y)} + e^{i(k_x x + k_y y)}] && (10)
\end{aligned}$$

We start with the momentum equation,

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{\boldsymbol{u}} \cdot \nabla u_b) + (\boldsymbol{u}_b \cdot \nabla \tilde{u}) = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}(\nabla^2 \tilde{u}) \quad (11)$$

$$\frac{\partial \tilde{v}}{\partial t} + (\tilde{\boldsymbol{u}} \cdot \nabla v_b) + (\boldsymbol{u}_b \cdot \nabla \tilde{v}) = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re}(\nabla^2 \tilde{v}) \quad (12)$$

$$\frac{\partial \tilde{w}}{\partial t} + \cancel{(\tilde{\boldsymbol{u}} \cdot \nabla w_b)} + (\boldsymbol{u}_b \cdot \nabla \tilde{w}) = -\frac{\partial \tilde{p}}{\partial z} + Ri(\tilde{T}) + \frac{1}{Re}(\nabla^2 \tilde{w}) \quad (13)$$

We will input our ansatz for each equation, then project onto the function $\int_{-\infty}^{\infty} e^{iNk_x x + iMk_y y} dxdy$. Starting with equation (11), we will go in order:

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{\boldsymbol{u}} \cdot \nabla u_b) + (\boldsymbol{u}_b \cdot \nabla \tilde{u}) = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}(\nabla^2 \tilde{u})$$

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u}\frac{\partial u_b}{\partial x} + \tilde{v}\frac{\partial u_b}{\partial y}) + (u_b\frac{\partial \tilde{u}}{\partial x} + v_b\frac{\partial \tilde{v}}{\partial y}) = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{u}}{\partial z})$$

We will take this component by component.

$$\begin{aligned}
\tilde{u}\frac{\partial u_b}{\partial x} &= -2\tilde{u}Ak_x \sin(k_x x)\cos(k_y y)\\
&= -2\tilde{u}Ak_x(\frac{-i}{2}(e^{ik_x x} - e^{-ik_x x})\frac{1}{2}(e^{ik_y y} + e^{-ik_y y}))\\
&= \frac{\tilde{u}Ak_x}{2}(e^{ik_x x + ik_y y} - e^{-ik_x x + ik_y y} + e^{ik_x x - ik_y y} - e^{-ik_x x - ik_y y})\\
\tilde{v}\frac{\partial u_b}{\partial y} &= \tilde{v}(k_y\cos(k_y y) - 2Ak_y\cos(k_x x)\sin(k_y y))\\
&= \tilde{v}(k_y(\frac{1}{2}(e^{ik_y y} + e^{-ik_y y})) + \frac{iAk_y}{2}(e^{ik_x x} + e^{-ik_x x})(e^{ik_y y} - e^{-ik_y y}))\\
&= \frac{\tilde{v}k_y}{2}(e^{ik_y y} + e^{-ik_y y} + iA(e^{ik_x x + ik_y y} + e^{-ik_x x + ik_y y} + e^{ik_x x - ik_y y} + e^{-ik_x x - ik_y y}))\\
u_b\frac{\partial \tilde{u}}{\partial x} &= (\sin(k_y y) + A(2\cos(k_x x)\cos(k_y y))(imk_y \sum_{n,m} \hat{v}_{n,m}e^{\gamma})\\
&= (\frac{-i}{2}(e^{ik_y y} - e^{-ik_y y}) + A(\frac{1}{2}(e^{ik_x x} + e^{-ik_x x})(e^{ik_y y} + e^{-ik_y y})))(imk_y \sum_{n,m} \hat{u}_{n,m}e^{\gamma})\\
v_b\frac{\partial \tilde{u}}{\partial y} &= A(-0.91\cos(k_x x) + \sin(k_x x)\sin(k_y y)))(ink_x \sum_{n,m} \hat{u}_{n,m}e^{\gamma})\\
&= A(-\frac{0.91}{2}(e^{ik_x x} + e^{-ik_x x}) - \frac{1}{4}(e^{ik_x x} - e^{-ik_x x})(e^{ik_y y} - e^{-ik_y y}))(ink_x \sum_{n,m} \hat{u}_{n,m}e^{\gamma})
\end{aligned}$$

$$= A(-\frac{0.91}{2}(e^{ik_x x} + e^{-ik_x x}) - \frac{1}{4}(e^{ik_x x + ik_y y} - e^{-ik_x x + ik_y y} - e^{ik_x x - ik_y y} + e^{-ik_x x - ik_y y}))(ink_x \sum_{n,m} \hat{u}_{n,m} e^{\gamma})$$

Projecting onto the previously shown function, we find,

$$\tilde{u}\frac{\partial u_b}{\partial x} \to \frac{ik_x A}{2}(\hat{u}_{n-1,m-1} - \hat{u}_{n+1,m+1} + \hat{u}_{n-1,m+1} - \hat{u}_{n+1,m-1})$$
$$\tilde{v}\frac{\partial u_b}{\partial y} \to \frac{k_y}{2}((\hat{v}_{n,m-1} + \hat{v}_{n,m+1}) + iA(\hat{v}_{n-1,m-1} - \hat{v}_{n+1,m+1} + \hat{v}_{n+1,m-1} - \hat{v}_{n-1,m+1}))$$
$$u_b\frac{\partial \tilde{u}}{\partial x} \to \frac{ik_x}{2}(n(u_{n,m-1} - u_{n,m+1}) + iA((n-1)(u_{n-1,m-1} + u_{n-1,m+1}) + (n+1)(u_{n+1,m+1} + u_{n+1,m-1}))$$
$$v_b\frac{\partial \tilde{u}}{\partial y} \to ik_y A(\frac{-.91}{2}(m(u_{n-1,m} + u_{n+1,m}) - \frac{1}{4}((m-1)(u_{n-1,m-1} + u_{m,m-1}) + (m+1)(-u_{n+1,m+1} - u_{n-1,m+1})))$$

Combining these all together, we find that our momentum equation in the x direction is now,

$$\begin{gathered}
\lambda u_{n,m} + \frac{ik_x A}{2}(\hat{u}_{n-1,m-1} - \hat{u}_{n+1,m+1} + \hat{u}_{n-1,m+1} - \hat{u}_{n+1,m-1}) \\
+ \frac{k_y}{2}((\hat{v}_{n,m-1} + \hat{v}_{n,m+1}) + iA(\hat{v}_{n-1,m-1} - \hat{v}_{n+1,m+1} + \hat{v}_{n+1,m-1} - \hat{v}_{n-1,m+1})) \\
+ \frac{ik_x}{2}(\frac{n}{i}(\hat{u}_{n,m-1} - \hat{u}_{n,m+1}) + A((n-1)(\hat{u}_{n-1,m-1} + \hat{u}_{n-1,m+1}) + (n+1)(\hat{u}_{n+1,m+1} + \hat{u}_{n+1,m-1})) \\
+ik_y A(\frac{-.91}{2}(m(\hat{u}_{n-1,m} + \hat{u}_{n+1,m}) - \frac{1}{4}((m-1)(\hat{u}_{n-1,m-1} + \hat{u}_{m,m-1}) + (m+1)(-\hat{u}_{n+1,m+1} - \hat{u}_{n-1,m+1}))) = \\
-ink_x \hat{p}_{n,m} - \frac{\hat{u}_{n,m}}{Re}[-(nk_x)^2 - (mk_y)^2 - k_z^2]
\end{gathered} \tag{14}$$

A similar, but omitted calculation shows that the remaining y,z directions of the momentum equation, and the temperature and continuity equations are equivelently rewritten as:

$$\begin{gathered}
\lambda \hat{v}_{n,m} + iAk_x[\frac{-.91}{2}(\hat{u}_{n-1,m} - \hat{u}_{n+1,m}) + \frac{1}{4}(\hat{u}_{n-1,m-1} - \hat{u}_{n+1,m+1} + \hat{u}_{n+1,m-1} - \hat{u}_{n-1,m+1})] \\
- iA\frac{k_y}{4}[\hat{v}_{n-1,m-1} - \hat{v}_{n+1,m+1} + \hat{v}_{n-1,m+1} - \hat{v}_{n+1,m-1}] \\
+ \frac{ikx}{2}[\frac{n}{i}(\hat{v}_{n,m-1} + \hat{v}_{n,m+1}) + A((n-1)(\hat{v}_{n-1,m-1} + \hat{v}_{n-1,m+1}) + (n+1)(\hat{v}_{n+1,m+1} + \hat{v}_{n+1,m-1}))] \\
+ ik_y A[\frac{-.915m}{2}(\hat{v}_{n-1,m} + \hat{v}_{n+1,m}) \\
- \frac{1}{4}((m-1)(\hat{v}_{n-1,m-1} - \hat{v}_{n+1,m-1}) + (m+1)(v_{n+1,m+1} - \hat{v}_{n-1,m+1}))] = \\
-imk_y p_{n,m} - \frac{\hat{v}_{n,m}}{Re}[-(nk_x)^2 - (mk_y)^2 - k_z^2]
\end{gathered} \tag{15}$$

$$\begin{gathered}
\lambda \hat{w}_{n,m} \\
+ink_x[\frac{n}{2i}(\hat{w}_{n,m-1} - \hat{w}_{n,m+1}) + \frac{A}{2}((n-1)(\hat{w}_{n-1,m-1} + \hat{w}_{n-1,m+1}) + (n+1)(\hat{w}_{n+1,m-1} + \hat{w}_{n+1,m+1}))] \\
+ ik_y A[\frac{-.91m}{2}(\hat{w}_{n-1,m} + \hat{w}_{n+1,m}) \\
- \frac{1}{4}((m-1)(\hat{w}_{n-1,m-1} - \hat{w}_{n+1,m-1}) + (m+1)(\hat{w}_{n+1,m+1} - \hat{w}_{n-1,m+1}))] = \\
-ik_z p_{n,m} + Ri\hat{T}_{n,m} - \frac{\hat{w}_{n,m}}{Re}[-(nk_x)^2 - (mk_y)^2 - k_z^2]
\end{gathered} \tag{16}$$

$$
\begin{aligned}
&\lambda\hat{T}_{n,m} \\
&\quad + ik_x[\frac{n}{2i}(\hat{T}_{n,m-1} - \hat{T}_{n,m+1}) + \frac{A}{2}((n-1)(\hat{T}_{n-1,m-1} + \hat{T}_{n-1,m+1}) + (n+1)(\hat{T}_{n+1,m-1} + \hat{T}_{n+1,m-1}))] \\
&\quad + ik_yA[\frac{-.91m}{2}(\hat{T}_{n-1,m} + \hat{T}_{n,m+1}) \\
&\quad\; - \frac{1}{4}((m-1)(\hat{T}_{n-1,m-1} - \hat{T}_{n+1,m-1}) + (m+1)(\hat{T}_{n+1,m+1} - \hat{T}_{n-1,m+1}))] \\
&\quad\; + \hat{w}_{n,m} = -\frac{\hat{T}_{n,m}}{Pe}[-(nk_x)^2 - (mk_y)^2 - k_z^2]
\end{aligned}
\tag{17}
$$

$$
nk_x\hat{u}_{n,m} + mk_y\hat{v}_{n,m} + k_z\hat{w}_{n,m} \tag{18}
$$