§5.1 Complex QR alg. for Hermitian

$A \in \mathbb{C}^{m\times m}$, $A = A^*$ (A is non-defective)

① eig. vals : real

② eig vectors : a basis of $\mathbb{C}^m$

③ unitarily diagonalizable. i.e,

$$A = U D_\lambda U^*, \quad U: \text{unitary}$$

$\Longrightarrow$ We use the complex version of "UR" alg. (similar to the real version of QR)

$\Downarrow$

$$UR = A$$

U ↳ unitary

§5.2 Schur decomposition for Non-Hermitian matrices

Thm. Let $A \in \mathbb{C}^{m\times m}$

Then $\exists\ U$: unitary s.t

$$\boxed{A = UTU^*}$$ where

T = upper Δ

Rmk ① A is similar to T

$\therefore$ $\Lambda(A) = \Lambda(T)$

② $\Lambda(T) = \{ t_{ii} \mid t_{ii} = \text{diag}(T) \}$

special case ③ If A is diagonalizable, then (or $A = A^*$)

$T = D_\lambda$

Rmk. QR w/o shift

$R^{(n)} = Q^{(n)*} A^{(n)}$

do while error > large

$Q^{(n)} R^{(n)} = A^{(n)}$

$A^{(n+1)} = R^{(n)} Q^{(n)}$

end do

$A^{(n+1)} = Q^{(n)*} A^{(n)} Q^{(n)}$

$= \ldots$

$$= \underbrace{Q^{(n)*} Q^{(n-1)*} \cdots Q^{(1)*}}_{\tilde{Q}^{(n)*}} A^{(0)} \underbrace{Q^{(1)} \cdots Q^{(n)}}$$

$$= \tilde{Q}^{(n)*} A^{(0)} \tilde{Q}^{(n)}$$

A

$$\longrightarrow \underbrace{V^{-1} A V}_{= D_\lambda} \quad \text{in the limit}$$

§5.3 LAPACK

(i) Driver routines

(ii) basic computation routines

(iii) low-level auxiliary routines (BLAS)

Naming Conventions

S : single precision , real
D : double " , real
C : single , complex
Z : double , complex

(ex) DSYEV

Chapter 5 , SVD

$A\hat{V} = \hat{U}\hat{\Sigma}$

(1.2) The reduced SVD

Assumption
$\boxed{\operatorname{rank}(A) = r \le n}$

From compact SVD, $\underset{m\times n}{A}\,\underset{n\times r}{V} = \underset{m\times r}{U}\,\underset{r\times r}{\Sigma}$,

$$\underset{n\times n}{\hat{V}} = [v_1 | \cdots | v_r \,|\, \underbrace{v_{r+1} | \cdots | v_n}]$$

$$\underset{m\times r}{\hat{U}} = [\underbrace{u_1 | \cdots | u_r}_{\text{compact SVD}} \,|\, u_{r+1} | \cdots | u_n]$$

$$\rightarrow \boxed{A\hat{V} = \hat{U}\hat{\Sigma}}, \quad \underset{n\times n}{\hat{\Sigma}} = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

Since rank $(A) = r \le n$,

$$A v_i = 0 u_i, \quad r+1 \le i \le n$$
$$= 0$$

i.e. $\begin{cases} \ker(A) = \mathrm{Null}(A) = \{v_{r+1}, \dots, v_n\} \\ \mathrm{range}(A) = \{u_1, \dots, u_r\} \end{cases}$

§1.3. The Full SVD : $AV = U\Sigma$

i) $V = \hat{V}$ (same as in reduced SVD)

ii) $U = \begin{bmatrix} \hat{U} & u_{n+1} & \cdots & u_m \end{bmatrix}_{m\times m}$

(ii) $\Sigma = \underbrace{\left[\begin{array}{cccccc} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ \hline & & & & & \\ & & & 0 & & \\ & & & & & \end{array}\right]}_{m \times n}$ $\quad n \times n$

Rmk From QR decomp to SVD

$$\underbrace{Q_n \cdots Q_2 Q_1}_{Q^*} \overset{m \times n}{A} = R$$

$$\rightarrow \boxed{A = QR}$$

Rmk. SVD is NOT unique

ex If $\sigma_1 = \sigma_2$, then

$$A = [u_1 | u_2 | \cdots | u_m] \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ v_n \end{bmatrix}$$

$$= [u_2 | u_1 | u_3 | \cdots | u_m] \begin{bmatrix} \sigma_2 & & & & \\ & \sigma_1 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \hline v_1 \\ \hline v_3 \\ \hline \vdots \\ \hline v_n \end{bmatrix}$$

Ex Consider $A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{row swap}} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \overbrace{\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{column swap}}}^{P \, =}$$

Note

$AP \to$ column swap

$PA \to$ row swap

Ex $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\boxed{2}$ Properties of SVD

(i) SVD of real $A \in \mathbb{R}^{m \times n}$
has real orthogonal U & V

(ii) # of non-zero s-values $= \operatorname{rank}(A)$

(iii) If $\operatorname{rank}(A) = r \leq n$

$$\begin{cases} \operatorname{range}(A) = \operatorname{span}\{u_1, \dots, u_r\} \\ \ker(A) = \operatorname{null}(A) = \operatorname{span}\{v_{r+1}, \dots, v_n\} \end{cases}$$

(iv) $\|A\|_2 = \sigma_1$

$$\|A\|_F = \left(\sigma_1^2 + \sigma_2^2 + \dots + \sigma_r^2\right)^{\frac{1}{2}}$$

(v) $|\det(A)| = \sigma_1 \sigma_2 \cdots \sigma_r$

Pf $|\det(A)| = |\det(U \Sigma V^*)|$

$$\textcircled{?}\quad = \left|\underbrace{\det(U)}_{=1}\det(\Sigma)\det(V^*)\right|$$

$$= |\det(\Sigma)|$$

$$= \sigma_1 \cdots \sigma_r$$

ⓐ $1 = \det(I) = \det(U^*U)$

$$= \det(U^*)\det(U)$$

$$= (\det(U))^*\det(U)$$

$$= |\det(U)|^2 \qquad \therefore \ |\det(U)| = 1$$

(vi) $\sigma_i = \sqrt{\lambda_i}$, $\begin{cases} \sigma_i = \text{nonzero s. values of } A \\ \lambda_i = \text{〃 e.vals of } A^*A \end{cases}$

(vii) If $A = A^*$,

$$\sigma_i = |\lambda_i| = \operatorname{sgn}(\lambda_i)\,\lambda_i$$

Rmk. $A^{-1} = (U \Sigma V^*)^{-1}$

$= V \Sigma^{-1} U^*$

A: square & non-singular

$$= \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$$

$$= V P \left(P \Sigma^{-1} P \right) P U^*$$

$$P = \begin{bmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sigma_r} & & & \\ & \frac{1}{\sigma_{r-1}} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_1} \end{bmatrix}$$

Rmk. A^*A in terms of SVD?

$$A^*A = (\underbrace{U \Sigma V^*}_{})^* (U \Sigma V^*)$$

$$= V \Sigma^* \underbrace{U^* U}_{I} \Sigma V^*$$

$$= V \underbrace{\Sigma^* \Sigma}_{} V^*$$

$$= V \Sigma^2 V^*$$

$$= V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2 \end{bmatrix} V^*$$

$\therefore$ σ_i^2 : eigen values of A^*A

This proves (vi)