Final Project: Report

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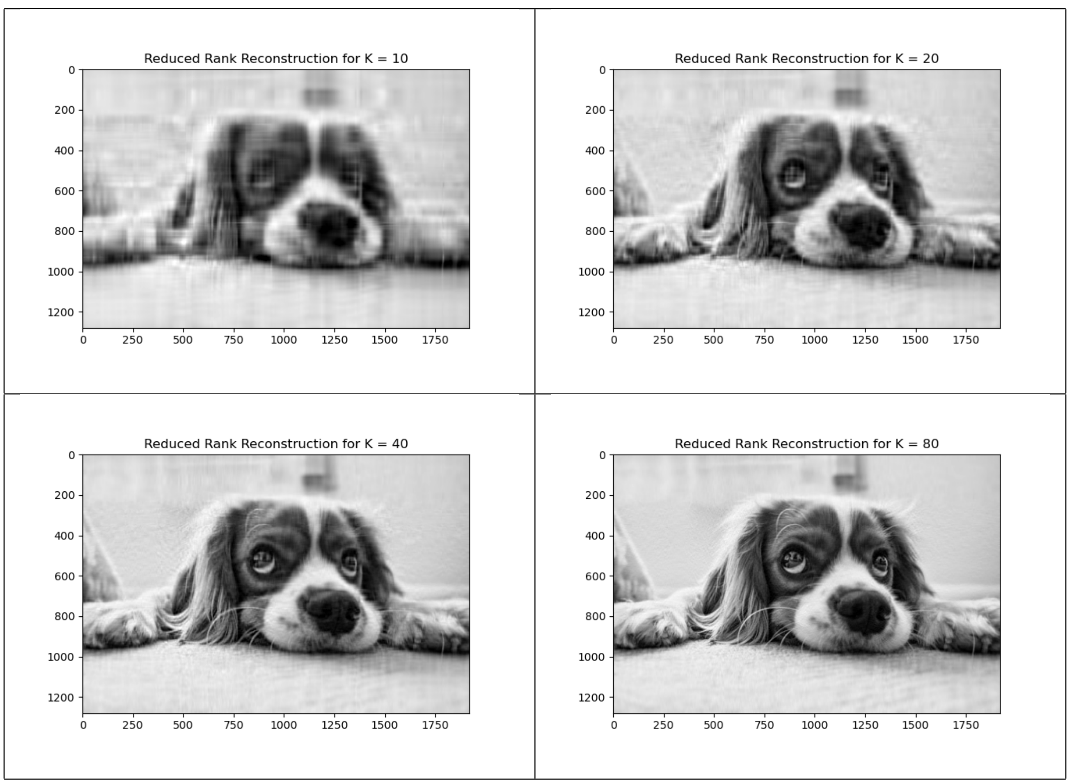
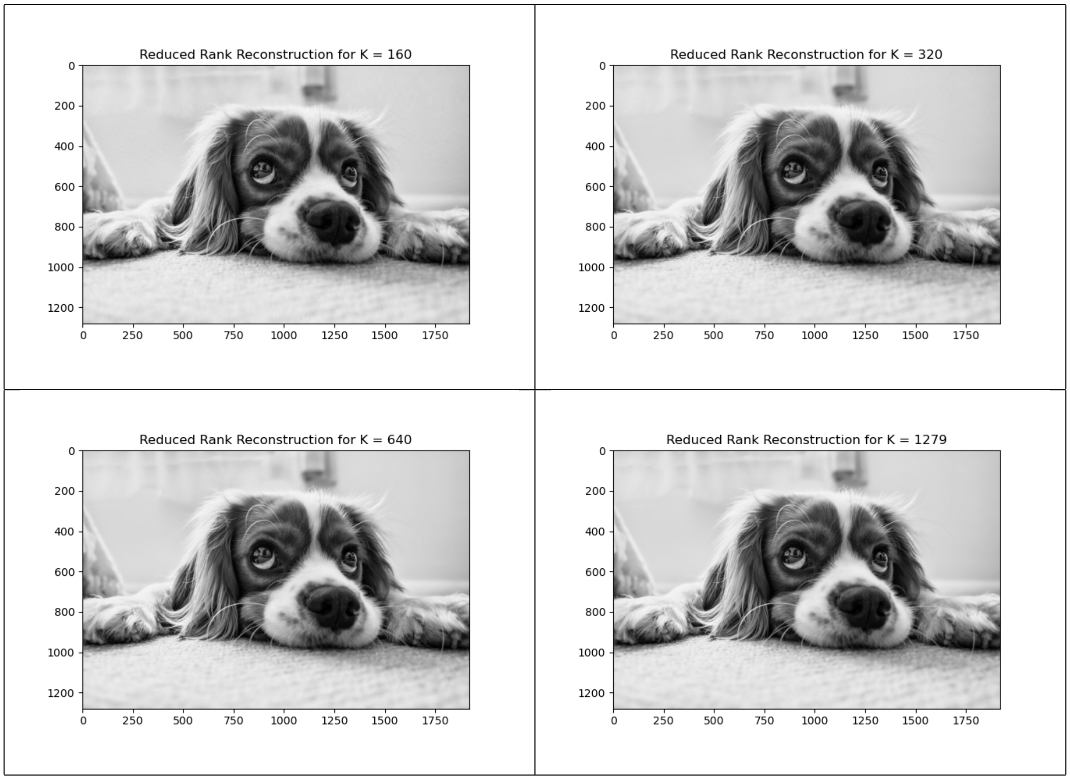
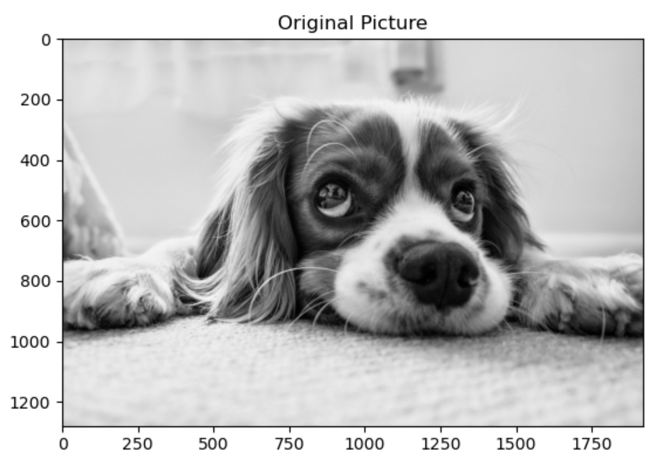
1 Part A: Reduced Rank Image Reconstruction

1. The first ten largest singular values are $\{\sigma_1 = 281897.27, \sigma_2 = 46571.71, \sigma_3 = 31487.79, \sigma_4 = 26436.71, \sigma_5 = 19631.55, \sigma_6 = 15569.22, \sigma_7 = 14390.26, \sigma_8 = 11254.04, \sigma_9 = 9660.39, \sigma_{10} = 9411.763\}$. Furthermore the "rest singular values" are as follows $\{\sigma_{10} = 9411.73, \sigma_{20} = 4167.18, \sigma_{40} = 2035.98, \sigma_{80} = 1193.12, \sigma_{160} = 758.44, \sigma_{320} = 456.38, \sigma_{640} = 189.13, \sigma_{1279} = 4.76\}$.

2. Here are the following eight images generated by the reduced rank approximation for the image.







3. It is easily noticable that as we reduce the rank of our appoximations more and more, that the quality of the image decreases. Some of the differences between rank approximations aren't as noticable as others. Between ranks of the reconstruction 40 and 80 there is a very noticable sacrifice in image quality. There seem to be smudges in the picture and a sort of typical graininess which is present in low resolution images. This is what is noticable here. Looking at the error reports of these approximations we find that we only find a normalized error below 10^{-3} for our rank 10 approximation. All other normalized errors are of order 10^{-3} or lower. The lowest being the rank 1279 error at 10^{-16} exactly the order of machine double precision.

2 Iterative Methods

2.1 Gauss-Jacobi/Gauss-Seidel

1. Both the Gauss-Jacobi and Gauss-Seidel methods involve an iterative method in which we recompute a vector x and after (hopefully) a finite number of iterations we will reach the desired level of error. They differ in the compution of the next x vector from the proceeding one. In Gauss-Jacobi, it is simply one line of matrix multiplcation and subtraction, i.e.

$$x^{(n)} = D^{-1}(b - Rx^{(n-1)}), \quad D = \mathrm{diag}(A), \quad R = A - D$$

   This is much easier to compute since $x^{(n)}$ only depends on the givens A, D, R, b and the previous iteration. A slightly more complex algorithm, Gauss-Seidel, is derived from this where we break R into an upper and lower matrix, i.e.

$$A = D + R + L, \quad D = \mathrm{diag}(A), \quad R = \text{upper}\,\Delta(A - D), \quad L = \text{lower}\,\Delta(A - D)$$

   Here the iterative method is

$$x^{(n)} = 0 \to x_i^{(n)} = \frac{1}{a_{ii}}(b_i - L_{i,1:i-1}x_{1:i-1}^{(n)} - R_{i,i+1:m}x_{i+1:m}^{(n-1)})$$

   Here we have that going down the elements of $x^{(n)}$ each element depends on the term before it. This means that rather than one line of matrix opertions, we have to express the iterative process for $x^{(n)}$ as a loop over the m elements of the vector. For each method, the convergence criterion is slightly different. For the Gauss-Jacobi method we must have that $T = D^{-1}R$ has eigenvalues such that, $\rho(T) < 1$. This will guarantee that as we take the number of iterations to infinity,

$$x^(n) = Tx^{(n-1)} + c = T^2x^{(n-2)} + Tc^{(n-2)} + c^{(n-1)}$$

$$x^{(n)} = T^n x^{(0)} + T^{n-1}c^{(0)} + \cdots + Tc^{(n-2)} + c^{(n-1)}$$

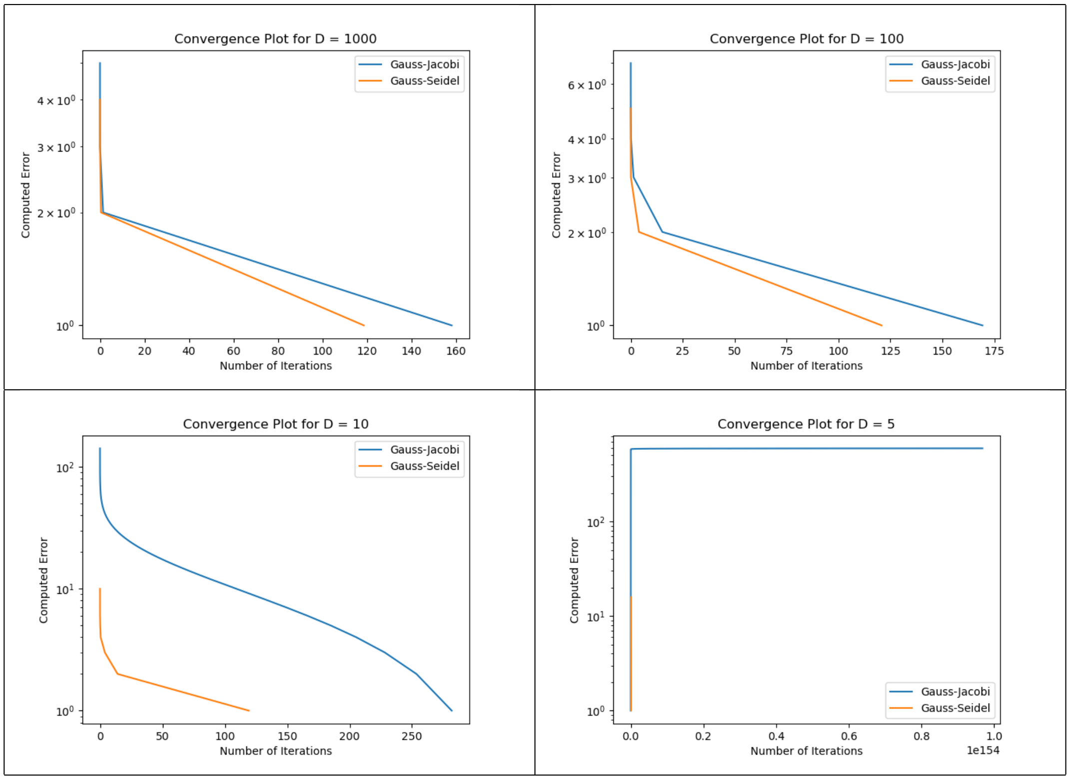
$$\lim_{n\to\infty} x^{(n)} = const.$$

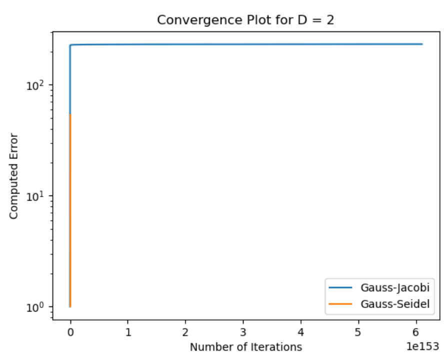
   The convergence criterion for Gauss-Seidel is similar, Gauss-Seidel will converge if Gauss-Jacobi converges and will converge faster, however there are cases where Gauss Seidel Converges but Gauss Jacobi does not. In the case of Gauss-Seidel a similar argument can be made but for $T = (D + L)^{-1}R$. We have,

$$x^{(n)} = Tx^{(n)} + c$$

   Notice that by Gergorin's theorem we will have that the eigenvalues of $D + L$ can be larger than the eigenvalues of D and similarly, the eigenvalues of R become constrained. Therefore we will have that the eigenvalues of T will be lower than in the case for Gauss-Jacobi and therefore will converge faster.

2. Here are the plots obtained for all values of D.





3. Some of the cases didn't converge, specifically $D = 2$ and $D = 5$ for the Gauss-Jacobi method. For both of these cases the number of iterations reached into the order of 10^2 and the final result is NaN, usually indicating an underflow. For the cases where both methods converge, we see that Gauss-Seidel converges more quickly. This matches the theoretical predictions that we went over in lecture yielding better convergence and stability for Gauss-Seidel with the trade off of not being able to parallelize. To investigate why this didn't converge, we look at the spectral radius of these matrices. Notice that for $D = 2$ we have that $D^{-1}R = \begin{bmatrix} 0 & \mathbf{\frac{1}{2}} \\ \mathbf{\frac{1}{2}} & 0 \end{bmatrix}$. For the case of $D = 5$, we have, $D^{-1}R = \begin{bmatrix} 0 & \mathbf{\frac{1}{5}} \\ \mathbf{\frac{1}{5}} & 0 \end{bmatrix}$. In both cases we have from Gershgorin's Theorem, for the eigenvalues of $D^{-1}R$,

$$|\lambda_{ii}| \leq 4.5, \quad |\lambda_{ii}| \leq 1.8$$

Notice that for both cases of T we have that the eigenvalues no not necessarily have to be below 1 and so the algorithm is not guaranteed to converge.

4. For the case of $A_{ii} = i$ we have that the Gauss-Jacobi method does not converge, but Gauss-Seidel does.

2.2 Conjugate Gradient

1. The Conjugate Gradient Method is a method similar to gradient descent where each iteration, there is a direction chosen and a step length along that direction such that each iteration the "function" will approach a local minimum. This "function" is chosen such that is contains the information of the system of equations that we are trying to solve. The reason we can pick such a method is because of the properties of martrices which this method applies to, specifically symmetric positive definite matrices. Why is it uaranteed to converge

2. *Proof.* To prove the two algorithms are identical we must look closely aat the differences between the two methods. First we notice that the "smart" method relies on two quantities called E and E_{new}. These quantities serve the purpose of holding onto the inner product of r with itself, also known as the two norm of r squared. We now look at the use of E and E_{new} specifically. Its most immediate use is the compuation of the error for the while loop, and for the computation of β. In the "basic" CG method, the while loop criterion is seeing if the two norm of r, the residual is below the tolerated error. We look at the use of $\sqrt{E}$ rather than $||r||_2$.

$$E = r^{(n-1)T} r^{(n-1)} = \sum_i^n r_i^{(n-1)2} = ||r||_2^2, \implies \sqrt{E} = ||r||_2$$

So we have that using $\sqrt{E}$ does not change the exit condition for the while loop. Next we look at the computation of β and α. The algorithm postulates that the following are equivalent,

$$E^{(n)} = ||r^{(n)}||_2^2 = r^{(n)T} r^{(n)}, \quad E^{(n-1)} = p^{(n-1)T} r^{(n-1)}, \quad \frac{E^{(n)}}{E^{(n-1)}} = -\frac{r^{(n)T} y^{(n)}}{p^{(n-1)T} y^{(n)}}$$

The rest of the proof follows from proving these two things. We attempt to prove through induction and start with the base case. We look at $E^{(0)}$ and $E^{(1)}$.

$$r^{(0)} = r^{(0)}, \quad p^{(0)} = r^{(0)}$$

$$p^{(0)T} r^{(0)} = r^{(0)T} r^{(0)} = E^{(0)}, \implies \text{Base case computation for } \alpha \text{ holds.}$$

$$\alpha^{(1)} = \frac{r^{(0)T} r^{(0)}}{r^{(0)T} A r^{(0)}}$$

$$r^{(1)} = r^{(0)} - \alpha^{(1)} A r^{(0)}$$

$$-\frac{r^{(1)T} y^{(1)}}{p^{(0)T} y^{(1)}} = -\frac{(r^{(0)} - \alpha^{(1)} A r^{(0)})^T A r^{(0)}}{r^{(0)} A r^{(0)}}$$

$$= -\frac{r^{(0)T} A r^{(0)} - \alpha^{(1)} r^{(0)} A^T A r^{(0)}}{r^{(0)} A r^{(0)}}$$

$$= -1 + \frac{r^{(0)T} r^{(0)}}{r^{(0)T} A r^{(0)}} \frac{r^{(0)} A^T A r^{(0)}}{r^{(0)} A r^{(0)}}$$

$$= -1 + \frac{(r^{(0)T} r^{(0)})(r^{(0)} A^T A r^{(0)})}{(r^{(0)} A r^{(0)})^2}$$

$$\frac{E^{(1)}}{E^{(0)}} = \frac{r^{(1)T} r^{(1)}}{r^{(0)T} r^{(0)}}$$

$$= \frac{(r^{(0)} - \alpha^{(1)} A r^{(0)})^T (r^{(0)} - \alpha^{(1)} A r^{(0)})}{r^{(0)T} r^{(0)}}$$

Notice that A is symmetric,

$$= \frac{r^{(0)T} r^{(0)} - 2\alpha^{(1)} r^{(0)T} A r^{(0)} + \alpha^{(1)2} r^{(0)T} A^T A r^{(0)}}{r^{(0)T} r^{(0)}}$$

Again, $\alpha^{(1)} = \frac{r^{(0)T}r^{(0)}}{r^{(0)T}Ar^{(0)}}$

$$= \frac{r^{(0)T}r^{(0)} - 2r^{(0)T}r^{(0)} + \left(\frac{r^{(0)T}r^{(0)}}{r^{(0)T}Ar^{(0)}}\right)^2 r^{(0)T}A^TAr^{(0)}}{r^{(0)T}r^{(0)}}$$

$$= -1 + \frac{(r^{(0)T}r^{(0)})(r^{(0)}A^TAr^{(0)})}{(r^{(0)}Ar^{(0)})^2}$$

So we have that,

$$\frac{E^{(1)}}{E^{(0)}} = -\frac{r^{(1)T}y^{(1)}}{p^{(0)T}y^{(1)}}$$

Therefore, the computation of β also holds for the base case. We now take the inductive step, and presume that, the following is given as true,

$$E^{(n-1)} = p^{(n-1)T}r^{(n-1)}, \quad \frac{E^{(n)}}{E^{(n-1)}} = -\frac{r^{(n)T}y^{(n)}}{p^{(n-1)T}y^{(n)}}$$

So we need to show that $E^{(n)} = p^{(n)T}r^{(n)}$, and that $\frac{E^{(n+1)}}{E^{(n)}} = -\frac{r^{(n+1)T}y^{(n+1)}}{p^{(n)T}y^{(n+1)}}$

$$p^{(n)} = r^{(n)} + \beta^{(n)}p^{(n-1)}$$

$$E^{(n)} = (r^{(n)} + \beta^{(n)}p^{(n-1)})^Tr^{(n)}$$

Next we notice a recursive definition for $p^{(n-1)}$.

$$= p^{(n-1)} = r^{(n-1)} + \beta^{(n-1)}p^{(n-2)}$$

If this is investigated thoroughly enough we can show that,

$$E^{(N)} = r^{(n)T}r^{(n)} + \sum_{i=0}^{n-1} \frac{r^{(n)T}r^{(n)}}{r^{(i)T}r^{(i)}} r^{(i)T}r^{(n)}$$

From here we take as a given that we have collected r such that for $i \neq j$, $r^{(j)T}r^{(i)} = 0$. Thus this part of proof concludes and we have that,

$$E^{(n)} = p^{(n)T}r^{(n)} = r^{(n)T}r^{(n)}$$

Next we look at $\frac{E^{(n+1)}}{E^{(n)}} = -\frac{r^{(n+1)T}y^{(n+1)}}{p^{(n)T}y^{(n+1)}}$. Let us look at the value of $y^{(n+1)}$ more closely.

$\square$

3. This method converges very quickly and with high accuracy for each value of D! In the table below we have the number of iterations needed for each method to converge to order 10^{-5} accuracy (note that though the tolerance parameter is given as 10^{-5} we have that the actual value of the resultant error is below 10^{-14} in only 2 iterations).

$D =$	2	5	10	100	1000
$N_{CG} =$	2	2	2	2	2
$N_{GJ} =$	N/A	N/A	142	7	5
$N_{GS} =$	53	16	10	5	4

4. Specifically the Diagonal preconditioner is not strong for these matrices. The reason is that since each diagonal element is the same we are effectively dividing the whole matric by the same value, thereby the matrix product $M^{-1}A$ is equivalent to a scalar multiplication, $M^{-1}A = \frac{1}{D}A$. This doesn't effetively change the eigenvalues of the system leading to an increase in the convergence rate.

5. Conjugate Gradient does converge. For the 10×10 case the algorithm converges within 10 iterations. The resultant error is of order 10^{-11}. For the 100×100 case the algorithm converges within 57 iterations and the error is of order 10^{-5}.

6. The convergence rate is much faster with the diagonal preconditioning. For the 10×10 case we have that the algorithm converged in 8 rather than 10 iterations. For the case of 100×100 the algorithm converged in 10 rather than 57 iterations, a massive improvement!