AM112/212A, Assignment #3

1. Let $L[\bullet]=c_2(x)\dfrac{d^2(\bullet)}{dx^2}+c_1(x)\dfrac{d(\bullet)}{dx}+c_0(x)(\bullet)$ with $c_0(x)$, $c_1(x)$, and $c_2(x)$ real.

   Consider the inner product $\langle u(x),v(x)\rangle=\int_a^b u(x)\overline{v}(x)dx$ and the function space

   $$C_{BC}[a,b]=\{u(x)|\{u(a)=0,u(b)=0\}$$

   Use integration by parts to show $L^*[\bullet]=\dfrac{d^2}{dx^2}(c_2(x)\bullet)-\dfrac{d}{dx}(c_1(x)\bullet)+c_0(x)(\bullet)$.

2. Solve the IBVP of the heat equation with a transportation term

$$\begin{cases} u_t=u_{xx}-u_x \\ u(0,t)=0\ , \quad u(L,t)=0 \\ u(x,0)=\mathrm{e}^{x/2} \end{cases} \qquad \text{(IBVP-P2)}$$

Hint: Use the results obtained in lecture.

3. Find a general solution of the Cauchy-Euler equation $x^2u''+xu'-\beta^2u=0, \beta\ge 0$.

   Try solutions of the form $u(x)=x^\alpha=\mathrm{e}^{\alpha\ln x}$. Discuss the cases of $\beta > 0$ and $\beta = 0$.

Hint: When α is a double root, two independent solutions are

$$u_1(x)=\mathrm{e}^{\alpha\ln x}=x^\alpha\ , \quad u_2(x)=(\ln x)\mathrm{e}^{\alpha\ln x}=(\ln x)x^\alpha.$$

Remark: Note that the coefficient of u'' disappear at $x = 0$. As a result, one of the two independent solutions diverges to ∞ at $x = 0$.

4. Use the results from Problem 3 to solve the BVP $\begin{cases} x^2u''+xu'-\beta^2u=0 \\ u(0)=\text{finite}\ , \ u(R)=u_R \end{cases}$.

Remark: When the coefficient of u'' disappears at $x = 0$, the BC at $x = 0$ is $u(0)$ = finite.

5. For $b > a > 0$, solve the Sturm-Liouville problem (SL-P5) in the case of $\lambda > 0$

$$\begin{cases} (xu_x)_x = -\lambda \dfrac{1}{x} u \\ u(a) = 0\,, \quad u(b) = 0 \end{cases} \qquad \text{(SL-P5)}$$

<u>Hint:</u> Use the procedure in lecture.

6. For $b > 1$, show that the Sturm-Liouville problem (SL-P6) does not have a non-trivial solution in the cases of $\lambda < 1/4$ or $\lambda = 1/4$.

$$\begin{cases} (x^2 u_x)_x = -\lambda u \\ u(1) = 0\,, \quad u(b) = 0 \end{cases} \qquad \text{(SL-P6)}$$

<u>Hint:</u> Use the procedure in lecture.

7. For $b > 1$, solve Sturm-Liouville problem (SL-P6) in the case of $\lambda > 1/4$.

<u>Hint:</u> Use the procedure in lecture.

8. Consider the Sturm-Liouville problem

$$\begin{cases} u_{xx} - x^2 u = -\lambda u \\ u_x(0) = 0\,, \quad u(1) = 0 \end{cases} \qquad \text{(SL-P8)}$$

Let $\{\lambda_0, \phi_0(x)\}$ be the lowest eigenvalue and eigenfunction. From Sturm-Liouville theory, $\phi_0(x)$ satisfies $u_x(0) = u(1) = 0$, and has no node. $u_0(x) = \cos(\frac{\pi}{2} x)$ also satisfies these properties. We use $u_0(x)$ to approximate $\phi_0(x)$. The Rayleigh principle gives

$$\lambda_0 = R(\phi_0) \approx R(u_0).$$

<u>Task:</u> Evaluate $R(u_0)$ and use the value to approximate λ_0.

<u>Remark:</u> From accurate numerical simulations, $\lambda_0 \approx 2.5969$. $R(u_0)$ should be close.