Homework 4: Report

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2 Cholesky Solution of the least-squares problem

1.

3 QR Solution of the least-squares problem

1.

4 Theory Problems

1. Show that is P is an orthogonal projector, then $\mathrm{I} - 2P$ is unitary.

   Proof. We begin with the definition of a unitary matrix. We have a unitary matrix Q is a matrix such that $Q^*Q = \mathrm{I}$. We now look for this quality in $(\mathrm{I} - 2P)$.

   $$(\mathrm{I} - 2P)^*(\mathrm{I} - 2P) = (\mathrm{I} - 2P^T)(\mathrm{I} - 2P) = (\mathrm{I} - 2P)(\mathrm{I} - 2P)$$

   The above quality $P^T = P$ is from the fact that P is an orthogonal projector. We also have the quality that $P^2 = P$.

   $$(\mathrm{I} - 2P)(\mathrm{I} - 2P) = \mathrm{I}(\mathrm{I} - 2P) - 2P(\mathrm{I} - 2P) = \mathrm{I} - 2P - 2P + 4P^2$$
   $$= \mathrm{I} - 4P + 4P = \mathrm{I}$$

   We have recovered the condition for unitary matrices. We can therefore declare $(\mathrm{I} - 2P)$ a unitary matrix. □

2. Let $P \in \mathbb{R}^{m\times m}$ be a nonzero projector.

   (a) Show that $||P||_2 \geq 1$, with equality if and only if P is an orthogonal projector.

   (b) Show that if P is an orthogonal projector, then P is semi-positive deinite with its eigenvalues either zero or 1.

   Proof. We look at the vector product definition of P. $P = xx^T$ for a unit vector x.

   $$(v, Pv) = v^T xx^T v = (v, x)(x, v) = (x, v)^2 \geq 0$$

   Since our choice of v was arbitary we have that P is semi-positive definite.
   Next we look at the eigenvalues of P. Say that we have an arbitary eigenvalue-eigenvector pair (λ, v) for P such that $v \neq \vec{0}$ (obviously). We have,

   $$Pv = \lambda v$$

   $$Pv = xx^T v = (x, v)x = \lambda v$$

Notice that (x, v) and λ are scalars. This implies that x and v are colinear but this was not an assumption made. Therefore we are left with two cases: v and x are colinear, or v and x are orthogonal. Let's look at the first case, $v = \alpha x$.

$$(x, v)x = \alpha x = \alpha\lambda x$$

$$x = \lambda x \implies \lambda = 1$$

We find that for all vectors colinear to x are eigenvectors with eigenvalue 1. We look at the other case. If x and v are orthogonal we have $(x, v) = 0$.

$$\vec{0} = \lambda v \implies \lambda = 0$$

Therefore all vectors orthogonal to x will be eigenvectors with $\lambda = 0$. $\square$

3. Let $A \in \mathbb{R}^{m\times m}$ with $m \geq n$, and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

   (a) Show that A has rank n if and only if all the diagonal entries of $\hat{R}$ are nonzero.

   (b) Suppose $\hat{R}$ has k nonzero diagonal entries for some k with $0 \leq k < n$. What does this imply about the rank of A? Exackly k? At least k? At most k? Give a precise answer and prove it.

4. Determine the (i) eigenvalues, (ii) determinant, and (iii) singular values of a Householder reflector. For the eigenvalues, give a geometric argument as well as an algebraic proof.

Proof. (i) Eigenvalues

We start with the definition of a householder reflector for a unit vector x. Take $H = \mathrm{I} - 2xx^T$ with an eigenvalue-eigenvector pair (λ, v) such that $Hv = \lambda v$.

$$Hv = (\mathrm{I} - 2xx^T)v = v - 2xx^Tv = v - 2(x, v)x = \lambda v$$

$$-2(x, v)x = (\lambda - 1)v$$

We again have a case where x and v are vectors connected by scalar arguments. We must have that x and v are colinear. We take the two cases, x and v are colinear, x and v are orthogonal.

$$v = \alpha x, \quad -2(x, v) = -2\alpha$$

$$-2\alpha = (\lambda - 1)\alpha$$

$$\lambda = -1$$

Therefore if x and v are colinear we have that v is an eigenvector of H and that its eigenvalue is $\lambda = -1$. We look at the next case, x and v are orthogonal, therefore $(x, v) = 0$.

$$-2(0)x = (\lambda - 1)v \implies \lambda - 1 = 0$$

$$\lambda = 1$$

Therefore we have that if x and v are orthogonal that the eigenvalue corresponding to v is equal to 1.

(ii) Determinant

Next we look at the determinant of H. We have that from exercise one that H is unitary (orthogonal) and symmetric. Therefore (going in one direction) that $H^{-1} = H^*$. This is because $H^*H = \mathrm{I} = H^{-1}H$. Next we also have that for any matrix A, $\det(A^*) = \overline{\det(A)}$. We also have that, $\det(A)\det(A^{-1}) = 1$.

$$\det(H^{-1}H) = \det(H^{-1})\det(H) = 1$$

$$\overline{\det(H)}\det(H) = 1$$

$$\det(H)^2 = 1 \implies \det(H) = \pm 1$$

(iii) Singular Values

We have from the proof in exercise one, we have that $H \in \mathbb{R}^{m \times m}$ is a unitary (orthogonal) matrix. We have therefore that H preserves the length of vectors under transformation. We also look at the singular value decomposition of H.

$$H = U\Sigma V^T, \quad ||Hv|| = ||v||, \forall v \in \mathbb{R}^m$$

Let us look at a specific vector v_i now such that v_i is i-th column vector of V.

$$||Hv|| = ||U\Sigma V^T v_i|| = ||u_i \sigma_{ii}||$$

We recover the scalar-vector product, $u_i\sigma_{ii}$ where u_i is the i-th column vector of U and σ_{ii} is the i-th diagonal element of Σ. We return to the fact that by the Singular Value Decomposition Theorem, that U, V are unitary, that is they are composed of orthogonal column vectors with norm of 1. Therefore we have,

$$||Hv|| = ||v_i|| = ||\sigma_{ii}u_i|| = |\sigma_{ii}|||u_i||$$

$$1 = |\sigma_{ii}|1, \implies \sigma_{ii} = \pm 1$$

Therefore since our choice of v was arbitrary among the column vectors of V we have that this example exhausts all singular values for H. Thus the singular values of H are ± 1. It can even be argued that the plus minus in this context does not matter. Since singular values are scalars which in a transformation from one vector basis to another scale the vector in the resulting basis. The vectors in the output basis are orthogonal so scaling one vector say by -1 would not make that basis linearly dependent. Therefore we claim that any u_i will absorb the sign of σ_{ii} (Also because of the fact that singular values are always positive). So it is as simpler to claim that,

$$\sigma_{ii} = 1.$$

Therefore the singular values of H are such that, $\sigma_{ii} = \sigma_i = 1$. $\square$

5. Let $A \in \mathbb{R}^{m \times n}$. Show that $\text{cond}(A^T A) = (\text{cond}(A))^2$.

   We start with