Homework 3

Gomez - Math 19B

Due: Feb 2nd, 2024

Problems cover sections 5.3 and 5.4 of the textbook (FTC P1 and P2). No problems will require substitution or more complicated antiderivative methods. Each antiderivative needed will take the form of a function which you should have had experience differentiating in Calc 1 (Math 19A/Math 11A). Useful equations are found at the end of the assignment. More antiderivative practice problems can be found in section 4.9 in the textbook, exercises 1-12.

1 FTC P1

Evaluate the integrals using FTC P1:

1.

$$\int_0^1 \frac{4dx}{x^2+1}$$

$$= 4\int_0^1 \frac{dx}{x^2+1}$$

$$= 4\arctan(x)\Big|_0^1$$

$$= 4\left(\frac{\pi}{4} - 0\right) = \pi$$

$\int_0^1 \frac{4dx}{x^2+1} = \pi$

2.

$$\int_2^4 (t^2 - t^{-2})dt$$

$$= \int_2^4 t^2 dt - \int_2^4 t^{-2} dt$$

$$= \frac{t^3}{3}\Big|_2^4 + t^{-1}\Big|_2^4$$

$$= \left(\frac{64}{3} - \frac{8}{3}\right) + \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{221}{12}$$

$\int_2^4 (t^2 - t^{-2})dt = \frac{221}{12}$

3.

$$\int_{\pi/4}^{2\pi} (\cos(u) + u)du$$

$$\int_{\pi/4}^{2\pi} \cos(u)du + \int_{\pi/4}^{2\pi} udu$$

$$= \sin(u)\Big|_{\pi/4}^{2\pi} + \frac{u^2}{2}\Big|_{\pi/4}^{2\pi}$$

$$= \left(0 - \frac{\sqrt{2}}{2}\right) + \left(2\pi^2 - \frac{\pi^2}{32}\right)$$

$$= \frac{63\pi^2}{32} - \frac{\sqrt{2}}{2}$$

$\int_{\pi/4}^{2\pi} (\cos(u) + u)du = \frac{63\pi^2}{32} - \frac{\sqrt{2}}{2}$

4.

$$\int_5^7 \left(\frac{1}{z} + 2\sec z \tan z\right) dz$$

$$= \int_5^7 \frac{1}{z} dz + 2\int_5^7 \sec(z)\tan(z)dx$$

$$= \ln(z)\Big|_5^7 + 2\sec(z)\Big|_5^7$$

$$= (\ln(7) - \ln(5)) + 2(\sec(7) - \sec(5))$$

$\int_5^7 \left(\frac{1}{z} + 2\sec z \tan z\right) dz = \ln\left(\frac{7}{5}\right) + 2(\sec(7) - \sec(5))$

5.

$$\int_{\ln(1)}^{\ln(4)} e^x dx$$

$$= e^x\Big|_{\ln(1)}^{\ln(4)}$$

$$= e^{\ln(4)} - e^{\ln(1)}$$

$$= 4 - 1 = 3$$

$\int_{\ln(1)}^{\ln(4)} e^x dx = 3$

6.

$$\int_1^3 \left(\csc^2(x) + \frac{1}{x\ln 3}\right) dx$$

$$= \int_1^3 \csc^2(x)dx + \int_1^3 \frac{dx}{x\ln 3}$$

$$= -\cot(x)\Big|_1^3 + \log_3(x)\Big|_1^3$$

$$= \cot(1) - \cot(3) + \log_3(\frac{3}{)} = \cot(1) - \cot(3) + 1$$

$\int_1^3 \left(\csc^2(x) + \frac{1}{x\ln 3}\right) dx = \cot(1) - \cot(3) + 1$

7. Evaluate the integral, $\int_0^\pi v(t)dt$, using $v(t)$ defined below:

$$v(t) = \left\{ \begin{array}{cc} \sin t & \text{if} \quad 0 \le x \le \frac{\pi}{2} \\ 2(t - \frac{\pi}{2}) + 1 & \text{if} \quad \frac{\pi}{2} < t \le \pi \end{array} \right\}$$

$$\int_0^\pi v(t)dt = \int_0^{\pi/2} \sin(t)dt + \int_{\pi/2}^{\pi} 2(t - \frac{\pi}{2}) + 1dt$$

$$= -\cos(t)\Big|_0^{\pi/2} + t^2\Big|_{\pi/2}^{\pi} + (1-\pi)t\Big|_{\pi/2}^{\pi}$$

$$= -(0-1) + \left(\pi^2 - \frac{\pi^2}{4}\right) + (1-\pi)\left(\pi - \frac{\pi}{2}\right)$$

$$= 1 + \frac{3\pi^2}{4} + \frac{\pi}{2} - \frac{\pi^2}{2} = 1 + \frac{\pi^2}{4} + \frac{\pi}{2}$$

$\int_0^\pi v(t)dt = 1 + \frac{\pi^2}{4} + \frac{\pi}{2}$

2 FTC P2

8. Find $F(0)$, $F'(1)$, and $F'(3)$, where

$$F(x) = \int_0^x \sqrt{t^2 + t}dt$$

$$F(0) = \int_0^0 \sqrt{t^2 + t}dt = 0$$

$$F'(1) = f(1) = \sqrt{1^2 + 1} = \sqrt{2}$$

$$F'(3) = f(3) = \sqrt{3^2 + 3} = 2\sqrt{3}$$

$F(0) = 0, F'(1) = \sqrt{2}, F'(3) = 2\sqrt{3}$

9. Find $H(-2)$, $H'(-2)$, and $H'(5)$, where

$$H(x) = \int_{-2}^x \frac{du}{u^2 + 1}$$

$$H(-2) = \int_{-2}^{-2} \frac{du}{u^2 + 1} = 0$$

$$H'(-2) = \frac{1}{(-2)^2 + 1} = \frac{1}{5}$$

$$H'(5) = \frac{1}{(5)^2 + 1} = \frac{1}{26}$$

$H(-2) = 0, H'(-2) = \frac{1}{5}, H'(5) = \frac{1}{26}$

10. 5.4 Exercise 20: Calculate the derivative

$$\frac{d}{dx}\int_1^x \sin{(t^2)}dt$$

$$\frac{d}{dx}\int_1^{g(x)} \sin{(t^2)}dt = \sin((g(x))^2) \cdot g'(x),\ g(x) = x$$

$$= \sin(x^2)$$

$\frac{d}{dx}\int_1^x \sin{(t^2)}dt = \sin(x^2)$

11. Show that $F(x) = x^2e^{\tan(x)} + C$ satisfies,

$$F(x) = \int_1^x 2xe^{\tan(x)} + x^2\sec^2(x)e^{\tan(x)}dx$$

Proof. By FTC P2, if $F(x)$ is an antiderivative which satisfies the integral we should have that its derivative recovers the function inside the integral, i.e.

$$F'(x) = 2xe^{\tan(x)} + x^2 \sec^2(x)e^{\tan(x)}$$

So we take the derivative of $F(x)$:

$$\frac{d}{dx}F(x) = \frac{d}{dx}x^2 e^{\tan(x)} + C$$

$$= e^{\tan(x)}\frac{d}{dx}x^2 + x^2\frac{d}{dx}e^{\tan(x)} + \frac{d}{dx}C$$

$$= e^{\tan(x)}(2x) + x^2(\sec^2(x)e^{\tan(x)}) + 0$$

$$\frac{d}{dx}F(x) = 2xe^{\tan(x)} + x^2 \sec^2(x)e^{\tan(x)} = f(x)$$

$\square$

See proof above.

Equations on next page

3 Fundamental Theorem of Calculus P1 and P2 equations:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{1}$$

$$F(x) = \int_a^x f(s)ds \tag{2}$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_c^x f(s)ds = f(x) \tag{3}$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(s)ds = f(g(x))g'(x) - f(h(x))h'(x) \tag{4}$$

4 Net Change and Displacement Equations:

$$\text{Net Change in } s(t) \text{ on } [t_1, t_2] = \int_{t_1}^{t_2} s'(t)dt \tag{5}$$

Integral of Velocity

$$\text{Displacement on } [t_1, t_2] = \int_{t_1}^{t_2} v(t)dt \tag{6}$$

$$\text{Total Distance traveled during } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)|dt \tag{7}$$

5 Properties of Integrals:

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx \tag{8}$$

$$\int Cf(x)dx = C\int f(x)dx,\ C = \text{const.} \tag{9}$$

Additivity Theorem for Adjacent Integrals

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ \ a < c < b \tag{10}$$