Section 4

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UCSC Math-19B

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Plan for Today

Topics to Cover
- Figuring out working groups
- Work and Energy
- Numerical Approximations

Section Activity 4
- 4 questions

Upcoming Assignments
- Homework 4 (Due Fri, Feb. 9^{th})
- Project 1 (Due Tues, Feb 20^{th}).

Scan this to submit working groups form!

GET INTO WORKING GROUPS ON CANVAS, ignore that llink below
https://forms.gle/tGbYvzhE8Eyqxw9c6

Learning Outcomes

- Understanding physically motivated integration problems
- Differentiating and applying Numerical Integration Techniques

Work and Energy Integrals

Physics is a language spoken in calculus often consisting of derivatives, integrals, and differential equations.
Recall that we can relate acceleration $a(t)$, velocity $v(t)$, and position $s(t)$ through derivatives.

$$a(t) = \frac{dv(t)}{dt}, \quad v(t) = \frac{ds(t)}{dt}$$

Similarly, we can express Work and Energy in terms of integrals. We have,

$$\text{Work, } W = \int_{\mathcal{D}} F dx$$

Where $\mathcal{D}$ represents the domain over which you integrate (e.g. $x \in \mathcal{D} := [a, b]$) and $F(x)$ represents a force that varies in x.

Work and Energy

We can look at this relationship a little more closely. By the fundamental theorem of calculus we should recover force $F(x)$ by taking the derivative of Work.

$$\frac{d}{dx}W = \frac{d}{dx}\int_a^x F(s)ds = F(x)$$

Another thing to notice is the units of force and work. In physics, forces are often measures in Newtons (N). By integrating in x (i.e. meters, m), we obtain a new scientific unit, Joules (J).

$$N = \frac{kg \cdot m}{s^2}$$

$$J = N \cdot m = \frac{kg \cdot m^2}{s^2}$$

Numerical Methods for Integrals

There are a couple new methods for approximating integrals and quantifying their errors shown in section 7.1 in the book.

- Trapezoidal Rule T_N
- Simpson's Rule S_N
- Error Bound, $\mathrm{Error}(A_N)$

Trapezoidal Rule

$$A_{Trap} = \frac{1}{2}(w)(h_1 + h_2) = \frac{1}{2}\Delta x(f(x_{i-1}) + f(x_i)) = A_i$$

$$\Delta x = \frac{b-a}{N}, \quad x_i = a + i\Delta x$$

$$T_N = \sum_{i=1}^{N} A_i = \frac{\Delta x}{2}\sum_{i=1}^{N}(f(x_{i-1}) + f(x_i))$$

Error Bound for T_N

$$\text{Error}(T_N) \leq \frac{K_2(b-a)^3}{12N^3}$$

Where K_2 is defined to be equal to be the maximum of $|f''(x)|$ on $[a, b]$.

Simpson's Rule S_N

Simpson's Rule is obtained from a combination of the Trapezoidal Rule and the Midpoint Approximation.

$$S_N = \frac{1}{3}T_{N/2} + \frac{2}{3}M_{N/2}, \quad N \text{ even}$$

$$\Delta x = \frac{b-a}{N}, \quad y_i = f(a + i\Delta x)$$

$$S_N = \frac{1}{3}\Delta x[y_0 + 4y_1 + 2y_2 + \cdots + 2y_{N-2} + 4y_{N-1} + y_N]$$

Error Bound

Error Bounds are also defined, with K_2 being the maximum value of the second derivative $f''(x)$ on $[a, b]$ and K_4 being defined the same but with the fourth derivative.

$$\text{Error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2}$$

$$\text{Error}(M_N) \leq \frac{K_2(b-a)^3}{24N^2}$$

$$\text{Error}(S_N) \leq \frac{K_4(b-a)^5}{180N^4}$$

$$K_2 \geq \max_{x\in[a,b]}\{f''(x)\}, \quad K_4 \geq \max_{x\in[a,b]}\{f^{(4)}(x)\}$$

Discussion Section Activity 4

Woah look, the TA is about to write the code on the board!

Better