AM 213A, Winter 2024
Homework 2 (100 points)

Posted on Fri, Jan 19, 2024
Due 11:59 pm, Thu, Feb 1, 2024

Submit your homework to Gradescope

- Use LaTeX or MS-words like text editors for homework. A scanned copy of handwritten solutions will be acceptable on an exceptional case-by-case only with permission from the instructor.
- Your report needs to have relevant discussions on each problem to describe what you demonstrate. In this coursework, do not simply copy and paste any screen outputs (e.g., screenshots) from your code execution and provide them as answers. Instead, discuss code results required for each problem and display them concisely with logical justification. For all coding problems, showing screen outputs only from your code execution is insufficient and will lose points.
- To disprove, you need to provide a counter-example.
- All homework submissions should meet the deadline. Late homework will be accepted under emergencies with permission from the instructor.
- Submit all code and reports using the following naming conventions:
  - The theory and computational report should be together in one PDF named as `LastnameFirstname_Report_hwX.pdf` where `X` is the homework number
  - The supporting code for each homework should be provided in a single compressed directory named as `LastnameFirstname_Code_hwX.tar.gz` or `LastnameFirstname_Code_hwX.zip`
    - Include only source files (e.g. `*.f90`, `*.c`, etc.) and the needed Makefile
    - Do **not** include object files, module files, executables, or data files.

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1. (6 pts) Let $A \in \mathbb{C}^{m\times m}$ be both upper-triangular and unitary. Show that A is a diagonal matrix. Does the same hold if $A \in \mathbb{C}^{m\times m}$ is both lower-triangular and unitary?

2. (15 pts) Prove the following in each problem.

(a) (5 pts) Let $A \in \mathbb{C}^{m\times m}$ be invertible and $\lambda \neq 0$ is an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

(b) (5 pts) Let $A, B \in \mathbb{C}^{m\times m}$. Show that AB and BA have the same eigenvalues.

(c) (5 pts) Let $A \in \mathbb{R}^{m\times m}$. Show that A and A^* have the same eigenvalues. (Hint 1: Use $\det(M) = \det(M^T)$ for any square matrix $M \in \mathbb{R}^{m\times m}$ in connection to the definition of characteristic polynomials. Hint 2: When a real-valued matrix A has a complex eigenvalue λ, then $\bar{\lambda}$ is also an eigenvalue of A.)

3. (8 pts) Let $A \in \mathbb{C}^{m\times m}$ be hermitian. Suppose that for nonzero eigenvectors of A, there exist corresponding eigenvalues λ satisfying $Ax = \lambda x$.'

(a) (4 pts) Prove that all eigenvalues of A are real.

(b) (4 pts) Let x and y be eigenvectors corresponding to distinct eigenvalues. Show that $(x, y) = 0$, i.e., they are orthogonal. (Hint: Use the result of Part (a).)

4. (6 pts) A matrix A is called *positive definite* if and only if $(Ax, x) > 0$ for all $x \neq 0$ in $\mathbb{C}^m$. Suppose A is Hermitian. Show that A is positive definite if and only if $\lambda_i > 0$, $\forall \lambda_i \in \Lambda(A)$, the spectrum of A.

(Hint 1: Use the following Theorem: If $A \in \mathbb{C}^{m\times m}$ is Hermitian, then A has real eigenvalues $\lambda_i, i = 1, \cdots, m$, not necessarily distinct, and m corresponding eigenvectors u_i form an orthonormal basis for $\mathbb{C}^m$. Hint 2: Now, you realize that for any arbitrary $x \neq 0$, you can write $x = \sum_{i=1}^{m} \alpha_i u_i$. Use Hint 1 to show $(Ax, x) = \sum_{i=1}^{m} \lambda_i |\alpha_i|^2$ and draw your conclusion.) Note that in the lecture, we discussed that a Hermitian matrix is non-defective, which implies that a Hermitian matrix has m linearly independent eigenvectors u_i and is diagonalizable with the eigenvector matrix. The above theorem states that, in addition, the linearly independent eigenvector matrix $U = [u_1|\ldots|u_m]$ "orthogonally diagonalizes" A. The theorem is referred to as the "Principal axis theorem."

5. (6 pts) Suppose A is unitary.

(a) (3 pts) Let (λ, x) be an eigenvalue-vector pair of A. Show λ satisfies $|\lambda| = 1$.

(b) (3 pts) Prove or disprove $||A||_F = 1$.

6. (6 pts) Let $A \in \mathbb{C}^{m\times m}$ be skew-hermitian, i.e., $A^* = -A$.

(a) (3 pts) Show that the eigenvalues of A are pure imaginary.

(b) (3 pts) Show that $I - A$ is nonsingular.

7. (5 pts) Show that $\rho(A) \leq ||A||$, where $\rho(A)$ is the spectral radius of A.

8. (6 pts) Let $A \in \mathbb{R}^{m\times m}$ and $Av_i = \alpha_i v_i$, $i = 1, \ldots, m$, where (α_i, v_i) is the eigenvalue-eigenvector pair of A for each i. Assume that A is symmetric, $A = A^T$ and the eigenvalues α_i are all distinct. Show that the solution to $Ax = b$, $x \neq 0$, can be written as

$$x = \sum_{i=1}^{m} \frac{v_i^T b}{v_i^T A v_i} v_i. \tag{1}$$

(Hint 1: Use the fact that symmetric matrices are non-defective, and non-defective matrices are diagonalizable. Hint 2: Use the fact that, for real symmetric matrices, the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other, i.e., $(v_i, v_j) = 0, i \neq j$.)

9. (6 pts) Let A be defined as an outer product $A = uv^*$, where $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$.

- **(a) (3 pts)** Prove or disprove $||A||_2 = ||u||_2||v||_2$.
- **(b) (3 pts)** Prove or disprove $||A||_F = ||u||_F||v||_F$. (Note: For a vector u, $||u||_F = ||u||_2$ by definition.)

10. (9 pts) Let $A, Q \in \mathbb{C}^{m\times m}$, where A is arbitrary and Q is unitary.

- **(a) (3 pts)** Show that $||AQ||_2 = ||A||_2$ (note that we did $||QA||_2 = ||A||_2$ in class).
- **(b) (6 pts)** Show that $||AQ||_F = ||QA||_F = ||A||_F$.

11. (8 pts) We say that $A, B \in \mathbb{C}^{m\times m}$ are unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m\times m}$.

- **(a) (4 pts)** Show that if A and B are unitarily equivalent, then they have the same singular values. (Hint. Note that every matrix has a SVD and their singular values are uniquely defined.)
- **(b) (4 pts)** Show that the converse of Part (a) is not necessarily true.

12. (9 pts) Find the relative condition number of the following functions and discuss if there is any concern of being ill-conditioned. If so, discuss when.

- **(a) (3 pts)** $f(x_1, x_2) = x_1 + x_2$
- **(b) (3 pts)** $f(x_1, x_2) = x_1 x_2$
- **(c) (3 pts)** $f(x) = (x - 2)^9$

13. (10 pts) Note that the function $f(x) = (x-2)^9$ in Part (c) in Problem 9 can also be expressed as $g(x) = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$. Note that, mathematically, the two functions f and g are identical. (Note: Use Matlab or Python for this problem, particularly for plotting purposes. Fortran coding is not necessary.)

- **(a) (3 pts)** Plot $f(x)$ by evaluating discrete function values of f at $1.920, 1.921, 1.922, \ldots, 2.080$, which are equally spaced with the distance of 0.001.
- **(b) (3 pts)** Over-plot $g(x)$ at the same set of discrete points in Part (a).
- **(c) (4 pts)** Draw your conclusion from your results of Part (c) in Prob. 11 and Parts (a) and (b) in this problem.