Homework 2: Report

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1. Let $A \in C^{m\times m}$ be both upper-triangular and unitary. Show that A is a diagonal matrix. Does the same hold if $A \in C^{m\times m}$ is both lower-triangular and unitary?

   Proof. **(Upper Triangular, by Induction)**

   Assume matrix $A \in \mathbb{C}^{m\times m}$ is unitary and is upper triangular such that,

$$A^*A = \mathrm{I}_m = AA^*$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix}$$

   Where A^* is the complex transpose matrix of A. We have then that A^* is of the form,

$$A^* = \begin{bmatrix} \overline{a_{11}} & 0 & \cdots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1m}} & \overline{a_{2m}} & \cdots & \overline{a_{mm}} \end{bmatrix}$$

   Then the product of matrix multiplication of A^* and A is then defined as B, (i.e. $B = A^*A$), and because A is a Unitary matrix, is equal to I.

   (Base Case):

   Now assume that the elements above the diagonal in A are non-zerp. Next examine the $(1,1)$ and $(2,1)$ cells of the matrix product, B. By the operation of Matrix multiplication we should have,

$$B(2,1) = a_{11}\cdot\overline{a_{12}} = \mathrm{I}_m(2,1) = 0$$

$$B(1,1) = a_{11}\cdot\overline{a_{11}} = \mathrm{I}_m(1,1) = 1$$

   From this, we know that $a_{11} \neq 0$ and $\overline{a_{11}} \neq 0$. But we have that the product of $a_{11}\cdot\overline{a_{12}} = 0$. Since we have that $a_{11} \neq 0$, we must therefore have that $\overline{a_{12}} = 0$ and by the definition of a complex conjugate, $a_{12} = 0$.

   (Inductive Step)

   We need to show that for a integer $k \leq m-1$ all of the columns of matrix A, $\vec{C}_i$, up to $\vec{C}_k$ is of the form,

$$\vec{C}_i = \begin{bmatrix} 0 \\ \vdots \\ a_{ii} \\ \vdots \\ 0 \end{bmatrix}$$

then $\vec{C}_{k+1}$ is also of the same form. We have that in the matrix product between A^* and A, B, then the i-th row of B is defined as the inner produt between the i-th row of A^*, $\vec{r}_i^*$ and the j-th column of A, $\vec{c}_j$.

$$B(i,:) = [(\vec{r}_i^*, \vec{c}_1), (\vec{r}_2^*, \vec{c}_2), \cdots, (\vec{r}_i^*, \vec{c}_j)] = \mathrm{I}_m(i,:) = [0, \cdots, 1, \cdots, 0]$$

Look at the $(k+1)$-th row of B. We have from the given form of the columns, $\{\vec{c}_1, \cdots, \vec{c}_k\}$,

$$(\vec{r}_{k+1}^*, \vec{c}_j) = \overline{a_{j(k+1)}} \cdot a_{jj} = \left\{ \begin{array}{lll} 0 & \text{if,} & j \neq k+1 \\ 1 & \text{if,} & j = k+1 \end{array} \right\}, i, j < k+1$$

We also have that each $a_{ii} \neq 0$. Thereby, for all $j < k+1$,

$$\overline{a_{j(k+1)}} = 0 \implies a_{j(k+1)} = 0$$

We now write the column, $\vec{c}_{k+1}$.

$$\vec{c}_{k+1} = \begin{bmatrix} 0 \\ \vdots \\ a_{(k+1)(k+1)} \\ \vdots \\ 0 \end{bmatrix}$$

Therefore, we have that $\vec{c}_{k+1}$ is of the same form as $\vec{c}_i, i \leq k$. By induction, each column of A is of this form. Therefore, A is a diagonal matrix!

□

Proof. **(Lower Triangular, by case of Upper Triangular)**

Assume as before, $A \in \mathbb{C}^{m \times m}$ is unitary and is lower triangular such that,

$$A^*A = \mathrm{I}_m = AA^*$$

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

Where A^* is the complex transpose matrix of A. We have then that A^* is of the form,

$$A^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ 0 & \overline{a_{22}} & \cdots & \overline{a_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{a_{mm}} \end{bmatrix}$$

We then define the matrix $C = A^*$, $C^* = A$. Notice that C is an upper triangular, unitary matrix. By the previous proof, C is a diagonal matrix. Notice all of its "off-diagonal" elements are zero. As a consequnce, all (i, j)-elements of C which are zero imply that (j, i)-elements of C^* are zero. Therefore, $C^* = A$ is a diagonal matrix.

□

2. Prove the following in each problem.

   (a) Let $A \in \mathbb{C}^{m \times m}$ be invertible and $\lambda \neq 0$ is an eigenvalue of A. Showthat λ^{-1} is an eigenvalue of A^{-1}.

Proof. Take any $A \in \mathbb{C}^{m\times m}$ to be invertible. Then we have inverse, A^{-1} exists such that,

$$AA^{-1} = \mathrm{I}_m = A^{-1}A$$

We also have by the fact that $\lambda \neq 0$ is an eigenvalue of A that,

$$\det(A - \lambda \mathrm{I}_m) = 0$$

We can substitute for I_m.

$$\det(A - \lambda \mathrm{I}_m) = \det(A - \lambda(A^{-1}A)) = \det(A)\det(\mathrm{I}_m - \lambda A^{-1}) = 0$$

$$\det(I_m - \lambda A^{-1}) = -\det(A^{-1} - \frac{1}{\lambda}\mathrm{I}_m) = 0$$

$$\det(A^{-1} - \frac{1}{\lambda}\mathrm{I}_m) = 0$$

Therefore, λ^{-1} is an eigenvalue of A^{-1}.

□

(b) Let $A, B \in \mathbb{C}^{m\times m}$. Show that AB and BA have the same eigenvalues.

Proof. Let A, B be square matrices as shown above. Now look at some eigenvalue of the matrix product AB, λ. We have by definition of an eigenvalue the following equality.

$$AB\vec{v} = \lambda\vec{v}$$

Now we multiply both vectors by the matrix B.

$$B(AB\vec{v}) = B(\lambda\vec{v})$$

$$(BA)(B\vec{v}) = \lambda(B\vec{v})$$

$$BA\vec{w} = \lambda\vec{w}$$

Therefore λ is also an eigenvalue of the matrix product BA. Since our choice of λ was arbitrary, we have that all eigenvalues of AB are eigenvalues of BA. □

(c) Let $A \in \mathbb{R}^{m\times m}$. Show that A and A^* have the same eigenvalues. (Hint 1: Use $\det(M) = \det(M^T)$ for any square matrix $M \in \mathbb{R}^{m\times m}$ in connection to the definition of characteristic polynomials. Hint 2: When a real-valued matrix A has a complex eigenvalue λ, then $\overline{\lambda}$ is also an eigenvalue of A.)

Proof. First look at an an arbitrary eigenvalue, λ, of A.

$$\det(A - \lambda\mathrm{I}) = 0$$

We examine the determinant of a conjugate transpose. Since determinant is the sum/difference of the products along the diagonals, we have that the determinant of the conjugate tranpose is equivalent to the complex conjugate of the determinant of the tranpose.

$$\det(A^*) = \overline{\det(A^T)} = \overline{\det(A)}$$

We then look at the definition of the characteristic polynomial.

$$\det(A - \lambda\mathrm{I}) = 0 \implies \overline{\det(A - \lambda\mathrm{I})} = 0 = \det[(A - \lambda\mathrm{I})^*] = \det(A^* - \overline{\lambda}\mathrm{I})$$

We have then that the complex conjugate of all eigenvalues of A are eigenvalues of A^*. We also notice that since A and A^* are real, that the characteristic polynomials are also real. Thus if we are to have a complex number as a root of a real polynomial, the complex conjugate must also be a root. Thereby, all complex λ and their conjugates $\overline{\lambda}$ are roots of both A and A^*. Moreover, if λ is not complex it is automatically a root of both A and A^*. □

3. Let $A \in \mathbb{C}^{m\times m}$ be hermitian. Suppose that for nonzero eigenvectors of A, there exist corresponding eigenvalues λ satisfying $Ax = \lambda x$.

   a Prove that all eigenvalues of A are real.

   Proof. We look at an arbitary eigenvalue of A.

$$Ax = \lambda x, x \in \mathbb{C}^m$$

   we multiple both sides by the conjugate transpose of x.

$$x^*(Ax) = x^*(\lambda x)$$

$$x^*Ax = \lambda(x^*x)$$

   We should notice that x^*x is a scalar with a real value. This is because each component of x is multiplied against its complex conjugate. Next we look at the dimensions and hermitian quantity of x^*Ax. We have that $x^* \in \mathbb{C}^{1\times m}$, otherwise known as a row vector. We also have, $Ax \in \mathbb{C}^m$. Thereby, the matrix product of x^* and Ax is a 1×1 quantity, a scalar! More importantly we have,

$$(x^*Ax)^* = x^*A^*(x^*)^* = x^*Ax$$

   So, x^*Ax is hermitian, or rather, x^*Ax is a real-valued scalar. We then have,

$$x^*Ax = \lambda(x^*x)$$

   Where both x^*Ax and x^*x are real valued, so consequently $\lambda \in \mathbb{R}$. $\square$

   b. Let x and y be eigenvectors corresponding to distinct eigenvalues. Show that (x, y) = 0, i.e., they are orthogonal. (Hint: Use the result of Part (a).)

   Proof. By the quaality that A is hermition, we have for any two vectors, $x, y \in \mathbb{C}^m$, that

$$(Ax, y) = x^*A^*y = x^*Ay = (x, Ay)$$

   Therefore we can say for distinct eigenvectors, v_1, v_2 ($v_1 \neq v_2$), with distinct eigenvalues, λ_1, λ_2 ($\lambda_1 \neq \lambda_2$),

$$(Av_1, v_2) - (v_1, Av_2) = 0$$
$$= (\lambda_1 v_1, v_2) - (v_1, \lambda_2 v_2) = \overline{\lambda_1} v_1^* v_2 - v_1^* \lambda_1 v_2$$
$$= (\overline{\lambda_1} - \lambda_2) v_1^* v_2 = 0$$

   There are two things to notice, first since all eigenvalues are real, $\overline{\lambda_1} = \lambda_1$. Second, by our construction of the problem, $\lambda_1 \neq \lambda_2$. Thereby, $(\overline{\lambda_1} - \lambda_2) \neq 0$. So,

$$v_1^* v_2 = 0 = (v_1, v_2)$$

   $\square$

4. A matrix A is called positive definite if and only if $(Ax, x) > 0$ for all $x \neq 0$ in $\mathbb{C}^m$. Suppose A is Hermitian. Show that A is positive definite if and only if $\lambda_i > 0, \forall \lambda_i \in \Lambda(A)$, the spectrum of A.

   Proof. By the property of A being hermitian, that we can write any vector, $x \in \mathbb{C}_m, x \neq \vec{0}$ as the linear combination of the orthonormal eigenvectors of A, u_i.

$$x = \alpha_1 u_1 + \cdots + \alpha_m u_m$$

   We then look the inner product, (Ax, x).

$$Ax = A(\alpha_1 u_1 + \cdots + \alpha_m u_m) = \lambda_1 \alpha_1 u_1 + \cdots + \lambda_m \alpha_m u_m$$

$$(Ax)^* = \overline{\lambda_1\alpha_1}u_1^* + \cdots + \overline{\lambda_m\alpha_m}u_m^*$$
$$(Ax, x) = (\overline{\lambda_1\alpha_1}u_1^* + \cdots + \overline{\lambda_m\alpha_m}u_m^*)(\alpha_1 u_1 + \cdots + \alpha_m u_m)$$

Here by the property of an orthonormal vector set, we have that $u_i^* u_j = 0$ if $i \neq j$ and $= 1$ if $i = j$.

$$(Ax, x) = \overline{\lambda_1\alpha_1}\alpha_1 + \cdots + \overline{\lambda_m\alpha_m}\alpha_m = \sum_{i=1}^{m} \lambda_i |\alpha_i|^2$$

Of course, $|\alpha_i|^2$ is a strictly positive value. So for $(Ax, x) < 0$ we need at least one $\lambda_i < 0$. In fact, it is the case that if even one $\lambda_i < 0$ that $(Ax, x) \not> 0$ for all $x \in \mathbb{C}^m$. To prove that $(Ax, x) > 0, \forall x \in \mathbb{C}^m$, we take the case of only the smallest λ_i, $\lambda_k < 0$ (i.e. $|\lambda_k| < |\lambda_i|, \forall \lambda_i \in (\Lambda(A) - \{\lambda_k\})$). We can show by counter-example

$$\lambda_k < 0, x \in \mathbb{C}^m, x = \alpha_1 u_1 + \cdots + \alpha_m u_m$$
$$(Ax, x) = \lambda_k |\alpha_k|^2 + \sum_{i=1, i\neq k}^{m} \lambda_i |\alpha_i|^2$$
$$\exists x_* \in \mathbb{C}^m, \text{ such that } |\alpha_k|^2 = \frac{1}{|\lambda_k|} \sum_{i=1, i\neq k}^{m} \lambda_i |\alpha_i|^2 + 1$$
$$(Ax_*, x_*) = \lambda_k < 0, \text{ by construction.}$$

We have then that if $(Ax, x) < 0$, then $\lambda_i < 0$, and if $\lambda_i < 0$, then $\exists x \in \mathbb{C}^m$ such that $(Ax, x) < 0$. So if A is positive definite if and only if all eigenvalues of A are positive. □

5. Suppose A is unitary.

   (a) Let (λ, x) be an eigenvalue-vector pair of A. Show λ satisfies $|\lambda| = 1$.

   Proof. Since A is unitary, we have that it preserves the angle and length of vectors under transformations. (i.e $(Ax, Ax) = (x, x)$ for any vector $x \in \mathbb{C}^m$). Thereby we have,

$$(Ax, Ax) = (\lambda x, \lambda x) = \overline{\lambda} x^* \lambda x = |\lambda|^2 x^* x = |\lambda|^2 (x, x)$$
$$(x, x) = (Ax, Ax) = |\lambda|^2 (x, x) \implies |\lambda|^2 = 1$$
$$|\lambda| = 1$$

□

   (b) Prove or disprove $||A||_F = 1$

   Proof. We have from the definition of the Frobenius Norm and since A is unitary,

$$||A||_F = \sqrt{\text{Tr}(A^* A)} = \sqrt{\text{Tr(I)}}$$

   Assume now that $\text{I} \in \mathbb{R}^{m\times m}$. Then, $\text{Tr(I)} = m$

$$||A||_F = \sqrt{m}$$

   Therefore, $||A||_F \neq 1$ unless, $A \in \mathbb{C}^{1\times 1}$ i.e. A is a scalar. In general though, for any $A \in \mathbb{C}^{m\times n}$ where $m, n > 1$, $||A||_F \neq 1$. □

6. Let $A \in \mathbb{C}^{m\times m}$ be skew-hermitian, i.e., $A^* = -A$.

   (a) Show that the eigenvalues of A are pure imaginary.

Proof. We look at the skew-hermition matrix A with (λ, x) being an eigenvalue-eigenvector pair $(Ax = \lambda x)$. We start by looking at the inner products $(Ax, x), (x, Ax)$.

$$\begin{aligned} &(Ax, x) - (x, Ax) \\ &= x^* A^* x - x^* Ax \\ &= -2x^*(Ax) \\ &= -2\lambda x^* x \end{aligned} \qquad \begin{aligned} &(Ax, x) - (x, Ax) \\ &= (\lambda x)^* x - x^* \lambda x \\ &= (\overline{\lambda} - \lambda) x^* x \end{aligned}$$

$$\begin{aligned} -2\lambda x^* x &= (\overline{\lambda} - \lambda) x^* x \\ -2\lambda &= \overline{\lambda} - \lambda \\ -\lambda &= \overline{\lambda} \implies \mathbb{R}(\lambda) = 0 \end{aligned}$$

Since our choice of x and lambda were arbitrary, we have that all eigenvalues of A are purely imaginary.

□

(b) Show that $I - A$ is nonsingular

Proof. To show that $\mathrm{I} - A$ is nonsingular we simply need to show that $\det(\mathrm{I} - A) \neq 0$. We have,

$$\det(\mathrm{I} - A) = (-1)^m \det(A - \mathrm{I})$$

We notice that $\det(A - \mathrm{I})$ looks very similar to the definition of the characteristic polynomial, $\det(A - \lambda \mathrm{I})$. We have then by the definition of a characteristic polynomial,

$$\det(\mathrm{I} - A) = (-1)^m \det(A - \mathrm{I}) = 0, \text{ if and only if } \lambda = 1 \text{ is an eigenvalue of } A.$$

We have from part (a) of the problem that all eigenvalues of A are pure imaginary, i.e. $\lambda \neq 1$. Therefore

$$\det(\mathrm{I} - A) = (-1)^m \det(A - \mathrm{I}) \neq 0$$

Therefore, $\mathrm{I} - A$ is nonsingular by definition. □

7. Show that $\rho(A) \leq ||A||$, where $\rho(A)$ is the spectral radius of A.

Proof. Start by taking the eigenvalue-eigenvector pair (λ_*, v) such that, $|\lambda_*| \geq |\lambda_i|, \lambda_i \in \Lambda(A)$. We have from the definition of the a matrix norm,

$$\begin{gathered} ||A|| = \sup_{x \in \mathbb{C}^m} \frac{||Ax||}{||x||} \geq \frac{||Av||}{||v||} \\ \frac{||Av||}{||v||} = \frac{||\lambda_* v||}{||v||} = \frac{|\lambda_*| ||v||}{||v||} = |\lambda_*| \\ ||A|| = \sup_{x \in \mathbb{C}^m} \frac{||Ax||}{||x||} \geq |\lambda_*| = \rho(A) \\ ||A|| \geq \rho(A) \end{gathered}$$

□

8. Let $A \in \mathbb{R}^{m \times m}$ and $Av_i = \alpha_i v_i, i = 1, ..., m$, where (α_i, v_i) is the eigenvalue-eigenvector pair of A for each i. Assume that A is symmetric, $A = A^T$ and the eigenvalues α_i are all distinct. Show that the solution to $Ax = b, x \neq 0$, can be written as,

$$x = \sum_{i=1}^{m} \frac{v_i^T b}{v_i^T A v_i} v_i$$

(Hint 1: Use the fact that symmetric matrices are non-defective, and non-defective matrices are diagonalizable. Hint 2: Use the fact that, for realsymmetric matrices, the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other, i.e., $(v_i, v_j) = 0, i \neq j$.)

Proof. Since we have that A is a real, symmetric matrix, it is true that its eigenvectors form an orthogonal basis which spans $\mathbb{R}^m$. Then we could write b as a linear combination of the eigenvalues of A.,

$$b = c_1 v_1 + \cdots + c_m v_m = A(d_1 v_1 + \cdots + d_m v_m), c_i = \alpha_i d_i$$

We now need to find the scalar coefficients d_i to obtain the correct linear combination. We next look at the inner products $(v_i, b), (v_i, Av_i)$. We have,

$$(v_i, b) = v_i^T(\alpha_1 d_1 v_1 + \cdots + \alpha_m d_m v_m) = \alpha_i d_i v_i^T v_i$$

The inner product of v_i with any $v_j, i \neq j$ is zero by orthogonality, so only the $v_i^T v_i$ term remains.

$$(v_i, Av_i) = v_i^T(Av_i) = v_i^T(\alpha_i v_i) = \alpha_i v_i^T v_i$$

We have then that, $\frac{(v_i,b)}{(v_i,Av_i)} = d_i$. Therefore we can now write,

$$x = \sum_{i=1}^{m} \frac{(v_i, b)}{(v_i, Av_i)} v_i = \sum_{i=1}^{m} \frac{v_i^T b}{v_i^T A v_i} v_i$$

□

9. Let A be defined as an outer product $A = uv^*$, where $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$.

(a) Prove or disprove $||A||_2 = ||u||_2||v||_2$

Proof. We have from the definition for the p-norm of a matrix $A \in \mathbb{C}^{\boldsymbol{\mu}_* \times n}$,

$$||A||_2 = \sup\left\{\frac{||Ax||_2}{||x||_2}\Big| x \in \mathbb{C}^n\right\}$$

$$||A||_2 = ||uv^*||_2 = \sup\left\{\frac{||uv^*x||_2}{||x||_2}\Big| x \in \mathbb{C}^n\right\}$$

Notice here, that v^*a produces a 1×1 matrix, a scalar. We have the by the property of matrix and vector norms,

$$||A||_2 = \sup\left\{\frac{||u\alpha||_2}{||x||_2}\Big| x \in \mathbb{C}^n\right\} = \sup\left\{\frac{||u||_2|\alpha|}{||x||_2}\Big| x \in \mathbb{C}^n\right\}, \ \alpha = |v^*x|$$

$$||A||_2 = ||u||_2 \sup\left\{\frac{|\alpha|}{||x||_2}\Big| x \in \mathbb{C}^n\right\}$$

We now look at the vector norm equality, the Hölder Inequality, for $p = q = 2$,

$$|\alpha| = |v^*x| \leq ||v||_2||x||_2$$

So,

$$\sup\left\{\frac{|\alpha|}{||x||_2}\Big| x \in \mathbb{C}^n\right\} \leq ||v||_2$$

We now chose $x = cv, c \in \mathbb{R}$, c constant. Therefore,

$$|(v, x)| = |v^*x| = |\alpha| = |v^*cv| = |c|||v||_2^2 = ||v||_2||x||_2$$

We have then that any x colinear to v gives us the supremum case. Therefore,

$$||A||_2 = ||u||_2 \left(\frac{||v||_2||x||_2}{||x|_2}\right) = ||u||_2||v||_2$$

□

(b) Prove or disprove $||A||_F = ||u||_F||v||_F$

Proof. We begin with the definition of the Frobenius Norm.

$$||A||_F = \sqrt{\mathrm{Tr}(AA^*)} = \sqrt{\mathrm{Tr}(uv^*vu^*)}$$

$$||A||_F = \sqrt{\mathrm{Tr}(u||v||_2^2u^*)} = ||v||_2\sqrt{\mathrm{Tr}(uu^*)}$$

$$||A||_F = ||v||_2\sqrt{||u||_2^2} = ||v||_2||u||_2 = ||v||_F||u||_F$$

□

10. Let $A, Q \in \mathbb{C}^{m\times m}$, where A is arbitrary and Q is unitary

(a) Show that $||AQ||_2 = ||A||_2$

Proof. We begin with the definition of a 2-norm for matrices.

$$||AQ||_2 = \sup\left\{\frac{||AQx||_2}{||x||_2}\Big| x \in \mathbb{C}^m\right\}$$

$$||AQ||_2 = \sup\left\{\frac{||Ay||_2}{||x||_2}\Big| x \in \mathbb{C}^m\right\},\ y = Qx, ||y||_2 = ||x||_2$$

We have by the property of unitary matrices that the length and angles of vectors are preserved under transformations. Thus, $||Qx||_2 = ||x||_2 = ||y||_2$. We then substitute,

$$||AQ||_2 = \sup\left\{\frac{||Ay||_2}{||y||_2}\Big| y \in \mathbb{C}^m\right\} = ||A||_2$$

□

(b) Show that $||AQ||_F = ||QA||_F = ||A||_F$.

Proof. We start with the definition of the frobenius norm.

$$||AQ||_F = \sqrt{\mathrm{Tr}((AQ)^*AQ)} = \sqrt{\mathrm{Tr}((QA)^*QA)} = ||QA||_F$$

$$||AQ||_F = \sqrt{\mathrm{Tr}(A^*Q^*QA)} = \sqrt{\mathrm{Tr}(A^*A)} = ||A||_F$$

□

11. We say that $A, B \in \mathbb{C}^{m\times m}$ are unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m\times m}$.

(a) Show that if A and B are unitarily equivalent, then they have the same singular values.

Proof. We start with the SVD of B.

$$B = U_B\Sigma V_B^T$$

$$A = QBQ^* = QU_B\Sigma V_B^TQ^* = U_A\Sigma V_A^T$$

□

(b) Show that the converse of Part (a) is not necessarily true

Proof. We start with a 2D case for simplicity to disprove the converse. Let us take A, B with the same singular values, i.e. they share the same matrix Σ in their singular value decompositions.

$$A = U_A \Sigma V_A^*$$

$$B = U_B \Sigma V_B^*$$

We have since $A, B \in \mathbb{C}^{2\times 2}$. That U_A, U_B, V_A, V_B are all unit basis for $\mathbb{C}^2$. We can then impose some choice of U_A and U_B. We chose for example,

$$U_A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$U_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

$$U_A = QU_B$$

$$U_A = Q$$

We now need to fix V_A, V_B which cannot be transformed by Q/Q^*. We have,

$$V_A^* = V_B^* Q^* \to V_A = QV_B$$

We chose $V_B = \mathrm{I}_2$ for simplicity. Therfore,

$$V_A = Q$$

We now have to chose a V_A that is not equal to Q. There exists some matrix A with the the choice of U_A so far and V_A,

$$V_A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies V_A \neq Q$$

Thus, A and B share the same singular values, but are not unitarily equivalent. $\square$

12. Find the relative condition number of the following functions and discuss if there is any concern of being ill-conditioned. If so, discuss when.

(a) $f(x_1, x_2) = x_1 + x_2$

$$J(x) = [1, 1]$$

$$k(x) = \frac{||J||_\infty ||x||_\infty}{||f(x)||_\infty}$$

$$k(x) = \frac{2\max\{x_1, x_2\}}{|x_1 + x_2|}$$

$$\lim_{x_1 \to -x_2} k(x) \to \infty$$

f is ill-conditioned when x_1 is very close to negative x_2.

(b) $f(x_1, x_2) = x_1 x_2$

$$J(x) = [x_2, x_1]$$

$$k(x) = \frac{|x_2 + x_1| \max\{x_2, x_1\}}{|x_2 x_1|}$$

$$\lim_{x_1 x_2 \to 0} k(x) = \infty$$

f is ill-conditioned when $x_1 x_2 << 1$.

(c) $f(x) = (x-2)^9$

$$f'(x) = 9(x-2)^8$$

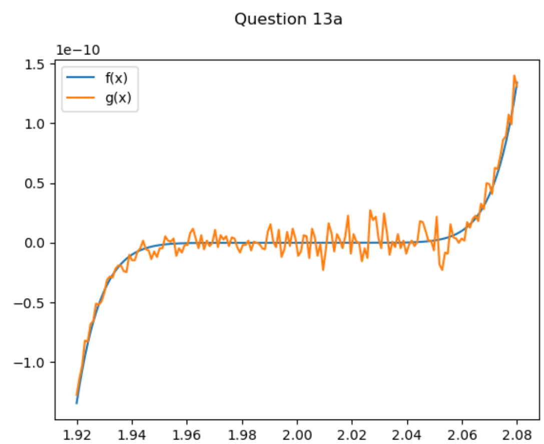
$$k(x) = \frac{9(x-2)^8|x|}{|(x-2)^9|} = \frac{9|x|}{|x-2|}$$

f is ill-conditioned near $x = 2$.

13. Note that the function $f(x) = (x-2)^9$ in Part (c) in Problem 9 can also be expressed as $g(x) = x9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$. Note that, mathematically, the two functions f and g are identical. (Note: Use Matlab or Python for this problem, particularly for plotting purposes. Fortran coding is not necessary.)

   (a) Plot f (x) by evaluating discrete function values of f at1.920, 1.921, 1.922, . . . , 2.080, which are equally spaced with the dis-tance of 0.001.

   (b) Over-plot g(x) at the same set of discrete points in Part (a).



   (c) Draw your conclusion from your results of Part (c) in Prob. 11 and Parts (a) and (b) in this problem.

   The plot shown very clearly demonstrates the stability of using $f(x)$ rather than $g(x)$ for numerics. It can be easily seen that $g(x)$ becomes very ragged when near $x = 2$ which $f(x)$ does not demonstrate at all. This is perhaps from numerical errors in the large amplitude of the terms in $g(x)$ as opposed to the single term in $f(x)$.