

AM 213A Midterm: Report

Dante Buhl

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1 Problem 1: Define the Following

1. A full rank matrix A and a rank deficient matrix B , for $A, B \in \mathbb{R}^{m \times n}$.

We have that A is such that, $\text{rank}(A) = \min(m, n) = k_A$. That is, A has k_A linearly independent columns. For B to be rank deficient we have that $\text{rank}(B) = k_b < \min(m, n)$. That is, B only has k_B linearly independent column vectors (less than the maximal number of linearly independent column vectors).

2. An orthogonal matrix Q and a unitary matrix U , where Q and U are square.

We have that $Q \in \mathbb{R}^{n \times n}$ such that, $Q^T Q = I$. It is also a fact that the columns of Q form a basis for \mathbb{R}^n . A unitary matrix U is very similar but for complex spaces. We have, $U \in \mathbb{C}^{n \times n}$, such that $U^* U = I$. The columns of U form a basis for \mathbb{C}^n .

3. Singular value decomposition of $A \in \mathbb{C}^{m \times n}$ with $\text{rank}(A) = k \leq \min(m, n)$

A singular value decomposition of a matrix A is of the following form.

$$A = U \Sigma V^T$$

Such that U, V are unitary matrices and Σ is an $m \times n$ matrix with k nonzero entries on the 1st k diagonal elements ordered from largest in absolute value to the lowest. These nonzero entries are the singular values of A . We have that A takes vectors from the span of V and returns vectors in the span of U . That is,

$$Av = \sigma u$$

4. Orthogonal projector $P \in \mathbb{R}^{m \times m}$

An orthogonal projector P is a matrix with the following properties,

$$P^2 = P, \quad P^T = P$$

There are two definitions which yield such a P .

$$P = \left\{ \begin{array}{c} A(A^T A)^{-1} A^T \\ Q Q^T \end{array} \right\}$$

P is a matrix which will project any vector onto the span of A/Q .

5. Defective matrix $A \in \mathbb{R}^{m \times m}$.

A defective matrix A is any matrix such that at least one of its eigenvalues has an algebraic multiplicity less than its geometric multiplicity. That is that an eigenvalue λ appears a root of the characteristic polynomial for A more times than the number of corresponding eigenvectors it has. This is equivalent to saying, A has less than m linearly independent eigenvectors. A does not span \mathbb{R}^m .

6. Relative condition number $\kappa(x_0)$ of a differentiable function $f(x) = Ax$ at $x = x_0$ and its upper bound when A is nonsingular

We have $\kappa(x_0)$ is defined to be the norm of the Jacobian of f times the norm of x_0 over the norm of $f(x_0)$. We have then,

$$\kappa(x_0) = \frac{\|J_0\| \|x_0\|}{\|f(x_0)\|} = \frac{\|A\| \|x_0\|}{\|Ax_0\|}$$

We then assume that A is invertible.

$$= \frac{\|A\| \|A^{-1}Ax_0\|}{\|Ax_0\|} \leq \frac{\|A\| \|A^{-1}\| \|Ax_0\|}{\|Ax_0\|} = \|A\| \|A^{-1}\|$$

Therefore by triangle inequality we have that the condition number is bound by $\|A\| \|A^{-1}\|$ when A is nonsingular (A is invertible because of this).

7. Condition number κ or $\text{cond}(A)$ of a nonsingular matrix A with $\text{rank}(A) = k$ in the 2-norm in terms of singular values.

We look at $\kappa(A)$. We have already that it is defined to be $\kappa(A) = \|A\| \|A^{-1}\|$. We now look at this value in the two norm. We have that the largest singular value corresponds to the transformation from one unit vector v to another unit vector u . We have by a property of unitary matrices (specifically that they preserve length under transformations),

$$\|A\|_2 = \sup_x \frac{\|Ax\|_2}{\|x\|_2} = \sup_x \frac{\|\Sigma x\|}{\|x\|} = \|\Sigma\|_2$$

It can be easily shown that the $\|\Sigma\|_2 = \sigma_1$ by the fact that $\sigma_1 > \sigma_i, \forall i > 1$. (If you take any x such that $x \neq \hat{e}_1, 0$ then, $\frac{\|\Sigma x\|}{\|x\|} \leq \frac{\sigma_1}{1}$). For similar reasons we have that the largest singular value of A^{-1} is going to be $\frac{1}{\sigma_k}$, we then have by the same logic that $\|A^{-1}\|_2 = \|\Sigma^{-1}\|_2 = \frac{1}{\sigma_k}$. So we have,

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_k}$$

8. Diagonalizable matrix $A \in \mathbb{R}^{m \times m}$. We have that A is a matrix such that it can be expressed as, $A = VDV^{-1}$. More specically, we have,

$$V^{-1}AV = D$$

where D is a diagonal matrix containing the eigenvalues of A , and V is an invertible matrix containing the eigenvectors of A which are linearly independent. Some other consequences are that A is non-defective.

9. Machine accuracy ϵ_{mach} in single and double precisions.

Machine accuracy is defined by two numbers. For single precision, we have that $\epsilon_{\text{mach}} = 10^{-7}$, and for double precision we have that $\epsilon_{\text{mach}} = 10^{-16}$.

10. Backward stable algorithm \tilde{f} for a problem f . What is the relation between the backward stability and accuracy?

A backward stable algorithm \tilde{f} for f is defined to be an algorithm such that,

$$\tilde{f}(X) = f(\tilde{X}), \quad \frac{\|\tilde{X} - X\|}{\|X\|} = O(\epsilon_{\text{mach}})$$

Where \tilde{X} is the approximated solution obtained by the algorithm. The relationship between backward stability and accuracy is given by a constant C .

$$\frac{\|\delta f\|}{\|f(x)\|} \leq \kappa(x) \frac{\|\delta x\|}{\|x\|} \leq \kappa(x) C \epsilon_{\text{mach}}$$

This is obtained by the argument that $\kappa(x)$ related $\|\delta f\|$ to $\|\delta x\|$ and that as a given for backwards stability $\frac{\|\delta x\|}{\|x\|}$ is of order ϵ_{mach} , i.e $= C \epsilon_{\text{mach}}$.

2 Problem 2 - Let $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$

1. Suppose A has nonnegative entries such that $\sum_{j=1}^n a_{ij} = 1$ for $1 \leq i \leq n$. Show that no eigenvalue of A has an absolute value greater than 1.

We start with the definition of an eigenvector/value pair and the inner product of (v, Av) .

$$\begin{aligned} Av &= \lambda v, & (v, Av) &= (v, \lambda v) \\ \lambda \|v\|_2^2 &= (v, Av) = v^T Av = v^T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned}$$

2. Suppose that $\sum_{j=1}^n |a_{ij}| < 1$ for each i . Prove that $B = I - A$ is invertible. (Hint: Using the Equivalence Theorem for a nonsingular matrix A , it suffices to show that the dimension of the null space of B is 0.)

3 Problem 3 - Let $A \in \mathbb{R}^{m \times m}$ be symmetric.

1. Define what it means to say A is symmetric positive definite.

If A is real, symmetric, positive definite we have the following properties of A . $A^* = A^T = A$, and $(x, Ax) > 0, \forall x \in \mathbb{R}^m$.

2. Show that the eigenvalues of a symmetric positive definite matrix A are positive.

Proof. We look at the inner product of the eigenvectors of A , with the matrix product between A and its eigenvector.

$$\begin{aligned} Av &= \lambda v, & \text{an eigenvalue eigenvector pair } (\lambda, v) \\ (v, Av) &= v^T Av = v^T(\lambda v) = \lambda \|v\|_2^2 > 0 \\ \|v\|_2^2 > 0 &\implies \lambda > 0 \end{aligned}$$

Since our choice of λ and v were arbitrary, we have that all eigenvalues of A are positive. \square

3. What can you say about the diagonal entries of a symmetric positive definite matrix A ? Justify your answer by proving or disproving it.

Claim: The diagonal entries of a symmetric positive definite matrix A are positive.

Proof. We begin by looking at the inner product with the unit basis vectors of \mathbb{R}^m .

$$\hat{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Where only the i -th element of \hat{e}_i is nonzero.

$$(\hat{e}_i, A\hat{e}_i) = \hat{e}_i^T \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} = a_{ii} > 0$$

Therefore the diagonal elements of A are all positive. \square

4 Problem 4 - Consider the following algorithm for a linear system with $A \in \mathbb{R}^{m \times m}$

1. What is this algorithm for? What are the anticipated outcomes? This is the Gaussian Elimination Algorithm without pivoting. The anticipated outcomes are a matrix R and B' such that R is upper triangular and B is a modified matrix in the same manner that A was manipulated to obtain R .
2. Count the floating point operations, including additions, subtractions, multiplications, and divisions in this algorithm.
3. The algorithm is subject to an issue when a_{jj} is zero or close to zero. Give a simple example of a 2×2 system that demonstrates the potential issue when a_{jj} is close to machine accuracy ϵ_{mach} .

This issue can be demonstrated with the matrix,

$$\begin{pmatrix} \epsilon & 1 \\ 1 & 1 \end{pmatrix}$$

When conducting Gaussian Elimination we have to include a term which is divided by ϵ which when close to machine precision, is extraordinarily large.

4. Describe the approach you learned in class, which can resolve the issue in (c). What is this approach called? Write a pseudo algorithm that can resolve the issue by modifying the given algorithm.

The technique learned in class is Pivoting. This allows us to choose the term that we divide by in the GE algorithm. The algorithm obtains an additional piece described by this. (before the if $a_{jj} = 0$ statement)

- (a) Find the element in the i -th column on or below the diagonal such that the absolute value of that element is the largest out of the absolute values of the rest of the values in that column (on or below the diagonal). Call this element a_{ki}
- (b) Swap the k -th rows of A and B with the i -th rows of A and B .
- (c) Proceed with the GE algorithm

After doing this, the algorithm does not have to divide by ϵ and instead the term becomes of order $\frac{1}{\epsilon}$ to order ϵ . This will result in a much more stable algorithm.

5 Problem 5 - Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$$

for the least squares problem $Ax \cong b$ with $0 < \epsilon < \sqrt{\epsilon_{\text{mach}}}$. Here ϵ_{mach} is the small value in machine accuracy and is numerical zero. In this problem, all your arithmetic manipulations should mimic the computer's finite precision handling.

1. Carry out to use the normal equation by first multiplying A^T on both sides to directly solve the linear system. Discuss what happens.

$$A^T A = \begin{bmatrix} 1 + \epsilon^2 & 1 \\ 1 & 1 + \epsilon^2 \end{bmatrix} \approx_{\text{mach}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We have that the matrix product becomes order of ϵ^2 which is actually smaller than order ϵ . It is so small, and by the given value of ϵ that $\epsilon^2 < \epsilon_{\text{mach}}$. Therefore all terms of order ϵ^2 will disappear from the system.

2. Check if the resulting $A^T A$ is symmetric positive definite (SPD) by directly using the definition of SPD. Conclude whether you can use the Cholesky factorization method or not for the resulting normal equation. Justify your answer.

Let us investigate with a dummy vector v .

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{aligned} (v, Av) &= v^T A v = v^T \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 \end{bmatrix} = v_1(v_1 + v_2) + v_2(v_1 + v_2) \\ &= v_1^2 + 2v_1 v_2 + v_2^2 = (v_1 + v_2)^2 > 0 \end{aligned}$$

Therefore, we have that $A^T A$ is symmetric positive definite. Because of this we can conclude that $A^T A$ has a cholesky factorization and we can use it for the resulting normal equation to the system.

3. In general, using the normal equation $A^T A x = A^T b$ for solving the least squares problem $Ax \approx b$ is not always ideal. Justify why it is not ideal by proving the condition number of the Gram matrix $A^T A$ is the square of the condition number of A , i.e $\kappa(A^T A) = (\kappa(A))^2$, for a full rank matrix $A \in \mathbb{R}^{m \times n}$, $m \geq n$. What is the implication of $\kappa(A^T A) = (\kappa(A))^2$.

Proof. We start with the singular value decomposition of A , with the notion in mind that U, V are unitary (orthogonal if $A \in \mathbb{R}^{m \times n}$).

$$A = U \Sigma V^T, \quad A^T = V \Sigma^T U^T$$

By definition we have that the condition number of A , $\kappa(A)$ is defined by the largest singular value of A and the largest singular value of A^{-1} . Let us now look at the singular values of $A^T A$ and $(A^T A)^{-1}$

$$\begin{aligned} A^T A &= V \Sigma^T U^T U \Sigma V^T \\ A^T A &= V \Sigma^2 V^T \end{aligned}$$

Therefore the singular values of $A^T A$ are the singular values of A squared.

$$(A^T A)^{-1} = V \Sigma^{-2} V^T$$

Therefore we have that the largest singular value of $(A^T A)^{-1}$ is $\frac{1}{\sigma_k^2}$, where σ_k is the smallest non-zero singular value of A .

$$\kappa(A^T A) = \sigma_1^2 \frac{1}{\sigma_k^2} = \frac{\sigma_1^2}{\sigma_k^2} = (\kappa(A))^2$$

The implication of this is that we will find that the condition number of $A^T A$ will be the square of the condition number of A . Therefore, if we have a nicely conditioned A it is the case that $A^T A$ might be very poorly conditioned. That is, if $\kappa(A) \approx 10^5$, $\kappa(A^T A) \approx 10^{10}$. This is often unacceptable in numerical methods.

□

6 Problem 6

The Schur decomposition theorem states that if $A \in \mathbb{R}^{m \times m}$, then $A = Q U Q^{-1}$ where Q is orthogonal and U is an upper triangular. Let $A \in \mathbb{R}^{m \times m}$ be normal, i.e., $A^T A = A A^T$

1. We start by looking at the matrix products,

$$A^T A = A A^T$$

Examining the first element in the product we have that,

$$a_{11}^2 = a_{11}^2 + a_{12}^2 + \cdots + a_{1m}^2$$

This implies that,

$$a_{12}^2 + \cdots + a_{1m}^2 = 0, \implies a_{12} = \cdots = a_{1m} = 0$$

We look at the next diagonal element,

$$a_{12}^2 + a_{22}^2 = a_{22}^2 + a_{23}^2 + \cdots + a_{2m}^2$$

We have by the same logic that, $a_{23} = \cdots = a_{2m} = 0$. Thus an inductive argument develops. From the matrix product we have that the following is given ,

$$a_{1,i}^2 + \cdots + a_{i,i}^2 = a_{i,i}^2 + a_{i,i+1}^2 + \cdots + a_{i,m}^2$$

Given that all elements above the diagonal and above the i-th row are zero, we have that,

$$a_{i,i+1} = \cdots = a_{i,m} = 0$$

Thus stating that the i-th row above the diagonal is also zero. We have already shown the base case. The next m rows follow. Thus A is diagonal!

2. We now use the Schur decomposition.

$$A^T A = A A^T, \quad A = Q U Q^{-1}$$

$$A^T A = Q^{-T} U^T Q^T Q U Q^{-1} = Q U Q^{-1} Q^{-T} U^T Q^T$$

Note that, $Q^T Q = I$ and that $Q^{-1} Q^{-T} = (Q^T Q)^{-1} = I$

$$Q^{-T} U^T U Q^{-1} = Q U U^T Q^T$$

We also have in the specific case, that $Q^T = Q^{-1}$ by the definition of an orthogonal matrix ($Q^T Q = I = Q^{-1} Q$)

$$(Q^T)^T U^T U Q^T = Q U U^T Q^T$$

$$Q U^T U Q^T = Q U U^T Q^T, \implies U^T U = U U^T$$

So we have that U is upper triangular and normal, we must have then that U is diagonal! Therefore the orthogonal matrix Q diagonalizes A .

$$A = Q U Q^{-1}, \implies Q^{-1} A Q = U, \text{ diagonal}$$

3. I would argue that A is non-defective on the basis that since A is diagonalizable A must be nondefective.
4. There is a condition that A is invertible. We look at A^{-1}

$$A = Q U Q^T, \implies A^{-1} = Q U^{-1} Q$$

Where since U is diagonal we must have that U^{-1} contains the reciprocal of each diagonal element of U 's diagonal. Notice that $A^{-1} A = I = A A^{-1}$. However we must note that if one of the eigenvalues (diagonal elements of U) are zero then U^{-1} contains infinity. Therefore the condition must be met that all of the eigenvalues of A (diagonal elements of U are non-zero).

7 Problem 7 -

1. To show that A_{i+1} is symmetric we must take the tranpose.

$$A_{i+1}^T = (L_i^T L_i)^T = L_i^T L_i$$

So A_{i+1} is symmetric. To show that it is positive definite we look at the inner product for some nonzero vector v .

$$(v, A_{i+1}v) = v^T L_i^T L_i v = \|L_i v\|_2^2 > 0$$

We have since the null space of L_i is zero we have that the two norm is non-zero. Hence, A_{i+1} is also positive definite.

2. We have,

$$\begin{aligned} A_1 &= L_1^T L_1, \quad A_0 = L_1 L_1^T \\ L_1 &= A_0 L_1^{-T} \\ A_1 &= L_1^T A_0 L_1^{-T} \\ A_{i+1} &= L_{i+1}^T \cdots L_1^T A_0 L_1^{-T} \cdots L_{i+1}^{-T} \\ B &= L_1^{-T} \cdots L_{i+1}^{-T} = (L_{i+1}^{-1} \cdots L_1^{-1})^T \end{aligned}$$

Thus A is similar to B

- 3.
- 4.

8 Problem 8 - Let $P \in \mathbb{R}^{m \times m}$ be an orthogonal projector

1. Show that P is positive semi-definite with its eigenvalues either zero or one. (Hint: A symmetric matrix is orthogonally diagonalizable.)

Proof. We start with a vector v and a vector w . Let us define the orthogonal projector $P = vv^T$. That is P will return the projection of a vector w onto the span of v . Therefore P should only have one non-zero eigenvector which is colinear to v . All other eigenvectors u_i must be orthogonal to v (i.e. $(u_i, v) = 0$) and are arbitrary. Notice that the projector should only return vectors colinear to v . So the eigenvalues for all eigenvectors not v , must have corresponding eigenvalues of 0. We also note that P is symmetric and thus has eigendecomposition/diagonalization.

$$P = VDV^{-1}, \quad V = \left[\frac{v}{\|v\|_2} \mid u_i \right], \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Take the matrices V and D to be composed of block matrices (not necessarily 2×1 and 2×2). Therefore,

$$\begin{aligned} (x, Px) &= x^T VDV^{-1}x = x^T VD \begin{bmatrix} (x, v) \\ (x, u_i) \end{bmatrix} \\ &= x^T v \frac{(x, v)}{\|v\|_2^2} = \frac{(x, v)^2}{\|v\|_2^2} \geq 0 \end{aligned}$$

Therefore we have that P is semi-positive definite and obviously have eigenvalues of 0 and 1. □

2. What can you say about the dimension of P if its eigenvalues are all distinct with algebraic multiplicity of 1?

We must have that P is a 2×2 matrix (only two eigenvalues 1, and 0 with two linearly independent vectors).

3. Construct $P \in \mathbb{R}^{2 \times 2}$ whose entries are all nonzero. Identify all possible choices of P . For each of your constructed P , show that it is positive semi-definite and its eigenvalues are either 0 and 1.

We chose some vector v with $v_1, v_2 \neq 0$ to construct P .

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$P = \frac{vv^T}{\|v\|_2^2} = \frac{1}{v_1^2 + v_2^2} \begin{bmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{bmatrix}$$

Notice that P is nonzero. Take $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$(x, Px) = \frac{1}{\|v\|_2^2} [x_1, x_2] \begin{bmatrix} x_1 v_1^2 + x_2 v_1 v_2 \\ x_1 v_1 v_2 + x_2 v_2^2 \end{bmatrix} = \frac{1}{\|v\|_2^2} (x_1^2 v_1^2 + 2x_1 x_2 v_1 v_2 + x_2^2 v_2^2) = \frac{1}{\|v\|_2^2} (x_1 v_1 + x_2 v_2)^2 \geq 0$$

$$P_P(\lambda) := \lambda^2 - \text{Tr}(P)\lambda + \det(P) = 0$$

$$\det(P) = \frac{1}{\|v\|_2^4} (v_1^2 v_2^2 - (v_1 v_2)^2) = 0$$

$$\text{Tr}(P) = \frac{v_1^2 + v_2^2}{v_1^2 + v_2^2} = 1$$

$$\lambda^2 - \lambda = 0, \implies \lambda = 1, 0$$