Homework 5: Report

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Feb. 26^{th} 2024

2 Numerical Problems

1.

3 Theory Problems

1. Consider the Householder matrix defined by

$$H = I - 2\frac{vv^T}{v^Tv}$$

(a) Show that for any nonzero vector v, the matrix is orthogonal and symmetric.

(b) Let a be any nonzero vector and let $v = a+\alpha e_1$, where $\alpha = \text{sign}(a_{11})||a||_2$. Show that $Ha = -\alpha e_1$ by direct calculation.

(c) Determine v and α that transforms,

$$H\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) Given the vector $a = (2, 3, 4)^T$, specify a Householder trans- formation that annihilates the third component of a.

(e) What are the eigenvalues of H for any nonzero vector x?

(a) *Proof.* Begin by looking at the transpose of H.

$$H^T = \left(\mathrm{I} - 2\frac{vv^T}{v^Tv}\right)^T$$

$$H^T = \mathrm{I} - \frac{2}{v^Tv}(vv^T)^T = \mathrm{I} - 2\frac{vv^T}{v^Tv} = H$$

So we can immediately see that H is symmetric. Next we look at H^TH.

$$H^TH = H^2 = \left(\mathrm{I} - 2\frac{vv^T}{v^Tv}\right)\left(\mathrm{I} - 2\frac{vv^T}{v^Tv}\right)$$

$$H^TH = \mathrm{I} - 2\frac{vv^T}{v^Tv} - 2\frac{vv^T}{v^Tv} + \frac{4}{(v^Tv)^2}vv^Tvv^T$$

$$H^TH = \mathrm{I} - 4\frac{vv^T}{v^Tv} + \frac{4v^Tv}{(v^Tv)^2}vv^T = \mathrm{I}$$

$$H^TH = \mathrm{I}$$

So we have that H is orthogonal as well. □

(b) *Proof.*

$$v = a + \alpha e_1, \quad v^T = a^T + \alpha e_1^T$$

$$H = \mathrm{I} - 2\frac{vv^T}{v^Tv}$$

$$= \mathrm{I} - \frac{2}{a^Ta + 2\alpha a_1 + \alpha^2}\left(aa^T + \left[\begin{array}{c} \alpha a^T \\ \hline 0 \\ \hline \vdots \\ 0 \end{array}\right] + \left[\begin{array}{c|c|c|c} \alpha a & 0 & \cdots & 0 \end{array}\right] + \begin{bmatrix} \alpha^2 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \right)$$

We now multiply by a.

$$Ha = a - \frac{2}{a^Ta + 2\alpha a_1 + \alpha^2}\left(aa^Ta + \left[\begin{array}{c} \alpha a^T \\ \hline 0 \\ \hline \vdots \\ 0 \end{array}\right] a + \left[\begin{array}{c|c|c|c} \alpha a & 0 & \cdots & 0 \end{array}\right] a + \begin{bmatrix} \alpha^2 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} a \right)$$

Recall that $a^Ta = \alpha^2$.

$$Ha = a - \frac{2}{\alpha^2 + 2\alpha a_1 + \alpha^2}\left(\alpha^2 a + \alpha^3 e_1 + \alpha a_1 a + \alpha^2 a_1 e_1\right)$$

$$Ha = a - \frac{\alpha}{\alpha(\alpha + a_1)}\left((\alpha + a_1)a + \alpha(\alpha + a_1)e_1\right)$$

$$Ha = a - a - \alpha e_1 = -\alpha e_1$$

$$Ha = -\alpha e_1$$

□

(c) *Proof.* We have evidently that $a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Therefore we have that $\alpha = -\sqrt{4}$ and $v = a + \sqrt{4}e_1$.

(Note that in this specific case, I am not using α as the exact modifier in the v vector but rather $-\alpha$.

$$v = \begin{bmatrix} 1 + \sqrt{4} \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

□

(d) *Proof.* hey □

2. The Schur decomposition theorem states that every square matrix $A \in \mathbb{C}^{m\times m}$ has a Schur Decomposition, $A = QUQ^*$, where Q is a unitary and U is upper triangular. Use this theorem to prove that, for an arbitrary norm $||||$,

$$\lim_{n\to\infty} ||A^n|| = 0 \Longleftrightarrow \rho(1) < 1$$

(Note: Show the claim first with the 2-norm or the Frobenius norm and use the fact that all norms are equivalent in a finite vector space.)

3. Let $A \in \mathbb{C}/\mathbb{R}^{m\times n}$ and $B \in \mathbb{C}/\mathbb{R}^{n\times m}$. Show that the matrices $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ have the same eigenvalues.

Proof. We start by looking at the determinant of block matrices. Take for example the matrix $\Gamma = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$.

$$\Gamma = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathrm{I} & B \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \mathrm{I} \end{bmatrix}$$

We look at the determinant of Γ.

$$\det(\Gamma) = \det\left(\begin{bmatrix} \mathrm{I} & 0 \\ 0 & D \end{bmatrix}\right) \det\left(\begin{bmatrix} \mathrm{I} & B \\ 0 & \mathrm{I} \end{bmatrix}\right) \det\left(\begin{bmatrix} A & 0 \\ 0 & \mathrm{I} \end{bmatrix}\right)$$

$$\det(\Gamma) = \det(D)\det(A)$$

The same is obviously true for a matrix Γ of the form, $\Gamma = \begin{bmatrix} A & 0 \\ B & D \end{bmatrix}$ with A invertible ($\Gamma = \begin{bmatrix} \mathrm{I} & 0 \\ BA^{-1} & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \mathrm{I} \end{bmatrix}$. We now look at the eigenvalues of $M_1 = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$. We look at the determinants,

$$\det(M_1 - \lambda\mathrm{I}) = \det(AB - \lambda\mathrm{I})\det(-\lambda\mathrm{I})$$

$$\det(M_2 - \lambda\mathrm{I}) = \det(-\lambda\mathrm{I})\det(BA - \lambda\mathrm{I})$$

Notice that if λ is an eigenvalue of its respective matrix then this determinant product must be equal to zero. We then take the case of each determinant can be zero. For M_1 we have that its eigenvalues are either zero, or the eigenvalues of AB by definition.

$$\det(M_1 - \lambda\mathrm{I}) = 0, \quad \det(AB - \lambda\mathrm{I}) = 0, \quad \det(-\lambda\mathrm{I}) = -\lambda = 0$$

Similarly we have that the eigenvalues for M_2 are either zero or the eigenvalues of BA. Notice from a different homework problem (hw2), we have that the matrix products AB and BA have the same eigenvalues. This is because,

$$ABv = \lambda v, \quad Bv = y, \implies \quad BAy = \lambda y$$

Finally both M_1 and M_2 have eigenvalues of zero and the eigenvalues of AB/BA.

□

4. Show that for a real-valued square matrix the Gerschgorin theorem also holds with the bounds r_i which are given by the partial column sums (instead of the partial row sums):

$$r_i = \sum_{i=1, i\neq j}^{m} |a_{i,j}|, \quad i = 1, \ldots, m$$

5. Use the Gerschgorin theorem to show that the following matrix has exactly one eigenvalue in each of the four circles: $|z - k| \leq 0.1, \quad k = 1, 2, 3, 4$.

$$A = \begin{bmatrix} 1.0 & 0.3 & 0.1 & 0.4 \\ 0.0 & 2.0 & 0.0 & 0.1 \\ 0.0 & 0.4 & 3.0 & 0.0 \\ 0.1 & 0.0 & 0.0 & 4.0 \end{bmatrix}$$

6. Let $A \in \mathbb{R}^{m\times m}$ be real and symmetric that is positive definite. Let $y \in \mathbb{R}^m$ be nonzero. Prove that the limit exists and is an eigenvalue of A.

$$\lim_{k\to\infty} \frac{y^T A^{k+1} y}{y^T A^k y}$$

Proof. First, since A is real and symmetric, we can decompose A into an eigenvector matrix, full with orthonormal eigenvectors which form a basis for $\mathbb{R}^m$ and a diagonal matrix with eigenvalues of A on the diagonal. Since its columns form a basis for $\mathbb{R}^m$ we have that we can express y as a linear combination of the eigenvectors of v: $y = c_1 v_1 + \cdots + c_m v_m$. We now define λ_* such that, $|\lambda_*| \geq |\lambda_i|, \forall \lambda_i \in \Lambda(A), c_* \neq 0$.

$$A = VDV^{-1}, \quad y = c_* v_* + c_1 v_1 + \cdots + c_m v_m$$

$$A^n = VD^n V^{-1}, \quad A^n y = c_* \lambda_*^n v_* + c_1 \lambda_1^n v_1 + \cdots + c_m \lambda_m^n v_m$$

$$A^n y = \lambda_*^n \left(c_* v_* + c_1 \frac{\lambda_1^n}{\lambda_*^n} v_1 + \cdots + c_m \frac{\lambda_m^n}{\lambda_*^n} v_m \right)$$

Note from our definition of λ_* we either have that $|\lambda_*| \geq |\lambda_i| \bigvee (|\lambda_* < |\lambda_i| \bigwedge c_i = 0)$. Thereby, we have that

$$\lim_{n \to \infty} A^n y = \lambda_*^n c_* v_*$$

Therefore we have,

$$\lim_{k \to \infty} \frac{y^T A^{k+1} y}{y^T A^k y} = \frac{\lambda_*^{k+1} c_* y^T v_*}{\lambda_*^k c_* y^T v_*} = \lambda_*$$

Where λ_* is the largest eigenvalue of A which has an eigenvector as part of the expansion of y into the basis of the eigenvectors of A. □

7. Let $A \in \mathbb{R}^{m \times m}$ be real with nonnegative entries such that

$$\sum_{j=1}^{m} a_{ij} = 1 \quad (1 \leq i \leq m)$$

Prove that no eigenvalue of A has an absolute value greater than 1.

8. Let $A \in \mathbb{R}^{m \times m}$ be a non-defective matric with its eigenvalues $\{\lambda_i\}_{i=1}^m$ and its singular values $\{\sigma_i\}_{i=1}^m$, satisfying

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$$

Let $\rho(a)$ be the spectral radius of A and $\text{cond}(A) = ||A||_2 ||A^{-1}||_2$ be the condition number of A. Let A be normal, i.e., $A^T A = AA^T$. Show that:

(a) $\sigma_i = |\lambda_i|, 1 \leq i \leq m$.

(b) $||A||_2 = |\lambda_1| = \rho(A)$.