AM112/212A, Assignment #2

1. Consider the initial boundary value problem (BVP-P1).

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0, y) = 0 , \quad u_x(L, y) = 0 \\ u(x, 0) = 0 , \quad u(x, H) = f(x) \end{cases} \qquad \text{(BVP-P1)}$$

(a) Use separation of variables to solve (BVP-P1) for a general $f(s)$.

(b) For $f(s) = s / L, s \in [0, L]$, find the solution of (BVP-P1).

Hint: Be careful with the case of $n = 0$ when solving $Y_n''(y) = \lambda_n Y_n(y)$.

2. Consider the initial boundary value problem (BVP-P2).

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = 0 , \quad u(L, y) = g(y) \\ u_y(x, 0) = 0 , \quad u_y(x, H) = 0 \end{cases} \qquad \text{(BVP-P2)}$$

Let $u^{(1)}(x, y; L, H, \{f(\bullet)\})$ be the solution of (BVP-P1) and $u^{(2)}(x, y; L, H, \{g(\bullet)\})$ the solution of (BVP-P2).

(a) Use change of variables to convert (BVP-P2) to the form of (BVP-P1) and use the result of Problem 1 to write $u^{(2)}(\ )$ in terms of the Fourier cosine coefficients of $g(\bullet)$.

(b) For $g(s) = s / H, s \in [0, H]$, use the result of (a) to find the solution of (BVP-P2).

3. For $f(s) = s / L, s \in [0, L]$, use separation of variables to solve (BVP-P3).

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0, y) = 0 , \quad u(L, y) = 0 \\ u(x, 0) = f(x) , \quad u(x, H) = 0 \end{cases} \qquad \text{(BVP-P3)}$$

Hint: The expansion of $f(x)$ in $\{X_n(x)\}$ was studied in Assignment 1.

4. Prove Theorem 1 in Lecture 4. Specifically, consider the IBVP below

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0\ , \quad u(L,t) = 0 \\ u(x,0) = f(x)\ , \quad u_t(x,0) = 0 \end{cases} \qquad \text{(IBVP-P4)}$$

We extend $f(x)$ to an odd function and then extend it periodically with period = $2L$. Let $F(x)$ be the resulting function after the odd+periodic extension.

Show that $u(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right)$ satisfies (IBVP-P4).

5. Consider the vibration of a string modeled by the IBVP of the wave equation with a dissipation term. We study the evolution of the n-th mode for small ε > 0.

$$\begin{cases} u_{tt} + \varepsilon u_t = c^2 u_{xx} \\ u(0,t) = 0\ , \quad u(L,t) = 0 \\ u(x,0) = \sin(\frac{n\pi}{L} x)\ , \quad u_t(x,0) = 0 \end{cases} \qquad \text{(IBVP-P5)}$$

Use separation of variables to solve (IBVP-P5). Observe that

- the ε term does not affect the eigenvalue problem; thus $X_n(x)$ is independent of ε;
- $T_n(t)$ is affected by ε; the oscillation frequency is close to $\frac{cn}{2L}$ when ε is small.

6. Classify the PDEs below as hyperbolic, parabolic or elliptic.

(a) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

(b) $2u_{xx} + 4u_{xy} + u_{yy} = 0$

(c) $u_{xx} + 2u_{xy} + 2u_{yy} = 0$

(d) $u_{xy} = 0$

7. Solve the final boundary value problem (FBVP-P7) of the Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = 0\,, \quad u(L, y) = 0 \\ u(x, H) = \varepsilon \sin(\frac{n\pi}{L} x)\,, \quad u_y(x, H) = 0 \end{cases} \qquad \text{(FBVP-P7)}$$

   Show that for any $y < H$, $u(x, y)$ is unbounded as $n \to \infty$.

Remark: A final boundary value problem of the Laplace Eq is ill-posed.

8. Solve the FBVP of the heat equation

$$\begin{cases} u_t = k u_{xx} \\ u(x, t_f) = \varepsilon \cos(\frac{(n - \frac{1}{2})\pi}{L} x) \\ u_x(0, y) = 0\,, \quad u(L, y) = 0 \end{cases} \qquad \text{(FBVP-P8)}$$

   Show that for any $t < t_f$, $u(x, t)$ is unbounded as $n \to \infty$.

Remark: Solving the heat equation backward in time is ill-posed.