Rmk $\mu = a_{mm}$

do while $\varepsilon >$ large

$\mu^{(n)} = a_{mm}$

$Q^{(n)} R^{(n)} = A^{(n)} - \mu^{(n)} I$

$A^{(n+1)} = R^{(n)} Q^{(n)} + \mu^{(n)} I$

end do

$R^{(n)} = Q^{(n)T} (A^{(n)} - \mu^{(n)} I)$

$\rightarrow A^{(n+1)} = Q^{(n)T} (A^{(n)} - \mu^{(n)} I) Q^{(n)} + \mu^{(n)} I$

$= Q^{(n)T} A^{(n)} Q^{(n)}$

$\rightarrow [A^{(n+1)}]_{mm} = a_{mm}^{(n+1)} = [Q^{(n)T} A^{(n)} Q^{(n)}]_{mm}$

$= q_m^T A^{(n)} q_m$ (RQ ~~fcn~~)

$\rightarrow \lambda_m$!

§ 4.2 Deflation

Rmk As $\mu = a_{mm} \longrightarrow \lambda_m$, $A \in \mathbb{R}^{m \times m}$

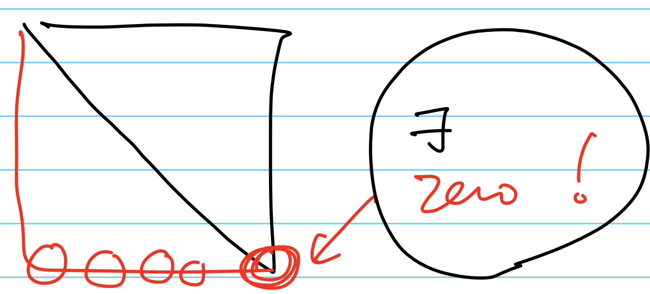
$A - \mu I$ become singular

$\longrightarrow$ let $\text{ran}(A - \mu I) = k < m$

$\longrightarrow$ $A - \mu I = QR$,

There are only k nonzero diagonal entries in R (using prob 3(a) HW 4)

Then $\rightarrow R =$

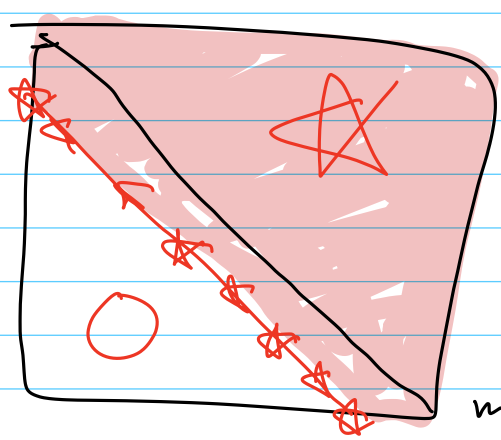


$\rightarrow$ in RQ, ∃ zero's in the last row

§4.3 The Hessenberg Form

Def. Hessenberg matrix A_H:

$A_H =$



$m \times m$

$$(A_H)_{ij} = 0 \quad \text{if} \quad i > j+1$$

Rmk. Given A,

$$A \xrightarrow{\text{similarity}} A_H \longrightarrow \text{QR on } A_H$$

faster!

Rmk $A = P A_H P^T$,

P: orthogonal matrix $(PP^T = P^TP = I)$

$\therefore P^T = P^{-1}$

$\rightarrow$ A & A_H: similar and have the same eigvals.

Rmk. The Hessenberg form is "preserved" by the QR alg.

$\boxed{\text{Claim 1}}$ Q is Hessenberg

$(A_H)_{ij} = 0 \quad \forall i > j+1$, Assume $QR = A_H$

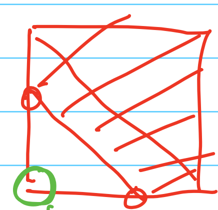
$$\to (A_H)_{ij} = \sum_{k=1}^{m} q_{ik} r_{kj} = \sum_{k=1}^{j} q_{ik} r_{kj}$$

Rmk: $r_{ij} = 0$, $i > j$

Then $\boxed{(A_H)_{ij} = 0 \quad \text{if} \quad i > j+1}$

$\Longleftrightarrow \boxed{q_{ik} = 0 \quad \text{if} \quad i > k+1}$

e.g. $\underset{\sim}{A_H}$ 3x3



$$\Rightarrow 0 = (A_H)_{31} = \sum_{k=1}^{1} q_{3k} r_{k1} = q_{31} r_{11}, \quad r_{11} \neq 0$$

$\therefore q_{31} = 0$

Claim 2: RQ is also Hessenberg.

$$(RQ)_{ij} = \sum_{k=1}^{m} r_{ik} q_{kj} = \sum_{k=i+1}^{m} r_{ik} q_{kj} \quad (*)$$

$k \geq i+1$

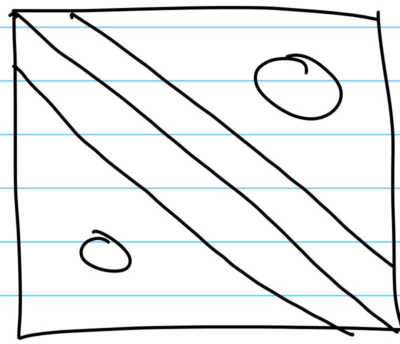
Let $i > j+1$

$\to k \geq i+1 > i > j+1$

$\because (*) = 0 \qquad \therefore (RQ)_{ij} = 0 \quad \forall i > j+1 \quad \blacksquare$

Rmk. If $A = A^T$, then

$A_H = PAP^T$ is also sym.

$\to$  tridiagonal!

Q $P = ?$

Householder $H_j \Longleftrightarrow v_j = \dfrac{1}{\|\tilde{v}_j\|} \overbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{jj} + s_j \\ a_{j+1,j} \\ \vdots \\ a_{mj} \end{bmatrix}}^{\tilde{v}_j}$ jth

$$s_j = \mathrm{sgn}(a_{jj}) \sqrt{\sum_{i=j}^{m} a_{ij}^2}$$

modify the original Householder H_j into

(i) $S_j = \text{sign}(a_{j+1,j}) \sqrt{\sum_{k=j+1}^{m} a_{kj}^2}$

(ii) $v_j = \dfrac{1}{\|\tilde{v}_j\|} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline a_{j+1,j} + S_j \\ a_{j+2,j} \\ \vdots \\ a_{mj} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow j\text{th} \\ \leftarrow (j+1)\text{th} \\ \\ \\ \\ \end{matrix}$ $\rightarrow \tilde{v}_j$

(iii) $H_j = I - 2 v_j v_j^T$ $\left(\text{sym } H_j = H_j^T \right)$

$= \left[\begin{array}{c|c} I_{j \times j} & \sim \\ \hline \sim & \sim \end{array}\right]$

Rule $A_H \Leftarrow H_j A H_j^T = H_j A H_j$

$H_j A = \left[\begin{array}{c|c} A_{1:j,1:j} & \sim \\ \hline \sim & \sim \end{array}\right]$

$$\rightarrow \quad H_j A H_j = \left[\begin{array}{c|c} A_{1:j,1:j} & \\ \hline & \end{array}\right] \left[\begin{array}{c|c} I & \\ \hline & \end{array}\right]$$

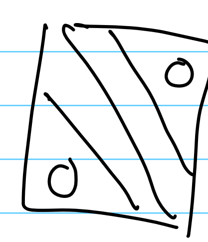
$$= \left[\begin{array}{c|c} A_{1:j,1:j} & \\ \hline & \end{array}\right]$$

$$\rightarrow \quad H_j A H_j \longrightarrow A_H$$

Summary

$A \in \mathbb{R}^{m\times m}$, $\quad A^T = A$

1. Hessenberg reduction
2. QR on A_H $\longrightarrow$ tridiagonal



3. QR w/ shift on A_H
4. deflation during QR

$\to \quad O\left(\frac{2}{3} m^3\right)$ hence cheaper than mat-mat multiplication

Algo, $\to$ back-stable

$\to \; \therefore \; \left| \lambda_j^{num} - \lambda_j^{exact} \right|$

$= O\left(\|A\| \, \varepsilon_{mach} \right)$

Rmk $\quad Ax = \lambda x$

$\to \quad x \lambda \simeq Ax$

$\to \quad x^T x \lambda = x^T A x$

$\to \quad \lambda = \dfrac{x^T A x}{x^T x}$ $\quad \blacksquare$

§5. Beyond real sym. matrices

→ A: real but not sym

→ eig vals of A : may not be real

→ eig vectors may not be orthogonal

$\cancel{A \overset{QR}{\longmapsto} D_\lambda}$

A was diagonalizable

$AV = VD_\lambda$

$A = VD_\lambda V^{-1}$

§5.1 Complex QR alg. for Hermitian matrices

If $A \in \mathbb{C}^{m \times n}$