Homework 3: Report

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1. The Schur decomposition theorem states that if $A \in C^{m\times m}$, then there exist a unitary matrix Q and an upper triangular matrix U such that $A = QUQ^{-1}$. Use the Schur decomposition theorem to show that a real symmetric matrix A is diagonalizable by an orthogonal matrix, i.e., $\exists$ an orthogonal matrix Q such that $Q^T AQ = D$, where D is a diagonal matrix with its eigenvalues in the diagonal.

   Proof. We begin with the Schur Decomposition Theorm for real symmetric matrix A. We have,

   $$A = QUQ^{-1} = QUQ^*$$

   By the property that A is real and symmetric, we have that $A^* = A$.

   $$A^* = A = QUQ^*$$

   $$Q^*A^*Q = U^* = Q^*AQ = U$$

   So we have, $U = U^*$. Since we have that U is upper triangular, that means that U^* is lower triangular. However, since they are equal to one another, we must have that they are both diagonal. Thereby we have that A is diagonalizable by an othorgonal matrix Q (a property of unitary matrices).

   $$A = QUQ^* \implies Q^{-1}AQ = U = D$$

   Since we have this property, we have that Q contains the eigenvectors of A and U is a diagonal matrix contianing its eigenvalues.

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