

# Homework 4: Report

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## 2 Cholesky Solution of the least-squares problem

1. How does the Cholesky Decomposition Work?

The cholesky decomposition is merely a special case of the LU factorization method. It is designed to create a matrix  $L$  such that  $LL^* = A$ . That is, we find that  $L$  is (with abuse of notation) a "square-root" of  $A$  in matrix form. The algorithm is very simple, we start with the  $l_{11}$  component of  $L$ . We have that  $l_{11} = \sqrt{a_{11}}$ . We proceed just like LU factorization except that the symmetry of  $A$  allows us to perform half of the computations necessary (this is due to the symmetry of the problem). You can throw away half of the matrix  $A$  (above the diagonal) upon entry and  $L$  wouldn't change.

2. The results of fitting a third degree polynomial are a frobenius norm of error at  $\|E\|_F = 0.19127$ . The coefficients build the following polynomial,

$$f(x) = 0.578 + 4.666x - 10.935x^2 + 7.514x^3$$

## 3 QR Solution of the least-squares problem

- 1.

## 4 Theory Problems

1. Show that if  $P$  is an orthogonal projector, then  $I - 2P$  is unitary.

*Proof.* We begin with the definition of a unitary matrix. We have a unitary matrix  $Q$  is a matrix such that  $Q^*Q = I$ . We now look for this quality in  $(I - 2P)$ .

$$(I - 2P)^*(I - 2P) = (I - 2P^T)(I - 2P) = (I - 2P)(I - 2P)$$

The above quality  $P^T = P$  is from the fact that  $P$  is an orthogonal projector. We also have the quality that  $P^2 = P$ .

$$\begin{aligned}(I - 2P)(I - 2P) &= I(I - 2P) - 2P(I - 2P) = I - 2P - 2P + 4P^2 \\ &= I - 4P + 4P = I\end{aligned}$$

We have recovered the condition for unitary matrices. We can therefore declare  $(I - 2P)$  a unitary matrix.  $\square$

2. Let  $P \in \mathbb{R}^{m \times m}$  be a nonzero projector.

(a) Show that  $\|P\|_2 \geq 1$ , with equality if and only if  $P$  is an orthogonal projector.

*Proof.* (i) without equality

We begin by looking at a projector  $P$  which transforms a vector onto the span of another unit vector  $v$ . That is  $P : \mathbb{R}^m \rightarrow \text{span}(v)$ . We look at the case of the definition of the two-norm for matrices.

$$\|P\|_2 = \sup_x \frac{\|Px\|_2}{\|x\|_2}$$

We now take the case  $x = v$ .

$$\|P\|_2 = \sup_x \frac{\|Px\|_2}{\|x\|_2} \geq \frac{\|Pv\|_2}{\|v\|_2}$$

We notice that  $v \in \text{span}(v)$ , so the transformation  $P$  is the identity for  $v$ . So,

$$\begin{aligned} \|P\|_2 &\geq \frac{\|Pv\|_2}{\|v\|_2} = \frac{\|v\|_2}{\|v\|_2} = 1 \\ \|P\|_2 &\geq 1 \end{aligned}$$

(ii) with equality ( $\implies$ )

We now look more closely at the definition of  $P$  and the singular value decomposition of  $P$ . If  $P$  is orthogonal we have that  $P = vv^T$  for some unit vector  $v \in \mathbb{R}^m$ . We also have by the singular value theorem, that a singular value will satisfy the following property,

$$Pv_i = \sigma_i u_i, \quad P = U\Sigma V^T$$

Where we have that  $v_i, u_i$  are the  $i$ -th column vectors of  $V, U$  respectively, and  $\sigma_i$  is the  $i$ -th diagonal element of  $\Sigma$ . Note that  $U, V$  are unitary and as a consequence its column vectors are orthogonal and have two-norm of 1. We look at some  $P = xx^T$  for a unit vector  $x$ .

$$\begin{aligned} Pv_i &= \sigma_i u_i \rightarrow xx^T v = \sigma_i u_i \\ (x, v_i)x &= \sigma_i u_i \end{aligned}$$

We notice that vectors  $x, u_i$  are related by scalars as a consequence of this definition of  $P$ . Therefore  $x$  and  $u_i$  must be colinear, however this is not guaranteed by our assumptions. We have a few consequences and cases. Either  $x$  and  $u_i$  are colinear, or they are not. We look at the case they are colinear,

$$x = \alpha u_i$$

We then must have that  $\alpha = \frac{\sigma_i}{(x, v_i)}$ . We then look at one of our prior assumptions. We have most importantly that  $\|x\|_2 = \|v_i\|_2 = \|u_i\|_2 = 1$ .

$$\|x\|_2 = |\alpha| \|u_i\|_2 = 1, \implies |\alpha| = 1$$

$$\sigma_i = \pm(x, v_i)$$

We need two more things. First, that singular values cannot be negative by definition. Second, we have that if both  $x, v_i$  are unit vectors, we cannot have that their inner product is greater than 1. Another way of expressing this is the geometrical interpretation that the dot product is  $x \cdot v_i = (x, v_i) = \|x\|_2 \|v_i\|_2 \cos(\theta)$ . If  $\|x\|_2 = \|v_i\|_2 = 1$ ,  $(x, v_i) = \cos(\theta) \leq 1$ . Finally, (and I mean it this time), we look at the case where  $Px = x$  we have that since  $x$  is a unit vector we recover an eigenvalue (in this case also a singular value) of  $P$ . Thereby we officially have,

$$0 < \sigma_i \leq 1, \sigma_1 = 1$$

We use a proof from a different homework problem (or maybe from the lecture note, I can't remember where) relating  $\|A\|_2 = \sigma_1$ , to show,

$$\|P\|_2 = \sigma_1 = 1$$

( $\Leftarrow$ ) If  $\|P\|_2 = 1$ , then  $P$  is an orthogonal projector (i.e.  $P^T = P$ )

We begin by looking at the definition of the two norm for matrices. We have,

$$\|P\|_2 = \sup_x \frac{\|Px\|_2}{\|x\|_2} = 1$$

Therefore the case exists such that we find,

$$\|Px\|_2 = \|x\|_2$$

We then introduce the fact that a two norm of a vector is the square root of the inner product of that vector with itself. That is  $\|v\|_2 = \sqrt{(v, v)}$ . Thus we have,

$$\sqrt{x^T P^T P x} = \sqrt{x^T x}$$

$$x^T P^T P x = x^T x$$

$$P^T P x = x = P P x, \quad \text{by definition of a projector}$$

$$P^T P x = P P x, \implies P^T y = P y$$

This implies that for any vector in the span of  $v$  (We take  $P : \mathbb{R}^m \rightarrow \text{span}(v)$ ), that  $P^T = P$ . Moreover we have that for any vector  $x$ ,  $P^T P = P^2$ . If we take  $P$  to be invertible, we have that  $P^T = P$ . Therefore, we have that  $P$  is idempotent and symmetric, making it an orthogonal projector. □

- (b) Show that if  $P$  is an orthogonal projector, then  $P$  is semi-positive definite with its eigenvalues either zero or 1.

*Proof.* We look at the vector product definition of  $P$ .  $P = x x^T$  for a unit vector  $x$ .

$$(v, P v) = v^T x x^T v = (v, x)(x, v) = (x, v)^2 \geq 0$$

Since our choice of  $v$  was arbitrary we have that  $P$  is semi-positive definite.

Next we look at the eigenvalues of  $P$ . Say that we have an arbitrary eigenvalue-eigenvector pair  $(\lambda, v)$  for  $P$  such that  $v \neq \vec{0}$  (obviously). We have,

$$P v = \lambda v$$

$$P v = x x^T v = (x, v) x = \lambda v$$

Notice that  $(x, v)$  and  $\lambda$  are scalars. This implies that  $x$  and  $v$  are colinear but this was not an assumption made. Therefore we are left with two cases:  $v$  and  $x$  are colinear, or  $v$  and  $x$  are orthogonal. Let's look at the first case,  $v = \alpha x$ .

$$(x, v) x = \alpha x = \lambda x$$

$$x = \lambda x \implies \lambda = 1$$

We find that for all vectors colinear to  $x$  are eigenvectors with eigenvalue 1. We look at the other case. If  $x$  and  $v$  are orthogonal we have  $(x, v) = 0$ .

$$\vec{0} = \lambda v \implies \lambda = 0$$

Therefore all vectors orthogonal to  $x$  will be eigenvectors with  $\lambda = 0$ . □

3. Let  $A \in \mathbb{R}^{m \times n}$  with  $m \geq n$ , and let  $A = \hat{Q} \hat{R}$  be a reduced  $QR$  factorization.

- (a) Show that  $A$  has rank  $n$  if and only if all the diagonal entries of  $\hat{R}$  are nonzero.

*Proof.* ( $\implies$ )  $A$  is rank  $n$  if all of the diagonal entries of  $\hat{R}$  are nonzero.

Let us look at the reduced QR factorization of  $A$  such that all diagonal entries of  $\hat{R}$  are nonzero.

$$A = \hat{Q}\hat{R} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \hat{R}$$

We have by construction of a QR factorization that the matrix  $\hat{Q}$  is composed of orthogonal unit column vectors  $q_i$ . Let us now look at the column vectors of  $A$ .

$$a_i = r_{1i}q_1 + \cdots + r_{ii}q_i$$

Notice that since all diagonal elements of  $\hat{R}$  are nonzero that we have that each  $a_i$  is immediately distinguished from  $a_{i-1}$  by the inclusion of the vector  $q_i$ . Let us start a small induction proof. Take the base case to demonstrate that  $a_1$  is linearly independent from  $a_2$ . By contradiction suppose that  $a_1, a_2$  are linearly dependent.

$$0 = c_1a_1 + c_2a_2 = c_1r_{11}q_1 + c_2r_{12}q_1 + c_2r_{22}q_2 = (c_1r_{11} + c_2r_{12})q_1 + c_2r_{22}q_2$$

$$0 = d_1q_1 + d_2q_2$$

However, since the vectors  $q_i$  are linearly independent, we require that  $d_1 = d_2 = 0$  since the vectors  $q_1, q_2$  are linearly independent. Immediately we notice that  $c_2$  must equal zero since  $r_{22} \neq 0$ . Therefore for the two to be linearly dependent we must have that  $c_1 \neq 0$ . A contradiction is reached, since  $d_1 = 0 = c_1r_{11} + 0 \implies r_{11} = 0$ . Thus we have that  $a_1, a_2$  are linearly independent.

Next we look at the inductive step. Take  $a_1, \dots, a_k$  to be linearly independent. Let us look at the set  $a_1, \dots, a_{k+1}$ . We have evidently, that,

$$c_1a_1 + \cdots + c_ka_k = d_1q_1 + \cdots + d_kq_k$$

Such that  $d_1q_1 + \cdots + d_kq_k = 0$  if and only if  $d_1 = \cdots = d_k = 0 = c_1 = \cdots = c_k$ . Let us now add  $c_{k+1}a_{k+1}$  and look at the linear dependence.

$$c_1a_1 + \cdots + c_ka_k + c_{k+1}a_{k+1}$$

$$(d_1 + c_{k+1}r_{1k})q_1 + \cdots (d_k + c_{k+1}r_{kk})q_k + c_{k+1}r_{k+1,k+1}q_{k+1} = 0$$

Again these vectors,  $q_i$ , are linearly independent so we must have that  $c_{k+1} = 0$  since  $r_{k+1,k+1} \neq 0$ . Therefore we are left with

$$d_1q_1 + \cdots + d_kq_k = 0$$

We already have that to satisfy this,  $d_1 = \cdots = d_k = 0$ , thereby we have immediately that the vectors  $a_1, \dots, a_{k+1}$  are linearly independent. Therefore, by inductive argument we have that all  $n$  vectors  $a_i$  constructed this way from the reduced QR factorization will be linearly independent. As a corollary to this finding, we find that  $A$  is rank  $n$  by the definition of rank and it being that  $A$  is composed of  $n$  linearly independent column vectors.

( $\Leftarrow$ ) All diagonal entries of  $\hat{R}$  are non-zero if  $A$  is rank  $n$ .

Assume by the way of contradiction that both  $A$  is rank  $n$  and that  $\hat{R}$  has at least one diagonal entry,  $r_{kk} = 0$ . We look at the construction and linear dependence of the column vectors of  $A$ . Look specifically at  $a_k$ . We have,

$$a_k = r_{1k}q_1 + \cdots + r_{k-1,k}q_{k-1} + r_{kk}q_k$$

$$c_1a_1 + \cdots + c_ka_k = 0$$

Notice that since  $r_{kk} = 0$   $a_k$  is only constructed of  $q_1, \dots, q_{k-1}$ .

$$c_1a_1 + \cdots + c_ka_k = (c_1r_{11} + \cdots + c_kr_{1k})q_1 + \cdots + (c_{k-1}r_{k-1,k-1} + c_kr_{k-1,k})q_{k-1} = 0$$

We must have again that,  $d_1 = \dots = d_{k-1} = 0$ . We then chose  $c_k = 1$  for simplicity and obtain a system of equations.

$$c_1 r_{11} + \dots + r_{1k} = \dots = c_{k-1} r_{k-1,k-1} + r_{k-1,k} = 0$$

Notice that we have  $k - 1$  equations with  $k - 1$  unknowns, so we are guaranteed a solution exists such that at least one  $c_i \neq 0$ . i.e.

$$c_{k-1} = -\frac{r_{k-1,k}}{r_{k-1,k-1}}$$

Therefore we have that the set of column vectors  $a_1, \dots, a_k$  are linearly dependent. Therefore, we have at most that  $A$  is rank  $n - 1$  (Take  $\{a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n\}$  and check their dependency. They may be linearly independent!). Therefore we have reached a contradiction. If  $A$  is full rank (rank  $= n$ ) we cannot have that any diagonal elements of  $\hat{R}$  are zero as it would reduce the rank of  $A$  by at least one. If  $A$  is full rank,  $\hat{R}$  must have nonzero diagonal entries. □

- (b) Suppose  $\hat{R}$  has  $k$  nonzero diagonal entries for some  $k$  with  $0 \leq k < n$ . What does this imply about the rank of  $A$ ? Exacktkly  $k$ ? At least  $k$ ? At most  $k$ ? Give a precise answer and prove it.

*Proof.* (Case: rank  $k$ )

The goal is to show with two cases that such a matrix can be constructed with rank  $n - 1$  and one with rank  $k$ . Therefore stating that the rank of  $A$  is at least  $k$ . Let us look at the case where  $\hat{R}$  is a matrix composed of zeros entirely except for  $k$  entries along the diagonal.

$$A = \hat{Q}\hat{R}$$

$$A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$$

Notice that only  $k$  column vectors of  $A$  are nonzero by this construction of  $A$ , and  $\hat{R}$ . Therefore the column vectors which are zero vectors are not linearly independent with each other nor the nonzero column vectors of  $A$ . So we must have that there are  $k$  linearly independent column vectors in  $A$ . Therefore  $A$  is rank  $k$ . To demonstrate this formally we have

$$c_1 a_1 + \dots + c_n a_n = \sum_{i, r_{ii} \neq 0} c_i r_{ii} q_i = 0$$

We must have by the linear independence of  $q_i$  that  $c_i r_{ii} = 0$  for this to be true, but then  $c_i = 0$ . Since there are  $k$  terms in this sum, there are therefore  $k$  linearly independence vectors in  $A$ .

(Case: rank  $n - 1$ )

We next take a case for  $\hat{R}$  that will produce  $A$  rank  $n - 1$ . We chose an  $\hat{R}$ , complete with  $k$  nonzero entries on the diagonal and zero's above the diagonal for those  $k$  columns. For the columns with zero's on the diagonal we demonstrate a particular form for them. For the first column with a zero on the diagonal, the form is not very important. Suppose this is column  $i$ . Look at the next column with a zero on the diagonal, suppose it is column  $j$ . Let column  $j$ ,  $r_j$  be of the following form.

$$r_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_{i,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

These columns are such that if column  $c_j$  was in the  $i - th$  column rather than the  $j - th$  it would resemble an diagonal matrix with nonzero diagonals except for the very last column with one zero

on the diagonal (lets denote this column  $r_z$ ). That is, if we permuted the columns of  $\hat{R}$  we could obtain a matrix  $\hat{R}'$  such that only one column of  $\hat{R}'$  has a diagonal entry of zero. This would produce a matrix  $A'$  with the corresponding columns permuted in the same way. Notice however, that  $A'$  has the same rank as  $A$ . That is, it contains the same column vectors, just in a different order. Notice that besides the one column with a zero along the diagonal (lets call this column  $a_z$ ), we have that  $\hat{R}'$  is a diagonal matrix. Therefore we have the columns of  $A'$  are such that,

$$a'_i = r'_{ii}q_i, \quad a_z = r'_{1z}q_1 + \cdots + r'_{z-1z}q_{z-1}$$

We have that  $r'_{ii} \neq 0$ , so

$$\sum_{1 \leq i \leq n, i \neq z} c_i a'_i = \sum_{1 \leq i \leq n, i \neq z} d_i q_i = 0, \quad (d_i \propto c_i), \quad \text{iff } c_i = 0, \quad \forall i$$

Notice that this linear combination (sum) has  $n - 1$  terms in it, therefore we have that  $A'$  is rank  $n - 1$  and therefore so is  $A$ . This ultimately implies that the rank of  $A$  is bounded on the lower end by  $k$  and on the upper end by  $n - 1$ .  $\square$

4. Determine the (i) eigenvalues, (ii) determinant, and (iii) singular values of a Householder reflector. For the eigenvalues, give a geometric argument as well as an algebraic proof.

*Proof.* (i) Eigenvalues

We start with the definition of a householder reflector for a unit vector  $x$ . Take  $H = I - 2xx^T$  with an eigenvalue-eigenvector pair  $(\lambda, v)$  such that  $Hv = \lambda v$ .

$$Hv = (I - 2xx^T)v = v - 2xx^T v = v - 2(x, v)x = \lambda v$$

$$-2(x, v)x = (\lambda - 1)v$$

We again have a case where  $x$  and  $v$  are vectors connected by scalar arguments. We must have that  $x$  and  $v$  are colinear. We take the two cases,  $x$  and  $v$  are colinear,  $x$  and  $v$  are orthogonal.

$$v = \alpha x, \quad -2(x, v) = -2\alpha$$

$$-2\alpha = (\lambda - 1)\alpha$$

$$\lambda = -1$$

Therefore if  $x$  and  $v$  are colinear we have that  $v$  is an eigenvector of  $H$  and that its eigenvalue is  $\lambda = -1$ . We look at the next case,  $x$  and  $v$  are orthogonal, therefore  $(x, v) = 0$ .

$$-2(0)x = (\lambda - 1)v \implies \lambda - 1 = 0$$

$$\lambda = 1$$

Therefore we have that if  $x$  and  $v$  are orthogonal that the eigenvalue corresponding to  $v$  is equal to 1.

(ii) Determinant

Next we look at the determinant of  $H$ . We have that from exercise one that  $H$  is unitary (orthogonal) and symmetric. Therefore (going in one direction) that  $H^{-1} = H^*$ . This is because  $H^*H = I = H^{-1}H$ . Next we also have that for any matrix  $A$ ,  $\det(A^*) = \overline{\det(A)}$ . We also have that,  $\det(A)\det(A^{-1}) = 1$ .

$$\det(H^{-1}H) = \det(H^{-1})\det(H) = 1$$

$$\overline{\det(H)}\det(H) = 1$$

$$\det(H)^2 = 1 \implies \det(H) = \pm 1$$

(iii) Singular Values

We have from the proof in exercise one, we have that  $H \in \mathbb{R}^{m \times m}$  is a unitary (orthogonal) matrix. We have therefore that  $H$  preserves the length of vectors under transformation. We also look at the singular value decomposition of  $H$ .

$$H = U\Sigma V^T, \quad \|Hv\| = \|v\|, \forall v \in \mathbb{R}^m$$

Let us look at a specific vector  $v_i$  now such that  $v_i$  is  $i$ -th column vector of  $V$ .

$$\|Hv\| = \|U\Sigma V^T v_i\| = \|u_i \sigma_{ii}\|$$

We recover the scalar-vector product,  $u_i \sigma_{ii}$  where  $u_i$  is the  $i$ -th column vector of  $U$  and  $\sigma_{ii}$  is the  $i$ -th diagonal element of  $\Sigma$ . We return to the fact that by the Singular Value Decomposition Theorem, that  $U, V$  are unitary, that is they are composed of orthogonal column vectors with norm of 1. Therefore we have,

$$\begin{aligned} \|Hv\| &= \|v_i\| = \|\sigma_{ii} u_i\| = |\sigma_{ii}| \|u_i\| \\ 1 &= |\sigma_{ii}| \cdot 1, \implies \sigma_{ii} = \pm 1 \end{aligned}$$

Therefore since our choice of  $v$  was arbitrary among the column vectors of  $V$  we have that this example exhausts all singular values for  $H$ . Thus the singular values of  $H$  are  $\pm 1$ . It can even be argued that the plus minus in this context does not matter. Since singular values are scalars which in a transformation from one vector basis to another scale the vector in the resulting basis. The vectors in the output basis are orthogonal so scaling one vector say by  $-1$  would not make that basis linearly dependent. Therefore we claim that any  $u_i$  will absorb the sign of  $\sigma_{ii}$  (Also because of the fact that singular values are always positive). So it is as simpler to claim that,

$$\sigma_{ii} = 1.$$

Therefore the singular values of  $H$  are such that,  $\sigma_{ii} = \sigma_i = 1$ . □

5. Let  $A \in \mathbb{R}^{m \times n}$ . Show that  $\text{cond}(A^T A) = (\text{cond}(A))^2$ .

*Proof.* We start with the singular value decomposition of  $A$ .

$$\begin{aligned} A &= U\Sigma V^T, \quad U, V \text{ unitary} \\ A^T A &= V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T \end{aligned}$$

Notice that this is a singular value decomposition for  $A^T A$  since  $V, V^T$  are unitary matrices and  $\Sigma^2$  is a diagonal matrix with positive or zero entries along the diagonal. We look at the fact that the condition number of a matrix  $A$  is proportional to the two norms of  $A$  and  $A^{-1}$ .

$$\text{cond}(A^T A) = \|A^T A\|_2 \cdot \|(A^T A)^{-1}\|_2$$

We will also use the fact that the two-norm of a matrix is equal to its largest singular value. We now look for  $(A^T A)^{-1}$ .

$$\begin{aligned} (A^T A)^{-1} (A^T A) &= I \\ U_1 U_2 U_3 V \Sigma^2 V^T &= I \end{aligned}$$

Very evidently from this assumption we can pick three matrices to invert  $A^T A$ . We take  $U_3 = V^T$ ,  $U_2 = \Sigma^{-2}$  (this inverse exists because  $\Sigma$  is diagonal),  $U_1 = V$  (assuming that  $V$  is invertible). Thus we have,

$$(A^T A)^{-1} = V \Sigma^{-2} V^T$$

Notice that this is also a singular value decomposition for  $(A^T A)^{-1}$  since both  $V, V^T$  are unitary and  $\Sigma^{-2}$  is still diagonal. Notice however the largest singular values for  $A^T A, (A^T A)^{-1}$  are  $\sigma_1^2, \frac{1}{\sigma_k^2}$  respectively. Therefore we go back to the condition number.

$$\text{cond}(A^T A) = \|A^T A\|_2 \cdot \|(A^T A)^{-1}\|_2 = \sigma_1^2 \frac{1}{\sigma_k^2} = \left( \frac{\sigma_1}{\sigma_k} \right)^2 = (\text{cond}(A))^2$$

This last bit ( $\text{cond}(A) = \frac{\sigma_1}{\sigma_k}$ ) is taken from a proof in lecture (I don't know where but its fairly evident using a singular value decomposition in almost exactly the same way as we are presenting this argument). □