Dante Bull

Work below

AM112/212A **Exam 1**

Instructions:

1. This is an open response exam using the venue of canvas quiz. The exam problems are given below. Complete the problems by yourself. You can use books and notes. But do not seek any help from others.

   Clearly mark each problem, and box the final answer for each part in each problem as shown below.

   Problem X:

   (a) Find a general solution of $u''(x) = 2u(x)$.

   (b) Solve the IVP $u''(x) = 2u(x)$, $u(0) = 0$, $u'(0) = \sqrt{2}$

   ...

   Part (a): $\boxed{\text{Final answer: } u(x) = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}}$

   Part (b): $\boxed{\text{Final answer: } u(x) = \frac{1}{2} e^{\sqrt{2}x} - \frac{1}{2} c_2 e^{-\sqrt{2}x}}$

2. You have ONE attempt with a time limit of 100 minutes (including the time for scanning and submission). **DO NOT click the "Take the Quiz" button until you are ready and have the next 100 minutes dedicated to take the exam.**

3. Complete the problems on paper. Scan all pages of your exam paper into ONE pdf and name the pdf using your own name. **Do NOT name it "Exam_1"** (this is to prevent you from accidentally uploading the list of exam problems as your answer!). Upload the scanned pdf as your answer in canvas.

4. If you miss the 100 min submission deadline by a few minutes, upload the scanned pdf in the comment of your exam.

→ **Go to the next page for exam problems.**

Important note:

- The problems below have similarities. **But they have different types of boundary conditions and different functions. Be careful!**
- You may re-use, without re-deriving them, any results we already obtained in lectures, in homework assignment(s), or in exam(s).

Problem 1 (4 points):

Consider the initial boundary value problem (IBVP) below.

$$\begin{cases} u_t = 2u_{xx} \\ u(x,0) = e^x - \sin(x) \\ u_x(0,t) = 0\ ,\ u(\pi,t) = 0 \end{cases} \qquad \text{(IBVP-P1)}$$

a) Clearly write out the associated eigenvalue problem.

b) Clearly write out the eigenvalues and eigenfunctions.

Do NOT solve the IBVP.

Problem 2 (4 points):

Consider the IBVP below.

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = e^{-x}\cos(x) \\ u(0,t) = 5\ ,\ u_x(L,t) = -1 \end{cases} \qquad \text{(IBVP-P2)}$$

a) Find the steady state $u_\infty(x)$.

b) Let $u_h(x,t) = u(x,t) - u_\infty(x)$. Clearly write out the IBVP for $u_h(x,t)$.

Do NOT solve the IBVP.

→ Go to the next page for more exam problems.

Problem 3 (4 points):

Consider the expansion $f(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{(n-\frac{1}{2})\pi}{L} x), \quad x \in [0, L]$.

a) Clearly write out the formula for calculating coefficient c_n.

b) Evaluate coefficient c_n for function $f(x) \equiv 1, \ x \in [0, L]$.

Problem 4 (8 points):

Solve the IBVP below.

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = 0 \\ u(0,t) = -2 \,, \ u(L,t) = 0 \end{cases} \qquad \text{(IBVP-P4)}$$

Problem 1 (4 points):

Consider the initial boundary value problem (IBVP) below.

$$\begin{cases} u_t = 2u_{xx} \\ u(x,0) = e^x - \sin(x) \\ u_x(0,t) = 0 \,, \ u(\pi,t) = 0 \end{cases} \qquad \text{(IBVP-P1)}$$

a) Clearly write out the associated eigenvalue problem.

b) Clearly write out the eigenvalues and eigenfunctions.

Do NOT solve the IBVP.

By Sep. of Vars.

$u_T = 2u_{xx} \rightarrow \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$

$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

$T(t) = e^{-\lambda k t}$

a. $\begin{bmatrix} -\sin(0) & \cos(0) \\ \cos(\sqrt{\lambda}\pi) & \sin(\sqrt{\lambda}\pi) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ \cos(\sqrt{\lambda}\pi) & \sin(\sqrt{\lambda}\pi) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\downarrow$

$C_2 = 0$

$C_1 \cos(\sqrt{\lambda}\pi) = 0$

$\cos(\sqrt{\lambda}\pi) = 0$

$\sqrt{\lambda}\pi = \frac{\pi}{2} + n\pi$

$\sqrt{\lambda} = n + 1/2$

$\lambda = (n + 1/2)^2$

b. $\lambda = (n + 1/2)^2$

Eigenfunctions $= \{\cos((n+1/2)\pi\}$

Problem 2 (4 points):

Consider the IBVP below.

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = e^{-x}\cos(x) \\ u(0,t) = 5\,,\ u_x(L,t) = -1 \end{cases} \qquad \text{(IBVP-P2)}$$

a) Find the steady state $u_\infty(x)$.

b) Let $u_h(x,t) = u(x,t) - u_\infty(x)$. Clearly write out the IBVP for $u_h(x,t)$.

Do NOT solve the IBVP.

$u_\infty(x) = C_1 x + C_0$

$u_\infty(0) = 5 = C_0$

$u_\infty'(L) = -1 = C_1$

$\boxed{u_\infty(x) = 5 - x}$

IBVP for u_h

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = e^{-x}\cos(x) - 5 \\ u(0,t) = 0\,,\ u_x(L,t) = 0 \end{cases}$$

All "u"s here are u_h

Problem 3 (4 points):

Consider the expansion $f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(n-\frac{1}{2})\pi}{L}x\right), \quad x \in [0, L]$.

a) Clearly write out the formula for calculating coefficient c_n.

b) Evaluate coefficient c_n for function $f(x) \equiv 1,\ x \in [0, L]$.

a) $\boxed{C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)dx}$

b) $C_n = \frac{2}{L}\int_0^L \sin\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)dx = -\frac{2}{\pi(n-\frac{1}{2})}\cos(\ )\Big|_0^L$

$\cos((n-\frac{1}{2})\pi) = 0$

$\cos(0) = 1$

$\boxed{C_n = \frac{2}{\pi(n-\frac{1}{2})}}$

Problem 4 (8 points):

Solve the IBVP below.

$$\begin{cases} u_t = k u_{xx} \\ u(x,0)=0 \\ u(0,t)=-2,\ u(L,t)=0 \end{cases} \qquad \text{(IBVP-P4)}$$

$u_\infty(x) = C_1 x + C_2$ $\qquad T(t) = e^{-\lambda k t}$ $\qquad u_n(x,t) = X(x)T(t)$

$C_2 = -2,\ C_1 = \frac{2}{L}$ $\qquad X(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$u_\infty(x) = 2\left(\frac{x}{L} - 1\right)$ $\qquad \lambda = \frac{n^2\pi^2}{L^2}$

$u(x,t) = X(x)T(t) - u_\infty(x)$

$u_n(x,0) = 2\left(1 - \frac{x}{L}\right)$

$u_n(0,t) = 0 \qquad u_n(L,t) = 0$

$$b_n = \frac{2}{L}\int_0^L 2\left(1-\frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{4}{L}\left[\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx - \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{4}{L}\left[\frac{-L}{n\pi}(\cos(n\pi)-1) + \frac{1}{n\pi}\left(x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L - \int_0^L \cos\left(\frac{n\pi x}{L}\right)dx\right)\right.$$

$$= \frac{4}{L}\left[\frac{-L}{n\pi}((-1)^n - 1) + \frac{1}{n\pi}\left(L(-1)^n\right) - \underbrace{\frac{1}{n\pi}\left(\frac{L}{n\pi}(\sin(n\pi)-\sin(0))\right)}_{=0}\right]$$

$$= \frac{4}{n\pi}\left[-(-1)^n + 1 + (-1)^n\right]$$

$$b_n = \frac{4}{n\pi}$$

$$\boxed{u(x,t) = \underbrace{2\left(\frac{x}{L}-1\right)}_{u_\infty} + \underbrace{\sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}}_{u_n}}$$