Alg. Simultaneous Iter (SI)

"power"

$Q = I$

do while error > large

$Z = AQ$

$QR = Z$

end do

$Q^{(0)} = I$

do while error > large

$Z^{(n)} = AQ^{(n-1)}$

$Q^{(n)} R^{(n)} = Z^{(n)}$

end do

SI

Rmk. error > large

$\| D_\lambda^{(k+1)} - D_\lambda^{(k)} \| >$ threshold

Step1 : $Z^{(1)} = AI$

$Q^{(1)} R^{(1)} = Z^{(1)} \longrightarrow Q^{(1)} = Z^{(1)} R^{(1)^{-1}}$

$= AIR^{(1)^{-1}}$

Step2 : $Z^{(2)} = AQ^{(1)} \longrightarrow$ $\boxed{\begin{aligned} Z^{(2)} &= AAIR^{(1)^{-1}} \\ &= A^2 I R^{(1)^{-1}} \end{aligned}}$

$$Q^{(2)} R^{(2)} = Z^{(2)}$$

$$\Downarrow$$

$$\boxed{Z^{(2)} R^{(1)} = A^2 I}$$

$$\Downarrow$$

$$\boxed{Q^{(2)} R^{(2)} R^{(1)} = A^2 I}$$

Pattern: $A^n I = Q^{(n)} \underbrace{R^{(n)} R^{(n-1)} \cdots R^{(1)}}$

$$= Q^{(n)} R \quad (*)$$

pf We prove $(*)$ using induction

i) $n=1$: $\boxed{Q^{(1)} R^{(1)} = A I} = Z^{(0)}$

ii) Assume it's true for $n-1$, i.e,

$$A^{n-1} I = \underline{Q^{(n-1)} R^{(n-1)} \cdots R^{(1)}}$$

iii) For n :

$$A^n I = A A^{n-1} I$$

$$= \underbrace{A Q^{(n-1)}} R^{(n-1)} \cdots R^{(1)}$$

$$= \underline{Z^{(n)}} \boxed{R^{(n-1)} \cdots R^{(1)}}$$

$$= Q^{(n)} R^{(n)} R^{(n-1)} \cdots R^{(1)}$$

Rmk. $Q^{(n)}$ contains a set of orthonormal column vectors forming a basis for span (A^n)

Rmk ① The convergence rate

$\approx$ the rate of Power Iter

(☹ slow)

② Let $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$

Then

The convergence rate for λ_1

$\approx$ the convergence rate for λ_m

and slow on intermediate λ_i

(TBD)

$\boxed{3.2}$ The QR algorithm w/o shift

Alg. QR alg.

```
do while error > large
    Q R = A
    A = R Q
end do
```

```
do while error > large
    Q^(n) R^(n) = A^(n)
    A^(n+1) = R^(n) Q^(n)
end do
```

Details

$$
\begin{aligned}
A^{(n+1)} &= R^{(n)} Q^{(n)} \\
&= Q^{(n)T} A^{(n)} Q^{(n)} \\
&= Q^{(n)T} Q^{(n-1)T} A^{(n-1)} Q^{(n-1)} Q^{(n)} \\
&= \cdots \\
&= Q^{(n)T} Q^{(n-1)T} \cdots Q^{(1)T} A \underbrace{Q^{(1)} \cdots Q^{(n)}}_{(\equiv \breve{Q}^{(n)})} \\
&= \left(\breve{Q}^{(n)}\right)^T A \left(\breve{Q}^{(n)}\right) \\
&\xrightarrow[\text{in the limit}]{} V^{-1} A V = D_\lambda
\end{aligned}
$$

Rmk

$\breve{Q}^{(n)} \longrightarrow V$ in the limit

Thm SI $\Longleftrightarrow$ QR

① QR is very simple

② $A \longmapsto D_\lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$

when $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$

③ $\underset{\underset{\displaystyle Q^{(n)}}{\parallel}}{Q} \longmapsto V = \begin{bmatrix} v_1 | \cdots | v_m \end{bmatrix}$

$v_i \perp v_j, \ \forall i \neq j$

$\|v_i\| = 1$

$A v_i = \lambda_i v_i, \ \forall i$

Alg QR with eigenvectors

$V = I$

do while error > large

$QR = A$

$A = RQ$

$V = VQ$

end do

$V = I Q^{(1)}$

$V = V Q^{(2)}$

$= Q^{(1)} Q^{(2)}$

$\to V \to Q^{(1)} Q^{(2)} \cdots Q^{(n)}$

Ex $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Step 1: $A = QR = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Step 2: $RQ = IA = A$

$\longrightarrow$ QR fails !

Rmk. We know RQ is fast !

We can do this by considering "QR alg. with shifts"

$\boxed{4}$ Improving QR alg.

4.1 QR with shifts

(i) RQ ftn r ; $r(v) = \mu$

$\longrightarrow$ II $(\mu) = X$

inverse iter algo.

(ii) μ ; any specific choice

Can it give all good convergence rate for all eigen vectors?

$\Longrightarrow$ "Simultaneous Inverse Iter" (SII)

☆ QR with shift

do while error > large

$\mu = a_{mm}$ (shift)

$QR = A - \mu I$

$A = RQ + \mu I$

end do

Thm — Duality Thm

QR w/o shift $\Longleftarrow$ $\boxed{SI}$ $\Longleftrightarrow$ unshifted SII on $A^{-1}P$

$Q = I$

do while error > large

$\quad Z = AQ$

$\quad QR = Z$

end do

Outcome

$$A^n I = Q^{(n)} R$$

$Q = I$

do while error > large

$\quad Z = (A^{-1}P)\,Q$

$\quad QR = Z$

end do

Outcome

$$(A^{-1}P)^n = Q^{(n)} R$$

Here, P = back-to-front I

e.g., $P_{4\times4} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$

Alg. QR with shift (Part 1)

do while error > large
$\mu = a_{mm}$ (select shift)
$QR = A - \mu I$
$A = RQ + \mu I$
end do

$Q^{(n)} R^{(n)} = A^{(n)} - \mu I$

Rak. $\mu = a_{mm}$