AM 213A Midterm Exam

[5 pm, Thu, Feb 15, 2024] and [5 pm, Sat, Feb 17, 2024]

The exam time is in Pacific Time
Submit your work to Gradescope (not to Canvas)

1. Please work on all problems. Write your logical explanations and conclusions in each step of your solutions, explicitly and clearly.

2. Use LaTeX or MS-words like text editors. A scanned copy of handwritten solutions will be acceptable on an exceptional case-by-case with permission from the instructor.

2. NO CHEATING! This is an open-book, open-note, take-home exam. You are welcome to use the lecture notes, our textbook, and other textbooks. However, consulting with other resources, including help from humans, AI, and online solutions, will not be tolerated. Such students will receive a zero score and those activities will be reported to the AM and the student's departments. Any two or more very similar exam papers will also be treated as cheating and reported. This means that if you find online solutions, you should be concerned about copying them into your report because doing so could potentially make your solutions look similar to someone else's solutions.

3. To disprove, you need to provide a counter-example with clear logical explanations.

4. All your work should be **submitted to Gradescope within the 48-hour window** unless you are approved for special accommodation. Late submissions will be accepted only in an emergency situation with permission from the instructor.

5. Submit your report in a **single file** using the naming convention,

`FirstnameLastname_midtermReport.pdf`.

Problem 1 (10 points; 1 pt each) Define the following:

a. A full rank matrix $\mathbf{A}$ and a rank deficient matrix $\mathbf{B}$, for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$.

b. An orthogonal matrix $\mathbf{Q}$ and a unitary matrix $\mathbf{U}$, where $\mathbf{Q}$ and $\mathbf{U}$ are square.

c. Singular value decomposition of $\mathbf{A} \in \mathbb{C}^{m \times n}$ with $\text{rank}(\mathbf{A}) = k \le \min(m, n)$.

d. Orthogonal projector $\mathbf{P} \in \mathbb{R}^{m \times m}$.

e. Defective matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$.

f. Relative condition number $\kappa(\mathbf{x}_0)$ of a differentiable function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ at $\mathbf{x} = \mathbf{x}_0$ and its upper bound when $\mathbf{A}$ is nonsingular.

g. Condition number κ or $\text{cond}(\mathbf{A})$ of a nonsingular matrix $\mathbf{A}$ with $\text{rank}(\mathbf{A}) = k$ in the 2-norm in terms of singular values.

h. Diagonalizable matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$.

i. Machine accuracy ϵ_{mach} in single and double precisions.

j. Backward stable algorithm $\tilde{f}$ for a problem f. What is the relation between the backward stability and accuracy?

Problem 2 (6 points) Let $\mathbf{A} = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n \times n}$.

a. [3 pts] Suppose $\mathbf{A}$ has nonnegative entries such that $\sum_{j=1}^{n} a_{ij} = 1$ for $1 \leq i \leq n$. Show that no eigenvalue of $\mathbf{A}$ has an absolute value greater than 1.

b. [3 pts] Suppose that $\sum_{j=1}^{n} |a_{ij}| < 1$ for each i. Prove that $\mathbf{B} = \mathbf{I} - \mathbf{A}$ is invertible. (Hint: Using the Equivalence Theorem for a nonsingular matrix $\mathbf{A}$, it suffices to show that the dimension of the null space of $\mathbf{B}$ is 0.)

Problem 3 (6 points) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be symmetric.

a. [2 pts] Define what it means to say $\mathbf{A}$ is symmetric positive definite.

b. [2 pts] Show that the eigenvalues of a symmetric positive definite matrix $\mathbf{A}$ are positive.

c. [2 pts] What can you say about the diagonal entries of a symmetric positive definite matrix $\mathbf{A}$? Justify your answer by proving or disproving it.

Problem 4 (10 points) Consider the following algorithm for a linear system with $\mathbf{A} \in \mathbb{R}^{m\times m}$:

```
do j = 1 to m-1

    if a_jj = 0 then
        stop
    endif

    do i = j+1 to m

        r_i = r_i - r_j a_ij/a_jj  ![r_i and r_i are the ith and jth rows of A, respectively]

        b_i = b_i - b_j a_ij/a_jj

    enddo
enddo
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a. [2 pts] What is this algorithm for? What are the anticipated outcomes?

b. [3 pts] Count the floating point operations, including additions, subtractions, multiplications, and divisions in this algorithm.

c. [3 pts] The algorithm is subject to an issue when a_{jj} is zero or close to zero. Give a simple example of a 2×2 system that demonstrates the potential issue when a_{jj} is close to machine accuracy ε_{mach}.

d. [2 pts] Describe the approach you learned in class, which can resolve the issue in (c). What is this approach called? Write a pseudo algorithm that can resolve the issue by modifying the given algorithm.

Problem 5 (20 points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix} \tag{1}$$

for the least squares problem $\mathbf{A}\mathbf{x} \simeq \mathbf{b}$ with $0 < \varepsilon < \sqrt{\varepsilon_{mach}}$. Here, ε_{mach} is the small value in machine accuracy and is numerical zero. In this problem, all your arithmetic manipulations should mimic the computer's finite precision handling.

a. [6 pts] Carry out to use the normal equation by first multiplying $\mathbf{A}^T$ on both sides to directly solve the linear system. Discuss what happens.

b. [7 pts] Check if the resulting $\mathbf{A}^T\mathbf{A}$ is symmetric positive definite (SPD) by *directly* using the definition of SPD. Conclude whether you can use the Cholesky factorization method or not for the resulting normal equation. Justify your answer.

c. [7 pts] In general, using the normal equation $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ for solving the least squares problem $\mathbf{A}\mathbf{x} \simeq \mathbf{b}$ is not always ideal. Justify why it is not ideal by proving the condition number of the Gram matrix $\mathbf{A}^T\mathbf{A}$ is the square of the condition number of $\mathbf{A}$, i.e., $\kappa(\mathbf{A}^T\mathbf{A}) = \Big(\kappa(\mathbf{A})\Big)^2$, for a full rank matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $m \geq n$. What is the implication of $\kappa(\mathbf{A}^T\mathbf{A}) = \Big(\kappa(\mathbf{A})\Big)^2$?

Problem 6 (15 points) The Schur decomposition theorem states that if $\mathbf{A} \in \mathbb{R}^{m\times m}$, then $\mathbf{A} = \mathbf{QUQ}^{-1}$ where $\mathbf{Q}$ is orthogonal and $\mathbf{U}$ is an upper triangular.

a. [5 pts] Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be normal, i.e., $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$. Show if $\mathbf{A}$ is *upper* triangular normal matrix, then $\mathbf{A}$ diagonal.

b. [6 pts] Now, let $\mathbf{A}$ be a general matrix (i.e., not just being upper). Show that $\mathbf{A}$ is orthogonally diagonalizable, i.e., there exist $\mathbf{D}$ diagonal and $\mathbf{Q}$ orthogonal such that $\mathbf{A} = \mathbf{QDQ}^{-1}$.

c. [2 pts] Do you think $\mathbf{A}$ is defective or non-defective? Justify your answer.

d. [2 pts] What can you say about $\mathbf{A}^{-1}$? Is $\mathbf{A}$ always invertible or is the invertibility under a certain condition?

Problem 7 (18 points) Suppose $\mathbf{A} \in \mathbb{R}^{m \times m}$ is symmetric and positive definite. Consider the following iteration where $\mathbf{L}_i$ is lower triangular with nonzero diagonal elements:

Algorithm:

$\mathbf{A}_0 = \mathbf{A}$

`for` $i = 0, \cdots$

  $\mathbf{L}_i \mathbf{L}_i^T = \mathbf{A}_i$ **[Cholesky Decomposition]**

  $\mathbf{A}_{i+1} = \mathbf{L}_i^T \mathbf{L}_i$

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a. [6 pts] Show that $\mathbf{A}_{i+1}$ is also symmetric and positive definite.

b. [6 pts] Show that $\mathbf{A}_{i+1}$ is similar to $\mathbf{A}_0$, i.e., $\mathbf{A}_{i+1} = \mathbf{B}^{-1}\mathbf{A}_0\mathbf{B}$ for some non-singular matrix $\mathbf{B}$. You should identify what $\mathbf{B}$ is.

c. [6 pts] Carry out the first one step of the iteration (i.e., $i = 0$) by explicitly identifying $\mathbf{L}_0$ to compute $\mathbf{A}_1$ for a matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{2}$$

Problem 8 (15 points) Let $\mathbf{P} \in \mathbb{R}^{m\times m}$ be an orthogonal projector.

a. [6 pts] Show that $\mathbf{P}$ is positive semi-definite with its eigenvalues either zero or one. (Hint: A symmetric matrix is orthogonally diagonalizable.)

b. [3 pts] What can you say about the dimension of $\mathbf{P}$ if its eigenvalues are all distinct with algebraic multiplicity of 1?

c. [6 pts] Construct $\mathbf{P} \in \mathbb{R}^{2\times 2}$ whose entries are all nonzero. Identify all possible choices of $\mathbf{P}$. For each of your constructed $\mathbf{P}$, show that it is positive semi-definite and its eigenvalues are either 0 and 1.