CHAPTER 2 : MHD EQNS

MAXWELL'S EQNS → MHD EQNS

Maxwell's equations describe ELECTROMAGNETISM (EM)
→ how ELECTRIC and MAGNETIC fields interact

James Clerk Maxwell (1831-1879) took existing experimental laws and unified them
Also figured out that EM waves and light were same thing

Maxwell's eqns are confusing since there are many different ways of writing them down (that are equivalent, of course) and some people prefer certain variables over others.
→ integral form vs differential (or point) form

Variables (alphabetically)

- $\underline{B}$ - magnetic field (magnetic flux density)
- $\underline{D}$ - electric displacement (electric flux density)
- $\underline{E}$ - electric field
- $\underline{H}$ - magnetic intensity ("mag field strength?")
- $\underline{j}$ - electric current density (elec curr per unit area) } sources!
- ρ_e - charge density (charge per unit volume) }

"Constants"

- ε - (dielectric) permittivity } "of free space" ⇒ vacuum
- μ - permeability }
- σ - conductivity

ε → property of medium that determines strength of electric field for a given charge and geometry. $\varepsilon \uparrow$ ⇒ more charge needed for same $\underline{E}$
$\varepsilon > \varepsilon_0$ (vacuum) ($\varepsilon_{water} \sim 81\varepsilon_0$; take $\varepsilon \sim \varepsilon_0$)

μ → $\mu \uparrow$ ⇒ given value of electric current produces stronger $\underline{B}$ $\quad \mu > \mu_0$
($\mu_{iron} \sim 1000\,\mu_0$; take $\mu \sim \mu_0$)

Constitutive relations $\quad \underline{H} = \dfrac{\underline{B}}{\mu} \qquad \underline{D} = \varepsilon \underline{E}$ $\quad$ *CAN BE MORE COMPLEX!

(Like viscosity in fluids; stress → strain)

(i.e can forget about $\underline{H}$, $\underline{D}$ really!)

Integral form dA = vector surface area dL = vector line segment

1/ $\oint_S \underline{E} \cdot d\underline{A} = q/\varepsilon_0$ (q = total enclosed charge) GAUSS'S LAW FOR ELECTRICITY

2/ $\oint_S \underline{B} \cdot d\underline{A} = 0$ GAUSS'S LAW FOR MAGNETISM

3/ $\oint_C \underline{E} \cdot d\underline{L} = -\frac{\partial}{\partial t} \phi_B$ FARADAY'S LAW OF INDUCTION

$\phi_B = \oint \underline{B} \cdot d\underline{A}$ (total) Magnetic flux

4/ $\oint_C \underline{B} \cdot d\underline{L} = \mu_0 j + \underbrace{\mu_0 \varepsilon_0}_{=1/c^2} \frac{\partial}{\partial t} \underbrace{\oint_S \underline{E} \cdot d\underline{A}}_{\phi_E \text{ (total) electric flux}}$ AMPERE'S LAW

DIFFERENTIAL ("AT A POINT") FORM

1/ $\underline{\nabla} \cdot \underline{E} = \rho_E / \varepsilon_0$ charge density ($\underline{\nabla} \cdot \underline{D} = \rho_E$) GAUSS

2/ $\underline{\nabla} \cdot \underline{B} = 0$

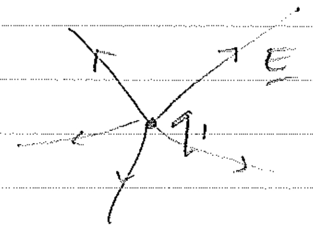
3/ $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ FARADAY

4/ $\underline{\nabla} \times \underline{B} = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ AMPERE

$\left(\underline{\nabla} \times \underline{H} = j + \frac{\partial D}{\partial t} \right)$

Derivations (quick!)

1/ charge q_1 (+ve) $\Rightarrow$



radiating E

Bring a test particle to point @ distance r, charge q_2

$\Rightarrow$ Coulomb force between charges

$$F = \frac{q_1 q_2}{4\pi \varepsilon r^2}$$

Electric field = force/unit charge = $\dfrac{q'}{4\pi\varepsilon r^2}$

Electric flux through a small area dA : $d\phi_E = \underline{E} \cdot d\underline{A}$

Total flux through a spherical region $\phi_E = \oint_S \underline{E} \cdot d\underline{A}$

$$= \underline{E} \; 4\pi r^2 \quad (\text{const } r)$$

$$= \frac{q'}{4\pi\varepsilon r^2} \cdot 4\pi r^2 = \frac{q'}{\varepsilon}$$

(Note surface can be any surface because now can add surfaces of any shape if they do not contain charge since contrib $= q/\varepsilon = \emptyset$)

If more charges contained, $\phi_E = \sum_i q_i / \varepsilon$ (beyond charges)

or $\boxed{\oint_S \underline{E} \cdot d\underline{A} = \dfrac{q_{\text{enclosed}}}{\varepsilon}}$

Notice $\oint_S \underline{E} \cdot d\underline{A} = \int_V \nabla \cdot \underline{E} \; dV$

and $q_{\text{enclosed}} = \int_V \rho_E \, dV$

$\Rightarrow \boxed{\nabla \cdot \underline{E} = \rho_E / \varepsilon}$

2/ Same for "magnetic charges" (magnetic "poles")

Magnetic flux (total)

$$\Phi_B = \oint_S \underline{B} \cdot d\underline{A} = \sum_i (\text{mag poles}) = \emptyset$$

since $\nexists$ no mag monopoles! (only N-S magnets!)

and again $\oint_S \underline{B} \cdot d\underline{A} = \int_V \nabla \cdot \underline{B} \, dV = \emptyset$

$$\Rightarrow \boxed{\nabla \cdot \underline{B} = 0}$$

3/ Induction! Move a magnet in and out of a coil

$\Rightarrow$ current in coil

Existence of current $\Rightarrow$ existence of a {potential difference / voltage / EMF (note NOT a force!!)}

Expts $\Rightarrow$ EMF $\mathcal{E} \propto \partial B/\partial t$, $\propto A$ area of coil $\Rightarrow \mathcal{E} \propto (-) \dfrac{\partial \Phi_B}{\partial t}$

Magnetic flux $\Phi_B = \underline{B} \cdot d\underline{A}$

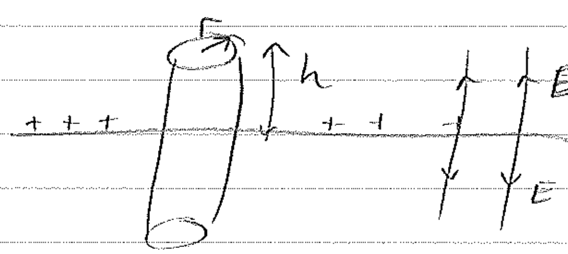
$$\Rightarrow \boxed{\frac{\partial \Phi_B}{\partial t} = \frac{\partial}{\partial t}(\underline{B} \cdot d\underline{A}) = -\mathcal{E}} \quad (3A)$$

All this just establishing EMF $= \oint E \cdot dl$

note J = Newtons/Coulombs = volts/meter

$\oint E \cdot dl = V$

Charge on a plate: charge per unit area $= \sigma$



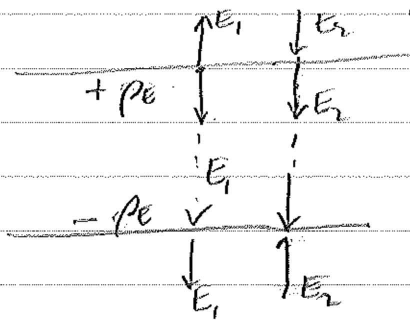
Gauss $\Rightarrow \oint^{\text{total}} E \cdot dA = \sum$ charges in cylinder

$$\Rightarrow \underbrace{2}_{\text{two ends}} \cdot 2\pi r^2 E = \underbrace{2\pi r^2 \rho_E}_{\text{total charge in cylinder}} / \varepsilon \Rightarrow \boxed{E = \frac{\rho_E}{2\varepsilon}}$$

Sides $\uparrow\uparrow$ to field $\Rightarrow$ do not contribute

Electric field indep of distance from plate

2 charged plates :



$\leftarrow$ Total $E = E_1 - E_2 = \emptyset$.

$\leftarrow$ Total $E = E_1 + E_2 = \dfrac{\rho_E}{\varepsilon}$ } Electric field constant (3B)

$\leftarrow$ Total $E = E_1 - E_2 = \emptyset$

Voltage = potential energy acting on a charge to move it from one point to another
= work done /per unit charge to move test charge between two points.
= electric potential difference per unit charge

PE = work done = force × distance

$PE_\infty = F_\infty \cdot \infty$ but $F_\infty = \emptyset$ since $F_\infty = \left.\dfrac{q_1 q_2}{4\pi\varepsilon r^2}\right|_{r\to\infty}$

$PE_r = \dfrac{q_1 q_2}{4\pi\varepsilon r^2} \cdot r = \dfrac{q_1 q_2}{4\pi\varepsilon r}$

$\Rightarrow$ Voltage, $V = PE/\text{unit charge} = \dfrac{q}{4\pi\varepsilon r} = E\,r$ ← distance ← force/unit charge

$E = V/r$

So in parallel plates, electric field constant $\Rightarrow$ voltage increases with distance from $r=0$

So $V = E\,r$ or more correctly in vectors

$V = \underline{E} \cdot d\underline{L}$

$\Rightarrow$ $\boxed{\text{total EMF} = \oint \underline{E} \cdot d\underline{L}}$ if in a loop, say. (3C)

Combining (3A) and (3C) $\rightarrow$

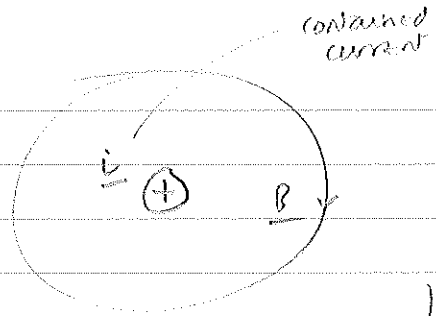
$$\boxed{\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\partial}{\partial t}\oint_s \underline{B} \cdot d\underline{A} = \oint_c \underline{E} \cdot d\underline{L}}$$

Note $\oint_c \underline{E} \cdot d\underline{L} = \oint_s (\nabla \times \underline{E}) \cdot d\underline{A}$ and removing integral

$\Rightarrow$

$$\boxed{\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}}$$

4/



Experimentally, found current in a wire induced a magnetic field around the wire.

Found magnetic intensity H drops $\sim 1/r$

$$\boxed{H \propto i/r}$$

Consider circle around wire radius r; $B = H$ = const at fixed $r \Rightarrow$

$$\oint_C \underline{H} \cdot d\underline{l} = \oint_C \frac{\underline{B}}{\mu_0} \cdot d\underline{l} = \frac{B}{\mu_0} 2\pi r = i$$

$$\Rightarrow \underline{B} = \frac{i}{2\pi r \mu_0} \quad \text{and} \quad \boxed{\oint_C \underline{B} \cdot d\underline{l} = \mu_0 i}$$

Stokes Th $\Rightarrow$ (current density)

$$\oint_S (\underline{\nabla} \times \underline{B}) \cdot d\underline{S} = \oint_S \mu_0 \underline{j} \cdot d\underline{S} = \mu_0 i$$

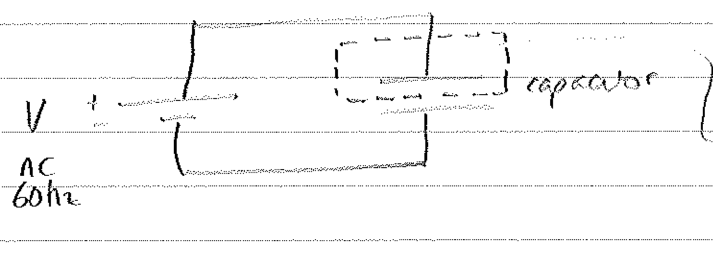
$$\Rightarrow \boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{j}}$$

Maxwell noticed an inconsistency:

Take divergence of above $\Rightarrow$ $\underline{\nabla} \cdot \underline{j} = 0$

Is this true?

OK for many circuits but what about, e.g., a capacitor?



A/C $\Rightarrow$ current flows!

$\hookrightarrow$ Test loop: NOT divergence free as current flows in wire but not in air of capacitor!!

Instead of current, electric field varies within capacitor due to varying charge. Since time-varying B gives rise to a curl of $\underline{E}$, why not the symmetric opposite: time-varying $\underline{E}$ gives rise to curl of B?

Faraday

$$\frac{\partial \underline{D}}{\partial t} = \underline{j}_D \quad \text{or} \quad \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \underline{j}_D$$

DISPLACEMENT CURRENT

Therefore now

$$\nabla \times \underline{B} = \mu_0 \left[\underline{j} + \underline{j}_D \right]$$

$$\boxed{\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}}$$

More precisely:

In capacitor,

$$\mu_0 \underline{i} = \mu_0 \frac{dq}{dt} = \mu_0 \frac{\varepsilon_0 \, dA}{\varepsilon_0 \, dA} \frac{dq}{dt} = \mu_0 \varepsilon_0 dA \frac{d}{dt}\left(\frac{q}{\varepsilon_0 dA}\right)$$

$$= \mu_0 \varepsilon_0 \, dA \frac{dE}{dt} = \mu_0 \varepsilon_0 \frac{d}{dt}(\underline{E} \cdot d\underline{A})$$

$$= \mu_0 \varepsilon_0 \frac{d}{dt}(d\Phi_E)$$

$\Phi_E = \oint_S d\Phi_E = \oint \underline{E} \cdot d\underline{A}$

Summing over area →

$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d}{dt} \Phi_E$$

Then

$$\oint_S \nabla \times \underline{B} \cdot d\underline{A} = \int_S \mu_0 \underline{j} \cdot d\underline{A} + \mu_0 \varepsilon_0 \int_S \frac{d}{dt}(\underline{E} \cdot d\underline{A})$$

$$\Rightarrow \nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{d\underline{E}}{dt}$$

somethings to notice about maxwell's eqns:

$$\nabla \cdot \underline{E} = \rho_E / \varepsilon_0 \qquad \nabla \cdot \underline{B} = 0$$
$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \nabla \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

→ they are LINEAR! v. handy. Superposable solutions

→ notice $\mu_0 \varepsilon_0 = 1/c^2$ where c = speed of light

→ equations here are written in SI units.
other units often used (esp by astronomers) with factors of 4π floating around.

→ symmetries / contrasts:
→ crucial difference between $\underline{E}$ and $\underline{B}$:
charge can be singular whereas no mag monopoles
→ changing $\underline{B}$ can create $\underline{E}$;
changing $\underline{E}$ can create $\underline{B}$
⇒ electromagnetic waves. Wave speed $= c$
(unifies electromagnetism & optics!)

→ equations are relativistically correct
→ start point for special theory of relativity.

→ describe ALL electromagnetic phenomena!

A note on units (see Priest p 436 Appendix 2)

Many different ways of measuring electromag quantities over history
- em quantities measured in "electrostatic units (esu)" or "electromagnetic units (emu)"
- μ, ϵ dimensionless or dimensionful!
- mass, length : centimetres & grams (cgs)
  metres & kilograms (mks)
- "rationalised" → removes geometric 4π factors

2 main sets:

1/ Gaussian cgs : → Length, mass, time fundamental units (c,g,s)
→ q, $\underline{E}$, $\underline{P}$ in esu → "statcoulomb, ..."
→ $\underline{j}$, $\underline{B}$, $\underline{M}$ in emu → "abamp, ..."
→ ϵ, μ dimensionless ↳ $\underline{B}$ in GAUSS
→ eqns have 4π's in them

2/ Rationalised mks — part of SI units ✓
→ Length, mass, time (m k s) and current (amp) fundamental
→ $\mu_0 = 4\pi \times 10^{-7}$ N amp^{-2}
$\epsilon_0 = 8.854 \times 10^{-12}$ $A^2 s^2 N^{-1} m^{-1}$
→ charge in Coulombs (C); force in Newtons (kg m s^{-2})
→ $\underline{B}$ in TESLA ($= 1$ N $A^{-1} m^{-1}$)
→ $\underline{E}$ in Volts / m $= 1$ N C^{-1}
→ eqns have no 4π's !!

(*) We are using rationalised mks.
However, people still often talk about $\underline{B}$ in GAUSS!
(Esp astrophysicists)

Note 1 GAUSS = 10^{-4} TESLA
10^4 GAUSS = 1 TESLA

Max — 8.75 —

UNITS FOR MAGNETIC PROPERTIES

Quantity	Symbol	Gaussian & cgs emu [a]	Conversion factor, C [b]	SI & rationalized mks [c]
Magnetic flux density, magnetic induction	B	gauss (G) [d]	10^{-4}	tesla (T), Wb/m^2
Magnetic flux	Φ	maxwell (Mx), G·cm^2	10^{-8}	weber (Wb), volt second (V·s)
Magnetic potential difference, magnetomotive force	U, F	gilbert (Gb)	$10/4\pi$	ampere (A)
Magnetic field strength, magnetizing force	H	oersted (Oe),[e] Gb/cm	$10^3/4\pi$	A/m [f]
(Volume) magnetization [g]	M	emu/cm^3 [h]	10^3	A/m
(Volume) magnetization	$4\pi M$	G	$10^3/4\pi$	A/m
Magnetic polarization, intensity of magnetization	J, I	emu/cm^3	$4\pi \times 10^{-4}$	T, Wb/m^2 [i]
(Mass) magnetization	σ, M	emu/g	1 $4\pi \times 10^{-7}$	A·m^2/kg Wb·m/kg
Magnetic moment	m	emu, erg/G	10^{-3}	A·m^2, joule per tesla (J/T)
Magnetic dipole moment	j	emu, erg/G	$4\pi \times 10^{-10}$	Wb·m [i]
(Volume) susceptibility	χ, κ	dimensionless, emu/cm^3	4π $(4\pi)^2 \times 10^{-7}$	dimensionless henry per meter (H/m), Wb/(A·m)
(Mass) susceptibility	χ_ρ, κ_ρ	cm^3/g, emu/g	$4\pi \times 10^{-3}$ $(4\pi)^2 \times 10^{-10}$	m^3/kg H·m^2/kg
(Molar) susceptibility	χ_{mol}, κ_{mol}	cm^3/mol, emu/mol	$4\pi \times 10^{-6}$ $(4\pi)^2 \times 10^{-13}$	m^3/mol H·m^2/mol
Permeability	μ	dimensionless	$4\pi \times 10^{-7}$	H/m, Wb/(A·m)
Relative permeability [j]	μ_r	not defined		dimensionless
(Volume) energy density, energy product [k]	W	erg/cm^3	10^{-1}	J/m^3
Demagnetization factor	D, N	dimensionless	$1/4\pi$	dimensionless

a. Gaussian units and cgs emu are the same for magnetic properties. The defining relation is $B = H + 4\pi M$.
b. Multiply a number in Gaussian units by C to convert it to SI (e.g., $1\ \text{G} \times 10^{-4}\ \text{T/G} = 10^{-4}\ \text{T}$).
c. SI (*Système International d'Unités*) has been adopted by the National Bureau of Standards. Where two conversion factors are given, the upper one is recognized under, or consistent with, SI and is based on the definition $B = \mu_0(H + M)$, where $\mu_0 = 4\pi \times 10^{-7}$ H/m. The lower one is not recognized under SI and is based on the definition $B = \mu_0 H + J$, where the symbol I is often used in place of J.
d. 1 gauss $= 10^5$ gamma (γ).
e. Both oersted and gauss are expressed as cm$^{-1/2}$·g$^{1/2}$·s^{-1} in terms of base units.
f. A/m was often expressed as "ampere-turn per meter" when used for magnetic field strength.
g. Magnetic moment per unit volume.
h. The designation "emu" is not a unit.
i. Recognized under SI, even though based on the definition $B = \mu_0 H + J$. See footnote *c*.
j. $\mu_r = \mu/\mu_0 = 1 + \chi$, all in SI. μ_r is equal to Gaussian μ.
k. $B \cdot H$ and $\mu_0 M \cdot H$ have SI units J/m^3; $M \cdot H$ and $B \cdot H/4\pi$ have Gaussian units erg/cm^3.

R. B. Goldfarb and F. R. Fickett, U.S. Department of Commerce, National Bureau of Standards, Boulder, Colorado 80303, March 1985
NBS Special Publication 696 For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, DC 20402

Non-relativistic approximation:

For most applications (not all astrophysics) expect typical velocity $u \ll c \rightarrow u/c \ll 1$
non-relativistic

$\nabla \times E = -\frac{\partial B}{\partial t} \rightarrow \frac{E}{L} \sim \frac{B}{T}$

E - typical E size
B = " B
L = typical length
T = typical time

Now in $\nabla \times B = \mu_0 j + \frac{1}{c^2}\frac{\partial E}{\partial t}$

Sizes: $\frac{B}{L}$ (1) $\qquad \frac{1}{c^2}\frac{E}{T} = \frac{1}{c^2 T}\left(\frac{BL}{T}\right) = \frac{1}{c^2}\frac{BL}{T^2}$ (2)

Ratio $\frac{(2)}{(1)} = \frac{BL/c^2T^2}{B/L} = \frac{(L^2/T^2)}{c^2} = \frac{U^2}{c^2} \ll 1$

Therefore approximate by dropping Maxwell correction to Ampere's law leaving

$$\boxed{\nabla \times B = \mu_0 j}$$

Sometimes THIS is called "the MHD approximation"
(not the way I think of it)

Note → this removes possibility of electromag waves
→ eqns are Galilean invariant but not Lorentz invariant (stronger; c const in all frames)
→ now we have decoupled E except in Faraday Law (and B)
→ solar convection zone $u = 10^3$ m/s
earth's core $u \approx$ m/hr } cf $c = 3\times10^8$ m/s
→ now $\nabla \cdot j = 0 \Rightarrow$ local accumulations of charge are negligible and currents flow in closed circuits

Max — 10

Note: This is already non-relativistic
Relativistic: $\underline{E}' = \gamma(\underline{E} + \underline{u} \times \underline{B})$
$\underline{B}' = \gamma(\underline{B} - \frac{\underline{u} \times \underline{E}}{c^2})$
$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$
$j = \sigma E'$
NonRelativistic: $\gamma \to 1$

Lorentz Force

Force acting on a moving charge in presence of both $\underline{E}$ and $\underline{B}$

$$\underline{F} = q(\underline{E} + \underline{u} \times \underline{B}) \qquad \left(\frac{\underline{F}}{q} = \underline{E}' \text{ NR}\right)$$

electric force — magnetic force

For a continuous charge distribution in motion

$$d\underline{F} = dq(\underline{E} + \underline{u} \times \underline{B})$$

force on small charge segment dq

Divide by volume of charge distribution $dV \to$

$$\underline{f} = \rho_e(\underline{E} + \underline{u} \times \underline{B})$$

force density (i.e. force per unit vol) — most fluid eqns "per unit vol"

charge density

$$= \rho_e \underline{E} + \rho_e \underline{u} \times \underline{B}$$

$$\underline{f} = \rho_e \underline{E} + \underline{j} \times \underline{B} \qquad \underline{j} = \rho_e \underline{u} \text{ current density}$$

Note

→ Total force $\underline{F} = \int_V (\rho \underline{E} + \underline{j} \times \underline{B})\, dV$

→ can be derived from Faraday, assuming moving loop

$$\oint_C d\underline{l} \cdot \frac{\underline{F}}{q} = -\frac{d}{dt} \oint_{S(t)} \underline{B} \cdot d\underline{A}$$

Leibnitz Th. Boundary moving

$$= -\oint_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{A} + \oint_{C(t)} \underline{u} \times \underline{B} \cdot d\underline{l}$$

$$\oint_C \frac{\underline{F}}{q} \cdot d\underline{l} = \oint_C \underline{E} \cdot d\underline{l} + \oint_C \underline{u} \times \underline{B} \cdot d\underline{l}$$

from Maxwell-Faraday

$$\Rightarrow \underline{F}/q = \underline{E} + \underline{u} \times \underline{B} \quad //$$

Ohm's Law

To close the system, need one further eqn to relate the current to the electric and magnetic fields

Ohm's Law is an empirically determined law:

$$\boxed{\underline{j}' = \sigma \underline{E}'}$$

where the dashes mean "measured in frame moving with conductor"

$\underline{j}'$ = current density

$\underline{E}' = (\underline{F}/q)'$

σ = electrical conductivity $\left(\frac{1}{\text{Ohm m}}\right)$ (related to "resistance"; more later)

From Lorentz force $\underline{E}' = \underline{E} + \underline{u} \times \underline{B}$ in a fixed frame

and $\underline{j}' = \underline{j}$ in fixed frame, so

$$\boxed{\underline{j} = \sigma (\underline{E} + \underline{u} \times \underline{B})}$$

Note

→ Probably used to Ohm's Law "$V = I\rho$":

For wire length L, cross-section A

$$A |\underline{j}| = I$$

If emf is V volts $\Rightarrow |\underline{E}| = V/L$ (volts/meter)

So

$$|\underline{j}| = \sigma |\underline{E}| \Rightarrow \frac{I}{A} = \sigma \frac{V}{L} \Rightarrow V = I \underbrace{\left(\frac{L\sigma}{A}\right)}_{R}$$

→ Ohm's Law has different status to Maxwell's eqns

→ NOT exact

→ approximation

→ can be much more complex depending on physics

→ Ohm's Law in this form leads to MHD

→ more complicated forms → PLASMA PHYSICS !! (Priest page 76)

→ MHD works OK for fully-ionised plasmas (more on validity of MHD later)

Induction Equation

① $\nabla \times E = -\frac{\partial B}{\partial t}$

② $\nabla \times B = \mu_0 j$

③ $j = \sigma(E + u \times B)$

①③ $\rightarrow \nabla \times B = \mu_0 \sigma (E + u \times B)$

Take curl: $\nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times B\right) = \nabla \times E + \nabla \times (u \times B)$

Eliminate E with ①

$$\nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times B\right) = -\frac{\partial B}{\partial t} + \nabla \times (u \times B)$$

or $\frac{\partial B}{\partial t} = \nabla \times (u \times B) - \nabla \times (\eta \nabla \times B)$

where $\eta = \frac{1}{\mu_0 \sigma}$ MAGNETIC DIFFUSIVITY $[m^2/s]$

If η = constant $\frac{\partial B}{\partial t} = \nabla \times (u \times B) - \eta \nabla \times (\nabla \times B)$

$\nabla \times \nabla \times B = \nabla(\nabla \cdot B) - \nabla^2 B$ vector identity)

but $\nabla \cdot B = 0$

So

$$\boxed{\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \eta \nabla^2 B}$$

INDUCTION EQN

Approximations to get here are "MHD approximations"

May — 13 —

Note

→ eqn solely for $\underline{B}$!
Given velocity, describes evolution of $\underline{B}$
Eliminated electric field (and current); have not used equation for charge
$\underline{B}$ is primary field
→ can calculate $\underline{j}$ from $\nabla \times \underline{B}$
→ can calculate $\underline{E}$ from Ohm's law $\underline{E} = \underline{j}/\sigma - \underline{u} \times \underline{B}$
→ can calculate charge from $\rho_E = \nabla \times \underline{E}$

→ $\underline{\nabla} \times \underline{B} = 0$ is just a constraint
It is preserved by the induction eqn :
$\nabla \cdot \{\text{induction eqn}\} \rightarrow \frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{B}) = \underline{\nabla} \cdot (\text{curls} \ldots) = \emptyset$

Lorentz Force (again)

If the induction eqn were the only addition to hydrodynamics, then life would be sweet and easy!!
→ The induction equation tells us how the velocity field $\underline{u}$ acts on the magnetic field
→ Conversely, the magnetic field can act back on that velocity field macroscopically through the Lorentz force.
→ It is this "duality" that creates all the v. subtle "fun" in MHD!

$$\underline{F} = \underbrace{\rho_e \underline{E}}_{\text{electrostatic force}} + \underbrace{\underline{j} \times \underline{B}}_{\text{magnetic force}}$$

sometimes just this bit = "Lorentz force" (for reasons below)
("Laplace force" if for current carrying wire)

$$\text{But } \frac{|\rho_e \underline{E}|}{|\underline{j} \times \underline{B}|} = \frac{|(\varepsilon_0 \underline{\nabla} \times \underline{E}) \cdot \underline{E}|}{\left|\frac{\underline{\nabla} \times \underline{B}}{\mu_0} \times \underline{B}\right|} = \frac{\varepsilon_0 \mu_0 \frac{1}{L}\left(\frac{BL}{T}\right)^2}{\frac{1}{L} B^2} = \varepsilon_0 \mu_0 \left(\frac{L^2}{T^2}\right)$$

$$= u^2/c^2 \ll 1 \text{ if non-relativistic}$$

⇒ for non-relativistic, macroscale Lorentz force

$$F \simeq \underline{j} \times \underline{B}$$

or

$$F = \frac{1}{\mu_0} (\underline{\nabla} \times \underline{B}) \times \underline{B}$$

This force slots into the body forces on the RHS of the momentum equation in the Navier-Stokes eqns:

$$\rho \frac{D\underline{u}}{Dt} = \rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}\right) = -\underline{\nabla} p$$

pressure

body forces

$$- \rho \nabla \Phi_g + \underline{j} \times \underline{B}$$

diffusion

$$+ \mu\left[\nabla^2 \underline{u} + \frac{1}{3}\nabla(\nabla\cdot\underline{u})\right]$$

Summary of MHD equations

Electrically-conducting fluid moving at velocity $\underline{u}(\underline{x},t)$ in a magnetic field $\underline{B}(\underline{x},t)$ in the limit $|\underline{u}| \ll c$

Continuity eqn (cons of mass) $$\frac{D\rho}{Dt} + \rho(\underline{\nabla}\cdot\underline{u}) = 0$$

$$\left(\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u}) = 0 \right)$$

Conservation of momentum $$\rho \frac{D\underline{u}}{Dt} = -\nabla p - \rho \nabla \Phi_g + \frac{1}{\mu_0}(\underline{\nabla}\times\underline{B})\times\underline{B}$$

$$+ \mu\left[\nabla^2 \underline{u} + \frac{1}{3}\nabla(\nabla\cdot\underline{u}) \right]$$

Conservation of energy (heat eqn) $$\rho T \frac{DS}{Dt} = -\mathcal{L}$$

(S = entropy per unit mass ; $\mathcal{L}$ = energy loss function) — sources and sinks

$$\left(\rho\left[\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} \right] = -\mathcal{L} \qquad e = \text{internal energy} \right)$$

Constitutive relations

eg ideal gas law : $p = R\rho T$ — R gas constant

isothermal : $T = T_0$

adiabatic : $p = \rho^{\gamma}$ $\qquad \gamma = c_p/c_v$

Induction equation $$\frac{\partial \underline{B}}{\partial t} = \nabla\times(\underline{u}\times\underline{B}) + \eta \nabla^2 \underline{B}$$

Magnetic constraint $$\underline{\nabla}\cdot\underline{B} = 0$$

Max — 15.5 —

Validity of eqns / assumptions made:

1. Plasma treated as continuum — lengthscale of variations much larger than internal plasma lengths e.g. ion gyroradius

2. Plasma in thermodynamic equilibrium with distribution functions close to Maxwellian $\Rightarrow$ timescales >> collision times
   lengthscales >> mean free paths

3. Constants η, μ, k are uniform; isotropic

4. Eqns are written in an inertial frame

5. Relativistic effects are ignored:
   flow speed (& sound speed & Alfvén speed) $\ll c$

6. Simple form of Ohm's Law is OK: $\underline{j} = \sigma(\underline{E} + \underline{u} \times \underline{B})$

7. Plasma is treated as a single fluid:
   astrophysics → high temp → ionised: electrons are free
   Plasma (ionised gas) = electrons (−), ions (+), neutrals
   obvs huge # of electrons/ions ⇒ statistical approach
   → PDE of distrib fns for density of particles → VLASOV EQN
   Integrate ("take moments of") Vlasov over macroscale ⇒ "velocity"
   → eqns for each species → multifluid
   → velocity is "centre of mass"
   For fully-ionised plasmas, this level of complexity can be ignored

Incompressible equations

1, $\nabla \cdot \underline{u} = 0$

2, $\rho \frac{D\underline{u}}{Dt} = -\nabla p - \cancel{\nabla \Phi_g} + \underline{j} \times \underline{B} + \mu \nabla^2 \underline{u}$

3, $\frac{\partial \underline{B}}{\partial t} = (\underline{B} \cdot \nabla)\underline{u} + \cancel{(\nabla \cdot \underline{B})}\underline{u} - (\underline{u} \cdot \nabla)\underline{B} - \cancel{(\nabla \cdot \underline{u})}\underline{B} + \eta \nabla^2 \underline{B}$

$\longrightarrow \quad \frac{\partial \underline{B}}{\partial t} + (\underline{u} \cdot \nabla)\underline{B} = (\underline{B} \cdot \nabla)\underline{u} + \eta \nabla^2 \underline{B}$

$\longrightarrow \quad \frac{D\underline{B}}{Dt} = (\underline{B} \cdot \nabla)\underline{u} + \eta \nabla^2 \underline{B}$

Solenoidal field

$\nabla \cdot \underline{B} = 0$ Consequences:

→ $\underline{\nabla} \cdot \underline{B} = 0$ is the constraint of no magnetic monopoles

→ it's a constraint not really an equation
$\underline{\nabla} \cdot \underline{B} = 0$ initially ⇒ $\underline{\nabla} \cdot \underline{B} = 0$ for all time
i.e.
$$\frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{B}) = 0$$
(Take divergence of induction
$$\underline{\nabla} \cdot \left\{ \frac{\partial \underline{B}}{\partial t} = \nabla \times u \times B + \nabla \times (\eta \nabla \times B) \right\}$$
$$\Rightarrow \frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{B}) = \nabla \cdot \{\text{curls}\} = \emptyset$$
)

→ However, DO have to satisfy this constraint!
Not easy to do as above in numerical schemes WARNING!

→ Perhaps better way:
Since $\underline{\nabla} \cdot \underline{B} = 0$ can ALWAYS express $\underline{B}$ as a curl
$$\underline{B} = \underline{\nabla} \times \underline{A} \qquad \text{since } \nabla \cdot (\nabla \times \underline{X}) = \emptyset$$
$\underline{A}$ = VECTOR POTENTIAL
Only defined up to addition of a gradient of a scalar
since $\underline{A}$ and $\underline{A} + \nabla\phi$ give same $\underline{B}$
(since $\nabla \times (\underline{A}) = \nabla \times (\underline{A} + \nabla\phi)$ since $\nabla \times \nabla\phi = 0$)

One common way of doing this is a
POLOIDAL - TOROIDAL DECOMPOSITION!

$$\underline{B} = \underline{B}_P + \underline{B}_T = \nabla \times \nabla \times (P\underline{r}) + \nabla \times (T\underline{r})$$
$$P = P(\underline{x}, t) \qquad T = T(\underline{x}, t)$$
$\underline{r}$ is any vector (e.g. $\hat{z}$ in Cartesian; $\hat{r}$ in spherical)

Insert in equations and solve for P, T instead of $\underline{B}$ ⇒ guarantees $\nabla \cdot \underline{B} = 0$

Notice this reduces calculation of $\underline{B} = (B_x, B_y, B_z)$ 3 components, to calculation of P, T 2 components. This is because not all 3 components of $\underline{B}$ are independent → they are connected by $\nabla \cdot \underline{B} = 0$!

Exercise: Prove in Cartesian!

Notice symmetry! curl of poloidal vector is toroidal
curl of toroidal vector is poloidal

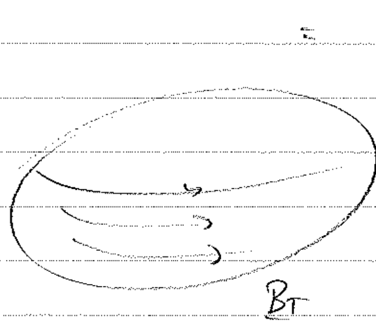
Cartesian: $\underline{B} = \underbrace{\nabla \times T\hat{z}}_{B_T} + \underbrace{\nabla \times \nabla \times P\hat{z}}_{B_p} + \overline{B}_x(z)\,\hat{x} + \overline{B}_y(z)\,\hat{y}$

must account for mean fields separately in Cartesian!!

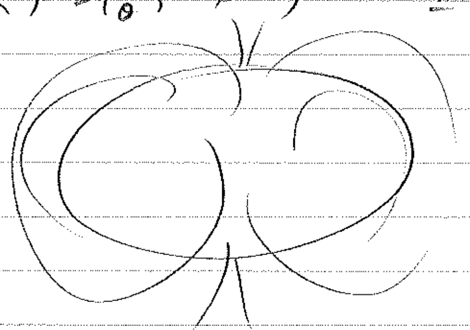
Spherical: $\underline{B} = \nabla \times T\,\hat{r} + \nabla \times \nabla \times P\,\hat{r} = \underline{B}_T + \underline{B}_p$

Note: ⇒ toroidal vectors tangent to spheres $\underline{r} \cdot \underline{B}_T = 0$
⇒ curl of pol vectors " $\underline{r} \cdot \nabla \times \underline{B}_p = 0$

Axisymmetric fields (perhaps most clear, useful usage)
Indep of azimuthal angle ϕ
$P = P(r, \theta) \qquad T = T(r, \theta)$
Sph polars ⇒
$\underline{B}_T = (0, 0, B_{T\phi})$ azimuthal field only
$B_\phi = -\partial T/\partial\theta$
$\underline{B}_p = \nabla \times (0, 0, A_\phi)$
curl of toroidal $\quad A_\phi = -\partial P/\partial\theta$
$= (B_{pr}, B_{p\theta}, \emptyset)$ no axisymm component



B_T



B_p

2D Cartesian

$$\underline{B} = \left(B_x(x,y,t),\ B_y(x,y,t)\ ,\ \emptyset\right)$$

$\Rightarrow$

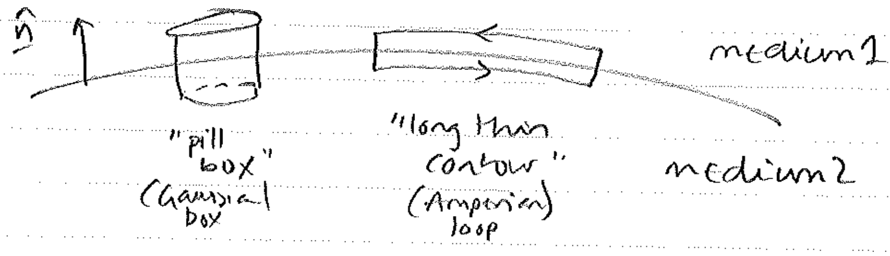
$$\underline{B} = \underline{\nabla} \times A(x,y,t)\,\underline{k}$$

Compare to fluid streamfunction.

BOUNDARY CONDITIONS

→ need boundary conditions for magnetic stuff just as you do for velocity, temperature etc

Consider a boundary, or interface, between two media, with normal $\hat{\underline{n}}$

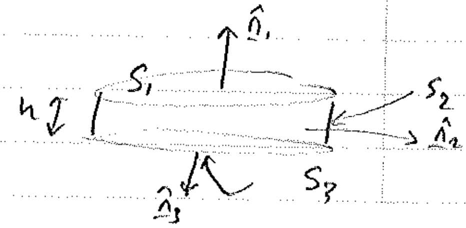


1) $\underline{\nabla} \cdot \underline{B} = 0$ or in integral form

$$\int_V \underline{\nabla} \cdot \underline{B} \, dV = 0$$

$$\Rightarrow \int_S \underline{B} \cdot d\underline{S} = 0 \quad \text{for surface } S \text{ of vol } V$$

Apply to Gaussian pill box:



$$\int_S \underline{B} \cdot d\underline{S} = \int_{S_1} \underline{B}_1 \cdot \hat{\underline{n}}_1 \, ds$$

$$+ \int_{S_2} \underline{B}_2 \cdot \hat{\underline{n}}_2 \, ds + \int_{S_3} \underline{B}_3 \cdot \hat{\underline{n}}_3 \, ds = 0$$

As $h \to 0$, contribution from $S_2 \to 0$
and

$$\int_{S_1} \underline{B} \cdot \hat{\underline{n}}_1 \, ds + \int_{S_3} \underline{B} \cdot \hat{\underline{n}}_3 \, ds = 0$$

But $\hat{\underline{n}}_3 = -\hat{\underline{n}}_1$

$$\Rightarrow \int_{S_1} \underline{B}_1 \cdot \hat{\underline{n}}_1 \, ds - \int_{S_3} \underline{B}_3 \cdot \hat{\underline{n}}_1 \, ds = 0$$

As $ds \to 0$ and since S_1 and S_3 are same size

$$\int_{S_1} \left(\underline{B}_1 \cdot \hat{\underline{n}}_1 - \underline{B}_3 \cdot \overset{n\_1}{\hat{\underline{n}}_3} \right) dS = 0$$

$$\Rightarrow \quad \underline{B}_1 \cdot \hat{\underline{n}}_1 = \underline{B}_3 \cdot \overset{n\_1}{\hat{\underline{n}}_3}$$

i.e. the normal component of the magnetic field is <u>continuous</u>

This is often written as a <u>JUMP CONDITION</u> as

$$\left[\underline{B} \cdot \hat{\underline{n}} \right] = 0$$

$0 \Rightarrow$ continuous – no jump

implies jump

2) $\underline{j} = \mu_0 (\nabla \times \underline{B})$

$\Rightarrow \quad \nabla \cdot \underline{j} = 0$

So we can do all the same again for $\underline{j}$ and get

$$\left[\underline{j} \cdot \hat{\underline{n}} \right] = 0$$

normal component of $\underline{j}$ is continuous

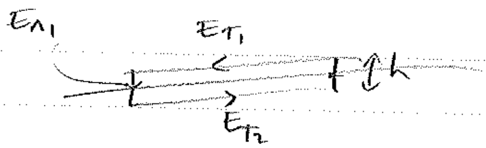
3) What about tangential components?

Faraday's Law

$$-\frac{d}{dt} \int_S \underline{B} \cdot d\underline{s} = \oint_C \underline{E} \cdot d\underline{l}$$

Consider the Amperian "long thin" contour.
As the length of the short sides $\to 0$, then $dS \to 0$
and the integral on the left hand side $\to 0$
i.e. the flux through the shrinking surface $\to 0$
(assuming time deriv remains finite)

$$\Rightarrow \quad \oint_C \underline{E} \cdot d\underline{l} = 0$$



$$\oint_c E \cdot dl = -\int E_{T_1} \cdot dl_1 - \int \cancel{E_{n_1} \cdot dh}$$
$$+ \int E_{T_2} \cdot dl_2 + \int \cancel{E_{n_2}} \cdot dh$$

$\to 0$ as $h \to 0$

$\Rightarrow \quad E_{T_1} - E_{T_2} = \emptyset$ Tangential components of E are continuous.

Vectorially, as a jump

$$[E \times \hat{n}] = \emptyset$$

Using Ohm's law, $j = \sigma(E + u \times B)$

$\Rightarrow$

$$\left[\left(\frac{j}{\sigma} - (u \times B)\right) \times \hat{n}\right] = \emptyset$$

i.e. no NORMAL velocity

Notice for fixed position (but possibly sliding moving) rigid wall $\to u$

$\uparrow w = 0 \to$ fluid $u = U$

then $[(u \times B) \times \hat{n}] = \emptyset$ (identically, not a b.c.)

$$\Rightarrow \quad [j/\sigma \times \hat{n}] = \emptyset$$

i.e. if there is a jump in σ (conductivity) between media

$$\boxed{j_{T_1}/\sigma_1 = j_{T_2}/\sigma_2}$$

$\Rightarrow$ jump in tangential current by relative size of conductivity σ

4/. Ampere's Law: Apply to "long thin contour":

$$\int_S \nabla \times \frac{\underline{B}}{\mu} \cdot d\underline{s} = \int_S \underline{j} \cdot d\underline{S} = I$$

↑ enclosed current

$$\rightarrow \oint_{C \text{ "long thin"}} \frac{\underline{B}}{\mu} \cdot d\underline{l} = I$$

$\Rightarrow$ as before $\left[\dfrac{\underline{B}}{\mu} \times \hat{\underline{n}}\right] = \underline{j}_s$ surface current density (I/dL)

If no surface current,

$$\left[\frac{\underline{B}}{\mu} \times \hat{\underline{n}}\right] = \emptyset$$

tangential $\frac{B}{\mu}$ is continuous

If μ changes between media

$$\frac{B_{T_1}}{\mu_1} = \frac{B_{T_2}}{\mu_2}$$

jump in tangential field by relative sizes of μ permeability.

Notice a lot of the times, we assume that the permeability and conductivity are essentially fixed (constant) e.g. $\mu \sim \mu_0$ always