Visualization of the singular value decomposition:

Consider a sqauare shape with directed axis as shown below

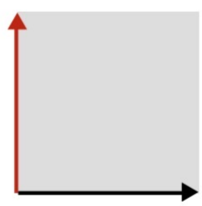

Figure 1: A square with orienting arrows.

A linear transformation, M on this square can either pull, push, shear, rotate or flip the square.

See below:

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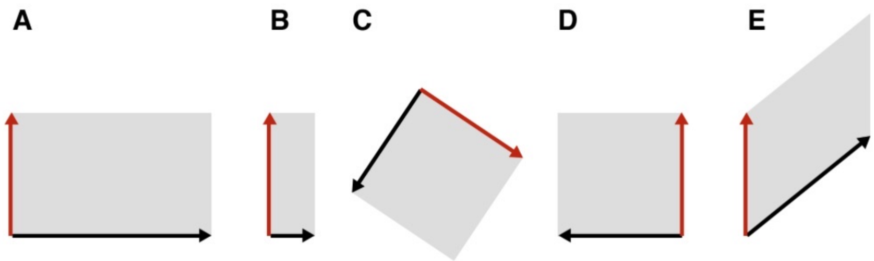

Figure 2: Our original square under different types of transformations: (A) stretched, (B) compressed, (C) rotated, (D) reflected or flipped, and (E) sheared.

The only constraint is that the transformation is linear.

So, a linear transformation can:

* shear the square

* stretch, compress, flip without shearing

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In the figure below, A shears the square, but B, does not

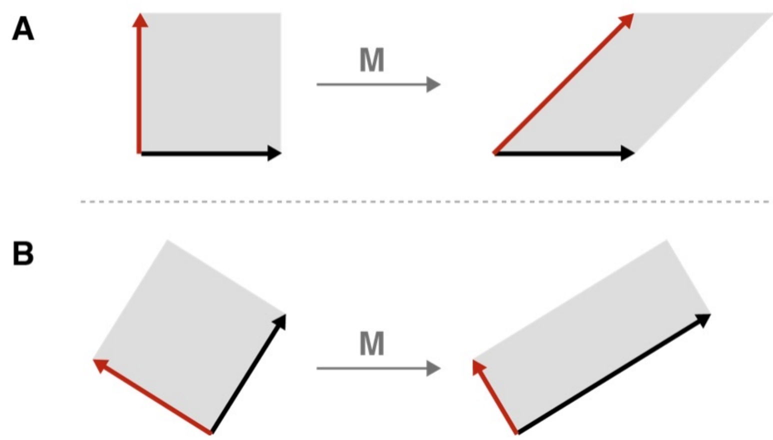

Figure 4: The geometric essence of the SVD: any linear transformation $\mathbf{M}$ of our square (A) can be thought of as simply stretching, compressing, or reflecting that square, provided we rotate the square before and after (B).

If we are allowed to rotate the square first, then any linear transformation will result in a rectangle without shear

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This is the essence of an SVD.
Any linear transformation can be thought of as stretching, compression, or flipping a square as long as we are allowed to rotate it first.

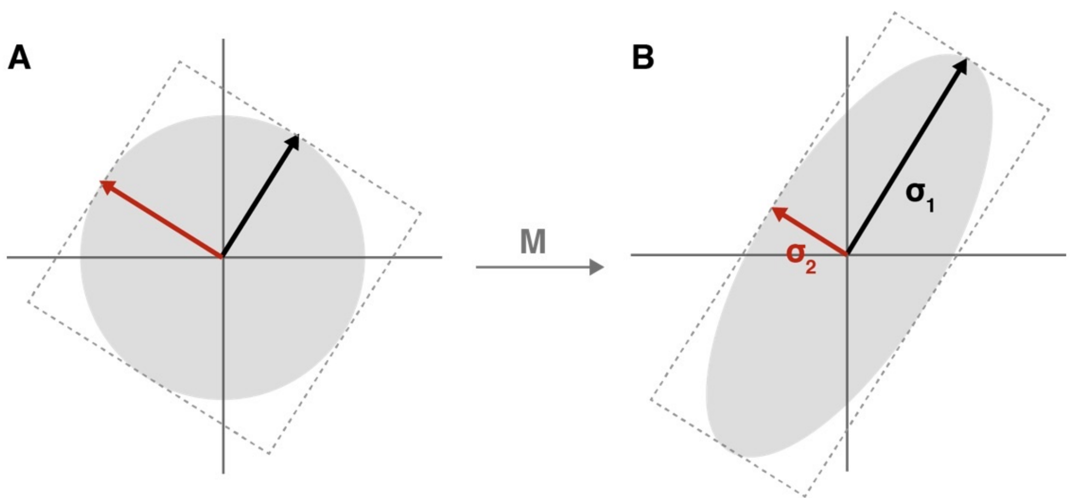

Figure 5: (A) An oriented circle; if it helps, imagine that circle inscribed in our original square. (B) Our circle transformed into an ellipse. The length of the major and minor axes of the ellipse have values σ_1 and σ_2 respectively, called the *singular values*.

So, the singular value σ_1 & σ_2 of a 2×2 matrix, M, is the major and minor axis of the ellipse sketched by all the vectors inside the circle that undergoes linear transformation with M.

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Consider a pair of orthogonal unit vectors, in 2D, v_1 & v_2.

Applying transformation, M, to v_1 & v_2, we get:

Mv_1 & Mv_2

We write:

$Mv_1 = \sigma_1 u_1$ unit vector $\quad Mv_2 = \sigma_2 u_2$ unit vector

Consider any vector x in 2D:

$$x = (x \cdot v_1)\, v_1 + (x \cdot v_2)\, v_2$$

$$Mx = (x \cdot v_1)\, M v_1 + (x \cdot v_2)\, M v_2$$

$$Mx = (x \cdot v_1)\, \sigma_1 u_1 + (x \cdot v_2)\, \sigma_2 u_2$$

Change order

$$Mx = (u_1 \sigma_1)(v_1 \cdot x) + (u_2 \sigma_2)(v_2 \cdot x)$$

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From here, we can write:

$$M = u_1 \sigma_1 v_1^T + u_2 \sigma_2 v_2^T$$

What does the SVD do:

For any vector x, the transformation of x with M is: $\boxed{Mx}$

$$M = U \Sigma V^T$$

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So : $Mx = \boxed{U}\,\boxed{\Sigma}\,\boxed{V^T x}$

Stretches the the components of x with σ_i

rotation (and reflection if $\det(V^T) = -1$)

(another rotation and reflection)