AM 275 - Magnetohydrodynamics: Homework 4

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Problem 1:

a.) We begin here with the linearized perturbation equations taken from the lecture, but now including the Coriolis force. We will ignore the centripedal force here as claim that it acts only on the base flow and not the perturbations. make a better excuse.

$$\rho_0 \left(\frac{\partial \boldsymbol{u}'}{\partial t}\right) + 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u}' = -\nabla\left(p' + \frac{1}{\mu_0}\boldsymbol{B}'\cdot\overline{\boldsymbol{B}}\right) + \frac{1}{\mu_0}(\overline{\boldsymbol{B}}\cdot\nabla)\boldsymbol{B}', \quad \nabla\cdot\boldsymbol{u}' = 0$$
$$\frac{\partial \boldsymbol{B}'}{\partial t} = (\overline{\boldsymbol{B}}\cdot\nabla)\boldsymbol{u}', \quad \nabla\cdot\boldsymbol{B}' = 0$$

From these equations, we proceed with the usual wave-like ansatz whereby each quantity (vector and scalar) is of the form,

$$q' = \tilde{q}e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$$

where $\tilde{q}$ is generally a complex scalar (or vector) and the exponential term follows Euler's identity. We give specific attention to the complex components of any general q'. We have very specifically, that conditions on that quantity (for example, the divergence of $\boldsymbol{u}'$ and boundary conditions on the relevant characteristics of the flow) are only considered in real space. That is, we would care very rigorously about the $\mathrm{Re}\left[\nabla\cdot\boldsymbol{u}\right] = 0$ and not necessarily about the imaginary component (Note that in this specific case there is an extra i floating around and can be factored out so it happens that both the real and imaginary components of the divergence of $\boldsymbol{u}'$ must be zero). We have as a consequence that this simplifies these equations to an algebraic dilema rather than that of a partial differential equation. These equations then have the form,

$$-i\rho_0\omega\tilde{\boldsymbol{u}} + 2\rho_0\boldsymbol{\Omega}\times\tilde{\boldsymbol{u}} = -i\boldsymbol{k}\left(\tilde{p} + \frac{1}{\mu_0}\overline{\boldsymbol{B}}\cdot\tilde{\boldsymbol{B}}\right) + \frac{i}{\mu_0}\left(\overline{\boldsymbol{B}}\cdot\boldsymbol{k}\right)\tilde{\boldsymbol{B}}, \quad i\boldsymbol{k}\cdot\tilde{\boldsymbol{u}} = 0$$
$$-i\omega\tilde{\boldsymbol{B}} = i(\overline{\boldsymbol{B}}\cdot\boldsymbol{k})\tilde{\boldsymbol{u}}, \quad i\boldsymbol{k}\cdot\tilde{\boldsymbol{B}}$$

In order to proceed, we take a pseudo-curl in this vector space. Note that since we have taken $\nabla = i\boldsymbol{k}$, and thus we take $i\boldsymbol{k}\times(\cdot)$ for the momentum and induction equation.

$$\rho_0\omega(\boldsymbol{k}\times\tilde{\boldsymbol{u}}) + 2i\rho_0(\boldsymbol{k}\times(\boldsymbol{\Omega}\times\tilde{\boldsymbol{u}})) = -\frac{1}{\mu_0}\left(\overline{\boldsymbol{B}}\cdot\boldsymbol{k}\right)(\boldsymbol{k}\times\tilde{\boldsymbol{B}})$$
$$-i\omega(\boldsymbol{k}\times\tilde{\boldsymbol{B}}) = i(\overline{\boldsymbol{B}}\cdot\boldsymbol{k})(\boldsymbol{k}\times\tilde{\boldsymbol{u}})$$

Here the "curl" of the Coriolis term is given specific attention (whereby the NRL plasma formulary 2019 is referenced) and we have,

$$\rho_0\omega(\boldsymbol{k}\times\tilde{\boldsymbol{u}}) - 2i\rho_0(\boldsymbol{k}\cdot\boldsymbol{\Omega})\tilde{\boldsymbol{u}} = -\frac{1}{\mu_0}\left(\overline{\boldsymbol{B}}\cdot\boldsymbol{k}\right)(\boldsymbol{k}\times\tilde{\boldsymbol{B}})$$
$$\omega(\boldsymbol{k}\times\tilde{\boldsymbol{B}}) = -(\overline{\boldsymbol{B}}\cdot\boldsymbol{k})(\boldsymbol{k}\times\tilde{\boldsymbol{u}})$$

Now, upon first inspection, we see something quite odd. We have (and this must be satisfied) that $\boldsymbol{k} \times \tilde{\boldsymbol{u}}$ must be colinear to $\tilde{\boldsymbol{u}}$. This first appears non-sensicle as multivariate calculus has taught us that the cross product of two vectors must be perpendicular to both those vectors. Here, we point to the consideration of these complex quantities representing real (physical) quantities at the end of the day. We can imagine different criterium for the orthogonality of these complex vectors, that is we must chose a specific inner product to satisfy orthogonality. Since we care about the real component of the inner product we must have that,

$$\begin{aligned}
\mathrm{Re}\left[\langle \tilde{\boldsymbol{u}}, \boldsymbol{k} \times \tilde{\boldsymbol{u}} \rangle\right] &= 0 \\
\mathrm{Re}\left[\langle \tilde{\boldsymbol{u}}, \boldsymbol{k} \times \tilde{\boldsymbol{u}} \rangle\right] &= \mathrm{Re}\left[(\tilde{\boldsymbol{u}}_R^T - i\tilde{\boldsymbol{u}}_I^T)((\boldsymbol{k} \times \tilde{\boldsymbol{u}}_R) + i(\boldsymbol{k} \times \tilde{\boldsymbol{u}}_I))\right] \\
&= \mathrm{Re}\left[i\tilde{\boldsymbol{u}}_R \cdot (\boldsymbol{k} \times \tilde{\boldsymbol{u}}_I) - i\tilde{\boldsymbol{u}}_I \cdot (\boldsymbol{k} \times \tilde{\boldsymbol{u}}_R)\right] \\
&= 0
\end{aligned}$$

and yet the magnitude/modulus/absolute-value of this cross product is non-zero!! It is for this precise reason that we have that $\tilde{\boldsymbol{u}}$ can be colinear to $\boldsymbol{k} \times \tilde{\boldsymbol{u}}$ (note the i attached to $\tilde{\boldsymbol{u}}$ in the momentum / vorticity equation). Thus we proceed with the notion that we have some wiggle room in an initially seemingly obsurb statement. We continue with the substitution made in class.

$$\begin{aligned}
\rho_0\omega(\boldsymbol{k} \times \tilde{\boldsymbol{u}}) - 2i\rho_0(\boldsymbol{k} \cdot \boldsymbol{\Omega})\tilde{\boldsymbol{u}} &= \frac{1}{\omega\mu_0}\left(\overline{\boldsymbol{B}} \cdot \boldsymbol{k}\right)^2 (\boldsymbol{k} \times \tilde{\boldsymbol{u}}) \\
\rho_0\omega^2(\boldsymbol{k} \times \tilde{\boldsymbol{u}}) - 2i\omega\rho_0(\boldsymbol{k} \cdot \boldsymbol{\Omega})\tilde{\boldsymbol{u}} - \frac{1}{\mu_0}\left(\overline{\boldsymbol{B}} \cdot \boldsymbol{k}\right)^2 (\boldsymbol{k} \times \tilde{\boldsymbol{u}}) &= 0
\end{aligned}$$

Finally we take the inner product of this entire equation with $(\boldsymbol{k} \times \tilde{\boldsymbol{u}})/|\boldsymbol{k} \times \tilde{\boldsymbol{u}}|^2$. We have then,

$$\omega^2 - 2\omega(\boldsymbol{k} \cdot \boldsymbol{\Omega})\frac{\langle i\tilde{\boldsymbol{u}}, \boldsymbol{k} \times \tilde{\boldsymbol{u}} \rangle}{|\boldsymbol{k} \times \tilde{\boldsymbol{u}}|^2} - \frac{1}{\rho_0\mu_0}\left(\overline{\boldsymbol{B}} \cdot \boldsymbol{k}\right)^2 = 0$$

In order to resolve this equation, we simply need to resolve the two inner products. It simplifies in the following manner:

$$\begin{aligned}
\langle i\tilde{\boldsymbol{u}}, \boldsymbol{k} \times \tilde{\boldsymbol{u}} \rangle &= (-\tilde{\boldsymbol{u}}_I - i\tilde{\boldsymbol{u}}_R) \cdot (\boldsymbol{k} \times \tilde{\boldsymbol{u}}_R + i\boldsymbol{k} \times \tilde{\boldsymbol{u}}_I) \\
&= -\tilde{\boldsymbol{u}}_I \cdot \boldsymbol{k} \times \tilde{\boldsymbol{u}}_R + \tilde{\boldsymbol{u}}_R \cdot \boldsymbol{k} \times \tilde{\boldsymbol{u}}_I \\
&= -\boldsymbol{k} \cdot (\tilde{\boldsymbol{u}}_R \times \tilde{\boldsymbol{u}}_I) + \boldsymbol{k} \cdot (\tilde{\boldsymbol{u}}_I \times \tilde{\boldsymbol{u}}_R) \\
&= 2\boldsymbol{k} \cdot (\tilde{\boldsymbol{u}}_I \times \tilde{\boldsymbol{u}}_R)
\end{aligned}$$

Where the second to last line is taken from line (1) of the NRL plasma formulary ($A \cdot B \times C = B \cdot C \times A$), and the last line can be shown algebraicly by the anti-symmetry of the cross product. We can simplify this further using some geometric arguments. Namely, we have that both $\tilde{\boldsymbol{u}}_I$ and $\tilde{\boldsymbol{u}}_R$ are perfectly orthogonal to $\boldsymbol{k}$ and this is taken from the divergence free condition on the flow. This means that $\boldsymbol{k}$ must be parallel to $\tilde{\boldsymbol{u}}_I \times \tilde{\boldsymbol{u}}_R$ (this is further aided by the fact that $\tilde{\boldsymbol{u}}_I$ is perpendicular to $\tilde{\boldsymbol{u}}_R$, which can be shown from the navier stokes equation). We can therefore use the norms of these vectors to determine the numerator. We have,

$$\begin{aligned}
2\boldsymbol{k} \cdot (\tilde{\boldsymbol{u}}_I \times \tilde{\boldsymbol{u}}_R) &= \pm 2|\boldsymbol{k}||\tilde{\boldsymbol{u}}_I \times \tilde{\boldsymbol{u}}_R| \\
&= \pm 2|\boldsymbol{k}||\tilde{\boldsymbol{u}}_I||\tilde{\boldsymbol{u}}_R|\sin(\theta) \\
&= \pm 2|\boldsymbol{k}||\tilde{\boldsymbol{u}}_I||\tilde{\boldsymbol{u}}_R|
\end{aligned}$$

where here the sine of theta must be 1 because (as I will show later) $\tilde{\boldsymbol{u}}_I$ and $\tilde{\boldsymbol{u}}_R$ are orthogonal ($\theta = 1$) and therefore their cross product's norm is their two norms multiplied together.

Next, we simplify the denominator

And thus we have according to these equations,

$$\omega^2 \pm 2\frac{\omega}{|\boldsymbol{k}|}(\boldsymbol{k} \cdot \boldsymbol{\Omega}) - \frac{1}{\rho_0\mu_0}\left(\overline{\boldsymbol{B}} \cdot \boldsymbol{k}\right)^2 = 0$$

b.) Next we are to determine the roots of this quadratic, i.e. find the values of ω in a specific limit. We have of course that

$$\omega = \frac{-b \pm \sqrt{b^2 - 4c}}{2}, \quad b = \pm 2\frac{(\boldsymbol{k} \cdot \boldsymbol{\Omega})}{|\boldsymbol{k}|}, \quad c = -\frac{1}{\rho_0 \mu_0}\left(\overline{\boldsymbol{B}} \cdot \boldsymbol{k}\right)^2$$

Specifically if we consider $\sqrt{c} \ll b$ then we have that,

$$\omega = \frac{-b \pm |b|\sqrt{1+\epsilon}}{2} \approx \frac{1}{2}\left(-b \pm |b|\left(1 + \frac{\epsilon}{2}\right) + O(\epsilon^2)\right) = \begin{cases} -b, & + \\ \frac{|b|\epsilon}{4}, & - \end{cases}$$

where we have specifically that $0 < \epsilon \ll 1$ and it is given by the following,

$$\epsilon = -\frac{4c}{b^2} = \frac{4\left(\overline{\boldsymbol{B}} \cdot \boldsymbol{k}\right)^2/(\mu_0\rho_0)}{4(\boldsymbol{k} \cdot \boldsymbol{\Omega})^2/|\boldsymbol{k}|^2}$$

$$\omega \approx \begin{cases} \mp 2(\boldsymbol{k} \cdot \boldsymbol{\Omega})/|\boldsymbol{k}|, & + \\ \frac{1}{2}\frac{\left(\overline{\boldsymbol{B}}\cdot\boldsymbol{k}\right)^2/(\mu_0\rho_0)}{(\boldsymbol{k}\cdot\boldsymbol{\Omega})/|\boldsymbol{k}|}, & - \end{cases}$$