AM 275 - Magnetohydrodynamics:

Dante Buhl

February 7, 2025

Problem 1: Show that magnetic helicity if conserved

Proof. Show that the following quantity is conserved with time.

$$h_m = \int_V \boldsymbol{A} \cdot \boldsymbol{B} dV, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\frac{\partial h_m}{\partial t} = \frac{\partial}{\partial t}\left(\int_V \boldsymbol{A} \cdot \boldsymbol{B} dV\right)$$

$$\frac{\partial h_m}{\partial t} = \int_V \frac{\partial}{\partial t}\left(\boldsymbol{A} \cdot \boldsymbol{B}\right) dV + \int_{\partial V} (\boldsymbol{A} \cdot \boldsymbol{B})(\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) dS$$

where V is a closed volume in the domain of the fluid and ∂V is the boundary surface of that closed volume. According to the Reynolds transport theorem we have the following time derivative of the magnetic helicity. We can follow up by taking the time derivatives of $\boldsymbol{A} \cdot \boldsymbol{B}$.

$$\begin{aligned}
\frac{\partial h_m}{\partial t} &= \int_V \frac{\partial}{\partial t}\left(\boldsymbol{A} \cdot \boldsymbol{B}\right) dV + \int_{\partial V} (\boldsymbol{A} \cdot \boldsymbol{B})(\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) dS \\
&= \int_V \frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{B}}{\partial t} dV + \int_{\partial V} (\boldsymbol{A} \cdot \boldsymbol{B})(\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) dS \\
&= \int_V \boldsymbol{B} \cdot \nabla \phi + \boldsymbol{A} \cdot (-(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \boldsymbol{B}(\nabla \cdot \boldsymbol{u})) + \int_{\partial V} (\boldsymbol{A} \cdot \boldsymbol{B})(\boldsymbol{u} \cdot \hat{\boldsymbol{n}}) dS \\
&= \int_V \boldsymbol{B} \cdot \nabla \phi + \boldsymbol{A} \cdot (-(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \boldsymbol{B}(\nabla \cdot \boldsymbol{u})) + (\nabla \cdot (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{u}) dV \\
&= \int_V \boldsymbol{B} \cdot \nabla \phi - \boldsymbol{A} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{A} \cdot \boldsymbol{B})(\nabla \cdot \boldsymbol{u}) + (\nabla \cdot (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{u}) dV \\
&= \int_V \boldsymbol{B} \cdot \nabla \phi - \boldsymbol{A} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - \boldsymbol{u} \cdot \nabla(\boldsymbol{A} \cdot \boldsymbol{B}) dV \\
&= \int_V \boldsymbol{B} \cdot \nabla \phi - \boldsymbol{B} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} dV
\end{aligned}$$

where we use $\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \times \boldsymbol{B} + \nabla \phi$ from the lecture note and the Davidson text.

□

Problem 2: Solve using Cauchy solutions