AM 275 - Magnetohydrodynamics:

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February 14, 2025

Problem 1: Show that magnetic helicity is conserved

Proof. In order to show that the magnetic helicity is preserved over a material volume with a bounding surface with $\boldsymbol{B} \cdot \hat{\boldsymbol{n}} = 0$ everywhere on the surface. We assume that the field lines (as well as the material volume) are frozen into the flow. This proof will follow the proof in the Davidson text. Let us consider the time evolution of the magnetic helicity,

$$\begin{aligned}
\frac{\partial h_m}{\partial t} &= \frac{\partial}{\partial t}\int_V \boldsymbol{A}\cdot\boldsymbol{B}dV \\
&= \int_V \frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) + \nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u}dV \\
&= \int_V \boldsymbol{A}\cdot\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{B}\cdot\frac{\partial A}{\partial t} + \nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u}dV \\
&= \int_V \boldsymbol{A}\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \boldsymbol{B}\cdot\nabla\phi + \nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u}dV
\end{aligned}$$

Here the Davidson text very quickly rewrites what remains from the induction equation. I will show this derivation in a little more detail.

$$\begin{aligned}
\boldsymbol{A}\cdot(\nabla\times(\boldsymbol{u}\times\boldsymbol{B})) = \boldsymbol{A}\cdot(\nabla\times\boldsymbol{F}) &= \boldsymbol{F}\cdot\nabla\times\boldsymbol{A} - \nabla\cdot(\boldsymbol{A}\times\boldsymbol{F}) \\
&= (\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B} - \nabla\cdot(\boldsymbol{A}\times\boldsymbol{u}\times\boldsymbol{B}) \\
&= -\nabla\cdot(\boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B})) \\
&= -\nabla\cdot((\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u} - (\boldsymbol{A}\cdot\boldsymbol{u})\boldsymbol{B}) \\
&= -\nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u} + \nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{u})\boldsymbol{B}
\end{aligned}$$

In addition to the following identity, $\boldsymbol{B}\cdot\nabla\phi = \nabla\cdot(\phi\boldsymbol{B}) - \phi(\nabla\cdot\boldsymbol{B})$, we find that this now reduces our original integral to the following ,

$$\begin{aligned}
\frac{\partial h_m}{\partial t} &= \int_V (-\nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u} + \nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{u})\boldsymbol{B}) + \nabla\cdot(\phi\boldsymbol{B}) + \nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u}dV \\
&= \int_V \nabla\cdot(\phi + \boldsymbol{A}\cdot\boldsymbol{u})\boldsymbol{B}dV
\end{aligned}$$

The final step is to use the divergence theorem to write this as a surface integral. We then use fact that we have chosen a frozen into the flow field preserves the fact that $\boldsymbol{B}\cdot\hat{\boldsymbol{n}} = 0$ everywhere along the boundary in time to state that the surface integral must be zero. That is, magnetic helicity is preserved in these conditions: over a material volume whose bounding surface is composed entirely of magnetic field lines and both the material volume and field lines are frozen into the flow. $\square$

Problem 2: Solve using Cauchy solutions

1. $\boldsymbol{u} = \langle \sin(z), \cos(z), 0\rangle\,, \quad \boldsymbol{B}(\boldsymbol{x}, 0) = \langle y, z, x\rangle$

We begin with the Cauchy solution.

$$\boldsymbol{x}(\boldsymbol{a},0) = (a_1, a_2, a_3)$$
$$\frac{\partial \boldsymbol{x}}{\partial t} = (\sin(z), \cos(z), 0)$$
$$\boldsymbol{x}(\boldsymbol{a},t) = (a_1 + t\sin(a_3), a_2 + t\cos(a_3), a_3)$$
$$\boldsymbol{B}(\boldsymbol{a},t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\boldsymbol{B}(\boldsymbol{a},0) = \begin{bmatrix} 1 & 0 & t\cos(a_3) \\ 0 & 1 & -t\sin(s_3) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ a_1 \end{bmatrix} = (a_2 + a_1 t\cos(a_3), a_3 - a_1 t\sin(a_3), a_1)$$

2. $\boldsymbol{u} = \langle \sin(z), \cos(z), 1\rangle, \quad \boldsymbol{B}(\boldsymbol{x},0) = \langle 1,1,1\rangle$

$$\boldsymbol{x}(\boldsymbol{a},0) = (a_1, a_2, a_3)$$
$$\frac{\partial \boldsymbol{x}}{\partial t} = (\sin(z), \cos(z), 1)$$
$$\boldsymbol{x}(\boldsymbol{a},t) = (a_1 + \cos(a_3) - \cos(a_3+t), a_2 - \sin(a_3) + \sin(a_3+t), a_3+t)$$
$$\boldsymbol{B}(\boldsymbol{a},t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\boldsymbol{B}(\boldsymbol{a},0) = \begin{bmatrix} 1 & 0 & \sin(a_3+t) - \sin(a_3) \\ 0 & 1 & -\cos(a_3) + \cos(a_3+t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= (1 + \sin(a_3+t) - \sin(a_3), 1 + \cos(a_3+t) - \cos(a_3), 1)$$

3. $\boldsymbol{u} = \langle \sin(z), \cos(z), 1\rangle, \quad \boldsymbol{B}(\boldsymbol{x},0) = \langle x, y, -2z\rangle$

$$\boldsymbol{x}(\boldsymbol{a},0) = (a_1, a_2, a_3)$$
$$\frac{\partial \boldsymbol{x}}{\partial t} = (\sin(z), \cos(z), 1)$$
$$\boldsymbol{x}(\boldsymbol{a},t) = (a_1 + \cos(a_3) - \cos(a_3+t), a_2 - \sin(a_3) + \sin(a_3+t), a_3+t)$$
$$\boldsymbol{B}(\boldsymbol{a},t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\boldsymbol{B}(\boldsymbol{a},0) = \begin{bmatrix} 1 & 0 & \sin(a_3+t) - \sin(a_3) \\ 0 & 1 & -\cos(a_3) + \cos(a_3+t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ -2a_3 \end{bmatrix}$$
$$= (a_1 - 2a_3(\sin(a_3+t) - \sin(a_3)), a_2 - 2a_3(\cos(a_3+t) - \cos(a_3)), -2a_3)$$

4. $\boldsymbol{u} = \langle y, -x, 0\rangle, \quad \boldsymbol{B}(\boldsymbol{x},0) = \langle x, -y, 0\rangle$

$$\boldsymbol{x}(\boldsymbol{a},0) = (a_1, a_2, a_3)$$
$$\frac{\partial \boldsymbol{x}}{\partial t} = (y, -x, 0)$$
$$x' = -y, \quad y' = x$$
$$\frac{d^2x}{dt} = -x, \quad \frac{d^2y}{dt} = -y$$
$$x(t) = x_1\cos(t) + x_2\sin(t), \quad y(t) = y_1\cos(t) + y_2\sin(t)$$
$$x(0) = a_1, \quad x'(0) = -a_2, \quad y(0) = a_2, \quad y'(0) = a_1$$
$$x_1 = a_1, \quad x_2 = -a_2, \quad y_1 = a_2, \quad y_2 = a_1$$
$$\boldsymbol{x}(\boldsymbol{a},t) = (a_1\cos(t) - a_2\sin(t), a_2\cos(t) + a_1\sin(t), a_3)$$
$$\boldsymbol{B}(\boldsymbol{a},t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\boldsymbol{B}(\boldsymbol{a},0) = \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ -a_2 \\ 0 \end{bmatrix} = (a_1\cos(t) + a_2\sin(t), a_1\sin(t) - a_2\cos(t), 0)$$