AM 112 - Exam 3:

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Problem 4:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \cos(ct)\sin(x)$$
$$u(0,t) = 0, \quad u(\pi,t) = 0$$
$$u(x,0) = 0, \quad u_t(x,0) = 0$$

We begin with a separation of variables approach, i.e. $u = X(x)T(t)$ and proceed to obtain the following ODEs and BC.

$$X'' = \lambda X, \quad X(0) = 0, \quad X(\pi) = 0 \tag{1}$$
$$X(x) = \sin(nx), \quad \lambda = n^2 \tag{2}$$

Using the eigenfunction expansion approach to obtain an ODE for time reveals,

$$T_n'' \sin(nx) = -c^2 n^2 T_n \sin(nx) + \cos(ct)\sin(x) \tag{3}$$
$$\int_0^\pi T'' \sin(nx)\sin(mx)dx = \int_0^\pi \left(-c^2 n^2 T \sin(nx) + \cos(ct)\sin(x)\right)\sin(mx)dx \tag{4}$$
$$T_1'' = -c^2 T_1 + \cos(ct) \tag{5}$$

Thus, only one of the spatial eigenmodes is forced and will be non-zero in our solution. We continue to solve this ODE using the Laplace transform.

$$L[T_1''](s) = c^2 L[T](s) + L[\cos(t)](s) \tag{6}$$
$$(s^2 + c^2)L[T](s) = L[\cos(t)](s) \tag{7}$$
$$L[T](s) = \frac{1}{s^2 + c^2} L[\cos(t)](s) \tag{8}$$
$$T(t) = L^{-1}\left[\frac{1}{s^2 + c^2} L[\cos(t)](s)\right] \tag{9}$$

Applying the convolution theorem for Laplace transforms we have,

$$T(t) = \frac{1}{c}\int_0^t \sin(c(t-u))\cos(cu)du \tag{10}$$
$$= \frac{1}{2c}\int_0^t \sin(ct) + \sin(c(t-2u))du \tag{11}$$
$$= \frac{1}{2c} u\sin(ct)\Big|_0^t + \frac{1}{4c^2}\cos(ct - 2cu)\Big|_0^t \tag{12}$$
$$= \frac{1}{2c} t\sin(ct) + \frac{1}{4c^2}\left(\cos(-ct) - \cos(ct)\right) \tag{13}$$
$$= \frac{1}{2c} t\sin(ct) \tag{14}$$

Thus the solution would be,

$$u(x,t) = \frac{1}{2c} t \sin(ct) \sin(x) \tag{15}$$

This solution can be validated by applying the initial conditions and checking if it satisfies the forced differential equation.

$$u(x,0) = \frac{1}{2c}(0)\sin(x) = 0 \tag{16}$$

$$u_t(x,t) = \frac{1}{2c}\left(\sin(ct) + ct\cos(ct)\right)\sin(x) \tag{17}$$

$$u_t(x,0) = \frac{1}{2c}\left(0 + c(0)\right)\sin(x) = 0 \tag{18}$$

Next we consider the original differential equation,

$$\begin{aligned} u_{tt} &= \frac{1}{2c}\left(c\cos(ct) + c\cos(ct) - c^2 t\sin(ct)\right)\sin(x) && (19)\\ &= \cos(ct)\sin(x) - \frac{ct}{2}\sin(t)\sin(x) && (20)\\ &= F(x,t) + c^2 u_{xx} && (21) \end{aligned}$$

Therefore, we have that this solution solves the forced PDE with the given boundary conditions.

4.1 The posted solution

In the posted solutions, the solution is,

$$u(x,t) = \left(\frac{1}{2c^2}\sin(ct) - \frac{1}{2c}t\cos(ct)\right)\sin(x)$$

This solution successfully satisfies the IC, but after expanding the second derivative in time we find that it may not satisfy the PDE.

$$u_t = \frac{1}{2c}\cos(ct)\sin(x) - \frac{1}{2c}(\cos(ct) - ct\sin(ct))\sin(x) = -\frac{1}{2}t\sin(ct)\sin(x) \tag{22}$$

$$u_{tt} = -\frac{ct}{2}\cos(ct)\sin(x) \neq c^2\left(\frac{1}{2c^2}\sin(ct) - \frac{1}{2c}t\cos(ct)\right)\sin(x) + \cos(ct)\sin(x) \tag{23}$$