AM112 Solutions of Exam 1

Important note:

- The problems below may have some similarities. **But they have different types of boundary conditions and different functions. Be careful!**
- You may re-use, without re-deriving them, any results we already obtained in lectures, in homework assignment(s), or in exam(s).

Problem 1 (4 points):

Consider the expansion $f(x)=\sum_{n=1}^{\infty} c_n \cos(\frac{(n-\frac{1}{2})\pi}{L}x), \quad x\in[0,L]$.

a) Write out the formula for calculating coefficient c_n.

b) Evaluate coefficient c_n for function $f(x) \equiv 1$, $x \in [0, L]$.

Solution:

a) The formula for calculating coefficient c_n. is

$$\boxed{c_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{(n-\frac{1}{2})\pi}{L}x)\,dx}$$

b) For function $f(x) \equiv 1$, $x \in [0, L]$, coefficient c_n is

$$c_n = \frac{2}{L}\int_0^L \cos(\frac{(n-\frac{1}{2})\pi}{L}x)\,dx = \frac{2}{(n-\frac{1}{2})\pi}\sin(\frac{(n-\frac{1}{2})\pi}{L}x)\Big|_0^L = \frac{2}{(n-\frac{1}{2})\pi}\sin(n\pi-\tfrac{1}{2}\pi)$$

$$\boxed{c_n = \frac{-2(-1)^n}{(n-\frac{1}{2})\pi}}$$

Problem 2 (4 points):

Consider the IBVP below.

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = \cos(x) \\ u(0,t) = -3\,,\ u_x(L,t) = 2 \end{cases} \qquad \text{(IBVP-P2)}$$

a) Find the steady state $u_\infty(x)$.

b) Let $v(x,t) \equiv u(x,t) - u_\infty(x)$. Write out the IBVP for $v(x, t)$.

Do NOT solve the IBVP.

Solution:

a) The steady state $u_\infty(x)$ satisfies $u_\infty''(x) = 0\,,\ u_\infty(0) = -3\,,\ u_\infty'(L) = 2$.

Solving for $u_\infty(x)$, we obtain

$$\boxed{u_\infty''(x) = -3 + 2x}$$

b) The IBVP for $v(x, t)$ is

$$\boxed{\begin{cases} v_t = k v_{xx} \\ v(x,0) = \cos(x) + 3 - 2x \\ v(0,t) = 0\,,\ v_x(L,t) = 0 \end{cases}}$$

Problem 3 (4 points):

Consider the IBVP below.

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = e^x \sin(x) \\ u_x(0,t) = 0\,,\ u(\pi,t) = 0 \end{cases} \qquad \text{(IBVP-P1)}$$

a) Write out the eigenvalue problem in separation of variables and the solution of the eigenvalue problem (eigenvalues and eigenfunctions).

b) Write out a general solution of $u_t = k u_{xx}$, $u_x(0,t) = 0$, $u(\pi,t) = 0$.

Do NOT enforce the IC.

Solution:

(a) Substituting $u(x, t) = X(x)\, T(t)$ into the differential equation, we get

$$X(x)T'(t) = X''(x)T(t) \quad \Rightarrow \quad \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

The eigenvalue problem is $\boxed{\begin{cases} X'' = -\lambda X \\ X'(0) = 0\,,\ X(\pi) = 0 \end{cases}}$

The eigenvalues and eigenfunctions are

$$\boxed{\lambda_n = (n - \tfrac{1}{2})^2, \quad X_n(x) = \cos((n - \tfrac{1}{2})x), \quad n = 1, 2, \ldots}$$

(b) A general solution of $u_t = k u_{xx}$, $u_x(0,t) = 0$, $u(\pi,t) = 0$ is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \cos((n - \tfrac{1}{2})x) e^{-k(n-\frac{1}{2})^2 t}}$$

Problem 4 (8 points):

Solve the IBVP below.

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = 1 \\ u(0,t) = 1\,,\ u(L,t) = 4 \end{cases} \qquad \text{(IBVP-P4)}$$

Be sure to add back $u_\infty(x)$ in your final answer.

Solution:

The steady state $u_\infty(x)$ is $u_\infty(x) = 1 + 3\dfrac{x}{L}$

Let $v(x,t) = u(x,t) - u_\infty(x)$. The IBVP for $v(x, t)$ is

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = -3\dfrac{x}{L} \\ u(0,t) = 0\,,\ u(L,t) = 0 \end{cases} \qquad \text{(IBVP-P4H)}$$

A general solution of $u_t = ku_{xx}$, $u(0,t) = 0$, $u(L,t) = 0$ is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$$

Imposing the IC: $u(x, 0) = -3x/L$ ==> $\dfrac{-3x}{L} = \sum_{n=1}^{\infty} b_n \sin(\dfrac{n\pi}{L}x)$

We already know $\dfrac{x}{L} = \sum_{n=1}^{\infty} \dfrac{-2(-1)^n}{n\pi} \sin(\dfrac{n\pi}{L}x)$ ==> $b_n = \dfrac{6(-1)^n}{n\pi}$.

The solution of (IBVP-P4H) is $v(x,t) = 6\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n\pi} \sin(\dfrac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$.

The solution of (IBVP-P4) is

$$\boxed{u(x,t) = u_\infty(x) + v(x,t) = 1 + \frac{3x}{L} + 6\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}}$$