

# AM 275 - Magnetohydrodynamics:

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## Problem 1: Magnetic Field Lines

Here is a figure 1 of the magnetic field lines, and following is a plot of the field lines.

## Problem 2: Oscillating Magnetic Field Diffusion

Solve the given PDE for a magnetic field in a semi-infinite volume ( $z > 0$ ) with high resistivity ( $\eta$ ), and the following boundary condition,

$$\mathbf{B}(z = 0, t) = (B_0 e^{-i\omega t}, 0, 0)$$

*Proof.* We begin by demonstrating that since this is a linear problem, we have that the conditions at  $z = 0$ , and  $t = 0$  (taken by setting  $t = 0$  in the BC), that the  $y$  and  $z$ -components of the magnetic field will be constant ( $= 0$ ) for all  $t \geq 0$ .

$$\begin{aligned}\frac{\partial B_y}{\partial t}(t = 0) &= \eta \nabla^2 B_y(t = 0) = 0 \\ \frac{\partial B_z}{\partial t}(t = 0) &= \eta \nabla^2 B_z(t = 0) = 0\end{aligned}$$

and so they must remain zero for all time. Thus we have shown that the problem is reduced to the following equation,

$$\begin{aligned}\frac{\partial B_x}{\partial t} &= \eta \nabla^2 B_x, \quad B_x(z, t) \\ \frac{\partial B_x}{\partial t} &= \eta \frac{\partial^2 B_x}{\partial z^2}, \quad B_x(z = 0, t) = B_0 e^{-i\omega t}, \quad B_x(z \rightarrow \infty, t) = 0, \quad B_x(z, t = 0) = B_0\end{aligned}$$

We begin by solving using a separation of variables solution in  $z$  and  $t$ .

$$B_x = Z(z)T(t), \quad \frac{T'}{\eta T} = \frac{Z''}{Z} = -c^2$$

here in order to satisfy the boundary conditions, we will take  $c$  to be complex valued, i.e.  $c = \alpha + i\beta$ . We begin solving these ODEs starting with  $Z$ .

$$\begin{aligned}Z'' &= -c^2 Z \implies Z(z) = c_1 e^{icz} + c_2 e^{-icz} \\ Z(z) &= e^{-\beta z} (a_1 \cos(\alpha z) + b_1 \sin(\alpha z)) + e^{\beta z} (a_2 \cos(\alpha z) + b_2 \sin(\alpha z)) \\ Z(0) &= 1 \implies a_1 + a_2 = 1 \\ \lim_{z \rightarrow \infty} Z(z) &= 0 \implies a_2 = b_2 = 0 \\ Z(z) &= e^{-\beta z} (\cos(\alpha z) + b_1 \sin(\alpha z))\end{aligned}$$

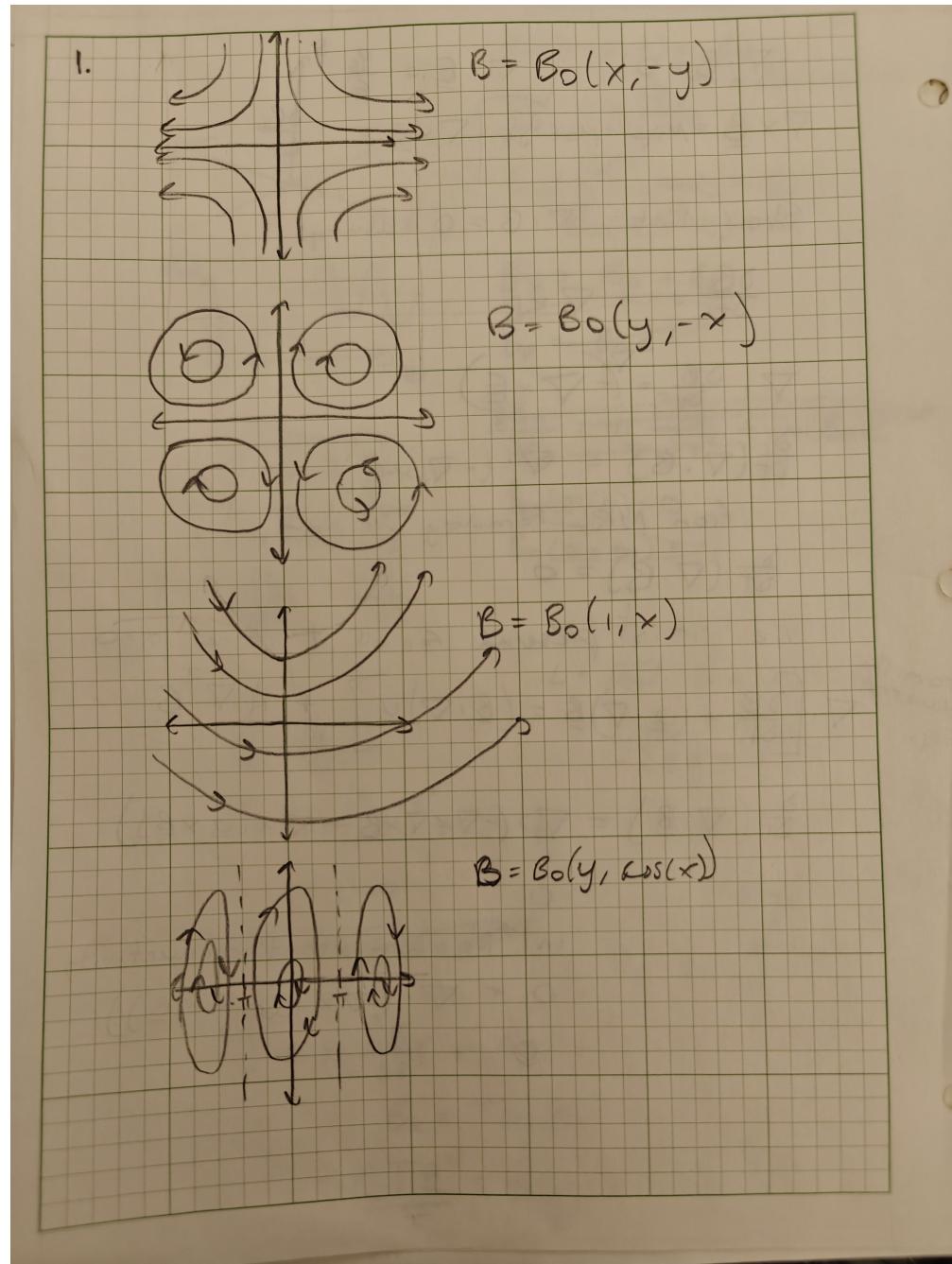


Figure 1: Drawing of magnetic field lines

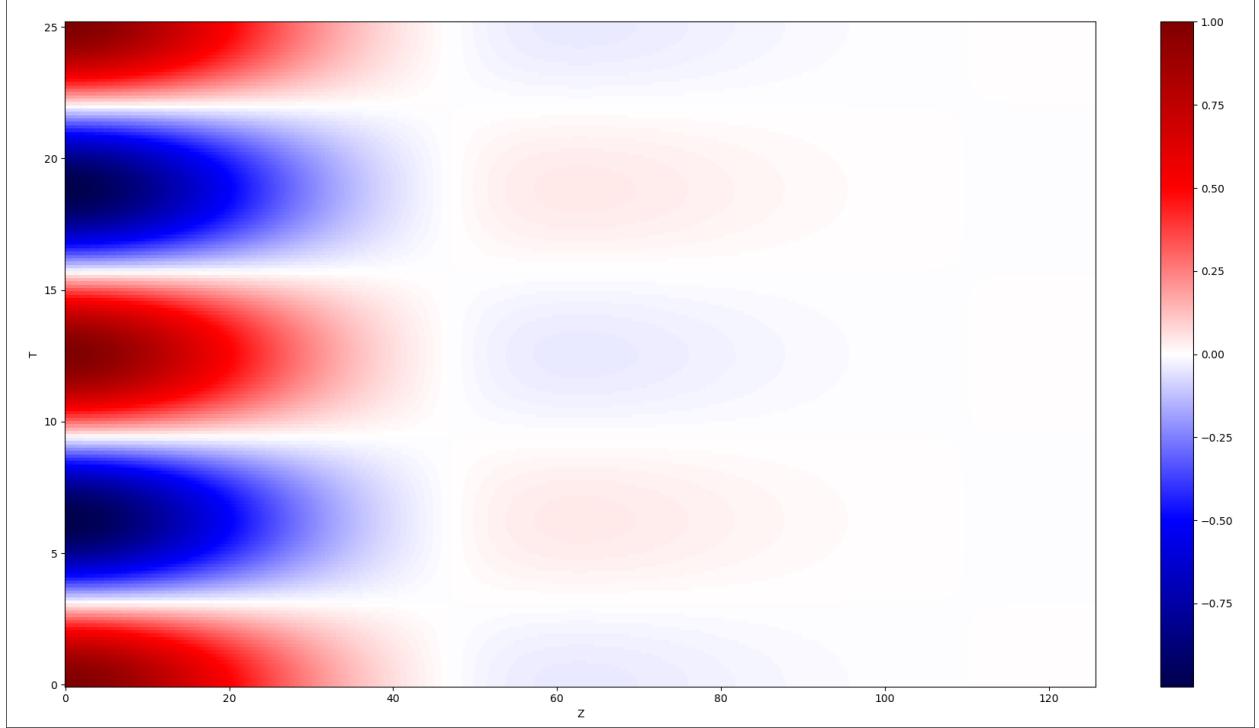


Figure 2: Plot of solution for problem 2 obtained using python using  $\eta = 100$ ,  $\omega = 0.5$ , and  $B_0 = 1$ .

Next we move to the ODE for time. We have,

$$\begin{aligned} T' = -c^2\eta T &\implies T = T_0 e^{-c^2\eta t}, \quad -c^2 = \beta^2 - \alpha^2 - i2\alpha\beta \\ T &= T_0 e^{-(\beta^2 - \alpha^2)\eta t} (\cos(2\alpha\beta\eta t) - i \sin(2\alpha\beta\eta t)) \\ T(t=0) &= T_0 = B_0 \end{aligned}$$

Next we put these two separable solutions together and solve for the last BC,

$$\begin{aligned} u(z=0, t) &= B_0 e^{-(\beta^2 - \alpha^2)\eta t} (\cos(2\alpha\beta\eta t) - i \sin(2\alpha\beta\eta t)) = B_0 e^{-i\omega t} \\ &\implies \beta^2 - \alpha^2 = 0, \quad \alpha = \pm\beta \\ 2\alpha\beta\eta t &= \pm 2\alpha^2\eta t = \omega t \\ \alpha &= \sqrt{\frac{\omega}{2\eta}} \end{aligned}$$

With the eigenvalue solved, we can then write the solution for  $B_x$ .

$$B_x = B_0 \exp\left(-\sqrt{\frac{\omega}{2\eta}}z - i\omega t\right) \left(\cos\left(\sqrt{\frac{\omega}{2\eta}}z\right) + \sin\left(\sqrt{\frac{\omega}{2\eta}}z\right)\right)$$

Notice that this solution oscillates on a wavelength of  $\lambda = 2\pi/k_z = 2\pi/\sqrt{\omega/2\eta} = 2\pi\sqrt{2\eta/\omega}$ , and the exponential decay occurs on a scale height of  $L_z = \sqrt{2\eta/\omega}$ .  $\square$