AM 275 - Magnetohydrodynamics:

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Problem 1: Magnetic Field Lines

Here is a sketch of the magnetic field lines, and following is a plot of the field lines using python.

Problem 2: Oscillating Magnetic Field Diffusion

Solve the given PDE for a magnetic field in a semi-infinite volume ($z > 0$) with high resistivity (η), and the following boundary condition,

$$\boldsymbol{B}(z=0,t) = \left(B_0 e^{-i\omega t}, 0, 0\right)$$

Proof. We begin by demonstrating that since this is a linear problem, we have that the conditions at $z = 0$, and $t = 0$ (taken by setting $t = 0$ in the BC), that the y and z-components of the magnetic field will be constant ($= 0$) for all $t \geq 0$.

$$\frac{\partial B_y}{\partial t}(t=0) = \eta \nabla^2 B_y(t=0) = 0$$
$$\frac{\partial B_z}{\partial t}(t=0) = \eta \nabla^2 B_z(t=0) = 0$$

and so they must remain zero for all time. Thus we have shown that the problem is reduced to the following equation,

$$\frac{\partial B_x}{\partial t} = \eta \nabla^2 B_x, \quad B_x(z,t)$$
$$\frac{\partial B_x}{\partial t} = \eta \frac{\partial^2 B_x}{\partial z^2}, \quad B_x(z=0,t) = B_0 e^{-i\omega t}, \quad B_x(z\to\infty, t) = 0, \quad B_x(z, t=0) = B_0$$

We begin by solving using a separation of variables solution in z and t.

$$B_x = Z(z)T(t), \quad \frac{T'}{\eta T} = \frac{Z''}{Z} = -c^2$$

here in order to satisfy the boundary conditions, we will take c to be complex valued, i.e. $c = \alpha + i\beta$. We begin solving these ODEs starting with Z.

$$Z'' = -c^2 Z \implies Z(z) = c_1 e^{icz} + c_2 e^{-icz}$$
$$Z(z) = e^{-\beta z}\left(a_1 \cos(\alpha z) + b_1 \sin(\alpha z)\right) + e^{\beta z}\left(a_2 \cos(\alpha z) + b_2 \sin(\alpha z)\right)$$
$$Z(0) = 1 \implies a_1 + a_2 = 1$$
$$\lim_{z\to\infty} Z(z) = 0 \implies a_2 = b_2 = 0$$
$$Z(z) = e^{-\beta z}\left(\cos(\alpha z) + b_1 \sin(\alpha z)\right)$$

Next we move to the ODE for time. We have,

$$T' = -c^2\eta T \implies T = T_0 e^{-c^2\eta t}, \quad -c^2 = \beta^2 - \alpha^2 - i2\alpha\beta$$
$$T = T_0 e^{-(\beta^2-\alpha^2)\eta t}\left(\cos(2\alpha\beta\eta t) - i\sin(2\alpha\beta\eta t)\right)$$
$$T(t=0) = T_0 = B_0$$

Next we put these two separable solutions together and solve for the last BC,

$$u(z=0,t) = B_0 e^{-(\beta^2-\alpha^2)\eta t}\left(\cos(2\alpha\beta\eta t) - i\sin(2\alpha\beta\eta t)\right) = B_0 e^{-i\omega t}$$
$$\implies \beta^2 - \alpha^2 = 0, \quad \alpha = \pm\beta$$
$$2\alpha\beta\eta t = \pm 2\alpha^2\eta t = \omega t$$
$$\alpha = \sqrt{\frac{\omega}{2\eta}}$$

With the eigenvalue solved, we can then write the solution for B_x.

$$B_x = B_0 \exp\left(-\sqrt{\frac{\omega}{2\eta}}z - i\omega t\right)\left(\cos\left(\sqrt{\frac{\omega}{2\eta}}z\right) + \sin\left(\sqrt{\frac{\omega}{2\eta}}z\right)\right)$$

Notice that this solution oscillates on a wavelength of $\lambda = 2\pi/k_z = 2\pi/\sqrt{\omega/2\eta} = 2\pi\sqrt{2\eta/\omega}$, and the exponential decay occurs on a scale height of $L_z = \sqrt{2\eta/\omega}$. $\square$