CHAPTER 1 : HD EQNS

HYDRODYNAMICS - REVIEW

- FLUID - something that "flows"
  - deforms continuously when force applied
  - no "elasticity" (no "memory" of shape)

- Categorisation
  → stresses are forces/unit area on a blob of fluid

  → all fluids "support" (resist) NORMAL stresses to a certain extent
    → some more than others : INCOMPRESSIBLE / COMPRESSIBLE

  → some fluids "support" SHEAR stresses more than others
    → VISCOUS / INVISCID

  → 2 types of fluids
    → LIQUIDS : virtually incompressible
    → GASES : compressible

- Theory of fluids : Continuum hypothesis
  → ignore molecular issues via averaging around a point using sufficient volume to average out molecular variation

- Eulerian vs Lagrangian approach
  → Eulerian - watch a fixed point
  → Lagrangian - go with the flow

Any function $f = f(x, y, z, t)$ $\quad \underline{x} = (x(t), y(t), z(t))$

Total derivative

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

or $$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z}$$

often $\frac{Df}{Dt}$

$$\Rightarrow \boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + (\underline{u}\cdot\nabla)f}$$

MATERIAL DERIVATIVE

Lagrangian derivative in Eulerian co-ordinates

Eulerian derivative

piece that compensates for Lagrangian movement (by $\underline{u}$) of fluid parcel in a inhomogeneous background $\nabla f \neq 0$

- Eqns : Mass conservation

Eulerian volume V, surface S

density (mass per unit vol)

$$\text{Total mass} = M = \int_V \rho \, dV$$

fixed vol

Change in mass $\Leftrightarrow$ must be flow (flux) of mass into/out of volume

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\int_V \rho \, dV = -\int_S \rho \, \underline{u} \cdot \hat{\underline{n}} \, ds$$

increase in mass

flux IN through surface

flow of "mass"

through surface

"mass" flux

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla\cdot(\rho\underline{u}) \, dV$$

inside since boundaries fixed

DIVERGENCE TH

$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\underline{u}\right) dV = 0$$

since arbitrary volume $\Rightarrow$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\underline{u} = 0}$$

CONTINUITY EQN

or $\quad \frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0$

- Incompressible : $\rho = \rho_0$ constant
  $\Rightarrow \underline{\nabla} \cdot \underline{u} = 0$

- EQNS : MOMENTUM EQN

Lagrangian volume : Newton's 2nd Law : rate of change of mom = force

$$\frac{D}{Dt}\left\{ \underbrace{\int_{V(t)} \rho \underline{u} \, dV}_{\text{TOTAL MOMENTUM}} \right\} = \underbrace{\int_V \underline{F} \rho \, dV}_{\text{BODY FORCES}} + \underbrace{\int_S \underline{\underline{\tau}} \cdot \underline{n} \, dS}_{\text{SURFACE FORCES}}$$

$\frac{D}{Dt}$: rate of change (Lagrangian)

$\underline{F}$: force per unit mass e.g. g

BODY FORCES (act on volume)

$\underline{\underline{\tau}}$: stress TENSOR

SURFACE FORCES (acts on surfaces)

$$\Rightarrow \int_{V(t)} \rho \frac{D\underline{u}}{Dt} \, dV = \int_V \underline{F} \rho \, dV + \int_V \underline{\nabla} \cdot \underline{\underline{\tau}} \, dV$$

$\rho \frac{D\underline{u}}{Dt}$: NOT obvious! since $V = V(t)$ Need Reynolds Transport Th and use of continuity

$\underline{\nabla} \cdot \underline{\underline{\tau}}$: Divergence Th

$$\Rightarrow \rho \frac{D\underline{u}}{Dt} = \rho \underline{F} + \underline{\nabla} \cdot \underline{\underline{\tau}}$$

$$\left(\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j} \right)$$

CAUCHY EQN OF MOTION

stress tensor?

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}$$

$-p\delta_{ij}$: pull out a piece that looks like STATIC PRESSURE (THERMODYNAMIC)

$$p = -\frac{1}{3}\tau_{ii}$$

σ_{ij}: the rest! DEVIATORIC STRESS TENSOR

→ Relate deviatoric stress tensor to velocity tensor $\partial u_i/\partial x_j$ and note that

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{antisymmetric}}$$

symmetric — STRAIN RATE TENSOR — e_{ij} — DEFORMATION

antisymmetric — ROTATION TENSOR — $\frac{1}{2}r_{ij}$ — rotation; no deformation = SOLID BODY ROTATION (no strain)

→ After a bunch of stuff, reduce

$$\sigma_{ij} = A_{ijkl}\, e_{kl}$$

↖ 81 coefficients!

to

$$\sigma_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)$$

$\frac{\partial u_k}{\partial x_k}$ ← $\nabla\cdot\underline{u}$

μ: one coefficient! DYNAMIC VISCOSITY

$$\boxed{\tau_{ij} = -p\,\delta_{ij} + 2\mu\left(e_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)}$$ CONSTITUTIVE RELATION

→ Stick this in CAUCHY'S EQN ⟹

$$\boxed{\rho\frac{D\underline{u}}{Dt} = \rho\underline{F} - \underline{\nabla}p + \mu\left\{\nabla^2\underline{u} + \frac{1}{3}\underline{\nabla}(\underline{\nabla}\cdot\underline{u})\right\}}$$ NAVIER-STOKES EQN

- $\rho\frac{D\underline{u}}{Dt}$: Lagrangian acceln (in Eulerian co-ords)
- $\rho\underline{F}$: body forces
- $-\underline{\nabla}p$: pressure gradient
- $\nabla^2\underline{u}$: viscous diffusion
- $\frac{1}{3}\underline{\nabla}(\underline{\nabla}\cdot\underline{u})$: part that compensates for non isotropic τ_{ii}

→ INCOMPRESSIBLE : $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + \underline{F} + \nu\nabla^2\underline{u}$

$\nu = \frac{\mu}{\rho_0}$ KINEMATIC VISCOSITY (a diffusivity)

→ Note : BIG difference visc vs inviscid!

- REDUCED EQNS

→ $\underline{u} = 0 \rightarrow \nabla p = \rho \underline{F}$

Typically, $\underline{F} = \underline{g} \Rightarrow \underline{\nabla} p = \rho \underline{g}$ $\quad \frac{\partial p}{\partial z} = -\rho g$

HYDROSTATIC EQN

→ Incompressible, inviscid, steady

$$\cancel{\frac{\partial \underline{u}}{\partial t}} + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) - \underline{u} \times \underline{\omega} = -\underline{\nabla}\left(\frac{p}{\rho_0}\right) + \underline{\nabla}\left(\underline{g} \cdot \underline{x}\right)$$

$$\Rightarrow \quad -(\underline{u} \times \underline{\omega}) + \nabla\left(\tfrac{1}{2}\underline{u}^2 + \frac{p}{\rho_0} - \underline{g} \cdot \underline{x}\right) = \emptyset$$

Dot with $\begin{cases} \underline{u} \\ \underline{\omega} \end{cases}$: $\quad \frac{1}{2}\underline{u}^2 + \frac{p}{\rho_0} + gz = \text{const}^{t}$ on $\begin{cases} \text{streamlines} \\ \text{vortex lines} \end{cases}$

BERNOULLI

EQNS : VORTICITY

Vorticity : $\underline{\omega} = \nabla \times \underline{u}$ "spin" of fluid element

N-S : $\dfrac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\dfrac{1}{\rho}\nabla p + \underline{F} + \nu\left\{\nabla^2\underline{u} + \nabla(\nabla\cdot\underline{u})\right\}$

$$\frac{\partial \underline{u}}{\partial t} + \underbrace{\nabla\left(\tfrac{1}{2}\underline{u}^2\right) - (\underline{u}\times\underline{\omega})}_{(\underline{u}\cdot\nabla)\underline{u}} = \quad ''$$

HANDY TRICK

Take curl →

$$\frac{\partial \underline{\omega}}{\partial t} - \nabla\times(\underline{u}\times\underline{\omega}) = \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\underline{F} + \nu\nabla^2\underline{\omega}$$

$$\underbrace{+(\underline{u}\cdot\nabla)\underline{\omega}}_{\frac{D\underline{\omega}}{Dt}} - (\underline{\omega}\cdot\nabla)\underline{u} - \underline{u}\,(\cancel{\nabla\cdot\underline{\omega}}) + \underline{\omega}\,(\nabla\cdot\underline{u})$$

$\nabla\cdot\underline{\omega} \to \emptyset$

$\nabla\cdot\underline{u}$: incompressible?

$(\underline{\omega}\cdot\nabla)\underline{u}$: VORTEX STRETCHING AND TILTING

$\frac{1}{\rho^2}\nabla\rho\times\nabla p$: BAROCLINIC GENERATION OF VORTICITY

$\nabla\times\underline{F}$: PRODUCTION BY NON-CONSERVATIVE FORCES

$\nu\nabla^2\underline{\omega}$: DIFFUSION

→ KELVIN / HELMHOLTZ THEOREMS :

If inviscid, barotropic, conservative forces
then vortex lines (tubes) FROZEN-IN to flow

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \nabla\times(\underline{u}\times\underline{\omega})}$$

EQNS : ROTATING CO-ORDINATE FRAME

Any vector $\underline{P}$

$$\left(\frac{d\underline{P}}{dt}\right)_{fixed} = \left(\frac{d\underline{P}}{dt}\right)_{rotating} + (\underline{\Omega} \times \underline{P})$$

$\Rightarrow$

$$\left(\frac{d\underline{u}_F}{dt}\right)_F = \left(\frac{d\underline{u}_R}{dt}\right)_R + 2\left(\underline{\Omega} \times \underline{u}_R\right)_R + \underbrace{\underline{\Omega} \times (\underline{\Omega} \times \underline{r})_R}_{= -\Omega^2 \underline{R}}$$

$\underline{R}$ distance from rotation axis

$\Rightarrow$ Eqns

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \times \underline{u} - \Omega^2 \underline{R} = -\frac{1}{\rho}\nabla p + \frac{\underline{F}}{\rho} + \nu \nabla^2 \underline{u} \ldots$$

CORIOLIS CENTRIFUGAL

fictitious forces due to being in rotating frame

absorb into effective gravity

$\rightarrow$ Vorticity :

$$\frac{\partial \underline{\omega}}{\partial t} + \nabla \times (\underline{\omega} \times \underline{u}) + \nabla \times (2\underline{\Omega} \times \underline{u}) = \ldots$$

$$\rightarrow \frac{D\underline{\omega}}{Dt} = \left([\underline{\omega} + 2\underline{\Omega}] \cdot \underline{\nabla}\right)\underline{u} + \ldots$$

vortex stretching / tilting $\longrightarrow$ vorticity generation from existing vorticity

extraction of vorticity from BACKGROUND rotation of frame (vorticity out of none!)

EQNS : ENERGY

→ Mechanical: $\underline{u}\cdot$ (Nav-Stokes) →

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2\right) + \nabla\cdot\left(\underline{u}\,\tfrac{1}{2}\rho u^2\right) = \rho\underline{u}\cdot\underline{F} + \underbrace{\underline{u}\cdot(\nabla\cdot\underline{\tau})}$$

$$\nabla\cdot(\underline{u}\cdot\underline{\tau}) + p(\nabla\cdot\underline{u}) + \Phi$$

total rate of work by body forces

total rate of work by stresses

rate of work by volume expansion

VISCOUS DISSIPATION.

→ Thermal energy

1^{st} Law of thermodynamics

"Rate of change of stored energy = rate of work done + rate of heat added"

$$\frac{D}{Dt}\int_{V(t)}\rho\left(e+\tfrac{1}{2}u^2\right)dV = \int_V \rho\,\underline{F}\cdot\underline{u}\;dV + \int_S \underline{u}\cdot\underline{\tau}\;d\underline{S} - \int_S \underline{q}\cdot d\underline{S}$$

$\nabla\cdot(\tau\cdot u)$ heat inflow

→ $\int_V \nabla\cdot\underline{q}\;dV$

$$\rightarrow \boxed{\rho\frac{D}{Dt}\left(e+\tfrac{1}{2}u^2\right) = \rho\underline{g}\cdot\underline{u} + \nabla\cdot(\underline{\tau}\cdot\underline{u}) - \nabla\cdot\underline{q}}$$

→ Subtract mechanical ⇒

$$\rho\frac{De}{Dt} = \underbrace{-\nabla\cdot q}_{\text{convergence of heat}} - \underbrace{p(\nabla\cdot\underline{u})}_{\text{compressional heating}} + \underbrace{\Phi}_{\text{viscous heating}}$$

$\Phi = 2\mu\left[e_{ij} - \tfrac{1}{3}\nabla\cdot\underline{u}\,\delta_{ij}\right]^2$ positive fn of μ

2^{nd} Law of thermodynamics : $T\,dS = de + p\,dV$

→

$$\rho\frac{DS}{Dt} = \underbrace{-\left(\nabla\cdot\frac{q}{T}\right)}_{\text{(reversible)}} + \frac{k}{T^2}|\nabla T|^2 + \frac{\Phi}{T}$$

⇒ $\mu, k > 0$