Magnetic field generation in electrically conducting fluids

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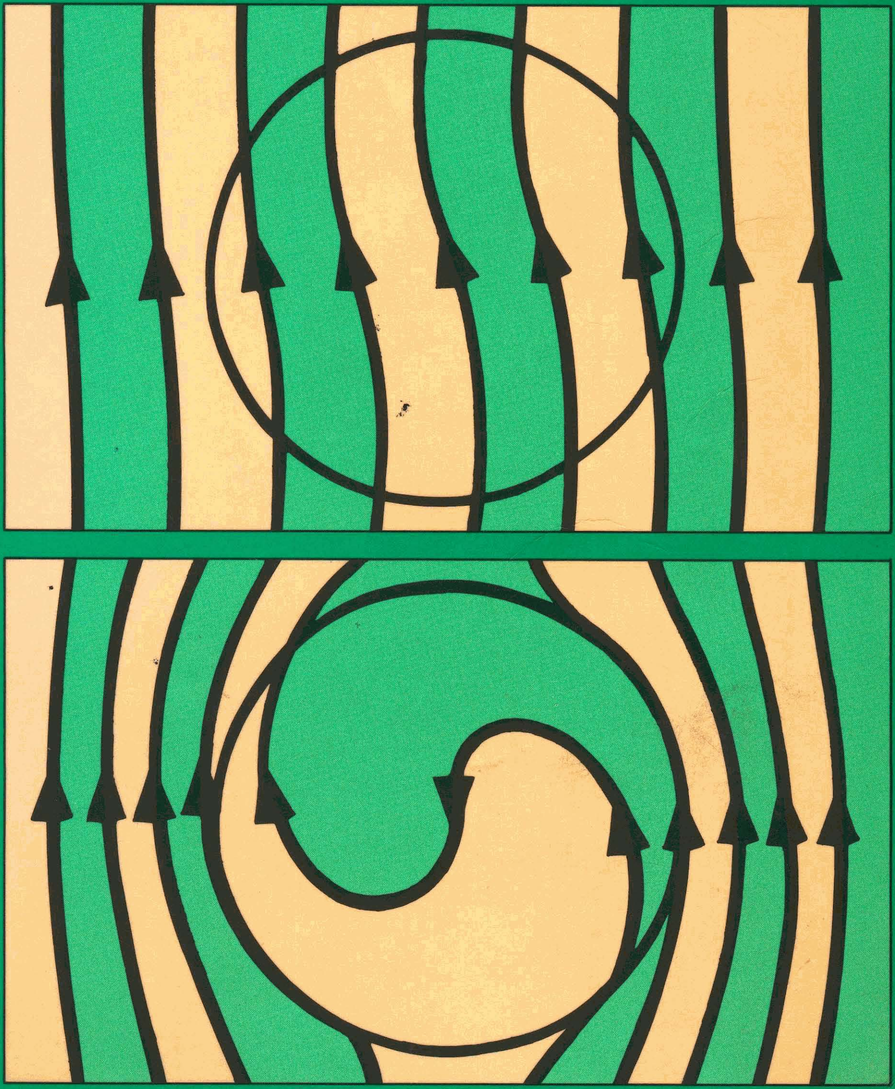


Similarly, with $\mathbf{B}_T = B(s, z)\mathbf{i}_\varphi$, $\mathbf{u}_T = u_\varphi(s, z)\mathbf{i}_\varphi$, we have

$$\begin{aligned} \nabla \wedge (\mathbf{u}_P \wedge \mathbf{B}_T) &= -\mathbf{i}_\varphi s(\mathbf{u}_P \,.\, \nabla)(s^{-1}B), \\ \nabla \wedge (\mathbf{u}_T \wedge \mathbf{B}_P) &= \mathbf{i}_\varphi s(\mathbf{B}_P \,.\, \nabla)(s^{-1}u_\varphi), \end{aligned} \tag{3.42}$$

so that (3.37) becomes

$$\partial B/\partial t + s(\mathbf{u}_P \,.\, \nabla)(s^{-1}B) = s(\mathbf{B}_P \,.\, \nabla)(s^{-1}u_\varphi) + \lambda(\nabla^2 - s^{-2})B. \tag{3.43}$$

Again there is a source term in the equation for B, but now it is variation of the angular velocity $\omega(s, z) = s^{-1}u_\varphi(s, z)$ along a $\mathbf{B}_P$-line which gives rise, by field distortion, to the generation of toroidal field. This phenomenon will be studied in detail in § 3.11 below.

Sometimes it is convenient to use the flux-function $\chi(s, z) = sA(s, z)$ (see (2.47)). From (3.41), the equation for χ is

$$\partial\chi/\partial t + (\mathbf{u}_P \,.\, \nabla)\chi = \lambda \mathrm{D}^2\chi, \tag{3.44}$$

where

$$\mathrm{D}^2\chi = s(\nabla^2 - s^{-2})(s^{-1}\chi) = (\nabla^2 - 2s^{-1}\partial/\partial s)\chi. \tag{3.45}$$

The operator D^2, known as the Stokes operator, occurs frequently in problems with axial symmetry. In spherical polars (r, θ, φ) it takes the form

$$\mathrm{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}. \tag{3.46}$$

Note, from (3.45), that

$$\mathrm{D}^2\chi = \nabla \,.\, \mathbf{f} \quad \text{where } \mathbf{f} = \nabla\chi - 2s^{-1}\chi\mathbf{i}_s. \tag{3.47}$$

3.7. Field distortion by differential rotation

By *differential rotation*, we shall mean an incompressible velocity field axisymmetric about, say, Oz, and with circular streamlines about this axis. Such a motion has the form (in cylindrical polars)

$$\mathbf{u} = \omega(s, z)\mathbf{i}_z \wedge \mathbf{x}. \tag{3.48}$$

If $\nabla\omega = 0$, then we have rigid body rotation which clearly rotates a magnetic field without distortion. If $\nabla\omega \neq 0$, lines of force are in general distorted in a way that depends both on the appropriate value of R_m (§ 3.3) and on the orientation of the field relative to the vector $\mathbf{i}_z$. The two main possibilities are illustrated in fig. 3.2. In (*a*),

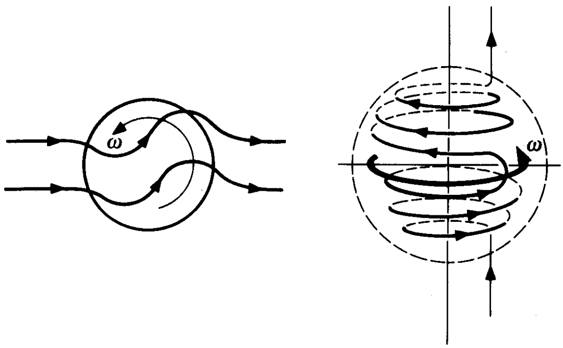


Fig. 3.2 Qualitative action of differential rotation on an initially uniform magnetic field; (*a*) rotation vector $\omega\mathbf{k}$ perpendicular to field; (*b*) rotation vector $\omega\mathbf{k}$ parallel to field.

ω is a function of s alone, and the **B**-field lies in the $x-y$ plane perpendicular to $\mathbf{i}_z$; the effect of the motion, neglecting diffusion, is to wind the field into a tight double spiral in the $x-y$ plane. In (*b*), $\omega=\omega(r)$, where $r^2=s^2+z^2$, and **B** is initially axisymmetric and poloidal; the effect of the rotation, neglecting diffusion, is to generate a toroidal field, the typical **B**-line becoming helical in the region of differential rotation.

Both types of distortion are important in the solar context, and possibly also in the geomagnetic context, and have been widely studied. We discuss first in the following two sections the type (*a*) distortion (first studied in detail by Parker, 1963), and the important related phenomenon of flux expulsion from regions of closed streamlines.

3.8. Effect of plane differential rotation on an initially uniform field

Suppose then that $\omega=\omega(s)$, so that the velocity field given by (3.48) is independent of z, and suppose that at time $t=0$ the field $\mathbf{B}(\mathbf{x}, 0)$ is uniform and equal to $\mathbf{B}_0$. We take the axis Ox in the direction of $\mathbf{B}_0$. For $t > 0$, $\mathbf{B} = -\mathbf{i}_z \wedge \nabla A$, where, from (3.38), A satisfies

$$\partial A/\partial t + \omega(s)(\mathbf{x} \wedge \nabla A)_z = \lambda \nabla^2 A. \tag{3.49}$$

It is natural to use plane polar coordinates defined here by

$$x = s \cos \varphi, \qquad y = s \sin \varphi, \tag{3.50}$$

in terms of which (3.49) becomes

$$\partial A/\partial t + \omega(s)\, \partial A/\partial \varphi = \lambda \nabla^2 A. \tag{3.51}$$

The initial condition $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0$ is equivalent to

$$A(s, \varphi, 0) = B_0 s \sin \varphi, \tag{3.52}$$

and the relevant solution of (3.51) clearly has the form

$$A(s, \varphi, t) = \mathrm{Im}\, B_0 f(s, t)\, \mathrm{e}^{\mathrm{i}\varphi}, \tag{3.53}$$

where

$$\frac{\partial f}{\partial t} + \mathrm{i}\omega(s) f = \lambda\left(\frac{1}{s}\frac{\partial}{\partial s} s \frac{\partial}{\partial s} - \frac{1}{s^2}\right) f, \tag{3.54}$$

and

$$f(s, 0) = s. \tag{3.55}$$

The initial phase

When $t = 0$, the field $\mathbf{B}$ is uniform and there is no diffusion; it is therefore reasonable to anticipate that diffusion will be negligible during the earliest stages of distortion. With $\lambda = 0$ the solution of (3.54) satisfying the initial condition (3.55) is $f(s, t) = s e^{-\mathrm{i}\omega(s)t}$, so that from (3.53)

$$A(s, \varphi, t) = B_0 s \sin(\varphi - \omega(s)t). \tag{3.56}$$

This solution is of course just the Lagrangian solution $A(\mathbf{x}, t) = A(\mathbf{a}, 0)$, since for the motion considered, the particle whose coordinates are (s, φ) at time t originated from position $(s, \varphi - \omega(s)t)$ at time zero. The components of $\mathbf{B} = -\mathbf{i}_z \wedge \nabla A$ are now given by

$$\left.\begin{aligned} B_s &= s^{-1}\, \partial A/\partial \varphi = B_0 \cos(\varphi - \omega(s)t), \\ B_\varphi &= -\partial A/\partial s = -B_0 \sin(\varphi - \omega(s)t) \\ &\qquad + B_0 s \omega'(s) t \cos(\varphi - \omega(s)t). \end{aligned}\right\} \tag{3.57}$$

If $\omega'(s)=0$, i.e. if the motion is a rigid body rotation, then as expected the field is merely rotated with the fluid. If $\omega'(s)\neq 0$, the φ-component of $\mathbf{B}$ increases linearly with time as a result of the stretching process.

We may estimate from (3.56) just for how long the effects of diffusion are negligible; for from (3.56)

$$\lambda \nabla^2 A = -\lambda B_0 s^{-2}(s^3\omega')'t \cos(\varphi-\omega(s)t) \\ -\lambda B_0 s\omega'^2 t^2 \sin(\varphi-\omega(s)t), \qquad (3.58)$$

while

$$\omega\, \partial A/\partial\varphi = B_0\omega s \cos(\varphi-\omega(s)t). \qquad (3.59)$$

For the purpose of making estimates, suppose that $\omega(s)$ is a reasonably smooth function, and let

$$\omega_0 = \max|\omega(s)|, \qquad \omega_0/s_0 = \max|\omega'(s)|. \qquad (3.60)$$

It is clear that $\lambda\nabla^2 A$ is negligible compared with $\omega\,\partial A/\partial\varphi$ provided the coefficients of both the cosine and the sine terms on the right of (3.58) are small compared with the coefficient $B_0\omega s = O(B_0\omega_0 s_0)$ of the cosine term in (3.59); this leads to the conditions

$$\omega_0 t \ll R_m \quad \text{and} \quad \omega_0 t \ll R_m^{1/2}, \qquad (3.61)$$

where $R_m = \omega_0 s_0^2/\lambda$ is the appropriate magnetic Reynolds number. If $R_m \ll 1$, then the more stringent condition is $\omega_0 t \ll R_m$, so that diffusion is negligible during only a small fraction of the first rotation period. If $R_m \gg 1$ however, the condition $\omega_0 t \ll R_m^{1/2}$ is the more stringent, but, even so, diffusion is negligible during a large number of rotations. Note that in this case the field is greatly intensified before diffusion intervenes; from (3.57), when $R_m \gg 1$ and $\omega_0 t = O(R_m^{1/2})$, B_φ is dominated by the part linear in t which gives

$$|\mathbf{B}|_{\max} = O(R_m^{1/2})B_0. \qquad (3.62)$$

This gives an estimate of the maximum value attained by $|\mathbf{B}|$ before the process is influenced by diffusion.

The estimates (3.61b) and (3.62) differ from estimates obtained by E. N. Parker[3] (1963) who observed that when $R_m \gg 1$, the radial distance between zeros of the field B_φ is (from (3.57)) $\Delta s = O(s_0/\omega_0 t)$, so that the time characteristic of field diffusion is

$$t_d = O((\Delta s)^2/\lambda) = O(s_0^2/\lambda\omega_0^2 t^2). \tag{3.63}$$

Parker argued that diffusion should be negligible for all $t \ll t_d$, a condition that becomes

$$\omega_0 t \ll R_m^{1/3}, \tag{3.64}$$

in contrast to (3.61*b*); the corresponding maximum value of $|\mathbf{B}|$ becomes

$$|\mathbf{B}|_{\max} = O(R_m^{1/3})B_0, \tag{3.65}$$

in contrast to (3.62). One can equally argue however that diffusion should be negligible for so long as $t_d \ll t_{in}$ where t_{in} is a time characteristic of the induction process; defining this by

$$t_{in} = |\mathbf{B}|/|\nabla \wedge (\mathbf{u} \wedge \mathbf{B})| \tag{3.66}$$

explicit evaluation from (3.57) gives

$$t_{in} = O(\omega_0^{-1}). \tag{3.67}$$

The condition $t_d \ll t_{in}$ restores the estimates (3.61*b*) and (3.62). The difference between $O(R_m^{1/2})$ and $O(R_m^{1/3})$ is not very important for modest values of R_m, but becomes significant if $R_m > 10^6$, say.

The ultimate steady state

It is to be expected that when $t \to \infty$ the solution of (3.54) will settle down to a steady form $f_1(s)$ satisfying

$$\frac{i\omega(s)}{\lambda} f_1 = \left(\frac{1}{s}\frac{d}{ds}s\frac{d}{ds} - \frac{1}{s^2}\right) f_1, \tag{3.68}$$

[3] Weiss (1966) also obtained estimates similar to those obtained by Parker, but for a velocity field consisting of a periodic array of eddies; the estimates (3.61) and (3.62) should in fact apply to this type of situation also. The numerical results presented by Weiss for values of R_m up to 10^3 are consistent with (3.65) rather than (3.62); but it is difficult to be sure that the asymptotic regime (for $R_m \to \infty$) has been attained. The matter perhaps merits further study.

and that, provided $\omega(s) \to 0$ as $s \to \infty$, the outer boundary condition should be that the field at infinity is undisturbed, i.e.

$$f_1(s) \sim s \quad \text{as} \quad s \to \infty. \tag{3.69}$$

The situation is adequately illustrated by the particular choice[4]

$$\frac{\omega(s)}{\lambda} = \begin{cases} k_0^2 & (s < s_0), \\ 0 & (s > s_0). \end{cases} \tag{3.70}$$

where k_0 is constant. The solution of the problem (3.68)–(3.70) is then straightforward:

$$f_1(s) = \begin{cases} s + Cs^{-1} & (s > s_0), \\ DJ_1(ps) & (s < s_0), \end{cases} \tag{3.71}$$

where $p = (1 - \mathrm{i})k_0/\sqrt{2}$. The constants C and D are determined from the conditions that B_s and B_φ (and hence f_1 and f_1') should be continuous across $s = s_0$; these conditions yield

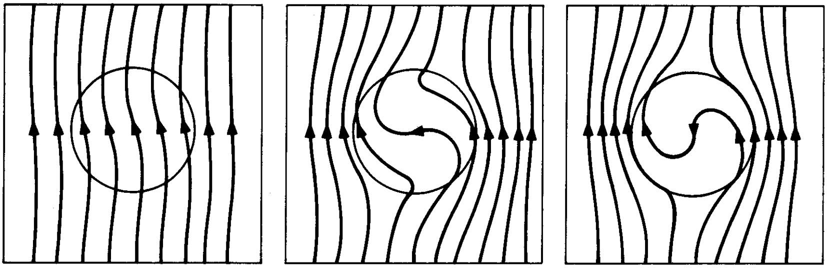
$$D = \frac{2}{pJ_0(ps_0)}, \qquad C = \frac{s_0(2J_1(ps_0) - ps_0J_0(ps_0))}{pJ_0(ps_0)}, \tag{3.72}$$

and this completes the formal determination of $f_1(s)$ from which A and hence B_s and B_φ may be determined. The **B**-lines are drawn in fig. 3.3 for $R_m = 1$, 10 and 25; note the increasing degree of distortion as R_m increases.

The nature of the solution is of particular interest when $R_m \gg 1$; in this situation $|ps_0| \gg 1$, and the asymptotic formulae

$$J_0(z) \sim (2/\pi z)^{1/2} \sin(z + \pi/4), \quad J_1(z) \sim -(2/\pi z)^{1/2} \cos(z + \pi/4) \tag{3.73}$$

[4] Note that for this discontinuous choice, there can be no initial phase of the type discussed above; diffusion must operate as soon as the motion commences to eliminate the incipient singularity in the magnetic field on $s = s_0$. Note also that it is only the variation with s of the ratio ω/λ that affects the ultimate field distribution; in particular if $\omega = 0$ for $s > s_0$, then λ may be an arbitrary (strictly positive) function of s for $s > s_0$ without affecting the situation.



(a) $R_m = 1$ (b) $R_m = 10$ (c) $R_m = 25$

Fig. 3.3 Ultimate steady state field distributions for three values of R_m; when R_m is small the field distortion is small, while when R_m is large the field tends to be excluded from the rotating region. The sense of the rotation is anticlockwise. [Curves computed by R. H. Harding.]

may be used both in (3.72) with $z = ps_0$, and in (3.71)[5] with $z = ps$. After some simplification the resulting formula for A from (3.53) takes the form

$$A \sim \begin{cases} B_0\left(s - \dfrac{s_0^2}{s}\right)\sin\varphi + \dfrac{2B_0 s_0^2}{k_0 s}\sin\left(\varphi + \dfrac{\pi}{4}\right) & (s > s_0), \\ \left(\dfrac{2B_0}{k_0}\right)\exp\left(-\dfrac{k_0(s_0 - s)}{\sqrt{2}}\right)\sin\left(\varphi + \dfrac{k_0(s_0 - s)}{\sqrt{2}} + \dfrac{\pi}{4}\right) & (s < s_0). \end{cases} \tag{3.74}$$

In the limit $R_m = \infty (k_0 = \infty)$, this solution degenerates to

$$A \sim \begin{cases} B_0\left(s - \dfrac{s_0^2}{s}\right)\sin\varphi & (s > s_0), \\ 0 & (s < s_0). \end{cases} \tag{3.75}$$

The lines of force $A = \text{cst.}$ are then identical with the streamlines of an irrotational flow past a cylinder. In this limit of effectively infinite conductivity the field is totally excluded from the rotating region $s < s_0$; the tangential component of field suffers a discontinuity across the surface $s = s_0$ which consequently supports a current sheet.

This form of field exclusion is related to the skin effect in conventional electromagnetism. Relative to axes rotating with

[5] There is a small neighbourhood of $s = 0$ where, strictly, the asymptotic formulae (3.73) may not be used, but it is evident from the nature of the result (3.74) that this is of no consequence.

angular velocity ω_0, the problem is that of a field rotating with angular velocity $-\omega_0$ outside a cylindrical conductor. (As observed in the footnote on p. 58, the conductivity is irrelevant for $s > s_0$ in the steady state so that we may treat the medium as insulating in this region.) A rotating field may be decomposed into two perpendicular components oscillating out of phase, and at high frequencies these oscillating fields are excluded from the conductor. The same argument of course applies to the rotation of a conductor of any shape in a magnetic field, when the medium outside the conductor is insulating; at high rotation rate, the field is always excluded from the conductor when it has no component parallel to the rotation vector.

The additional terms in (3.74) describe the small perturbation of the limiting form (3.75) that results when the effects of finite conductivity in the rotating region are included. The field does evidently penetrate a small distance δ into this region, where

$$\delta = O(k_0^{-1}) = O(R_m^{-1/2}) s_0. \tag{3.76}$$

The current distribution (confined to the region $s < s_0$) is now distributed through a layer of thickness $O(\delta)$ in which the field falls to an effectively zero value. The behaviour is already evident in the field line pattern for $R_m = 25$ in fig. 3.3(c).

The intermediate phase

The full time-dependent problem described by (3.54) and (3.55) has been solved by R. L. Parker (1966) for the case of a rigid body rotation $\omega = \omega_0$ in $s < s_0$ and zero conductivity ($\lambda = \infty$) in $s > s_0$. In this case, there are no currents for $s > s_0$ so that $\nabla^2 A = 0$ in this region (for all t), and hence (cf. 3.71a)

$$f(s, t) = s + C(t) s^{-1}. \tag{3.77}$$

This function satisfies

$$f + s\, \partial f/\partial s = 2s, \tag{3.78}$$

and so continuity of f and $\partial f/\partial s$ across $s = s_0$ provides the boundary condition

$$f + s_0\, \partial f/\partial s = 2s_0 \quad \text{on } s = s_0 \tag{3.79}$$

for the solution of (3.54). Setting $f = f_1(s) + g(s, t)$, the transient function $g(s, t)$ may be found as a sum of solutions separable in s

and t. The result (obtained by Parker by use of the Laplace transform) is

$$g(s, t) = \sum_{n=1}^{\infty} \frac{4s_0 \exp(-i\omega_0 t - (\omega_0 t \sigma_n^2/R_m)) J_1(\sigma_n s/s_0)}{\sigma_n^2(1+(\sigma_n^2/R_m))J_1(\sigma_n)}, \tag{3.80}$$

where σ_n is the n^{th} zero of $J_0(\sigma)$.

The lines of force $A = \text{cst.}$ as computed by Parker for $R_m = 100$ and for various values of $\omega_0 t$ during the first revolution of the cylinder are reproduced in fig. 3.4. Note the appearance of closed

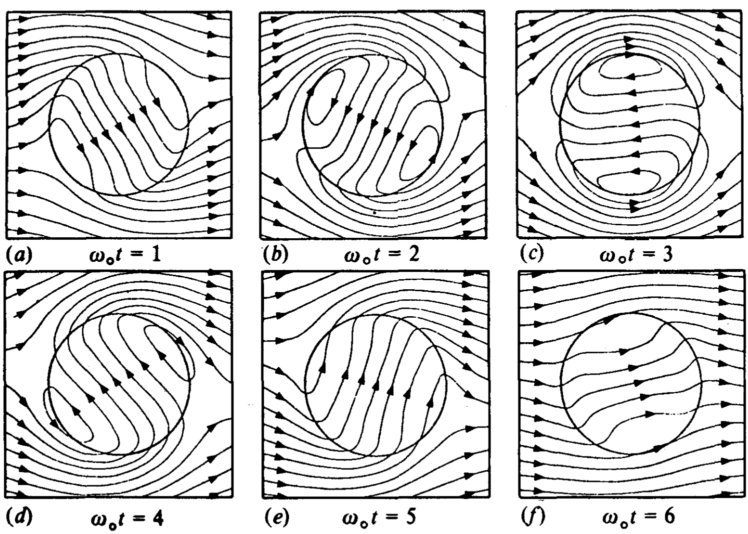


Fig. 3.4 Development of lines of force $A = \text{cst.}$ due to rotation of cylinder with angular velocity ω_0; the sense of rotation is clockwise. The sequence (a)–(f) shows one almost complete rotation of the cylinder, with magnetic Reynolds number $R_m = \omega_0 a^2/\lambda = 100$. (From Parker, 1966.)

loops[6] when $\omega_0 t \approx 2$ and the subsequent disappearance when $\omega_0 t \approx 5$; this process is clearly responsible for the destruction of flux within the rotating region. The process is repeated in subsequent revolu-

[6] This manifestation of diffusion effects at a time rather earlier than the discussion preceding (3.61) would suggest is presumably attributable (at least in part) to the assumed discontinuity of ω at $s = s_0$.

tions, flux being repeatedly expelled until the ultimate steady state is reached. Parker has in fact shown that, when $R_m = 100$, closed loops appear and disappear during each of the first fifteen revolutions of the cylinder but not subsequently. He has also shown that the number of revolutions during which the closed loop cycle occurs increases as $R_m^{3/2}$ for large R_m; in other words it takes a surprisingly long time for the field to settle down in detail to its ultimate form.

3.9. Flux expulsion for general flows with closed streamlines

A variety of solutions of (3.38) have been computed by Weiss (1966) for steady velocity fields representing either a single eddy or a regular array of eddies. The computed lines of force develop in much the same way as described for the particular flow of the previous section, closed loops forming and decaying in such a way as to gradually expel all magnetic flux from any region in which the streamlines are closed. The following argument (Proctor, 1975), analogous to that given by Batchelor (1956) for vorticity, shows why the field must be zero in the final steady state in any region of closed streamlines in the limit of large R_m (i.e. $\lambda \to 0$).

We consider a steady incompressible velocity field derivable from a stream function $\psi(x, y)$:

$$\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x, 0). \tag{3.81}$$

In the limit $\lambda \to 0$ and under steady conditions, (3.38) becomes $\mathbf{u} \,.\, \nabla A = 0$, and so A is constant on streamlines, or equivalently

$$A = A(\psi). \tag{3.82}$$

If λ were exactly zero, then any function $A(x, y)$ of the form (3.82) would remain steady. However, the effect of non-zero λ is to eliminate any variation in A across streamlines. To see this, we integrate the exact steady equation

$$\mathbf{u} \,.\, \nabla A \equiv \nabla \,.\, (\mathbf{u}A) = \lambda \nabla^2 A \tag{3.83}$$

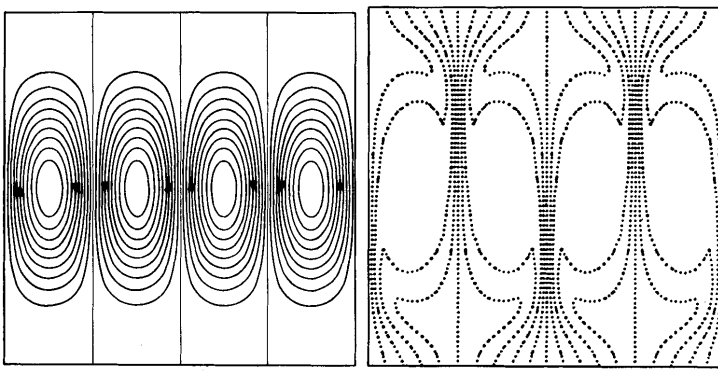
over the area inside any closed streamline C. Since $\mathbf{n} \,.\, \mathbf{u} = 0$ on C, where $\mathbf{n}$ is normal to C, the left-hand side integrates to zero, while the right-hand side becomes (with s representing arc length)

$$\oint_C \lambda \mathbf{n} \,.\, \nabla A \, \mathrm{d}s = \lambda A'(\psi) \oint_C (\partial\psi/\partial n) \, \mathrm{d}s = \lambda K_C A'(\psi), \tag{3.84}$$

where K_C is the circulation round C. It follows that $A'(\psi) = 0$; hence $A = \text{cst.}$, and so $\mathbf{B} \equiv 0$ throughout the region of closed streamlines.

We have seen in § 3.8 that the flux does in fact penetrate a distance $\delta = O(l_0 R_m^{-1/2})$ into the region of closed streamlines, where l_0 is the scale of this region. Within this thin layer, the diffusion term in (3.83) is $O(\lambda A/\delta^2)$ and this is of the same order of magnitude as the convective term $\mathbf{u} \, . \, \nabla A = O(u_0 A/l_0)$.

The phenomenon of flux expulsion has interesting consequences when a horizontal band of eddies acts on a vertical magnetic field. Fig. 3.5, reproduced from Weiss (1966), shows the steady state field



(*a*) (*b*)

Fig. 3.5 Concentration of flux into ropes by a convective layer ($R_m = 10^3$); (a) streamlines $\psi = \text{cst.}$ where ψ is given by (3.85); (b) lines of force of the resulting steady magnetic field. (From Weiss, 1966.)

structure when

$$\psi(x, y) = -(u_0/4\pi l_0)(1-(4y^2/l_0^2))^4 \sin(4\pi x/l_0), \qquad (3.85)$$

and when $R_m = u_0 l_0/\lambda = 10^3$. The field is concentrated into sheets of flux along the vertical planes between neighbouring eddies. These sheets have thickness $O(R_m^{-1/2})$, and the field at the centre of a sheet is of order of magnitude $R_m^{1/2} B_0$ where B_0 is the uniform vertical field far from the eddies; this result follows since the total vertical magnetic flux must be independent of height. This behaviour is comparable with that described by the flux rope solution (3.25), particularly if $\beta = 0$ in that solution, when the 'rope' becomes a 'sheet'.

3.10. Expulsion of poloidal fields by meridional circulation

We consider now the axisymmetric analogue of the result obtained in the preceding section. Let $\mathbf{u}$ be a steady poloidal axisymmetric velocity field with Stokes stream function $\psi(s, z)$ and let $\mathbf{B}$ be a poloidal axisymmetric field with flux-function $\chi(s, z, t)$. Then from (3.44), we have

$$\mathrm{D}\chi/\mathrm{D}t \equiv \partial\chi/\partial t + \mathbf{u}\,.\,\nabla\chi = \lambda\,\mathrm{D}^2\chi, \tag{3.86}$$

with the immediate consequence that when $\lambda = 0$, in Lagrangian notation, $\chi(\mathbf{x}, t) = \chi(\mathbf{a}, 0)$. In a region of closed streamlines in meridian planes, steady conditions are therefore possible in the limit $R_m = \infty$ only if

$$\chi = \chi(\psi(s, z)). \tag{3.87}$$

Again, as in the plane case, the effect of weak diffusion is to eliminate any variation of χ as a function of ψ. This may be seen as follows.

Using $\nabla\,.\,\mathbf{u} = 0$ and the representation (3.47) for $\mathrm{D}^2\chi$, the exact steady equation for χ may be written

$$\nabla\,.\,(\mathbf{u}\chi) = \lambda\nabla\,.\,(\nabla\chi - 2s^{-1}\chi\mathbf{i}_s). \tag{3.88}$$

Let C be any closed streamline in the $s - z$ (meridian) plane, and let S and $\mathcal{T}$ be the surface and interior of the torus described by rotation of C about Oz. Then $\mathbf{u}\,.\,\mathbf{n} = 0$ on S, and integration of (3.88) throughout $\mathcal{T}$ leads to

$$\int_S \mathbf{n}\,.\,\nabla\chi\;\mathrm{d}S = \int_S 2s^{-1}\chi\mathbf{i}_s\,.\,\mathbf{n}\;\mathrm{d}S. \tag{3.89}$$

With $\chi = \chi(\psi)$, and noting that $\mathbf{i}_s\,.\,\mathbf{n} = \mathbf{i}_z\,.\,\mathbf{t}$ on S, where $\mathbf{t}$ is a unit vector tangent to C, (3.89) gives

$$\chi'(\psi)\int \mathbf{n}\,.\,\nabla\psi\;\mathrm{d}S = 4\pi\chi\oint_C \mathbf{i}_z\,.\,\mathrm{d}\mathbf{x} = 4\pi\chi\oint_C \mathrm{d}z = 0, \tag{3.90}$$