AM 275 - Magnetohydrodynamics: Lecture Notes

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Lecture 2: Hydrodynamics Review

What is a fluid?

- Its flows!
- it deforms continuously

Categorization of fluids:

- Compressible v. incompressible
- viscous v. inviscid
- + many more

Eularian

Rate of change at a given point, no bother for where the fluid goes.

$$\frac{\partial}{\partial t}(\cdot)$$

Lagrangian

Follows the particle, introduces the advection term

$$\boldsymbol{u} \cdot \nabla (\cdot)$$

Mass Conservation

In order to conserve mass we consider an arbitrary eularian volume (i.e. the volume doesn't move with the flow). We then find the total mass which is equal to the integral of the density over the volume, and then consider the flux of mass through the boundary (change in mass over time). Using the divergence theorem, we then have a conservation equation for mass.

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_D \rho dV &= \int_{\partial D} \rho \boldsymbol{u} \cdot \eta dA \\
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} &= 0 \\
\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} &= 0 \\
\frac{D\rho}{Dt} &= -\rho \nabla \cdot \boldsymbol{u}
\end{aligned}
$$

More importantly, if we consider an incompressible fluid, i.e. $\rho = \rho_0$, we have very specifically,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

Stresses

Stresses can be divided into two catagories, body forces and surface forces. Body forces are forces such as gravity and the electric force, which surface forces are forces such as normal force and friction.

Newtons Second Law

Newton's second law

$$\frac{\partial p}{\partial t} = \sum_i F_i$$

where p here is the momentum of a fluid parcel. In actuality, the momentum can be written as $p = \int_D \rho \boldsymbol{u} dV$. So,

$$
\begin{aligned}
\frac{D}{Dt}\int_D \rho \boldsymbol{u} dV &= \int_D \rho \boldsymbol{g} dV + (\text{other body forces}) + \nabla \cdot \boldsymbol{\tau} \\
\rho \frac{D\boldsymbol{u}}{Dt} &= \rho F + \nabla \cdot \boldsymbol{\tau}
\end{aligned}
$$

where $\boldsymbol{\tau}$ is the stress tensor acting on the fluid parcel. Surface forces are then introduced into this stress tensor. First and foremost, surface pressure is introduced along the stress tensor.

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}$$

where σ_{ij} is the deviatoric stress tensor and is responsible for the off-diagonal components of the stress tensor. Some components are the velocity gradient tensors, $\frac{\partial u_i}{\partial x_j}$ and $\frac{\partial u_j}{\partial x_i}$. Each of these has a symmetric compoenent and an antisymmetric componenet.

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

The first term is labeled symmetric and denoted e_{ij} while the second component is the rotation componenet.

Lecture 3: Continuing Review of the Kinematic Equations

3.1 Obtaining the Navier-Stokes equation

Decomposition of the deviatoric stress tensor reveals a 4th order tensor with 81 components.

$$\boldsymbol{\sigma}_{ij} = \boldsymbol{A}_{ijkl}\boldsymbol{e}_{kl}$$

In order to reduce the complexity of the system, we make some assumptions about the tensor $\boldsymbol{A}_{ijkl}$. First, we state that this tensor must be isotropic, i.e. that it doesn't care about the direction of the stress with respect to the coordinate system it is in. We have,

$$\boldsymbol{A}_{ijkl} = \mu\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \mu'\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \mu''\boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk}$$

Next, we assume that this tensor must be symmetric. This reduces the complexity down to two coefficients, μ, the viscosity, and μ' which is the bulk viscosity.

In order to obtain the Navier-Stokes equation, we require the Stokes assumption which postulates that the diagonal components of the deviatoric stress tensor are zero, i.e. $\boldsymbol{\sigma}_{ii} = 0$.

$$\boldsymbol{\sigma}_{ij} = 2\mu\left(\boldsymbol{e}_{ij} - \frac{1}{3}\left(\nabla\cdot\boldsymbol{u}\right)\boldsymbol{\delta}_{ij}\right)$$
$$\boldsymbol{\tau}_{ij} = -p\boldsymbol{\delta}_{ij} + 2\mu\left(\boldsymbol{e}_{ij} - \frac{1}{3}\left(\nabla\cdot\boldsymbol{u}\right)\boldsymbol{\delta}_{ij}\right)$$

Therefore, when we take the divergence of this stress tensor we obtain the Navier-Stokes equation:

$$\rho\frac{D\boldsymbol{u}}{Dt} = \rho\boldsymbol{F} - \nabla p + \mu\left[\nabla^2\boldsymbol{u} + \frac{1}{3}\nabla\left(\nabla\cdot\boldsymbol{u}\right)\right]$$

Of course, when working in an incompressible framework (i.e. $\nabla\cdot\boldsymbol{u} = 0$), we have that part of the diffusive term disappears from the equation, resulting in the commonly used equation:

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{F} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{u} \tag{2}$$

Additional terms are included in this equation as necessary to model relevant physics of various fluid systems. For example, if in a rotating frame we include the coriolis force $2\Omega(\boldsymbol{e}_\Omega \times \boldsymbol{u})$, if some component of the fluid is stratified we need some buoyancy forcing $T/N^2\boldsymbol{e}_z$. And most relevant, there might be magnetic forces which affect the fluid, in which case we obtain the MHD equations.

3.2 Vorticity equation

Vorticity is a quantity related to the fluid field which can be very important to the scientific study of fluid dynamics. The vorticity is obtained by taking the curl of the velocity field.

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

The vorticity has an evolution-advection equation just as the velocity field does, and in fact the vorticity equation is obtained by taking the curl of the Navier-Stokes equations.

$$\nabla \times (2)$$

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\,\boldsymbol{u} + \boldsymbol{\omega}\,(\nabla \cdot \boldsymbol{u}) + \nabla \times \boldsymbol{F} - \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nu\nabla^2\boldsymbol{\omega}$$

If the flow is incompressible, one of the vortex stretching terms disappears. Generally, the first two right hand terms are vortex stretching/tilting/speed-up terms. Then the pressure and density gradient cross product is the baroclinicity term, the curl of $\boldsymbol{F}$ is the forcing of vorticity, and finally, we have a viscous diffusion of vorticity which behaves similarly to the diffusion of velocity.
Baroclinicity is perhaps the most unintuitive term in this equation, and it simply represents the creation of rotation in the fluid when there is a disalignment between the pressure and density gradients in the fluid.
Some fluid dynamicists prefer to study the vorticity equation, especially for rotating flows where voricites and cyclones are common phenomenon in the flow field.

3.3 Rotation

In the presence of rotation, the coriolis force becomes relevant as the motion of a fluid particle is deflected due to the rotation of the cordinate frame. That is, our equations are modified such that,

$$\frac{\partial \boldsymbol{q}}{\partial t}_F = \frac{\partial \boldsymbol{q}}{\partial t}_R + 2\boldsymbol{\Omega} \times \boldsymbol{q} - \boldsymbol{\Omega}^2 \boldsymbol{R}$$

This also introduces an additional term to the vorticity equation which looks like, $+\left(2\boldsymbol{\Omega} \cdot \nabla\right)\boldsymbol{u}$.

Lecture 4: Conservation of Energy and Maxwell's equations

4.1 Conservation of Energy

The equation of state chosen for a particular problem is a source of physics which affects the solutions of a given PDE. The incompressible equation of state is used very commonly as an equation of state. Another common one is the ideal gas law $pV = \rho RT$.

In order to understand the origin and importance of the equation of state, the laws of thermodynamics are needed.

The first law of thermodynamics states,

$$\frac{\partial e}{\partial t} = \frac{\partial W}{\partial t} + \frac{\partial Q}{\partial t}$$

where e is the internal energy, W is work done on the system, and Q is heat flux into the system. However, for a fluid flow taken from a Lagrangian perspective, we must modify this law of thermodynamics. It must include the energy given by the velocity field.

$$\frac{D}{Dt}\int_D \rho\left(e + \frac{1}{2}\boldsymbol{u}^2\right) dV = \int_D \rho \boldsymbol{F} \cdot \boldsymbol{u} dV + \int_{\partial D} \boldsymbol{\tau} \cdot \boldsymbol{u} dS - \int_{\partial D} \boldsymbol{q} \cdot dS$$
$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}\boldsymbol{u}^2\right) = \rho \boldsymbol{F} \cdot \boldsymbol{u} + \nabla(\boldsymbol{\tau} \cdot \boldsymbol{u}) - \nabla \cdot \boldsymbol{q}$$

Next, we obtain a mechanical energy equation by dotting $\boldsymbol{u}$ by the Navier-Stokes equation and adding

$\boldsymbol{u}^2/2 \cdot \frac{D\rho}{Dt}$

$$\frac{D\rho\boldsymbol{u}^2/2}{Dt} = \rho\boldsymbol{F}\cdot\boldsymbol{u} - \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) + \dots$$
$$\boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) = \Phi = 2\mu\left[\boldsymbol{e}_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\boldsymbol{\delta}_{ij}\right]$$

Finally, we obtain an energy equation with a positive definite dissipation term Φ which acts purely to remove energy from the system.

$$\rho\frac{De}{Dt} = -\nabla\cdot\boldsymbol{q} - p\,(\nabla\cdot\boldsymbol{u}) + \Phi$$

The Second law of Thermodynamics also plays an important role in the conservation of energy. The ssecond law makes statements about the entropy of a system, S.

$$dS = \frac{dq}{T}$$
$$TdS = de + pdV$$
$$T\frac{dS}{dt} = \frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}$$
$$\rho\frac{DS}{Dt} = -\frac{\nabla\cdot\boldsymbol{q}}{T} + \frac{k}{T^2}|\nabla T|^2 + \mu\frac{\Phi}{T}$$

Essentially, since both $|\nabla T|^2$ and Φ are positive definite terms and their coefficients are positive definite, it must be that the entropy of a system can only increase "on average" (curse the statisticians).

4.2 Introducing Maxwell's Equations

Electricity and Magnetism are very closely related to one another and governed by a main set of governing equations. The main variables which we consider are a position vector, $\boldsymbol{x}$, a velocity field, $\boldsymbol{u}$, density ρ, pressure p, time t, temperature T, magnetic field (magnetic flux density) $\boldsymbol{B}$, magnetic field strength $\boldsymbol{H}$, electric field $\boldsymbol{E}$, electric displacement $\boldsymbol{D}$, electric current density $\boldsymbol{j}$, and charge density ρ_e.
Alongside these variables, we have constants describing components of electropmangetism: permitivity ε, permeability μ, and conductivity σ. Permitivity describes the charge requirement for a specific electric field, i.e. large ε implies a larger charge is needed for a specific electric field. Permeability describes the current requirement for a specific magnetic field, i.e. large μ implies a smaller current is needed to obtain a specific magnetic field.

Consitutive relationships describe the relationships between specific electromagnetic quantities.

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu}, \text{ for an isotropic permeability}$$
$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \text{ for an isotropic permitivity}$$

where generally, we take $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ where q_0 is taken from a vacuum.

Now we write Maxwell's equations in their differential form:

$$\nabla \cdot \boldsymbol{B} = 0, \text{ Gauss' law for magnetism} \qquad \nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\varepsilon_0}, \text{ Gauss' law}$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \text{ Faraday's law} \qquad \nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right), \text{ Ampere's Law of induction}$$

They can be written in their integral form as well:

$$\oint_{\partial D} \boldsymbol{B} \cdot \boldsymbol{dA} = 0 \qquad \oint_{\partial D} \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0}$$
$$\oint_{L} \boldsymbol{E} \cdot \boldsymbol{dL} = -\frac{\partial \phi_B}{\partial t} \qquad \oint_{L} \boldsymbol{B} \cdot \boldsymbol{dL} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

where $\phi_B = \int_S \boldsymbol{B} \cdot \boldsymbol{dA}$ is the total magnetic fluid, and $\phi_E = \int_S \boldsymbol{E} \cdot \boldsymbol{dA}$ is the total electric flux.

We cover their derivations in a brief sense also. Consider a positive point charge which creates an electric field. This imposes a force acting on any other point charge in the field. This force is called Coulomb force given by $F = q_1 q_2/(4\pi\varepsilon r^2)$. Thus we have a given electric field of strength $E/q = q_1/(4\pi\varepsilon r^2)$. We obtain the total electric flux ϕ_E

$$d\phi_E = \boldsymbol{E} \cdot \boldsymbol{dA}$$
$$\phi_E = \oint_S \frac{q_1}{4\pi\varepsilon r^2} \cdot \boldsymbol{dA}$$
$$\phi_E = \frac{q_1}{4\pi\varepsilon r^2} \oint_S \boldsymbol{dA}$$
$$\phi_E = \frac{q_e}{\varepsilon}$$

where q_e in the final equation is given by the sum of all point charges enclosed in the closed volume, i.e. $q_e = \sum_i q_i$. Notice that q_e can be thought of as mass for point charges, i.e. the integral of the charge density equals the total charge similar to how the integral of mass density equals the total mass. It can be represented as a sum of point charges because point cahrges are discrete and do not usually exist in a continuum.

Finally, using the divergence theorem,

$$\oint_S \boldsymbol{E} \cdot \boldsymbol{dA} = \int_V \nabla \cdot \boldsymbol{E} dV$$
$$q_e = \int_V \rho_E dV$$
$$\nabla \cdot \boldsymbol{E} = \frac{\rho_E}{\varepsilon}$$

Similarly, the same proof holds for Gauss' law of magnetism, only that monopoles do not exist in magnetic fields, i.e. every source must have a sink. Therefore, for an arbitraty volume it must be that the divergence of the magnetic field must be zero:

$$\nabla \cdot \boldsymbol{B} = 0$$

Lecture 5: Derivation of Maxwell's Equations: Continued

5.1 Faraday's Law of Induction

The laws of electrodynamics are empiracle. Faraday realized that the EMF, proportional to $\frac{\partial \boldsymbol{B}}{\partial t}$ and also the area of the coil. This led them to deduce that EMF should be proportional to $\frac{\partial \phi_B}{\partial t}$. Note that EMF represents the amount of work done per unit charge to move a charge from one place to another, i.e. the electric potential difference. It has the units of Nm/C (Newton meters per Coulomb).

$$\begin{aligned} \text{EMF} = \oint_C \boldsymbol{E} \cdot \boldsymbol{dL} &= -\frac{\partial \phi_B}{\partial t} \\ &= -\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot \boldsymbol{dA} \end{aligned}$$

where here the RHS integral is not necessarily over a closed surface (because the same integral over a closed surface must be zero).

5.2 Ampere's Law

If there is a current moving through a wire, imagine a cross section going through the wire (i.e. going through the page), there is a magnetic field around the wire. The magnetic field can be described using the following integral form.

$$\oint_L \boldsymbol{H} \cdot \boldsymbol{dL} = \boldsymbol{i}$$
$$\oint_L \frac{\boldsymbol{B}}{\mu_0} \cdot \boldsymbol{dL} = \boldsymbol{i}$$
$$\oint_L \boldsymbol{B} \cdot \boldsymbol{dL} = \mu_0 \boldsymbol{i}$$
$$\int_S \nabla \times \boldsymbol{B} \cdot \boldsymbol{dA} = \mu_0 \oint_A \boldsymbol{j} \cdot \boldsymbol{dA}$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

Here is where Maxwell's contribution to Ampere's law is notable. Ampere assumed that $\nabla \cdot \boldsymbol{j} = 0$, where Maxwell noticed that in some scenarios, this is not necessarily true. Thus, he modified the equation to include displacement currents.

$$\frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j}_0$$
$$\frac{\partial \varepsilon \boldsymbol{E}}{\partial t} = \boldsymbol{j}_0$$
$$\nabla \times \boldsymbol{B} = \boldsymbol{j} + \boldsymbol{j}_0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

This was significant because $\mu_0 \varepsilon_0 \propto 1/c^2$ (where c is the speed of light), and this implied connections to electromagnetic radiation (doublecheck this). More importantly, these equations are linear and relativistically correct (not sure what exactly this means).

5.3 Units of Electrodynamics (c.f. Priest p436)

Electrostatis units are denoted "esu". Electromagnetic units are denoted "emu": e, m respectively. The Gaussian cgs system utilizes the standard units of centimeters, grams, seconds, in addition to the electrostatic units statcoulomb, q, and electromagnetic units, "abAmp". The Gaussian cgs representation of the governing equations often have an extra factor of 4π in the equations.

In general, we will use the Rationalized MKS system (standard SI system). Where the default length, mass, time, is given in meters, kilograms, and seconds. In addition, current is given in amps. The variables, $\mu_0 = 4\pi \cdot 10^{-7} NA^{-2}$ and $\varepsilon_0 = 8.8 \cdot 10^{-12} A^2 s^2 N^{-1} m^{-1}$ have dimension, charges are given in Coulombs, forces are given in Newtons. The magnetic field is given by Teslas $T = NA^{-1}m^{-1}$, and the electric field is given by V/m (Volts per meter).

5.4 From Maxwell's Equation to MHD

Generally, for MHD we will be working in a non-relativistic approximation (i.e. typical velocities are much less than the speed of light, $U \ll c$). Let us consider the equations and their typical unit scales,

$$\begin{aligned}
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\frac{\boldsymbol{E}}{L} &\ \frac{\boldsymbol{B}}{T} \\
\nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \\
\frac{\boldsymbol{B}}{L} &\ \frac{1}{c^2}\frac{\boldsymbol{B}L}{T^2}
\end{aligned}$$

If we manipulate the last line of this equation, we find that $L^2/T^2? = c^2$ is the leading balance of Ampere's law, and therefore we neglect the relativistic term of Maxwell's equations.
In order to connect Maxwell's equations to fluid dynamics, we must consider the Lorentz force.

$$\begin{aligned}
\boldsymbol{F} &= q\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) \\
\frac{d\boldsymbol{F}}{dV} &= \frac{dq}{dV}\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) \\
\frac{d\boldsymbol{F}}{dV} &= \rho_E\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) \\
\frac{d\boldsymbol{F}}{dV} &= \rho_E \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} \\
\boldsymbol{F} &= \int_V \rho_E \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} dV
\end{aligned}$$

Next we must consider Ohm's Law, which describes the current and electric field as movign with the conductor (Lagrangian perspective) (denoted with $'$).

$$\boldsymbol{j}' = \sigma \boldsymbol{E}'$$

and thus we are able to simplify the equations to become,

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}$$

$$\boldsymbol{j}' = \boldsymbol{j}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

Taking the curl of this equation leads to the following,

$$\begin{aligned}
\nabla \times \left(\frac{\nabla \times \boldsymbol{B}}{\mu_0 \sigma} \right) &= \nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\nabla \times \eta \nabla \times \boldsymbol{B} &= -\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\frac{\partial \boldsymbol{B}}{\partial t} &= -\nabla \times (\eta \nabla \times \boldsymbol{B}) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})
\end{aligned}$$

which is the induction equation. If we take η to be constant, we can write with the derivative identity, $\nabla \times \nabla \times \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

Notice that we obtain an equation solely for $\boldsymbol{B}$ which has taken into account all of Maxwell's equations. This tells us that we really only have to care about the magnetic field, and can obtain the electric field as a consequence of our solution. For example, $\boldsymbol{j} = \nabla \times \boldsymbol{B}/\mu_0$, $\boldsymbol{E} = \boldsymbol{j}/\sigma - \boldsymbol{u} \times \boldsymbol{B}$, and $\rho_E = \varepsilon_0(\nabla \cdot \boldsymbol{E})$.

We can interpret the terms in this equation as well. On the LHS we have a typical rate of change of the magnetic field. On the RHS we have first the induction term, and the diffusion of the magnetic field $\boldsymbol{B}$.

We also notice that the linearity of this equation depends primarily on the relationship between $\boldsymbol{u}$ and $\boldsymbol{B}$. If, for example, $\boldsymbol{u}$ is a function of $\boldsymbol{B}$ then the induction equation is not linear. If the induction equation is linear, then the equation is generally regarded as the kinematic induction equation. If the equation is nonlinear, then it is generally regarded as a dynamic equation of induction.

In general, the Lorentz force is vital to determining which dynamical regime we are in for the velocity and magnetic fields. Consider again the Lorentz force,

$$\boldsymbol{F} = \rho_E \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}$$

where the first RHS term is the electrostatic component and the second RHS term is the magnetic component. Generally, we compare the order of each term in the equation.

$$\begin{aligned}
\frac{|\rho_E \boldsymbol{E}|}{|\boldsymbol{j} \times \boldsymbol{B}|} &\propto \frac{|\varepsilon_0 \nabla \cdot \boldsymbol{E}\boldsymbol{E}|}{|(\nabla \times \boldsymbol{B})\boldsymbol{B}/\mu_0|} \\
&\propto \frac{\varepsilon_0 \mu_0 \boldsymbol{E}^2/L}{\boldsymbol{B}^2/L} \\
&\propto \varepsilon_0 \mu_0 \left(\frac{L}{T}\right)^2 = \frac{U^2}{c^2}
\end{aligned}$$

Therefore we are able to deduce that the Lorentz force in a non-relativistic regime, can be approximated as:

$$\begin{aligned}
\boldsymbol{F} &\propto \boldsymbol{j} \times \boldsymbol{B} \\
&\propto \frac{1}{\mu_0}(\nabla \times \boldsymbol{B})\boldsymbol{B}
\end{aligned}$$

With this Lorentz force as a body force, we write the Navier Stokes Equation

$$\begin{aligned}
\rho\frac{D\boldsymbol{u}}{Dt} &= -\nabla p + \mu \left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u})\right] + \rho \boldsymbol{F} + \rho\,(\boldsymbol{j} \times \boldsymbol{B}) \\
\frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{u}) &= 0 \\
\frac{De}{Dt} &= \dots \\
&\text{An equation of state} \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}
\end{aligned}$$