AM 275 - Magnetohydrodynamics: Homework 1

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Problem 1:

Show that

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} + p e_{kk} - 2\mu \left[e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \right]^2 .$$

Proof. First, we begin with the derivative identity

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

and in order to simplify this statement, take τ_i to be the i-th row vector of τ, we have:

$$\sum_i u_i \nabla \cdot \tau_i = \sum_i \nabla \cdot u_i \tau_i - \tau_i \cdot \nabla u_i$$

Already we have shown the first RHS term originates from the derivative identity, whereas the other terms must originate from $-\sum_i \tau_i \cdot \nabla u_i$. Thus, we investigate this term in more detail.

$$-\sum_i \tau_i \cdot \nabla u_i = \sum_i \left[p + \frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right] \delta_{ij} \cdot \nabla u_i - 2\mu e_i \cdot \nabla u_i$$

where e_{kk} is written as $\nabla \cdot \boldsymbol{u}$ and e_i is the i-th row of e (as in e_{ij}). Notice that $\sum_i \delta_{ij} \cdot \nabla u_i = \nabla \cdot \boldsymbol{u}$, and therefore,

$$\begin{aligned} -\sum_i \tau_i \cdot \nabla u_i &= \left[p + \frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right] (\nabla \cdot \boldsymbol{u}) - 2\mu \sum_i e_i \cdot \nabla u_i \\ &= p(\nabla \cdot \boldsymbol{u}) + \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 - 2\mu \sum_i e_i \cdot \nabla u_i \end{aligned}$$

Thus we recover the second RHS term, pe_{kk}. Now we must show the rest of $-\sum_i \tau_i \cdot \nabla u_i$ recovers the last term of the RHS. We write the decomposition of e_i.

$$\begin{aligned}
-2\mu \sum_i e_i \cdot \nabla u_i &= -\mu \sum_i \left(\nabla u_i + \frac{\partial \boldsymbol{u}}{\partial x_i} \right) \cdot \nabla u_i \\
&= -\mu \sum_i |\nabla u_i|^2 + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\
&= -\mu |\nabla \boldsymbol{u}|^2 - \mu \sum_i \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\
&= -\mu |\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_j}{\partial x_i}
\end{aligned}$$

Now we must show by the transitive propery that,

$$\frac{2}{3}\mu (\nabla \cdot \boldsymbol{u})^2 - \mu |\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} = -2\mu \left[e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \right]^2_{ll}$$

We begin by writing the double inner product of these second order tensors (necessary in order to obtain a scalar) (also sorry about the indices, I couldn't decide which letters I wanted to stick with in the long run)

$$\begin{aligned}
-2\mu \left[e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \right]^2_{ll} &= -2\mu \left[(e_{ij}^2)_{ll} - \frac{2}{3} (\nabla \cdot \boldsymbol{u}) e_{ll} + \frac{1}{9} (\nabla \cdot \boldsymbol{u})^2 \delta_{ll} \right] \\
&= -2\mu \left[(e_{im} \cdot e_{mj})_{ll} - \frac{2}{3} (\nabla \cdot \boldsymbol{u})^2 + \frac{1}{3} (\nabla \cdot \boldsymbol{u}^2) \right] \\
&= -\frac{\mu}{2} \left(\frac{\partial u_i}{\partial x_m} \frac{\partial u_m}{\partial x_j} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} + \frac{\partial u_i}{\partial x_m} \frac{\partial u_j}{\partial x_m} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_j}{\partial x_m} \right)_{ll} + \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 \\
&= -\frac{\mu}{2} \left(\nabla u_i \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} + \nabla u_i \cdot \nabla u_i + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \right) + \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 \\
&= -\frac{\mu}{2} \left(2 |\nabla \boldsymbol{u}|^2 + 2 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right) + \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 \\
&= \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 - \mu |\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}
\end{aligned}$$

Therefore, we have shown that

$$\begin{aligned}
u_i \frac{\partial \tau_{ij}}{\partial x_j} &= \frac{\partial u_i \tau_{ij}}{\partial x_j} + p (\nabla \cdot \boldsymbol{u}) + \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 - \mu |\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \\
&= \frac{\partial u_i \tau_{ij}}{\partial x_j} + p e_{kk} - 2\mu \left[e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \right]^2
\end{aligned}$$

where $[\cdot]^2$ implies a tensor double inner product (c.f. §3.5, "Tensor Calculus Made Simple," Sochi 2016) where first a (tensor) inner product is taken and the resultant second order tensor is contracted to become a scalar. $\square$

Problem 2:

2.1 Show that the imcompressible induction equation is

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

Proof. We begin by writing the (non-diffusive) induction equation and the corresponding derivative identity.

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{u}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}$$

Using this substitution and keeping in mind that $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{u} = 0$ we obtain,

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

□

2.2 Show that the compressible induction equation can be written as

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}}{\rho}\right) + (\boldsymbol{u} \cdot \nabla)\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right)\boldsymbol{u}$$

Proof. We begin by taking the compressible induction equation and multiplying by $1/\rho$.

$$\frac{1}{\rho}\frac{\partial \boldsymbol{B}}{\partial t} + \frac{1}{\rho}(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = \frac{1}{\rho}(\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - \frac{1}{\rho}\boldsymbol{B}(\nabla \cdot \boldsymbol{u})$$

Then, we use the product rule derivative identity to change some of the derivatives. We have,

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}}{\rho}\right) + \frac{\boldsymbol{B}}{\rho^2}\frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \nabla)\frac{\boldsymbol{B}}{\rho} + \frac{\boldsymbol{B}}{\rho^2}(\boldsymbol{u} \cdot \nabla)\rho = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right)\boldsymbol{u} - \frac{\boldsymbol{B}}{\rho}(\nabla \cdot \boldsymbol{u})$$

Here we consider the conservation of mass equation which for compressible fluids is written as,

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} = 0 \\
\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} + \frac{1}{\rho^2}(\nabla \cdot \rho \boldsymbol{u}) = 0 \\
\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} + \frac{1}{\rho^2}\left(\rho(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\rho\right) = 0 \\
\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} + \frac{1}{\rho^2}(\boldsymbol{u} \cdot \nabla)\rho = -\frac{1}{\rho}(\nabla \cdot \boldsymbol{u}).
\end{aligned}
$$

BNotice that we can take this equation, multiply it by $\boldsymbol{B}$ and subtract it from the induction equation. This leaves us with,

$$
\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}}{\rho}\right) + (\boldsymbol{u} \cdot \nabla)\frac{\boldsymbol{B}}{\rho} = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right)\boldsymbol{u}
$$

□

Problem 3:

3.1 Derive the induction equation given that σ is not necessarily constant

Proof. Let us begin with Ohm's law as we have written in lecture.

$$
\boldsymbol{j} = \boldsymbol{j}' = \sigma \boldsymbol{E}'
$$

$$
\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}
$$

$$
\begin{aligned}
\nabla \times \boldsymbol{B} = \mu_0 \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \\
\nabla \times \left(\frac{1}{\mu_0 \sigma}\nabla \times \boldsymbol{B}\right) = \nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\frac{1}{\mu_0 \sigma}(\nabla \times \nabla \times \boldsymbol{B}) - \frac{1}{\mu_0 \sigma^2}\nabla\sigma \times (\nabla \times \boldsymbol{B}) = -\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B} + \frac{1}{\mu_0 \sigma^2}((\nabla \boldsymbol{B})^T \cdot \nabla\sigma - (\nabla\sigma \cdot \nabla)\boldsymbol{B}) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})
\end{aligned}
$$

This can then be simplified keeping in mind that $\nabla \cdot \boldsymbol{B} = 0$, and especially if the flow is incompresible, to the following:

$$
\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + \frac{1}{\mu_0 \sigma^2}(\nabla\sigma \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \frac{1}{\mu_0 \sigma^2}((\nabla \boldsymbol{B})^T \cdot \nabla\sigma) + \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B}
$$

Essentially we see the appearence of two new terms if the conductivity is not constant. First, the advection of $\boldsymbol{B}$ by the gradient of conductivity, and then some weird term related to $\nabla \boldsymbol{B}^T$ on the RHS. □

Problem 4:

4.1 Show that initial conditions of the divergence of the magnetic field are preserved for Maxwell's equations

Proof. In order to show this, we must first assume that the temporal and spatial derivatives can be taken in any order, i.e. $\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$. We proceed by taking the dot product of Faraday's law,

$$\nabla \cdot \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \cdot (-\nabla \times \boldsymbol{E})$$
$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

where the RHS is zero because the divergence of a curl is always zero. Thus if we have that $\nabla \cdot \boldsymbol{B} = 0$ at $t = 0$, it will always be zero. □

4.2 Show that initial conditions of the divergence of the magnetic field are preserved for the induction equation

Proof. A similar proof can be written from the perspective of the induction equation. Let us write a form of the induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B}\right) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

where σ is not necessarily a constant and the fluid is not necessarily incompressible. Similarly, we take the divergence of this equation and obtain,

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = \nabla \cdot \left(\nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B}\right) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})\right)$$
$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

since again, the divergence of a curl is always zero. Therefore, from the perspective of the induction equation, we have that $\nabla \cdot \boldsymbol{B} = 0$ will be maintained for all $t > 0$ if $\nabla \cdot \boldsymbol{B} = 0$ at $t = 0$. □