RESEARCH ARTICLE | NOVEMBER 01 1991

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Phys. Fluids 3, 2644–2651 (1991)
https://doi.org/10.1063/1.858153

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(Received 6 March 1991; accepted 28 June 1991)

Three-dimensional flows of an inviscid incompressible fluid and an inviscid subsonic compressible gas are considered and it is demonstrated how the WKB method can be used for investigating their stability. The evolution of rapidly oscillating initial data is considered and it is shown that in both cases the corresponding flows are unstable if the transport equations associated with the wave which is advected by the flow have unbounded solutions. Analyzing the corresponding transport equations, a number of classical stability conditions are rederived and some new ones are obtained. In particular, it is demonstrated that steady flows of an incompressible fluid and an inviscid subsonic compressible gas are unstable if they have points of stagnation.

I. INTRODUCTION

Investigating the stability of a flow of an inviscid incompressible fluid and an inviscid compressible gas is important for developing an understanding of the most fundamental features of the fluid or gas motion. For over a century, this problem has attracted the attention of numerous researchers; nevertheless, some important questions concerning the stability of general flows are still open. In the present paper we describe some recent results concerning local stability conditions in hydrodynamics and gas dynamics reported in Ref. 1 as well as in Refs. 2–4, and demonstrate how they could be applied to the investigation of the stability of flows having stagnation points. We study the stability of vortex rings with swirl in a companion paper.[5]

We base our approach on the Wentzel–Kramers–Brillouin (WKB) method. An important distinction of our method from many others is that we consider the initial value problem for the linearized Euler equations and the linearized equations of gas dynamics, rather than the corresponding spectral problems. This approach allows us to consider the stability of time-dependent basic flows and investigate not only exponentially growing instabilities but also instabilities having algebraic growth. Our main result, a local stability condition, can be stated as follows. A basic flow of an incompressible fluid or a subsonic compressible gas is unstable if the transport equations associated with the wave which is advected by the flow have unbounded solutions. This instability condition is very effective from a practical point of view because it allows one to estimate from below the growth rate of solutions of the initial value problem for a partial differential equation in terms of the growth rate of solutions of the initial value problems for ordinary differential equations. It can effectively be applied to the investigation of many flows of interest and can provide many classical stability conditions and a number of new ones. Our result is a natural generalization of the well-known result of Eckhoff and Storesletten concerning the stability of certain particular flows of a compressible gas; some stability results using similar ideas have been recently obtained by Friedlander and Vishik (see below).

It is well known that in order to find WKB solutions of a given differential equation, one needs to solve two different problems: (a) construct formal asymptotic solutions, (b) prove that these solutions are close to actual solutions of the equation in question. The methods for solving these two problems are completely different. Usually the first problem is much simpler than the second one. Nevertheless, in general it is impossible to find global formal solutions because of ray intersection and the existence of caustics. Fortunately, in the problems of hydrodynamics and gas dynamics, rays that are important for our consideration do not intersect because they coincide with streamlines of the basic flow, and we are able to find a global asymptotic solution localized near these rays. Moreover, using the so-called energy inequalities, we can prove that these solutions are close to actual ones. As a result, we are able to estimate the growth rate of actual solutions in terms of the behavior of the leading-order terms of the corresponding asymptotic solutions. These terms are governed by the so-called transport equations, which are ordinary differential equations along rays (streamlines of the flow), so that we obtain a local stability condition for general time-dependent, three-dimensional flows. Note that recently this result was extended to the nonlinear case by one of the present authors.[6]

It is widely believed that short wavelength instabilities are responsible for the transition from large-scale coherent structures to three-dimensional spatial chaos; see, e.g., important papers by Orszag and Kells[7] and Orszag and Patera,[8] and a recent review by Bayly *et al.*[9] In order to describe such instabilities one can use solutions with complicated time dependence. The idea of using such solutions in hydrodynamic stability theory can be traced back to Kelvin[10] and Orr.[11] Until recently, little attention was paid to this idea, but in the last few years it was reexamined by Craik and Criminale.[12] General localized instabilities of the Craik–Criminale type are known under the name of broadband instabilities. They have been used by Bayly[13,14] in order to confirm numerical results by Pierrehumbert[15] indicating that flows having elliptic stagnation points are unstable and to describe the behavior of general quasi-two-dimensional flows. Quite recently, important results concerning broadband instabilities have been obtained by Friedlander and Vishik[16] who estimated from below the growth rate of small perturbations of a three-dimensional flow, and proved that flows having hyperbolic stagnation points are unstable.

Applying Eckhoff's theory of stability of symmetric hyperbolic systems[17] based on the generalized progressing wave expansion to gas dynamics, Eckhoff and Storesletten[18,19] studied the stability of helical flows and azimuthal shear flows and showed that for these flows streamlines form a family of rays along which instabilities governed by a set of ordinary differential equations can develop. However, we were not able to extend their equations to more general flows.

It is worth noting that in a different context the WKB method has been used in hydrodynamic stability theory by several researchers, see, e.g., Refs. 20–23. In these papers the authors were concerned either with basic flows with slowly varying parameters, or with modes with fixed frequency or wave number, and used more complicated eikonal equations for the phase than our Eq. (7). In contrast to these authors, we do not restrict ourselves with modes having a particular frequency and wave number and this is the main reason why we are able to consider the stability problem for general three-dimensional flows.

Unstable modes that are localized near magnetic lines of force are well known in magnetohydrodynamics;[24–28] they are discussed in detail in Ref. 29. It can be shown[27,28] that, in the framework of the ideal magnetohydrodynamic model, magnetic lines of force form a family of rays for the so-called ballooning modes which are considered to be the cause of the most dangerous instabilities. In our present study we broadly use the technique borrowed from magnetohydrodynamics[27,28] and extend it to hydrodynamics and gas dynamics.

II. LOCAL STABILITY CONDITIONS FOR AN INCOMPRESSIBLE FLUID

We start with the incompressible fluid case and consider a basic flow which is assumed to be uniform at infinity. Spatially periodic flows and asymptotically parallel flows between two parallel plates can easily be considered as well. We impose this condition in order to be able to get an energy estimate for the remainder in the corresponding WKB expansion of the solution of the linearized problem (see below). The velocity $\mathbf{V}(t,\mathbf{x})$, the uniform density R, and the pressure $P(t,\mathbf{x})$ in the basic flow are related via the Euler equations

$$\frac{D\mathbf{V}}{Dt}+\frac{1}{R}\nabla P=0, \tag{1a}$$

$$\nabla\cdot\mathbf{V}=0. \tag{1b}$$

We call the basic flow in question linearly stable if any solution of the initial value problem for the linearized Euler equations supplied with regularity conditions at infinity is bounded in time, otherwise we call it unstable. The corresponding initial value problem can be written as follows:

$$\frac{D\mathbf{v}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{v}+\frac{1}{R}\nabla p=0, \tag{2a}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2b}$$

$$\mathbf{v}(0,\mathbf{x})=\mathbf{v}_0(\mathbf{x}), \tag{3}$$

with $\mathbf{v}_0$, which is different from zero only in some ball, sufficiently smooth and divergence free. Here the rate of deformation tensor $\partial\mathbf{V}/\partial\mathbf{x}$ has elements $\partial V^i/\partial x^j$. Below we assume that the problem (2) and (3) is always solvable and has a unique solution.

In order to obtain the stability condition mentioned above we consider rapidly oscillating localized initial data of the form

$$\mathbf{v}_0(\mathbf{x})=\exp[i\Phi_0(\mathbf{x})/\epsilon]\hat{\mathbf{v}}_0(\mathbf{x}), \tag{4}$$

where

$$\hat{\mathbf{v}}_0(\mathbf{x})=\nabla\times[\alpha_0(\mathbf{x})\nabla\Phi_0(\mathbf{x})]=\nabla\alpha_0(\mathbf{x})\times\nabla\Phi_0(\mathbf{x}), \tag{5}$$

where ϵ is a small parameter, so that at the initial time $\hat{\mathbf{v}}_0\cdot\nabla\Phi_0=0$. Such a choice of the initial data allows us to present the solution of the initial value problem (2) and (3) in the WKB form,

$$(\mathbf{v},p)=\exp(i\Phi/\epsilon)[(\hat{\mathbf{v}}^{(0)},\hat{p}^{(0)})+\epsilon(\hat{\mathbf{v}}^{(1)},\hat{p}^{(1)})]+\epsilon[\mathbf{w}(\epsilon),q(\epsilon)]. \tag{6}$$

It is well known that the only wave in an incompressible fluid is one which is advected by the basic flow. The eikonal equation for the phase Φ is therefore

$$\frac{D\Phi}{Dt}=\frac{\partial\Phi}{\partial t}+\mathbf{V}\cdot\nabla\Phi=0, \tag{7}$$

with the corresponding initial data

$$\Phi(0,\mathbf{x})=\Phi_0(\mathbf{x}). \tag{8}$$

Taking the gradient of the eikonal equation and the initial data we obtain the initial value problem for the local wave vector $\nabla\Phi$,

$$\frac{D\nabla\Phi}{Dt}+\left(\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\right)^T\nabla\Phi=0, \tag{9}$$

$$\nabla\Phi(0,\mathbf{x})=\nabla\Phi_0(\mathbf{x}). \tag{10}$$

Substituting expression (6) into Eqs. (2) and the initial condition (3) and expanding the result in powers of ϵ we obtain

$$\hat{p}^{(0)}\nabla\Phi=0, \tag{11a}$$

$$\hat{\mathbf{v}}^{(0)}\cdot\nabla\Phi=0, \tag{11b}$$

$$i\frac{1}{R}\hat{p}^{(1)}\nabla\Phi=-\left(\frac{D\hat{\mathbf{v}}^{(0)}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\hat{\mathbf{v}}^{(0)}+\frac{1}{R}\nabla\hat{p}^{(0)}\right), \tag{12a}$$

$$i\hat{\mathbf{v}}^{(1)}\cdot\nabla\Phi=-\nabla\cdot\hat{\mathbf{v}}^{(0)}, \tag{12b}$$

$$\frac{D\mathbf{w}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{w}+\frac{1}{R}\nabla q=-\exp\left(\frac{i\Phi}{\epsilon}\right)\left(\frac{D\hat{\mathbf{v}}^{(1)}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\hat{\mathbf{v}}^{(1)}+\frac{1}{R}\nabla\hat{p}^{(1)}\right), \tag{13a}$$

$$\nabla\cdot\mathbf{w}=-\exp(i\Phi/\epsilon)\nabla\cdot\hat{\mathbf{v}}^{(1)}, \tag{13b}$$

$$\hat{\mathbf{v}}^{(0)}(0,\mathbf{x})=\hat{\mathbf{v}}_0(\mathbf{x}),\quad \hat{\mathbf{v}}^{(1)}(0,\mathbf{x})=0,\quad \hat{\mathbf{w}}(0,\mathbf{x})=0. \tag{14}$$

Equations (11a) and (11b) show that $\hat{p}^{(0)}=0$, while $\hat{\mathbf{v}}^{(0)}$ is an arbitrary vector orthogonal to $\nabla\Phi$. In order to simplify the notation below we denote $\hat{\mathbf{v}}^0$ by $\mathbf{a}$. The amplitude $\mathbf{a}$ can be found from the so-called transport equation which is the component of Eq. (12a) perpendicular to $\nabla\Phi$. This equation can be written as follows

$$\frac{D\mathbf{a}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}-\left(\frac{D\mathbf{a}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}\right)\cdot\nabla\Phi\frac{\nabla\Phi}{|\nabla\Phi|^2}=0. \qquad (15)$$

Differentiating the orthogonality condition and using Eq. (9) for the wave-vector evolution we obtain

$$\frac{D\mathbf{a}}{Dt}\cdot\nabla\Phi=-\mathbf{a}\cdot\frac{D\nabla\Phi}{Dt}=\mathbf{a}\cdot\left(\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\right)^T\nabla\Phi=\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}\cdot\nabla\Phi. \qquad (16)$$

Using this identity we rewrite the transport equation as follows

$$\frac{D\mathbf{a}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}-2\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}\cdot\nabla\Phi\frac{\nabla\Phi}{|\nabla\Phi|^2}=0. \qquad (17)$$

This equation should be supplied with the initial condition of the form

$$\mathbf{a}(0,\mathbf{x})=\hat{\mathbf{v}}_0(\mathbf{x}). \qquad (18)$$

Notice that Eqs. (9) and (17) guarantee that the scalar product $\mathbf{a}\cdot\nabla\Phi$ is conserved along streamlines of the basic flow. By virtue of our choice of the initial conditions this scalar product vanishes initially, so that it vanishes at any time.

The eikonal equation (7) and the transport equation (17) can easily be rewritten in the characteristic form. The corresponding characteristic equations and the initial conditions can be written as follows:

$$\frac{dt}{d\tau}=1, \qquad (19a)$$

$$\frac{d\mathbf{x}}{d\tau}=\mathbf{V}, \qquad (19b)$$

$$t(0)=0, \qquad (20a)$$

$$\mathbf{x}(0)=\mathbf{x}_0, \qquad (20b)$$

$$\frac{d\omega}{d\tau}=-\frac{\partial\mathbf{V}}{\partial t}\cdot\mathbf{k}, \qquad (21a)$$

$$\frac{d\mathbf{k}}{d\tau}=-\left(\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\right)^T\mathbf{k}, \qquad (21b)$$

$$\omega(0)=\omega_0=-\mathbf{V}(0,\mathbf{x}_0)\cdot\nabla\Phi_0(\mathbf{x}_0), \qquad (22a)$$

$$\mathbf{k}(0)=\mathbf{k}_0=\nabla\Phi_0(\mathbf{x}_0); \qquad (22b)$$

$$\frac{d\mathbf{a}}{d\tau}=-\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}+2\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}\cdot\mathbf{k}\frac{\mathbf{k}}{|\mathbf{k}|^2}, \qquad (23)$$

$$\mathbf{a}(0)=\mathbf{a}_0=\hat{\mathbf{v}}_0(\mathbf{x}_0). \qquad (24)$$

Equations (19b) and (20b) determine the streamline passing through the point $\mathbf{x}_0$, Eqs. (21b) and (22b) determine the evolution along this streamline of the local wave vector having the original orientation $\mathbf{k}_0$, while Eqs. (23) and (24) govern the evolution of the amplitude $\mathbf{a}$ which is originally equal to $\mathbf{a}_0$. The corresponding solutions are denoted by $\mathbf{x}(\tau;\mathbf{x}_0)$, $\mathbf{k}(\tau;\mathbf{x}_0,\mathbf{k}_0)$, and $\mathbf{a}(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0)$ in order to emphasize their parametric dependence on $\mathbf{x}_0$, $\mathbf{k}_0$, and $\mathbf{a}_0$. The phase Φ and the amplitude $\mathbf{a}$ can be written in terms of these solutions as follows:

$$\Phi[t,\mathbf{x}(t;\mathbf{x}_0)]=\Phi_0(\mathbf{x}_0), \qquad (25)$$

$$\mathbf{a}[t,\mathbf{x}(t;\mathbf{x}_0)]=\mathbf{a}[t;\mathbf{x}_0,\nabla\Phi_0(\mathbf{x}_0),\mathbf{a}_0(\mathbf{x}_0)]. \qquad (26)$$

It is shown below that for stagnation points the characteristic and transport equations can be studied in the most effective fashion in the standard Cartesian coordinates. This is not the case for general streamlines. Analysis of the corresponding equations becomes much more feasible in appropriate curvilinear coordinates. Fortunately, Eqs. (19), (21), and (23) are geometrically invariant. In general curvilinear coordinates these equations do not change their form if according to the rules of differential geometry we define the covector $d\mathbf{k}/d\tau$, the vector $d\mathbf{a}/d\tau$, and the rate of deformation tensor $\partial\mathbf{V}/\partial\mathbf{x}$ in the following way:

$$\left(\frac{d\mathbf{k}}{d\tau}\right)_i=\frac{dk_i}{d\tau}-\Gamma_{ij}^k V^j k_k, \qquad (27a)$$

$$\left(\frac{d\mathbf{a}}{d\tau}\right)^i=\frac{da^i}{d\tau}+\Gamma_{jk}^i V^j a^k, \qquad (27b)$$

$$\left(\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\right)^i_j=\frac{\partial V^i}{\partial x^j}+\Gamma_{jk}^i V^k, \qquad (27c)$$

where Γ_{ij}^k are the corresponding Christoffel symbols.

Knowing $\mathbf{a}$ we can specify $\hat{\mathbf{v}}^{(1)}$ and $\hat{p}^{(1)}$ as solutions of Eqs. (12),

$$\hat{\mathbf{v}}^{(1)}=i\frac{\nabla\cdot\mathbf{a}}{|\nabla\Phi|^2}\nabla\Phi,\quad \hat{p}^{(1)}=2iR\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}\cdot\frac{\nabla\Phi}{|\nabla\Phi|^2}. \qquad (28)$$

Let us now estimate the remainder on any fixed time interval $0\leqslant t\leqslant T$. Using the expression (28) we can write Eqs. (13) for the remainder in the following form:

$$\frac{D\mathbf{w}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{w}+\frac{1}{R}\nabla q=\mathbf{F}, \qquad (29a)$$

$$\nabla\cdot\mathbf{w}=G, \qquad (29b)$$

where

$$\mathbf{F}=\exp(i\Phi/\epsilon)\hat{\mathbf{F}},$$

$$\hat{\mathbf{F}}=-i\left[\frac{D}{Dt}\left(\frac{\nabla\cdot\mathbf{a}}{|\nabla\Phi|^2}\nabla\Phi\right)+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\left(\frac{\nabla\cdot\mathbf{a}}{|\nabla\Phi|^2}\nabla\Phi\right)+2\nabla\left(\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}\cdot\frac{\nabla\Phi}{|\nabla\Phi|^2}\right)\right], \qquad (30a)$$

$$G=\exp(i\Phi/\epsilon)\hat{G},\quad \hat{G}=-i\nabla\cdot[((\nabla\cdot a)/|\nabla\Phi|^2)\nabla\Phi]. \qquad (30b)$$

It is worth noting that

$$\begin{aligned}\int_{R^3}G\,d\mathbf{x}&=-i\int_{R^3}\exp\left(\frac{i\Phi}{\epsilon}\right)\nabla\cdot\left(\frac{\nabla\cdot\mathbf{a}}{|\nabla\Phi|^2}\nabla\Phi\right)d\mathbf{x}\\&=-\frac{1}{\epsilon}\int_{R^3}\exp\left(\frac{i\Phi}{\epsilon}\right)\nabla\Phi\cdot\left(\frac{\nabla\cdot\mathbf{a}}{|\nabla\Phi|^2}\nabla\Phi\right)d\mathbf{x}\\&=-\frac{1}{\epsilon}\int_{R^3}\exp\left(\frac{i\Phi}{\epsilon}\right)\nabla\cdot\mathbf{a}\,d\mathbf{x}\\&=\frac{i}{\epsilon^2}\int_{R^3}\exp\left(\frac{i\Phi}{\epsilon}\right)\nabla\Phi\cdot\mathbf{a}\,d\mathbf{x}=0, \end{aligned}\qquad (31)$$

so that G can be written as a divergence of a rapidly decreasing vector field as required by Eq. (29b). Taking the divergence of Eq. (29a) and using Eq. (29b) we obtain after some algebra

$$-\frac{1}{R}\Delta q = 2\frac{\partial \mathbf{V}}{\partial \mathbf{x}}:\frac{\partial \mathbf{w}}{\partial \mathbf{x}} - \nabla\cdot\mathbf{F} + \frac{DG}{Dt}, \tag{32}$$

where : denotes the convolution of two tensors. We invert the Laplacian and write q in the form

$$\frac{1}{R}q = \frac{1}{4\pi|\mathbf{x}|} * \left(2\frac{\partial \mathbf{V}}{\partial \mathbf{x}}:\frac{\partial \mathbf{w}}{\partial \mathbf{x}} - \nabla\cdot\mathbf{F} + \frac{DG}{Dt}\right), \tag{33}$$

where * denotes the convolution of two functions. Noting that

$$\frac{DG}{Dt} = \frac{i}{\epsilon}\frac{D\Phi}{Dt}\exp\left(\frac{i\Phi}{\epsilon}\right)\hat{G} + \exp\left(\frac{i\Phi}{\epsilon}\right)\frac{D\hat{G}}{Dt}$$
$$= \exp\left(\frac{i\Phi}{\epsilon}\right)\frac{D\hat{G}}{Dt}, \tag{34}$$

we can estimate the squared norm of q as follows

$$\frac{1}{R^2}(q,q) \leqslant c_1(\mathbf{w},\mathbf{w}) + c_2(\hat{\mathbf{F}},\hat{\mathbf{F}}) + c_3\left(\frac{D\hat{G}}{Dt},\frac{D\hat{G}}{Dt}\right), \tag{35}$$

where (,) denotes the standard scalar product in the space of square integrable functions, and $c_1 = c_1(T)$, $c_2 = c_2(T)$, $c_3 = c_3(T)$ are some positive T-dependent constants. Now we are in a position to use the standard technique for deriving energy inequalities, see, e.g., Ref. 30. From Eqs. (29) it easily follows that

$$\frac{d}{dt}(\mathbf{w},\mathbf{w}) = -\left\{\left[\frac{\partial \mathbf{V}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{V}}{\partial \mathbf{x}}\right)^T\right]\mathbf{w},\mathbf{w}\right\}$$
$$+ (\mathbf{F},\mathbf{w}) + (\mathbf{w},\mathbf{F}) + \frac{1}{R}(G,q) + \frac{1}{R}(q,G), \tag{36}$$

so that by virtue of the inequality (35) we can write

$$\frac{d}{dt}(\mathbf{w},\mathbf{w}) \leqslant c_1'(\mathbf{w},\mathbf{w}) + c_2'(\hat{\mathbf{F}},\hat{\mathbf{F}})$$
$$+ c_3'\left(\frac{D\hat{G}}{Dt},\frac{D\hat{G}}{Dt}\right) + c_4'(\hat{G},\hat{G}), \tag{37}$$

where c_1', c_2', c_3', c_4' are appropriate positive T-dependent constants. Integration of this inequality using the fact that $\mathbf{w}(0) = 0$ yields an estimate for $(\mathbf{w},\mathbf{w})$ that is uniform in ϵ on the fixed time interval $0 \leqslant t \leqslant T$:

$$(\mathbf{w},\mathbf{w}) \leqslant \int_0^t \exp[c_1'(t-\nu)]\left[c_2'[\hat{\mathbf{F}}(\nu),\hat{\mathbf{F}}(\nu)]\right.$$
$$\left. + c_3'\left(\frac{D\hat{G}(\nu)}{Dt},\frac{D\hat{G}(\nu)}{Dt}\right) + c_4'[\hat{G}(\nu),\hat{G}(\nu)]\right]d\nu. \tag{38}$$

This estimate is similar to the classical energy estimate for an incompressible fluid (see, e.g., Ref. 31).

We therefore conclude that the initial value problem (2) and (3) with the initial data (4) has the global WKB-type solution of the form

$$(\mathbf{v},p) = \exp\left(\frac{i\Phi}{\epsilon}\right)\left[(\mathbf{a},0) + \frac{i\epsilon}{|\nabla\Phi|^2}\right.$$
$$\left.\times\left((\nabla\cdot\mathbf{a})\nabla\Phi, 2R\frac{\partial \mathbf{V}}{\partial \mathbf{x}}\mathbf{a}\cdot\nabla\Phi\right)\right]$$
$$+ \epsilon[\mathbf{w}(\epsilon),q(\epsilon)], \tag{39}$$

with the phase Φ which is advected by the basic flow, the amplitude $\mathbf{a}$ which is governed by the transport equation along streamlines, and the remainder $[\mathbf{w}(\epsilon),q(\epsilon)]$ which is bounded uniformly in ϵ on any fixed time interval.

Assuming that at the initial time our velocity field is localized in a small vicinity of the point $\mathbf{x}_0$ we prove that the basic flow is unstable if the solution $\mathbf{a}(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0)$ is unbounded. Considering all possible initial points, orientations of the wave front, and polarizations of the wave amplitude, we obtain the local stability condition sought. Namely, we prove that the basic flow is unstable if

$$\sup_{\substack{\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0 \\ |\mathbf{k}_0|=1,\ |\mathbf{a}_0|=1,\ \mathbf{a}_0\cdot\mathbf{k}_0=0}} \overline{\lim_{\tau\to\infty}}|\mathbf{a}(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0)| = \infty, \tag{40}$$

and thus obtain a necessary stability condition and a sufficient instability condition in the incompressible case. Let us assume that for certain $\mathbf{x}_0$, $\mathbf{k}_0$, and $\mathbf{a}_0$ such that $|\mathbf{k}_0| = 1$, $|\mathbf{a}_0| = 1$, $\mathbf{a}_0\cdot\mathbf{k}_0 = 0$ the corresponding solution $\mathbf{a}(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0)$ is unbounded. We choose the initial data for the linearized Euler equations in the form (4) and (5) with Φ_0 and α_0 defined as follows:

$$\Phi_0(\mathbf{x}) = \mathbf{k}_0\cdot(\mathbf{x}-\mathbf{x}_0)\chi(|\mathbf{x}-\mathbf{x}_0|/2\delta), \tag{41a}$$

$$\alpha_0(\mathbf{x}) = \delta^{-3/2}\mathbf{l}_0\cdot(\mathbf{x}-\mathbf{x}_0)\chi(|\mathbf{x}-\mathbf{x}_0|/\delta), \tag{41b}$$

where $\mathbf{l}_0 = \mathbf{k}_0\times\mathbf{a}_0$, χ is a smooth cutoff function such that $\chi(s) = 1$ when $|s| \leqslant 1$ and $\chi(s) = 0$ when $|s| \geqslant 2$, and δ is an auxiliary small parameter. It is clear that for any $N > 0$ we can find T which is so large and δ which is so small that

$$[\mathbf{a}(T),\mathbf{a}(T)]/(\hat{\mathbf{v}}_0,\hat{\mathbf{v}}_0) > 2N. \tag{42}$$

Keeping T and δ fixed we can choose ϵ which is so small that both the first-order term $\exp(i\Phi/\epsilon)(\hat{\mathbf{v}}^{(1)},\hat{p}^{(1)})$ and the remainder $(\mathbf{w},q)$ are dominated by the leading-order term $\exp(i\Phi/\epsilon)(\hat{\mathbf{v}}^{(0)},\hat{p}^{(0)})$, so that

$$[\mathbf{v}(T),\mathbf{v}(T)]/[\mathbf{v}(0),\mathbf{v}(0)] > N. \tag{43}$$

Thus, for suitable initial data the ratio $[\mathbf{v}(T),\mathbf{v}(T)]/[\mathbf{v}(0),\mathbf{v}(0)]$ can be made arbitrarily large and the basic flow in question is unstable because the corresponding solution of the linearized Eulers equations is not bounded in time.

It is worth noting that for a steady flow the existence of an exponentially increasing amplitude $\mathbf{a}(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0)$ implies the existence of an unstable eigenvalue for the corresponding spectral problem.

III. LOCAL STABILITY CONDITIONS FOR A COMPRESSIBLE GAS

Let us now consider a basic subsonic flow of a compressible gas that is uniform at infinity. It is worth noting that in the compressible case we can consider not only asymptotically uniform and spatially periodic flows but flows in bounded domains as well. The velocity $\mathbf{V}(t,\mathbf{x})$, the density $R(t,\mathbf{x})$, and the pressure $P(t,\mathbf{x})$ satisfy the following equations:

$$\frac{D\mathbf{V}}{Dt} + \frac{1}{R}\nabla P = 0, \tag{44a}$$

$$\frac{DR}{Dt} + R\nabla\cdot\mathbf{V} = 0, \tag{44b}$$

$$\frac{DP}{Dt}+\gamma P\nabla\cdot\mathbf{V}=0, \tag{44c}$$

where γ is the adiabaticity index. We denote by $\mathbf{v}$, ρ, and p small perturbations of the velocity, density, and pressure. Following Eckart[32] we introduce new variables m and n such that $\rho=(R/C)(m+n)$, $p=(\gamma P/C)n$, where $C=(\gamma P/R)^{1/2}>0$ is the local sound speed, and write the linearized equations of motion in the form of a symmetric hyperbolic system

$$\frac{D\mathbf{v}}{Dt}+\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{v}-\frac{\nabla P}{RC}m+C\nabla n+\left(\frac{(\gamma-1)\nabla P}{CR}-\nabla C\right)n =0, \tag{45a}$$

$$C\frac{D}{Dt}\left(\frac{m}{C}\right)+\mathbf{v}\cdot C\nabla\ln\left(\frac{R}{P^{1/\gamma}}\right)=0, \tag{45b}$$

$$C\frac{D}{Dt}\left(\frac{n}{C}\right)+C\nabla\cdot\mathbf{v}+\mathbf{v}\cdot C\nabla\ln(P^{1/\gamma})=0. \tag{45c}$$

The corresponding initial conditions are

$$[\mathbf{v}(0,\mathbf{x}),m(0,\mathbf{x}),n(0,\mathbf{x})]=[\mathbf{v}_0(\mathbf{x}),m_0(\mathbf{x}),n_0(\mathbf{x})]. \tag{46}$$

Below we assume that the problem (45) and (46) is solvable and has a unique solution for any localized and sufficiently smooth initial data. Although the Eckart transform is cumbersome it is very convenient for our purposes because due to the fact that the system (45) is written in the symmetric hyperbolic form we can obtain an energy estimate for the remainder in the corresponding WKB expansion in a completely standard fashion (see below).

We choose rapidly oscillating initial data of the form

$$[\mathbf{v}_0(\mathbf{x}),m_0(\mathbf{x}),n_0(\mathbf{x})] =\exp[i\Phi_0(\mathbf{x})/\epsilon][\hat{\mathbf{v}}_0(\mathbf{x}),\hat{\beta}_0(\mathbf{x}),0], \tag{47}$$

where

$$\hat{\mathbf{v}}_0(\mathbf{x})=P^{-1/\gamma}\nabla\times[\alpha_0(\mathbf{x})\nabla\Phi_0(\mathbf{x})] =P^{-1/\gamma}\nabla\alpha_0(\mathbf{x})\times\nabla\Phi_0(\mathbf{x}). \tag{48}$$

By analogy with the incompressible case we can demonstrate that in the case in question the corresponding initial value problem has the WKB-type solution of the form

$$(\mathbf{v},m,n)=\exp\left(\frac{i\Phi}{\epsilon}\right)\frac{1}{\sqrt{J}}\left\{(\mathbf{a},Cb,0)+\frac{i\epsilon}{|\nabla\Phi|^2}\times\left[\left(\nabla\cdot\mathbf{a}+\mathbf{a}\cdot\nabla\ln\frac{P^{1/\gamma}}{J^{1/2}}\right)\times\nabla\Phi,0,\frac{1}{C}\left(2\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}-\frac{\nabla P}{R}b\right)\cdot\nabla\Phi\right]\right\} +[\mathbf{w}(\epsilon),q_1(\epsilon),q_2(\epsilon)], \tag{49}$$

where Φ is the phase that is advected by the fluid; J is the Jacobian of the shift along trajectories of the basic flow; $J=\partial\mathbf{x}(t;\mathbf{x}_0)/\partial\mathbf{x}_0$; $\mathbf{a}$ and b are the amplitudes that can be found from the corresponding transport equation; and $[\mathbf{w}(\epsilon),q_1(\epsilon),q_2(\epsilon)]$ is the remainder, which is bounded uniformly in ϵ on any fixed time interval. It is worth noting that in contrast with the incompressible case in the case in question the uniform boundness of the remainder can be established by means of the standard technique; see, e.g., Ref. 30.

The phase Φ is the solution of the initial value problem (7) and (8). The corresponding characteristic equations (19) and (21) and the initial conditions (20) and (22) remain unchanged. By virtue of the well-known Liouville theorem, J is the solution of the following initial value problem:

$$\frac{dJ}{d\tau}=(\nabla\cdot\mathbf{V})J,\quad J(0)=1. \tag{50}$$

The transport equations for the amplitudes $\mathbf{a}$ and b, and the corresponding boundary conditions can be written as follows:

$$\frac{d\mathbf{a}}{d\tau}=-\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}+\frac{1}{2}(\nabla\cdot\mathbf{V})\mathbf{a}+2\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{a}\cdot\mathbf{k}\frac{\mathbf{k}}{|\mathbf{k}|^2} +\frac{1}{R}\left(\nabla P-\nabla P\cdot\mathbf{k}\frac{\mathbf{k}}{|\mathbf{k}|^2}\right)b, \tag{51a}$$

$$\frac{db}{d\tau}=-\mathbf{a}\cdot\nabla\ln\left(\frac{R}{P^{1/\gamma}}\right)+\frac{1}{2}(\nabla\cdot\mathbf{V})b, \tag{51b}$$

$$\mathbf{a}(0)=\mathbf{a}_0=\hat{\mathbf{v}}_0(\mathbf{x}_0),\quad b(0)=b_0=(1/C)\beta_0(\mathbf{x}_0). \tag{52}$$

We denote the solution of this initial value problem through $[\mathbf{a}(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0,b_0),b(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0,b_0)]$. It can be shown that the basic flow of an inviscid compressible gas is unstable if

$$\sup_{\substack{\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0,b_0\\ |\mathbf{k}_0|=1,\ |[\mathbf{a}_0,b_0]|=1,\ \mathbf{a}_0\cdot\mathbf{k}_0=0}}\ \overline{\lim_{\tau\to\infty}}|[\mathbf{a}(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0,b_0),b(\tau;\mathbf{x}_0,\mathbf{k}_0,\mathbf{a}_0,b_0)]|=\infty. \tag{53}$$

This observation provides a necessary stability condition and a sufficient instability condition for subsonic flows of a compressible gas.

IV. INSTABILITY OF STAGNATION POINTS

Let us now demonstrate how the local stability conditions described above can be used in order to investigate the stability of certain flows of interest. First, we consider the stability of a steady flow of an incompressible fluid with the velocity $\mathbf{V}(\mathbf{x})$, the density R, and the pressure $P(\mathbf{x})$ having a nondegenerate stagnation point $\mathbf{X}$ such that $\mathbf{V}(\mathbf{X})=0$ and $\partial\mathbf{V}/\partial\mathbf{x}(\mathbf{X})\neq0$. In order to simplify our notation below we will denote the matrix $\partial\mathbf{V}/\partial\mathbf{x}(\mathbf{X})$ by L. Choosing the point $\mathbf{X}$ as the initial data for Eq. (19b) we obtain the trajectory consisting of the point $\mathbf{X}$ itself. The corresponding Eqs. (21b) and (23) depend only on elements of the matrix L, they are affected neither by the behavior of the flow away from the stagnation point in question nor by the regularity conditions at infinity. These equations coincide with the classical Kelvin equations describing propagation of plane waves in a flow with linear velocity profile $\mathbf{V}(\mathbf{x}) = L(\mathbf{x}-\mathbf{X})$, and can be solved explicitly.[12,13,16]

The equilibrium conditions impose strict limitations on the form of the matrix L, see, e.g., Ref. 12. In order to formulate these conditions in a convenient form we decompose L into its symmetric part L_+ describing straining, and antisymmetric part L_- describing rotation, $L = L_+ + L_-$. It follows from $\nabla\cdot\mathbf{V}=0$ that

$$\operatorname{tr} L = \operatorname{tr} L_+ = 0, \tag{54a}$$

and from the force balance condition $\mathbf{V}\cdot\nabla\mathbf{V} = -\nabla P/R$ that

$$L_+ L_- = -L_- L_+. \tag{54b}$$

Noting that $L_-\mathbf{x} = \mathbf{f}\times\mathbf{x}$ for an appropriate vector $\mathbf{f}$ and using Eqs. (54) we obtain

$$L_+\mathbf{f} = 0. \tag{55}$$

It is clear that there exist the following possibilities: (i) the matrix L_+ is nondegenerate and $\mathbf{f}=0$, (ii) the matrix L_+ has an eigenvalue $\lambda=0$ with the corresponding normalized eigenvector $\mathbf{e}$, and $\mathbf{f}$ is parallel to $\mathbf{e}$. Using this observation we can easily distinguish the following cases.

(a) $L = L_+$ has three nonzero real eigenvalues $\lambda_1,\lambda_2,\lambda_3$, $\lambda_1+\lambda_2+\lambda_3=0$, $\lambda_1>0$, $\lambda_3<0$, with the corresponding orthonormal eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and

$$L\mathbf{x} = \lambda_1\mathbf{x}\cdot\mathbf{e}_1\mathbf{e}_1 + \lambda_2\mathbf{x}\cdot\mathbf{e}_2\mathbf{e}_2 + \lambda_3\mathbf{x}\cdot\mathbf{e}_3\mathbf{e}_3. \tag{56a}$$

Near the stagnation point $\mathbf{X}$ streamlines are three-dimensional hyperbolas.

(b) L has an eigenvalue $\lambda_1=0$ with the corresponding normalized eigenvector $\mathbf{e}_1$ and two real nonzero eigenvalues λ_2,λ_3, $\lambda_{2,3}=\pm\lambda$, $\lambda>0$, with the corresponding normalized real eigenvectors $\mathbf{e}_2$ and $\mathbf{e}_3$ orthogonal to $\mathbf{e}_1$, and

$$L\mathbf{x} = \lambda\mathbf{x}\cdot\mathbf{g}_2\mathbf{e}_2 - \lambda\mathbf{x}\cdot\mathbf{g}_3\mathbf{e}_3, \tag{56b}$$

where $\mathbf{g}_2 = \mathbf{e}_3\times\mathbf{e}_1/(\mathbf{e}_1\times\mathbf{e}_2\cdot\mathbf{e}_3)$, $\mathbf{g}_3 = \mathbf{e}_1\times\mathbf{e}_2/(\mathbf{e}_1\times\mathbf{e}_2\cdot\mathbf{e}_3)$. In the case in question streamlines near the stagnation point are two-dimensional hyperbolas lying in planes orthogonal to $\mathbf{e}_1$.

(c) L has a degenerate eigenvalue $\lambda=0$ of multiplicity three with two eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and one adjoint vector $\mathbf{e}_3$ that are orthonormal, and

$$L\mathbf{x} = \mathbf{x}\cdot\mathbf{e}_3\mathbf{e}_2. \tag{56c}$$

This case corresponds to the pure shear flow case considered by Kelvin himself.

(d) L has an eigenvalue $\lambda_1=0$ with the corresponding normalized eigenvector $\mathbf{e}_1$ and two pure imaginary eigenvalues λ_2, λ_3, $\lambda_{2,3}=\pm i\lambda$, $\lambda>0$, with the corresponding normalized complex eigenvectors $\mathbf{e}_2$ and $\mathbf{e}_3$, which are orthogonal to $\mathbf{e}_1$, and

$$L\mathbf{x} = (\lambda/\mu)\mathbf{x}\cdot\mathbf{g}_3\mathbf{g}_2 - \lambda\mu\mathbf{x}\cdot\mathbf{g}_2\mathbf{g}_3, \tag{56d}$$

where $\mu<1$ is a parameter characterizing ellipticity of the flow near the stagnation point $\mathbf{X}$, and orthonormal real vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ have the form $\mathbf{g}_2 = [i(\mu^2+1)^{1/2}/2](\mathbf{e}_2-\mathbf{e}_3)$, $\mathbf{g}_3 = [(\mu^2+1)^{1/2}/2\mu]\times(\mathbf{e}_2+\mathbf{e}_3)$. In the case in question streamlines are two-dimensional ellipses lying in planes orthogonal to $\mathbf{e}_1$. It is clear that $\mu=1$ corresponds to rigid body rotation. Stretching of the time variable allows us to set $\lambda=1$ and to write $L\mathbf{x}$ as

$$L\mathbf{x} = (1/\mu)\mathbf{x}\cdot\mathbf{g}_3\mathbf{g}_2 - \mu\mathbf{x}\cdot\mathbf{g}_2\mathbf{g}_3, \tag{56d'}$$

The hyperbolic cases (a) and (b) and the shear flow case (c) can be treated analytically, while the elliptic case (d) requires some simple computations (the case of rigid body rotation can be treated analytically as well).

In case (a) we choose $\mathbf{k}_0=\mathbf{e}_1$, $\mathbf{a}_0=\mathbf{e}_3$. Straightforward computation shows that

$$\mathbf{k}(\tau;\mathbf{X},\mathbf{e}_1) = \exp(-\lambda_1\tau)\mathbf{e}_1,$$
$$\mathbf{a}(\tau;\mathbf{X},\mathbf{e}_1,\mathbf{e}_3) = \exp(-\lambda_3\tau)\mathbf{e}_3, \tag{57}$$

so that we obviously have an exponential instability.

In case (b) we take $\mathbf{k}_0=\mathbf{g}_2$, $\mathbf{a}_0=\mathbf{e}_3$ and obtain

$$\mathbf{k}(\tau;\mathbf{X},\mathbf{g}_2) = \exp(-\lambda\tau)\mathbf{g}_2,$$
$$\mathbf{a}(\tau;\mathbf{X},\mathbf{g}_2,\mathbf{e}_3) = \exp(\lambda\tau)\mathbf{e}_3, \tag{58}$$

thus demonstrating that we have an exponential instability in the case in question as well.

In case (c) we take $\mathbf{k}_0=\mathbf{e}_1$, $\mathbf{a}_0=\mathbf{e}_3$, and obtain the following expressions for $\mathbf{k}(\tau)$ and $\mathbf{a}(\tau)$:

$$\mathbf{k}(\tau;\mathbf{X},\mathbf{e}_1) = \mathbf{e}_1, \quad \mathbf{a}(\tau;\mathbf{X},\mathbf{e}_1,\mathbf{e}_3) = -\tau\mathbf{e}_2+\mathbf{e}_3. \tag{59}$$

It is clear that the norm of $\mathbf{a}(\tau)$ is increasing in time (although as a power function rather than an exponent) and the shear flow in question is algebraically unstable. This result seems to be in contradiction with the well-known Rayleigh stability theorem, but this contradiction is only apparent because this theorem deals with exponential instabilities only.

In the elliptic case (d) the search for an increasing $\mathbf{a}(\tau)$ requires some numerical work. We decompose the vectors $\mathbf{k}$ and $\mathbf{a}$ in the orthonormal basis $\mathbf{e}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$, and write them as (k_1,k_2,k_3) and (a^1,a^2,a^3). Without loss of generality we can choose $\mathbf{k}_0$ to be

$$(k_{10},k_{20},k_{30}) = (\cos\theta,0,\sin\theta), \tag{60}$$

with the angle θ, $0<\theta<\pi/2$, which is specified below, and find the corresponding $\mathbf{k}(\tau)$,

$$[k_1(\tau),k_2(\tau),k_3(\tau)] = (\cos\theta,\mu\sin\theta\sin\tau,\sin\theta\cos\tau). \tag{61}$$

It is clear that the end of the vector $\mathbf{k}(\tau)$ moves along an ellipse in the plane $k_1=k_{10}$ perpendicular to $\mathbf{e}_1$, and θ is the maximum angle between $\mathbf{e}_1$ and $\mathbf{k}$. Substituting this expression for $\mathbf{k}$ into the transport equation (23) we obtain the following system of equations for the scaled variables $d^2=a^2$, $d^3=a^3/\mu$:

$$\frac{dd^2}{d\tau} = -2\mu\frac{k_2k_3}{|k|^2}d^2 - \frac{k_1^2-k_2^2+k_3^2}{|k|^2}d^3, \tag{62a}$$

$$\frac{dd^3}{d\tau} = \frac{k_1^2+k_2^2-k_3^2}{|k|^2}d^2 + \frac{2}{\mu}\frac{k_2k_3}{|k|^2}d^3. \tag{62b}$$

The scaling is necessary in order to obtain coefficients which depend analytically on μ (notice that the ratio k_2/μ is independent of μ). The first component of the amplitude a^1 can be expressed in terms of d^2,d^3 via the incompressibility condition

$$a^1(\tau) = -\mu \tan\theta \sin\tau d^2(\tau) - \mu \tan\theta \cos\tau d^3(\tau). \tag{63}$$

This observation allows us to restrict ourselves to Eqs. (62). If this equation has an unbounded solution d^2, d^3, corresponding to some initial data d_0^2, d_0^3, then the transport equation has an unbounded solution corresponding to the initial data $a_0^1 = -\mu \tan\theta d_0^3, d_0^2, \mu d_0^3$ and the flow in question is unstable. Equations (62) have 2π-periodic coefficients and can be solved via the Floquet method. Any solution can be presented as a superposition of Floquet modes of the form

$$[d^2(\tau), d^3(\tau)] = \exp(\sigma\tau)[D^2(\tau), D^3(\tau)], \tag{64}$$

where D^2 and D^3 are periodic functions, and the Floquet exponent σ is defined in such a way that $\exp(2\pi\sigma)$ is an eigenvalue of the corresponding monodromy matrix $M(\mu,\theta)$ parametrically depending on μ and θ. It is clear that for given μ and θ Eqs. (62) have an unbounded solution if the corresponding monodromy matrix $M(\mu,\theta)$ has an eigenvalue ν such that $|\nu| > 1$ or if it has an eigenvalue $|\nu| = 1$ of multiplicity two with a nontrivial Jordan cell corresponding to it. Noting that the average of the trace of the matrix in Eqs. (62) over $0 \leqslant \tau \leqslant 2\pi$ is zero, we use the Liouville theorem to show that the determinant of M is unity. Thus, eigenvalues of M can be found from the equation

$$\nu^2 - \operatorname{tr} M(\mu,\theta)\nu + 1 = 0, \tag{65}$$

and M has an eigenvalue ν with an absolute value greater than unity if $|\operatorname{tr} M(\mu,\theta)| > 2$ and eigenvalue $\nu = \pm 1$ of multiplicity two when $\operatorname{tr} M(\mu,\theta) = \pm 2$. Numerically it is very easy to compute $\operatorname{tr} M(\mu,\theta)$ for $0 < \mu < 1$ and $0 < \theta < \pi/2$, because of the analytic dependence of the coefficients of Eqs. (62) on the parameters μ and θ.

We start with the case of rigid body rotation and put $\mu = 1$. It is well known that in this case the functions D^2 and D^3 are linear combinations of $\sin\tau$ and $\cos\tau$ and the monodromy matrix can be found explicitly. Its trace can be written as $\operatorname{tr} M(1,\theta) = 2\cos(4\pi\cos\theta)$, so that $\operatorname{tr} M(1,\theta) \leqslant 2$ and the equality takes place when $\theta = \pi/3$. In this case the monodromy matrix coincides with the unity matrix and all solutions of Eqs. (62) are bounded. It means that in the case of rigid body rotation our local stability condition is marginally stable.

Now we consider the general case and assume that $0 < \mu < 1$. In this case we numerically find $\operatorname{tr} M(\mu,\theta)$. Our computations, which are similar to the computations of Refs. 13 and 33, show that for any $0 < \mu < 1$ one can find two angles $\theta_-(\mu)$ and $\theta_+(\mu)$ such that $\operatorname{tr} M(\mu,\theta) > 2$ when $\theta_-(\mu) < \theta < \theta_+(\mu)$, so that all elliptic stagnation points are unstable.

Finally we conclude that any steady flow of an incompressible fluid having a nondegenerate point of stagnation is unstable unless this point lies on the axis of rigid body rotation.

As a simple corollary of the above stated result we can prove that the classical Hill vortex is unstable due to the presence of two hyperbolic points lying on the axis of symmetry and infinitely many elliptic points lying on the minor axis.

In some cases we can expand the area of applicability of our result by means of a Galilean transformation. For example, for a plane parallel flow between two plates the corresponding velocity field can be written in the form

$$\mathbf{V}(\mathbf{x}) = [V^1(x^3), 0, 0], \tag{66a}$$

so that in the frame of reference moving with a constant velocity $\mathbf{U} = (U^1, 0, 0)$ parallel to the x^1 axis the velocity field remains steady. It can be written as follows:

$$\widetilde{\mathbf{V}}(\mathbf{x}) = [V^1(x^3) - U^1, 0, 0], \tag{66b}$$

where the tilde denotes the velocity field in the moving frame. Our analysis applied to the field $\widetilde{\mathbf{V}}$ shows that its points of stagnation are unstable. On the other hand, it is clear that any point in the original flow is a point of stagnation in an appropriate moving frame. Thus, we come to the conclusion that any point in a plane parallel flow is unstable in some moving frame. Nevertheless, this instability is not devastating because the corresponding modes grow in time algebraically [notice that we always deal with case (c) for plane parallel flows]. Similarly, we can show that for quasi-two-dimensional flows, i.e., for three-dimensional flows with the velocity field $\mathbf{V}(\mathbf{x})$, which is independent of the vertical coordinate x^3, any point $\mathbf{X}$ such that the velocity $\mathbf{V}(\mathbf{X})$ is vertical is unstable in an appropriate moving frame of reference.

The above stated result concerning the instability of stagnation points in incompressible flows is very pessimistic. It is natural to ask if the compressibility can have a stabilizing effect. The answer is no. Indeed, let us consider a nondegenerate stagnation point $\mathbf{X}$ in the flow of a compressible gas with the velocity $\mathbf{V}(\mathbf{x})$, the density $R(\mathbf{x})$, and the pressure $P(\mathbf{x})$. It can easily be shown that at the stagnation point $\mathbf{X}$ the matrix $\partial\mathbf{V}/\partial\mathbf{x}$ has the same properties as in the incompressible case and $\nabla P(\mathbf{X}) = 0$, $\nabla\cdot\mathbf{V}(\mathbf{X}) = 0$. Thus Eq. (51a) does not contain the b term and coincides with its incompressible counterpart. It means that all instabilities existing in the incompressible case exist in the compressible case as well. In other words, any steady subsonic flow of a compressible gas having a nondegenerate point of stagnation is unstable unless this point lies on the axis of rigid body rotation.

V. CONCLUDING REMARKS

The local stability conditions derived in Secs. II and III can be applied for studying the stability of vortex rings with swirl. It is shown in Sec. III that, in general, vortex rings without swirl are unstable with respect to localized modes. It is interesting and practically important to know if swirl has a stabilizing effect on localized modes (detailed discussion of the stability of vortex rings can be found in review papers[34,35]). Our preliminary computations indicate that the answer is positive. Because of space constraints, we cannot go into details and refer the reader to our companion paper.[5]

Let us briefly mention some results concerning the stability of general streamlines in vortex rings with swirl. We start with the idealized case of helical flows. Such flows are not asymptotically uniform but this difficulty can easily be overcome. For two-dimensional circular motions of an incompressible fluid ("pure" vortices in the terminology of Ref. 34) our stability condition is equivalent to the classical Rayleigh's centrifugal stability condition. For general circular columnar vortices (helical flows) of an incompressible fluid our stability condition yields the sufficient instability condition recently discovered by Leibovich and Stewartson.[36] It is worth noting that columnar vortices can be unstable not only with respect to exponentially growing perturbations but also with respect to perturbations growing algebraically. We obtain two different conditions guaranteeing instability with respect to these two types of perturbations. For columnar vortices in question these two conditions coincide, but for toroidal vortices this is not the case. For circular columnar vortices of a compressible gas we obtain a sufficient instability condition that was formulated by Eckhoff and Storesletten.[18] For two-dimensional incompressible flows with convex streamlines we obtain Bayly's stability condition.[13] Finally, for general axisymmetric toroidal vortex rings with swirl (both incompressible and compressible) we obtain two conditions guaranteeing instability with respect to exponentially growing perturbations and algebraically growing perturbations and demonstrate that in general algebraically growing perturbations are more unstable. To the best of our knowledge these conditions are new. They are too cumbersome to be presented here and we refer the interested reader to our paper.[5] It is worth emphasizing that in spite of the fact that the analytic expressions for our stability conditions are quite complex, they involve only certain integrals over stream surfaces and can effectively be applied for the numerical study of the stability of a given vortex ring.

Remark: The method described in the present paper can be used in order to analyze the stability of stagnation points located on the boundary of the domain occupied by an inviscid fluid.

ACKNOWLEDGMENTS

This research has been supported in part by National Science Foundation under Grant No. DMS-9100327, and by the Air Force Office of Scientific Research under Grant No. AFOSR-91-0044.

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