# AM 275 - Magnetohydrodynamics: Homework 1

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## Problem 1:

Show that

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} + p e_{kk} - 2\mu \left[ e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \right]^2.$$

*Proof.* First, we begin with the derivative identity

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

and in order to simplify this statement, take $\tau_i$ to be the i-th row vector of $\tau$, we have:

$$\sum_i u_i \nabla \cdot \tau_i = \sum_i \nabla \cdot u_i \tau_i - \tau_i \cdot \nabla u_i$$

Already we have shown the first RHS term originates from the derivative identity, whereas the other terms must originate from $-\sum_i \tau_i \cdot \nabla u_i$. Thus, we investigate this term in more detail.

$$-\sum_i \tau_i \cdot \nabla u_i = \sum_i \left[ p + \frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right] \delta_{ij} \cdot \nabla u_i - 2\mu e_i \cdot \nabla u_i$$

where $e_{kk}$ is written as $\nabla \cdot \boldsymbol{u}$ and $e_i$ is the i-th row of $e$ (as in $e_{ij}$). Notice that $\sum_i \delta_{ij} \cdot \nabla u_i = \sum_i \frac{\partial u_i}{\partial x_i} = \nabla \cdot \boldsymbol{u}$, and therefore,

$$\begin{aligned} -\sum_i \tau_i \cdot \nabla u_i &= \left[ p + \frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right] (\nabla \cdot \boldsymbol{u}) - 2\mu \sum_i e_i \cdot \nabla u_i \\ &= p(\nabla \cdot \boldsymbol{u}) + \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 - 2\mu \sum_i e_i \cdot \nabla u_i \end{aligned}$$

Thus we recover the second RHS term, $pe_{kk}$. Now we must show the rest of $-\sum_i \tau_i \cdot \nabla u_i$ recovers the last term of the RHS. We write the decomposition of $e_i$.

$$\begin{aligned}
-2\mu \sum_i e_i \cdot \nabla u_i &= -\mu \sum_i \left( \nabla u_i + \frac{\partial \boldsymbol{u}}{\partial x_i} \right) \cdot \nabla u_i \\
&= -\mu \sum_i |\nabla u_i|^2 + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\
&= -\mu |\nabla \boldsymbol{u}|^2 - \mu \sum_i \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\
&= -\mu |\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_j}{\partial x_i}
\end{aligned}$$

Now we must show by the transitive propery that,

$$\frac{2}{3}\mu(\nabla \cdot \boldsymbol{u})^2 - \mu|\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = -2\mu \left[ e_{ij} - \frac{1}{3} e_{kk}\delta_{ij} \right]_{ll}^2$$

We begin by writing the double inner product of these second order tensors (necessary in order to obtain a scalar) (also sorry about the indices, I couldn't decide which letters I wanted to stick with in the long run)

$$\begin{aligned}
-2\mu \left[ e_{ij} - \frac{1}{3} e_{kk}\delta_{ij} \right]_{ll}^2 &= -2\mu \left[ (e_{ij}^2)_{ll} - \frac{2}{3}(\nabla \cdot \boldsymbol{u}) e_{ll} + \frac{1}{9}(\nabla \cdot \boldsymbol{u})^2 \delta_{ll} \right] \\
&= -2\mu \left[ (e_{im} \cdot e_{mj})_{ll} - \frac{2}{3}(\nabla \cdot \boldsymbol{u})^2 + \frac{1}{3}(\nabla \cdot \boldsymbol{u}^2) \right] \\
&= -\frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_m}\frac{\partial u_m}{\partial x_j} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} + \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_j}{\partial x_m} \right)_{ll} + \frac{2}{3}\mu(\nabla \cdot \boldsymbol{u})^2 \\
&= -\frac{\mu}{2} \left( \nabla u_i \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} + \nabla u_i \cdot \nabla u_i + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \right) + \frac{2}{3}\mu(\nabla \cdot \boldsymbol{u})^2 \\
&= -\frac{\mu}{2} \left( 2|\nabla \boldsymbol{u}|^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \right) + \frac{2}{3}\mu(\nabla \cdot \boldsymbol{u})^2 \\
&= \frac{2}{3}\mu(\nabla \cdot \boldsymbol{u})^2 - \mu|\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}
\end{aligned}$$

Therefore, we have shown that

$$\begin{aligned}
u_i \frac{\partial \tau_{ij}}{\partial x_j} &= \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \boldsymbol{u}) + \frac{2}{3}\mu(\nabla \cdot \boldsymbol{u})^2 - \mu|\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \\
&= \frac{\partial u_i \tau_{ij}}{\partial x_j} + pe_{kk} - 2\mu \left[ e_{ij} - \frac{1}{3} e_{kk}\delta_{ij} \right]^2
\end{aligned}$$

where $[\cdot]^2$ implies a tensor double inner product (c.f. §3.5, "Tensor Calculus Made Simple," Sochi 2016) where first a (tensor) inner product is taken and the resultant second order tensor is contracted to become a scalar. $\square$

# Problem 2:

## 2.1 Show that the imcompressible induction equation is

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

*Proof.* We begin by writing the (non-diffusive) induction equation and the corresponding derivative identity.

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{u}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}$$

Using this substitution and keeping in mind that $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{u} = 0$ we obtain,

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

□

## 2.2 Show that the compressible induction equation can be written as

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}}{\rho}\right) + (\boldsymbol{u} \cdot \nabla)\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right)\boldsymbol{u}$$

*Proof.* We begin by taking the compressible induction equation and multiplying by $1/\rho$.

$$\frac{1}{\rho}\frac{\partial \boldsymbol{B}}{\partial t} + \frac{1}{\rho}(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = \frac{1}{\rho}(\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - \frac{1}{\rho}\boldsymbol{B}(\nabla \cdot \boldsymbol{u})$$

Then, we use the product rule derivative identity to change some of the derivatives. We have,

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}}{\rho}\right) + \frac{\boldsymbol{B}}{\rho^2}\frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \nabla)\frac{\boldsymbol{B}}{\rho} + \frac{\boldsymbol{B}}{\rho^2}(\boldsymbol{u} \cdot \nabla)\rho = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right)\boldsymbol{u} - \frac{\boldsymbol{B}}{\rho}(\nabla \cdot \boldsymbol{u})$$

Here we consider the conservation of mass equation which for compressible fluids is written as,

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} = 0 \\
\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} + \frac{1}{\rho^2}(\nabla \cdot \rho \boldsymbol{u}) = 0 \\
\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} + \frac{1}{\rho^2}\left(\rho(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\rho\right) = 0 \\
\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} + \frac{1}{\rho^2}(\boldsymbol{u} \cdot \nabla)\rho = -\frac{1}{\rho}(\nabla \cdot \boldsymbol{u}).
\end{aligned}
$$

BNotice that we can take this equation, multiply it by $\boldsymbol{B}$ and subtract it from the induction equation. This leaves us with,

$$
\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}}{\rho}\right) + (\boldsymbol{u} \cdot \nabla)\frac{\boldsymbol{B}}{\rho} = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right)\boldsymbol{u}
$$

□

# Problem 3:

## 3.1 Derive the induction equation given that $\sigma$ is not necessarily constant

*Proof.* Let us begin with Ohm's law as we have written in lecture.

$$
\boldsymbol{j} = \boldsymbol{j}' = \sigma \boldsymbol{E}'
$$

$$
\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}
$$

$$
\begin{aligned}
\nabla \times \boldsymbol{B} = \mu_0 \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \\
\nabla \times \left(\frac{1}{\mu_0 \sigma}\nabla \times \boldsymbol{B}\right) = \nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\frac{1}{\mu_0 \sigma}(\nabla \times \nabla \times \boldsymbol{B}) - \frac{1}{\mu_0 \sigma^2}\nabla \sigma \times (\nabla \times \boldsymbol{B}) = -\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B} + \frac{1}{\mu_0 \sigma^2}((\nabla \boldsymbol{B})^T \cdot \nabla \sigma - (\nabla \sigma \cdot \nabla)\boldsymbol{B}) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})
\end{aligned}
$$

This can then be simplified keeping in mind that $\nabla \cdot \boldsymbol{B} = 0$, and especially if the flow is incompresible, to the following:

$$
\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + \frac{1}{\mu_0 \sigma^2}(\nabla \sigma \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \frac{1}{\mu_0 \sigma^2}((\nabla \boldsymbol{B})^T \cdot \nabla \sigma) + \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B}
$$

Essentially we see the appearence of two new terms if the conductivity is not constant. First, the advection of $\boldsymbol{B}$ by the gradient of conductivity, and then some weird term related to $\nabla \boldsymbol{B}^T$ on the RHS. $\square$

# Problem 4:

## 4.1 Show that initial conditions of the divergence of the magnetic field are preserved for Maxwell's equations

*Proof.* In order to show this, we must first assume that the temporal and spatial derivatives can be taken in any order, i.e. $\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$. We proceed by taking the dot product of Faraday's law,

$$\nabla \cdot \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \cdot (-\nabla \times \boldsymbol{E})$$
$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

where the RHS is zero because the divergence of a curl is always zero. Thus if we have that $\nabla \cdot \boldsymbol{B} = 0$ at $t = 0$, it will always be zero. $\square$

## 4.2 Show that initial conditions of the divergence of the magnetic field are preserved for the induction equation

*Proof.* A similar proof can be written from the perspective of the induction equation. Let us write a form of the induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B}\right) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

where $\sigma$ is not necessarily a constant and the fluid is not necessarily incompressible. Similarly, we take the divergence of this equation and obtain,

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = \nabla \cdot \left(\nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B}\right) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})\right)$$
$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

since again, the divergence of a curl is always zero. Therefore, from the perspective of the induction equation, we have that $\nabla \cdot \boldsymbol{B} = 0$ will be maintained for all $t > 0$ if $\nabla \cdot \boldsymbol{B} = 0$ at $t = 0$. $\square$