Pre-candidacy notes:

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Contents

1 Billant & Chomaz Papers

1.1 Experimental evidence for a new instability of a vertical columnar vortex pair in a strongly stratified fluid (2000)

- The first paper in a series of papers by Billant and Chomaz describing and investigating the properties of the so called "zigzag" instability present in the Lamb-Chaplygin vortex pair (a counterrotating vortex dipole).
- This paper demonstrated the existence of such an instability from experimental findings at sufficient stratification. For insufficient stratification $Fr \geq 0.2 \pm 0.01$, the ellipitcal instability appears to be the dominant instability and after its gravitational collapse, the vortex pair appears to irregularly zigzag into layer formation.
- From what can be observed from the zigzag instability is that it doesn't perturb the horizontal cross-section structure of the vortex column, only its vertical structure. It is positted that this phenomenon may be responsible for the layering phenomenon demonstrated in many stratified flows.
- Over a long enough time frame the original vortex pair column ends up divided into pancake dipole segments in the vertical direction, obtaining what is usually described as pancake eddies in the flow.

1.2 Self-similarity of strongly stratified inviscid flows (2001)

- Posits the scaling of an intrinsic vertical length scale of strongly stratified flows, $l_z \propto U/N$.
- Third paper which describes the "Zig-Zag" instability. Two previous papers conducted linear stability analysis of the instability.
- Zig-zag instability is self-similar with respect to $k_z U/N$ which implies that the dominant vertical wavenumber of the flow is proportional to Fr.

1.3 Three-dimensional stability of vertical columnar vortex pair in a stratified fluid

- This paper conducted a numerical stability analysis on the linearized equations using mean-pertubation flow separation. They found for flows with sufficient stratification that the primary instability of the counterrotating vortex pair was the "zig-zag" instability in which the entire vortex column was destabilized and oscillationed ide to side with a typical scale height, later found to be proportional to the froude number. For insufficient stratification, the elliptical instability was the dominant instabtility.
- Among their findings is the approximate scaling that the root mean squared $u_z' \propto 1/Fr$ and $p' \propto Fr$ (normalized by the rms horizontal velocity). Furthermore, these numerical findings for the growth rate of the zig-zag instability concur with the experimental results within reasonable error.
- Their nondimensionalization involved

$$Fr = \frac{U_{\text{prop}}}{NR}$$

where U_{prop} is the propagation speed of the vortex pair, and R is given by the dipole radius. This is similar to the non-dimensionalization from Chini et al, in which the unit velocity and lengthscale are given by the typical horizontal flow (i.e. horizontal forcing which is order 1 in both U and L).

2 Hattori & Hirota Papers

2.1 Stability of two-dimensional Taylor-Green vortices in rotating stratified fluids (2023)

- Conducted a local stability analysis as well as DNS and analyzed the data using modal stability analysis.

- Linear Stability analysis is conducted on a linearized and inviscid version of the governing equations.
- Both the DNS and LSA begin with a base flow composed of Taylor-Green vortices, which are arranged in a grid lattice.
- 5 instabilities are identified from the LSA, each with a different mechanism and different instability/resonance conditions.
- Linear Stability analysis found that the pure hyperbolic instability is often the fastest growing istability as also the most realizable. Variation of the input rossby and froude numbers reveals characteristics of other secondary instabilities which vary with vertical wavenumber, and radius from vortex centers (as well as input parameters).

2.2 Modal stability analysis of arrays of stably stratified vortices (2021)

- This paper conducted a modal stability analysis for nonrotating stratified Taylor-Green (and Stuart) vortices. They find three vital hyperbolic instabilities which are primarily affected by stratification: the pure hypebolic, strato-hyperbolic, and mixed-hyperbolic instability.
- One of the distinguishing metrics for these instabilities are two integral quantities s_1, s_2.

$$s_1 = \frac{\int_{|\Phi| \leq 0.1b} \omega_z'^2 dV}{\int_V \omega_z^2 dV}$$
$$s_2 = \frac{\int_V \left(\nabla \cdot_h \boldsymbol{u}_h'\right)^2 dV}{\int_V \omega_z^2 dV}$$

  The stratohyperbolic instability is characterized by zero or near zero values of s_1 (< 0.1). The pure hyperbolic instability is characterized by low values of s_2 (< 0.1) and intermediate values of s_1 (~ 0.3). Finally, the mixed hyperbolic instability is seemingly unbounded by either s_1 or s_2 (c.f. Figure 7 for visual representation).

- The modal stability analysis shows that the fastest growing mode and instability of the taylor green vortices in the stratified case is always the mixed hyperbolic instability and the fastest growing mode is consistenly near $k_z L_0 \approx 10$ (multiplied by L_0 to non-dimensionalize k_z).
- This paper also discusses and demonstrates the modal stuctures of these instabilities. Most important of these characteristics is its vertical structure, however, there are modes with nontrivial structures such as the mixed-hyperbolic "spiral-mode" which very closely resembles the structure of the vertical velocity from my own rotating stratified simulations (within the stable cyclones in the flow).

2.3 Suzuki et al (2018), Strato-hyperbolic instability: a new mechanism of instability in stably stratified vortices

- Need to read this, makes comparisons between the stratohypberolic and zigzag instability

3 Miyazaki and Fukumoto

3.1 Three-dimensional instability of strained vortices in a stably stratified fluid (1992)

- This paper conducts a linear stability analysis of "unbounded strained vortices". The linear stability analysis is derived analytically and then solved numerically by a floquet problem. The primary investigation is into the elliptical instability of "Pierrehumbertt type". Two other instability modes are noted which depend distinctly on the buoyancy frequency N.

3.2 Elliptical instability in a stably stratified rotating fluid (1993)

- This paper studies the elliptical instability present for vorticies with an aspect ratio between the major and minor axes of the ellipse created by the streamlines of the flow $\varepsilon \neq 1$.
- The problem assumed the flow is inviscid, incompressible, and nondiffusive. Their nondimensionalization relies on the following:

$$Ro = \frac{U}{\Omega L} = \frac{\gamma}{\Omega}, \quad Fr = \frac{U}{NL} = \frac{\gamma^2 L}{g}$$

  the paper may have a typo in the definition for Ro

- The problem is solved numerically using a linearized mean-perturbation equation. The perturbation equations assume an ansantz of fourier modes with a floquet multiplier and then are numerically intergrated to determine the growth rate for varying rotation and stratification.
- The paper divides this study into sections where first the effects of stratification are ignored (similar to the 1992 papers), and then rotation is ignored. Finally, they cover some cases where both rotation and stratification are included and characterize phenomenon of the elliptical instability.
- In the purely rotating case, specifically for strong cyclones, we see that there is a subharmonic and superharmonic mode of the elliptical instability where the superharmonic mode is very weak and exists for a very small range of vertical wavenumbers. For strong anti-cyclones, we find that the superharmonic mode disappears and the subharmonic mode becomes very realizable as horizontal disturbances destabilize the vortex.
- When the anticyclones are weaker ($|Ro|$ is larger), the elliptical instability's growth rate resembles that for the cyclones.
- Weak cyclones/anticyclones in the presence of stratification and roation are very stable to the elliptical instabtility. Roughly we can find if a particular strained vortex is unstable by seeing if its initial ε_0^* has any unstable modes. If not, it may be the case that that particular vortex is stable in a specific configuration of Fr and Ro.
- mention their conclusion

4 Linden Papers

4.1 The stability of vortices in a rotating, stratified fluid

- experimental setup where two layer vortices are created in a closed volume. A rotating tank whose center is aligned with the axis of rotation, is spun up to a steady state and then a cylinder immersed at a depth H in the center is removed which contains dye. When spun up, this cylinder will be rotating, and form a vortex in the flow (either at the surface or at the bottom according to the density regime ($\rho_1 > \rho_2$ or $\rho_1 < \rho_2$)
- These closed volume vortices lose their intial structure over time, whereby the dyed vortex will disperse radially in either an axissymmetric or non-axissymmetric manner (need to clarify what causes this)
- Essentially, the disparity between the density layers causes the vortex to become unstable and move (either upwards to downwards and then radially) according to the stratification. This movement is opposed by the conservation of angular momentum, and we see either cyclonic or anticyclonic movements within the vortex according to whether the vortex expands or contracts radially.
- Non-axissymmtric movements are then induced due to a baroclinic instability which fueled by potential energy stored in the density gradient and by the kinetic energy contained in the horizontal shear in the flow.

- The number of vortices shed radially from the original vortex is given by a modal configuration n, which increases with the depth ratio of the vortex. That is, for smaller vortex depth ratios, the fastest growing unstable mode is $n = 2$, whereas for larger depth ratios the fastest growing unstable mode is $n = 3$ (three shed vortices).

- The shed vortices are in a dipole configuration, similar to lamb-Chaplygin vortex pairs.

5 Herring and Metias

5.1

- stuff

6 Waite and Bartello

6.1

- stuff

7 GFD Group (Garaud, Chini, Shah, Caulfield . . .)

7.1 Exploiting self-organized criticality in strongly stratified turbulence (2021)

- Developed a multiscale model for strongly stratified flows wherein an aspect ratio α is used to describe scale separation of horizontal and vertical motions recovering that $l_z \propto Fr$ as posited by

7.2 Cope et al. 2020

7.3 Shah et al. 2023

7.4 Layering, Instabilities, and Mixing in Turbulent Stratified Flows, Caulfield 2021

- This is a review paper aimed at highlighting revcent findings in density stratified flows as well as the importance of relevant length scales, energy conversion mechanisms, and thermal/compoisitional mixing in the flow.

- This paper dicusses many aspects of stratified flow. Some of the most fundamental quantities which are still trying to be understood are the mixing efficiency and turbulent flux coefficient as they depend on flow parameters. The complication with this conceptualization, is that it is unclear which flow paremeters are the right ones, and more importnatly if the flow parameters exhibit dependence on one another.

- Caulfield also details several relevant lengthscales intrinsic to stratified turbulence. Among these scales are a typical horizonal length scale of the background flow l_h, the vertical length scale from the anisotropic flow field l_v, the Ozmidov Length scale l_O which describes the length scale seperating anisotropic dynamics from the isotropic dynamics, and finally the Kolmorogov length scale l_K which describes the smallest length scale where isotropic turbulence occurs. According to Caulfield, in the "Layered Anisotropic Stratified Turbulence" (LAST) regime there is an implicit asymptotic ordering of these length scales given by,

$$l_K \ll l_O \ll l_v \ll l_h$$

8 Praud, Sommeria, Fincham

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9 Guo, Taylor, Zhou

9.1 Zigzag instability of columnar Taylor-Green vortices in a strongly stratified fluid

- This paper performs a linear stability analysis of the primary instability of the Taylor-Green vortex columns. They find that this instability is the zig-zag instability, whereby it develops with a vertical lengthscale and wavenumber which scale with Fr^{-1} as shown by Billant and Chomaz in their 2001 papers on the same instability.
- They also detail in a similar manner to the Hattori papers, the modal structure of this fastest growing unable mode. Note that this confirms that the mixed hyperbolic instability presented by the Hattori papers (modal stability analysis of 2d taylor green vortices 2023) is in fact the zigzag instability, which is reliably the fastest growing instability for the 2D Taylor-Green vortices in a stratified fluid.
- The findings of this paper support the claim that the zig-zag instability is a fundamental instability which transfers energy from vertically invariant modes to three-dimensional turbulence characterized by the Froude number.
- Notes on the study, this linear stability analysis shows only the development of instability from the vortex columns and does not include the effect of rotation in their study.
- They compute the most dominant lengthscale of the flow according to an integrl over wavenumbers of the kinetic energy fo the flow, i.e. grouping of energy is what they use to declare the dominant vertical lengthscale of the instabiltiy (c.f. eqn 3.4)
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