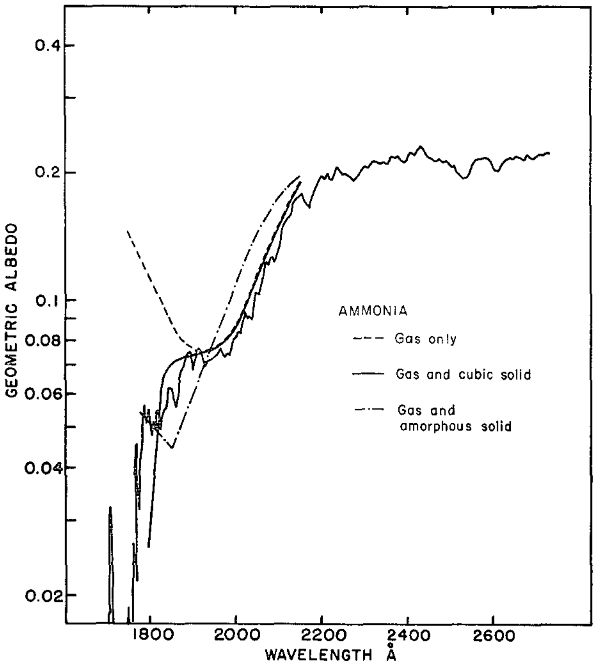

FIG. 1. Experimental (jagged line) and calculated (smooth lines) Jovian geometric albedos.

cloud particles and atmospheric gases. Both amorphous and cubic crystal ices were considered. Which form occurs depends upon whether the ice forms above or below 125K. The cubic crystal forms above 125K. If one assumes only gaseous ammonia, the albedo curve can be fitted nicely down to 1950 Å, but at shorter wavelengths there is a considerable discrepancy between theory and experiment. Such a curve is shown as a dashed line in Fig. 1 for an amount[1] of 1.6×10^{-3} cm-atm of NH_3. If a 8.5 μ layer of cubic crystal ammonia is added to the gas, strong absorption occurs below 1900 Å, and a reasonably good fit to the data is obtained. This curve is shown in Fig. 1 as a solid line. If amorphous ice is used with the gas, the agreement is poor. A sample calculation for the amorphous ice, also plotted in Fig. 1, shows that the trend of the curve is not correct below 1900 Å.

A mixture of ammonia gas and cloud particles of almost pure solid NH_3 with a cubic structure is quite consistent with current Jovian atmospheric models. For example, since Lewis (1969) and others predict pure solid NH_3 cloud tops at a temperature level near 150K, the cubic ice form would be expected. It is thought that this is a much more likely candidate for the absorption below 2100 Å than the hydrocarbons suggested by Greenspan and Owen (1967).

[1] No attempt should be made to attach a physical significance to the amounts of substances used in our over-simplified model.

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Geostrophic Turbulence

JULE G. CHARNEY
Dept. of Meteorology, Massachusetts Institute of Technology, Cambridge

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ABSTRACT

A theory is presented for the spectra of horizontal velocity and temperature at high wavenumbers in three-dimensional quasi-geostrophic flow. The theory predicts a minus third power dependence on both the horizontal and vertical wavenumbers for the spectra of both the kinetic energy and the temperature variance, with amplitudes determined by the pseudo-potential vorticity transfer function. It also predicts equipartition among the components of kinetic energy and available potential energy. Comparisons of the predicted with the observed spectra of kinetic energy and temperature are cited. There is approximate agreement, notably in the prediction of equipartition.

1. Introduction

Theoretical studies of atmospheric predictability (Lorenz, 1969; Leith, 1971) have emphasized the importance of the equilibrium energy spectrum of the atmosphere as determining the rate at which uncertainties at small scales propagate to larger scales and ultimately affect the principal energy bearing scales. Observations (Horn and Bryson, 1963; Wiin-Nielsen, 1967; Julian *et al.*, 1970) and model calculations (Manabe *et al.*, 1970; Wellck *et al.*, 1970) indicate an approximate K^{-3} dependence of the kinetic energy on the longitudinal wavenumber K between 7 and 20, i.e., for wavelengths in the range 4000–1500 km at middle latitudes. This spectral behavior is so different from the Kolmogoroff $-5/3$ law for the inertial subrange of three-dimensional homogeneous isotropic turbulence that it suggests an entirely different mechanism of turbulent scale interaction. Since the early work of Rossby *et al.* (1939), the flow of the atmosphere at a mid-level has come to be regarded as approximately two-dimensional. For such flows it has been shown by Onsager (1949), Lee (1951), Batchelor (1953), and especially Fjørtoft (1953) that vorticity conservation places a strong constraint on the nature of the scale interactions. The mean-squared vorticity (enstrophy) as well as the kinetic energy must be conserved, and this prevents the kind of energy cascade toward high wavenumbers that is produced in three-dimensional flow by the stretching of the vortex tubes. In two dimensions, Fjørtoft found that a transfer of energy from one wavenumber to a higher one must be accompanied by a transfer of still more energy toward a lower wavenumber. This circumstance led Kraichnan (1967) to postulate inertial subranges for two-dimensional turbulence in which energy injected in a given wavenumber band is transferred uniformly to lower wavenumbers while enstrophy is transferred uniformly to higher wavenumbers. The transfers of enstrophy in the first range and energy in the second are identically zero. A similarity argument then gives a Kolmogoroff $k^{-5/3}$ law for the kinetic energy per unit scalar wavenumber k in the first range and a k^{-3} law in the second. Kraichnan's hypotheses were approximately confirmed in numerical experiments carried out by Lilly (1969, 1971a,b).

Since the publication of Kraichnan's theory, the k^{-3} behavior of atmospheric turbulence at wavelengths smaller than the baroclinic instability excitation wavelength has been ascribed to the approximate two-dimensional character of the motions. But one must ask: Are the motions in the atmosphere, especially in the range $K=7-20$, two dimensional in any real sense? Observations show that the main synoptic-scale motions in the atmosphere are decidedly baroclinic, with vertical variations in velocity, temperature, etc., through the depth of the troposphere as great as the horizontal variations. In general, changes in the vertical vorticity component are as much due to vertical stretching of the vortex tubes of the earth's rotation as to horizontal advection. The similarity between the observed spectrum and that of two-dimensional flow would thus seem to be somewhat fortuitous.

There is, however, a deeper similarity. Two-dimensional inviscid flow is governed by the conservation of a single scalar invariant, the vorticity. Three-dimensional quasi-geostrophic flow is governed by the conservation of two scalar invariants, potential vorticity and potential temperature (Charney, 1948). A systematic scaling analysis permits these two inviscid-adiabatic constants of the motion to be combined into a single constant which will be called the "pseudo-potential vorticity," and whose conservation law completely determines the motion (Charney, 1960; Charney and Stern, 1961). In the next section it will be shown that conservation of the pseudo-potential vorticity also forbids an energy cascade under certain conditions, and in the following section that it leads in a natural way to the k^{-3} law at wavenumbers higher than the excitation wavenumber for both the horizontal kinetic energy and the temperature variance.

2. Energy cascades in quasi-geostrophic flow

The proof to be presented has already been given by Charney (1966) but in a publication which is not easily accessible. Following the scaling analysis presented by Charney and Stern for geostrophic flow on a β plane, we introduce the geostrophic streamfunction

$$\psi=\frac{p-\bar{p}}{f_0\bar{\rho}}, \tag{1}$$

in terms of which the geostrophic relationship becomes

$$\mathbf{V}=\mathbf{k}\times\nabla\psi, \tag{2}$$

and the hydrostatic equation

$$\ln\theta-\ln\bar{\theta}\approx\frac{\theta-\bar{\theta}}{\bar{\theta}}\approx\frac{f_0}{g}[\psi_z-(\ln\bar{\theta})_z\psi]\approx\frac{f_0}{g}\psi_z, \tag{3}$$

or

$$\ln T-\ln\bar{T}\approx\frac{T-\bar{T}}{\bar{T}}\approx\frac{f_0}{g}[\psi_z-(\ln\bar{T})_z\psi]\approx\frac{f_0}{g}\psi_z. \tag{4}$$

In the above p is pressure, ρ density, T temperature, θ potential temperature, $|\mathbf{V}|$ the horizontal velocity, ∇ the horizontal gradient operator, z the vertical coordinate, $\mathbf{k}$ a vertical unit vector, g the acceleration of gravity, $f_0=2\Omega\sin\phi_0$ the Coriolis parameter at the latitude ϕ_0, Ω the angular speed of the earth's rotation, and the bars denote horizontally averaged mean values depending on z. If y is the meridional distance from the latitude ϕ_0 and w the vertical velocity, the equation for the vertical component of vorticity in inviscid flow on the β plane becomes

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)(\nabla^2\psi+\beta y)=\frac{f_0}{\bar{\rho}}(\bar{\rho}w)_z, \tag{5}$$

where $\beta=2\Omega\cos\phi_0/a$ and a is the radius of the earth. The first law of thermodynamics for adiabatic motion may be written

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\psi_z+\frac{N^2}{f_0}w=0, \tag{6}$$

where

$$N^2(z)=g(\ln\bar{\theta})_z \tag{7}$$

is the square of the mean Brunt-Väisälä frequency.

Elimination of w between (5) and (6) gives

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\left[\nabla^2\psi+\frac{f_0^2}{\bar{\rho}}\left(\frac{\bar{\rho}}{N^2}\psi_z\right)_z+\beta y\right]=0, \tag{8}$$

which asserts that the quantity in brackets is conserved at the horizontal projection of the particle motion. This quantity will be called the "pseudo-potential vorticity" to distinguish it from the "potential vorticity" which is conserved at a particle.

It was shown by Charney and Stern that the above equation is subject to the scale restrictions

$$\left.\begin{aligned}\frac{k_H U_k}{f_0}&<\mathrm{O}(1)\\ \frac{f_0^2/k_H{}^2}{g/k_V}&<\mathrm{O}(1)\\ \frac{k_V}{k_H}&\leqslant\mathrm{O}(\mathrm{Ro})^2\\ (ak_H)^{-1}&\leqslant\mathrm{O}(\mathrm{Ro})\\ k_V&>\mathrm{O}(\ln\bar{\theta})_z\end{aligned}\right\}, \tag{9}$$

where k_H and k_V are characteristic horizontal and vertical wavenumbers, and U_k is a characteristic horizontal velocity corresponding to the scales k_H and k_V. Not by accident these inequalities define bands of k_H and k_V which include the spectral ranges of interest.

To fix ideas we suppose the flow to be periodic in the zonal direction and to be contained between two vertical walls at the latitudes y_1 and y_2. We write (8) in the form

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)L(\psi)+\beta\psi_x=0, \tag{10}$$

where x is the eastwardly directed zonal coordinate, and $L(\psi)$ is defined by

$$L(\psi)\equiv\nabla^2\psi+\frac{f_0^2}{\bar{\rho}}\left(\frac{\bar{\rho}}{N^2}\psi_z\right)_z. \tag{11}$$

Multiplication of (10) by $-\bar{\rho}\psi$ and integration over x, y, z gives the equation for the total energy E:

$$\frac{dE}{dt}\equiv\frac{d}{dt}\int\int\int\frac{1}{2}\left(\nabla\psi\cdot\nabla\psi+\frac{f_0^2}{N^2}\psi_z{}^2\right)\bar{\rho}dxdydz=0. \tag{12}$$

The second term in brackets is the geostrophic form of the available potential energy. Similarly, multiplication by $\bar{\rho}L(\psi)$, integration by parts, and utilization of the boundary condition obtained from (6) by setting $w=0$ at $z=0$ gives

$$\frac{dF}{dt}\equiv\frac{d}{dt}\int\int\int[L(\psi)]^2\bar{\rho}dxdydz$$

$$=\beta\int\int\left(\bar{\rho}\frac{f_0{}^2}{N^2}\psi_x\psi_z\right)_{z=0}dxdy=0, \tag{13}$$

providing $\psi_z=0$ at $z=0$, i.e., if the ground is an isentropic or isothermal surface. Let us first suppose that this is the case. Then both E and F are constants of the motion, and we find that

$$2E+\int\int\int\psi L(\psi)\bar{\rho}dxdydz$$

$$=-\int\int\left(\bar{\rho}\frac{f_0^2}{N^2}\psi\psi_z\right)_{z=0}dxdy=0. \tag{14}$$

Now L is a self-adjoint elliptic operator with a complete orthonormal set of eigenfunctions ψ_m and eigenvalues λ_m ($m=1,2,\cdots$) satisfying the homogeneous boundary conditions $(\psi_m)_z=0$ at $z=0$ and $\bar{\rho}|\nabla\psi_m|^2\to0$ or $\bar{\rho}|\psi_m\psi_{mz}|\to0$ at $z\to\infty$. The latter correspond to vanishing energy density or energy flow. These eigenfunctions may be so ordered that λ_m is a positive, nondecreasing function of m which tends toward infinity like $(m^{\frac{1}{3}})^2$. Moreover, the nodal surfaces of these eigenfunctions divide the fluid into m subdomains whose volumes approach zero like $(m^{\frac{1}{3}})^{-3}$. Thus, the eigenfunctions are completely analogous to Fourier series, and if we can show that energy cannot cascade toward higher m, it will be equivalent to showing that it cannot cascade into smaller scales.

By virtue of the completeness property, we may set

$$\psi=\sum_1^\infty a_m\psi_m, \tag{15}$$

where

$$L(\psi_m)=-\lambda_m\psi_m. \tag{16}$$

Substituting the above into (13) and (14) and utilizing the orthonormality property,

$$\iiint \bar{\rho}\psi_m\psi_n dxdydz = \delta_n^m, \tag{17}$$

we obtain

$$\left.\begin{aligned} 2E &= \sum_1^\infty \lambda_m a_m^2 \equiv \sum_1^\infty b_m = \text{constant} \\ 2F &= \sum_1^\infty \lambda_m^2 a_m^2 \equiv \sum_1^\infty \lambda_m b_m = \text{constant} \end{aligned}\right\}. \tag{18}$$

It then follows that

$$\sum_M^\infty b_m < \frac{1}{\lambda_M}\sum_M^\infty \lambda_m b_m < \frac{F}{\lambda_m},$$

i.e., that $\sum_M^\infty b_m$ approaches zero with increasing M, and an energy cascade is impossible. All the other theorems pertaining to energy exchange among spectral components in two-dimensional flow may now be shown to apply to three-dimensional quasi-geostrophic flow as well, but now it is the geostrophic constraint, not the two-dimensionality, that prevents the cascade.

3. Spectral properties of quasi-geostrophic flow

The theorem just derived is not strictly valid if the ground is not an isentropic or isothermal surface, for in that case the right-hand integrals in (13) and (14) do not vanish. Surface temperature gradients permit the formation of velocity and temperature discontinuities and therefore an energy cascade.[1] Nevertheless, we shall postulate that, in general, small-scale motions at some distance from the bottom boundary are not appreciably influenced by the boundary condition on ψ and behave as if this condition were homogeneous—in the same manner as small-scale turbulence bounded by walls is assumed in the Kolmogoroff theory to be independent of the walls if sufficiently far from them. We are less justified in making this assumption in the present case because of the existence of fronts which penetrate into the fluid. (Quasi-geostrophic fronts appear to form only at boundaries.) But if these fronts are sufficiently sporadic, random, and of sufficiently small amplitude, they will merely produce a weak k^{-2} spectrum superimposed on the spectrum the principal motions dictate.

We therefore suppose that energy is prevented from flowing from large to small scales but not from small to large scales. A formal analogy with Kraichnan's arguments for two-dimensional flow may then be established.

[1] In a similar manner, a free surface in two-dimensional flow may be shown to permit an energy cascade.

For this purpose we introduce the substitutions

$$d\zeta = \frac{N}{f_0}dz, \tag{19}$$

$$\chi = \left[\frac{\bar{\rho}(z)}{\bar{\rho}(0)}\right]^{\frac{1}{2}}\psi, \tag{20}$$

into (10) and (11), and obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)G(\chi) + \beta\chi_x = 0, \tag{21}$$

where

$$\mathbf{V} = e^{\zeta/(2\lambda)}\mathbf{k}\times\nabla\chi, \tag{22}$$

$$G(\chi) \equiv \chi_{xx} + \chi_{yy} + \chi_{\zeta\zeta} - (\ln\bar{\theta})_\zeta\left(\chi_\zeta + \frac{\chi}{2\lambda}\right) - \frac{\chi}{4\lambda^2}, \tag{23}$$

providing $\bar{\rho}$ is approximated by

$$\bar{\rho} = \bar{\rho}_0 e^{-z/H}, \tag{24}$$

H being defined as the scale height RT_m/g corresponding to the volume-averaged mean temperature T_m, and $\lambda = HN/f_0$ is the Rossby radius of deformation corresponding to the vertical scale H.

The following additional assumptions are made:

1) The horizontal scale of the turbulence, as well as the vertical scale in the stretched coordinate ζ, are small in comparison with 2λ. (This implies that the scales are small in comparison with the baroclinic excitation scales.)

2) The excitation energies are so high that advection of relative vorticity dominates advection of the earth's vorticity: if U_k is the energy at horizontal wavenumber k_H, then $k_H^2 U_k \gg \beta$. Hence the β term in (21) may be ignored and the energy cannot be dispersed by Rossby waves.

3) The scale of variation of N in the stretched coordinate is larger than the vertical scale of the turbulence.

4) Away from boundaries the turbulence is locally homogeneous and isotropic in horizontal planes, except possibly for an irrelevant Gallilean translation with the mean flow. This is permitted by assumption 2) which results in an invariance of (21) under rotation about a vertical axis.

5) The nonlinear interactions are local in wavenumber space, that is, the wave components interact only with components of comparable wavenumber. This assumption is made by both Kolmogoroff and Kraichnan but is by no means obvious. Kraichnan (1967) and Lilly (1969) have considered the consequences of non-localness.

6) The input energy is high enough to guarantee an inertial subrange for which the Reynolds numbers are so high that no appreciable internal viscous dissipa-

tion occurs. Moreover, boundary friction is assumed not to affect the internal turbulent motion at sufficiently high wavenumbers.

As a consequence of assumptions 1)–6), the exponent in (22), the zero- and first-order terms in (23), and the β term in (21) may be ignored. They will only influence the large-scale flows which, by assumption 4), can only produce a translation which leaves unaffected the energy relationships. The pseudo-potential vorticity equation is thereby reduced to

$$\chi_t=\frac{\partial(\triangle\chi,\chi)}{\partial(x,y)}, \tag{25}$$

where

$$\triangle\chi\equiv\chi_{xx}+\chi_{yy}+\chi_{\zeta\zeta}, \tag{26}$$

and results in the energy equation

$$E=\int\int\int\frac{f_0}{2N}[(\chi_x)^2+(\chi_y)^2+(\chi_\zeta)^2]dxdyd\zeta=\text{constant}. \tag{27}$$

We now invoke assumption 4), which may be interpreted as stating that χ_x, χ_y and χ_ζ are random, uncorrelated functions of x and y for each ζ whose statistics are similar and homogeneous in x and y. Since (25) is invariant under vertical translation, we also assume that $\nabla\chi$ is not correlated with $\nabla\chi_\zeta$ as well as χ_ζ. It then follows that $\nabla\chi$ is homogeneous in ζ as well as in x and y. Consequently, χ_ζ must also be homogeneous in ζ, and we may assume that for some scale factor σ the vector $(\chi_x,\chi_y,\sigma\chi_\zeta)$ is homogeneous and isotropic in x, y *and* ζ. Since all physical parameters have been removed from (25), and because of the three-dimensional symmetry of $\triangle\chi$, it is difficult to see how this factor can be other than unity. We therefore postulate that this is so, and, following Kraichnan, assume that the total scalar energy spectrum $E(k)$ depends only on the pseudo-potential vorticity transfer function η. Dimensional considerations then lead to the expression

$$E(k)=C\eta^{\frac{2}{3}}k^{-3}, \tag{28}$$

where

$$k^2=(k_1)^2+(k_2)^2+(k_3)^2 \tag{29}$$

is the scalar wavenumber, and k_1, k_2 and k_3 are the component wavenumbers. Because of the isotropy, the energy spectrum for each of the components χ_x, χ_y and χ_ζ has the same form. Thus, there is equipartition of energy among the x and y components of the kinetic energy and the available potential energy. We note from (27) that it is the energy per unit *volume*, weighted by N^{-1}, which is locally independent of height.

4. Discussion

The mechanism by which vertical homogeneity at large wavenumbers is established would seem to be through direct interaction of moderate scales with larger scales and through vertical propagation of Rossby wave energy. If we ignore the horizontal Reynolds' stresses, the eddy energy equation may be written

$$\tfrac{1}{2}[(\chi_x')^2+(\chi_y')^2+(\chi_z')^2]_t=\overline{\chi_\zeta'\mathbf{V}'}\cdot\overline{\mathbf{V}}_\zeta-N(\overline{\chi'w'})_\zeta, \tag{30}$$

where the primes denote perturbations and the bars horizontal averages. It states that the eddy energy is increased by conversion of mean flow potential energy to perturbation potential energy and by convergence of the vertical energy flux due to the vertical pressure work. When the horizontal scales have been reduced to the point where the β effect as well as the horizontal Reynolds' stress is no longer appreciable, only the first mechanism remains. Then the only parameters on which the eddy energies can depend at statistical equilibrium are $|\mathbf{V}_\zeta|$ and the wavenumber k_H, suggesting energy spectra of the form

$$\left.\begin{aligned}K(k_H)&=C_1|\mathbf{V}_\zeta|^2k_H^{-3}\\P(k_H)&=C_2|\mathbf{V}_\zeta|^2k_H^{-3}\end{aligned}\right\} \tag{31}$$

for the kinetic and potential energies, respectively. It follows that for increasingly smaller vertical and horizontal scales the energies will tend to become homogeneous. At sufficiently small scales the interaction with the mean flow will cease, and the dependence on $|\mathbf{V}_\zeta|$ will end. The energies will then become partitioned equally among the two components of kinetic energy and the potential energy, and the spectrum will satisfy a universal relation of the form (28).

It has been assumed that internal and boundary dissipation do not affect the spectra in the range of wavenumbers under consideration. In the quasi-geostrophic theory, surface friction may be incorporated by altering the surface boundary condition to provide for convergence in the frictional boundary layer (Charney and Eliassen, 1949). The resultant vertical pumping does work on the fluid and thereby contributes a dissipative term to Eq. (12) which then becomes, in the new variables,

$$\frac{d}{dt}\int\int\int\frac{1}{2\lambda}[(\chi_x)^2+(\chi_y)^2+(\chi_\zeta)^2]dxdyd\zeta=-\sqrt{2}f_0\mathrm{E}^{\frac{1}{2}}\int\int\tfrac{1}{2}[(\chi_x)^2+(\chi_y)^2]_{\zeta=0}dxdy, \tag{32}$$

where E is the Ekman number, $\nu/(f_0H^2)$, and the eddy viscosity ν is assumed constant. Since energy is prevented from cascading to high wavenumbers, the bulk of the energy dissipation occurs through surface friction at low wave numbers.[2]

The dissipation of pseudo-potential enstrophy, unlike energy, is not affected by surface friction away from

[2] By hypothesis, dissipation in frontal zones is excluded.

boundaries. This is because the horizontal, as well as the three-dimensional integral of $[L(\psi)]^2$ in Eq. (13), are conserved in the absence of β effects. Any dissipation must be due to internal vertical diffusion of momentum and heat. If we assume that the coefficients of eddy viscosity and heat conduction are constant and equal, Eq. (25) becomes

$$\triangle\chi_t=\frac{\partial(\triangle\chi,\chi)}{\partial(x,y)}+\nu^*\triangle\chi_{\zeta\zeta}, \tag{33}$$

where $\nu^*=\nu(N/f_0)^2$. Multiplying by $\triangle\chi$, we obtain the dissipation equation

$$\frac{d}{dt}\int\int\int\tfrac{1}{2}(\triangle\chi)^2dxdyd\zeta=-\nu^*\int\int\int(\triangle\chi_\zeta)^2dxdyd\zeta +\nu^*\int\int(\triangle\chi\triangle\chi_\zeta)_{\zeta=0}dxdy. \tag{34}$$

The integral at the ground does not in general vanish unless the ground is isentropic. Thus, it seems possible for diffusion to increase the pseudo-potential enstrophy. However, diffusive effects will become important only when the Reynolds' number, $\mathrm{Re}=U_k/(k\nu^*)$, is of order unity. Setting $(U_k)^2=O[kE(k)]=O(\eta^{\frac{2}{3}}k^{-2})$, we find that $\mathrm{Re}\leqslant O(1)$ when $k\geqslant O(\eta^{1/6}\nu^{*-\frac{1}{2}})$. Setting $\nu\approx10^5$ cm^2 sec, $N/f_0\approx10^2$, and estimating $\eta^{\frac{1}{3}}$ from observed spectra to be 10^{-5} sec^{-1}, we get $k\geqslant10^{-7}$ cm^{-1} or $K\geqslant45$ at 45°. Hence, appreciable dissipation does not occur until the scales are so small that the flow is no longer geostrophic.

The estimate $(U_k)^2=O(\eta^{\frac{2}{3}}k^{-2})$ also gives $\beta/(k_H\eta^{\frac{1}{3}})$ for the ratio $\beta/(k_H{}^2U_k)$. This ratio then becomes small for $k_H>10^{-8}$ cm^{-1} or $K>4$. Hence, we are justified in ignoring β effects at the higher wavenumbers.

We note finally that the properties of isotropy and equipartition are not dependent on the postulates of localness or uniformity of the pseudo-potential enstrophy transfer function required for the establishment of the k^{-3} dependence. They are due essentially to the symmetry of the $\triangle$ operator in (25). The horizontal scale, $k_H{}^{-1}$, is simply the Rossby radius of deformation, $N/(f_0k_v)$, corresponding to the vertical scale $k_v{}^{-1}$.

Saffman (1971) has pointed out that the observed kinetic energy spectra appear to be as consistent with a k^{-4} as with a k^{-3} dependence, and Deem and Zabusky (1971) have found a k^{-4} dependence in numerical simulations of two-dimensional turbulent flows. Saffman attributes this behavior to the property that near-discontinuities develop in the vorticity. A random collection of vorticity discontinuities separated by regions with small vorticity gradient will give a k^{-2} behavior for enstrophy and a k^{-4} behavior for kinetic energy. If it should be found that the pseudo-potential vorticity also exhibits such near-discontinuities, Saffman's argument could be generalized to three-dimensional quasi-geostrophic flow to account for a k^{-4} behavior.

The data to be shown in the next section do not distinguish clearly between generalizations of Kraichnan's and Saffman's hypotheses. We have not shown spectra for hemispheric wavenumbers >20 because of lack of confidence in the measured and calculated values, but the values derived from numerical simulations do show an increasing slope toward higher wavenumbers in a log-log energy-wavenumber diagram. This suggests the possibility that the Kraichnan mechanism operates at relatively lower wavenumbers and the Saffman mechanism at higher wavenumbers. At still higher wavenumbers one might expect genuine frontal discontinuities to produce a k^{-2} dependence, which ultimately becomes superceded by the Kolmogoroff $k^{-5/3}$ dependence.

5. Comparison with observation

We present comparisons of observed and computed one-dimensional spectra along circles of latitude. If "$\langle\ \rangle$" denotes an average over a sufficiently large volume V in the x, y, ζ space, and X is the Fourier transform of χ, we have

$$[\langle(\chi_x)^2\rangle,\langle(\chi_y)^2\rangle,\langle(\chi_\zeta)^2\rangle] =\frac{1}{V}\int\int\int_{-\infty}^{+\infty}[k_1{}^2,k_2{}^2,k_3{}^2]|X|^2dk_1dk_2dk_3, \tag{35}$$

$$\langle(\chi_x)^2\rangle+\langle(\chi_y)^2\rangle+\langle(\chi_\zeta)^2\rangle=\frac{4\pi}{V}\int_0^\infty k^4|X|^2dk =2\int_0^\infty E(k)dk, \tag{36}$$

where $X=X(k)$ by isotropy. Carrying out the integrations in (35) with respect to k_2 and k_3, we find

$$\langle(\chi_y)^2\rangle=\langle(\chi_\zeta)^2\rangle=\int_0^\infty F_1(k_1)dk_1 \tag{37}$$

because of the isotropy, and

$$\langle(\chi_x)^2\rangle=\int_0^\infty G_1(k_1)dk_1, \tag{38}$$

where

$$F_1(k_1)=\int_{k_1}^\infty\frac{k^2-k_1{}^2}{k^3}E(k)dk, \tag{39}$$

$$G_1(k_1)=\int_{k_1}^\infty\frac{k_1{}^2}{k^3}E(k)dk=-k_1\frac{dF_1}{dk_1}. \tag{40}$$

Setting $E=Ak^{-n}$, we get $F_1=2n^{-1}(n+2)^{-1}Ak_1{}^{-n}$ and $G_1=2(n+2)^{-1}Ak_1{}^{-n}$, showing that the longitudinal spectral function for the transverse velocity v is n times greater than that for the longitudinal velocity u or for χ_ζ. This result has been verified for u and v observationally by Leith (*loc. cit*) on the assumption that $n=3$.

The hypotheses of Section 3 permit several immediate comparisons with observation. They of course predict a k^{-3} spectrum for the kinetic energy of the horizontal velocity and for its x and y components. They also predict a k^{-3} spectrum for the temperature variance. Such a spectral behavior was observed by Kao and Wendell (1970) and Kao (1970) in a longitudinal spectral analysis of winter and summer 500-mb data at 20, 40, 60 and 80N. The data were taken from routine objective analyses prepared by the National Meteorological Center. The spectra at 60 and 80N show a greater decrease of spectral energy with increasing wavenumber but are not reliable because the data were not adequate to resolve the high wavenumber components. The 20N spectra are also inaccurate because of lack of data. We expect equipartition among the power spectra of χ_x, χ_y and χ_ζ. Transferring back to ψ and z, this implies that the two-dimensional kinetic energy spectrum E_k should equal *twice* the available potential energy spectrum E_p [cf. Eq. (12)]. In particular, the longitudinal spectra of $n^{-1}(\psi_x)^2, (\psi_y)^2$ and $(f^2/N^2)(\psi_z)^2$ should be equal. Utilizing (4), we find that $g^2/(N^2\bar{T}^2)$ is the factor that converts the temperature variance spectrum to the power spectrum of the u velocity component. At 500 mb and 40N this factor is found to be about 1.1×10^5 cm^2 sec^{-2} (°K)$^{-2}$ in both summer and winter. Fig. 1 shows the variance spectrum of $(g/N)\times(T-\bar{T})/\bar{T}$ for the winter and summer of 1964 calculated by Kao as functions of the hemispheric wavenumber $K=k_1$. The corresponding power spectra of u and v are superimposed as dashed and dotted lines, respectively. It will be seen that there is approximate equipartition for hemispheric wavenumbers >6. We note that the actual slopes in the log-log diagram are closer to -4 than to -3, but this may be due to lack of data or to excessive smoothing in the NMC analysis. Ten-day averages of velocity and temperature spectra computed from an NCAR six-level model prediction for a period of ten days with a grid resolution of $2\frac{1}{2}°$ in latitude and longitude were kindly made available to the author by Wellck and Washington of NCAR. These spectra are shown in Fig. 2 for the levels 1.5, 3.5, 7.5, and 10.5 km at 40N in winter. The temperature variance spectra have been multiplied by the factor $g^2/(N^2\bar{T}^2)$, whose zonally averaged values were found to be 0.7×10^5, 0.7×10^5, 1.2×10^5 and 4.0×10^5 at the respective levels. We see that there is equipartition and that the spectra do have an approximate k^{-3} behavior.

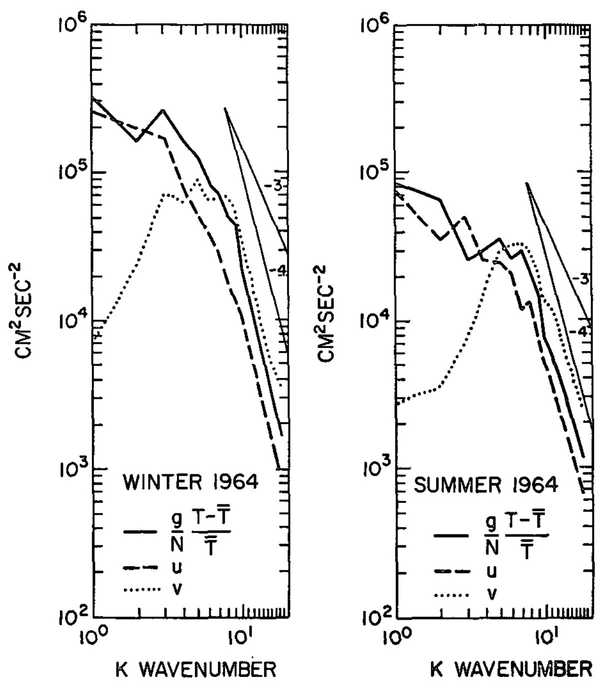

FIG. 1. The observed power spectra of u, v, and $(g/N)(T-\bar{T})/\bar{T}$ at 40N and 500 mb for winter and summer 1964 as a function of the hemispheric wavenumber $K=k_1$.

It is thought that the above theory should also apply to the oceans in regions of strong baroclinic excitation, such as in the region of the Gulf Stream meanders. No observations are yet available for comparisons.

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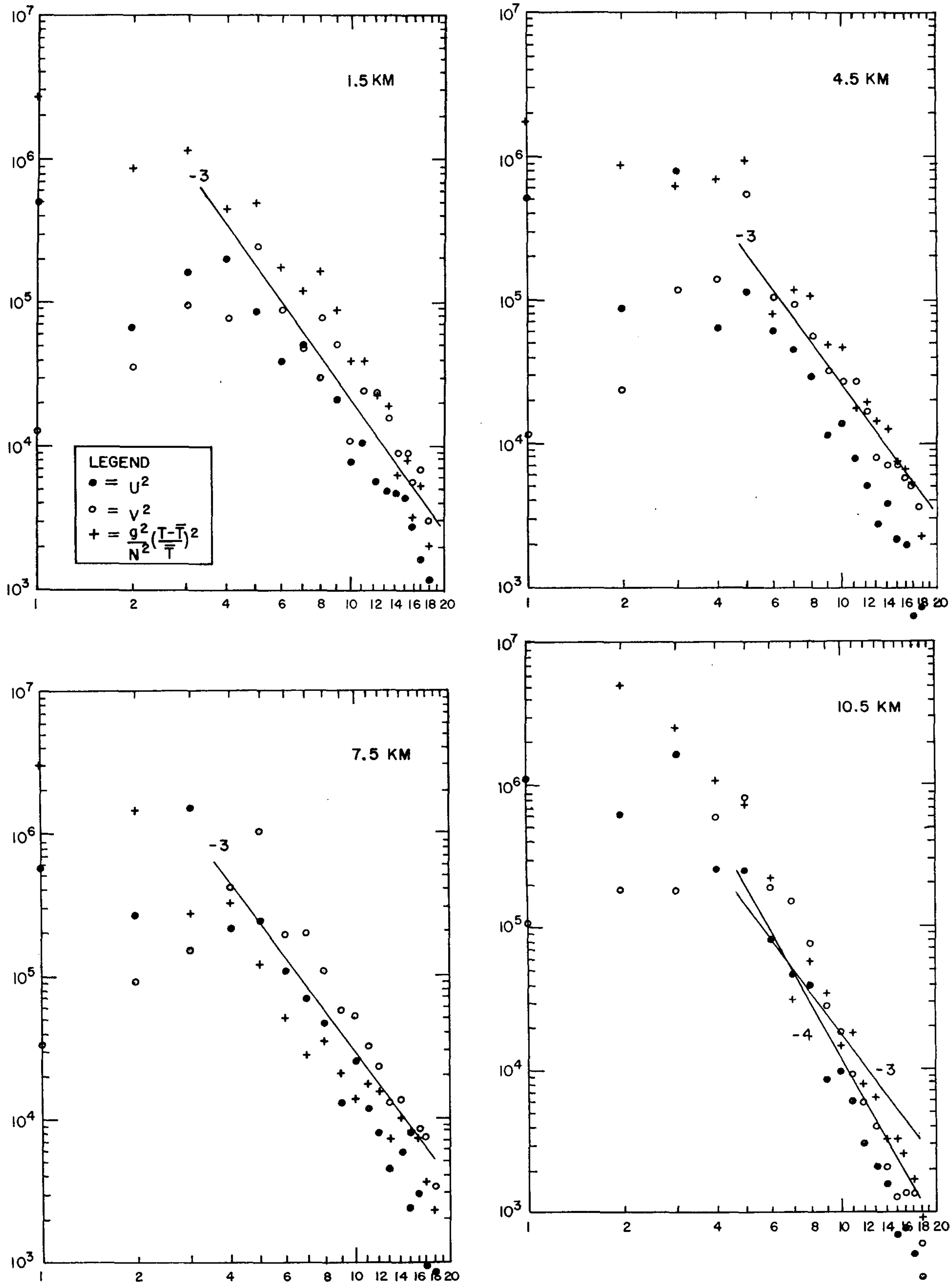

FIG. 2. Power spectra of u, v, and $(g/N)(T-\bar{T})/\bar{T}$ for the levels 1.5, 4.5, 7.5 and 10.5 km at 40N in winter derived from numerical simulations with the NCAR six-level model as a function of the hemispheric wave number $K=k_1$.

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Nitrous Oxide: A Natural Source of Stratospheric NO

MICHAEL B. MCELROY AND JOHN C. MCCONNELL

Center for Earth and Planetary Physics, Harvard University, Cambridge, Mass.

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ABSTRACT

Supersonic transport planes currently under development will cruise in the stratosphere and there is concern about possible environmental effects. In particular, NO emitted by these aircraft may catalytically affect atmospheric ozone. Here we investigate an important natural source of NO, the reaction $O(^1D) + N_2O \rightarrow 2NO$, and compare the natural source with estimates for the source due to a fleet of 500 planes cruising for an average of 7 hr a day. The natural and artificial inputs above 15 km are of comparable magnitude. The natural source corresponds to a net production of NO, averaged over the globe, of about 2×10^7 molecules cm^{-2} sec^{-1}, and offers a yardstick for judging the possible significance of any artificial input. Additional sources of stratospheric NO, due to downward diffusion from the ionosphere and upward transport from the earth's surface, are discussed but have not been quantitatively estimated at this time.

1. Introduction

Crutzen (1971) has argued that odd oxides of nitrogen may play a significant role in the chemistry of ozone in the atmosphere below 45 km. They act to catalyze its destruction and, according to Johnston (1971), this catalysis may be of crucial importance in the context of the supersonic transport plane (SST).

A major uncertainty concerns the abundance of nitrogen oxides present normally in the atmosphere at SST cruise levels. Nitrogen oxides are formed at high levels in the atmosphere by reactions involving charged particles and are transported downward into the stratosphere by eddy processes. Oxides of nitrogen are also produced at the earth's surface, a source which is likewise expected to contribute significantly to the stratospheric input.

Detailed analyses of airglow data (Meira, 1971) indicate downward fluxes of NO at 80 km in the range 1–5×10^8 molecules cm^{-2} sec^{-1} (Strobel *et al.*, 1970), but the fraction of this flux which eventually reaches the stratosphere cannot be reliably estimated at the present time due to the probable importance of slow-loss processes for odd nitrogen atoms. Strobel (1971) indicates that the concentration of odd nitrogen atoms at 30 km, which results from downward flow, is uncertain by perhaps three orders of magnitude, and the flux at this level is similarly ill-defined.

Bates and Hays (1967) estimate a globally averaged source of NO and NO_2 equal to about 2×10^9 cm^{-2} sec^{-1} due to combustion of crude oil, coal, natural gas and the primary metal industry. Junge (1963) discusses an additional source due to bacterial activity in the soil. The fraction of the surface production which penetrates the stratosphere depends on a variety of factors. The dominant loss of odd nitrogen atoms in the troposphere is believed due to rainout of NO_2 and soluble photochemical products such as HNO_2 and HNO_3.