AM 275 - Magnetohydrodynamics: Lecture Notes

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Contents

Lecture 2: Hydrodynamics Review

What is a fluid?

- Its flows!
- it deforms continuously

Categorization of fluids:

- Compressible v. incompressible
- viscous v. inviscid
- + many more

Eularian

Rate of change at a given point, no bother for where the fluid goes.

$$\frac{\partial}{\partial t}(\cdot)$$

Lagrangian

Follows the particle, introduces the advection term

$$\boldsymbol{u}\cdot\nabla(\cdot)$$

Mass Conservation

In order to conserve mass we consider an arbitrary eularian volume (i.e. the volume doesn't move with the flow). We then find the total mass which is equal to the integral of the density over the volume, and then consider the flux of mass through the boundary (change in mass over time). Using the divergence theorem, we then have a conservation equation for mass.

$$\frac{\partial}{\partial t}\int_D \rho dV = \int_{\partial D} \rho \boldsymbol{u} \cdot \eta dA$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} = 0$$
$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = 0$$
$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \boldsymbol{u}$$

More importantly, if we consider an incompressible fluid, i.e. $\rho = \rho_0$, we have very specifically,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

Stresses

Stresses can be divided into two catagories, body forces and surface forces. Body forces are forces such as gravity and the electric force, which surface forces are forces such as normal force and friction.

Newtons Second Law

Newton's second law

$$\frac{\partial p}{\partial t} = \sum_i F_i$$

where p here is the momentum of a fluid parcel. In actuality, the momentum can be written as $p = \int_D \rho \boldsymbol{u} dV$. So,

$$\frac{D}{Dt}\int_D \rho \boldsymbol{u} dV = \int_D \rho \boldsymbol{g} dV + (\text{other body forces}) + \nabla \cdot \boldsymbol{\tau}$$
$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho F + \nabla \cdot \boldsymbol{\tau}$$

where $\boldsymbol{\tau}$ is the stress tensor acting on the fluid parcel. Surface forces are then introduced into this stress tensor. First and foremost, surface pressure is introduced along the stress tensor.

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}$$

where σ_{ij} is the deviatoric stress tensor and is responsible for the off-diagonal components of the stress tensor. Some components are the velocity gradient tensors, $\frac{\partial u_i}{\partial x_j}$ and $\frac{\partial u_j}{\partial x_i}$. Each of these has a symmetric compoenent and an antisymmetric componenet.

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

The first term is labeled symmetric and denoted e_{ij} while the second component is the rotation componenet.

Lecture 3: Continuing Review of the Kinematic Equations

3.1 Obtaining the Navier-Stokes equation

Decomposition of the deviatoric stress tensor reveals a 4th order tensor with 81 components.

$$\boldsymbol{\sigma}_{ij} = \boldsymbol{A}_{ijkl}\boldsymbol{e}_{kl}$$

In order to reduce the complexity of the system, we make some assumptions about the tensor $\boldsymbol{A}_{ijkl}$. First, we state that this tensor must be isotropic, i.e. that it doesn't care about the direction of the stress with respect to the coordinate system it is in. We have,

$$\boldsymbol{A}_{ijkl} = \mu\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \mu'\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \mu''\boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk}$$

Next, we assume that this tensor must be symmetric. This reduces the complexity down to two coefficients, μ, the viscosity, and μ' which is the bulk viscosity.

In order to obtain the Navier-Stokes equation, we require the Stokes assumption which postulates that the diagonal components of the deviatoric stress tensor are zero, i.e. $\boldsymbol{\sigma}_{ii} = 0$.

$$\boldsymbol{\sigma}_{ij} = 2\mu\left(\boldsymbol{e}_{ij} - \frac{1}{3}\left(\nabla\cdot\boldsymbol{u}\right)\boldsymbol{\delta}_{ij}\right)$$
$$\boldsymbol{\tau}_{ij} = -p\boldsymbol{\delta}_{ij} + 2\mu\left(\boldsymbol{e}_{ij} - \frac{1}{3}\left(\nabla\cdot\boldsymbol{u}\right)\boldsymbol{\delta}_{ij}\right)$$

Therefore, when we take the divergence of this stress tensor we obtain the Navier-Stokes equation:

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{F} - \nabla p + \mu \left[\nabla^2 \boldsymbol{u} + \frac{1}{3} \nabla \left(\nabla \cdot \boldsymbol{u} \right) \right]$$

Of course, when working in an incompressible framework (i.e. $\nabla \cdot \boldsymbol{u} = 0$), we have that part of the diffusive term disappears from the equation, resulting in the commonly used equation:

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{F} - \frac{1}{\rho_0} \nabla p + \nu \nabla^2 \boldsymbol{u} \tag{2}$$

Additional terms are included in this equation as necessary to model relevant physics of various fluid systems. For example, if in a rotating frame we include the coriolis force $2\Omega(\boldsymbol{e}_\Omega \times \boldsymbol{u})$, if some component of the fluid is stratified we need some buoyancy forcing $T/N^2 \boldsymbol{e}_z$. And most relevant, there might be magnetic forces which affect the fluid, in which case we obtain the MHD equations.

3.2 Vorticity equation

Vorticity is a quantity related to the fluid field which can be very important to the scientific study of fluid dynamics. The vorticity is obtained by taking the curl of the velocity field.

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

The vorticity has an evolution-advection equation just as the velocity field does, and in fact the vorticity equation is obtained by taking the curl of the Navier-Stokes equations.

$$\nabla \times (2)$$

$$\frac{D\boldsymbol{\omega}}{Dt} = \left(\boldsymbol{\omega} \cdot \nabla \right) \boldsymbol{u} + \boldsymbol{\omega} \left(\nabla \cdot \boldsymbol{u} \right) + \nabla \times \boldsymbol{F} - \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nu \nabla^2 \boldsymbol{\omega}$$

If the flow is incompressible, one of the vortex stretching terms disappears. Generally, the first two right hand terms are vortex stretching/tilting/speed-up terms. Then the pressure and density gradient cross product is the baroclinicity term, the curl of $\boldsymbol{F}$ is the forcing of vorticity, and finally, we have a viscous diffusion of vorticity which behaves similarly to the diffusion of velocity.

Baroclinicity is perhaps the most unintuitive term in this equation, and it simply represents the creation of rotation in the fluid when there is a disalignment between the pressure and density gradients in the fluid.

Some fluid dynamicists prefer to study the vorticity equation, especially for rotating flows where voricites and cyclones are common phenomenon in the flow field.

3.3 Rotation

In the presence of rotation, the coriolis force becomes relevant as the motion of a fluid particle is deflected due to the rotation of the cordinate frame. That is, our equations are modified such that,

$$\frac{\partial \boldsymbol{q}}{\partial t}_F = \frac{\partial \boldsymbol{q}}{\partial t}_R + 2\boldsymbol{\Omega} \times \boldsymbol{q} - \boldsymbol{\Omega}^2 \boldsymbol{R}$$

This also introduces an additional term to the vorticity equation which looks like, $+ (2\boldsymbol{\Omega} \cdot \nabla)\, \boldsymbol{u}$.

Lecture 4: Conservation of Energy and Maxwell's equations

4.1 Conservation of Energy

The equation of state chosen for a particular problem is a source of physics which affects the solutions of a given PDE. The incompressible equation of state is used very commonly as an equation of state. Another common one is the ideal gas law $pV = \rho RT$.

In order to understand the origin and importance of the equation of state, the laws of thermodynamics are needed.

The first law of thermodynamics states,

$$\frac{\partial e}{\partial t} = \frac{\partial W}{\partial t} + \frac{\partial Q}{\partial t}$$

where e is the internal energy, W is work done on the system, and Q is heat flux into the system. However, for a fluid flow taken from a Lagrangian perspective, we must modify this law of thermodynamics. It must include the energy given by the velocity field.

$$\frac{D}{Dt} \int_D \rho \left(e + \frac{1}{2} \boldsymbol{u}^2 \right) dV = \int_D \rho \boldsymbol{F} \cdot \boldsymbol{u} dV + \int_{\partial D} \boldsymbol{\tau} \cdot \boldsymbol{u} dS - \int_{\partial D} \boldsymbol{q} \cdot dS$$
$$\rho \frac{D}{Dt} \left(e + \frac{1}{2} \boldsymbol{u}^2 \right) = \rho \boldsymbol{F} \cdot \boldsymbol{u} + \nabla (\boldsymbol{\tau} \cdot \boldsymbol{u}) - \nabla \cdot \boldsymbol{q}$$

Next, we obtain a mechanical energy equation by dotting $\boldsymbol{u}$ by the Navier-Stokes equation and adding

$\boldsymbol{u}^2/2 \cdot \frac{D\rho}{Dt}$

$$\frac{D\rho\boldsymbol{u}^2/2}{Dt} = \rho\boldsymbol{F}\cdot\boldsymbol{u} - \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) + \dots$$
$$\boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) = \Phi = 2\mu\left[\boldsymbol{e}_{ij} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\boldsymbol{\delta}_{ij}\right]$$

Finally, we obtain an energy equation with a positive definite dissipation term Φ which acts purely to remove energy from the system.

$$\rho\frac{De}{Dt} = -\nabla\cdot\boldsymbol{q} - p\,(\nabla\cdot\boldsymbol{u}) + \Phi$$

The Second law of Thermodynamics also plays an important role in the conservation of energy. The ssecond law makes statements about the entropy of a system, S.

$$dS = \frac{dq}{T}$$
$$TdS = de + pdV$$
$$T\frac{dS}{dt} = \frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}$$
$$\rho\frac{DS}{Dt} = -\frac{\nabla\cdot\boldsymbol{q}}{T} + \frac{k}{T^2}|\nabla T|^2 + \mu\frac{\Phi}{T}$$

Essentially, since both $|\nabla T|^2$ and Φ are positive definite terms and their coefficients are positive definite, it must be that the entropy of a system can only increase “on average” (curse the statisticians).

4.2 Introducing Maxwell’s Equations

Electricity and Magnetism are very closely related to one another and governed by a main set of governing equations. The main variables which we consider are a position vector, $\boldsymbol{x}$, a velocity field, $\boldsymbol{u}$, density ρ, pressure p, time t, temperature T, magnetic field (magnetic flux density) $\boldsymbol{B}$, magnetic field strength $\boldsymbol{H}$, electric field $\boldsymbol{E}$, electric displacement $\boldsymbol{D}$, electric current density $\boldsymbol{j}$, and charge density ρ_e.
Alongside these variables, we have constants describing components of electropmangetism: permitivity ε, permeability μ, and conductivity σ. Permitivity describes the charge requirement for a specific electric field, i.e. large ε implies a larger charge is needed for a specific electric field. Permeability describes the current requirement for a specific magnetic field, i.e. large μ implies a smaller current is needed to obtain a specific magnetic field.

Consitutive relationships describe the relationships between specific electromagnetic quantities.

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu}, \text{ for an isotropic permeability}$$
$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \text{ for an isotropic permitivity}$$

where generally, we take $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ where q_0 is taken from a vacuum.

Now we write Maxwell's equations in their differential form:

$$\nabla \cdot \boldsymbol{B} = 0, \text{ Gauss' law for magnetism} \qquad \nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\varepsilon_0}, \text{ Gauss' law}$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \text{ Faraday's law} \qquad \nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right), \text{ Ampere's Law of induction}$$

They can be written in their integral form as well:

$$\oint_{\partial D} \boldsymbol{B} \cdot \boldsymbol{dA} = 0 \qquad \oint_{\partial D} \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{q}{\varepsilon_0}$$
$$\oint_L \boldsymbol{E} \cdot \boldsymbol{dL} = -\frac{\partial \phi_B}{\partial t} \qquad \oint_L \boldsymbol{B} \cdot \boldsymbol{dL} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

where $\phi_B = \int_S \boldsymbol{B} \cdot \boldsymbol{dA}$ is the total magnetic fluid, and $\phi_E = \int_S \boldsymbol{E} \cdot \boldsymbol{dA}$ is the total electric flux.

We cover their derivations in a brief sense also. Consider a positive point charge which creates an electric field. This imposes a force acting on any other point charge in the field. This force is called Coulomb force given by $F = q_1 q_2/(4\pi\varepsilon r^2)$. Thus we have a given electric field of strength $E/q = q_1/(4\pi\varepsilon r^2)$. We obtain the total electric flux ϕ_E

$$d\phi_E = \boldsymbol{E} \cdot \boldsymbol{dA}$$
$$\phi_E = \oint_S \frac{q_1}{4\pi\varepsilon r^2} \cdot \boldsymbol{dA}$$
$$\phi_E = \frac{q_1}{4\pi\varepsilon r^2} \oint_S \boldsymbol{dA}$$
$$\phi_E = \frac{q_e}{\varepsilon}$$

where q_e in the final equation is given by the sum of all point charges enclosed in the closed volume, i.e. $q_e = \sum_i q_i$. Notice that q_e can be thought of as mass for point charges, i.e. the integral of the charge density equals the total charge similar to how the integral of mass density equals the total mass. It can be represented as a sum of point charges because point cahrges are discrete and do not usually exist in a continuum.

Finally, using the divergence theorem,

$$\oint_S \boldsymbol{E} \cdot \boldsymbol{dA} = \int_V \nabla \cdot \boldsymbol{E} dV$$
$$q_e = \int_V \rho_E dV$$
$$\nabla \cdot \boldsymbol{E} = \frac{\rho_E}{\varepsilon}$$

Similarly, the same proof holds for Gauss' law of magnetism, only that monopoles do not exist in magnetic fields, i.e. every source must have a sink. Therefore, for an arbitraty volume it must be that the divergence of the magnetic field must be zero:

$$\nabla \cdot \boldsymbol{B} = 0$$

Lecture 5: Derivation of Maxwell's Equations: Continued

5.1 Faraday's Law of Induction

The laws of electrodynamics are empiracle. Faraday realized that the EMF, proportional to $\frac{\partial \boldsymbol{B}}{\partial t}$ and also the area of the coil. This led them to deduce that EMF should be proportional to $\frac{\partial \phi_B}{\partial t}$. Note that EMF represents the amount of work done per unit charge to move a charge from one place to another, i.e. the electric potential difference. It has the units of Nm/C (Newton meters per Coulomb).

$$\begin{aligned} \text{EMF} = \oint_C \boldsymbol{E} \cdot \boldsymbol{dL} &= -\frac{\partial \phi_B}{\partial t} \\ &= -\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot \boldsymbol{dA} \end{aligned}$$

where here the RHS integral is not necessarily over a closed surface (because the same integral over a closed surface must be zero).

5.2 Ampere's Law

If there is a current moving through a wire, imagine a cross section going through the wire (i.e. going through the page), there is a magnetic field around the wire. The magnetic field can be described using the following integral form.

$$\oint_L \boldsymbol{H} \cdot \boldsymbol{dL} = \boldsymbol{i}$$
$$\oint_L \frac{\boldsymbol{B}}{\mu_0} \cdot \boldsymbol{dL} = \boldsymbol{i}$$
$$\oint_L \boldsymbol{B} \cdot \boldsymbol{dL} = \mu_0 \boldsymbol{i}$$
$$\int_S \nabla \times \boldsymbol{B} \cdot \boldsymbol{dA} = \mu_0 \oint_A \boldsymbol{j} \cdot \boldsymbol{dA}$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

Here is where Maxwell's contribution to Ampere's law is notable. Ampere assumed that $\nabla \cdot \boldsymbol{j} = 0$, where Maxwell noticed that in some scenarios, this is not necessarily true. Thus, he modified the equation to include displacement currents.

$$\frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j}_0$$
$$\frac{\partial \varepsilon \boldsymbol{E}}{\partial t} = \boldsymbol{j}_0$$
$$\nabla \times \boldsymbol{B} = \boldsymbol{j} + \boldsymbol{j}_0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

This was significant because $\mu_0 \varepsilon_0 \propto 1/c^2$ (where c is the speed of light), and this implied connections to electromagnetic radiation (doublecheck this). More importantly, these equations are linear and relativistically correct (not sure what exactly this means).

5.3 Units of Electrodynamics (c.f. Priest p436)

Electrostatis units are denoted "esu". Electromagnetic units are denoted "emu": e, m respectively. The Gaussian cgs system utilizes the standard units of centimeters, grams, seconds, in addition to the electrostatic units statcoulomb, q, and electromagnetic units, "abAmp". The Gaussian cgs representation of the governing equations often have an extra factor of 4π in the equations.

In general, we will use the Rationalized MKS system (standard SI system). Where the default length, mass, time, is given in meters, kilograms, and seconds. In addition, current is given in amps. The variables, $\mu_0 = 4\pi \cdot 10^{-7} NA^{-2}$ and $\varepsilon_0 = 8.8 \cdot 10^{-12} A^2 s^2 N^{-1} m^{-1}$ have dimension, charges are given in Coulombs, forces are given in Newtons. The magnetic field is given by Teslas $T = NA^{-1}m^{-1}$, and the electric field is given by V/m (Volts per meter).

5.4 From Maxwell's Equation to MHD

Generally, for MHD we will be working in a non-relativistic approximation (i.e. typical velocities are much less than the speed of light, $U \ll c$). Let us consider the equations and their typical unit scales,

$$\begin{aligned}
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\frac{\boldsymbol{E}}{L} &\ \frac{\boldsymbol{B}}{T} \\
\nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \\
\frac{\boldsymbol{B}}{L} &\ \frac{1}{c^2}\frac{\boldsymbol{B}L}{T^2}
\end{aligned}$$

If we manipulate the last line of this equation, we find that $L^2/T^2? = c^2$ is the leading balance of Ampere's law, and therefore we neglect the relativistic term of Maxwell's equations.

In order to connect Maxwell's equations to fluid dynamics, we must consider the Lorentz force.

$$\begin{aligned}
\boldsymbol{F} &= q\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) \\
\frac{d\boldsymbol{F}}{dV} &= \frac{dq}{dV}\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) \\
\frac{d\boldsymbol{F}}{dV} &= \rho_E\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) \\
\frac{d\boldsymbol{F}}{dV} &= \rho_E \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} \\
\boldsymbol{F} &= \int_V \rho_E \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} dV
\end{aligned}$$

Next we must consider Ohm's Law, which describes the current and electric field as movign with the conductor (Lagrangian perspective) (denoted with $'$).

$$\boldsymbol{j}' = \sigma \boldsymbol{E}'$$

and thus we are able to simplify the equations to become,

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}$$

$$\boldsymbol{j}' = \boldsymbol{j}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

Taking the curl of this equation leads to the following,

$$\begin{aligned}
\nabla \times \left(\frac{\nabla \times \boldsymbol{B}}{\mu_0 \sigma} \right) &= \nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\nabla \times \eta \nabla \times \boldsymbol{B} &= -\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\frac{\partial \boldsymbol{B}}{\partial t} &= -\nabla \times (\eta \nabla \times \boldsymbol{B}) + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})
\end{aligned}$$

which is the induction equation. If we take η to be constant, we can write with the derivative identity, $\nabla \times \nabla \times \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

Notice that we obtain an equation solely for $\boldsymbol{B}$ which has taken into account all of Maxwell's equations. This tells us that we really only have to care about the magnetic field, and can obtain the electric field as a consequence of our solution. For example, $\boldsymbol{j} = \nabla \times \boldsymbol{B}/\mu_0$, $\boldsymbol{E} = \boldsymbol{j}/\sigma - \boldsymbol{u} \times \boldsymbol{B}$, and $\rho_E = \varepsilon_0(\nabla \cdot \boldsymbol{E})$.

We can interpret the terms in this equation as well. On the LHS we have a typical rate of change of the magnetic field. On the RHS we have first the induction term, and the diffusion of the magnetic field $\boldsymbol{B}$.

We also notice that the linearity of this equation depends primarily on the relationship between $\boldsymbol{u}$ and $\boldsymbol{B}$. If, for example, $\boldsymbol{u}$ is a function of $\boldsymbol{B}$ then the induction equation is not linear. If the induction equation is linear, then the equation is generally regarded as the kinematic induction equation. If the equation is nonlinear, then it is generally regarded as a dynamic equation of induction.

In general, the Lorentz force is vital to determining which dynamical regime we are in for the velocity and magnetic fields. Consider again the Lorentz force,

$$\boldsymbol{F} = \rho_E \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}$$

where the first RHS term is the electrostatic component and the second RHS term is the magnetic component. Generally, we compare the order of each term in the equation.

$$
\begin{aligned}
\frac{|\rho_E \boldsymbol{E}|}{|\boldsymbol{j} \times \boldsymbol{B}|} &\propto \frac{|\varepsilon_0 \nabla \cdot \boldsymbol{E}\boldsymbol{E}|}{|(\nabla \times \boldsymbol{B})\boldsymbol{B}/\mu_0|} \\
&\propto \frac{\varepsilon_0 \mu_0 \boldsymbol{E}^2/L}{\boldsymbol{B}^2/L} \\
&\propto \varepsilon_0 \mu_0 \left(\frac{L}{T}\right)^2 = \frac{U^2}{c^2}
\end{aligned}
$$

Therefore we are able to deduce that the Lorentz force in a non-relativistic regime, can be approximated as:

$$
\begin{aligned}
\boldsymbol{F} &\propto \boldsymbol{j} \times \boldsymbol{B} \\
&\propto \frac{1}{\mu_0}(\nabla \times \boldsymbol{B})\boldsymbol{B}
\end{aligned}
$$

With this Lorentz force as a body force, we write the Navier Stokes Equation

$$
\begin{aligned}
\rho \frac{D\boldsymbol{u}}{Dt} &= -\nabla p + \mu \left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u})\right] + \rho \boldsymbol{F} + \rho\,(\boldsymbol{j} \times \boldsymbol{B}) \\
\frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{u}) = 0 \\
\frac{De}{Dt} &= \ldots \\
&\text{An equation of state} \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}
\end{aligned}
$$

Lecture 6: Boundary Conditions and Kinematics for MHD

6.1 Validity of MHD equations and assumptions

1. In MHD, we are attempting to treat a plasma as a fluid (a continuum), which is not necessarily always valid. Length scales of interest for MHD problems are much larger than lengthscales for plasma physics, i.e. ion gyroradius. If we look at a small scale problem, this assumption becomes much less valid. (NOTE: how does this resolve high Reynolds number flows, where the kolmogorov length scale approaches typical plasma physics length scales? Ask Nic next class).

2. The next consideration is how we represent plasma in the thermodynamic equilibrium. For example, we require typical timescales and lengthscales to be much larger than particle collision times and mean

free path lengths.

3. The constants η, μ, k are uniform, and isotropic, which is an assumption we will take for granted, but there exist fluids who don't satisfy these properties.

4. The equations are in an inertial frame

5. Non-relativistic flows (because we disregarded Maxwell's addition to Ampere's law). This requires that the flow speed is much less than the speed of light, c.

6. This derivation relies on a very simple version of Ohm's law, more complicated forms of this physical law will not necessarily recover the induction equation we have derived earlier.

7. Plasma is a single fluid, i.e. plasmas can have Consitutive parts which contribute to its total mass, and all of them don't necessarily behave the same. Ideally, we would have some statisitical framework for the composition and behavior of a plasma and incorporate that into our model.

To summarize what we have obtained so far, let us write the incompressible MHD equations,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{3}$$

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \boldsymbol{F} + \boldsymbol{j} \times \boldsymbol{B} + \mu \nabla^2 \boldsymbol{u} \tag{4}$$

$$\frac{D\boldsymbol{B}}{Dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \eta \nabla^2 \boldsymbol{B} \tag{5}$$

6.2 Consequences of Gauss's law for Magnetism

We have (and this can be shown easily) that if the property that the magnetic field is divergence free as an initial condition, we have that this property is maintained for all $t > 0$. As a consequence, we can write that $\boldsymbol{B}$ as a potential function, i.e. $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A}$ is a vector potential,

$$\boldsymbol{A} = \nabla \phi + A$$

An example of finding vector potentials in a spherical coordinate frame, is the decomposition of a poloidal and toroidal magnetic field:

$$\boldsymbol{B} = \boldsymbol{B}_P + \boldsymbol{B}_T = \nabla \times (\nabla \times (P\boldsymbol{r}) + \nabla \times (T\boldsymbol{r}))$$

where P and T are scalar functions which represent the poloidal and toroidal components of the magnetic field respectively. In a physical sense, we can think of the toroidal field being the axis-symmetric, azmuthal component, and the poloidal field being the meridional component.

6.3 Common Boundary Conditions for the MHD equations

Consider an interface between two mediums M_1 and M_2. Let us denote the normal vector to that interface $\hat{\boldsymbol{n}}$. We can imagine a cylindrical volume through the interface, which we will refer to as the "pill-box" (otherwise known as a Gaussian Box), and this is often used for considering fluxes through the boundary. We can imagine a contour line along the interface which is very thin and envelops a section of the interface, particularly with a direction along the contour. This could be a line integral or a surface integral condition for example. This is called "along the contour" (also known as an Amperion Loop).
The integral form of these boundary conditions can be written as,

$$\int_V \nabla \cdot \boldsymbol{B} dV = 0$$
$$\int_S \boldsymbol{B} \cdot \boldsymbol{dS} = 0$$

AS this applies to the gaussian pill-box scenario, which is composed of three surfaces, top and bottom S_1 and S_3, and the side S_2, eich of which has their own unit normal vectors. We consider,

$$\int_S \boldsymbol{B} \cdot \boldsymbol{dS} = \int_{S_1} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_1 dS_1 + \int_{S_1} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_2 dS_2 + \int_{S_3} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_3 dS_3 = 0$$

Next we consider the limit, where the height of the cylinder tends to zero (note that the cylinder is centered along the interface). Therefore, the cylinder is compressed to a circle on the surface of the interface, and specifically the integral becomes,

$$\int_S \boldsymbol{B} \cdot \boldsymbol{dS} = \int_{S_1} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_2 dS_2 = 0$$

In essence, this allows us to formulate a "jump condution" that there must be no discontinuity in the normal components of $\boldsymbol{B}$ or $\boldsymbol{j}$ through the surface of the interface. These boundary conditions are often expressed as,

$$[\boldsymbol{B} \cdot \hat{\boldsymbol{n}}] = 0, \quad [\boldsymbol{j} \cdot \hat{\boldsymbol{n}}] = 0$$

Next we must consider the tangential components, and this requires that we revisit Faraday's law,

$$-\frac{\partial}{\partial t}\int_S \boldsymbol{B}\cdot \boldsymbol{dS} = \oint_L \boldsymbol{E}\cdot \boldsymbol{dL}$$

If we consider the surface to be an Amperion loop and take the height of that loop to tend to zero, we have that the LHS of the given equation is zero, i.e.

$$\oint_L \boldsymbol{E}\cdot \boldsymbol{dL} = -\int_{\text{top}} \boldsymbol{E}_{T1}\cdot \boldsymbol{dL_1} - \int_{\text{left}} \boldsymbol{E}_{N1}\cdot \boldsymbol{dL_2} + \int_{\text{bottom}} \boldsymbol{E}_{T2}\cdot \boldsymbol{dL_3} + \int_{\text{right}} \boldsymbol{E}_{N2}\cdot \boldsymbol{dL_4} = 0$$

$$\boldsymbol{E}_{T1} = \boldsymbol{E}_{T2}$$

$$[\boldsymbol{E}\times \hat{\boldsymbol{n}}] = 0$$

we can then rewrite this using Ohm's law $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u}\times \boldsymbol{B})$,

$$\left[\left(\frac{\boldsymbol{j}}{\sigma} - \boldsymbol{u}\times \boldsymbol{B}\right)\times \hat{\boldsymbol{n}}\right] = 0$$

$$\left[\frac{\boldsymbol{j}}{\sigma}\times \hat{\boldsymbol{n}}\right] = 0$$

where the last line is obtained assuming that $\boldsymbol{u} = 0$ or that $\boldsymbol{u}$ is purely tangential to the boundary. This is indicating in essence, that the jump condition for $\boldsymbol{j}/\sigma$ must be satisfied across the interface, i.e. $\boldsymbol{j}_1/\sigma_1 = \boldsymbol{j}_2/\sigma_2$.

Another source of boundary conditions can come from Ampere's Law. We have,

$$\int_S \frac{\nabla\times \boldsymbol{B}}{\mu_0}\cdot \boldsymbol{dS} = \int_S \boldsymbol{j}\cdot \boldsymbol{dS} = I$$

where I is the total enclosed current.

$$\oint_L \frac{\boldsymbol{B}}{\mu_0}\cdot \boldsymbol{dL} = I$$

Taking the "along the contour" integral approach, we have,

$$\left[\frac{\boldsymbol{B}}{\mu_0}\times \hat{\boldsymbol{n}}\right] = \boldsymbol{j}_S$$

where $\boldsymbol{j}_S$ is the surface current density ($\approx I/dL$). If there is no surface current, the RHS of this jump condition goes to zero and we have similarly, $\boldsymbol{B}_1/\mu_1 = \boldsymbol{B}_2/\mu_2$.

6.4 Kinematics

In orde to have a kinematic understanding of the MHD equations, we must have an induitive understanding of the effects that magnetic field and field lines have on the flows they generate.

One can imagine, magnetic field lines which exist around a live wire. These field lines have the same property as streamlines in hydrodynamics, i.e. they give contours of constant field strength along the field line. These field lines must be parallel to one another. We can imagine a small change in distance along a field line ds (which is essentially an infinitesimal arc length). We can say then,

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{B}(\boldsymbol{x}, t_0)$$

where $\boldsymbol{B}(\boldsymbol{x}, t_0)$ is a snapshot of the magnetic field at $t = t_0$. We define $\boldsymbol{B}$ then as follows,

$$\boldsymbol{B} = \left\langle \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right\rangle$$

Consider for example, a magnetic field given by $\boldsymbol{B} = \langle y, x, 0 \rangle$. We would have then,

$$\begin{aligned}
\frac{dx}{\boldsymbol{B}_x} &= \frac{dy}{\boldsymbol{B}_y} = ds \\
\frac{dx}{y} &= \frac{dy}{x} = ds \\
\int x dx &= \int y dy \\
\frac{x^2}{2} &= \frac{y^2}{2} + C \\
x^2 - y^2 &= C_1
\end{aligned}$$

where C_1 is given by the initial conditions for x and y. We can imagine what the field lines might look for this magnetic field, and they happen to look like a saddle node in the $x - y$ plane.

Lecture 7: Kinematics for MHD: Continued

7.1 Magnetic Fieldlines

$$\frac{\partial \boldsymbol{x}}{\partial s}(s) = \boldsymbol{B}(\boldsymbol{x}, t_0), \quad \boldsymbol{x}(s = 0, t_0) = \boldsymbol{x}_0(t_0)$$

7.2 Magnetic Flux

$$d\boldsymbol{S} = \hat{\boldsymbol{n}} dS$$

We can define the Magnetic flux as the integral of $\boldsymbol{B}$ through the surface/interface.

$$\int_S \boldsymbol{B}\hat{\boldsymbol{n}} dS$$

7.3 Magnetic Flux Tubes

We have streamtubes for velocity, vortex tubes for vorticity, and magnetic flux tubes for magnetic fields. Consequently, we must have that the flux through a surface must be constant when that surface is advected some δt away from its original position, i.e.

$$\nabla \cdot \boldsymbol{B} = 0$$
$$\int_{S1} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_1 dS + \int_{S2} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_2 dS + \int_{S3} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_3 dS = 0$$
$$\int_{S1} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_1 dS = -\int_{S2} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}_2 dS$$

where $S1$ and $S2$ are the two surfaces at the end of the tube, and $S3$ is the surface connecting the perimeters of $S1$ and $S2$. We can think of the LHS of this equation as the flux coming in from the left, and the RHS as the flux leaving from the right (or vice versa as I have written the signs here).

7.4 Nondimensionalization

We remember first how to nondimensionalize the kinematic equations, first with the navier stokes equation.

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} + \mu \nabla^2 \boldsymbol{u}$$
$$\rho_0 \frac{U^2}{L} \frac{D\boldsymbol{u'}}{Dt'} = -\frac{P}{L} \nabla' p' + \frac{B^2}{\mu_0 L} (\boldsymbol{j'} \times \boldsymbol{B'}) + \mu \frac{U}{L^2} \nabla'^2 \boldsymbol{u'}$$
$$\frac{D\boldsymbol{u'}}{Dt'} = -\frac{P}{\rho_0 U^2} \nabla' p' + \frac{B^2}{\mu_0 \rho_0 U^2} (\boldsymbol{j'} \times \boldsymbol{B'}) + \frac{\mu}{\rho_0 L U} \nabla'^2 \boldsymbol{u'}$$

where here we chose P such that the fraction $PL/\rho_0 U^2$ is equal to 1, and define the Reynolds number $Re = U/\nu$, where $\nu = \mu/\rho_0$ and the Chandrasehkar number $Q = B^2/\mu_0 \rho_0 U^2$ which describes the signficance of the Lorentz force onto the dynamics of the system.

$$\frac{D\boldsymbol{u'}}{Dt'} = -\nabla' p' + \frac{1}{Re}\nabla'^2 \boldsymbol{u'}$$

Next we nondimensionalize the induction equation,

$$\begin{aligned}
\frac{D\boldsymbol{B}}{Dt} &= (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} + \eta\nabla^2\boldsymbol{B} \\
\frac{BU}{L}\frac{D\boldsymbol{B'}}{Dt'} &= \frac{BU}{L}(\boldsymbol{B'}\cdot\nabla')\boldsymbol{u'} + \frac{\eta B}{L^2}\nabla'^2\boldsymbol{B'} \\
\frac{D\boldsymbol{B'}}{Dt'} &= (\boldsymbol{B'}\cdot\nabla')\boldsymbol{u'} + \frac{\eta}{LU}\nabla'^2\boldsymbol{B'}
\end{aligned}$$

where we define the Magnetic Renyolds number, $Rm = UL/\eta$. Thus we obtain the nondimensionalized induction equation,

$$\frac{D\boldsymbol{B'}}{Dt'} = (\boldsymbol{B'}\cdot\nabla')\boldsymbol{u'} + \frac{1}{Rm}\nabla'^2\boldsymbol{B'}$$

Similar to the navier stokes equation, the Magnetic Reynolds number can be taken to have two extreme limits in which the dynamics of the Magnetic field is affected drastically. Most importantly, we have the limit where $Rm \gg 1$ in which magnetic diffusivity is negligable and the induction equation can be written as,

$$\frac{D\boldsymbol{B}}{Dt} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}$$

and conversely the opposite limit ($Rm \ll 1$) in which magnetic diffusivity is dominant, i.e.

$$\frac{\partial\boldsymbol{B}}{\partial t} = \frac{1}{Rm}\nabla^2\boldsymbol{B}.$$

In the diffusive case, the induction equation reduces to a heat-like differential equation in which we have periodic spatial modes which decay on a time scale dependent on the wavenumber of the spatial modes and the diffusion coefficient $1/Rm \sim L^2/\eta$.

Lecture 8: Kinematics for MHD: Continued

8.1 Charged Plate example

Let us imagine a magnetic field with initial conditions that looks like a step function

$$B_y(x, t = 0) = \begin{cases} B_0 & x > 0 \\ -B_0 & x < 0 \end{cases}$$

This implies there is a jump discontinuity in the problem, and let us solve for the time evolution of this problem in the diffusive limit, i.e. $Rm \ll 1$. We begin with the diffusive limit of the dimensional induction equation. We attempt to solve using a similarity solution since we have no useful time or length scales in the problem (i.e. infinite domain and no velocity field (advection/induction) leads us to not have a well defined non-dimensional equation).

$$\frac{\partial B_y}{\partial t} = \eta \nabla^2 B_y$$
$$B_y = B_0 f(\xi), \xi = \frac{x}{2\sqrt{\eta t}}$$
$$\frac{\partial \xi}{\partial t}\frac{\partial B_y}{\partial \xi} = \eta \left(\frac{\partial x}{\partial \xi}\right)^2 \frac{\partial^2 B_y}{\partial \xi^2}$$
$$B_0 \frac{\partial \xi}{\partial t}\frac{\partial f}{\partial \xi} = \frac{B_0}{4t}\frac{\partial^2 f}{\partial \xi^2}$$
$$-\frac{B_0 x \eta}{4\sqrt{\eta^3 t^3}}\frac{\partial f}{\partial \xi} = \frac{B_0}{4t}\frac{\partial^2 f}{\partial \xi^2}$$
$$-\frac{\partial f}{\partial \xi}\frac{\xi}{2} = \frac{\partial^2 f}{\partial \xi}$$

Integrating this equation twice reveals the following function,

$$f(\xi) = c_1 \int_0^{\xi} e^{-s^2} ds + c_2$$

We can then fit this function to the boundary conditions relevant to the problem. The conditions we consider are the following,

$$
\begin{gathered}
B_y(x \to \infty) = B_+, \quad B_y(x \to -\infty) = B_-, \quad B_y(x = 0) = 0 \\
f(0) = 0 \implies c_2 = 0 \\
f(\xi \to \infty) = c_1 \int_0^\infty e^{-s^2} ds = 1 \\
c_1 \frac{\sqrt{\pi}}{2} = 1 \\
B_y(\xi) = \frac{2B_0}{\sqrt{\pi}} \int_0^\xi e^{-s^2} ds
\end{gathered}
$$

We can use the fact that the incompressible vorticity equation is analogous to the induction equation. This prompts us to use solutions for the vorticity field as solutions to the induction equation. Solutions are not guaranteed to work, as the vorticity equation is guaranteed to be nonlinear, where as the induction equation can be defined to the linear or non-linear depending on the relevant scaling of the lorenz force and other factors.

8.2 Perfectly Conducting Limit

Let us make some assumptions about the fluid and conductivity related to the magnetic field. In a perfectly conducting limit, we have $Rm \ll 1$.

$$
\begin{aligned}
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \\
\frac{D\boldsymbol{B}}{Dt} &= (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \eta \nabla^2 \boldsymbol{B}
\end{aligned}
$$

According to Alfven's theorem, in an ideal (non-resistive; $\eta = 0$) fluid, magnetic field lines move as if they are frozen-in to the flow.

$$
\begin{aligned}
\frac{D\delta x}{Dt} &= \boldsymbol{u}(x + \delta x) - \boldsymbol{u}(x) \\
\frac{D\delta x}{Dt} &= \delta x \cdot \nabla \boldsymbol{u} + O(\delta x^2)
\end{aligned}
$$

More specifically, magnetic flux through any loop moving with the fluid is constant. That is, $\frac{\partial \phi_B}{\partial t} = 0$. This makes the intuitive idea that magnetic flux through a surface is conserved in time, but it relies on arguments which are not immediately obvious. Generally, ϕ_B can change position and the magnetic field is not constant.

$$\int_V \nabla \cdot \boldsymbol{B} = 0 = \oint_{S_V} \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS_V$$
$$= -\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS + \int_{S'} \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS' + \int_{S''} \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS'' = 0$$

Note that the surface S'' is represented by the cross product between the line vector on the perimeter of S and the flow field $\boldsymbol{u}$.

$$\boldsymbol{B} \cdot \boldsymbol{dS''} = \boldsymbol{B} \cdot (\boldsymbol{dL} \times \boldsymbol{u}) dt$$
$$= (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{dL} dt$$

and we can now write the following integral equation,

$$\int_{S'} \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS' = \int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS - dt \oint_C \boldsymbol{u} \times \boldsymbol{B} \cdot \boldsymbol{dL}$$

We can then denote the rate of change of the magnetic flux, we have

$$\frac{\partial \phi_B}{\partial t} = \int_S [\boldsymbol{B}(t + dt) - \boldsymbol{B}(t)] \cdot \boldsymbol{dS} - dt \oint_C \boldsymbol{u} \times \boldsymbol{B} \cdot \boldsymbol{dL}$$
$$\frac{\partial \phi_B}{\partial t} = \int_S \frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{dL} = 0$$

This is an equivalent statement to the Reynolds transport theorem or the Leibniz theorem. We can also show something similar with Faraday's law in integral form. Pick a curve C moving at a speed $\boldsymbol{u}$.

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = -\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$
$$\oint_C (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{dL} = \int_S i \left[-\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \right] \cdot \boldsymbol{dS}$$
$$\oint_C \boldsymbol{E}_r \cdot \boldsymbol{dL} = -\frac{\partial}{\partial t} \left(\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS \right)$$

We obtain a differential equation for the electric field in relative motion (EMF) which as two components which are a transformer EMF which is due to changes in $P(t)$ (static changes), and a motional EMF which is resultant from the motion of the curve.

$$\oint_C \boldsymbol{E}_r \cdot \boldsymbol{dL} = -\frac{\partial}{\partial t}\left(\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS\right), \quad \boldsymbol{j} = \sigma \boldsymbol{E}_r$$

$$\frac{1}{\sigma}\oint_C \boldsymbol{j} \cdot \boldsymbol{dL} = -\frac{\partial}{\partial t}\left(\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS\right)$$

$$\frac{\partial}{\partial t}\left(\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS\right) = 0, \quad \sigma \to \infty$$

Lecture 9: Kinematics for MHD: Continued

9.1 Additional Vorticity Analogies

The perfectly conducting limit of the induction equation, resembles the vorticity equation as discussed earlier. More statements can be made comparing the nature of vorticity and magnetism. Let us write the lagrangian form of the induction equation,

$$\frac{D\boldsymbol{B}}{Dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

where the RHS is generally thought of as a magnetic field stretching term. We can imagine a scenario in which the magnetic field on ly has a z-component, and the flow field has a verticla velocity which increases with z. We would have the,

$$\frac{DB_z}{Dt} = B_z \frac{\partial w}{\partial z}$$

And in this case, the field stretching would produce a stronger magnetic field, i.e. stretching and compression of magnetic fields can produce changes to the magnetic field strength.

9.2 Helicity

The Helicity of a fluid flow is given by the scalar quantity,

$$H_{\text{fluid}} = \int_V \boldsymbol{u} \cdot \boldsymbol{\omega} dV$$

The Helicity quantity has three topological components that it accounts for. First is the notion of "twist" for example following the edge surface of a cylinder which doesn't move parllel to the edge of the surface. Another is "writhe" which is if a vortex tube twists about itself, creating a sort of kink in the center of the tube. The last notion is “linkage” in which field lines or streamlines are linked together much like links in a chain are bound together.

The magnetic helicity can be defined using two methods,

$$H_m = \int_V \boldsymbol{A} \cdot \boldsymbol{B} dV$$

where A is the vector potential for $\boldsymbol{B}$, i.e. $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. We also have the current helicity which is given as,

$$H_c = \int_V \boldsymbol{B} \cdot \boldsymbol{j} dV.$$

These quantities represent different things and have different properties. We could consider an ideal fluid, and then we would find that the magnetic helicity is a constant quantity. “Cross helicity” is also conserved $\int_V \boldsymbol{B} \cdot \boldsymbol{u} dV$.

9.3 Additional Theorems for MHD

The Bondi-Gold theorem is a theorem states that in the perfectly conducting limit, we consider a material surface in which the magentic flux is constant, $\phi_B = C$ and since we are on a closed surface, it is equal to zero. We can define a closed null line going around this closed surface which divides the surface into two bits, S_1 and S_2 in which in turns allows us to write the total flux as the sum of two fluxes ϕ_1 and ϕ_2. We must have that $\phi_1 = -\phi_2$. We can consider the unsigned fllux, given by $|\phi_B| = \int_S |\boldsymbol{B} \cdot \hat{\boldsymbol{n}}| dS$. We must have that this unsigned flux is non-zero, but is still constant. This led MHD scholars to postulate the existence of invisible dynamos, in which the magnetic field can grow in magnetude and yet from the outside, we cannot see any change in the field strength.

Another popular theorem is known as Ferraro’s Law (of Isorotation). This theorem was published in 1937 and is accepted as a subset of Alfven’s theorem (though it was published before Alfven’s theorem in 1942). Let us consider and axissymmetric rotating star, with a steady flow field and no diffusion. In cylindrical polar coordinates, we have,

$$\boldsymbol{u} = r\Omega(r, z)\hat{\boldsymbol{\phi}}$$

where Ω is the rotation rate of the star dependent on height and radius. We can consider the magnetic field,

$$\boldsymbol{B} = \boldsymbol{B}_P(r,z,t) \cdot \left\langle \hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}} \right\rangle + \boldsymbol{B}_\phi(r,z,t) b\hat{s}\phi$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}_p + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}_p + \boldsymbol{B}_p(\nabla \cdot \boldsymbol{u})$$

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = -(\boldsymbol{u}_p \cdot \nabla_p + \boldsymbol{u}_\phi \cdot \nabla_\phi)\boldsymbol{B}_p + (\boldsymbol{B}_p \cdot \nabla_p + \boldsymbol{B}_\phi \cdot \nabla_{phi})\boldsymbol{u}_p + \boldsymbol{B}_p(\nabla_p \cdot \boldsymbol{u}_p + \nabla_\phi \cdot \boldsymbol{u}_\phi)$$

$$\frac{\partial \boldsymbol{B}_\phi}{\partial t} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}_\phi = (\boldsymbol{B}_p \cdot \nabla_p)\boldsymbol{u}_\phi$$

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{u}_\phi = 0 \implies (\boldsymbol{B} \cdot \nabla)\Omega = 0$$

This theorem essentially states that the stretching of field lines by differential rotation is not allowed in the scenario in which you are looking for a steady state solution.

9.4 Cauchy Solutions

$$\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a},t))$$

$$\delta\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a} + \delta\boldsymbol{a}, t) - \boldsymbol{x}(\boldsymbol{a},t)$$

$$= \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \cdot \delta\boldsymbol{a} + O(+)$$

$$\delta\boldsymbol{x} = \boldsymbol{J} \cdot \delta\boldsymbol{a}$$

We obtain,

$$\boldsymbol{B}(\boldsymbol{x}(\boldsymbol{a},t),t) = \boldsymbol{J}(\boldsymbol{a},t) \cdot \boldsymbol{B}(\boldsymbol{a},0)$$

This is the cauchy solution to our original equation. Let us consider an example. Take $\boldsymbol{B}(\boldsymbol{x},t)$ to be in a perfectly conducting fluid, where hte velocity field is defined to be $\boldsymbol{u} = \langle \sin(z), \cos(z), 0 \rangle$ and the field evolves from initial condition of $\boldsymbol{B}(\boldsymbol{x},0) = \langle 0,0,1 \rangle$. First we solve with the Cauchy method, i.e. following the particle paths.

$$\frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}$$

$$\boldsymbol{x}(t) = \langle t\sin(c_3) + c_1,\, t\cos(c_3) + c_2,\, c_3\rangle$$

$$\boldsymbol{x}(0) = \boldsymbol{a} \implies c_1 = a_1,\, c_2 = a_2,\, c_3 = a_3$$

$$\boldsymbol{x}(t) = \langle t\sin(a_3) + a_1,\, t\cos(a_3) + a_2,\, a_3\rangle$$

Furthermore, we can find the jacobian $(\frac{\partial x_i}{\partial a_j})$,

$$\boldsymbol{J} = \begin{bmatrix} 1 & 0 & t\cos(a_3) \\ 0 & 1 & -t\sin(a_3) \\ 0 & 0 & 1 \end{bmatrix}$$

and finally, $\boldsymbol{J} \cdot \boldsymbol{B}$

$$\boldsymbol{J} \cdot \boldsymbol{B} = \langle t\cos(a_3),\, -t\sin(a_3),\, 1\rangle$$

Now in order to put this back into an eularian form we have to write $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}, t)$ and to do this we write $z = a_3$ and find,

$$\boldsymbol{B}(\boldsymbol{x}, t) = \langle t\cos(z),\, t\sin(z),\, 1\rangle$$

Lecture 10: Kinematics for MHD: Continued

10.1 Comparing Cauchy solutions v.s. solving directly

Let us consider anothe example, with the same flow field $\boldsymbol{u} = \langle \sin(z), \cos(z), 0 \rangle$, and with initial magnetic field given by $\boldsymbol{B}_0 = \langle 0, y, -z \rangle$. Let us consider the Cauchy solution,

$$\boldsymbol{x}(\boldsymbol{a}, t) = \langle a_1 + t\cos(a_3), a_2 + t\sin(a_3), a_3 \rangle$$

$$\boldsymbol{B}(\boldsymbol{a}, t = 0) = \langle 0, a_2, -a_3 \rangle$$

$$\boldsymbol{B}(\boldsymbol{a}, t) = \begin{bmatrix} 1 & 0 & t\cos(a_3) \\ 0 & 1 & -t\sin(a_3) \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{B}(\boldsymbol{a}, 0)$$

$$\boldsymbol{B}(\boldsymbol{a}, t) = \langle -a_3 t\cos(a_3), a_2 + a_3 t\sin(a_3), -a_3 \rangle$$

Then we need to invert the form of our spatial solution, i.e. $\boldsymbol{x}(\boldsymbol{a}, t) \to \boldsymbol{a}(\boldsymbol{x}, t)$.

$$x = a_1 + t\sin(a_3) \implies a_1 = x - t\sin(z)$$

$$y = a_2 + t\cos(a_3) \implies a_2 = y - t\cos(z)$$

$$z = a_3 \implies a_3 = z$$

compared to solving this problem directly, the Cauchy solution can offer a much more simplified approach.

10.2

Let us consider an example with a sinusoidal shear flow acting in my flow, i.e. $\boldsymbol{u} = \langle \sin(z), 0, 0 \rangle$, and an initial magnetic field $\boldsymbol{B}_0 = \langle 0, 0, 1 \rangle$.

$$\boldsymbol{x}(\boldsymbol{a}, 0) = \langle a_1, a_2, a_3 \rangle \, \frac{d\boldsymbol{x}}{dt} = \boldsymbol{u} \implies \boldsymbol{x}(\boldsymbol{a}, t) = \langle a_1 + t\sin(a_3), a_2, a_3 \rangle$$

$$\boldsymbol{B}(\boldsymbol{a}, t) = \begin{bmatrix} 1 & 0 & t\cos(a_3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{B}(\boldsymbol{a}, 0) = \langle t\cos(z), 0, 1 \rangle$$

We can solve this including diffusion. The exact equations including diffusion are,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \sin(z)\frac{\partial \boldsymbol{B}}{\partial x} = B_z \cos(z)\boldsymbol{e}_x + \frac{1}{Rm}\nabla^2 \boldsymbol{B}$$
$$\boldsymbol{B} = \left\langle Rm\left(1 - e^{-t/Rm}\right)\cos(z), 0, 1\right\rangle$$

This recovers the cauchy solution if you take the limit as $Rm \to \infty$ and using L'Hopital's rule once.

$$\lim_{Rm\to\infty} Rm\left(1 - e^{-t/Rm}\right) = \frac{1 - e^{-t/Rm}}{Rm^{-1}}$$
$$= \lim_{Rm\to\infty} \frac{-\frac{t}{Rm^2}e^{-t/Rm}}{-\frac{1}{Rm^2}} = \lim_{Rm\to\infty} te^{-t/Rm}$$
$$= t$$

10.3 Magnetic Potential Solutions

Let us consider a solution to the induction equation using a potential function to generate a magnetic field, i.e. consider $\boldsymbol{B} = \left\langle \frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0\right\rangle = \nabla \times A\boldsymbol{e}_z$. Let us consider this field around a wire, which is going in the z-direction.

$$\frac{\partial}{\partial t}(\nabla \times A\boldsymbol{e}_z) = \nabla \times (\boldsymbol{u} \times A\boldsymbol{e}_z) + \eta\nabla^2(\nabla \times A\boldsymbol{e}_z)$$
$$\frac{\partial A}{\partial t} = [\boldsymbol{u} \times \nabla \times A\boldsymbol{e}_z]_z + \eta\nabla^2 A$$
$$\frac{DA}{Dt} = \eta\nabla^2 A$$

Notice that we obtain a scalar advection diffusion equation for the magnetic potential function, A. This advection problem doesn't consider three dimensional phenomenon such as stretching and tilting. If we are in search of a steady state solution to this PDE, we can set $\frac{\partial}{\partial t}$ to zero, and consider a magnetic reynolds number defined using a length scale given by the radius from the wire.

10.4 Prandtl-Bachelor Theorem

In two dimensions, if we have circular (or at least closed loop) streamlines, we can consider different differential regimes for a 2D magnetic field $\boldsymbol{B} = \boldsymbol{B}(x, y) = \nabla \times A\boldsymbol{e}_z$. In the scalar equation obtained for A, we can make a claim that in an ideal and steady scenario, A must be constant along streamlines, i.e. $A = A(\psi)$.

Lecture 11: Dynamics for MHD

11.1 The Lorentz Force

The dynamic regimes for MHD involve the Navier-Stokes equation, specifically the Lorentz force.

$$\rho\left(\frac{D\boldsymbol{u}}{Dt}\right) = -\nabla p + \rho\boldsymbol{g} + \boldsymbol{j}\times\boldsymbol{B} + \mu\left(\nabla^2\boldsymbol{u} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{u})\right)$$

We must remember that the lorentz force is defined as $\boldsymbol{j}\times\boldsymbol{B} = (\nabla\times\boldsymbol{B})\times\boldsymbol{B}$. which is perpendicular to both $\boldsymbol{B}$ and the curl of $\boldsymbol{B}$. We can attempt to simplify the lorentz force using a vector identity. Let us remember,

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\left(\frac{\boldsymbol{u}^2}{2}\right) - \boldsymbol{u}\times\boldsymbol{\omega}$$

Similarly we can write with the magnetic field,

$$\begin{aligned}
\boldsymbol{j}\times\boldsymbol{B} &= \frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nabla\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right)\\
&= \frac{1}{\mu_0}\nabla\cdot\boldsymbol{B}\boldsymbol{B} - \nabla\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right)\\
&= \frac{1}{\mu_0}\frac{\partial B_i B_j}{\partial x_j} - \nabla\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right)\\
&= \nabla\cdot m_{ij}
\end{aligned}$$

where m_{ij} is denoted as maxwell stresses. This yields two perspectives on the lorentz force. We can either think of it as a typical body force, i.e. the field acts on the fluid similar to gravity, or we can think of it as a stress which acts on the fluid. Either way, it is important to identify two components of the lorentz force which can be described as magnetic pressure $\nabla\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right)$, and magnetic tension $\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$. The tension component relies on some projection of $\nabla\boldsymbol{B}$ onto $\boldsymbol{B}$. The pressure component relies on some magnetic energy being non-constant. We can consider the tension component to be simplified as the following

$$\begin{aligned}
\boldsymbol{B} &= B\hat{\boldsymbol{s}}\\
\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} &= \frac{B}{\mu_0}\frac{\partial B\hat{\boldsymbol{s}}}{\partial s}\\
&= \frac{B}{\mu_0}\frac{\partial B}{\partial s}\hat{\boldsymbol{s}} + \frac{B^2}{\mu_0}\frac{\partial\hat{\boldsymbol{s}}}{\partial s}\\
&= \frac{\partial}{\partial s}\left(\frac{B^2}{\mu_0}\right)\hat{\boldsymbol{s}} + \frac{B^2}{\mu_0}\frac{\hat{\boldsymbol{n}}}{R_c}
\end{aligned}$$

where R_c^{-1} is defined by the quantity $|d\hat{\boldsymbol{s}}/ds|$ Notice the appearance of what looks similar to the magnetic pressure term in the tension component. Because typically, pressures are isotropic we can make the claim that the tension component will annihilate all pressure components in the fluid.

With this result, we can imagine some simplified dynamics which can arise between the velocity field $\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$. Consider a flow which perturbs a field line at a given point, we would expect there to be some magnetic tension acting back on the flow field against the direction of perturbation.

Another common scenario, is one which a bundle of field lines are close together. This creates an increase of magnetic pressure in that area, causing fluid to move away from that congregation of field lines.

Other quantities we can consider are the total pressure, i.e.

$$\begin{aligned} p &= p_{\text{gas}} + p_{\text{mag}} \\ &= p + \frac{1}{2\mu_0}\boldsymbol{B}^2 \end{aligned}$$

a plasma β:

$$\frac{p_{\text{gas}}}{p_{\text{mag}}} = \begin{cases} \gg 1 & \text{solar interior} \\ \ll 1 & \text{solar atmosphere} \end{cases}$$

Furthermore, magnetic pressure leads to the concept of magnetic buoyancy, i.e. a flux tube might have an internal gas density less than that of the gas density outside of the flux tube, inducing a buoyancy force on that flux tube. We also have the concept of magnetic tension causing Alfven waves,

$$\begin{aligned} \rho\frac{D\boldsymbol{u}}{Dt} &= -\nabla\left(p + \frac{\boldsymbol{B}^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \rho\boldsymbol{g} + \mu\,(\text{Diffusion}) \\ \frac{\partial\rho\boldsymbol{u}}{\partial t} &+ \nabla\cdot\left(\rho\boldsymbol{u}\boldsymbol{u} - \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B}\right) = \text{RHS} \end{aligned}$$

Let us consider some exampe problems with this equation. First, a uniform field $\boldsymbol{B} = \langle B_0, 0, 0\rangle$. We can decide there is neither magnetic pressure or tension acting on the fluid primarily because the field is constant and the specing between field lines is constant.

Second, we consider a unidirectional field $\boldsymbol{B} = \langle 0, B_0 e^x, 0\rangle$. We deduce that there would be magnetic pressure, of order $B_0^2 e^{2x}/\mu_0 \boldsymbol{e}_x$, but no magnetic tension since the gradients of $\boldsymbol{B}$ are orthogonal to $\boldsymbol{B}$. We confirm this by taking the curl of $\boldsymbol{B}$ which is a vector with only a z component, and then taking the cross product with $\boldsymbol{B}$ ($\boldsymbol{j}\times\boldsymbol{B}$) is a vector with only an x component and we find it is exactly equal to the term obtained by the magnetic pressure. Finally, a third scenario is the field $\boldsymbol{B} = B_0\langle -y, 1, 0\rangle$ which produces both magnetic tension and pressure which contribute to the lorentz force.

Lecture 12: Dynamics for MHD: Continued

12.1 Computing the Lorentz force

The following magnetic field produces field lines which generate a good deal of magnetic pressure and tension.

$$\boldsymbol{B} = \langle y, x, 0 \rangle$$

$$\boldsymbol{j} \times \boldsymbol{B} = \mathbf{0} \times \boldsymbol{B} = 0$$

Notice however, that the lorentz force is zero, and thus this is a force free field, namely a potential field. In order to understand why this happens we must look at the components of the lorentz force, individually. We have,

$$\frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = \frac{1}{\mu_0}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$$\nabla\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) = \frac{1}{2\mu_0}\begin{bmatrix} 2x \\ 2y \\ 0 \end{bmatrix}$$

and therefore the magnetic tension and pressure cancel each other out. Thus, there is no net force acting on the fluid in this particular state.

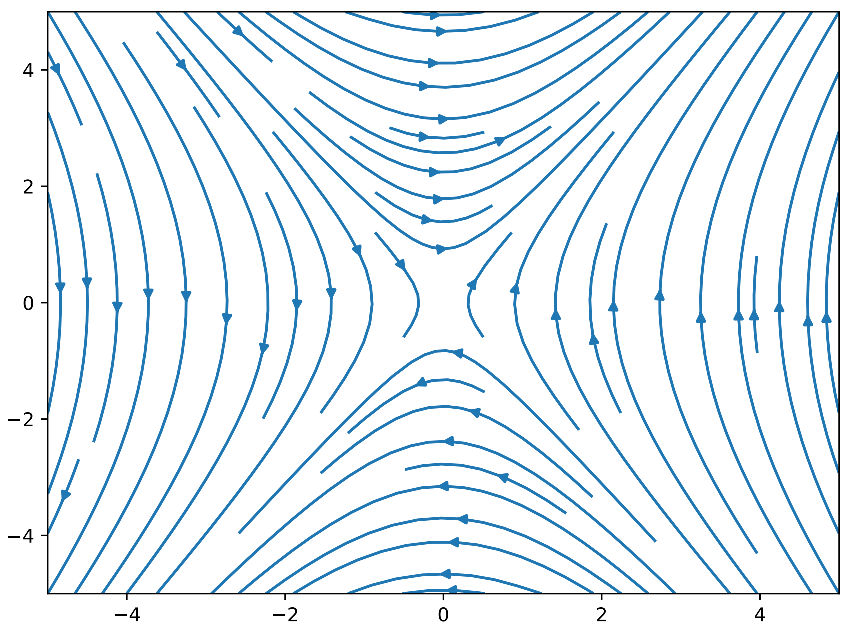

Figure 1: Plot of the magnetic field lines ($\boldsymbol{B} = \left\langle y, \alpha^2 x, 0 \right\rangle$) using streamplot from python. We can see that that nullclines are at a different angle due to the parameter α (> 1), which according to the lorentz force will cause a net force acting on the fluid (i.e. magnetic pressure and tension are no longer aligned).

We can consider a very similar field given by,

$$\boldsymbol{B} = \left\langle y, \alpha^2 x, 0 \right\rangle$$

$$\boldsymbol{j} \times \boldsymbol{B} = \left\langle -\alpha^2(1-\alpha^2)x, (1-\alpha^2)y, 0 \right\rangle$$

$$\frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = \frac{1}{\mu_0}\begin{bmatrix} \alpha^2 x \\ \alpha^2 y \\ 0 \end{bmatrix}$$

$$\nabla\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) = \frac{1}{2\mu_0}\begin{bmatrix} \alpha^4 2x \\ 2y \\ 0 \end{bmatrix}$$

in which the nullclines of the field are adjusted and therefore we do have a net lorentz force. This phenomenon is very interesting and often referred to as Magnetic Reconnection which is the complex phenomenon of how the nullclines of magnetic fields are moved around by outside circumstances.

12.2 Magnetic Energy (Density)

We can obtain an equation for the magnetic energy density similar to how a kinetic energy equation is obtained for the velocity field.

$$E_M = \int_V \frac{|\boldsymbol{B}^2|}{2\mu_0} dV$$
$$\frac{\boldsymbol{B}}{\mu_0} \cdot \left(\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \right)$$
$$\frac{\partial}{\partial t} \left(\frac{|\boldsymbol{B}|^2}{2\mu_0} \right) = -\frac{\boldsymbol{B}}{\mu_0} \cdot \nabla \times \boldsymbol{E}$$
$$\frac{\partial}{\partial t} \left(\frac{|\boldsymbol{B}|^2}{2\mu_0} \right) = -\frac{1}{\mu_0} \left(\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) + \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) \right)$$
$$\frac{\partial M}{\partial t} = -\frac{1}{\mu_0} \int_S (\boldsymbol{E} \times \boldsymbol{B}) \cdot \hat{\boldsymbol{n}} dS - \int_V \boldsymbol{u} \cdot (\boldsymbol{j} \times \boldsymbol{B}) - \frac{\boldsymbol{j}^2}{\sigma} dV$$

Here we can analyze each component of this magnetic energy density equation. We have the "Poynting vector" $\boldsymbol{E} \times \boldsymbol{B}$ which represents the flux of magnetic energy into the domain. We have the rate of magnetic energy loss by the mechanical work against the lorentz force $\boldsymbol{u} \cdot (\boldsymbol{j} \times \boldsymbol{B})$, and finally "Ohmic dissipation" $\boldsymbol{j}^2/\sigma$.

12.3 Magnetohydrostatics

We reintroduce the concept of balanced equations for hydrostatic balances, but in the specific context for MHD. We look at the zero-velocity steady state flow field. We must have some dominant balance in the navier stokes equation,

$$0 = -\nabla p + \rho \boldsymbol{g} + \boldsymbol{j} \times \boldsymbol{B}$$

This balance is denoted magnetohydrostatic balance and it relies on a very important assumption that the timescales of interest for this problem are much shorter than a diffusion timescale for the magnetic field. Note that the velocity field doesn't necessarily have to be zero, but it must satisfy some basic properies which make its relative magnitude negligable in the navier stokes equation.

Some common speeds used in physics scenarios are the following,

$$\left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}, \quad \text{speed of sound}$$
$$\left(\frac{B_0}{\mu_0 \rho_0}\right)^{1/2}, \quad \text{Alfven speed}$$
$$(2g_0 l_0)^{1/2}, \quad \text{gravity free fall speed}$$

where we must have u is much less than these quantities in order to preserve magentohydrostatic balance. Furhtermore there are several ways to obtain hydrostatic balance from the magnetohydrostatic balance. Namely, we could have $\boldsymbol{B} = 0$, $\boldsymbol{j} \times \boldsymbol{B} =$, or $\boldsymbol{j} = 0$ (in this case $\boldsymbol{B}$ can be written by a potential function $\boldsymbol{B} = \nabla A$).

Some examples of force free fields are Beltrami fields, where $\nabla \times \boldsymbol{B} = \alpha(x)\boldsymbol{B}$ which is of course perpendicular to $\boldsymbol{B}$ at all points in space. We find,

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = 0 = \nabla \cdot \alpha \boldsymbol{B}$$
$$= \alpha(\nabla \cdot \boldsymbol{B}) + \boldsymbol{B} \cdot \nabla \alpha \implies \boldsymbol{B} \cdot \nabla \alpha = 0$$

let us assume for now that α is some constant. We have the following,

$$\nabla \times (\nabla \times \boldsymbol{B}) = \alpha^2 \boldsymbol{B}$$
$$\nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = \alpha^2 \boldsymbol{B}$$
$$\nabla^2 \boldsymbol{B} + \alpha^2 \boldsymbol{B} = 0$$

which is a linear helmholtz equation for the magnetic field. Where as, solving for a force free field by $\boldsymbol{j} \times \boldsymbol{B} = 0$ is much more difficult due to nonlinearity.

Lecture 13: Magnetohydrostatics: Continued

13.1 Review of Magnetohydrostatic balance

We remember the equation for Magnetohydrostatic balance, whereby a steady state inviscid flow is given by the balance between pressure, buoyancy/gravity, and the lorentz force. Similarly, in the induction equation we only have magnetic diffusion affecting the magnetic field. Thereby, the magnetic field will decay and thus the magnetohydrostatic balance will erode and return to hydrostatic balance. Note that in order for hydrostatic balance to be useful, we must consider timescales much shorter than the magnetic diffusion timescale.

$$0 = -\nabla p + \rho \boldsymbol{g} + \boldsymbol{j} \times \boldsymbol{B}$$
$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B}$$

13.2 Non-forcefree solutions

Let us consider a simplified case to solve these equations whereby we have,

$$\boldsymbol{B} \cdot (0 = -\nabla p + \boldsymbol{j} \times \boldsymbol{B}), \quad \beta = \frac{p_{\text{gas}}}{p_{\text{mag}}}$$
$$0 = \boldsymbol{B} \cdot \nabla p$$
$$\boldsymbol{j} \cdot (0 = -\nabla p + \boldsymbol{j} \times \boldsymbol{B}) \to 0 = \boldsymbol{j} \cdot \nabla p$$

this tells us, as the original equation implies, that both $\boldsymbol{B}$ and $\boldsymbol{j}$ lie within surfaces of constant pressure, and therefore $\boldsymbol{j} \times \boldsymbol{B}$ is parallel to the gradients of pressure everywhere. Essentially this balance begs the physical intuition of pressure balancing the lorentz force.

13.3 The Theta-Pinch

$$\boldsymbol{j} = J\hat{\boldsymbol{\theta}}, \quad \boldsymbol{B} = b(r)\hat{\boldsymbol{r}}$$
$$\boldsymbol{j} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B} = \frac{1}{\mu_0} \frac{\partial b(r)}{\partial r} \hat{\boldsymbol{\theta}}$$
$$\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0} \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ 0 & -\frac{\partial b}{\partial r} & 0 \\ 0 & 0 & b \end{vmatrix}$$
$$= -\frac{b}{\mu_0} \frac{db}{dr} \hat{\boldsymbol{r}}$$

With some algebra, we can show that .... Thus we obtain the conclusion that at some point,

$$\frac{b^2}{2\mu_0} = p_0 \implies p_{\text{gas}} = 0$$

13.4 Linear Pinch

$$
\begin{aligned}
\boldsymbol{B} &= b(r)\hat{\boldsymbol{\theta}} \\
\boldsymbol{j} &= \frac{1}{\mu_0 r}\begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & rb(r) & 0 \end{vmatrix} \\
&= \frac{1}{\mu_0 r}\frac{\partial rb}{\partial r}\hat{\boldsymbol{z}} \\
\boldsymbol{j}\times\boldsymbol{B} &= \frac{1}{\mu_0 r}\begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ 0 & 0 & \frac{\partial rb}{\partial r} \\ 0 & b & 0 \end{vmatrix} \\
&= -\frac{b}{\mu_0 r}\frac{\partial rb}{\partial r}\hat{\boldsymbol{r}}
\end{aligned}
$$

In order to solve this equation we first write

This solves the problem inside the radius of a where there is current.

$$
\begin{aligned}
\boldsymbol{j} &= J_0\hat{\boldsymbol{z}} \\
\frac{\partial rb}{\partial r} &= J_0\mu_0 r \\
rb &= \frac{J_0\mu_0}{2}r^2 + c_1 \\
b(r) &= \frac{J_0\mu_0}{2}r + \frac{c_1}{r} \\
b_i(r) &= \frac{J_0\mu_0}{2}r, \quad r \le a
\end{aligned}
$$

Outside of this current region we have a different solution and must solve for continuity at the interface.

$$
\begin{aligned}
\frac{\partial rb}{\partial r} &= 0 \\
b_o(r) = \frac{c_1}{r} &\to \frac{J_0\mu_0 a^2}{2r}, \quad r > a
\end{aligned}
$$

We next want to understand how this solution affects the gas pressure in the system.

$$\frac{dp}{dr} = -\frac{b}{\mu_0 r}\frac{d(rb)}{dr}\hat{\boldsymbol{r}}$$

$$\vdots$$

$$p = \begin{cases} p_{\text{atmos}} & r > a \\ -\frac{1}{4}J_0^2\mu_0(r^2 - a^2) + p_{\text{atmos}} & r \leq a \end{cases}$$

13.5 Cylindrical Pinch

The cylindrical pinch requires that the magnetic field has both an angular component and a vertical component, i.e.

$$\boldsymbol{B} = b_L\hat{\boldsymbol{\theta}} + b_T\hat{\boldsymbol{z}}$$

$$\boldsymbol{j} = J_T\hat{\boldsymbol{\theta}} + J_L\hat{\boldsymbol{z}}$$

$$-\frac{\partial p}{\partial r} - \frac{\partial}{\partial r}\left(\frac{b_L^2 + b_T^2}{2\mu_0}\right) - \frac{b_L^2}{\mu_0 r} = 0$$

We can solve for the amount of twist in the system using the equations for the fieldlines.

$$\frac{d\theta}{dz} = \frac{b_L}{rb_T}$$

$$\int d\theta = \frac{Lb_L}{rb_T}$$

13.6 Reverse Field Pinch

This idea stems from the Beltrami cylindrical geometry,

$$\boldsymbol{B} = (0, B_\theta, B_z)$$

$$\nabla^2\boldsymbol{B} + \alpha\boldsymbol{B} = 0$$

$$B_\theta = B_0 J_0(\alpha r), \quad \text{this is a bessels function of order zero}$$

$$B_z = B_0 J_1(\alpha r), \quad \text{this is a bessels function of order one}$$

Lecture 14: Dynamics for MHD: Continued

14.1 Steady State MHD

Let us consider a simplified dynamical regime in which we encounter a steady state flow, i.e. $\frac{\partial \boldsymbol{u}}{\partial t} = 0$. We have that the governing equations reduce to the following:

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho_0}\nabla p + \boldsymbol{j} \times \boldsymbol{B} + \nu \nabla^2 \boldsymbol{u}$$
$$\frac{D\boldsymbol{B}}{Dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \eta \nabla^2 \boldsymbol{B}$$
$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{B} = 0$$

Next we consider a specific flow field $\boldsymbol{u}$ called the Hartmann flow given by $\boldsymbol{u} = (u, 0, 0)$ and a specific magnetic field given by $\boldsymbol{B} = (0, 0, B_0)$. Let us investigate the dynamics that arise from this scenario. At $t = 0$ we have that $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial B_0}{\partial z} = 0$. Therefore,

$$0 = -\frac{1}{\rho_0}\nabla p + \frac{1}{\mu_0 \rho_0}\left(\frac{\partial b}{\partial z} - \frac{\partial B_0}{\partial x}\right)(B_0 \boldsymbol{e}_x - b\boldsymbol{e}_z) + \nu \frac{\partial^2 u}{\partial z^2}$$
$$u\frac{\partial b}{\partial x}\boldsymbol{e}_x + u\frac{\partial B_0}{\partial x}\boldsymbol{e}_z = B_0 \frac{\partial u}{\partial z}\boldsymbol{e}_x + \eta \nabla^2 B_0 \boldsymbol{e}_z + \eta \nabla^2 b \boldsymbol{e}_x$$
$$\frac{\partial b}{\partial x} + \frac{\partial B_0}{\partial z} = 0, \quad \frac{\partial u}{\partial x} = 0$$

This problem has boundary conditions $u(z+d) = 0$ and $u(z-d) = 0$ where d is half of the height of the tube, and there is no boundary in the x-direction (implicitly indicating it is an infinite tube). With these conditions, we cannot presume any dependence on x in the velocity field, nor the magnetic field perturbations b. Thus we obtain the solution given by,

$$u(z) = \left(\frac{p_0 + \sigma B_0 E_0}{\sigma B_0^2}\right)\left(1 - \frac{\cosh(Mz/d)}{\cosh(M)}\right)$$
$$\frac{db}{dz} = \frac{E_0 - uB_0}{\eta}$$
$$b(z) = \frac{E_0 z}{\eta} - \left(\frac{p_0 + \sigma B_0 E_0}{\eta \sigma B_0}\right)\left(z - \frac{d \sinh(Mz/d)}{M \cosh(M)}\right)$$

where M is the Hartmann number given by $M = B_0/\sqrt{\mu \mu_0 \eta}$ (measures lorentz force against viscous diffusivity and E_0 is the y-component of the given electric field) and μ is the permeability and μ_0 is the viscosity.

Lecture 15: Dynamics for MHD: Continued

15.1 Steady State MHD and Hartmann flows

As a result of studying Hartmann flows, we obtain the Hartmann number which describes the relative weight of the lorentz force to the viscous diffusion of the flow. Most importantly, like any non-dimensional number, we have two limits which describe extreme regimes of dynamics. We have either $M \ll 1$ or $M \gg 1$. Let us investigate the solution to the Hartmann flow problem in either regime.

$$
\begin{aligned}
\frac{\cosh Mz/d}{\cosh M} &\sim \frac{\left(1 + \frac{(Mz/d)^2}{2} + \cdots\right)}{\left(1 + \frac{M^2}{2} + \cdots\right)} \\
&\sim \left(1 + \frac{(Mz/d)^2}{2} + \cdots\right)\left(1 - \frac{M^2}{2} + \cdots\right) \\
&\sim 1 - \frac{M^2}{2} + \frac{(Mz/d)^2}{2} + \cdots \\
&\sim 1 - \frac{M^2}{2}\left(1 - \frac{z^2}{d^2}\right) + \cdots
\end{aligned}
$$

We can substitute this back into the solution for $u(z)$ we obtained in the last lecture. We have,

$$
\begin{aligned}
u(z) &= \left(\frac{p_0 + \sigma B_0 E_0}{\sigma B_0^2}\right)\left(\frac{M^2}{2}\left(1 - \frac{z^2}{d}\right)\right) \\
&= \frac{1}{2\mu}\left(p_0 + \sigma B_0 E_0\right)\left(d^2 - z^2\right)
\end{aligned}
$$

This is the given solution in the limit of $M \ll 1$. We can redo this computation in the limit where $M \gg 1$. In this scenario we have,

$$
\begin{aligned}
\cosh(x) &= \frac{e^x + e^{-x}}{2} \approx \frac{1}{2}e^{|x|} \\
\frac{\cosh Mz/d}{\cosh M} &\sim \begin{cases} e^{|Mz/d| - |M|} & \left|\frac{Mz}{d}\right| \gg 1 \\ O(e^{-M}) & \left|\frac{Mz}{d}\right| < O(1) \end{cases}
\end{aligned}
$$

we notice that the cosh term is small everywhere except where $(1 - |z/d|)$ is also small, i.e. close to the boundary. Thus our solution reduces to a constant flow field (with the constant given by the prefactor in front of the $1 - \cosh$ term), with a small boundary layer with a scale height given by d. We can rescale our

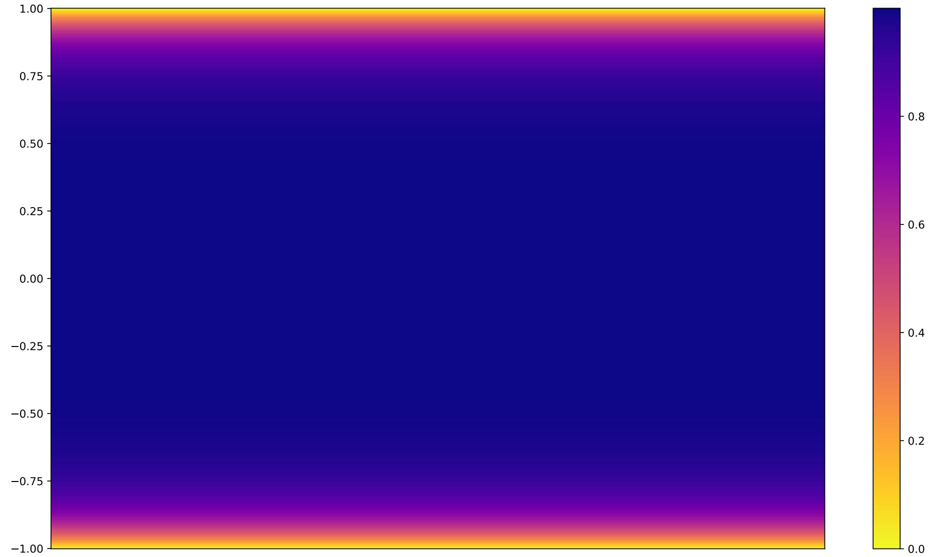

Figure 2: Plot of the Steady State Hartmann flow with $M = 10$ $d = 1$ which shows the boundary layers created in the flow

main variable, $z = d - \epsilon$, where ϵ is the distance from the boundary such that $0 < \epsilon \ll 1$. We have,

$$\cosh(Mz/d) = \cos\left(M(1 - \epsilon/d)\right) \sim e^{-M\epsilon/d}$$

$$u(\epsilon) \sim \left(\frac{p_0 + \sigma B_0 E_0}{\sigma B_0^2}\right)\left(\frac{M\epsilon}{d} + \cdots\right)$$

This solution gives a flow field which is asymptotically constant except within the boundary layers which are given by a scale height $\frac{d}{M}$. This can be seen in Figure 2.

15.2 Controlling flow and field with current

Let us consider an electric field in a given three-dimensional domain $y \in [-L, L]$. Let us consider, a simple Ohm's law given by $j = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ and a given electric field $\boldsymbol{E} = (0, E_0, 0)$, magnetic field $\boldsymbol{B} = (0, 0, B_0)$, and unidirectional flow $\boldsymbol{u} = u(z)\boldsymbol{e}_x$. Next we will perform an average in the z direction of the y component of the current.

$$I_y = \frac{1}{2d}\int_{-d}^{d} j_y dz = \sigma E_0 + \sigma B_0 \int_{-d}^{d} u(z) dz$$

$$= \sigma E_0 + \sigma B_0 \overline{u}$$

This integral has cases. For example, given an open circuit we have that $I_y = 0$ whereby $\bar{u} = E_0/B_0$. This will act as a flow meter. Another example is that of a short circuit, for example shorting the circuit by connecting the two ends with another fire. We will have $E_0 = 0$ and therefore $j_y = -u(z)B_0$. Let us consider the Lorentz force in the x-direction according to this scenario. We have,

$$(\boldsymbol{j} \times \boldsymbol{B})_x = -j_z B_y + j_y B_z = -u(z)B_0^2$$

which will act as a negative definite term and resist the flow field.
Another example could be introducing a small EMF (particularly $E_0 < \bar{u}B_0$). Then we have, $j_y < 0$ and specifically, we have created an electric generator, that is, the mechanical energy of the fluid drives a current and this generates electrical energy. Note, that in this scenario the Lorentz force would be negative as well. Finally, we could consider a large EMF (specifically larger than $\bar{u}B_0$). This would create a positive current and thus a positive Lorentz force.

15.3 Waves

In order to investigate waves in MHD, we must review waves in their most general conceptualization. For example, the wave equation is classically defined as,

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2}$$

which is a hypervolid PDE and is governed by a parameter c which is generally thought of as the phase speed of the wave. The wave equation has some very general solutions, for example D'Alembert's solution for the wave equation given by $u(x,t) = f(x - ct) + f(x + ct)$ which is obtained by rewriting the wave equation as the superposition of two transport equations ($\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}$). Most important about waves is their typical solutions. The periodicity of waves often yields solutions of the form sin and cos, i.e. fourier eigenfunctions. This often simplifies the wave equation into an eigenvalue problem which can be solved for multiple BC forms and problem setups. Additionally, we can use Euler's identity to write general solutions as,

$$u(x,t) = \sum_{k,\omega} A_{k,\omega} e^{i(kx - \omega t)}$$

D'Alemberts solution is non-dispersive (for c constant). However, there are many disperive waves, i.e. they might not have constant phase and group speeds. If we return to the wave equation and use an ansatz to solve for the angular velocity and wavenumber we obtain a dispersion relation,

$$\frac{\partial \eta}{\partial t} + c\frac{\partial \eta}{\partial x} + \alpha\frac{\partial^3 \eta}{\partial x^3} = 0 - i\omega + c(ik) - \alpha i k^3 = 0 \implies \frac{w}{k} = c - k^2$$

This is the phase speed, and in constrast the group speed is given by $\frac{\partial \omega}{\partial k}$. These are general wave properties and each of them apply to the waves studied in other contexts. Specifically in MHD, there are several restoring forces which can be used to generate waves. Alfven waves are generated by magnetic tension. Compressional Alfven waves are generated by magnetic pressure. Surface/internal gravity waves are generated by gravity/stratification. Acoustic waves ar generated by pressure gradients. Inertial waves are typically generated by the coriolis force.

Lecture 16: Waves in MHD: Continued

16.1 Elastic Waves on a String

Let us consider a string which has elasticity that is plucked. We can consider a 2D problem where the string only oscillates up and down (not forward and backwards). The restoring force becomes the tension in the string rather than gravity. Thus any displacements from equilibrium will be acted on by tension. We can immagine the force acting on either ends of an infinitesimal length of string ds. On either end we have T_1 and α_1 and T_2 and α_2 which are the amplitide of tension and the angle with respect to the horizontal that the tension acts in. In the scenario where there is no Left-Right motion we have,

$$T_2 \cos(\alpha_2) = T_1 \cos(\alpha_1)$$

and in the case where we have Up-Down motion (i.e. a standing wave),

$$\begin{aligned}
T_2 \sin(\alpha_2) - T_1 \sin(\alpha_1) &= \frac{\rho \delta x}{T}\frac{\partial^2 y}{\partial t^2} \\
\tan(\alpha_2) - \tan(\alpha_1) &= \frac{\rho \delta x}{T}\frac{\partial^2 y}{\partial t^2} \\
\frac{1}{\delta x}\left(\frac{\partial y}{\partial x}x|_{x+\delta x} - \frac{\partial y}{\partial x}x|_x\right) &= \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \\
\frac{\partial^2 y}{\partial x^2} &= \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

Thus we obtain a wave equation for a transverse wave. We could consider the case where rather than a physical string, we consider the perturbation of magnetic field lines, in which case the restoring force is magnetic tension. In this case we replace the dummy variable T with magnetic tension.

$$\frac{\partial^2 y}{\partial t^2} = \frac{B_0^2}{\rho\mu_0}\frac{\partial^2 y}{\partial x^2}, \quad v_A = \sqrt{\frac{B_0^2}{\rho\mu_0}}$$

We can obtain this result with a formal analysis. Consider an incompressible, undamped, IDEAL, fluid without the affects of gravity nor rotation.

$$\begin{gathered}
\rho_0\left(\frac{D\boldsymbol{u}}{Dt}\right) = -\nabla p + \boldsymbol{j}\times\boldsymbol{B}, \quad \nabla\cdot\boldsymbol{u} = 0 \\
\frac{D\boldsymbol{B}}{Dt} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \quad \nabla\cdot\boldsymbol{B} = 0
\end{gathered}$$

Let us assume a basic state for the flow, which we take to be magnetohydrostatic balance. Thus we have,

$$0 = -\nabla p + \boldsymbol{j}\times\boldsymbol{B}$$

and consider the perturbations to the flow, i.e. $\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}'$, and specifically $\overline{\boldsymbol{u}} = 0$.

$$\begin{gathered}
\rho_0\left(\frac{D\boldsymbol{u}'}{Dt}\right) = -\nabla p' + \overline{\boldsymbol{j}}\times\boldsymbol{B}' + \boldsymbol{j}'\times\overline{\boldsymbol{B}} + \boldsymbol{j}'\times\boldsymbol{B}', \quad \nabla\cdot\boldsymbol{u}' = 0 \\
\frac{\partial}{\partial t}(\overline{\boldsymbol{B}} + \boldsymbol{B}') = \nabla\times(\boldsymbol{u}'\times\overline{\boldsymbol{B}}) + \nabla\times\boldsymbol{u}'\times\boldsymbol{B}', \quad \nabla\cdot\overline{\boldsymbol{B}} + \boldsymbol{B}' = 0
\end{gathered}$$

After linearizing with the fact that $|\boldsymbol{B}'| \ll |\overline{\boldsymbol{B}}|$ and taking $\overline{\boldsymbol{B}}$ to be steady, we have

$$\begin{gathered}
\rho_0\left(\frac{\partial\boldsymbol{u}'}{\partial t}\right) = -\nabla p' + \overline{\boldsymbol{j}}\times\boldsymbol{B}' + \boldsymbol{j}'\times\overline{\boldsymbol{B}}, \quad \nabla\cdot\boldsymbol{u}' = 0 \\
\frac{\partial\boldsymbol{B}'}{\partial t} = \nabla\times(\boldsymbol{u}'\times\overline{\boldsymbol{B}}), \quad \nabla\cdot\boldsymbol{B}' = 0
\end{gathered}$$

We can simplify the problem more if we simply take $\overline{\boldsymbol{B}}$ to be a constant field.

$$\begin{gathered}
\rho_0\left(\frac{\partial\boldsymbol{u}'}{\partial t}\right) = -\nabla p' + \boldsymbol{j}'\times\overline{\boldsymbol{B}}, \quad \nabla\cdot\boldsymbol{u}' = 0 \\
\frac{\partial\boldsymbol{B}'}{\partial t} = \nabla\times(\boldsymbol{u}'\times\overline{\boldsymbol{B}}), \quad \nabla\cdot\boldsymbol{B}' = 0
\end{gathered}$$

Then using a vector identity to simplify the triple vector product,

$$\rho_0 \left(\frac{\partial \boldsymbol{u'}}{\partial t} \right) = -\nabla \left(p' + \frac{1}{\mu_0} \boldsymbol{B'} \cdot \overline{\boldsymbol{B}} \right) + \frac{1}{\mu_0} (\overline{\boldsymbol{B}} \cdot \nabla) \boldsymbol{B'}, \quad \nabla \cdot \boldsymbol{u'} = 0$$
$$\frac{\partial \boldsymbol{B'}}{\partial t} = (\overline{\boldsymbol{B}} \cdot \nabla) \boldsymbol{u'}, \quad \nabla \cdot \boldsymbol{B'} = 0$$

thus we obtain lineared, momentum and induction equations which contain a typical magnetic pressure and tension term (products of $\overline{B}$ and $\boldsymbol{B'}$) and in the induction equation a typical linearized evolution equation. In order to get rid of the pressure terms, we will take the curl of the two obtained equations. We have,

$$\rho_0 \left(\frac{\partial \boldsymbol{\omega'}}{\partial t} \right) = (\overline{\boldsymbol{B}} \cdot \nabla) \boldsymbol{j'}$$
$$\frac{\partial \boldsymbol{j'}}{\partial t} = \mu_0 (\overline{\boldsymbol{B}} \cdot \nabla) \boldsymbol{\omega'}$$

Now can can combine these equations by taking additional derivatives and then use substitution to obtain.

$$\frac{\partial^2 \boldsymbol{\omega'}}{\partial t^2} = \frac{1}{\mu_0 \rho_0} (\overline{\boldsymbol{B}} \cdot \nabla)^2 \boldsymbol{\omega'}$$
$$\frac{\partial^2 \boldsymbol{j'}}{\partial t^2} = \frac{1}{\mu_0 \rho_0} (\overline{\boldsymbol{B}} \cdot \nabla)^2 \boldsymbol{j'}$$

Further investigation into the RHS reveals that for a uniform vertical magnetic field we have, $(\overline{\boldsymbol{B}} \cdot \nabla)^2 = B_0^2 \frac{\partial^2}{\partial z^2}$. Thus we have found a wave equation with a well defined wavespeed for transverse waves of the perturbations to magnetic field lines.

16.2 Searching for Wavelike Solutions from the linearized equations

Let us investigate this problem from the perspective of an ansatz considering the equations have constant coefficients.

$$q' = \tilde{q} e^{i\boldsymbol{k} \cdot \boldsymbol{x} - \omega t}$$

Let us plug this into our linearized equations,

$$-\rho_0 \omega \tilde{\boldsymbol{u}} = -\boldsymbol{k} \left(\tilde{p} + \frac{1}{\mu_0} \tilde{\boldsymbol{B}} \cdot \overline{\boldsymbol{B}} \right) + \frac{1}{\mu_0} (\overline{\boldsymbol{B}} \cdot \boldsymbol{k}) \tilde{\boldsymbol{B}}, \quad \boldsymbol{k} \cdot \tilde{\boldsymbol{u}} = 0$$
$$-\omega \tilde{\boldsymbol{B}} = (\overline{\boldsymbol{B}} \cdot \boldsymbol{k}) \tilde{\boldsymbol{u}}, \quad \boldsymbol{k} \cdot \tilde{\boldsymbol{B}} = 0$$

Similar to the last setup, we will take the curl of these two equations $\nabla\times = \boldsymbol{k}\times$,

$$-\rho_0\omega\boldsymbol{k}\times\tilde{\boldsymbol{u}} = \frac{1}{\mu_0}(\overline{\boldsymbol{B}}\cdot\boldsymbol{k})\boldsymbol{k}\times\tilde{\boldsymbol{B}}$$
$$-\omega\boldsymbol{k}\times\tilde{\boldsymbol{B}} = (\overline{\boldsymbol{B}}\cdot\boldsymbol{k})\tilde{\boldsymbol{u}}$$

Now using substitutions we can show,

$$-\rho_0\omega\boldsymbol{k}\times\frac{-\omega\tilde{\boldsymbol{B}}}{\overline{\boldsymbol{B}}\cdot\boldsymbol{k}} = \frac{1}{\mu_0}(\overline{\boldsymbol{B}}\cdot\boldsymbol{k})\boldsymbol{k}\times\tilde{\boldsymbol{B}}$$
$$\frac{\rho_0\omega^2}{\overline{\boldsymbol{B}}\cdot\boldsymbol{k}}\boldsymbol{k}\times\tilde{\boldsymbol{B}} = \frac{1}{\mu_0}(\overline{\boldsymbol{B}}\cdot\boldsymbol{k})\boldsymbol{k}\times\tilde{\boldsymbol{B}}$$
$$\omega^2 = \frac{1}{\rho_0\mu_0}(\overline{\boldsymbol{B}}\cdot\boldsymbol{k})^2 = \frac{1}{\rho_0\mu_0}|\overline{\boldsymbol{B}}|^2|\boldsymbol{k}|^2\cos^2(\theta)$$

This is a dispersion relation for these MHD waves. We notice a few things. First and foremost, the wavespeed is fastest when the wavevector is parallel to $\overline{\boldsymbol{B}}$. Secondly, there is no propagation when the wavevector is perpendicular to $\overline{\boldsymbol{B}}$. These statements are originate from the geometry of the dot product. We can also consider the phase and group speed for these waves,

$$c = \frac{\omega}{\boldsymbol{k}} = v_A^2|\boldsymbol{k}|\cos^2(\theta)$$
$$c_g = \frac{\partial\omega}{\partial\boldsymbol{k}} = 2v_A^2\frac{\boldsymbol{k}^2}{|\boldsymbol{k}|}\cos^2(\theta)$$

An interesting way to visualize this dispersion relation is a polar diagram of the dispersion relation.