

For the next bit of analysis,  
we are going to assume that  
the data comes from a dynamical  
system:

$$\frac{d\bar{u}}{dt} = F(\bar{u})$$

Solving this system, leads to

$$\bar{u}(t_1), \bar{u}(t_2) \dots \bar{u}(t_n)$$

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We define a data matrix

$$\bar{X} = \begin{bmatrix} 1 & 1 & \dots \\ u(t_1) & u(t_2) & \dots \\ | & | & \\ | & | & \end{bmatrix}$$

The singular value decomposition

deals with approximating the data matrix  $\bar{X}$ .

# Singular Value Decomposition

We would now exploit notation a little bit

$$\bar{X} = \begin{bmatrix} | & | & & | \\ x_0 & x_1 & \dots & x_m \\ | & | & & | \end{bmatrix}$$

$\bar{X} \in \mathbb{C}^{n \times m}$   $\rightarrow$  time measurements  
 $\downarrow$   $\rightarrow$  spatial observation  
mostly real numbers, but think Schrodinger's equation

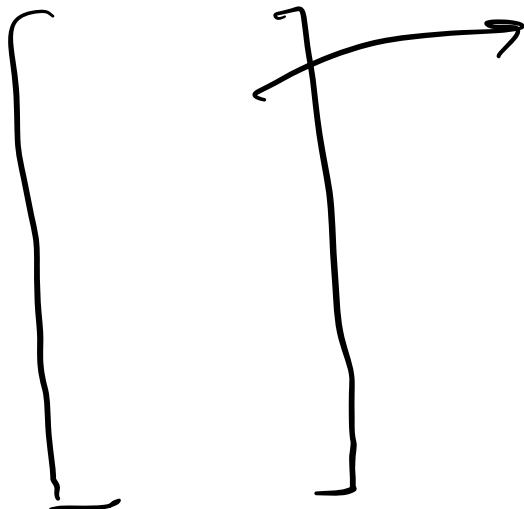
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Note: More often than not,

$x_0$  is a long vector and

$\bar{X}$  is a skinny matrix, since we don't have enough temporal measurements.

So  $\overline{X}$ , in reality looks like:

$\bar{X} =$   Tail and skinny

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The SVD decomposition is a matrix factorization:

$$\bar{X} = \underbrace{U}_{n \times n} \underbrace{\Sigma}_{n \times m} \underbrace{V^*}_{m \times m}$$

Singular values

right Singular vectors

unitary  
orthonormal  
basis  
(left singular  
vectors)

these are  
orthonormal  
as well

$\Sigma$  has at most  
m non zero elements if  $n \geq m$

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Note: \* Unitary means  $A^T = A^*$

\* So  $AA^* = A^*A = I$

For  $n \geq m$ , we can re-write the SVD:

$$\bar{X} = \begin{bmatrix} \hat{U} & \hat{U}^\perp \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} \begin{bmatrix} \hat{V}^* \end{bmatrix}$$

Dimensions:  $\bar{X}$  is  $n \times m$ .  $\hat{U}$  is  $n \times m$ ,  $\hat{U}^\perp$  is  $n \times (n-m)$ .  $\hat{\Sigma}$  is  $m \times m$ .  $\hat{V}^*$  is  $m \times m$ .

$$\bar{X} = \underbrace{\hat{U}}_{n \times m} \underbrace{\hat{\Sigma}}_{m \times m} \underbrace{\hat{V}^*}_{m \times m} \rightarrow \text{memory efficient}$$

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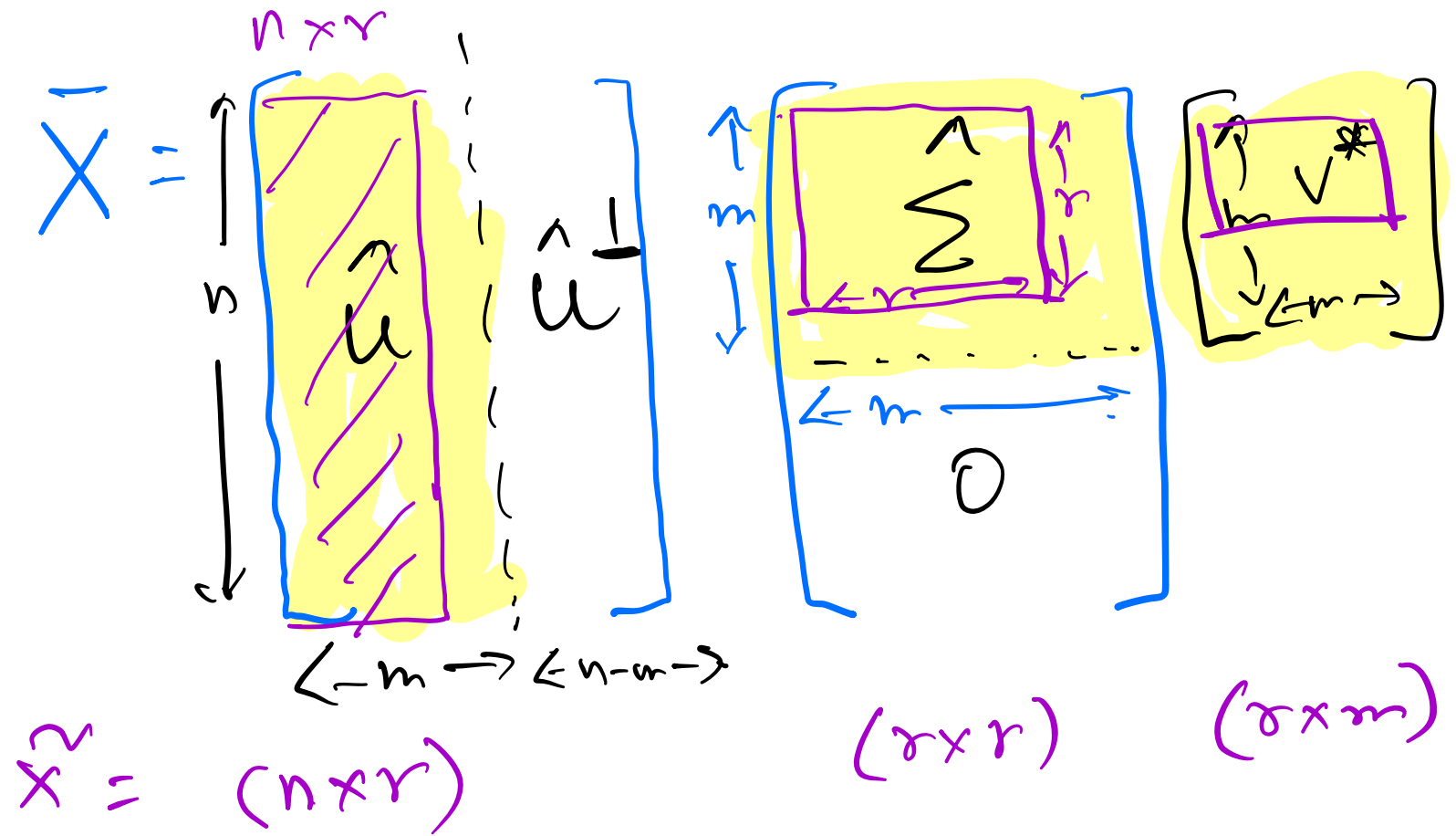
The rank of the data matrix  $\bar{X}$  is the number of non zero singular values.

Using the same idea as that of economy SVD, we can now define an optimal  $r$ -rank approximation.

$$\tilde{X} = \sum_{k=1}^r \sigma_k u_k v_k^*$$

↙ dyadic summation

$$= \underbrace{\sigma_1 u_1 v_1^*}_{\text{rank-1}} + \underbrace{\sigma_2 u_2 v_2^*}_{\text{rank-1}} + \dots$$



A-priori error analysis:

$$\|X - \tilde{X}\|_F^2 = \sum_{k=r+1}^m \sigma_k^2$$



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### Relationship with Eigenvalue decomposition

Consider the correlation matrix:  $XX^*$

$$X = [u] [\hat{\Sigma}] [v^*]$$

$$X^* = [v] [\hat{\Sigma}^* \ 0] [u^*]$$

$$XX^* = [u] [\hat{\Sigma}^2 \ 0] [u^*]$$

Because  $v^* v = I$

So  $\begin{bmatrix} \hat{\Sigma}^2 & 0 \\ 0 & 0 \end{bmatrix}$  are the eigenvalues

of  $XX^*$ . So, we can say:

eigenvalues of  $XX^*$  are the square  
of the singular values of  $X$

Projection on the singular  
vectors:

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Let's go back to the  
data matrix,  $X$

$$X = \begin{pmatrix} x_0 & x_1 & \dots & x_m \end{pmatrix}$$

$\downarrow \quad \quad \quad \leftarrow m$

Projected data matrix

$$\hat{X} = U_{r \times n}^T X$$

Here, we take the first ' $r$ ' columns of the left singular vector and project the data on these vectors.

So  $\hat{X}$  is of size  $r \times m$

So, we have reduced the spatial dimension of the system from " $n$ " to " $r$ "