AM 275 - Magnetohydrodynamics: Homework 1

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Problem 1:

Show that

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} + p e_{kk} - 2\mu \left[e_{ij} - \frac{1}{3} e_{kk}\delta_{ij} \right]^2.$$

Proof. First, we begin with the derivative identity

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

and in order to simplify this statement, take τ_i to be the i-th row vector of τ, we have:

$$\sum_i u_i \nabla \cdot \tau_i = \sum_i \nabla \cdot u_i \tau_i - \tau_i \cdot \nabla u_i$$

Already we have shown the first RHS term originates from the derivative identity, whereas the other terms must originate from $-\sum_i \tau_i \cdot \nabla u_i$. Thus, we investigate this term in more detail.

$$-\sum_i \tau_i \cdot \nabla u_i = \sum_i \left[p + \frac{2}{3}\mu \nabla \cdot \boldsymbol{u} \right] \delta_{ij} \cdot \nabla u_i - 2\mu e_i \cdot \nabla u_i$$

where e_{kk} is written as $\nabla \cdot \boldsymbol{u}$ and e_i is the i-th row of e (as in e_{ij}). Notice that $\sum_i \delta_{ij} \cdot \nabla u_i = \nabla \cdot \boldsymbol{u}$, and therefore,

$$\begin{aligned}
-\sum_i \tau_i \cdot \nabla u_i &= \left[p + \frac{2}{3}\mu \nabla \cdot \boldsymbol{u} \right] (\nabla \cdot \boldsymbol{u}) - 2\mu \sum_i e_i \cdot \nabla u_i \\
&= p(\nabla \cdot \boldsymbol{u}) + \frac{2}{3}\mu (\nabla \cdot \boldsymbol{u})^2 - 2\mu \sum_i e_i \cdot \nabla u_i
\end{aligned}$$

Thus we recover the second RHS term, $p e_{kk}$. Now we must show the rest of $-\sum_i \tau_i \cdot \nabla u_i$ recovers the last term of the RHS. We write the decomposition of e_i.

$$\begin{aligned}
-2\mu \sum_i e_i \cdot \nabla u_i &= -\mu \sum_i \left(\nabla u_i + \frac{\partial \boldsymbol{u}}{\partial x_i} \right) \cdot \nabla u_i \\
&= -\mu \sum_i |\nabla u_i|^2 + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\
&= -\mu |\nabla \boldsymbol{u}|^2 - \mu \sum_i \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\
&= -\mu |\nabla \boldsymbol{u}|^2 - \mu \left((\nabla \cdot \boldsymbol{u})^2 + 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} \right) \right)
\end{aligned}$$

Now we must show by the transitive propery that,

$$\frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2 - \mu|\nabla\boldsymbol{u}|^2 - \mu\left((\nabla\cdot\boldsymbol{u})^2 + 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right)\right) = -2\mu\left[e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right]^2$$

$$-\frac{1}{3}(\nabla\cdot\boldsymbol{u})^2 - |\nabla\boldsymbol{u}|^2 - 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right) = -2\left[e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right]^2$$

$$-2\left[e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right]^2 = -2\left[e_{ij}^2 - \frac{2}{3}(\nabla\cdot\boldsymbol{u})e_{ij} - \frac{1}{9}(\nabla\cdot\boldsymbol{u})^2\boldsymbol{I}\right]$$

$\square$