AM112, Assignment #1

1. Solve $y''(t)+\varepsilon y'(t)+y(t)=0$ for a general solution for small ε > 0.

   Observe that the solution oscillates while the amplitude decays slowly.

2. Show that $\int_0^L \sin(\frac{(n-\frac{1}{2})\pi}{L}x)\sin(\frac{(m-\frac{1}{2})\pi}{L}x)\,dx = \begin{cases} 0, & n\neq m, n>0, m>0 \\ L/2, & n=m>0 \end{cases}$.

<u>Hint:</u> Use $\sin\alpha\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$.

<u>Remark:</u> This result demonstrates that eigenfunctions $\left\{\sin(\frac{(n-\frac{1}{2})\pi}{L}x), n=1,2,\ldots\right\}$ are orthogonal to each other.

3. We expand $f(x)$ in eigenfunctions $\left\{\sin(\frac{(n-\frac{1}{2})\pi}{L}x), n=1,2,\ldots\right\}$.

   Suppose the expansion has the form: $f(x)=\sum_{n=1}^{\infty}c_n\sin(\frac{(n-\frac{1}{2})\pi}{L}x)$.

(a) Multiply both sides by $\sin(\frac{(m-\frac{1}{2})\pi}{L}x)$ and integrate over $[0, L]$ to derive

$$c_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{(n-\frac{1}{2})\pi}{L}x)\,dx$$

(b) Use this formula to expand $f(x) = x/L, x \in [0, L]$.

4. Use separation of variables to solve (IBVP-P4) with $f_2(x) \equiv 1, x \in [0, L]$..

$$\begin{cases} u_t = ku_{xx}, & \text{differential equation (DE)} \\ u(x,0)=f_2(x), & \text{initial condition (IC)} \\ u(0,t)=0, \quad u_x(L,t)=0, & \text{homogeneous BCs (HBCs)} \end{cases} \qquad \text{(IBVP-P4)}$$

5. Solve (IBVP-P5) with $f_3(x) \equiv x/L$, $x \in [0, L]$.

$$\begin{cases} u_t = k u_{xx}\,, & \text{differential equation (DE)} \\ u(x,0) = f_3(x)\,, & \text{initial condition (IC)} \\ u(0,t) = -2\,,\quad u_x(L,t) = 1/L\,, & \text{boundary conditions (BCs)} \end{cases} \qquad \text{(IBVP-P5)}$$

<u>Hint:</u> Convert it to an IBVP with homogeneous BCs.

6. Solve the ODE eigenvalue problem

$$X''(x) = -\lambda X(x)\,,\quad X'(0) = 0\,,\quad X(L) = 0 \qquad \text{(EIG-P6)}$$

7. (a) Show the orthogonality of eigenfunctions $\{X_n(x)\}$ obtained in Problem 6.

   (b) Derive the coefficient formula for the expansion of $f(x)$ in $\{X_n(x)\}$.

   (c) Use the formula to expand $f(x) = x/L$, $x \in [0, L]$ in $\{X_n(x)\}$.

8. Use separation of variables to solve (IBVP-P8) below with $f_4(x) = x/L$, $x \in [0, L]$.

$$\begin{cases} u_t = k u_{xx}\,, & \text{differential equation (DE)} \\ u(x,0) = f_4(x)\,, & \text{initial condition (IC)} \\ u_x(0,t) = 0\,,\quad u(L,t) = 0\,, & \text{homogeneous BCs (HBCs)} \end{cases} \qquad \text{(IBVP-P8)}$$