For the next bit of analysis, we are going to assume that the data comes from a dynamical sytem:

$$\frac{d\bar{u}}{dt} = F(\bar{u})$$

Solving this sytem, leads to

$\bar{u}(t_1), \bar{u}(t_2) \ldots\ldots \bar{u}(t_n)$

We define a data matrix

$$\bar{X} = \begin{bmatrix} | & | & \\ u(t_1) & u(t_2) & \cdots \\ | & | & \end{bmatrix}$$

The singular value decomposition deals with approximating the data matrix $\bar{X}$.

Singular Value Decomposition

We would now exploit notation a little bit

$$\bar{X} = \begin{bmatrix} | & | & & | \\ x_0 & x_1 & \cdots & x_m \\ | & | & & | \end{bmatrix}$$

$$\bar{X} \in \mathbb{C}^{n \times m}$$

m → time measurements

n → Spatial observation

$\mathbb{C}$ → mostly real numbers, but think Schrodinger's equation

Note: More often than not,

$\begin{array}{c} | \\ x_0 \\ | \end{array}$ is a long vector and

$\bar{X}$ is a skinny matrix, since we don't have enough temporal measurements.

So $\bar{X}$, in reality looks like:

$$\bar{X} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Tall and skinny

Page 5

The SVD decomposition is a matrix factorization:

$$X = \underbrace{U}_{n\times n}\ \underbrace{\Sigma}_{n\times m}\ \underbrace{V^*}_{m\times m}$$

singular values

right singular vectors

≥ 0

unitary orthonormal basis (left singular vectors)

these are orthonormal as well

Σ has at most m nonzero elements if $n \geq m$

Note: * Unitary means $A^{-1} = A^*$

* So $AA^* = A^*A = I$

For $n \geq m$, we can re-write the SVD:

$$\bar{X} = n\left[\begin{array}{c|c} \underbrace{\hat{U}}_{m} & \underbrace{\hat{U}^{\perp}}_{n-m} \end{array}\right] \left[\begin{array}{c} m\;\underbrace{\hat{\Sigma}}_{m} \\ \hline 0 \end{array}\right] \left[\begin{array}{c} m\;\underbrace{V^*}_{m} \end{array}\right]$$

$$\bar{X} = \underbrace{\hat{U}}_{n \times m}\,\underbrace{\hat{\Sigma}}_{m \times m}\,\underbrace{V^*}_{m \times m} \longrightarrow \text{memory efficient}$$

The rank of the data matirx X is the number of nonzeroes singular values.

Using the same idea as that of economy SVD, we can now define an optimal r-rank approximation.

$$\tilde{X} = \sum_{k=1}^{r} \sigma_k u_k v_k^*$$

dyadic summation

$$= \underbrace{\sigma_1 u_1 v_1^*}_{\text{rank-1}} + \underbrace{\sigma_2 u_2 v_2^*}_{\text{rank-1}} + \cdots$$

$n \times r$

$$\bar{X} = \left[\; \underbrace{\hat{U}}_{\leftarrow m \rightarrow} \;\middle|\; \underbrace{\hat{U}^{\perp}}_{\leftarrow n-m \rightarrow} \;\right] \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} \left[\; V^{*} \;\right]$$

n m r $\leftarrow r \rightarrow$ $\leftarrow m \rightarrow$ m $\leftarrow m \rightarrow$

$\tilde{X} = (n \times r)$ $(r \times r)$ $(r \times m)$

A-priori error analysis:

$$\| X - \tilde{X} \|_F^{\,2} \sum_{k=r+1}^{m} \sigma_k^2 .$$

Relationship with Eigenvalue decomposition

Consider the correlation matrix: XX^*

$$X = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} \begin{bmatrix} V^* \end{bmatrix}$$

$$X^* = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \hat{\Sigma} & 0 \end{bmatrix} \begin{bmatrix} U^* \end{bmatrix}$$

$$XX^* = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \hat{\Sigma}^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U^* \end{bmatrix}$$

Because $V^*V = I$

So $\begin{bmatrix} \hat{\Sigma}^2 & 0 \\ 0 & 0 \end{bmatrix}$ are the eigenvalues

of XX^*. So, we can say:

eigenvalues of XX^* are the square of the singular values of X

Projection on the singular vectors:

Let's go back to the data matrix, X

$$X = n\updownarrow \underbrace{\begin{pmatrix} | & | & & | \\ x_0 & x_1 & \cdots & x_m \\ | & | & & | \end{pmatrix}}_{\longleftarrow\, m \,\longrightarrow}$$

Projected data matrix $\hat{X} = U^T_{r \times n} X$

Here, we take the first 'r' columns of the left singular vector and project the data on these vectors.

So $\hat{X}$ is of size $r \times m$

So, we have reduced the spatial dimension of the system from "n" to "r"