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ABC flows then and now

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We review cellular space-periodic dynamos without scale separation, starting with early work in the 1980s on *ABC* flows with prescribed steady velocity fields $\mathbf{u} = (A\sin z + C\cos y, B\sin x + A\cos z, C\sin y + B\cos x)$. These led naturally to work done in the 1990s together with Mike Proctor on two-dimensional time dependent versions which gave strong numerical evidence for the existence of fast dynamos growing on the flow turnover timescale. Similar calculations were subsequently perfomed for a spherical shell geometry jointly with with Rainer Hollerbach. Also in the 1990s other studies began to take into account the back reaction of the Lorentz force when the flow rather than being prescribed was instead allowed to evolve in response to a forcing of the above *ABC* form. The dynamos that resulted were mostly filamentary and showed a disconcerting tendency to equilibrate with total magnetic energy much less than total kinetic energy in the low diffusivity limit relevant for astrophysics. The remarkable discovery by Archontis in 1999 of a non-filamentary dynamo with almost equal magnetic and kinetic energies showed that the unfavourable scalings for the filamentary case can be overcome; this dynamo used an *ABC* forcing with the cosines left out. Since then several authors have been struggling with partial success to understand just how this state of affairs comes about. Most recently efforts have been made to produce other examples of this type of dynamo, to investigate why the Archontis case is robust over a wide range of magnetic Prandtl numbers ν/η, and above all to understand its remarkable stability at very low diffusivities when non-magnetic flows are almost always unstable.

1 Introduction: Fast dynamos and the *ABC* flows

The fast dynamo story began with a paper by Vainshtein and Zeldovich (1972). They made the point that for the kinematic dynamo problem where the velocity field $\mathbf{u}$ is prescribed in the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \ ,$$

with $\nabla \cdot \mathbf{B} = 0$, the problem for the magnetic field $\mathbf{B}$ is linear with two natural time scales, the turnover time and the electromagnetic diffusion time. Solutions either grow or decay exponentially unless one is exactly at the transition where the growth rate has real part zero. For a dynamo, we need at least one growing solution. If this solution grows on the diffusive timescale, and if one uses laminar values for the magnetic diffusivities according to Spitzer's well-known formula or some equivalent, one obtains a growth time which is longer than the age of the Universe for most astrophysical objects larger than a small star. It is therefore apparently necessary to find dynamos which grow on the turnover time scale rather than the diffusive one. In 1972 none of these so-called "fast dynamos" had been found; the only known dynamos were "slow", and operated on the diffusive time scale. This caused the Russian school of dynamo theorists to mount a campaign to find fast ones.

For Zeldovich, the rope dynamo was a self-evident example showing that fast dynamos were possible. Imagine a thin toroidally directed loop of magnetic field embedded in a highly conducting fluid. Stretch it out so that it becomes twice as long; assuming flux freezing this doubles the field strength. Then twist the loop half a turn to get a figure of eight. Finally fold one half of this back on the other to get a loop with the same size as the original but twice as strong a field. Each time this cycle is repeated there is a further doubling; if we define the time unit to be the cycle period, the growth rate is $\log 2$. What could be simpler than this stretch-twist-fold example?

The above argument completely ignores diffusion; the issue remains whether the same result holds in the presence of an arbitrarily small but non-zero diffusivity. For true dynamo action (as opposed to temporary

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amplification of the field), the relevant limiting process is to take $t \to \infty$ *before* letting the magnetic diffusivity $\eta \to 0$, rather than the other way round. With no diffusion, the induction equation is formally the same as the equation describing the action of the flow on an oriented line element, leading to the assertion that if the line element grows exponentially on the flow turnover timescale, then so will the magnetic field—both have the same Cauchy solution. Exponential growth of line elements typically means the flow is chaotic, and we thus anticipate that chaotic flows will be good candidates for fast dynamo action. See Galloway (2003) for a fuller discussion of this story; the definitive paper that settled the issue is due to Klapper and Young (1995), who established the need for the topological entropy of the flow to be positive in order that fast dynamo action is possible. A fascinating examination of a real flow akin to the rope dynamo is found in Moffatt and Proctor (1985), where it is shown that complexities set in after two or three turnovers in the neighbourhood of the crossover point of the fold/twist, leading to extreme uncertainty in what will occur as $t \to \infty$. The paper also gives interesting spectral results relating to the intermittency of any fast dynamo.

Chaotic flows in real fluids are mathematically very difficult to come to grips with; progress in dealing with both toy models such the Baker's Map and more realistic flows in the dynamo context is described fully in the book by Childress and Gilbert (1995). Fully developed turbulent flows in particular cannot be described by a formula which can be used as the velocity field in a dynamo calculation; for such flows the practice has been to resort to turbulent closures, leading to dynamos based on mean-field electrodynamics (Krause and Rädler 1980). However, Arnold (1965) and Hénon (1966) found a family of periodic 3-D flows with chaotic streamlines that had the simple form

$$\mathbf{u} = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)\,.$$

The case where one of A, B or C (say B) is zero is non-chaotic (ie integrable), and was used for studying dynamo action by Roberts (1970), who obtained slow dynamos; these were further investigated by Soward (1987), who found that if one allowed the $B = 0$ dynamo to adopt its optimal wavenumber in the x-direction, the growth rate scaled as $\log\log R_m/\log R_m$, still slow, yet tantalisingly close to being fast. (Here R_m is a non-dimensional measure of the conductivity.) When $ABC \neq 0$ the flow domain contains chaotic regions whose relative size depends on the precise values of the ratios $A : B : C$; furthermore, if A^2, B^2 and C^2 can form a triangle each periodic cube contains 8 stagnation points, otherwise there are none. The chaotic regions are manifest in Poincaré sections of the particle paths; Dombre *et al.* (1986) provide many examples of these, and give an extensive analysis of the dynamical systems aspects of the flow family.

Because they are generically chaotic yet have a formula that is simply evaluated, it is straightforward

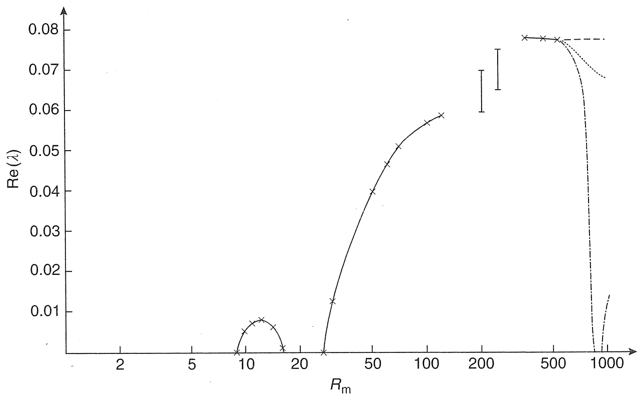

Figure 1. Growth rate real part for the kinematic $A = B = C = 1$ dynamo. Note the two windows of dynamo action, which have different symmetries. The low R_m window was discovered by Arnold and Korkina and is oscillatory; above $R_m = 18$ this eigenfunction decays. The higher R_m window is oscillatory with a longer period at onset; by $R_m = 300$ the period is so long as to be imperceptible. Beyond $R_m = 500$ it is not clear what will happen; some possibilities are sketched.

to use these flows for finding numerical solutions of the induction equation. This was first done by Arnold and Korkina (1983), who used an inverse iteration method to find the eigenvalue λ of an assumed $e^{\lambda t}$ time dependence for the magnetic field. They found dynamo action, but were limited to magnetic Reynolds numbers below around 15 because of lack of computer power. The $A = B = C = 1$ case was chosen for study, largely owing to the richness of its symmetry structure. Galloway and Frisch (1986) extended these calculations using a timestepping method and a much larger computer, and were able to reach R_m values up to 400. The results for the real part of the growth rate of the fastest-growing mode are shown as a function of R_m in Figure 1; the Arnold and Korkina mode appears between around 8 and 18, but thereafter decays. Another growing mode with different symmetries appears around $R_m = 27$ and persists up to $R_m = 400$; somewhere in between, the complex part of the growth rate becomes zero. For fast dynamo action we expect to see the growth rate asymptoting to a constant positive value as $R_m \to \infty$, yet there is little sign of such a state of affairs—the whole picture is suggestive but inconclusive. The eigenfunction in the second dynamo window is shown in Figure 2; one sees the field is organised into filaments centred on stagnation points. Field is swept in by the flow on the 2-D stable manifolds and stretched out along the 1-D unstable ones. Four of the stagnation points have this vacuum-cleaner like quality and four behave in the opposite way, attracting field in one dimension and repelling it in two. Analysis in the neighbourhood of the former type of stagnation point shows that the fields have a cigar-like shape with characteristic sizes $R_m^{-1/2}$ in two directions and order 1 in the third (Galloway and Zeligovsky 1994). This filamentary structure is typical of many dynamos and is consistent with the spectral results of Moffatt and Proctor (1985). We will return to this issue later.

The (1,1,1) case was also studied by Gilbert (1991), for the case where the diffusivity is formally set to zero, corresponding to infinite R_m. The field can be calculated from the Cauchy solution to the induction equation, using push-pull maps. There are marked differences between this case and the large but finite R_m results of the numerical calculations with diffusion, highlighting the difficulties with the ordering of the limits referred to earlier. Childress and Gilbert (1995) give an extensive discussion of this issue.

It turns out that from the point of view of dynamo action the choice $A = B = C = 1$ is quite a bad one; Arnold was seduced by the richness of the additional symmetries, but too much symmetry can actually be a bad thing , as shown by Cowling's theorem which precludes axisymmetric steady dynamos. In addition, Poincaré sections for the (1,1,1) case show quite small chaotic regions, so that the relative space available for exponential stretching is very limited. In a search for better examples, Galloway and O'Brian (2003) experimented with other values and found that $A : B : C = 5 : 2 : 2$ was much more convincing as a fast

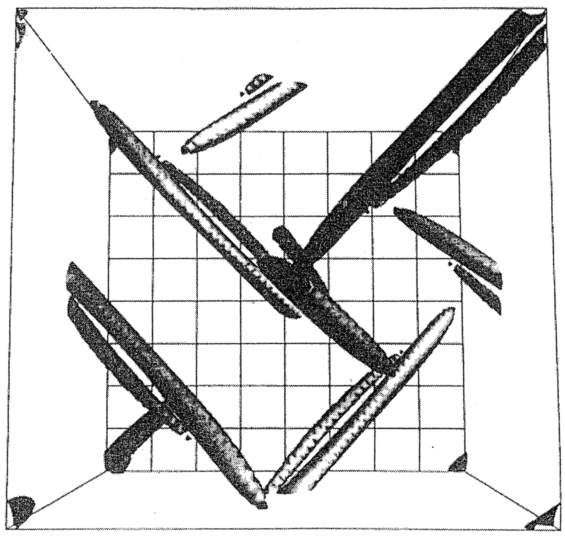

Figure 2. Nature of the eigenfunction for the kinematic $A = B = C = 1$ dynamo in the high R_m (= 100) window. Shown is an isosurface with $|\mathbf{B}|^2 = 0.17B_{max}^2$. The two cigars visible in the top right hand corner are centred on the stagnation point at $7\pi/4, 7\pi/4, 7\pi/4,$

dynamo candidate, with the growth rate of around 0.208 hardly changing as R_m was increased from 200 to 800. This case has no stagnation points and much larger chaotic regions, with magnetic eigenfunctions which are sheet-like rather than filamentary.

Other calculations of ABC dynamos, some including a degree of scale separation, were computed by Lau and Finn (1993), Dorch (2000), and Archontis *et al.* (2003). The most recent work will be summarised in the closing section.

2 Time-dependent kinematic flows

The computational expense of even kinematic fully three-dimensional dynamo calculations restricted investigations to R_m values too low to give compelling evidence for the existence of fast dynamos. To address this, Galloway and Proctor (1992) looked again at the two-dimensional case studied by Roberts (1970) and Soward (1987). These dynamos lack chaos and are inevitably slow, but if a time-dependent velocity field is used, chaos can be restored. We take an ABC flow with (say) $B = 0$, so that the flow has no x-dependence. As G.O. Roberts observed, magnetic modes with a dependence of the assumed form e^{ikx} evolve independently from one another for each prescribed k. One can thus pick a k and evolve it with a 2-D code; the overhead in adding a time-dependence to the velocity field is minimal. Different choices of k can be used to bracket the value which has the fastest growth rate, to any desired degree of accuracy.

We took two different forms for the time dependence, modulating the phase of the A-terms and the C-terms either in phase with one another (LP flow), or $\pi/2$ out of phase (CP flow). Whilst both are probably fast dynamos, the numerical evidence from the CP case is much more convincing. The latter has velocity field

$$\mathbf{u} = \left(A\sin(z+\sin t) + C\cos(y+\cos t), A\cos(z+\sin t), C\sin(y+\cos t)\right),$$

and the effect of the phase modifying terms is to shake the stagnation points around in circles. For large R_m the optimum k and the growth rate settle down to 0.57 and 0.297 respectively. Stroboscopic plots of the particle paths with a 2π time interval show that the flow has large chaotic regions; this is shown

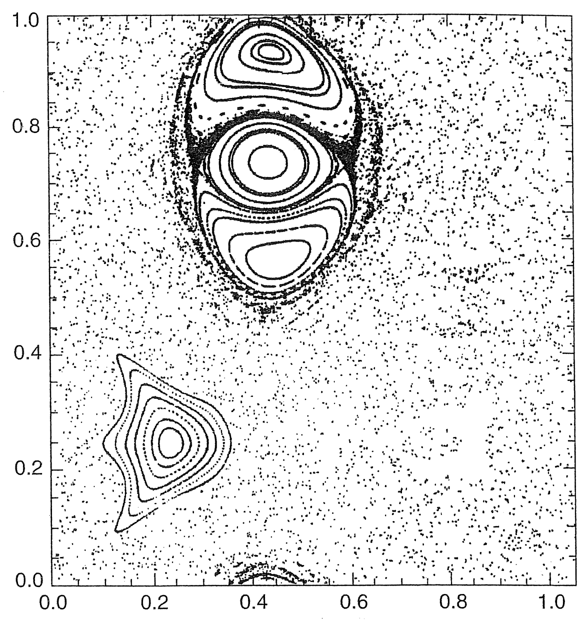

Figure 3. Poincaré section for the CP flow in the (y, z) plane: the x-component of $\mathbf{u}$ has been ignored. A dot is drawn every time a particle trajectory has advanced another 2π in time, with (y, z) values taken modulo 2π in space. Several initial conditions have been followed forward in order to show both the chaotic regions and the KAM tori. From Galloway (2003)

in Figure 3, which includes both regular and chaotic orbits (the former of which were absent from the Conference T-shirt!) The growth rate as a function of R_m is shown in Figure 4, along with that of the LP flow where the $\cos t$ terms are replaced by $\sin t$, and with that of the 3-D steady flow $\mathbf{u} = (\sin z, \sin x, \sin y)$ which is also a probable fast dynamo and which has an interesting life of its own, to be discussed in the next section. Note the much higher R_m values attainable for the 2-D flows; since our paper several authors have used more powerful computers to reach still higher R_m and have confirmed the apparent fastness of the CP dynamo. A related way of introducing chaos via time-dependence is to take $A = A_0 \cos^2 t$, $C = C_0 \sin^2 t$, ie amplitude rather than phase modulation. This was done by Otani (1993); the results are very similar and also give rise to a dynamo which is apparently fast (an abstract of this work appeared as early as 1989 and is referred to in the paper).

The fastest growing eigenfunction for the high R_m CP dynamo is given in Figure 5. This shows a series of extended field striations "queuing up" to approach the moving stagnation points along their stable manifolds, and makes clear that the mechanism producing the chaos and stretching the magnetic field is heteroclinic tangling. Successive stitches of field are subject to increasingly extreme excursions alternating first to one side and then to the other as the stagnation point is approached, like a mad version of the zig-zag setting on a sewing machine. Ponty *et al.* (1995) used Melnikov's method to analyse this mechanism for the case where a small parameter ϵ was placed before the $\cos t$ and $\sin t$. Several authors have used this dynamo as a basis for other calculations; Mike Proctor and I unwittingly started a minor industry!

The artificiality of the periodic geometry spurred us to seek examples in spherical shell geometry, in collaboration with Rainer Hollerbach. The same trick of separating out one dimension works here; if we solve the kinematic induction equation using an axisymmetric flow, assuming a dependence $e^{im\phi}$ in longitude for the magnetic field, each integer m value evolves independently. A two-dimensional time-stepping method can thus be used for the r, θ dependence for each specified m (in practice we only used $m = 1$). Further simplifications ensue if just one Legendre polynomial $P_n(\cos\theta)$ is used in the specification of the velocity field, because the evolution of magnetic mode n depends only on modes $n-1, n$ and $n+1$. Even then it was very computationally expensive to run at high enough R_m; details of solutions can be found in Hollerbach *et al.* (1995) and Hollerbach *et al.* (1998). The specific flow used in the former paper was

$$\mathbf{u} = \nabla \times [r^{-1} f(r,t) \sin\theta \cos\theta \mathbf{e}_\phi] + r\omega(r,t) \sin\theta \mathbf{e}_\phi\,,$$

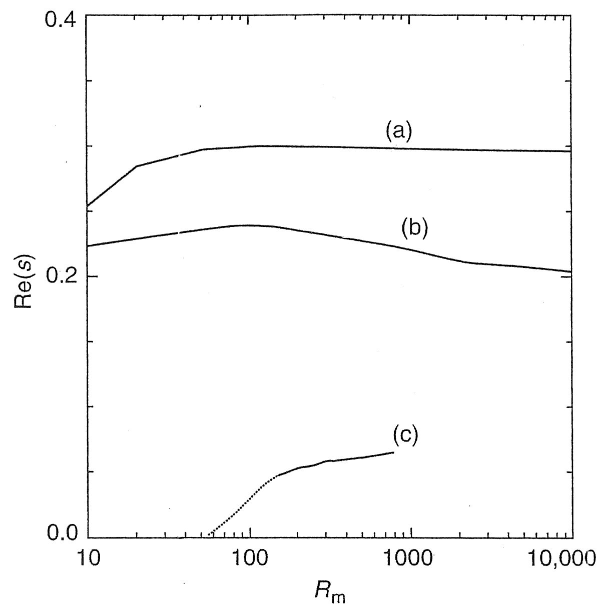

Figure 4. Real part of the growth rate as a function of R_m for the CP flow with $k = 0.57$ (a). The growthrate of the LP flow is shown in (b), and (c) gives the growthrate for the "sines" flow $\mathbf{u} = (\sin z, \sin x, \sin y)$. From Galloway and Proctor (1992)

where $f(r,t) = 0.5(r-0.5)(r-1.5)\sin[4\pi r + \sin(0.25\pi t)]$ describes the meridional motion and $\omega(r,t) = 2\sin[4\pi r + \sin(0.25\pi t)]$ the rotation. This motion is confined to the spherical shell $0.5 \le r \le 1.5$, and has time period 8.

A plot of the largest growth rate versus R_m (omitted here) shows less conclusive evidence for fastness than the corresponding plot for the CP flow; there is a gradual increase from around 0.1 to 0.2 as R_m increases from 1000 to 500,000, but even at such a huge R_m asymptopia[1] has not been convincingly attained. The associated eigenfunction is shown in Figure 6, for two values of R_m. One sees evidence of the heteroclinic tangling mechanism, but the magnetic structures are cramped and small in relative volume, especially at higher R_m.

Time-dependent *ABC* flows were revisited in an interesting paper by Brummell *et al.* (2001). These authors applied the time-dependent phase shift idea to the fully three-dimensional flow family, again dealing just with the (1,1,1) case. Since these calculations were nonlinear, they will be described in the next section, but their initial phases are kinematic and give dynamo action with a measurable growth rate suggestive of fast dynamo action.

3 Including the dynamics

With the existence of fast dynamos established, attention turned to the effects of nonlinearity and the back reaction of the Lorentz force $\nabla \times \mathbf{B}/\mu_0$ on the velocity. The game now changes from the prescription of a velocity field to the provision of a force, possibly intended to produce a certain velocity field in the absence of a magnetic field although we shall see things are not that simple in practice. The first numerical treatment of the *ABC* problem was due to Galanti *et al.* (1992), who used pseudospectral methods to solve the full incompressible MHD problem with equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

[1]an expression due to Mike Proctor

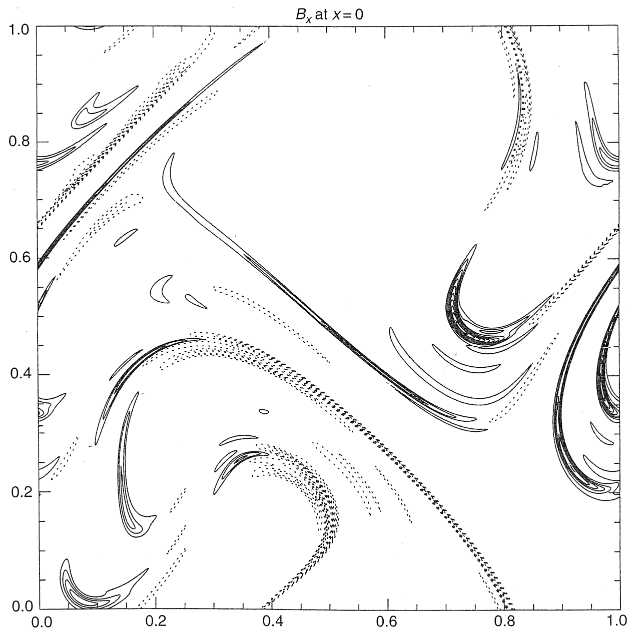

Figure 5. Structure of the fastest-growing eigenfunction for $R_m = 2000$ for the CP flow with $k = 0.57$. Shown are the contours at one instant of the x-component of the magnetic field at $x = 0$. (From Galloway and Proctor (1992)

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla(p/\rho + \frac{1}{2}u^2) + \mathbf{j} \times \mathbf{B} + \mathbf{F} + \nu \nabla^2 \mathbf{u} \, ,$$

$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0$, $\mu_o \mathbf{j} = \nabla \times \mathbf{B}$, and a force

$$\mathbf{F} = \nu(A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x) \ .$$

With no magnetic field, there is a solution to the momentum equation with a velocity identical in form to the force—the viscous and forcing terms balance exactly because the *ABC* flows are Beltrami flows with $\mathbf{u} \times (\nabla \times \mathbf{u}) \equiv 0$. However, when the viscosity is small enough, corresponding to a Reynolds number of greater than 13.5 in the (1,1,1) case, the flow is unstable (Galloway and Frisch 1987) and evolves into something else. The subsequent evolution is time-dependent in a very interesting way, and was investigated by Podvigina and Pouquet (1994), but from the dynamo point of view it is frustrating because the flow changes from the one whose non-linear magnetic equilibration was being sought—effectively the time-dependent switching of the velocity field hijacks the problem. The only way to avoid this is to compute at kinetic Reynolds numbers Re of less than 13.5, and Figures 7 and 8 show results for $Re = 5, R_m = 400$. There is evolution to a statistically steady but time-dependent state where the total energy swaps between magnetic field and velocity. The former is reminiscent of the kinematic case in that it has a filamentary

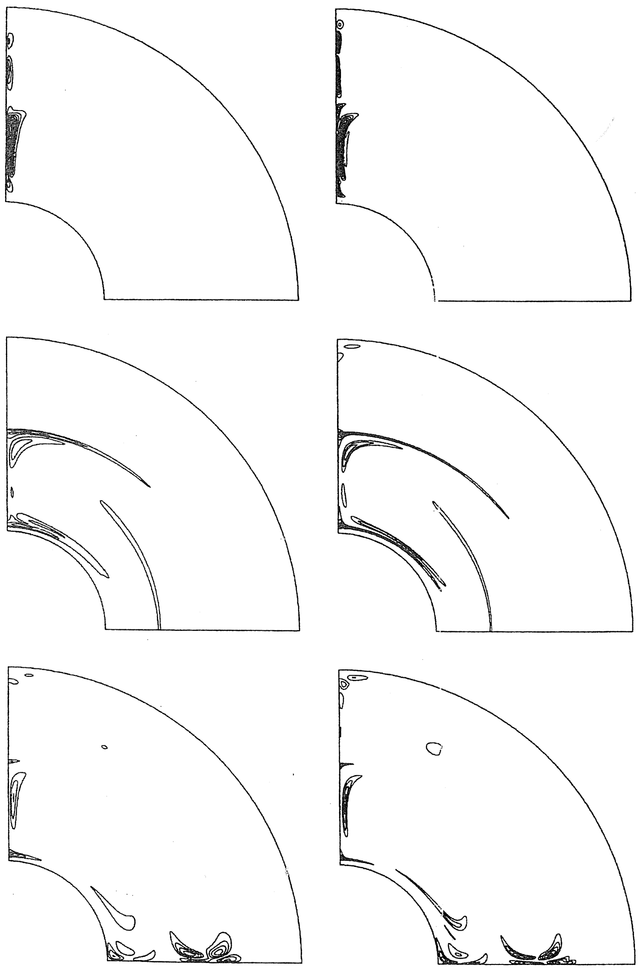

Figure 6. Contours of field components B_r (top), B_θ (centre), and B_ϕ (bottom), for the $m = 1$ fast dynamo candidate. Left: $R_m = 10000$, right $R_m = 40000$. (From Hollerbach *et al.* (1995)

structure and is concentrated near the stagnation points.

The comparable magnitudes of the total magnetic and kinetic energies imply a very effective dynamo, and one can ask if this persists when both Reynolds numbers are large, as is the case in astrophysics. There has been controversy over whether strong-field dynamos can exist in this limit; such dynamos are apparently required to explain the fields observed in many stars and galaxies, and doubts have been cast over whether traditional mean-field dynamos can achieve this (see e.g. Vainshtein and Cattaneo (1992); the debate has continued and the issue is still largely unresolved). For the 1:1:1 ABC case, if the field is filamentary scaling arguments based on dissipation rates can be given to show that the ratio of total magnetic to kinetic energies tends to zero as the diffusivities are reduced (Galloway 2003). This leads to the question of whether non-filamentary dynamos with strong fields are actually possible, given the key role played by stagnation points as natural places for the field to accumulate.

Before answering this affirmatively, we return to the calculations of Brummell *et al.* (2001). These authors used an ABC forcing as for the standard case studied by Galanti *et al.* (1992), but with all variables x, y, z replaced by $x + \epsilon \sin \Omega t$ etc. This is designed to give the 3-D analog of the LP flow of Galloway and Proctor (1992). However, they fix the force but not the velocity, so they are also plagued by the fact that the velocity changes to something other than the LP-type flow on a dynamical time scale. Thus what they report as LP-type kinematic dynamos are indeed kinematic, but they are due to a more complicated and elusive flow. Results were provided for various values of ϵ between 0 and 1; increasing ϵ corresponds to larger chaotic regions, as illustrated by their Poincaré sections, which however are calculated for the 3-D LP forcing function, rather than the velocity field which evolves in the calculation. This is an important but frustrating point, and it is not clear how it affects their kinematic conclusions. It would be interesting to repeat the solutions of just the induction equation with the velocity frozen into the 3-D LP form.

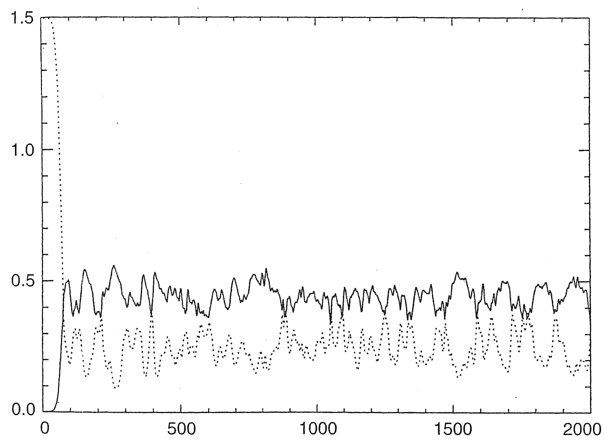

Figure 7. The ABC dynamo with (1,1,1) forcing, for $\nu = 1/5, \eta = 1/400$. The solid line represents the total magnetic energy, the dashed the total kinetic energy. Time is given in units of the flow turnover timescale.

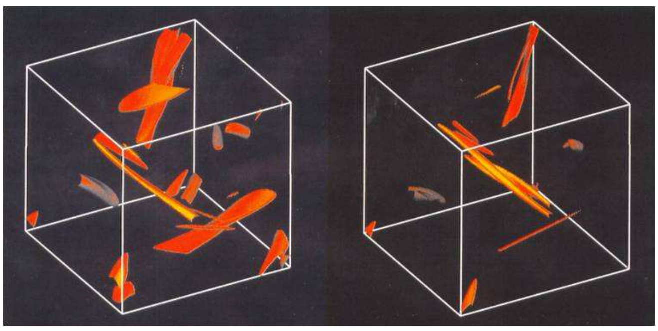

Figure 8. The ABC dynamo with (1,1,1) forcing, for $\nu = 1/5, \eta = 1/400$. Shown are isosurfaces where the magnetic energy attains 20% of its maximum value at that instant, for two separate times approximately 40 turnover units apart. Note the filamentary structure.

Once the field becomes strong enough to be dynamically important, further evolution takes place and the velocity and magnetic fields reorganise into a configuration where the field decays away, leaving behind a non-dynamo flow with a complicated time dependence. This flow is still being sustained against viscous dissipation by the 3-D LP forcing, but its structural correlation with the latter is not at all apparent. The force has to drive a flow; this starts off as a kinematic dynamo but when the magnetic field becomes a nuisance dynamically, the flow abandons its dynamo aspirations and metamorphoses into a different form. Perhaps if the calculations had been run much longer, more episodes of growth and subsequent decay of magnetic field might have occurred—but the computations were much too expensive to permit this. This work is intriguing but raises as many questions as it answers.

Are non-filamentary dynamos at all possible for small magnetic and viscous diffusivities? Until 2000 numerical evidence suggested not, and as discussed earlier, this poses difficulties for the generation of astrophysically significant fields. In that year, Archontis (2000) reported, as a part of his PhD thesis, on a study with an *ABC*-type forcing with no cosines. Such a velocity field was known to be a good fast dynamo candidate (Galloway and Proctor 1992), but these numerical calculations treated the dynamics too. With a driving force $\mathbf{F} = \nu(\sin z, \sin x, \sin y)$, the solution to the momentum equation with no $\mathbf{B}$ is in fact *not* $\mathbf{u} = (\sin z, \sin x, \sin y)$, because unlike an *ABC* flow the nonlinear term is not zero and must be balanced somehow. The solution must evolve into something else.

However, if a seed magnetic field is added, satisfying $\nabla \cdot \mathbf{B} = 0$, something quite remarkable happens—amazing enough to warrant more calculations to confirm and clarify what is going on (Cameron and Galloway 2006a). The "something else" is a kinematic dynamo, and generates field which reaches a dynamically significant level. Figure 9 shows the evolution of the total kinetic and magnetic energies for diffusivities $\nu = \eta = 1/100$. The solution reaches a plateau representing an apparent fluctuating but statistically steady state, with the total energies differing by a factor of order 1.5, and a magnetic field which is presumably filamentary. After a long time in this state there is an unexpected jump to another configuration with total energies which are almost equal. In fact the velocity and magnetic fields are *locally* almost exactly equal (or opposite, there is a symmetry) when expressed in Alfvénic units, and become more and more so as the diffusivities tend to zero. In this limit both are close, but not asymptotically equal, to $0.5(\sin z, \sin x, \sin y)$—there is a residual part of the solution in other Fourier modes, at the level of a few percent. The switch to the Alfvénic state is less unexpected when the evolution of the normalised cross-helicity is included on the plot; one sees that during the preceding fluctuating state, the magnetic field and the velocity have been slowly aligning themselves on the diffusive timescale. This will be revisited later.

There are two minor differences in the way the problems of Archontis (2000) and Cameron and Galloway (2006a) were specified. The latter took an incompressible fluid and a forcing which was constant in time. The former took a fully compressible fluid and modulated the forcing with a time-dependent feedback

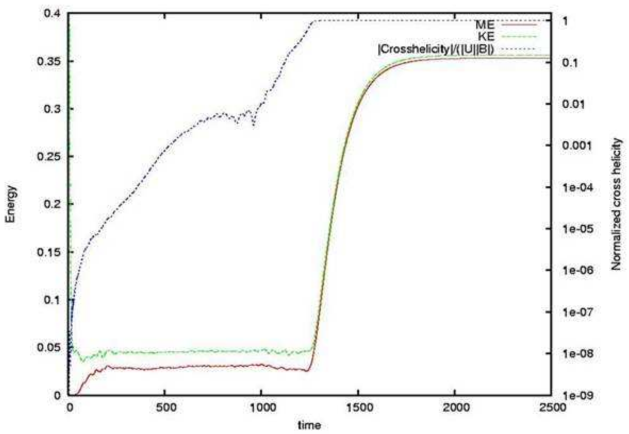

Figure 9. The evolution of total magnetic (lower curve) and kinetic (middle curve) energies for the Archontis dynamo described in the text, over a time long compared with both turnover and diffusion timescales. Also shown in the upper curve is the normalised cross-helicity, better referred to as the magnetic alignment. From Cameron and Galloway (2006a)

factor designed to maintain the energies at a constant level. The first issue seems to make little difference, but the second has the effect that the feedback factor can occasionally knock the solution off a branch which is otherwise steady. This is apparently caused because the modulation uses the flow turnover timescale, which is the same as the Alfvénic timescale in the $\mathbf{u} = \pm\mathbf{B}$ state. This can lead to resonance effects; perversely the attempt to drive the evolution to a steady state can actually have the opposite outcome. (In unpublished work we have reproduced this with our incompressible code.) There is a more recent description of the compressible calculations in Archontis *et al.* (2007).

The incompressible constant-force calculations display solutions which have $\mathbf{u}$ and $\mathbf{B}$ almost exactly aligned or anti-aligned with one another, and the degree of alignment tends to zero with the diffusivities. The amazing regularity of the solution is shown in Figure 10, which gives the web of connections between various stable and unstable manifolds of selected stagnation points. Most of our calculations were performed with $\nu = \eta$, and we verified that the solution is stable down to diffusivities of 1/2000. Since then, Gilbert *et al.* (2011) have shown this persists to 1/10000. Given that non-magnetic flows are almost invariably unstable and turbulent at such high Reynolds numbers, this is remarkable and immediately prompts the question whether such solutions are common in MHD or whether we happen to have stumbled on a freak.

4 Discussion and conclusion

There are now a number of computed dynamos which are apparently fast, and the question of existence can be regarded as established. Whether it was ever the issue that was claimed is debatable, and depends on how seriously one is prepared to take growth times based on laminar diffusivities. The length of the latter were the driving force behind much of the early research, and it is interesting to compare Zeldovich's insistence on this point with his readiness to accept the flux-expelling role of turbulent diffusion in other contexts—the two points of view are fundamentally inconsistent. If one takes the Schatzmannian point of view that all astrophysical flows are at such high Reynolds numbers that they are certain to be unstable, and that instabilities beget further instabilities a la Kolmogorov, one can argue that the effective diffusivities are much greater than the laminar values, which are therefore irrelevant. The whole of mean-field theory is built on parametrisations of this viewpoint.

That there is still life in the kinematic ABC fast dynamo problem has recently been shown by Alexakis (2011), who conducted a detailed study over the full range of possible values of $A : B : C$, with R_m based on the normalisation $A^2 + B^2 + C^2 = 3$. The results are broadly organised into three classes: dynamos similar to the 1:1:1 case, dynamos similar to the integrable 2.5-dimensional case, and dynamos similar to the 5:2:2 case. The 5:2:2 case is confirmed to be the "fastest" dynamo, and results for the 1:1:1 case are

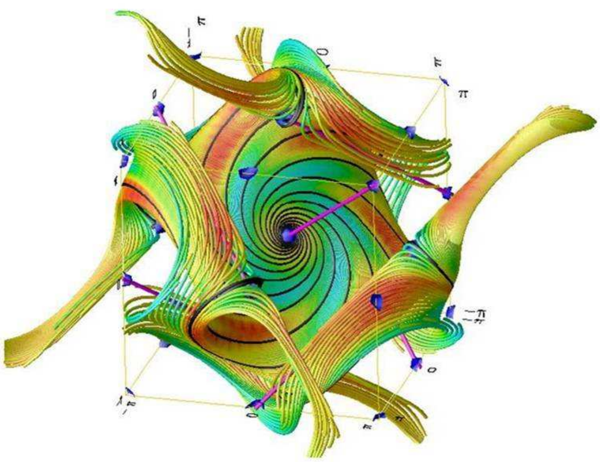

Figure 10. The strucure of the velocity and magnetic fields for the steady state of the Archontis dynamo; the two fields are indistinguishable in this picture calculated for diffusivities $\nu = \eta = 1/100$. Plotted are selected trajectories forming heteroclinic orbits that connect selected stagnation points. From Cameron and Galloway (2006a).

presented shedding new light on the high R_m story corresponding to our Figure 1.

When dynamics are included, there are two issues: the existence of solutions and their stability. In fact, if one is not fussy about how unrealistic a force is to be employed, there is a simple recipe whereby dynamo solutions can be constructed to order (Cameron and Galloway 2006a). Take any neutrally stable solution $\mathbf{B_0}$ to the kinematic problem with prescribed velocity $\mathbf{u_0}$ and magnetic diffusivity η_0. We can now generate an equilibrium solution to the full dynamo problem (including the momentum equation) for $\eta = \epsilon\eta_0$. This is $\mathbf{u} = \epsilon\mathbf{u}_0 + \mathbf{B}_0$, with $\mathbf{B} = \mathbf{B}_0$, and $\mathbf{F} = \epsilon^2(\mathbf{u}_0 \cdot \nabla\mathbf{u}_0) - \epsilon^2\nu\nabla^2\mathbf{u}_0 + 2\epsilon(\mathbf{B}_0 \cdot \nabla\mathbf{u_0})$. This is for $\nu = \eta$, but more general results are given in Cameron and Galloway (2006a), along with associated scaling transformations. Note that as $\epsilon \to 0$, we end up with an Alfvénic solution.

Variations on this theme were followed up in Cameron and Galloway (2006b), particularly for the Gibson (1968) three-sphere rotor dynamo which is an example that does not use a periodic geometry. In addition we looked there at a family of forcings

$$\mathbf{F} = \nu(\sin z + \epsilon\cos y, \sin x + \epsilon\cos z, \sin y + \epsilon\cos x)$$

where $\epsilon = 0$ gives the Archontis dynamo and $\epsilon = 1$ the (1,1,1) *ABC* dynamo. This forcing can also be modified according to the above recipe in order to search for steady solutions, enabling the study of the transition from non-filamentary to filamentary cases, as well as a few other issues not dealt with here. The resulting strong-field *ABC* dynamos are much less filamentary than those that used the standard forcing, but require more investigation to understand the extent to which they are stable.

Any pair $\mathbf{u} = \pm\mathbf{B}$ is a solution to the steady MHD equations with both diffusivities set to zero. The stability of the Archontis dynamo is less surprising in the light of a remarkable result due to Friedlander and Vishik (1995), who showed that *all* these solutions are neutrally stable. With small diffusivities, it is likely that some of these solutions are attracted to a nearby stable state. In a particular case such as the sines flow with $\mathbf{u} = \pm\mathbf{B} = 0.5(\sin z, \sin x, \sin y)$, the problem is to determine this nearby state. So far this has only been done numerically, and the reason why there are extra terms present at the few percent level must be to satisfy some kind of solvability condition. Both Cameron and Galloway (2006a) and Gilbert *et al.* (2011) have made unsuccessful attempts to understand how they come about, and to prove stability, but only heuristic arguments have so far been advoked. Both papers treat the importance of what is happening near the 1-D unstable manifolds connecting neighbouring stagnation points; the second represents the current state of our knowledge concerning this dynamo.

Some unfinished work remains in the pipeline. One challenge is to understand why the Archontis dynamo works well over a wide range of values of ν/η, the magnetic Prandtl number; Cameron and Galloway (2006a) confirm that values of 4 and 0.25 still give a $\mathbf{u} = \pm\mathbf{B}$ solution. By adding the momentum equation dotted with $\mathbf{B}$ to the induction equation equation dotted with $\mathbf{u}$ and converting various terms into surface integrals which vanish because of the periodicity, one can derive an evolution equation for the total cross-helicity:

$$\frac{d}{dt}\int_V \mathbf{u}\cdot\mathbf{B}\,dV = \int_V \mathbf{F}\cdot\mathbf{B}\,dV + \nu\int_V \mathbf{B}\cdot\nabla^2\mathbf{u}\,dV + \eta\int_V \mathbf{u}\cdot\nabla^2\mathbf{B}\,dV\ .$$

Given that the forces used are proportional to a diffusivity, this shows that the total cross-helicity can only change on a slow timescale, as borne out in Figure 9. This means the Archontis dynamo is slow! (It does not preclude patches of opposite alignment which evolve faster but cancel one another.) The equation also shows that it is only the $(\sin z, \sin x, \sin y)$ term in the Fourier series for $\mathbf{B}$ which is doing the alignment. By deriving in addition a total energy equation and assuming that $\mathbf{u}$ and $\mathbf{B}$ are both proportional to $(\sin z, \sin x, \sin y)$ (which is close but not exactly true), one can show that they must in fact be equal in magnitude (Galloway and Jones, in preparation). This is heuristic in the same sense as the stability arguments mentioned earlier. Also, for a forcing proportional to a fixed ν, the solution cannot remain Alfvénic once η is large enough for the value of R_m to fall below that necessary to satisfy the Backus-Proctor bound on dynamo action (Proctor 1977). David Lewis (private communication) investigated this by taking $\nu = \eta = 1/100$ and then increasing η; when η is around $1/13$, the dynamo turns off and the flow evolves into a turbulent time-dependent state.

All the dynamos that have been discussed here are examples designed to guide us in understanding fundamental dynamo processes. Often they are described by the deprecating expression "toy models". The hope is that the processes studied are actually involved in generating astrophysical magnetic fields. They can also be incorporated into direct numerical simulations of specific objects. In particular, the non-filamentary dynamos we have examined here are fascinating objects, and the obvious question is whether they are isolated freaks or whether such behaviour (with $\mathbf{u} \simeq \mathbf{B}$) can be expected to occur as a typical feature of an astrophysical dynamo, at least over some fraction of the flow domain. Otherwise we are stuck for better or worse with filamentary dynamos, with all their attendant difficulties referred to earlier.

One natural next step is to examine the numerical calculations done to date and see whether there is any evidence of alignment processes at work. The best evidence so far is found in the ASH-code simulations of faster-rotating Suns due to Brown *et al.* (2010); the solutions are referred to as wreath-building dynamos, and are clearly strongly influenced by differential rotation. There are hints in other calculations, but nothing as convincing as what is seen in the Archontis dynamo or similar cases. One important point about these is that they have boundary conditions which are compatible with an aligned outcome; the Archontis dynamo for instance is periodic. Any real astrophysical object will have boundary conditions which are not compatible, and it is possible that this can wreck any alignment mechanism not just near the boundaries, but more globally as well. In the Sun, the non-compatible boundary conditions are enough to ensure that the magnetic field seen at the photosphere is highly intermittent. The question is whether this persists throughout the Sun or whether somewhere, deeper down, there may be a region where alignment mechanisms are significant for field generation. The tachocline is the obvious place to consider, and in a paper submitted to Monthly Notices of the Royal Astronomical Society I have devised a model based on the Gailitis (1970) dynamo which is located primarily in this region and works by having the same magnitudes (in Alfvénic units) for the differential rotation and toroidal fields. In order that the magnetic field changes sign every 11 years whilst the differential rotation does not, this needs to have shocks between $\mathbf{u} = \mathbf{B}$ and $\mathbf{u} = -\mathbf{B}$ regions. The radical feature of this dynamo is that the surface manifestations of the solar cycle are entirely the waste product of a permanently circulating MHD configuration in the tachocline, which presents a bottom boundary condition to the convection zone changing in sign every 11 years.

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