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Stability of stagnation points in rotating flows

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The Lifschitz and Hameiri theory for short-wave instabilities is used to show that any steady inviscid plane flow subjected (or not) to a Coriolis with perpendicular angular velocity vector is unstable to three-dimensional perturbations if $\Phi(\boldsymbol{x}_0)<0$ on a stagnation point located at $\boldsymbol{x}_0$. Φ is the second invariant of the inertial tensor [Leblanc and Cambon, Phys. Fluids **9**, 1307 (1997)]. The particular cases of zero absolute $\boldsymbol{W}(\boldsymbol{x}_0)+2\boldsymbol{\Omega}=0$ and zero tilting $\boldsymbol{W}(\boldsymbol{x}_0)+4\boldsymbol{\Omega}=0$ vorticities are also considered. The criterion is applied to Chaplygin's non-symmetric dipolar vortex moving along a circular path, which is shown to be unstable. © *1997 American Institute of Physics.* [S1070-6631(97)00410-8]

I. INTRODUCTION

As an alternative to classical tools of hydrodynamic stability theory (such as spectral or energy methods), the ''geometrical optics'' (WKB) stability theory, developed and applied recently to fluid dynamics by Lifschitz and co-workers,[1–5] is a powerful tool to provide local stability criteria based on the geometrical properties of the basic flows. Thus, *in an inertial frame any steady (two- or three-dimensional) inviscid flow with a stagnation point is linearly and non-linearly unstable to short-wave perturbations* (except if the streamlines are circular in the vicinity of the stagnation point).[1,2,6] In a rotating flow however, short-wave instabilities may be killed.

The search of local stability criteria is not new, and in the context of centrifugal and/or Coriolis induced instabilities, some exact derivations were proposed independently for simple basic flows. One of the most popular is the ''Rayleigh'' criterion for basic flows with circular streamlines.[7] Synge's exact derivation shows that three-dimensionality is needed for instability to occur. The same conclusion holds for the ''Bradshaw–Richardson'' criterion for shear flows with parallel streamlines in a rotating frame, derived rigorously by Pedley.[8] For the class of unbounded quadratic flows (with hyperbolic, elliptical or parallel streamlines) in both a rotating and a non-rotating frame, Cambon and co-workers[9–11] derived, in the context of rapid distortion theory (RDT) of homogeneous turbulence, an exact stability criterion for the class of ''pressureless'' perturbations (with wave vector perpendicular to the plane of the basic flow). Similar conclusions also were established in the context of hydrodynamic stability theory.[12,13] Finally, the Rayleigh criterion has been extended by Bayly[14] to the more general class of flows having closed convex streamlines, and to circular vortices in rotating fluids by Kloosterziel and van Heijst.[15,16]

Recently, Leblanc and Cambon[17] pointed out that all these exact criteria for curvature and/or Coriolis induced instabilities may be expressed solely by the following (in Cartesian coordinates): a sufficient condition for instability to three-dimensional disturbances is that

$$\Phi(x,y)=-\tfrac{1}{2}\boldsymbol{S}:\boldsymbol{S}+\tfrac{1}{4}\boldsymbol{W_t}\cdot\boldsymbol{W_t}<0,$$

somewhere in the flow domain. It involves the ''tilting'' vorticity[11] $\boldsymbol{W_t}=\boldsymbol{W}+4\boldsymbol{\Omega}$ and the symmetric part $\boldsymbol{S}$ of the velocity gradient of the basic flow. Φ is the generalization of the Rayleigh discriminant and involves an additional curvature term when the problem is expressed in plane curvilinear coordinates. In agreement with Bayly,[14] they also showed that, when $\Phi<0$, a class of three-dimensional pressureless modes is excited. These modes are characterized by a wave vector aligned with the rotation axis, and when the physical problem exhibits a characteristic length-scale, they consist of short-wavelength eigenmodes strongly localized on streamlines. They suggest that this criterion is valid for any complex plane flow.

Using the geometrical optics stability theory, in the present paper we show simply that the pressureless problem makes sense, and that *the flow is unstable if* $\Phi(x_0,y_0)<0$ *on a stagnation point (if any) located at* (x_0,y_0). But a more general conclusion is not available from this theory. As an application, it is shown that Chaplygin's non-symmetric dipolar vortex moving along a circular path is unstable. Simple results may also be expressed for particular values of the rotation rate: for zero tilting vorticity $\boldsymbol{W}(\boldsymbol{x}_0)+4\boldsymbol{\Omega}=0$, it is shown that any stagnation point is unstable, whereas for zero absolute vorticity $\boldsymbol{W}(\boldsymbol{x}_0)+2\boldsymbol{\Omega}=0$, by examination of the vorticity equation, it may be concluded that short-wave instabilities are killed for pure-shear (with locally parallel streamlines) stagnation points and for circular or elliptical vortex cores. Note that all the results exposed here are natural extensions to stagnation points in an inhomogeneous flow (with non-constant velocity gradient) of the results obtained previously in the RDT context for homogeneous turbulence by Cambon and co-workers.[9–11]

II. STABILITY ANALYSIS

The problem is expressed in Cartesian coordinates for clarity, and the formalism is close to recent papers.[4,5] In a rotating frame with angular velocity vector $\boldsymbol{\Omega}=\Omega\boldsymbol{e}_z$, let $\boldsymbol{U}(x,y)$ be the (relative) velocity field of an inviscid steady *two-dimensional* basic flow, with (relative) vorticity $\boldsymbol{W}(x,y)=W(x,y)\boldsymbol{e}_z$. A *three-dimensional* localized disturbance,

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$$(\boldsymbol{u}',\pi')(\boldsymbol{x},t)=e^{i\phi(\boldsymbol{x},t)/\epsilon}(\boldsymbol{a},\pi)(\boldsymbol{x},t)+\epsilon e^{i\phi(\boldsymbol{x},t)/\epsilon}(\boldsymbol{a}_\epsilon,\pi_\epsilon)(\boldsymbol{x},t)+\epsilon(\boldsymbol{u}_r,\pi_r)(\boldsymbol{x},t) \tag{1}$$

is superimposed to the basic flow. The phase field $\phi(\boldsymbol{x},t)$ is a real-valued function, and ϵ is a small parameter. The velocity "envelope-polarization" $\boldsymbol{a}(\boldsymbol{x},t)$ is complex. Substitution into the linearized incompressible Euler equations,[11,17]

$$D_t\boldsymbol{u}'+\boldsymbol{S}\boldsymbol{u}'+\tfrac{1}{2}\boldsymbol{W_t}\times\boldsymbol{u}'=-\nabla\pi',\quad \nabla\cdot\boldsymbol{u}'=0,$$

where $D_t=\partial_t+\boldsymbol{U}\cdot\nabla$, $\boldsymbol{S}=\tfrac{1}{2}(\boldsymbol{L}+\boldsymbol{L}^T)$ and $\boldsymbol{L}=\nabla\boldsymbol{U}$, leads to an equation containing terms of various order in ϵ. Equating the different order terms yields

$$D_t\phi=0,\quad \pi=0,\quad \boldsymbol{k}\cdot\boldsymbol{a}=0,$$

where the wave vector $\boldsymbol{k}(\boldsymbol{x},t)=\nabla\phi$ verifies the "eikonal" equation,

$$D_t\boldsymbol{k}=-\boldsymbol{L}^T\boldsymbol{k}.$$

Introducing $\boldsymbol{\alpha}(\boldsymbol{x},t)$ and $\boldsymbol{v}(\boldsymbol{x},t)$, the projections of $\boldsymbol{k}(\boldsymbol{x},t)$ and $\boldsymbol{a}(\boldsymbol{x},t)$ on the (x,y)-plane, and let $k(\boldsymbol{x},t)=\boldsymbol{k}\cdot\boldsymbol{e}_z$ and $w(\boldsymbol{x},t)=\boldsymbol{a}\cdot\boldsymbol{e}_z$ be their spanwise component, the linearized equations now read[4,5]

$$\begin{aligned}&D_t\boldsymbol{\alpha}=-\boldsymbol{N}^T\boldsymbol{\alpha},\quad D_tk=0,\\ &D_t\boldsymbol{v}+\boldsymbol{M}\boldsymbol{v}=\boldsymbol{\alpha}\boldsymbol{\alpha}^T/|\boldsymbol{k}|^2(\boldsymbol{M}+\boldsymbol{N})\boldsymbol{v},\\ &D_tw=k\boldsymbol{\alpha}^T/|\boldsymbol{k}|^2(\boldsymbol{M}+\boldsymbol{N})\boldsymbol{v},\end{aligned} \tag{2}$$

where $\boldsymbol{N}(x,y)$ is the (x,y)-plane projection of $\boldsymbol{L}$ and

$$\boldsymbol{M}(x,y)=\boldsymbol{N}+\boldsymbol{C},\quad \boldsymbol{C}=\begin{pmatrix}0&-2\Omega\\2\Omega&0\end{pmatrix}. \tag{3}$$

$\boldsymbol{M}$ and $\boldsymbol{C}$ are, respectively, the "inertial" and "Coriolis" tensors of the basic flow.[17] The two eigenvalues of $\boldsymbol{M}$ are solutions of $\lambda^2+\Phi=0$. $\Phi=-\frac{1}{2}\operatorname{tr}(\boldsymbol{MM})$ is the second invariant of the inertial tensor. It may be rewritten as

$$\begin{aligned}\Phi(x,y)&=-\tfrac{1}{2}\boldsymbol{S}:\boldsymbol{S}+\tfrac{1}{4}\boldsymbol{W_t}\cdot\boldsymbol{W_t}\\&=(\Psi_{xx}-2\Omega)(\Psi_{yy}-2\Omega)-(\Psi_{xy})^2,\end{aligned} \tag{4}$$

where $\Psi(x,y)$ is the basic streamfunction and the subscripts denote partial differentiation. The eigenvalues are real and opposite at points (x,y) where $\Phi(x,y)<0$ and purely complex otherwise.

For a pure *spanwise* wave vector $\boldsymbol{\alpha}(\boldsymbol{x},t)=0$, it may be verified that $\phi=\phi(z)$ and $k=d\phi/dz$, whereas $w=0$ in order to ensure $\boldsymbol{k}\cdot\boldsymbol{a}=0$. The Eulerian system (2) is reduced to the "pressureless" equation,

$$D_t\boldsymbol{v}(\boldsymbol{x},t)+\boldsymbol{M}(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x},t)=0,$$

that evolves locally along the trajectories (or streamlines since steady) of the basic flow. Let $\boldsymbol{x}=\boldsymbol{\chi}(\boldsymbol{X},t)$ be the position of a fluid particle which was initially located at $\boldsymbol{\chi}(\boldsymbol{X},0)=\boldsymbol{X}$. Using a Lagrangian representation, the above equation reads as

$$\begin{aligned}&\partial_t\boldsymbol{\chi}(\boldsymbol{X},t)=\boldsymbol{U}'(\boldsymbol{X},t),\\&\partial_t\boldsymbol{v}'(\boldsymbol{X},t)+\boldsymbol{M}'(\boldsymbol{X},t)\boldsymbol{v}'(\boldsymbol{X},t)=0,\end{aligned} \tag{5}$$

where the prime denote the Lagrangian representation of any field f: $f'(\boldsymbol{X},t)=f(\boldsymbol{\chi}(\boldsymbol{X},t),t)=f(\boldsymbol{x},t)$. Note that $\boldsymbol{M}'(\boldsymbol{X},t)=\boldsymbol{M}(\boldsymbol{x})$ is generally time-dependent in a Lagrangian representation. Equation (5) forms a system of ordinary differential equations in time, that may be solved sequentially, with appropriate initial conditions. According to the geometrical optics stability theory,[1] instability in the Lagrangian sense occurs if the amplitude $|\boldsymbol{v}'(\boldsymbol{X},t)|$ increases unboundedly when $t\rightarrow\infty$. Furthermore, instability in a Lagrangian representation implies instability in an Eulerian description.

For a pure spanwise wave vector $\boldsymbol{\alpha}=0$, the following order term in the WKB expansion (1) reads as $\boldsymbol{v}_\epsilon=0$, $w_\epsilon=ik^{-1}\nabla\cdot\boldsymbol{v}$ and $\pi_\epsilon=0$. Moreover, since the problem is z-independent, k may be chosen constant and $\phi(z)=kz$, so that this kind of short-wave perturbation corresponds to the pressureless modes described by Bayly[14] or Leblanc and Cambon.[17] Generally, nothing can be concluded from (5). For example, if the streamlines are closed, $\boldsymbol{M}'(\boldsymbol{X},t)$ will be periodic in time, and a Floquet analysis is needed, as performed by Bayly.[14] However, if the basic flow exhibits a *stagnation point* $\boldsymbol{U}(\boldsymbol{x}_0)=0$ located at $\boldsymbol{x}_0=(x_0,y_0)$, then

$$\boldsymbol{x}_0=\boldsymbol{\chi}(\boldsymbol{X}_0,t)=\boldsymbol{X}_0,\quad \boldsymbol{M}'(\boldsymbol{X}_0,t)=\boldsymbol{M}'(\boldsymbol{X}_0)=\boldsymbol{M}(\boldsymbol{x}_0),$$

and from (5), $|\boldsymbol{v}'(\boldsymbol{X}_0,t)|$ will grow exponentially with growth rate $\sqrt{-\Phi(\boldsymbol{x}_0)}$, if the discriminant (4) verifies

$$\Phi(\boldsymbol{x}_0)<0. \tag{6}$$

Indeed, on a stagnation point,

$$\begin{aligned}\Phi'(\boldsymbol{X}_0)&=-\tfrac{1}{2}\boldsymbol{M}'(\boldsymbol{X}_0):\boldsymbol{M}'(\boldsymbol{X}_0)\\&=-\tfrac{1}{2}\boldsymbol{M}(\boldsymbol{x}_0):\boldsymbol{M}(\boldsymbol{x}_0)=\Phi(\boldsymbol{x}_0)\end{aligned}$$

is time-independent in the Lagrangian representation. This is of course a *sufficient* condition for instability.

Owing to the nature of the stagnation points, the bandwidths of instability are plotted schematically in Fig. 4 in Ref. 17: they are centered around the case of *zero tilting vorticity*,

$$\boldsymbol{W}(\boldsymbol{x}_0)+4\boldsymbol{\Omega}=0,$$

for which $\Phi(\boldsymbol{x}_0)=-\frac{1}{2}\boldsymbol{S}:\boldsymbol{S}$ is always negative, and *the flow is always unstable, whatever the nature of the stagnation point* (except if the streamlines are circular in the vicinity of the stagnation point). Furthermore, the Coriolis force changes the nature of instability: in the elliptical case, the Floquet analysis is no more needed for rotation rates verifying (6), because instability occurs for spanwise wave vectors, which are time-independent.[4,5,11,13,17]

The *non-rotating* case is recovered with $\boldsymbol{\Omega}=0$, $\boldsymbol{W_t}=\boldsymbol{W}$ and $\boldsymbol{M}=\boldsymbol{N}$. In that case, the condition is equivalent to say that the basic flow is unstable if it contains a hyperbolic stagnation point $(\Phi<0)$, but does not allow us to conclude for an elliptical stagnation point $(\Phi>0)$, since Φ contains no pressure information coming from a time-dependent (oblique) wave vector, which is needed for the "elliptical instability."[18,19] The present condition is then weaker than the Lifschitz and Hameiri stability condition for stagnation points of steady flows in an inertial frame.[1]

In a rotating flow however, the conclusion is not straightforward. Thus an elliptical stagnation point (in a vortex core for example) is stabilized by rotation for *zero absolute vorticity*,

$$\boldsymbol{W}(\boldsymbol{x}_0)+2\boldsymbol{\Omega}=0.$$

Indeed, introducing the leading order vorticity amplitude[3] $\boldsymbol{b}(\boldsymbol{x},t)=\boldsymbol{k}\times\boldsymbol{a}$, it may be verified that in a rotating frame, it is governed by

$$D_t\boldsymbol{b}=(\boldsymbol{L}^T-\boldsymbol{C})\boldsymbol{b}+((\boldsymbol{W}+2\boldsymbol{\Omega})\times\boldsymbol{k})\times\boldsymbol{a},$$

for *any* wave vector $\boldsymbol{k}(\boldsymbol{x},t)$. Then, on the stagnation point $\boldsymbol{x}_0=\boldsymbol{\chi}(\boldsymbol{X}_0,t)=\boldsymbol{X}_0$ at zero absolute vorticity, $\boldsymbol{b}$ verifies in a Lagrangian representation,

$$\partial_t\boldsymbol{b}'(\boldsymbol{X}_0,t)=\boldsymbol{L}'(\boldsymbol{X}_0)\boldsymbol{b}'(\boldsymbol{X}_0,t),$$

since $\boldsymbol{L}^T-\boldsymbol{C}=\boldsymbol{S}-(\boldsymbol{A}+\boldsymbol{C})=\boldsymbol{S}+\boldsymbol{A}=\boldsymbol{L}$ [with $\boldsymbol{A}=\frac{1}{2}(\boldsymbol{L}-\boldsymbol{L}^T)$] at zero absolute vorticity, leading to stability or instability according to the nature of the stagnation point. Thus *elliptical, circular and pure-shear stagnation points are exponentially stable to short-wave perturbations at zero absolute vorticity whereas a hyperbolic one is unstable.*

III. APPLICATION

As an application of these conditions, the stability of Chaplygin's dipole is now investigated. In the beginning of the century, Chaplygin discovered various inviscid solutions of two-dimensional vortex structures, that have escaped the attention of later investigators in this field. A review has been done recently by Meleshko and van Heijst.[20] Among them, Chaplygin discovered a non-symmetric dipolar vortex of radius a moving along a circular path. In a frame rotating uniformly with angular velocity $-\kappa/a^2$, the dipole is steady, and its dimensional streamfunction is given by Eq. (4.3) in Ref. 20. In dimensionless variables $r\to ar$, $\kappa\to av\kappa$, $\lambda\to av\lambda$ and $\Psi\to av\Psi$, the streamfunction reads

$$\Psi(r,\theta)=\begin{cases}\left(r-\dfrac{1}{r}\right)\sin\,\theta+\kappa\,\log\,r-\dfrac{\kappa}{2}(r^2-1), & r\geqslant 1,\\[2mm] 2\dfrac{J_1(br)}{bJ_0(b)}\sin\,\theta+\lambda\left(1-\dfrac{J_0(br)}{J_0(b)}\right), & r\leqslant 1.\end{cases}$$

J_0 and J_1 are the standard Bessel functions, and $b=3.8317$ is such that $J_1(b)=0$ The flow depends on two dimensionless parameters: $\lambda\geqslant 0$ is a measure of the asymmetry of the dipole in the inside flow, whereas $\kappa\geqslant 0$ is a measure of the angular velocity of the dipole: in the inertial frame, it rotates with the dimensionless angular velocity $-\kappa$. On the boundary $r=1$, the vorticity is discontinuous except if $\lambda=-2\kappa/b^2$, corresponding to the Flierl, Stern and Whitehead ''modon.''[21] Streamline patterns are plotted in Ref. 20 for various values of λ and κ. The flow contains two stagnation points of the hyperbolic kind located on the boundary at $r_0=1$ and $\theta_0=0,\pi$, and two elliptical stagnation points on the axis $\theta_0=\pm\pi/2$, for which the exact locations r_0 requires a numerical approach. However, the hyperbolic points are sufficient to conclude.

In plane curvilinear coordinates, the problem requires an additional curvature tensor.[1,17] The inertial tensor (3) reads now as

$$\boldsymbol{M}=\boldsymbol{N}+\boldsymbol{C}+\boldsymbol{C}',\quad \boldsymbol{C}'=\begin{pmatrix}0 & -V/r\\ V/r & 0\end{pmatrix},$$

and the discriminant (4) is

$$\Phi=\left(2\Omega+\frac{1}{r}\frac{\partial(rV)}{\partial r}\right)\left(2\Omega+\frac{2V}{r}-\frac{1}{r}\frac{\partial U}{\partial\theta}\right)-\left(\frac{\partial U}{\partial r}\right)^2,$$

with $U=r^{-1}\partial\Psi/\partial\theta$ and $V=-\partial\Psi/\partial r$.

The relative frame (in which Chaplygin's dipole is steady) rotates at the dimensionless angular velocity $\Omega=-\kappa$, and then, on the hyperbolic stagnation points $r_0=1,\ \theta_0=0,\pi$,

$$\Phi_E=-4,\quad \Phi_I=-4+2\kappa(2\kappa+\lambda b^2).$$

The subscripts refer, respectively, to the external and the inside solutions. The discontinuity in Φ on the stagnation points is due to the discontinuity of the vorticity distribution on the boundary. Without more considerations, *Chaplygin's dipolar vortex is unstable to three-dimensional perturbations*, since it is possible to construct an initial condition so that the instability will occur in the outside flow.[1] The discontinuity in Φ disappears when ($\lambda=-2\kappa/b^2$) and the *Flierl, Stern and Whitehead modon is unstable* too.

In fact this conclusion should have been made without any calculus since in an inertial frame, the outside flow is irrotational, so that in the relative frame rotating with the dipole, the outside flow has a zero absolute vorticity, for which a hyperbolic stagnation point is unstable.

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