AM 275 - Magnetohydrodynamics: Homework 1

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Problem 1:

Show that

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} + p e_{kk} - 2\mu \left[e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \right]^2.$$

Proof. First, we begin with the derivative identity

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

and in order to simplify this statement, take τ_i to be the i-th row vector of τ, we have:

$$\sum_i u_i \nabla \cdot \tau_i = \sum_i \nabla \cdot u_i \tau_i - \tau_i \cdot \nabla u_i$$

Already we have shown the first RHS term originates from the derivative identity, whereas the other terms must originate from $-\sum_i \tau_i \cdot \nabla u_i$. Thus, we investigate this term in more detail.

$$-\sum_i \tau_i \cdot \nabla u_i = \sum_i \left[p + \frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right] \delta_{ij} \cdot \nabla u_i - 2\mu e_i \cdot \nabla u_i$$

where e_{kk} is written as $\nabla \cdot \boldsymbol{u}$ and e_i is the i-th row of e (as in e_{ij}). Notice that $\sum_i \delta_{ij} \cdot \nabla u_i = \nabla \cdot \boldsymbol{u}$, and therefore,

$$\begin{aligned} -\sum_i \tau_i \cdot \nabla u_i &= \left[p + \frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right] (\nabla \cdot \boldsymbol{u}) - 2\mu \sum_i e_i \cdot \nabla u_i \\ &= p(\nabla \cdot \boldsymbol{u}) + \frac{2}{3} \mu (\nabla \cdot \boldsymbol{u})^2 - 2\mu \sum_i e_i \cdot \nabla u_i \end{aligned}$$

Thus we recover the second RHS term, $p e_{kk}$. Now we must show the rest of $-\sum_i \tau_i \cdot \nabla u_i$ recovers the last term of the RHS. We write the decomposition of e_i.

$$\begin{aligned} -2\mu \sum_i e_i \cdot \nabla u_i &= -\mu \sum_i \left(\nabla u_i + \frac{\partial \boldsymbol{u}}{\partial x_i} \right) \cdot \nabla u_i \\ &= -\mu \sum_i |\nabla u_i|^2 + \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\ &= -\mu |\nabla \boldsymbol{u}|^2 - \mu \sum_i \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \nabla u_i \\ &= -\mu |\nabla \boldsymbol{u}|^2 - \mu \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_j}{\partial x_i} \end{aligned}$$

Now we must show by the transitive propery that,

$$\frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2-\mu|\nabla\boldsymbol{u}|^2-\mu\sum_{ij}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}=-2\mu\left[e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}\right]^2_{ll}$$

We begin by writing the inner product of these second order tensors and then taking the contraction (necessary in order to obtain a scalar) (also sorry about the indices, I couldn't decide which letters I wanted to stick with in the long run)

$$\begin{aligned}
-2\mu\left[e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}\right]^2_{ll} &= -2\mu\left[(e_{ij}^2)_{ll}-\frac{2}{3}(\nabla\cdot\boldsymbol{u})e_{ll}+\frac{1}{9}(\nabla\cdot\boldsymbol{u})^2\delta_{ll}\right] \\
&= -2\mu\left[(e_{im}\cdot e_{mj})_{ll}-\frac{2}{3}(\nabla\cdot\boldsymbol{u})^2+\frac{1}{3}(\nabla\cdot\boldsymbol{u}^2)\right] \\
&= -\frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_m}\frac{\partial u_m}{\partial x_j}+\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}+\frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m}+\frac{\partial u_m}{\partial x_i}\frac{\partial u_j}{\partial x_m}\right)_{ll}+\frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2 \\
&= -\frac{\mu}{2}\left(\nabla u_i\cdot\frac{\partial\boldsymbol{u}}{\partial x_i}+\frac{\partial\boldsymbol{u}}{\partial x_i}\cdot\frac{\partial\boldsymbol{u}}{\partial x_i}+\nabla u_i\cdot\nabla u_i+\frac{\partial\boldsymbol{u}}{\partial x_i}\cdot\nabla u_i\right)+\frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2 \\
&= -\frac{\mu}{2}\left(2|\nabla\boldsymbol{u}|^2+2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)+\frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2 \\
&= \frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2-\mu|\nabla\boldsymbol{u}|^2-\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}
\end{aligned}$$

Therefore, we have shown that

$$\begin{aligned}
u_i\frac{\partial\tau_{ij}}{\partial x_j} &= \frac{\partial u_i\tau_{ij}}{\partial x_j}+p(\nabla\cdot\boldsymbol{u})+\frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2-\mu|\nabla\boldsymbol{u}|^2-\mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \\
&= \frac{\partial u_i\tau_{ij}}{\partial x_j}+pe_{kk}-2\mu\left[e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}\right]^2
\end{aligned}$$

where $[\cdot]^2$ implies a tensor "double dot product", where first a (tensor) inner product is taken and the resultant second order tensor is contracted to become a scalar. □