AM 275 - Magnetohydrodynamics:

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Problem 1:

a.) We begin solving this problem by simply setting $\boldsymbol{j} \times \boldsymbol{B} = 0$.

$$\boldsymbol{B} = \langle B_x, B_y, 0\rangle, \quad \boldsymbol{j} = \left\langle -\frac{\partial B_y}{\partial z}, \frac{\partial B_x}{\partial z}, 0 \right\rangle$$
$$\boldsymbol{j} \times \boldsymbol{B} = \left\langle 0, 0, -B_y\frac{\partial B_y}{\partial z} - B_x\frac{\partial B_x}{\partial z} \right\rangle$$
$$\boldsymbol{j} \times \boldsymbol{B} = 0 \implies \frac{1}{2}\frac{\partial}{\partial z}\left(B_x^2 + B_y^2\right) = 0 \implies B_x^2 + B_y^2 = \text{constant}$$

We have obtained the condition for this field to be force free, and in the proceeding problems it will become clear why this satisfies the Beltrami condition in Helmholtz form.

b.) Next we enforce the Beltrami property, i.e. $\nabla \times \boldsymbol{B} = \alpha\boldsymbol{B}$. We have,

$$\left\langle -\frac{\partial B_y}{\partial z}, \frac{\partial B_x}{\partial z}, 0 \right\rangle = \alpha \langle B_x, B_y, 0\rangle$$
$$\alpha B_x = -\frac{\partial B_y}{\partial z}, \quad \alpha B_y = \frac{\partial B_x}{\partial z}$$

c.) We will now show using substitution that this field satisfies both the Beltrami property in the Helmholtz form as well as the force-free condition derived in part a.

$$\frac{\partial}{\partial z}\left(\alpha B_x = -\frac{\partial B_y}{\partial z}, \quad \alpha B_y = \frac{\partial B_x}{\partial z}\right)$$
$$\alpha\frac{\partial B_x}{\partial x} = -\frac{\partial^2 B_y}{\partial z^2}, \quad \alpha\frac{\partial B_y}{\partial z} = \frac{\partial^2 B_x}{\partial z^2}$$
$$\alpha^2 B_y = -\frac{\partial^2 B_y}{\partial z^2}, \quad -\alpha^2 B_x = \frac{\partial^2 B_x}{\partial z^2}$$

We note that since B_x and B_y depend only on z, this exactly satisfies the Helmholtz equation for the Beltrami property. We notice a standard form for B_x and B_y given by,

$$B_x = a\cos(\alpha z) + b\sin(\alpha z), \quad B_y = c\cos(\alpha z) + d\sin(\alpha z)$$

Notice that given the condition found in part b, that we have a limited choice for the coefficients a, b, c, d. Specifically,

$$\alpha\left(a\cos(\alpha z) + b\sin(\alpha z)\right) = -\alpha c\sin(\alpha z) + \alpha d\cos(\alpha z) \implies a = d, \quad b = -c$$
$$B_x = d\cos(\alpha z) - c\sin(\alpha z), \quad B_y = c\cos(\alpha z) + d\sin(\alpha z)$$
$$B_x^2 + B_y^2 = d^2\cos^2(\alpha z) + c^2\sin^2(\alpha z) - 2cd\cos(\alpha z)\sin(\alpha z) + c^2\cos^2(\alpha z) + d^2\sin^2(\alpha z) + 2cd\cos(\alpha z)\sin(\alpha z)$$
$$= d^2 + c^2 = \text{constant}$$

Problem 2:

We begin this problem by writing the steady induction equation for an incompressible fluid. We have,

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} + \eta\nabla^2\boldsymbol{B}$$

For the given flow field $\boldsymbol{u} = U_0\,(-x, y, 0)$ and $\boldsymbol{B} = (0, B(x), 0)$, this reduces to,

$$\begin{aligned} -U_0 x\frac{\partial B}{\partial x} &= U_0 B + \eta\frac{\partial^2 B}{\partial x^2} \\ \eta\frac{d^2 B}{dx^2} + U_0 x\frac{dB}{dx} + U_0 B &= 0 \end{aligned}$$

This ODE can be solved in two manners. First, and probably the most general argument is that we can write $U_0(x\frac{dB}{dx} + B) = U_0\frac{d}{dx}(xB)$ from which you can directly integrate and then use integration by parts to show the solution. The alternate method is to presume an ansatz for B and then solve the resultant ODE obtained from that. We will do the latter,

$$\begin{gathered} B = e^{f(x)}, \quad B' = f'e^f, \quad B'' = f'^2 e^f + f'' e^f \\ \eta\left(f'^2 + f''\right)e^f + U_0 x f' e^f + U_0 e^f = 0 \\ \eta f'^2 + \eta f'' + U_0 x f' + U_0 = 0 \end{gathered}$$

Here we will attempt to solve using $f = Cx^\alpha$. This gives,

$$\eta C^2\alpha^2 x^{2\alpha-2} + \eta C\alpha(\alpha-1)x^{\alpha-2} + U_0 C\alpha x^\alpha + U_0 = 0$$

Notice that we have 4 different orders of x in this solution and yet it must be satisfied for all x. We must chose α such that there are actually only 2 different orders of x and then solve for the coefficient C. Notice that $\alpha = 2$ yields $2\alpha - 2 = \alpha$ and $\alpha - 2 = 0$. Thus we substitute $\alpha = 2$ into the ODE.

$$4\eta C^2 = -2U_0 C, \quad 2\eta C = -U_0$$

These two equations are thankfully linearly dependent, i.e. thery are rescaled versions of one another. Thus we find $C = -U_0/2\eta$. Finally, we have,

$$B(x) = B_0 e^{-U_0 x^2/2\eta}$$

Problem 3:

The solution of this problem derives first from expanding each term in the navier stokes equation from the presummed magnetohydrostatic balance. We have,

$$\begin{gathered} \boldsymbol{B} = (B_x, 0, B)\,, \quad \boldsymbol{j} = \frac{1}{\mu_0}\left(0, -\frac{\partial B}{\partial x}, 0\right), \quad \boldsymbol{j}\times\boldsymbol{B} = -\frac{B}{\mu_0}\frac{dB}{dx}\boldsymbol{e}_x + \frac{B_x}{\mu_0}\frac{dB}{dx}\boldsymbol{e}_z \\ p = \alpha\rho, \quad \nabla p = \alpha\frac{d\rho}{dx}\boldsymbol{e}_x \\ -\alpha\frac{d\rho}{dx} = \frac{B}{\mu_0}\frac{dB}{dx}, \quad \frac{B_x}{\mu_0}\frac{dB}{dx} = \rho g \end{gathered}$$

Notice here that we obtain a series of coupled ODEs for B and ρ. More importantly, we can use a derivative identity to write,

$$\frac{d\rho}{dx} = -\frac{1}{2\alpha\mu_0}\frac{dB^2}{dx} \implies \rho = -\frac{B^2}{2\alpha\mu_0} + c$$

Using the boundary conditions given ($B(0) = 0$ and $\rho(0) = \rho_0$) we find that $c = \rho_0$. Now we substitute into the other obtained equation,

$$\frac{B_x}{\mu_0}\frac{dB}{dx} = -\frac{gB^2}{2\alpha\mu_0} + \rho_0 g$$
$$\frac{dB}{dx} = -\frac{gB^2}{2\alpha B_x} + \frac{\mu_0\rho_0 g}{B_x}$$

Now, if one were to naively approach this ODE, they might have some trouble finding a solution. We, however, who are primed to find a solution of the form $\tanh(x)$, know that the derivative of $\tanh(x)$ is $1 - \tanh^2(x)$, and might notice some resemblance in the resultant ODE. We therefore write an ansatz of the form $B(x) = C\tanh(f(x))$.

$$B' = Cf'(1 - \tanh^2(f)) = Cf' - \frac{f'}{C}B^2$$
$$\frac{\mu_0\rho_0 g}{B_x} = Cf', \quad \frac{g}{2\alpha B_x} = \frac{f'}{C}$$
$$\frac{\mu_0\rho_0 g}{B_x} = \frac{C^2 g}{2\alpha B_x} \implies C = \sqrt{2\mu_0\alpha\rho_0}$$
$$f' = \frac{g\sqrt{\mu_0\rho_0}}{B_x\sqrt{2\alpha}} \implies f = \frac{g\sqrt{\mu_0\rho_0}}{B_x\sqrt{2\alpha}}x$$

Thus we have solved for B to satisfy the obtained ODE and find in conclusion,

$$B(x) = \sqrt{2\mu_0\alpha\rho_0}\tanh\left(\sqrt{\frac{\mu_0\rho_0}{2\alpha}}\frac{gx}{B_x}\right)$$