The *Pow*-der of Ani-*snow*-tropy

AM 214 | Fall 2022

Examining dissipative and non-dissipative dynamical systems of snowboards on moguls' slopes.

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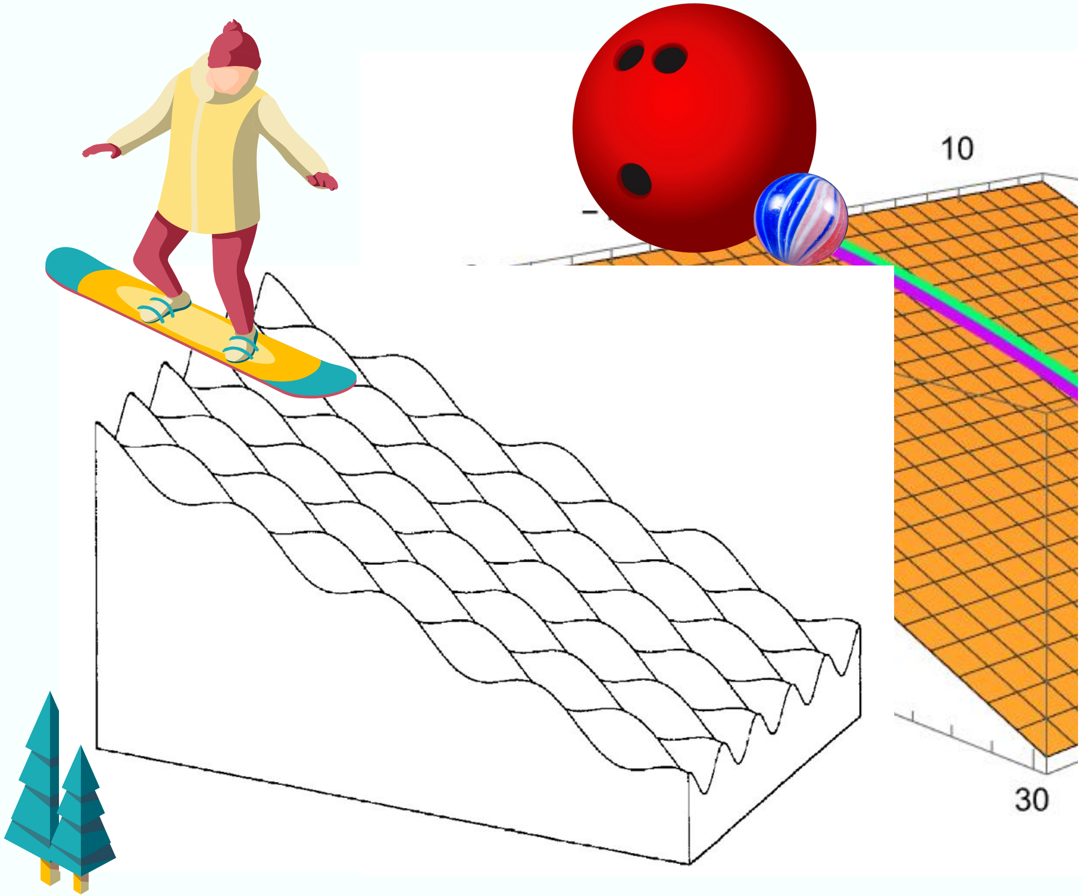
Presentation Contents

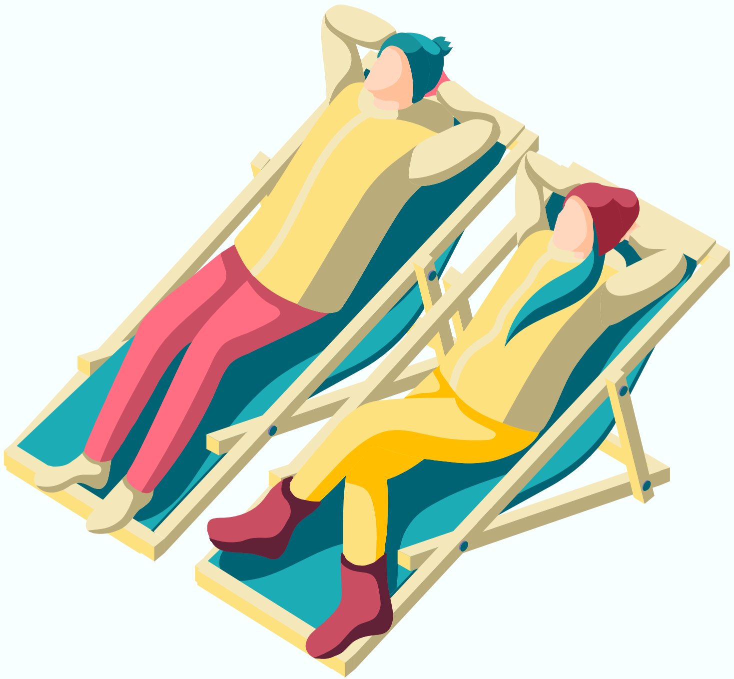
PART 1
Foundations

PART 1 – Foundations



$$\dot{x} = U$$
$$\dot{y} = V$$
$$\dot{z} = W$$
$$\dot{U} = a_{net,x}$$
$$\dot{V} = a_{net,y}$$
$$\dot{W} = a_{net,z}$$

PART 2
The System



PART 2 - The Dynamical System & The Derivations of its Equations

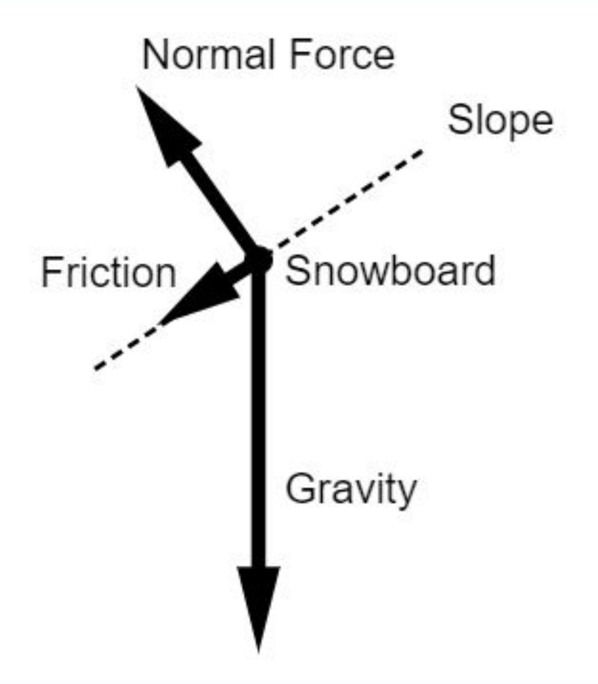
System is derived using a Free-Body Diagram and Newton's 2nd Law:

- Three Position Variables (X, Y, Z)
- Three Velocity Variables (U, V, W)

Forces acting on the snowboard:

- Gravity
- Friction
- Normal Force

Board and Sled treated as particles



PART 2 - The Dynamical System & The Derivations of its Equations

We begin our derivations by representing the components of acceleration on each axis. We also define a height function which creates the surface of the slope, H(x, y) (to be explained later). By extension H(x, y) = z

$$H(x, y) = -ax - b\cos(px)\cos(qy)$$

We denote the function, F, to be the vertical component of the normal force (along the Z-axis). We then scale the Gradient of H with F obtain the normal force in each direction.

$$a_{Net\ x} = -FH_X - cU$$

$$a_{Net\ y} = -FH_Y - cV$$

$$a_{Net\ Z} = -g + F - cW$$

Finally we account for friction as a constant c, which has the unit Hz. Multiplied by the velocity (m/s), it represents acceleration due to friction, always in the opposing direction of velocity.

System of Equations:

$$\dot{x} = U$$

$$\dot{y} = V$$

$$\dot{z} = W$$

$$\dot{U} = a_{net,x}$$

$$\dot{V} = a_{net,y}$$

$$\dot{W} = a_{net,z}$$

PART 2 - The Dynamical System & The Derivations of its Equations

We now look to define the function, F. We have that z = H(x, y) and therefore,

1. $$\dot{z} = \frac{dH(x,y)}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt}$$

Differentiate **H** w.r.t **Time**

2. $$\dot{z} = W = H_x U + H_y V$$

Differentiate W w.r.t Time

3. $$\ddot{z} = \dot{W} = H_x \dot{U} + H_y \dot{V} + H_{xx}U^2 + H_{xy}UV + H_{yy}V^2 + H_{xy}UV$$

4. $$\dot{W} = -H_x(FH_x + cU) - H_y(FH_y + cV) + H_{xx}U^2 + 2H_{xy}UV + H_{yy}V^2$$

Substitute eqn 4 and 5 together to obtain eqn 6

5. $$\dot{W} = -g + F - cW$$

6. $$-g + F - cW = FH_x^2 - cUH_x + FH_y^2 - cVH_y + H_{xx}U^2 + 2H_{xy}UV + H_{yy}V^2$$

Terms with c cancel due to substitution from eqn. 2

7. $$-g + F + FH_x^2 + FH_y^2 = H_{xx}U^2 + 2H_{xy}UV + H_{yy}V^2$$

Rearrange for F

8. $$F = \frac{g + H_{xx}U^2 + 2H_{xy}UV + H_{yy}V^2}{1 + H_x^2 + H_y^2}$$

PART 2 - The Dynamical System & The Derivations of its Equations

To finish modeling our system we need the partial derivatives of H and a definition of the parameters:

- a: slope of the hill; as x increases, H decreases (unitless)
- b: half of the height of moguls (meter)
- p: down-slope frequency of moguls (rad/meter)
- q: cross-slope frequency of moguls (rad/meter)

Partial Derivatives of H

$$H(x,y) = -ax - b\cos(px)\cos(qy)$$
$$H_x = -a + bp\sin(px)\cos(qy)$$
$$H_y = bq\cos(px)\sin(qy)$$
$$H_{xx} = bp^2\cos(px)\cos(qy)$$
$$H_{yy} = bq^2\cos(px)\cos(qy)$$
$$H_{xy} = -bpq\sin(px)\sin(qy)$$

PART 2 - The Dynamical System & The Derivations of its Equations

We can now write our system of equations in terms of our variables (X, Y, Z, U, V, W) and give them to a simulator. Notice however that since F only depends on X, Y, U, V, that we can chose to represent our system as a 4-variable system or a 6-variable system. Numerical Integrations will be shown later.

6-Var System

$$\begin{aligned}
\dot{x} &= U \\
\dot{y} &= V \\
\dot{z} &= W \\
\dot{U} &= -FH_x - cU \\
\dot{V} &= -FH_y - cV \\
\dot{W} &= -g + F - cW
\end{aligned}$$

4-Var System

$$\begin{aligned}
\dot{x} &= U \\
\dot{y} &= V \\
\dot{U} &= -FH_x - cU \\
\dot{V} &= -FH_y - cV
\end{aligned}$$

PART 2 - The Dynamical System & The Derivations of its Equations

We now introduce the concept of the Hypothetical Sled System which further simplifies the system mathematically:

- Motor and Brake system given to the sled
- Constant Southbouth Velocity (U)
- Reduces System to 3 variables

Note that the presence of less variables doesn't change the potentially chaotic nature of the system!

3-Var Sled System

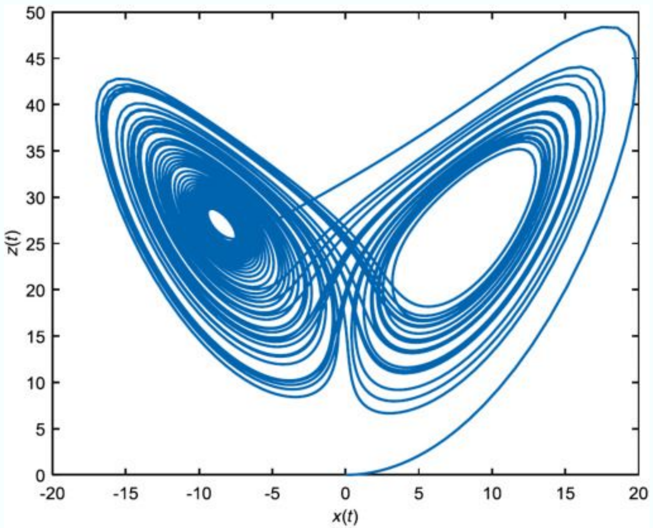
$$\dot{x} = 3.5$$
$$\dot{y} = V$$
$$\dot{U} = 0$$
$$\dot{V} = -FH_y - cV$$

PART 3
Dissipative Case

PART 3 – Dissipative Case

First, a definition of a dissipative system:

A system is *dissipative* if the volumes in phase space contract under the flow. This is in contrast to conservative systems, where volumes in phase space tend to remain area-preserving *or* expand.

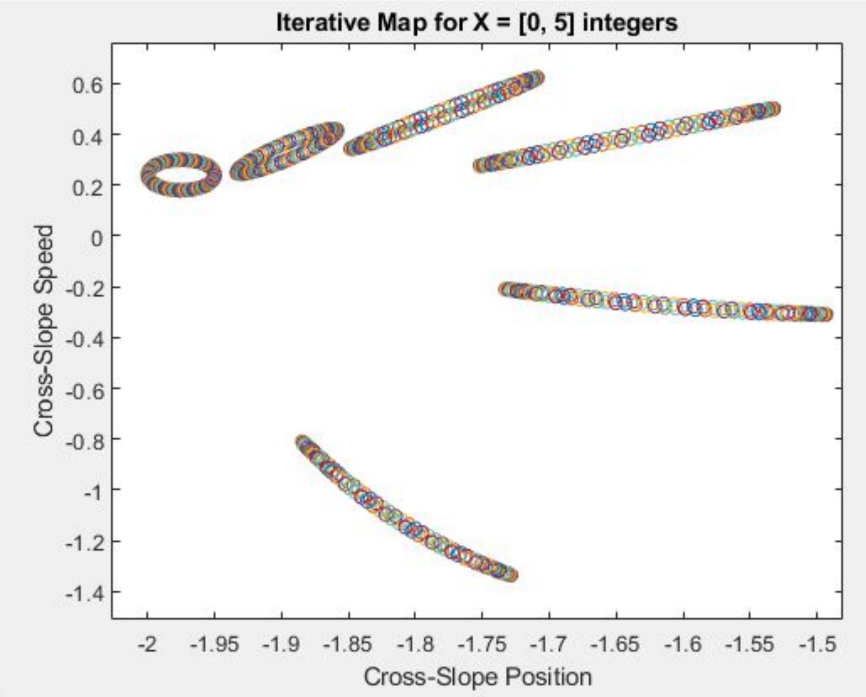


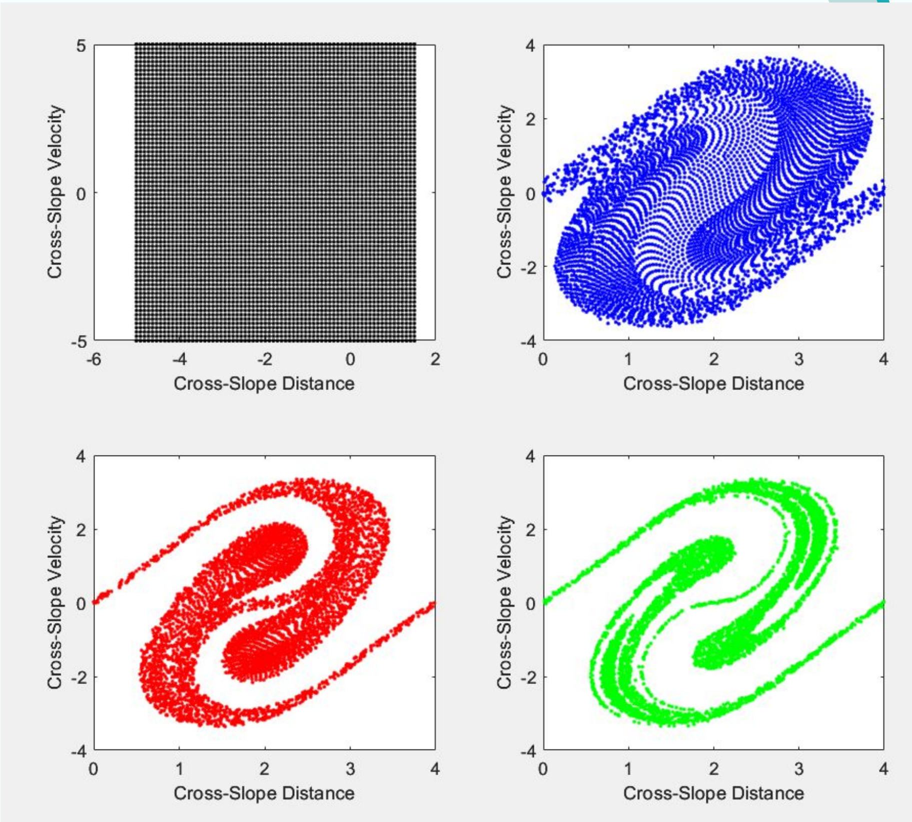
$$\nabla \cdot f = \frac{\partial}{\partial x}[\sigma(y-x)] + \frac{\partial}{\partial y}[rx - y - xz] + \frac{\partial}{\partial z}\left[xy - bz\right]$$

$$= -\sigma - 1 - b < 0$$

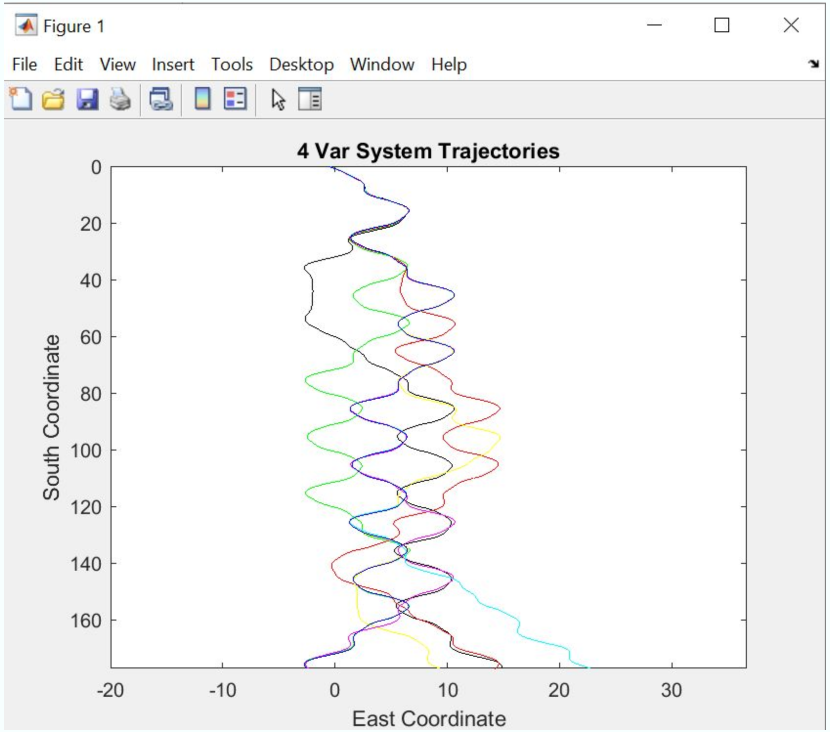
For example, the Lorenz system is dissipative.

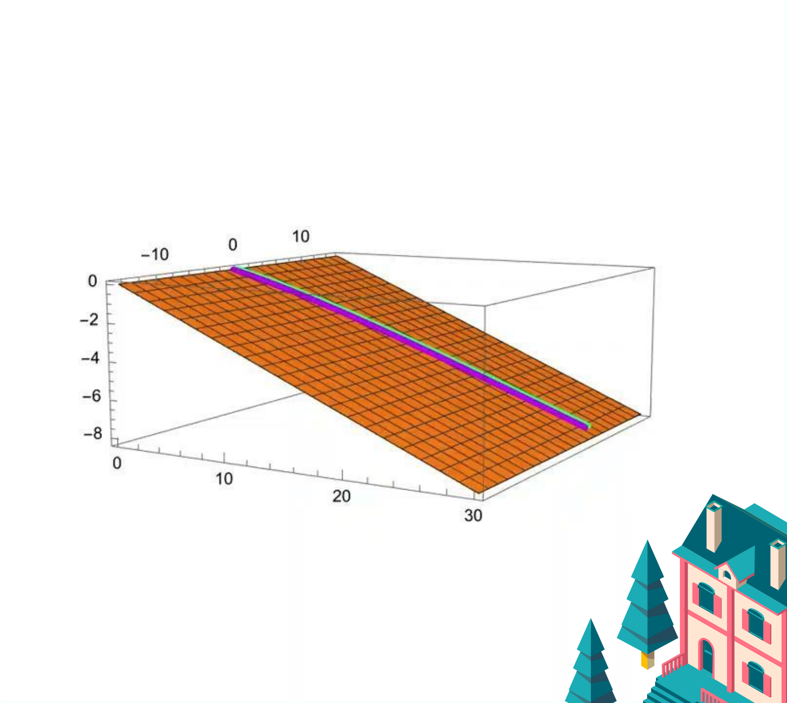
PART 3 - Dissipative Case





PART 3 - Dissipative Case





PART 4
Non-Dissipative Case

PART 4 - The Non-Dissipative Case

First, a definition of a non-dissipative system:

A system is non-*dissipative* in the case in which the volumes in phase space is conservative or has an expanding volume.

$$\sigma - 1 - b \geq 0$$

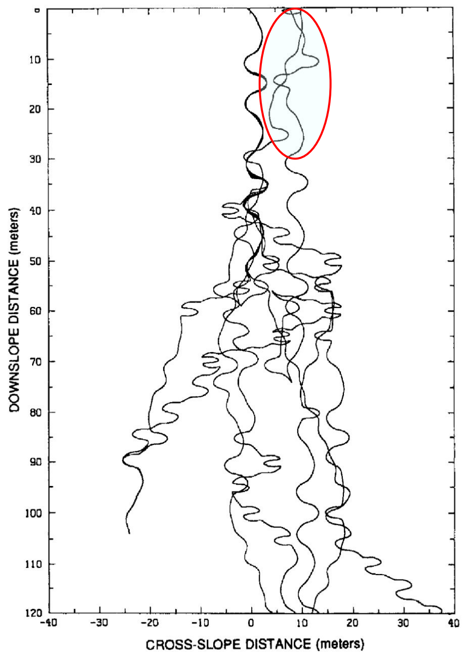
This is an example using the Lorenz system.

PART 4 - The Non-Dissipative Case

For our system of equations, non-dissipative cases can be observed when the system becomes conservative.

$$
\begin{aligned}
\dot{x} &= U \\
\dot{y} &= V \\
\dot{u} &= -FH_x - cU \\
\dot{v} &= -FH_y - cV
\end{aligned}
\quad \xrightarrow{\substack{a = 0 \\ c = 0}} \quad
\begin{aligned}
\dot{x} &= U \\
\dot{y} &= V \\
\dot{u} &= -FH_x \\
\dot{v} &= -FH_y
\end{aligned}
$$

PART 4 - The Non-Dissipative Case



Example of a Non-Dissipative Case

Notice how certain boards reverse their direction and begin moving upwards.

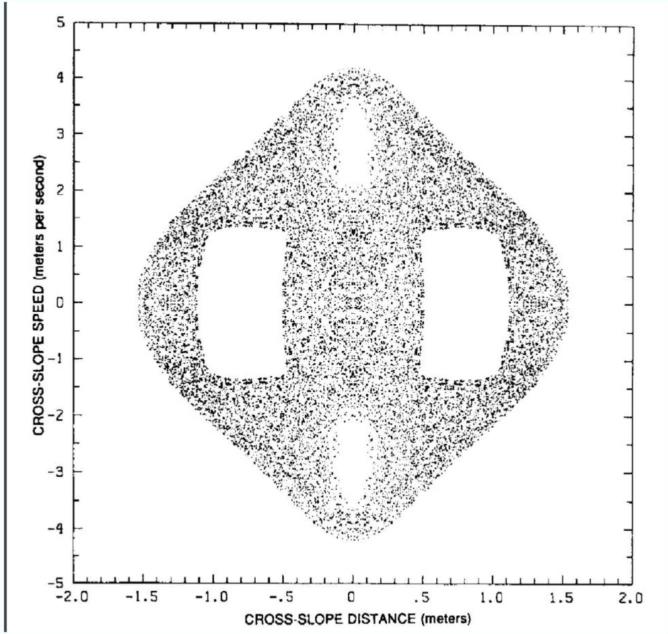
This reversal in direction is possible due to there being enough energy stored in the boards while also having no slope.

$a = 0$ No Slope
$c = 0$ No Friction

This is a Hamiltonian Graph provided by Lorenz

PART 4 - The Non-Dissipative Case

Why is there still movement with no slope?



This is the chaotic sea, a view of the velocity of varying initial conditions.

When we pick a point that is not in the sea, there is no velocity to carry the board so its has no movement.

PART 6 - Properties of the System

There exists a bifurcation relating to parameters a, b, p. We examine the fixed points:

$$\dot{x} = 0 \rightarrow U = 0$$
$$\dot{y} = 0 \rightarrow V = 0$$
$$\dot{U} = 0 \rightarrow -FH_x = 0$$
$$\dot{V} = 0 \rightarrow -FH_y = 0$$

When U = V = 0,

$$F = \frac{g}{1 + H_x^2 + H_y^2} \neq 0$$

Therefore, a fixed point requires:

$$\dot{U} = 0 \rightarrow H_x = 0$$
$$\dot{V} = 0 \rightarrow H_y = 0$$

We obtain the following result: if a/(bp) is greater than 1, then no fixed points exist in the system

$$H_x = 0 \rightarrow \frac{a}{bp} = sin(px)cos(qy)$$
$$H_y = 0 \rightarrow 0 = cos(px)sin(qy)$$

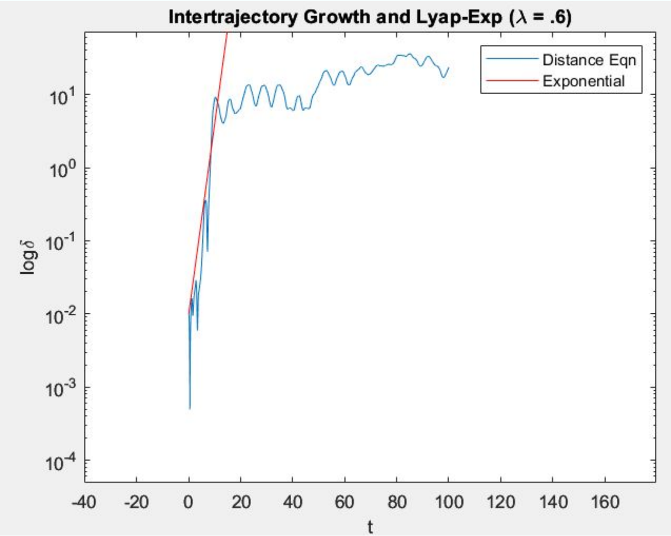
$$px = \frac{n\pi}{2}, \;\; qy = n\pi$$

$$\text{if } \frac{a}{bp} > 1, \text{ then there exist no solutions x, y.}$$

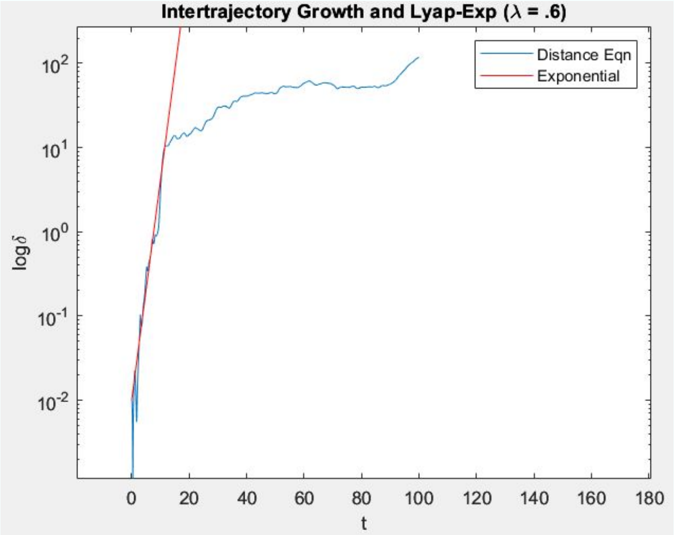
PART 6 – Properties of the System

Sensitive Dependence on Initial Conditions: a small change in initial conditions results in exponential growth between trajectories! Both graphs below start at an initial displacement of 1 cm

Dissipative System Growth



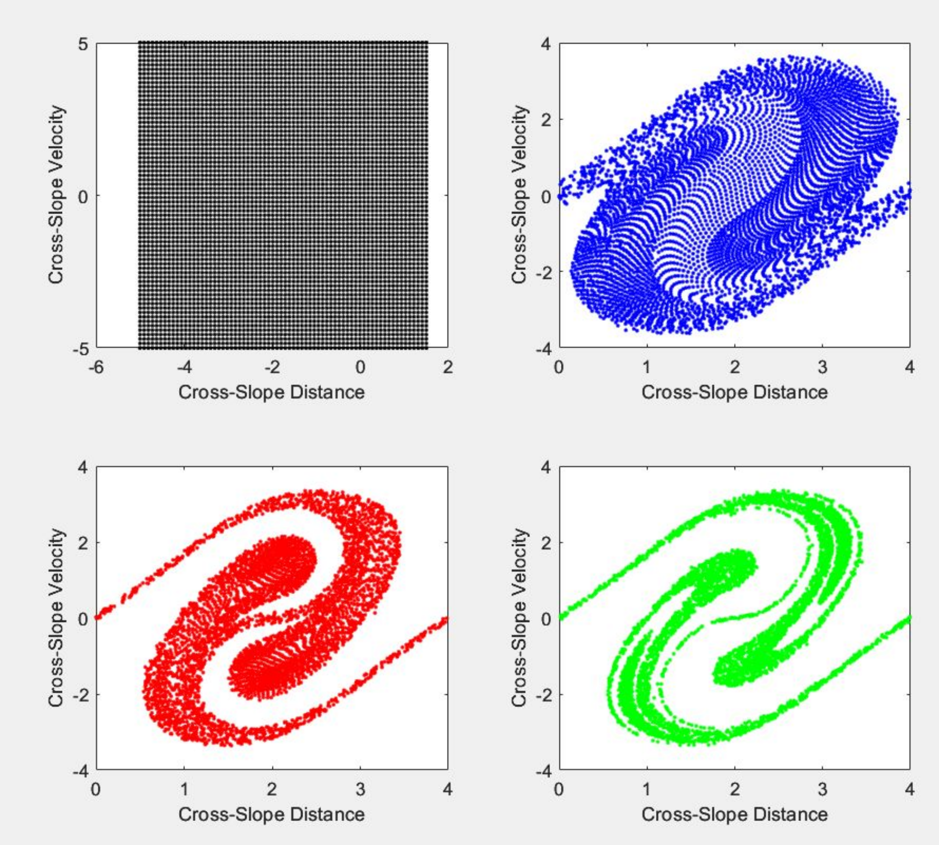
Conservative System Growth



PART 6 - Properties of the System: The Attractor

The System Exhibits the property of a strange attractor.

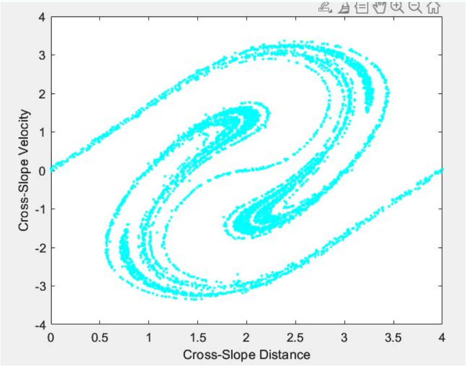
- Y, V tend to an attractor for relatively equal values of X, (same x coordinate on each mogul)
- Each graph is taken at another 5 meter increment of x (size of the moguls).
- For different chosen values of X, attractor takes on different shape

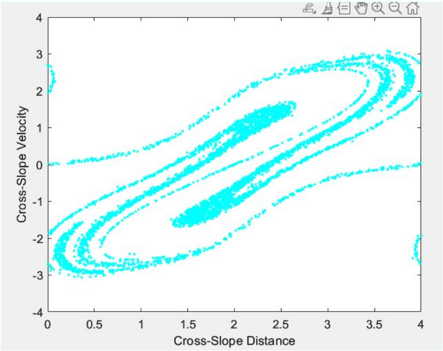


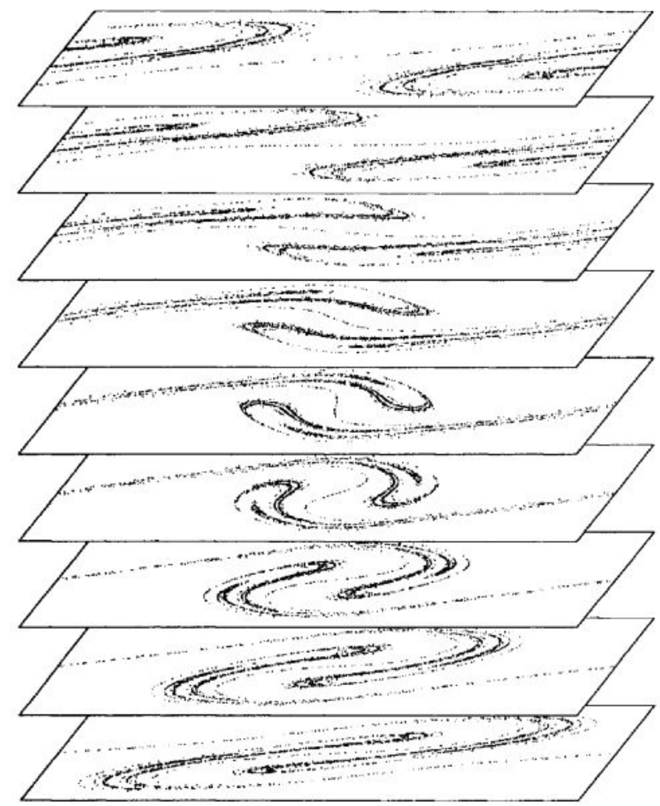
PART 6 - Properties of the System: The Attractor

The cross sections to the right are different cross sections of the attractor as it is a 3-dimensional attractor for a 4-variable system.

- A different value of X produce a new cross section of the attractor! (left x = 2.5m, right x = 3.5m)

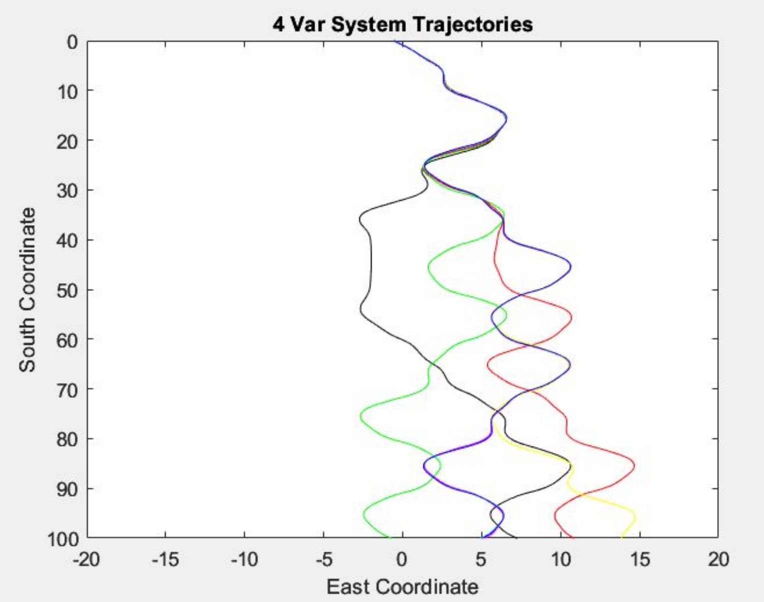




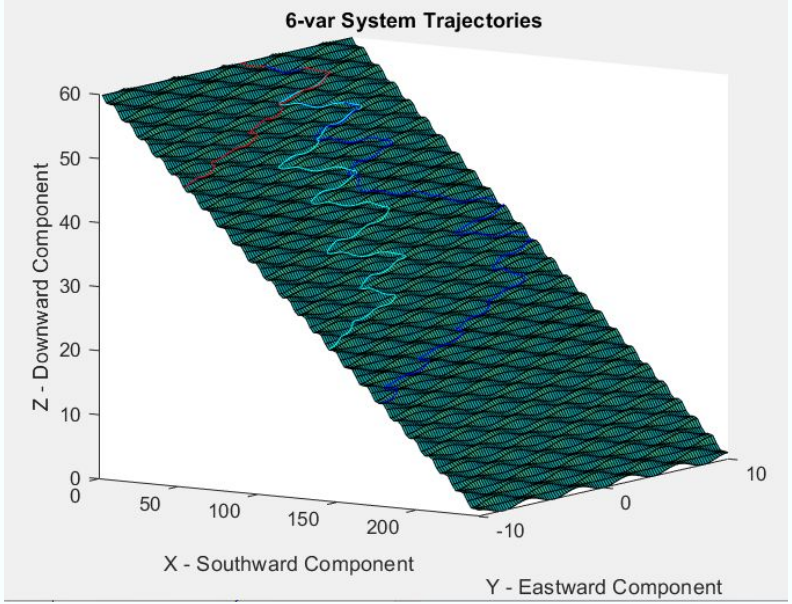


PART 7 - Numerical Experiments

These numerical integrations show several trajectories which start within 1 cm of each other and grow apart for both the 4-var and 6-var systems

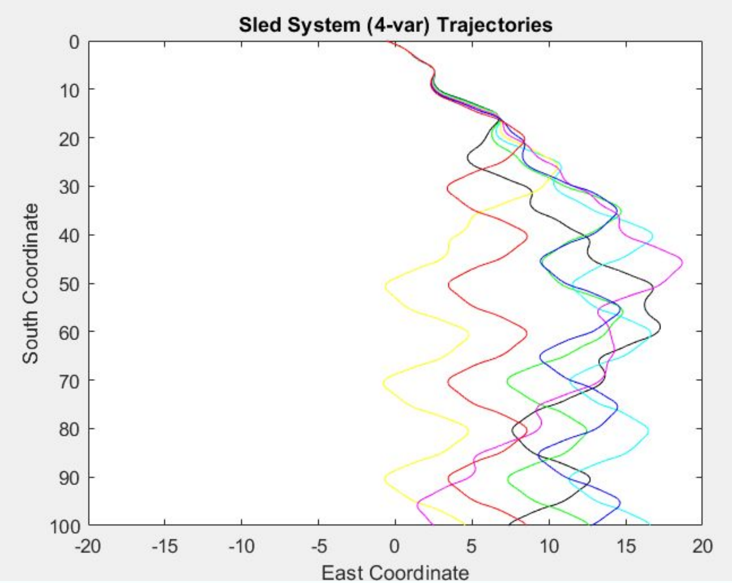


Parameters:
a = .25
b = .5
c = .5
p = pi/5
q = pi/2

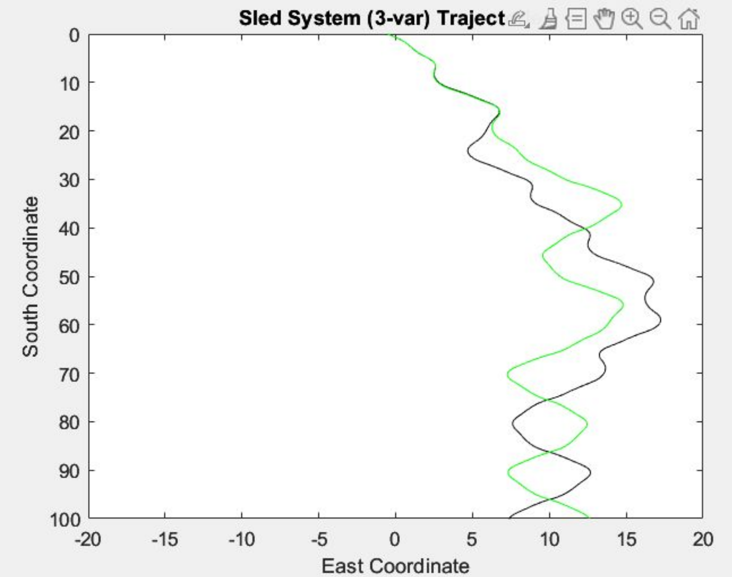


PART 7 - Numerical Experiments

These numerical integrations show several trajectories which start within 1 cm of each other and grow apart for both the 4-var and 6-var systems

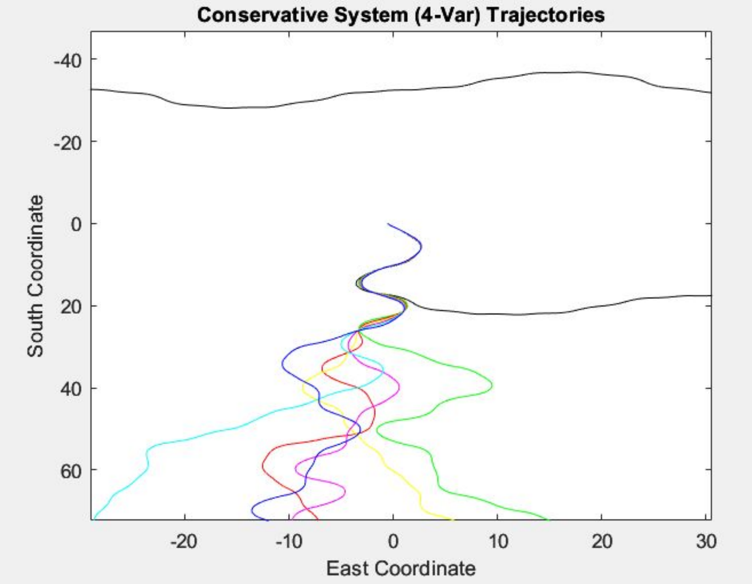


Parameters:
a = .25
b = .5
c = .5
p = pi/5
q = pi/2



PART 7 - Numerical Experiments

This numerical integrations show several trajectories which start within 1 cm of each other and grow apart for the conservative system

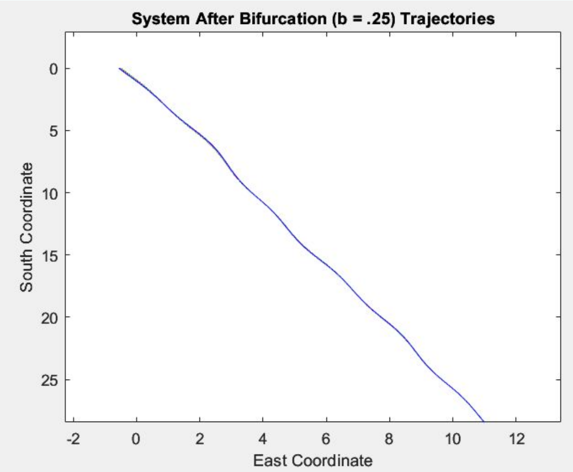


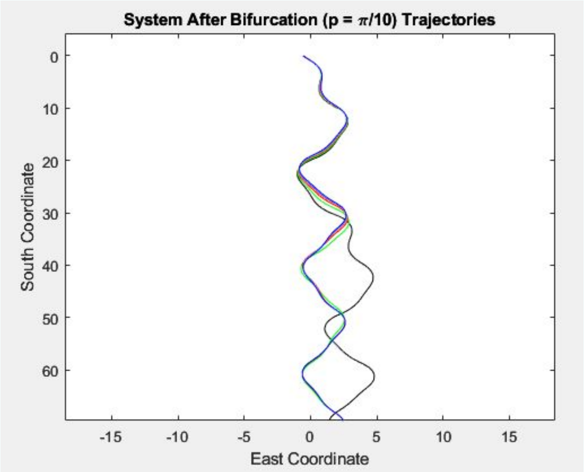
Cons. Sys:
a = c = 0
b = .5
p = pi/5
q = pi/2

- Trajectories can reverse direction!

Bifurcation in the System

As shown before there is a bifurcation relating to the variables a, b, p. We can demonstrate the bifurcation through a manipulation of either variable

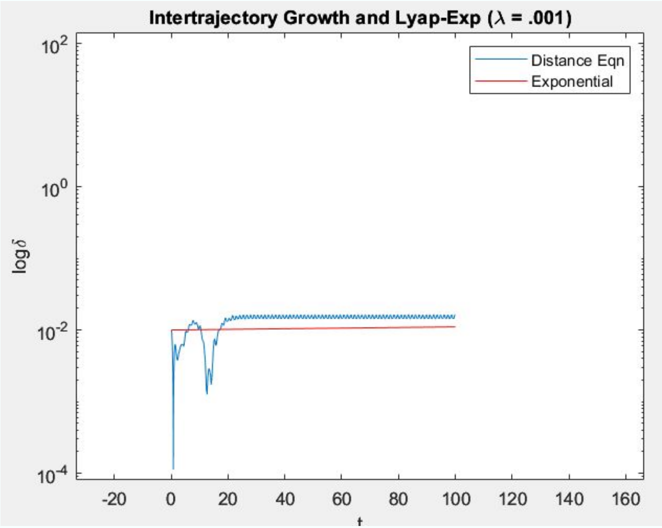




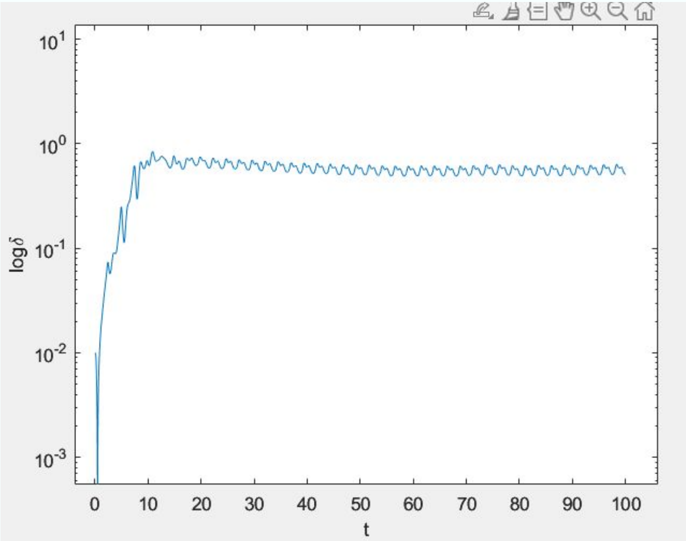
Bifurcation in the System

Of the two graphs below, the left shows the growth between adjacent starting trajectories in the bifurcations. **Both fail to grow a distance of more than a meter!**

B = .25



P = pi/10



Snowboarding on Different Planets! ❄
(and the moon 🌕)

Gravitational Constants

Moon 0.17 g ↔ 1.66 m/s^2

Mercury 0.38 g ↔ 3.724 m/s^2

Venus 0.9 g ↔ 8.820 m/s^2

Mars 0.38 g ↔ 3.724 m/s^2

Jupiter 2.53 g ↔ 24.794 m/s^2

Saturn 1.07g ↔ 10.486 m/s^2

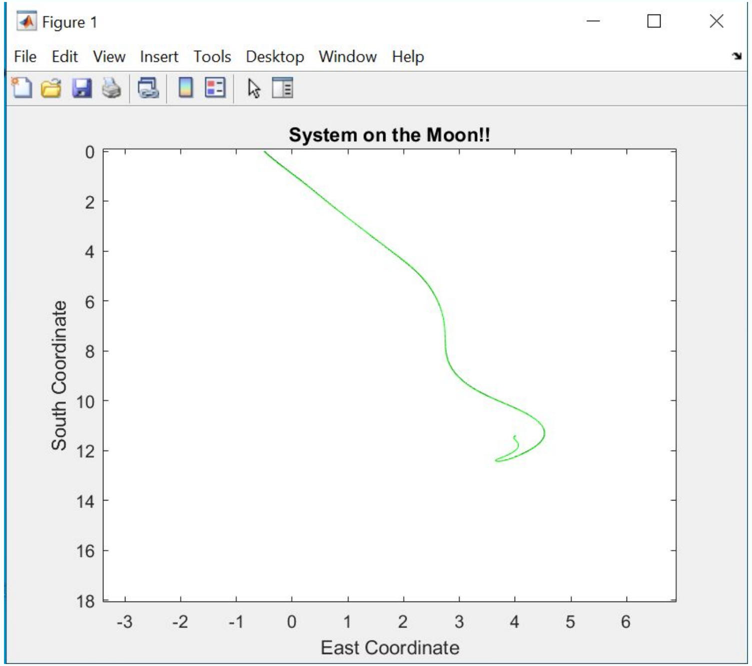
Uranus 0.89 g ↔ 8.722 m/s^2

Neptune 1.14 g ↔ 11.172 m/s^2

What happens when we travel to different planets?
(suspension of disbelief required)

Snowboarding on the Moon

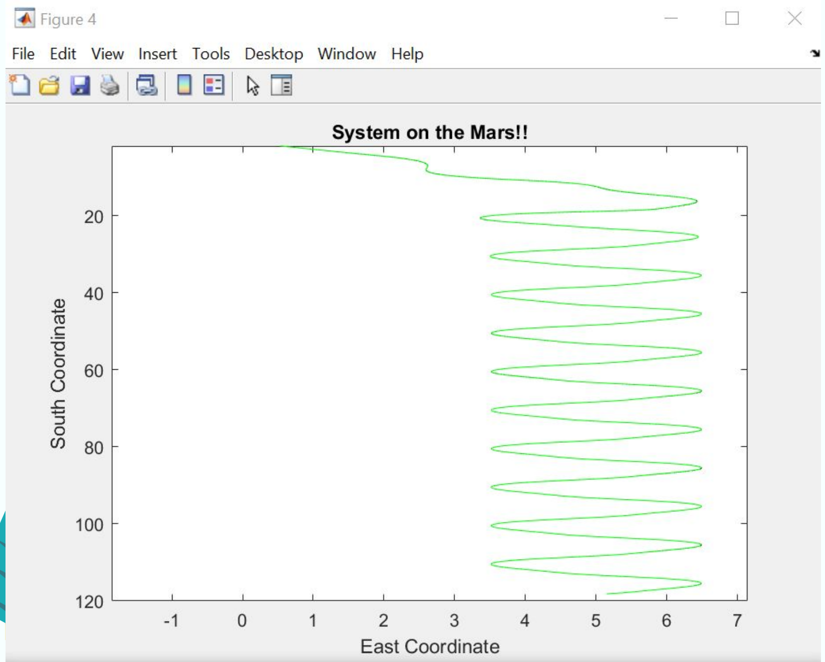
Moon 0.17 g $\leftrightarrow$ 1.66 m/s^2

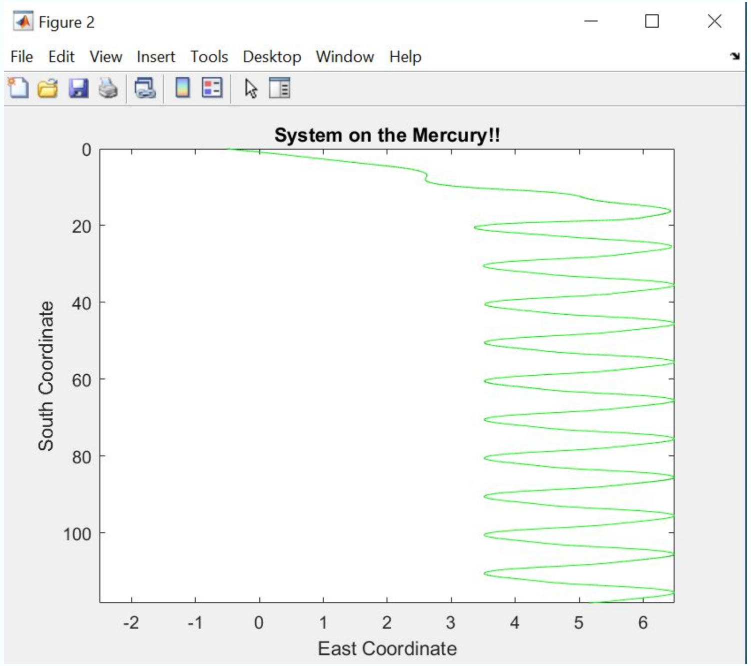


Snowboarding on Mercury, Mars

Mars 0.38 g ↔ 3.724 m/s^2

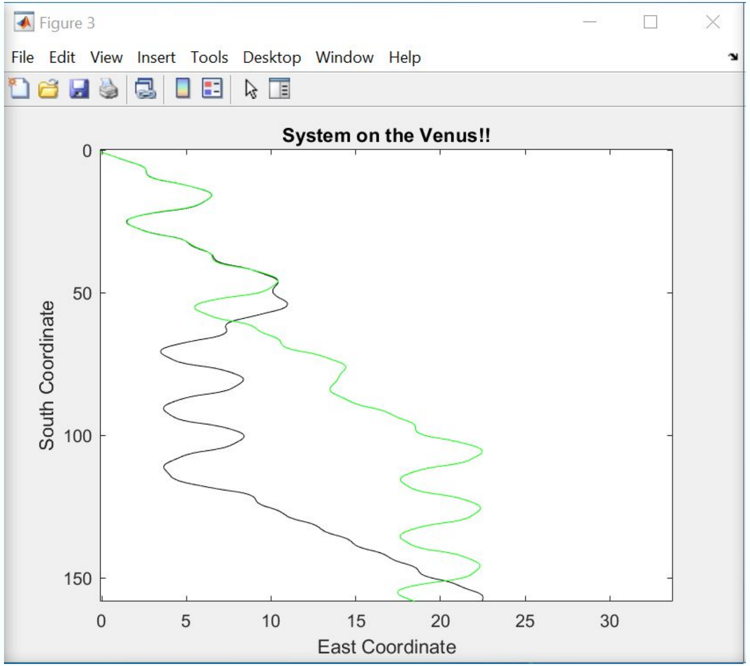
Mercury 0.38 g ↔ 3.724 m/s^2





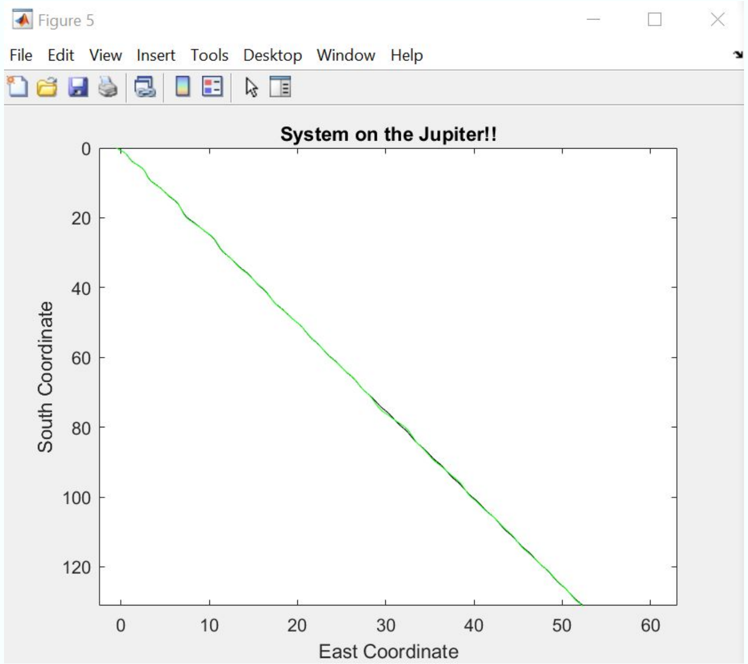
Snowboarding on the Venus

Venus 0.9 g $\leftrightarrow$ 8.820 m/s^2



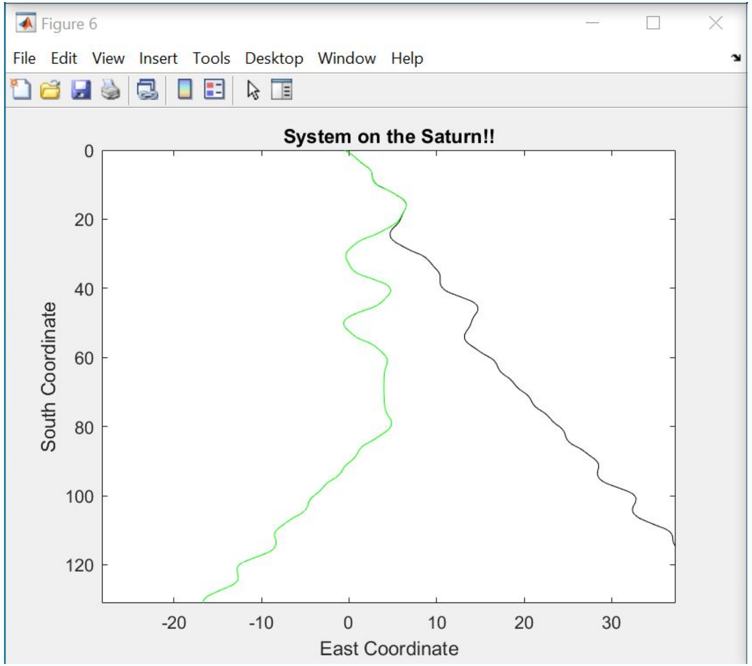
Snowboarding on the Jupiter

Jupiter 2.53 g ↔ 24.794 m/s^2

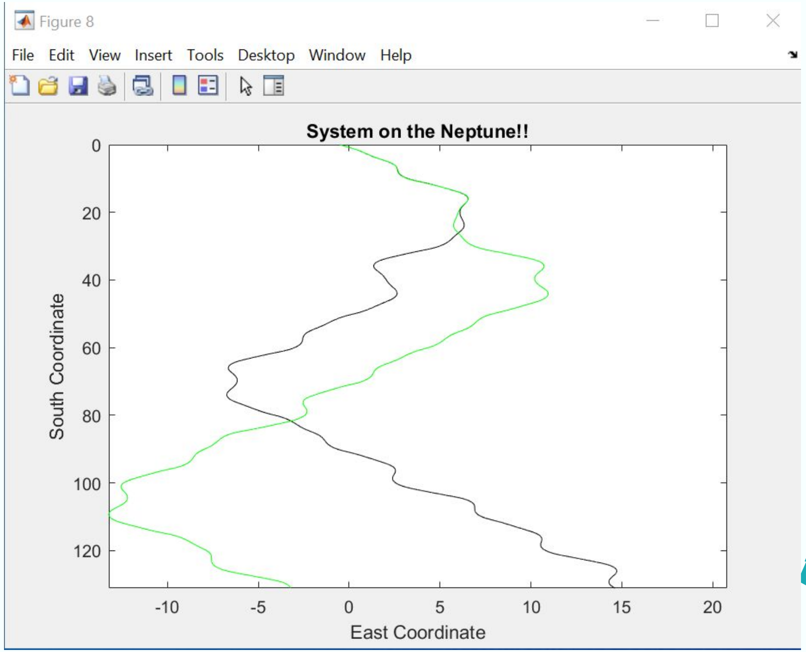


Snowboarding on Saturn and Neptune

Saturn 1.07g ↔ 10.486 m/s²

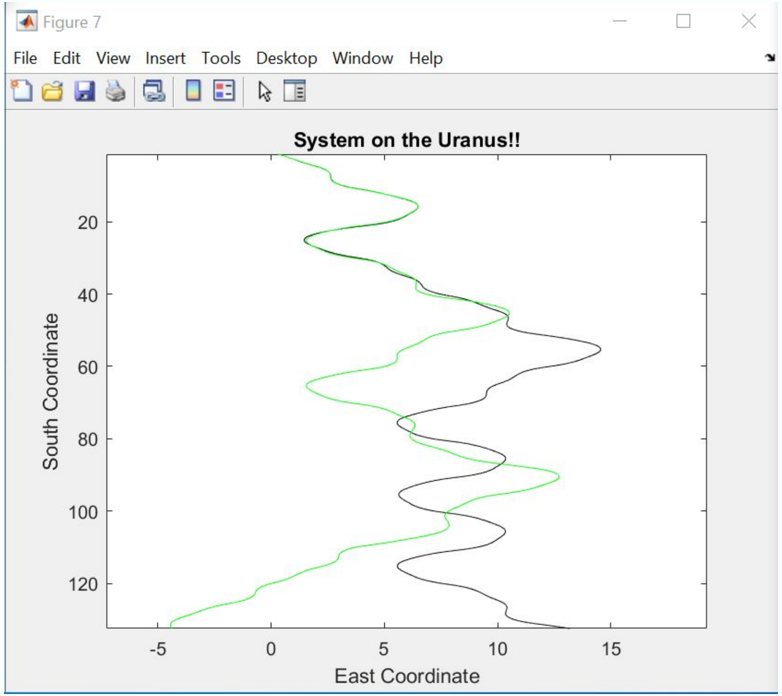


Neptune 1.14 g ↔ 11.172 m/s²



Snowboarding on the Uranus

Uranus 0.89 g ↔ 8.722 m/s²



Conclusion