Boxes and Balls: Quantifying Chaos

On the Dimension of Chaotic Attractors

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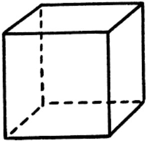
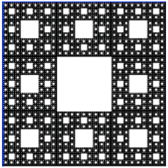
Why care about Dimension?

- Fractal Structures have non-integer dimension!! (What?)
- Dimension is a measure of the “roughness” of micro-structure.
- While studying Chaos, we find objects with elaborate micro-structure!!

Consider a shape that is almost perfectly flat, but has a near-infinitesimal thickness. What dimension would you assign it?



Cube - $D = 3$



Sierpinski Carpet

$D_C = \frac{\ln(8)}{\ln(3)} \approx 1.8928.$

Crash Course Dynamical Systems

- Dynamical Systems are systems of 1st order coupled differential equations.
- Fixed points are where the time derivatives are Zero.
- Jacobian (Trace-Determinant) analysis reveals information about fixed points.
- Distinct trajectories cannot intersect (except at fixed points).
- Parameters can cause bifurcations!

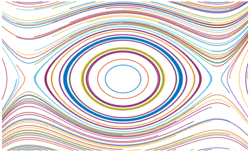
$$\dot{x}_1 = f_1(x_1, \cdots, x_n)$$

$$\dot{x}_n = f_n(x_1, \cdots, x_n)$$

General Form: Continuous, Autonomous, N-Dim

Example System:

$$\dot{x} = y$$
$$\dot{y} = -x - \epsilon x^3$$
$$\epsilon = -1$$



Trace-Determinant Analysis

- The eigenvalues and eigenvectors of the Jacobian linearization can be found using equations involving the Trace and Determinant.
- The eigenvalues give the exponetial behavior along its corresponding eigenvector.
- Eigenvalues can have imaginary values, creating spirals!!

$$\lambda^2 - T\lambda + Det = 0$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4Det}}{2}$$

This equation is used to find the eigenvalues of a 2D matrix.

$$\mathcal{J}(\vec{x_{FP}}) \cdot \xi = \lambda \xi$$

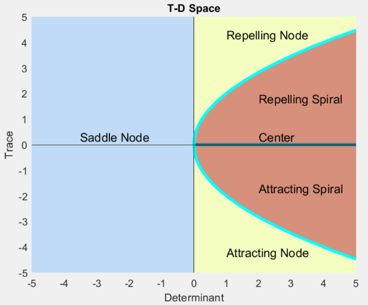
$$\mathcal{J}(\vec{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Jacobian Matrix of a Dynamical System

Eigenvalue λ_i operates in the direction of Eigenvector, ξ_i.

Regions of T-D Space

- For the 2D case!



A plot of T-D Space with $T = \pm 2\sqrt{Det}$ shown.

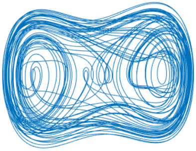
Saddle Node
$Det < 0 \rightarrow \lambda_1 > 0, \lambda_2 < 0$

Attracting or Repelling node
$\lvert T \rvert > 2\sqrt{Det}$
$\lambda_1, \lambda_2 > 0$, or $\lambda_1, \lambda_2 > 0$

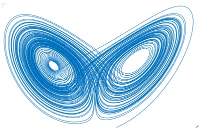
Attracting or Repelling Spiral
$\lvert T \rvert < 2\sqrt{Det}$
$Imag(\lambda_1) \neq 0, Imag(\lambda_2) \neq 0$

What are Chaotic Attractors?

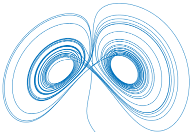
- In 3D+ Continuous Dyn. systems, chaos is possible!
- Dyn. Sys. can yield globally attracting geometrical objects, forcing infinite trajectories to converge to finite space.
- These “Attractors” have fine fractal-like structure
- Dimension measures the “smoothness” of a chaotic object!



Forced Double Well Oscillator



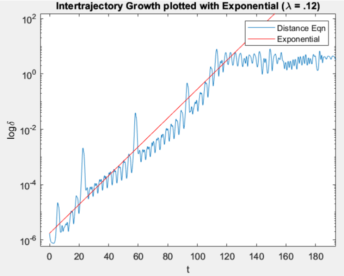
Lorenz Butterfly



Rikitake Attractor

Characteristics of Chaos

- Criteria 4 Chaos: Bounded, Deterministic, Aperiodic, Sensitive Dependence on IC.
- Lyapunov Exponents determine growth between adjacent trajectories.
- Usually the largest lyapunov exponent dominates the growth!



Intertrajectory Lyapunov Growth

On to the (Many) Fractal Dimensions!

- The Box-Counting / Capacity Dimension
  - The Hausdorff Dimension
- Natural Measure Dimensions
  - The Renyi Dimension
  - The Information / Correlation Dimension
  - The Pointwise Dimension
- The Lyapunov Dimension
- No really, there are a lot :)

The Box-Counting / Capacity Dimension, D_C

- Counts number of n-dimensional cubes of side length, ϵ, needed to cover an object in n-dimensional space.
- This relationship is analogous to an Power Law relation.

$$D_C = \frac{\ln(N(\epsilon))}{\ln(\frac{1}{\epsilon})}$$

N(ϵ) is number of "boxes" needed to cover the object.

$$N(\epsilon) \sim \epsilon^{-D_C}$$

The Power Law Relationship

The Hausdorff Dimension, $D_{\mathcal{H}}$

- Covers a set with n-dimensional spheres/balls.
- Very difficult to numerically calculate.
- Paper written by Ronnie Mainieri, "On the equality of Hausdorff and box counting dimensions".

$$\mathrm{l}_d(\epsilon) = \inf \sum_i \epsilon_i^d$$

$$\epsilon_i < \epsilon$$

$$\mathrm{l}_d = \lim_{\epsilon \to 0} \mathrm{l}_d(\epsilon)$$

Hausdorff proved there exists a value, d, such that below which, the limit approaches 0 and above which, approaches ∞. This value is $d = D_{\mathcal{H}}$.

The Natural Measure, $\mu(C)$

- The Natural Measure is a probability/fraction of what percentage of an object is in an area.

The natural measure of a cube, C, with initial condition, $\vec{x_0}$.

$$\mu(C, \vec{x_0}) = \lim_{t \to \infty} \mu(C, \vec{x_0}, t)$$

The Natural measure is given by the fraction:

$$\mu(C, \vec{x_0}) = \frac{\text{\# of Points in C}}{\text{Total \# of Points}}$$

The Renyi Dimension, D_q

- Counts the measure of the n-dim cubes, rather than the number needed!
- Parameter, q, is read as the "Order" of Renyi Entropy.
- For $q = 0$, $D_0 = D_C$

The q^{th}-order Renyi Dimension

$$D_q = \lim_{\epsilon \to 0} \left(\frac{1}{1-q} \cdot \frac{\ln(\sum_j [\mu(C_j)]^q)}{\ln(1/\epsilon)} \right)$$

Generally, the below relationship holds:

for $q_1 > q_2$

we have $D_{q_1} < D_{q_2}$

The Information and Correlation Dimension, $D_{\mathcal{I}}, D_{\mathcal{C}}$

- Special Cases of the Renyi Dimension.
- Information Dimension is obtained by the limit as $q \to 1$.
- The Correlation Dimension is obtained with q = 2.

Information Dimension Equation	Correlation Dimension Equation
$D_{\mathcal{I}} = D_1 = \lim_{\epsilon \to 0} \frac{\mathcal{I}(\epsilon)}{\ln(\frac{1}{\epsilon})}$	$D_{\mathcal{C}} = D_2 = \lim_{\epsilon \to 0} \frac{\ln(S(\epsilon))}{\ln(\epsilon)}$
$\mathcal{I}(\epsilon) = \sum_{i=1}^{N(\epsilon)} \mu(C) \ln(\mu(C))$	$S(\epsilon) = u(\epsilon - \lvert x_i - x_j \rvert)$ $u(y) = 1, y > 0$ $u(y) = 0, y < 0$
$\mathcal{I}(\epsilon)$ is also known as Shannon Entropy.	u is the unit-step function

The Pointwise Dimension

- Counts the natural measure covered by an n-dimensional sphere of radius, ϵ, as $\epsilon \to 0$.

The Pointwise Dimension

$$D_{\mathcal{P}}(x) = \lim_{\epsilon \to 0} \frac{\ln(\mu(B_\epsilon(x)))}{\ln(\epsilon)}$$

If $D_{\mathcal{P}}(x)$ is independent of x such that the measure, $\mu \neq 0$, then we call $D_{\mathcal{P}}(x) = D_{\mathcal{P}}$

The Lyapunov Dimension, D_L

- Uses the Lyapunov Numbers or Exponents to calculate the dimension
- Derived from Box-Counting Dimension!
- Doesn't rely on limits!

The Lyapunov Dimension

$$D_L = k + \frac{\lambda_1 + \cdots + \lambda_k}{|\lambda_{k+1}|}$$

$$k \text{ such that, } \sum_{i=1}^{k} \lambda_i > 0$$

Where λ_i indicates the i^{th} Lyapunov Exponent.

Also **The Lyapunov Dimension**

$$D_L = k + \frac{\ln(\Lambda_1 \cdots \Lambda_k)}{|\ln(\Lambda_{k+1})|}$$

$$\prod_{i=1}^{k} \Lambda_i > 1$$

$$\Lambda_i = e^{\lambda_i}$$

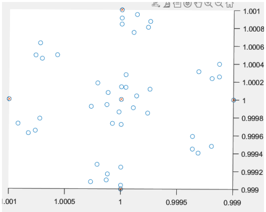
Λ_i is the i^{th} Lyapunov Number.

Finding the Lyapunov Numbers/Exponents

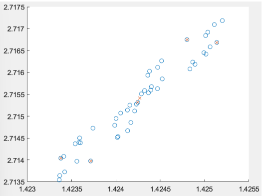
Strogatz describes the process as such, "Consider the evolution of an infinitesimal sphere of perturbed initial conditions. During its evolution, the sphere will become distorted into an infinitesimal ellipsoid. Let $\delta_k(t)$ be the k^{th} principal axis of the ellipsoid. Then,

$$\delta_k(t) \backsim \delta_k(0)e^{\lambda_k t}\text{ ''}.$$

$$\epsilon_k(t) = \epsilon e^{\lambda_k t}$$



Sphere of Initial Conditions



Perturbed Ellipsoid

The Variational Equation

- Extends a n-dim System into an $(n+n^2)$-dim system using the Jacobian Matrix.
- Varitaions can be orthognormalized to find the principle axes of a perturbed ellipse.

$$\begin{pmatrix} \dot{\delta}_{xx} & \dot{\delta}_{yx} & \dot{\delta}_{zx} \\ \dot{\delta}_{xy} & \dot{\delta}_{yy} & \dot{\delta}_{zy} \\ \dot{\delta}_{xz} & \dot{\delta}_{yz} & \dot{\delta}_{zz} \end{pmatrix} = \mathcal{J}(\vec{x}) \begin{pmatrix} \delta_{xx} & \delta_{yx} & \delta_{zx} \\ \delta_{xy} & \delta_{yy} & \delta_{zy} \\ \delta_{xz} & \delta_{yz} & \delta_{zz} \end{pmatrix}$$

The new dynamical system extended by the evolution of the variations, δ_{ij}.

$$\begin{aligned} \dot{x} &= \cdots \\ \dot{y} &= \cdots \\ \dot{z} &= \cdots \\ \dot{\delta}_{xx} &= \cdots \\ &\vdots \\ \dot{\delta}_{zz} &= \cdots \end{aligned}$$

Numerical Calculations

- We can program an algorithm to numerically compute the dimension of a chaotic attractor for multiple dimension definitions.
- We will use MatLab to write the code for our computations.

```
function CapDim(func, ei, estep, tspan, trashTime, IC)

[~, y] = ode45(func, tspan, IC);

figure(1)
tidx = trashTime:size(tspan, 2);
plot3(y(tidx,1), y(tidx,2), y(tidx,3))
xlabel('X')
ylabel('Y')
zlabel('Z')
title('Attractor Phase Portrait from IC')

Mspan = [];
minY = [];
for i = 1:3
    Mspan(i) = -min(y(tidx, i)) + max(y(tidx, i));
    minY(i) = min(y(tidx,i));
end
```

Numerical Calculation of Box-Dimension

- Numerically simulate the Dynamical System
- Define the ends of a grid using the max and min of X, Y, Z
- Choose an ϵ, partition the grid.
- Count the number of cubes, $N(\epsilon)$.
- Repeat for smaller ϵ, and graph $\ln(N(\epsilon))$ against $\ln(\frac{1}{\epsilon})$
- Find a slope of best fit, this approximates dimension by the power law relation!

Numerical Calculation of Inf./Renyi Dimension

- The same algorithm used for box-counting method is used here. Instead of counting a box, we count the natural measure
- Our results should be less than our calculations for the Box-Counting Dimension!

Numerical Calculation of Pointwise Dimension

- Generates the attractor. Then by using the mean and the midpoint of each coordinate axis, the attractor is centered about the origin.
- The X, Y, and Z coordinates are used to find a radius from the origin.
- The natural measure is taken by how many points are in a specific radius.

```
%Converting to Spherical Coordinates

SY = sqrt(y(:, 1).^2 + y(:, 2).^2 + y(:, 3).^2);
n = size(tspan, 2);
MU = zeros(size(nstep, 1)+1, 1);
E = zeros(size(nstep, 1)+1, 1);
e  = max(SY);
fact = 1/efact;

%Searching for Points in radius e
for i = 1:nstep
    [fx, ~] = find(SY <= e);
    mu = size(fx, 1)/n;
```

Numerical Calculation of Lyapunov Dimension

- Runs variational version of the Dyn. Sys.
- After fixed, Δt, the variations are orthonormalized and given back to the ODE integrator.
- Factors used in normalization approximate Lyapunov Exponents

```
%Creating Values for our two data points
for n = 1:nstep
    [~, y] = ode45(@rosslerV, [t t+tstep], IC);
    delta = y(:, 4:12);

    %Declaring our delta matrix for this iteration
    deltai = [delta(end,[1 4 7]); delta(end,[2 5 8]); delta(end,[3 6 9])];

    %Normalizing our X vector (all normalizing factors will be known as NF
    NF(1) = sqrt(sum(deltai(:, 1).^2));
    deltai(:, 1) = deltai(:, 1)/NF(1);

    %Calculating the Projection factor of X on Y
    Pfxy = sum(deltai(:, 1).*deltai(:, 2));

    %Subtracting the Projection of X from Y
    deltai(:, 2) = deltai(:, 2) - Pfxy*deltai(:, 1);

    %Normalizing Y
    NF(2) = sqrt(sum(deltai(:, 2).^2));
    deltai(:, 2) = deltai(:, 2)/NF(2);
```

We are done explaining things!!!! (That felt like forever)

- Please help yourself de-stress by looking at all of my pretty pictures and numbers to show you

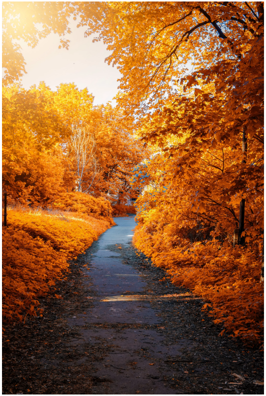
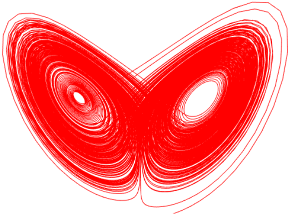


Figure 1: Pretty Picture

Figure 2: Pretty Numbers
3.14159265358979323 8...
2.718281828459045...
4.6692016091...

The Lorenz Attractor

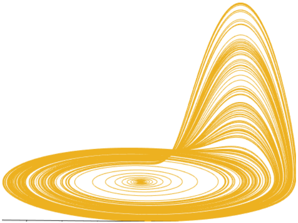


Phase Portrait

Dimensions

Capacity:	2.0303
Information:	2.0850
Pointwise:	1.9958
Lyapunov:	2.0563

The Rossler Attractor

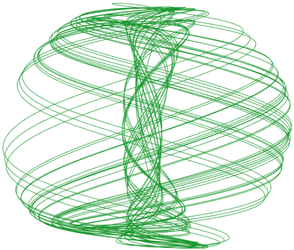


Phase Portrait

Dimensions

Capacity:	1.8806
Information:	1.8493
Pointwise:	2.4609
Lyapunov:	2.0071

The Aizawa Attractor

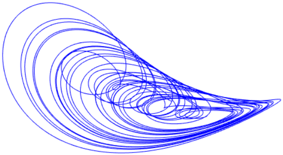


Phase Portrait

Dimensions

Capacity:	2.5628
Information:	2.4750
Pointwise:	3.6180
Lyapunov:	2.8389

The Sprott Linz F Attractor

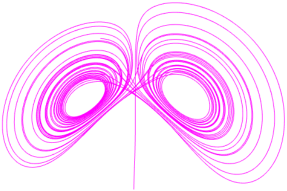


Phase Portrait

Dimensions

Capacity:	2.0554
Information:	1.7607
Pointwise:	2.624
Lyapunov:	2.2235

The Rikitake Attractor



Phase Portrait

Dimensions

Capacity:	1.9892
Information:	1.5236
Pointwise:	1.7055
Lyapunov:	2.0568

Here are some other sources where you can find some cool chaotic systems!!

Scan ME!!

NO, Scan MEEEE

Thank you!!

- A huge thanks to my mentor, John G. Pelias!!!!!
- A second thanks to DRP for this opportunity!!!!
- A third thanks to each of you for listening!!!
- A final thanks to all of my sources!!
- A post-final thanks to math for being interesting!

Bibliography

"Attractor Dimension." *Scholarpedia*, www.scholarpedia.org/article/Attractor_dimension. Accessed 10 Mar. 2023.

Bradley, Elizabeth. "The Variational Equation." *CSCI 4446/5446 Course materials*, U of Colorado, home.cs.colorado.edu/~lizb/chaos/variational-notes.pdf. Accessed 13 Mar. 2023.

Danforth, Chris. "Lecture 21: Numerical Calculation of Lyapunov Exponents." *Youtube*, 6 Apr. 2017, www.youtube.com/watch?v=YL4twBVKNK0&ab_channel=Chaos%2CFractals%2C%26DynamicalSystems. Accessed 9 Mar. 2023. Speech.

Farmer, J. Doyne, et al. "The Dimension of Chaotic Attractors." *Physica D: Nonlinear Phenomenon*, vol. 7, nos. 1-3, May 1983, pp. 153-80.

Franceschini, Valter, et al. "Characterization of the Lorenz Attractor by unstable Periodic Orbits." *Nonlinearity*, vol. 6, 1 Jan. 1993, pp. 251-58.

Govorukhin, Vasiliy. *Calculation Lyapunov Exponents for ODE*. Version 1.0.0.0. *MathWorks*, 18 Mar. 2004, www.mathworks.com/matlabcentral/fileexchange/4628-calculation-lyapunov-exponents-for-ode.

Gratrix, Sam, and John Elgin. "Pointwise Dimension of the Lorenz Attractor." *Physical Review Letters*, vol. 92, nos. 1-9, 8 Jan. 2004, https://doi.org/10.1103/PhysRevLett.92.014101. Accessed 13 Mar. 2023.

Hsu, Guan-Hsong, et al. "Strange Saddles and the Dimension of their Invariant Manifolds." *Physical Letters A*, vol. 127, no. 4, 22 Feb. 1988, pp. 199-204.

Karimi, A., and M. R. Paul. "Extensive Chaos in the Lorenz-96 Model." *Chaos*, vol. 20, no. 043105, Sept. 2010.

Mainieri, Ronnie. "On the equality of Hausdorff and box counting dimensions." *Chaos*, vol. 3, no. 119, Mar. 1993, pp. 119-25.

McGuinness, Mark J. "The Fractal Dimension of the Lorenz Attractor." *Physics Letters A*, vol. 99, no. 1, Nov. 1983, pp. 5-9.

Sandri, Marco. "Numerical Calculation of Lyapunov Exponents." *The Mathematica Journal*, vol. 6, no. 3, 1996, pp. 78-84.

Viswanath, Divikar. "The Fractal Property of the Lorenz Attractor." *Physica D: Nonlinear Phenomena*, vol. 190, nos. 1-2, Mar. 2004, pp. 115-28.

Last Source

I forgot to include this very important source!

Strogatz, Stephan. *Nonlinear Dynamics and Chaos*. 2nd ed., CRC Press, 2015.