1. Counting

a)

For each of the questions, there are $\binom{6}{1}$, or 6, possible results, since only one answer can be chosen in each question and the set of answers must be complete, meaning one answer must be selected. With two questions, there are a total of 6^2 sets of answers, since, for each answer to the first question, there are 6 possible answers to the second question. For a third question, there would be another 6 sets of answers for each set of two questions, and then again with the fourth question, resulting in a total of 6^4 ways to choose a complete set of answers.

b)

For the first question, there are 6 possible answers that can be chosen. However, unlike part (a), only 5 answers can be chosen for the second question, since the answer cannot be the same as the answer from the first question. The third question would only have 4 remaining options, and the fourth question would have 3 remaining options, resulting in a total of $6 * 5 * 4 * 3$, or 360 complete sets of answers.

c)

When selecting the first bead on the bracelet, you have 13 possible options. Then, you have 12 remaining options for the second bead, giving a total of $13 * 12$ possible combinations of these two beads. The number of beads you can select from decreases by one each time you add a new bead, until you are left with only 1 option for the final bead. The total number of possible arrangements for all the beads can be represented as $13 * 12 * 11 * ... * 1$, or simply 13!. However, this is only true if the beads are arranged in a straight line. When arranged in a circle, the "starting bead" is irrelevant, since there is no start or end. This would mean that, for each combination in a straight line, there are another 12 combinations that would be considered the same if they were on a bracelet. This can be shown by dividing the straight line result by 13, giving us $\frac{13!}{13}$, or 12!. There is also the fact that flipping the bracelet does not change the arrangement. For each of the current number of arrangements, there is a flipped version that should be considered to be the same. This can be shown by dividing by 2, giving us a final answer of $\frac{12!}{2}$ possible arrangements

d)

For each of the 6 flavors, Megan can select 1 of 10 toppings for the first topping, giving $6 * 10$ choices. Because she wants two different toppings, she can only select 1 of 9 toppings for the second topping, giving her $10*9$ choices for two toppings. But, because the order of the two toppings doesn't matter, this is divided by 2, giving $\frac{10*9}{2}$, which could instead be represented by $\binom{10}{2}$. This gives Megan a total of $6 * \binom{10}{2}$, or 270 possible choices.

e)

Because the books "Anna Karenina" and "O Menino Maluquinho" must stay together, we can consider them as the same object, but then multiply our answer by 2, since they can be switched. For n books, there would be $n!$ ways to arrange the books on a shelf, since the order of the books matters. Because we are treating "Anna Karenina" and "O Menino Maluquinho" as one object, we would say there are $(n-1)!$ arrangements. But, this would be multiplied by 2 to account for the order of "Anna Karenina" and "O Menino Maluquinho", giving a final result of $2((n-1)!)$ possible arrangements.

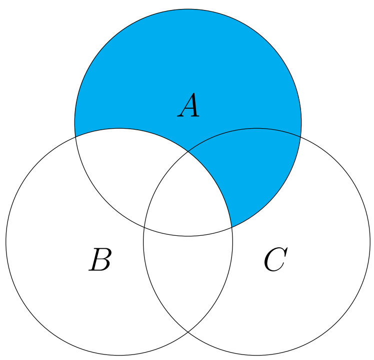
2. Sets

a)

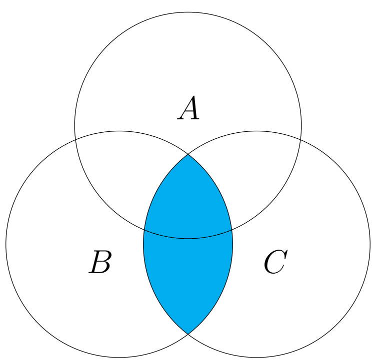
The number of sets that satisfy $S \subseteq \{a,b,c,d,e,f\}$ would be 2^6, since there are 6 elements in the set $\{a,b,c,d,e,f\}$. However, S must contain $\{a,b\}$, because $\{a,b\} \subseteq S$. To account for this, we can effectively remove $\{a,b\}$ from $\{a,b,c,d,e,f\}$ when finding the number of possible sets, since $\{c,d,e,f\}$ are the only items that are optional in S. Therefore, the number of possible sets that satisfy $\{a,b\}S \subseteq \{a,b,c,d,e,f\}$ is 2^4.

b)

First, we can do $A \setminus B$ to get the following:



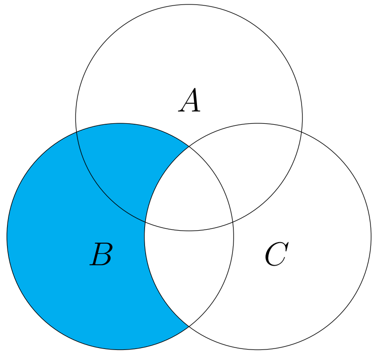
Next, we can do $B \cap C$ to get the following:



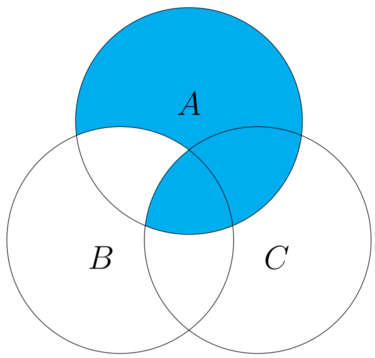
The union between these two, or $(A \setminus B) \cup (B \cap C)$ gives us the final result.

c)

The Venn diagram for $B \setminus C$ would be the following:



Subtracting this from A gives us the final result for $A \setminus (B \setminus C)$:



d)

To find the total number of paths, ignoring A and B, we can either do $\binom{18}{10}$ or $\binom{18}{8}$, which are equivalent to each other. This can be found by saying "how many ways are there to choose the 10 steps to the right from the 18 total steps", or "how many ways are there to choose the 8 steps up from the 18 total steps". We can then find the number of paths that go through A and/or B and subtract that from the total. To do this, we will first find the number of paths that go through just A. To get from the start to A takes 4 steps right and 1 up, or 5 total, giving us $\binom{5}{4}$ or $\binom{5}{1}$ paths to A. To get from A to the finish takes 6 steps right and 7 steps up, or 13 total, giving us $\binom{13}{6}$ or $\binom{13}{7}$ paths. Multiplying these will give us the total number of complete paths that go through A, $5 * \binom{13}{6}$. Now we can do the same with B. There are 7 steps right and 5 up, a total of 12, to get to B, giving us $\binom{12}{5}$ or $\binom{12}{7}$ paths. From B to the end there are 3 steps right and 3 up, a total of 6, giving us $\binom{6}{3}$ paths. Together, that's $\binom{12}{5} * \binom{6}{3}$ paths through B. However, adding these two would result in the paths that travel through both A and B to be counted twice. To fix this, we can simply subtract that amount. We know there are 5 paths from the start to A. A to B is 3 right 4 up, a total of 7, giving us $\binom{7}{3}$ or $\binom{7}{4}$ paths. We also know there are $\binom{6}{6}$ paths from B to the end. This results in $5 * \binom{7}{3} * \binom{6}{3}$ paths through both A and B. This means the number of paths that go through A and or B is $5 * \binom{13}{6} + \binom{12}{5} * \binom{6}{3} - 5 * \binom{7}{3} * \binom{6}{3}$. Finally, we subtract this from the total number of paths, giving the final result of $\binom{18}{8} - (5 * \binom{13}{6} + \binom{12}{5} * \binom{6}{3} - 5 * \binom{7}{3} * \binom{6}{3})$.

3. Proof Techniques

a)

Hypothesis: For all $n > 2$, $1 + 2 + ... + n$ is composite.
Our final value can either be represented as $1 + 2 + ...n$, or in reverse as $n + (n-1) + (n-2)... + 2 + 1$, because addition is commutative. If we add these two representations together by each value, we get $(n+1) + (n+1) + ...(n+1)$, with there being n values. Dividing this by 2 gives us a formula for the final result: $\frac{n(n+1)}{2}$.

Case 1: n is an even number:

> Because n is even, it can be divided by 2 and remain an integer. If we let a value $i = \frac{n}{2}$, $2i = n$, and the formula can be rewritten as $\frac{2i(2i+1)}{2}$, which simplifies to $i(2i+1)$. Substituting n back in gives $\frac{n}{2}(n+1)$. This shows that the final value has at least one divisor ($\frac{n}{2}$ and $(n+1)$) that is not itself or 1, meaning it is composite.

Case 2: n is an odd number:

> Because n is odd, $n-1$ must be even and can therefore be divided by 2. If we let a value $i = \frac{n-1}{2}, n = 2i+1$, and the formula can be rewritten as $\frac{(2i+1)(2i+2)}{2}$, which simplifies to $(2i+1)(i+1)$. Substituting n back in gives $n(\frac{n-1}{2}+1)$ or $n(\frac{n+1}{2})$. This shows that the final value has at least one divisor (n and $\frac{n+1}{2}$) that is not itself or 1, meaning it is composite.

b)

If the statement "if $|x| + x > 0$, x must be positive" is true, then by contrast, "if $|x| + x \leq 0$, x must be negative or 0" must also be true. Subtracting x from both sides of $|x| + x \leq 0$ gives us $|x| \leq -x$. Because $|x|$ will always be positive or 0, the only way for this inequality to be true is if $-x$ is also positive or 0, meaning x must be negative or 0. By contradiction, the original statement must also be true.

c)

Base Case: $n = 2$:

> $(1 - \frac{1}{4}) = \frac{3}{4}$

Induction Hypothesis:

> Assume that $(1 - \frac{1}{4}) * (1 - \frac{1}{9}) * ... * (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$ is true.

Inductive Step:

> For $(1 - \frac{1}{4}) * (1 - \frac{1}{9}) * ... * (1 - \frac{1}{(n+1)^2}) = \frac{n+2}{2n+2}$ to be true, it should also equal $(1 - \frac{1}{4}) * (1 - \frac{1}{9}) * ... * (1 - \frac{1}{n^2}) * (1 - \frac{1}{(n+1)^2})$ or $\frac{n+1}{2n} * (1 - \frac{1}{(n+1)^2})$.
> $\frac{n+1}{2n} * (1 - \frac{1}{(n+1)^2}) = \frac{n+1}{2n} - \frac{n+1}{2n(n+1)^2} = \frac{n+1}{2n} - \frac{1}{2n(n+1)} = \frac{(n+1)^2}{2n(n+1} - \frac{1}{2n(n+1} = \frac{n^2+2n}{2n(n+1)} = \frac{n+2}{2n+2}$

Conclusion:

Because the Base Case and the Inductive Step are both true, the original statement must also be true.

4. Calculus

a)

i. $\log_2(x) + \log_2(3 + x) = 2$
$\log_2(x^2 + 3x) = \log_2(4)$
$x^2 + 3x = 4$
$x = 1$

ii. $5e^{0.12x} = 10e^{0.08x}$
$e^{0.12x} = 2e^{0.08x}$
$ln(e^{0.12x}) = ln(2e^{0.08x})$
$0.12x = ln(2) + 0.8x$
$0.04x = ln(2)$
$x = 25ln(2)$

b)

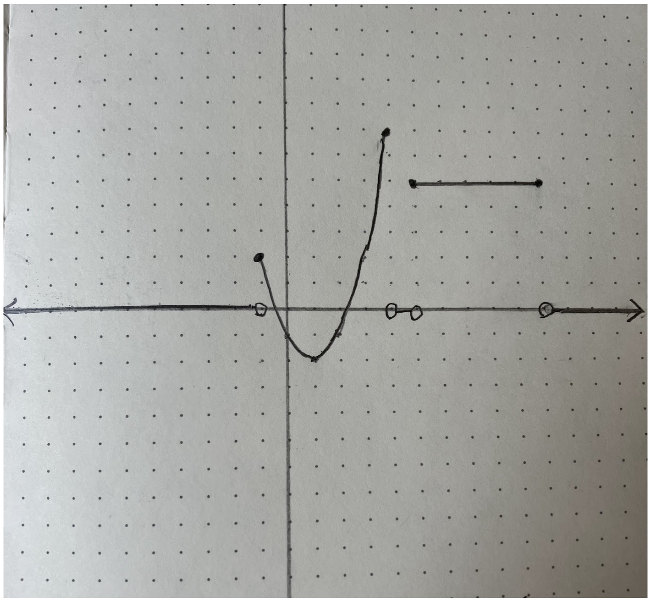
i. $f'(x) = 6x^2 + 6x^{-4} + 6$

ii. $f'(x) = -2(2x^3 - 2x^{-3} + 6x)(6x^2 + 6x^{-4} + 6)$

c)

$\int e^x - 3x^2 + 1dx = e^x - x^3 + x + C$
$(e^2 - 2^3 + 2 + C) - (e^0 + 0^2 + C) = e^2 - 8 + 2 + C - 1 - C = e^2 - 7$

d)

i. 

ii. $\int_1^4 cf(x)dx = c\int_1^4 f(x)dx = c(F(4) - F(1))$
$F(x)$ for $x \in [1,4] = \frac{1}{3}x^3 - x^2 - x + C$
$F(4) - F(1) = (\frac{4^3}{3} - 4^2 - 4 + C) - (\frac{1}{3} - 1 - 1 + C) = 3$
$3c = 42$
$c = 14$