Problem 1: Game of Dice

a)

i. The sample space Ω is the set of all possible combinations of rolling one green die and one white die. $\Omega = \{(1,3),(1,4),(1,5),(1,12),(2,3),(2,4),(2,5),(2,12),(9,3),(9,4),(9,5),(9,12),(10,3),(10,4),(10,5),(10,12)\}$
Because these are both fair dice, the probability of each of these combinations is equally likely. Since $|\Omega| = 16$, for any event $E \in \Omega$, $\Pr(E) = \frac{1}{16}$.

ii. The green die wins if it shows a higher number. The set of combinations where the green die wins $G = \{(9,3),(9,4),(9,5),(10,3),(10,4),(10,5)\}$. The probability of this event, $\Pr(G) = \frac{|G|}{|\Omega|} = \frac{6}{16} = \frac{3}{8}$.

b)

i. The sample space Ω would be the set of all possible sums from rolling two green dice.
$\Omega = \{2,3,4,10,11,12,18,19,20\}$
These sums could be represented by the pairs of rolls that produced them:
$\{(1,1),(1,2),(1,9),(1,10),(2,1),(2,2),(2,9),(2,10),(9,1),(9,2),(9,9),(9,10),(10,1),(10,2),(10,9),(10,10)\}$
With this list, certain pairs of rolls produce the same sum. For example: $\{(1,10),(10,1),(2,9),(9,2)\} = 11$. With the given pairs of rolls, we can see there are four ways to produce 11, two ways to produce 3, 10, 12, and 19, and only one way to produce 2, 4, 18, and 20. Out of a total of 16 possible rolls, this gives us the following probabilities:
$\Pr(\text{sum} = 2, 4, 18, \text{ or } 20) = \frac{1}{16}$
$\Pr(\text{sum} = 3, 10, 12, \text{ or } 19) = \frac{2}{16} = \frac{1}{8}$
$\Pr(\text{sum} = 11) = \frac{4}{16} = \frac{1}{4}$

ii. To find the missing probabilities, we simply multiply the probability of getting the green sum with the probability of getting the white sum. For example, $(11,8)$ would be $\frac{4}{16} * \frac{3}{16} = \frac{12}{256}$.

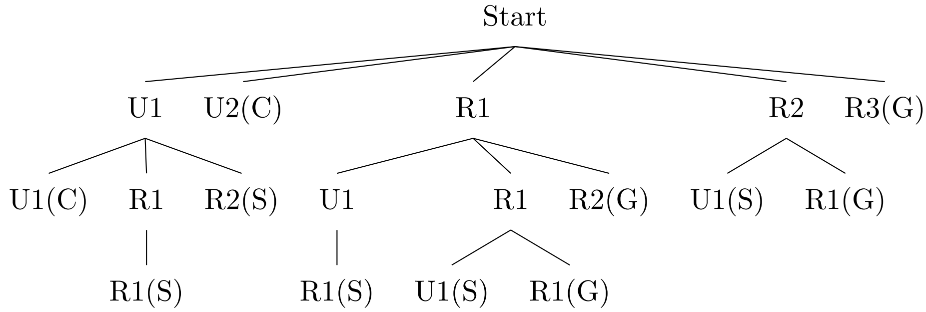
$G\backslash W$	8	9	24
11	12/256	8/256	
12	6/256	4/256	
18			1/256

To find the probability that green wins, we sum up all of the probabilities where the sum of the green dice is greater than the sum of the white dice.
(Per Row:)
$\Pr(\text{Green Wins}) = \frac{16}{256} + \frac{36}{256} + \frac{18}{256} + \frac{15}{256} + \frac{30}{256} + \frac{15}{256} = \frac{130}{256} = \frac{65}{128}$.

Problem 2: Berry Choice

a)

(For the sake of space, the moves have been labeled U1(Up One), with the ending berry labeled in parentheses.)



Because each move at each level is equally likely, the probability of a final position can be found by multiplying the probability of each layer. Then, for each berry, the probability of each final result can be summed.
Pr(Cherry): $\frac{1}{5} + \frac{1}{5} * \frac{1}{3} = \frac{4}{15}$.
Pr(Strawberry):$\frac{1}{5} * \frac{1}{3} + \frac{1}{5} * \frac{1}{3} * \frac{1}{1} + \frac{1}{5} * \frac{1}{3} * \frac{1}{1} + \frac{1}{5} * \frac{1}{3} * \frac{1}{2} + \frac{1}{5} * \frac{1}{2} = \frac{1}{3}$
Pr(Grape): $\frac{1}{5} + \frac{1}{5} * \frac{1}{3} + \frac{1}{5} * \frac{1}{2} + \frac{1}{5} * \frac{1}{3} * \frac{1}{2} = \frac{2}{5}$

b)

Base Case: $m = 1$:

> The possible first moves when $m = 1$ are $\{U1, R1, R2, ...R50\}$. With 51 different possible first moves, the probability of reaching the cherry on the first move is $\frac{1}{51}$, since only $U1$ reaches the cherry. If any other move is chosen, the rook will move to the right of the cherry and can no longer reach it, meaning the final probability for this case is $\frac{1}{51}$.

Induction Hypothesis:

> Assume Pr(Cherry)= $\frac{1}{51}$ for all boards from $m = 1$ up to and including $m = k$.

Inductive Step:

> When $m = k$, The list of starting moves is $\{U1, R1, R2, ..., R(k + 49)\}$When k is increased by 1, it is effectively adding another column to the left side of the board. This also adds the move $R(k + 50)$ to the list of starting moves. From the starting position on the $m = k + 1$ board, there is a probability of $\frac{k}{k+51}$ to make a move $\{R1, R2, ..., Rk\}$. All of these moves would place the rook in the starting position of a board of a smaller size. From that position, it is assumed that Pr(Cherry)= $\frac{1}{51}$ based on the hypothesis, meaning the probability of one of these moves to be made and then to reach the cherry is $\frac{k}{k+51} * \frac{k}{51} = \frac{k}{51(k+51)}$. However, there is also the $\frac{1}{k+51}$ chance that the move $U1$ is made from the starting position, which would also result in the cherry being reached, as it is no longer possible for the rook to move down and it will inevitably reach the cherry. Adding these probabilities together will give the total probability of reaching a cherry from a board of $m = k + 1$:
> $\frac{k}{51(k+51} + \frac{1}{k+51} = \frac{k+51}{51(k+51)} = \frac{1}{51}$

Problem 3: Desserts

a)

Alice gives a rating greater than three for the outcomes $\{cheesecake, tart\}$. The probability of this event is the sum of the probabilities for the two outcomes. $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

b)

The set of outcomes where $B = 3$ is $\{brownies, tart\}$. The probability of this event is the sum of the probabilities for the two outcomes. $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

c)

The set of outcomes where the minimum of B and A is an even number is $\{cheesecake, brownies\}$. The probability of this event is the sum of the probabilities for the two outcomes. $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

d)

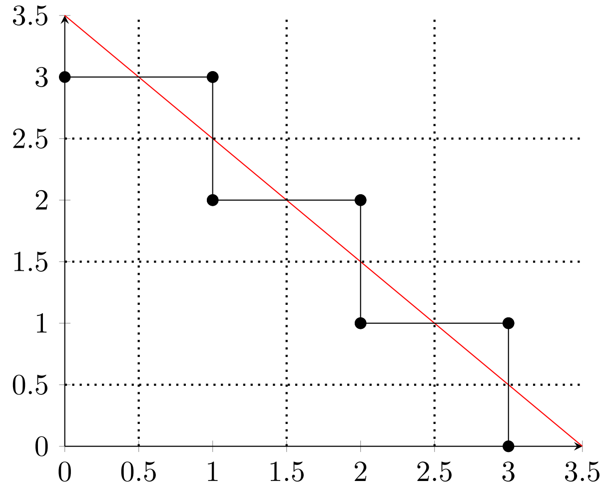
The event $A = B$ never occurs, meaning the set of outcomes is $\emptyset$, and therefore the probability is 0.

e)

$\Pr(m = 4 \text{ and } n = 2) = \frac{1}{4}$
$\Pr(m = 3 \text{ and } n = 4) = \frac{1}{3}$
$\Pr(m = 2 \text{ and } n = 3) = \frac{1}{4}$
$\Pr(m = 5 \text{ and } n = 3) = \frac{1}{6}$
For all other pairs of m and n, $\Pr(A = m \text{ and } B = n) = 0$.

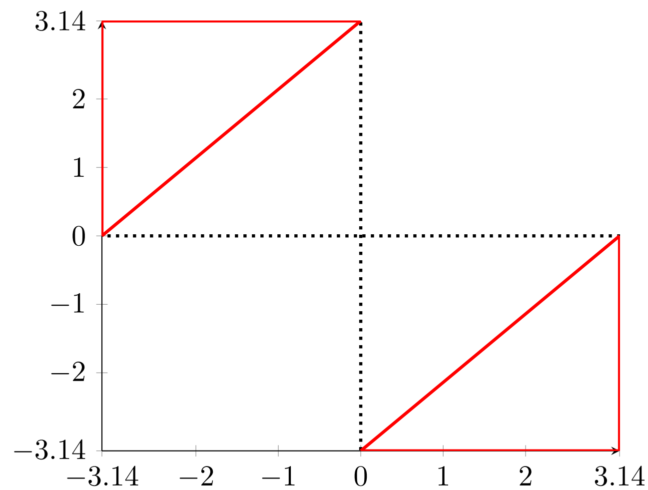
Problem 4: Geometric Method

a)



Based on the graph, we can see there are seven distict segments along the red line, three where both lengths are rounded up and four where both lengths are rounded down. The sum of the two is equal to three when both are rounded down, meaning the probability of this is $\frac{4}{7}$.

b)



The areas outlined by red represent the set of points where the friends are not dropped off in the same semi-circle. These areas combined take up $\frac{1}{4}$ of the total valid area, meaning the probability that the friends *are* dropped off in the same semi-circle is $\frac{3}{4}$.