1. Probability Spaces

a)

We know Pr[Ω] equals 1 by the Normalization Axiom, meaning the probability of each item in the set must add to 1. Therefore, 1-Pr[$\{a\}$]-Pr[$\{b\}$]-Pr[$\{c\}$]=Pr[$\{d\}$]. $1 - 0.3 - 0.05 - 0.05$ =Pr[$\{d\}$]=0.6.

b)

By the Normalization Axiom, the sum of each item in the set is 1, meaning $1 = 19p$. Therefore, $p = \frac{1}{19}$.

c)

Each item in the set Ω has to contain a color and a size, meaning $\Omega = \{SB, SW, MB, MW, LB, LW\}$. The experiment is choosing a uniformly random shirt, meaning the probabilities are solely based on the quantity of each shirt. Looking at just the colors, there are twice as many black shirts as white, so PR[$\{SB, MB, LB\}$]=$\frac{2}{3}$ and PR[$\{SW, MW, LW\}$]=$\frac{1}{3}$, which adds up to 1 due to the Normalization Axiom. For just the sizes, PR[$\{SB, SW\}$]=$\frac{1}{6}$, PR[$\{MB, MW\}$]=$\frac{1}{3}$, and PR[$\{LB, LW\}$]=$\frac{1}{2}$, since there are twice as many mediums as smalls and twice as many larges as smalls. These also sum to 1 by Normalization. Finally, we can multiply the probabilities for color and size to get the individual probabilities:
Pr[$\{SB\}$] $= \frac{1}{9}$, Pr[$\{SW\}$] $= \frac{1}{18}$, Pr[$\{MB\}$] $= \frac{2}{9}$, Pr[$\{MW\}$] $= \frac{1}{9}$, Pr[$\{LB\}$] $= \frac{1}{3}$, Pr[$\{LW\}$] $= \frac{1}{6}$

d)

i. There must be at least three shirts in the drawer, since there must be two black shirts and the probability of the event is less than one, meaning there is at least one non-black shirt in the drawer. We can represent the probability of picking one black shirt at random as $\frac{b}{t}$, where b is the number of black shirts and t is the total number of shirts in the drawer. The probability of picking two black shirts can be represented as $\frac{b}{t} * \frac{b-1}{t-1}$, which we know is equal to $\frac{1}{2}$. Having two black shirts out of three total would give a probability of $\frac{1}{3}$, meaning the ratio of two black shirts to one white shirt is too low. However, raising that ratio to 3:1, or three black shirts out of a total of four, gives $\frac{3}{4} * \frac{2}{3}$, which equals $\frac{1}{2}$. Therefore, the smallest possible number of shirts in the drawer is four, with three of them being black.

ii. If we let B represent black shirts and N represent non-black shirts, the sample space Ω for the experiment would be $\{(B, B), (B, N)\}$. The result (N, B) is not included since the shirts are chosen simultaneously so the order doesn't matter, and the result (N, N) is not possible since there is only one non-black shirt. We already know Pr[$\{(B, B)\}$] $= \frac{1}{2}$, therefore Pr[$\{BN\}$] $= \frac{1}{2}$ by the Normalization Axiom.

2. Probability and Set Operations

a)

$$\begin{aligned}
\Pr(A \cap B) + \Pr(A \cap \bar{B}) &= \Pr(A) \\
\Pr(A \cap \bar{B}) &= \Pr(A) - \Pr(A \cap B) && \text{Subtract } \Pr(A \cap B) \text{ from both sides} \\
\Pr(A \cap \bar{B}) &= \Pr(A \setminus B) && \text{Difference rule} \\
\Pr(A \setminus B) &= \Pr(A \setminus B) && \text{Definition of set difference}
\end{aligned}$$

b)

The largest possible value for $\Pr(A \cap B)$ would be $\frac{1}{2}$. This would be the value if $B \subseteq A$, meaning $\Pr(A \cap B) = \Pr(B)$.
The smallest possible value would be $\frac{1}{6}$. Because $\Pr(A)+\Pr(B) > 1$, A and B must overlap at some point by Normalization. If $\Pr(A + B) = 1$, meaning A and B are overlapping as little as possible, then $\Pr(A \cap B) = \Pr(A)+\Pr(B) - 1 = \frac{2}{3} + \frac{1}{2} - 1 = \frac{1}{6}$.

3. Cards

a)

There are 4 different suits and $\binom{52}{13}$ possible 13 card hands, with only one being valid per suit, giving a probability of $\frac{4}{\binom{52}{13}}$.

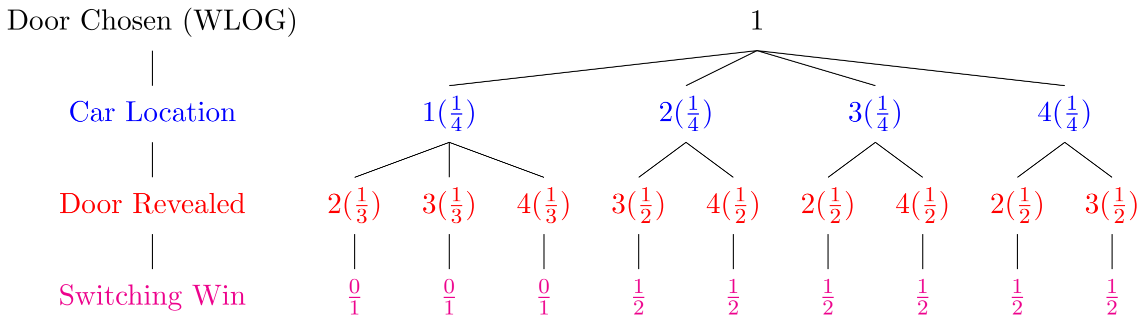
b)

This is the opposite of the previous question. By the Normalization axiom, the sum of these two must be 1, meaning the probability is $1 - \frac{4}{\binom{52}{13}}$

c)

First we can find the probability they are all one color. With 2 colors, the number of correct hands would be $2 * \binom{26}{13}$. Divided by the total number of hands gives the probability $\frac{2*\binom{26}{13}}{\binom{52}{13}}$. Therefore, the probability that they *aren't* all one color is $1 - \frac{2*\binom{26}{13}}{\binom{52}{13}}$. Because the 13 cards here contain more than one color, they must contain at least two suits, at least one red and at least one black. This means the final answer is $1 - \frac{2*\binom{26}{13}}{\binom{52}{13}}$.

4. Monty Hall with Four Doors



a)

Without switching, the probability that he wins is just the probability that the car is already behind door one: $\frac{1}{4}$.

b)

Because the doors are symmetrical, we are able to renumber the doors without changing the problem. Looking at the tree and treating it as if Mila originally picked door one, we can see that switching when the car is behind door one will always result in a loss. If the car is behind door two and door 3 is revealed, she would still only have a $\frac{1}{2}$ chance of winning. By going down the tree, we can see the probability she wins in this specific case is $\frac{1}{16}$. This probability is the same for the 5 remaining cases ((2,4),(3,2),(3,4),(4,2),(4,3)). Summing those probabilities $(6 * \frac{1}{16})$ gives the final probability of $\frac{3}{8}$.