Fall 2020 – 16-642 Manipulation, Estimation, and Control
Problem Set 1

Due: 30 September 2020

GUIDELINES:

- You must *neatly* write up your solutions and submit all required material electronically via canvas by the start of the lecture on the due date. Include any MATLAB scripts that were used in finding the solutions.
- You are encouraged to work with other students in the class, however you must turn in your own *unique* solution.
- Late Policy: If you do not turn your problem set in on time, you can turn it in up to 48 hours later but you will lose half of the points. After 48 hours, you will receive a zero.

1. All of the following problems refer to the following linear time invariant state space system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} x(t).$$

   (a) (5 points) Is the system stable? Explain your answer.

   (b) (5 points) Is the system controllable? Explain your answer.

   (c) (5 points) Let the initial state vector be

$$x_0 = x(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

   Using the MATLAB `expm` command, plot the output of the unforced system for $t \in [0, 2]$.

   (d) (5 points) Use the MATLAB `place` command to find the matrix K such that the matrix $A - bK$ contains the following set of eigenvalues:

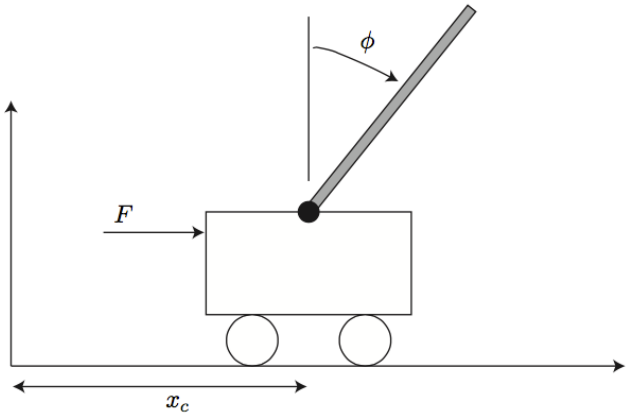
$$\{-1 + i, \ -1 - i, \ -2\}$$

   (e) (5 points) Let the initial state vector be

$$x_0 = x(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

   Use the MATLAB `expm` command to plot the output of the system under the feedback law $u(t) = -Kx(t)$ for $t \in [0, 10]$. Use the K found in the previous problem.

2. Consider the "pendulum on a cart" system:



The equations of motion for this system are

$$\gamma \ddot{x}_c - \beta \ddot{\phi} \cos\phi + \beta \dot{\phi}^2 \sin\phi + \mu \dot{x}_c = F$$

$$\alpha \ddot{\phi} - \beta \ddot{x}_c \cos\phi - D \sin\phi = 0,$$

where α, β, γ, D, and μ are physical constants determined by the masses of the cart and pendulum, pendulum length, and friction, and F is an externally applied force. All units are mks, and angles are expressed in radians.

(a) (10 points) Define the state vector

$$x = \begin{bmatrix} x_c \\ \phi \\ \dot{x}_c \\ \dot{\phi} \end{bmatrix}$$

and the input $u = F$. Write the cart/pendulum equations of motion as a nonlinear state space equation. *hint:* It may help to convert the system into standard mechanical form as an intermediate step, and note that the matrix

$$M = \begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix}$$

is always invertible.

(b) (10 points) Describe the set of equilibrium points for the system. Describe them mathematically (i.e., with equations) and in English (i.e., what do they physically mean).

(c) (5 points) Let $\gamma = 2$, $\alpha = 1$, $\beta = 1$, $D = 1$, and $\mu = 3$. The linearized system about the equilibrium point at $x = 0$ is

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t).$$

Compute the eigenvalues of A, and use them to say whatever you can about the stability of the equilibrium point at $x = 0$ for both the linearized system and original nonlinear system.

(d) (15 points) Assume that the entire state can be measured directly (i.e., there is no need for an observer). Letting $R = 10$ and

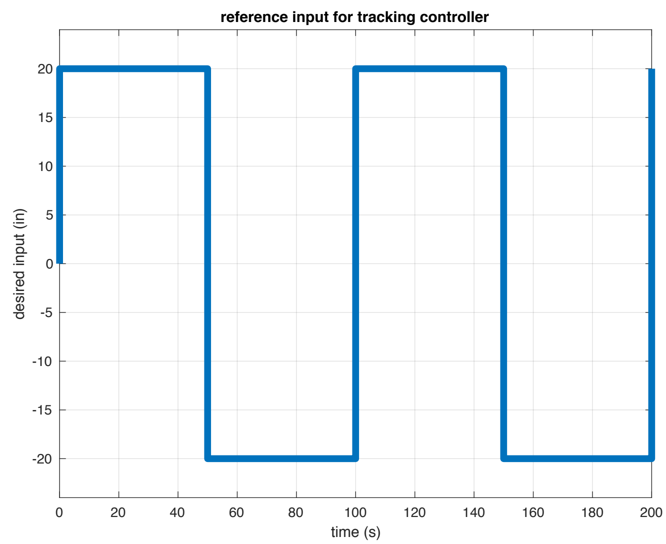
$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

use the MATLAB lqr command to find the corresponding optimal feedback control $u(t) = -K_c x(t)$. Using a timestep of $T = 0.01$ seconds and a final time of $t_f = 30$ seconds and an initial state of $x_0 = [0,\, 0.1,\, 0,\, 0]^T$, calculate and plot the state of the *linearized system* under the feedback control law above. You may either write your own 4th order Runge-Kutta routine to solve the state equation, or use the MATLAB function `ode45`. Repeat for $x_0 = [0,\, 0.5,\, 0,\, 0]^T$, $x_0 = [0\, 1.0886\, 0\, 0]^T$, and $x_0 = [0,\, 1.1,\, 0,\, 0]^T$.

(e) (5 points) Repeat part 2d using the full nonlinear state equations in the simulation (as opposed to the linearized state equations). Explain any differences you see in the results.

(f) (5 points) Assume you have an output y given by a sensor that measures the cart position *in inches*. Find the matrix C so that $y = Cx$.

(g) (15 points) Using the LQR controller designed above, create a tracking controller that allows you to specify a desired cart position trajectory. Test this tracking controller for a desired output y_d that is a square wave as shown in the picture below. Simulate the full nonlinear dynamics using $T = 0.01$ seconds, $t_f = 200$ seconds, and $x_0 = [0,\, 0,\, 0,\, 0]^T$. Plot the state vs. time. On a separate graph, make a plot that overlays the desired and actual outputs. Explain what is going on.



(h) (10 points) Choose new Q and R for the LQR controller to make the tracking controller better. Explain what you mean by "better", and describe your reasoning for how you changed Q and R. Simulate the system using the same parameters as above, and demonstrate your improvement by plotting the desired and actual outputs.