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1. (a) **Answer:**

The eigenvalues obtained are:

λ_1 = 7.6690 + 0.0000i
λ_2 = -0.3345 + 0.1361i
λ_3 = -0.3345 - 0.1361i

Explanation:

To determine system stability, we cross-reference the following checks:

1) System is stable if and only if Re(λ_i) ¡ 0 for all eigenvalues
2) System is unstable if and only if Re(λ_i) ¿ 0 for some i
3) Generally, $x_e = 0$ is stable if and only if both of the following hold
(i) Re(λ_i) $\leq$ 0 for all i
(ii) every eigenvalue with Re(λ_i) = 0 has an associated Jordan block of order 1.

Therefore the system is unstable since not all Re(λ_i) ¡ 0 and Re(λ_i) ¿ 0 for some i

(b) **Answer:**

To determine controllability, we use MATLAB scripts Co = ctrb (A,B) to compute the controllability matrix, followed by unco = length(A) - rank(Co) to determine the number of uncontrollable states.

Using MATLAB, the controllability matrix was computed as:

$$\text{Co} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 7 \end{bmatrix}$$

and unco = 0, which concludes that we have zero uncontrollable states, therefore the system in controllable.

(c) **Plot of unforced system output for t =[0,2]:**

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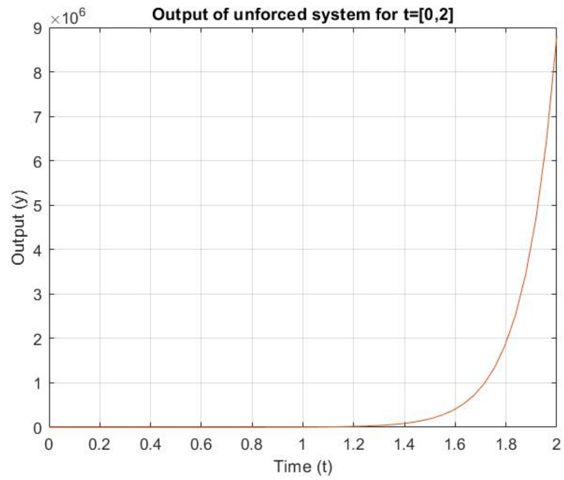


(d) **Answer:**

Matrix K = $\begin{bmatrix} 11 & 60 & 88 \end{bmatrix}$

(e) **Answer:**

We represent the given matrices as the general form:
$\dot{x}$(t) = A.x(t) + B.u(t)

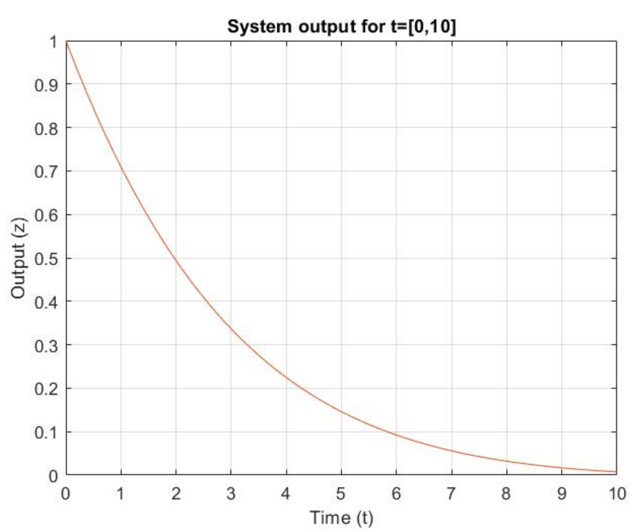
Using a matrix, E, of eigenvectors of A such that, x = E z
With the matrix of eigenvectors, we get $\tilde{A} = E^T$ A E

This allows us to get a diagonal matrix of eigenvalues:

$$\dot{Z} = \begin{bmatrix} 7.669 & 0 & 0 \\ 0 & -0.335 + 0.136i & 0 \\ 0 & 0 & -0.3345 - 0.136i \end{bmatrix} \text{Z}$$

which translates to the output function, $Z_n = C * e^{\lambda t} * X_0$

Plot of system output under feedback law $u(t) = -Kx(t) fort = [0, 10]$



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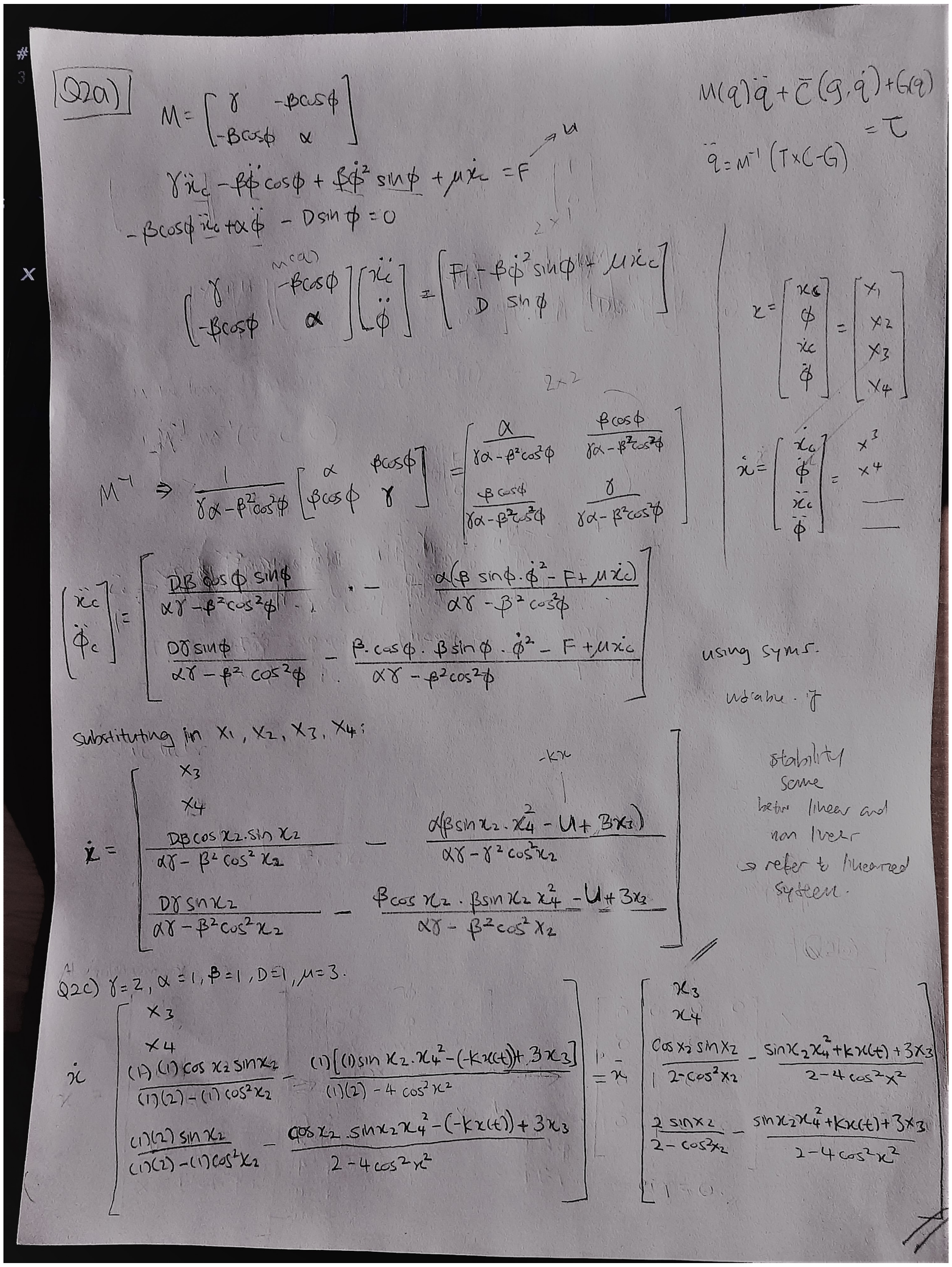
2. (a) **Answer:**



Q2a)

$$M = \begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix}$$

$$M(q)\ddot{q} + \bar{C}(q,\dot{q}) + G(q) = \tau$$

$$\ddot{q} = M^{-1}(\tau \times C - G)$$

$$\gamma\ddot{x}_c - \beta\ddot{\phi}\cos\phi + \beta\dot{\phi}^2\sin\phi + \mu\dot{x}_c = F \;\; (u)$$

$$-\beta\cos\phi\,\ddot{x}_c + \alpha\ddot{\phi} - D\sin\phi = 0$$

$$\begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix}\begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} F - \beta\dot{\phi}^2\sin\phi + \mu\dot{x}_c \\ D\sin\phi \end{bmatrix}$$

$$x = \begin{bmatrix} x_c \\ \phi \\ \dot{x}_c \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{\phi} \\ \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \_\_ \\ \_\_ \end{bmatrix}$$

$$M^{-1} \Rightarrow \frac{1}{\gamma\alpha - \beta^2\cos^2\phi}\begin{bmatrix} \alpha & \beta\cos\phi \\ \beta\cos\phi & \gamma \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\gamma\alpha-\beta^2\cos^2\phi} & \frac{\beta\cos\phi}{\gamma\alpha-\beta^2\cos^2\phi} \\ \frac{\beta\cos\phi}{\gamma\alpha-\beta^2\cos^2\phi} & \frac{\gamma}{\gamma\alpha-\beta^2\cos^2\phi} \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x}_c \\ \ddot{\phi}_c \end{bmatrix} = \begin{bmatrix} \frac{D\beta\cos\phi\sin\phi}{\alpha\gamma-\beta^2\cos^2\phi} - \frac{\alpha(\beta\sin\phi\cdot\dot{\phi}^2 - F + \mu\dot{x}_c)}{\alpha\gamma-\beta^2\cos^2\phi} \\ \frac{D\gamma\sin\phi}{\alpha\gamma-\beta^2\cos^2\phi} - \frac{\beta\cdot\cos\phi\cdot\beta\sin\phi\cdot\dot{\phi}^2 - F + \mu\dot{x}_c}{\alpha\gamma-\beta^2\cos^2\phi} \end{bmatrix}$$

using syms.

stability same between linear and non linear → refer to linearized system.

Substituting in x_1, x_2, x_3, x_4:

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{D\beta\cos x_2 \sin x_2}{\alpha\gamma-\beta^2\cos^2 x_2} - \frac{\alpha(\beta\sin x_2\cdot x_4^2 - u + 3x_3)}{\alpha\gamma-\gamma^2\cos^2 x_2} \\ \frac{D\gamma\sin x_2}{\alpha\gamma-\beta^2\cos^2 x_2} - \frac{\beta\cos x_2\cdot\beta\sin x_2\, x_4^2 - u + 3x_3}{\alpha\gamma-\beta^2\cos^2 x_2} \end{bmatrix}$$

Q2c) $\gamma=2, \alpha=1, \beta=1, D=1, \mu=3$.

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{(1)(1)\cos x_2\sin x_2}{(1)(2)-(1)\cos^2 x_2} - \frac{(1)[(1)\sin x_2\cdot x_4^2 - (-kx(t)) + 3x_3]}{(1)(2) - 4\cos^2 x^2} \\ \frac{(1)(2)\sin x_2}{(1)(2)-(1)\cos^2 x_2} - \frac{\cos x_2 \sin x_2 x_4^2 - (-kx(t)) + 3x_3}{2-4\cos^2 x^2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{\cos x_2 \sin x_2}{2-\cos^2 x_2} - \frac{\sin x_2 x_4^2 + kx(t) + 3x_3}{2-4\cos^2 x^2} \\ \frac{2\sin x_2}{2-\cos^2 x_2} - \frac{\sin x_2 x_4^2 + kx(t) + 3x_3}{2-4\cos^2 x^2} \end{bmatrix}$$

(b) **Answer:**

The inverterd pendulum system has two equlibria, one at $\phi = 0$ (saddle point equilibrium with rod pointing up toward the ceiling) and the other at $\phi = \pi$ (stable equilibrium with rod ponting down toward the ground). We can set this as kϕ, where k is a real integer. At these equilibria, the angular velocity of the rod and the velocity of the cart are both zero.
The state of, x, exists for real dimensions n. The input, u, exists for real dimensions m. The output, y, exists for real dimensions p. The function, f, maps $\mathbb{R}^n$ x $\mathbb{R}^m$ to the derivate of the state vector $\mathbb{R}^n$.

A point $\bar{x}$ is called an equilibrium point if there is a specific $\bar{u} \in \mathbb{R}^m$ called the equilibrium input such that f($\bar{x}, \bar{u}$) = 0_n. For the system $\dot{x}$(t) = f (x(t),u(t)), starting from the initial condi-

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tion $x(t_0) = \bar{x}$ and applying the input u(t) $\bar{u}$ for all t $\geq t_0$, the resulting solution x(t) satisfies x(t) = $\bar{x}$ for all t $\geq t_0$.

Mathematically,
$\dot{x}$(t) = f (x(t),u(t))
y(t) = h(x(t))
where:
$x \in \mathbb{R}^n$
u$\in \mathbb{R}^m$
y$\in \mathbb{R}^p$
f :R^n x $\mathbb{R}^m \rightarrow \mathbb{R}^n$

At the equilibrium points, $\phi \rightarrow k\pi$, where k is any integer.
$\dot{\phi} = 0$ and $\dot{x} = 0$

(c) **Answer:**

The linearized system about the equilibrium point at x=0 is:

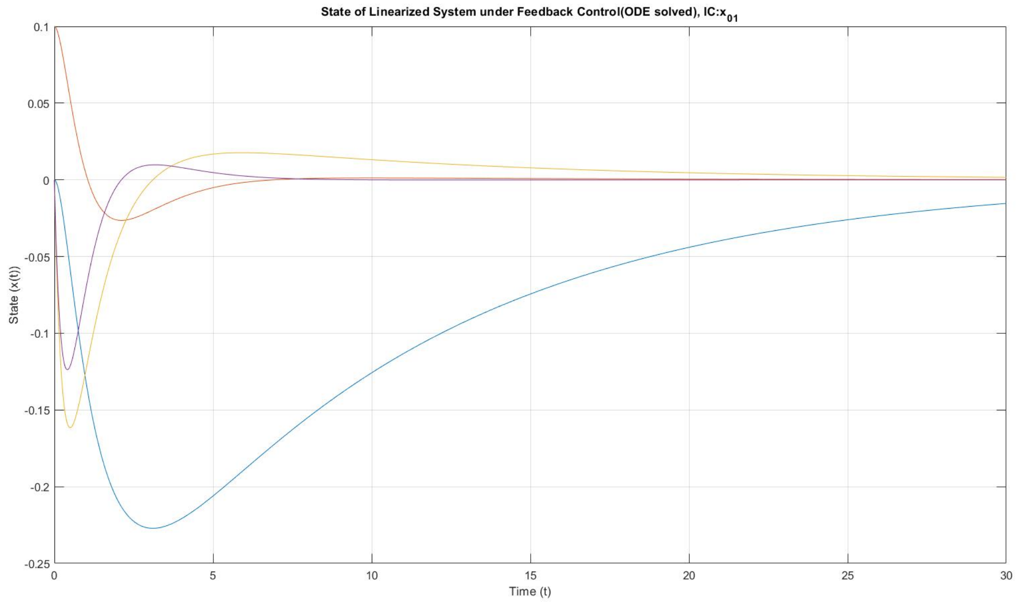
$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} \text{x(t)} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{u(t)}$$

The eigenvalues A for the linearized system are:
λ_1 = 0
λ_2 = -3.330
λ_3 = 1.128
λ_4 = -0.798

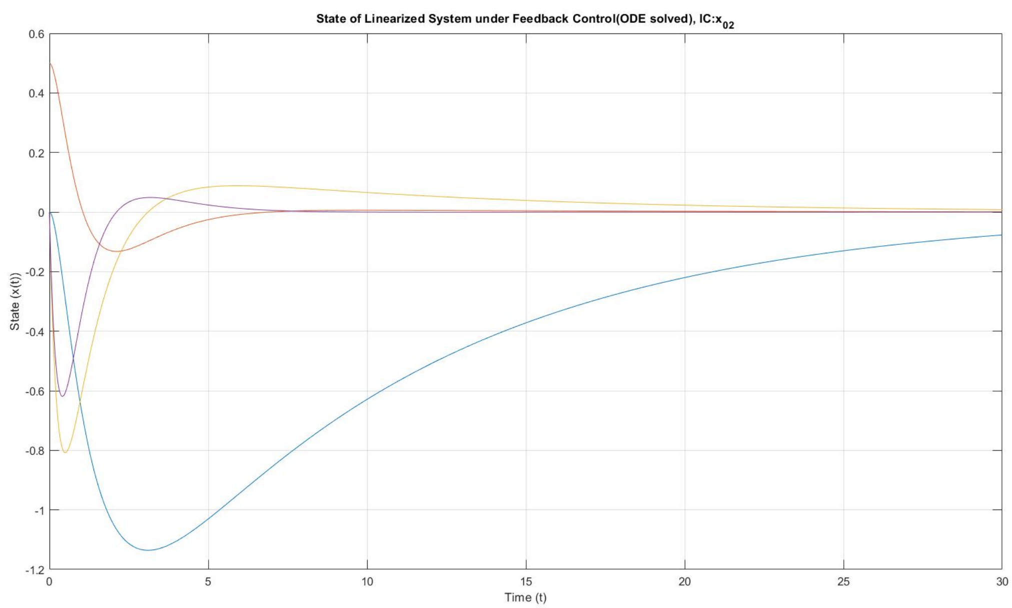
Both the linearized and non-linear systems are unstable as there is at least one eigenvalue > 0 in the linearized system,
The stability of the non-linear system corresponds to its linearzied form; the nonlinear system is stable if and only if it's linearized form is stable or asymptotically stable. The linearized system is controllable as its rank is 4, therefore the non-linear system shares the same characteristic of its linearized form.

(d) **Answer:**
Using the MATLAB lqr command, the linearized system's values of K are: -0.3162, 10.2723, -6.7875, 9.2183

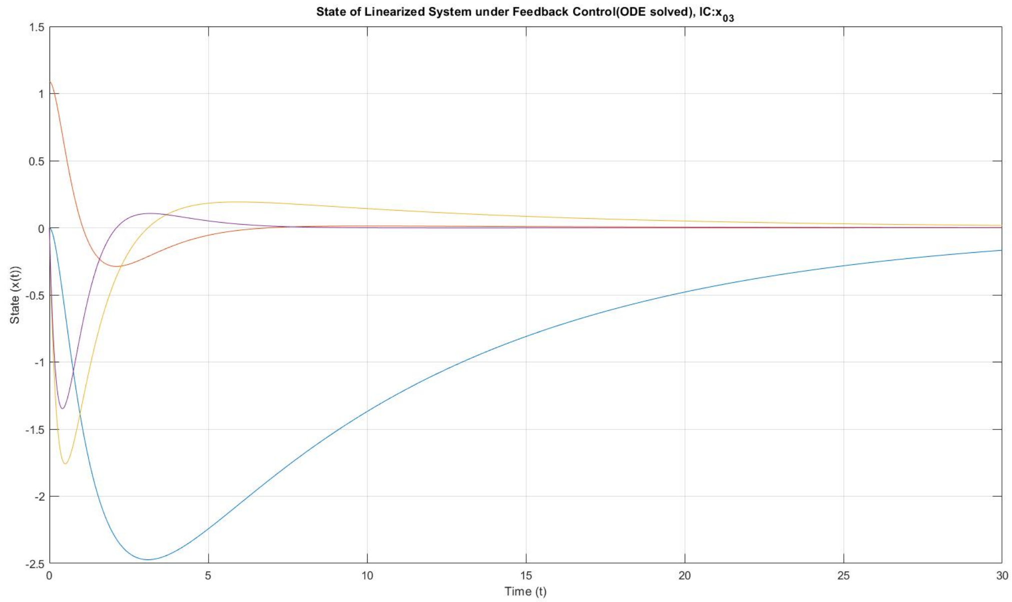
for initial state $x_0 = [0, 0.1, 0, 0]^T$



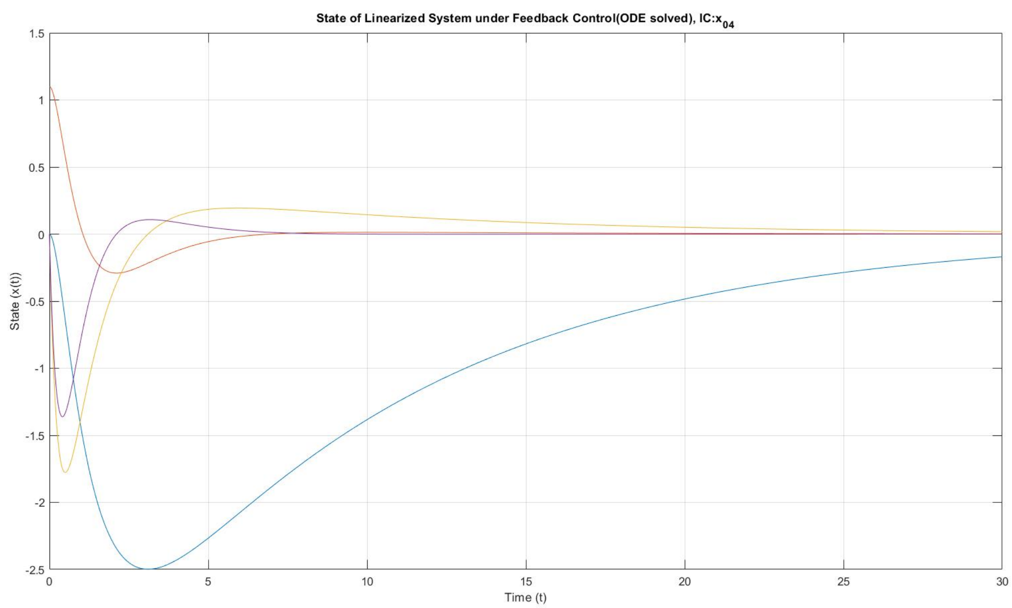
for initial state $x_0 = [0, 0.5, 0, 0]^T$



for initial state $x_0 = [0, 1.0886, 0, 0]^T$



for initial state $x_0 = [0, 1.1, 0, 0]^T$



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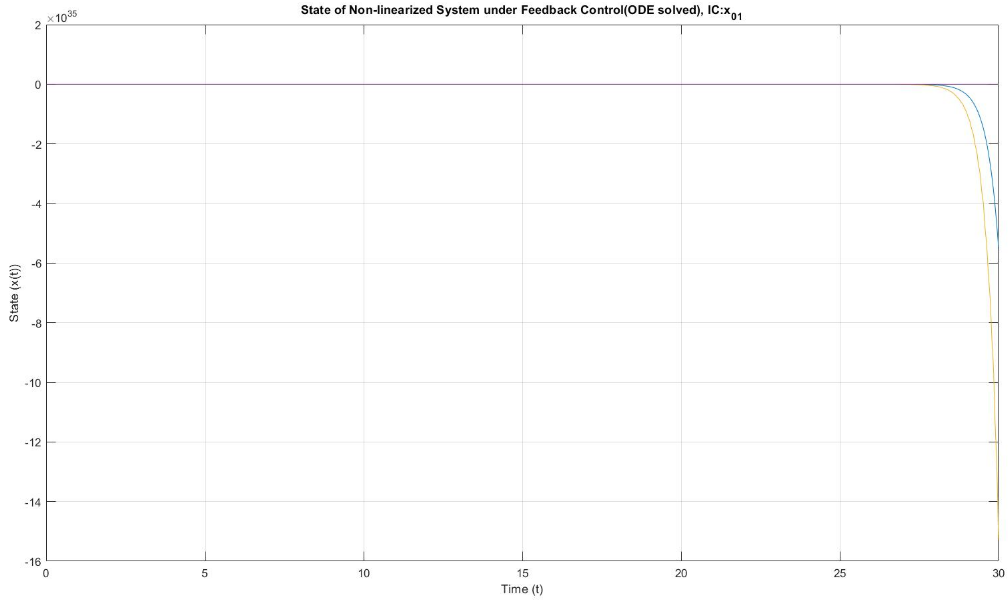
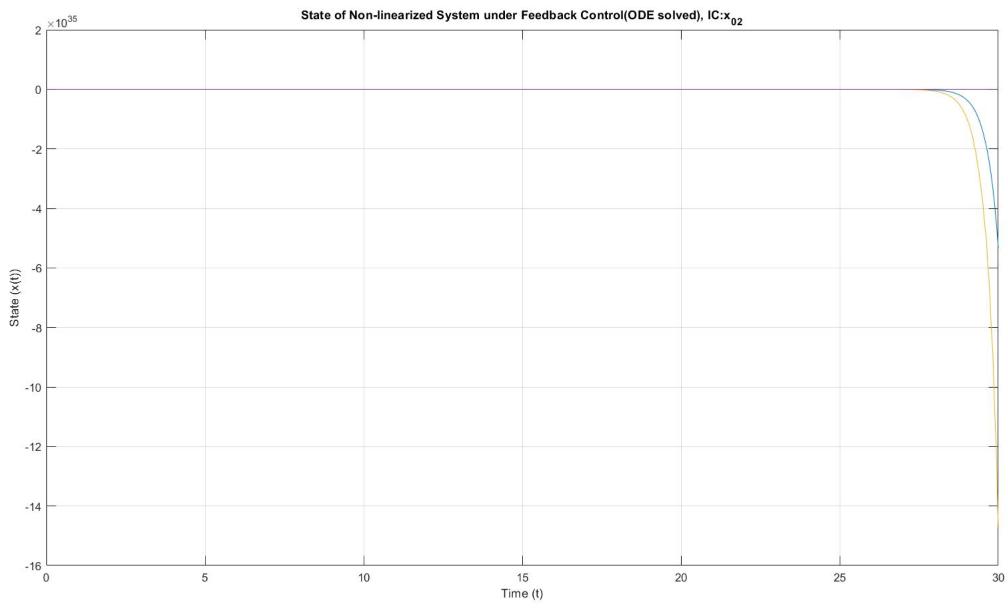
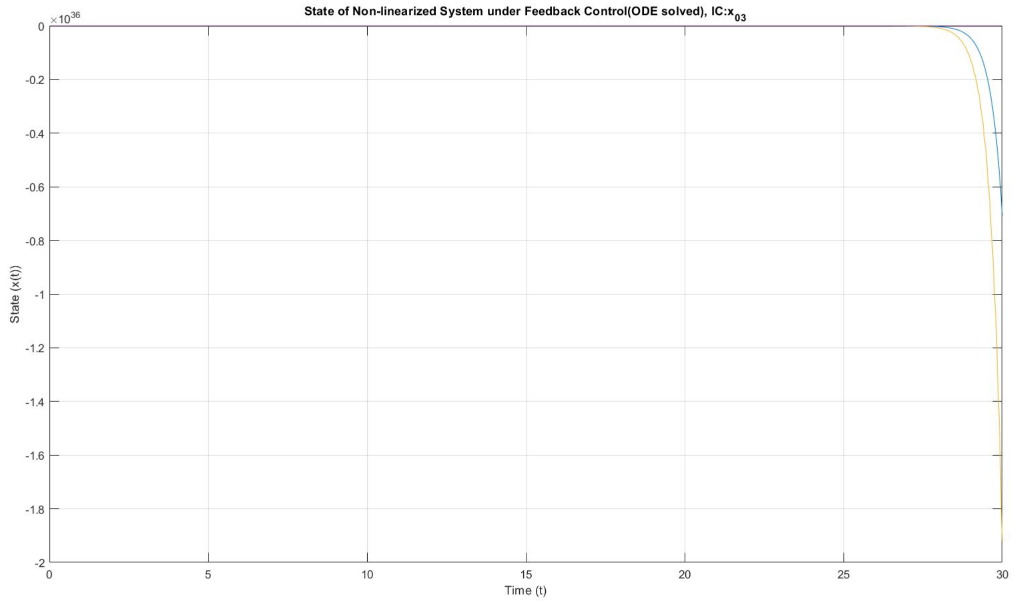
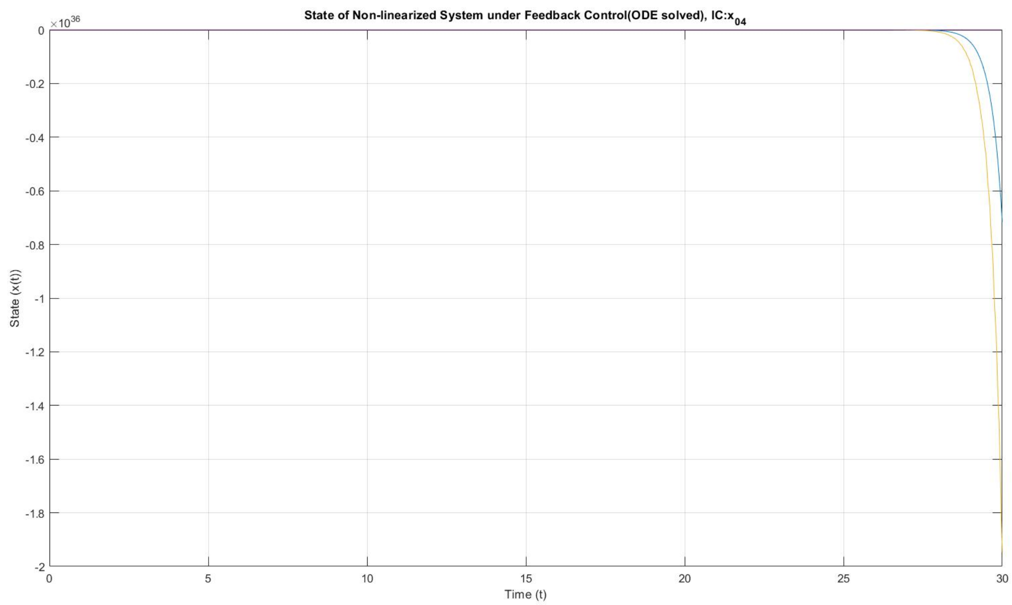
(e) **Explanation:** the linearized system's values of K are: -0.3162, 10.2723,-6.7875, 9.2183
We use these values of K to substitute into the u = - Fx term in the nonlinear equation obtained in 2a.



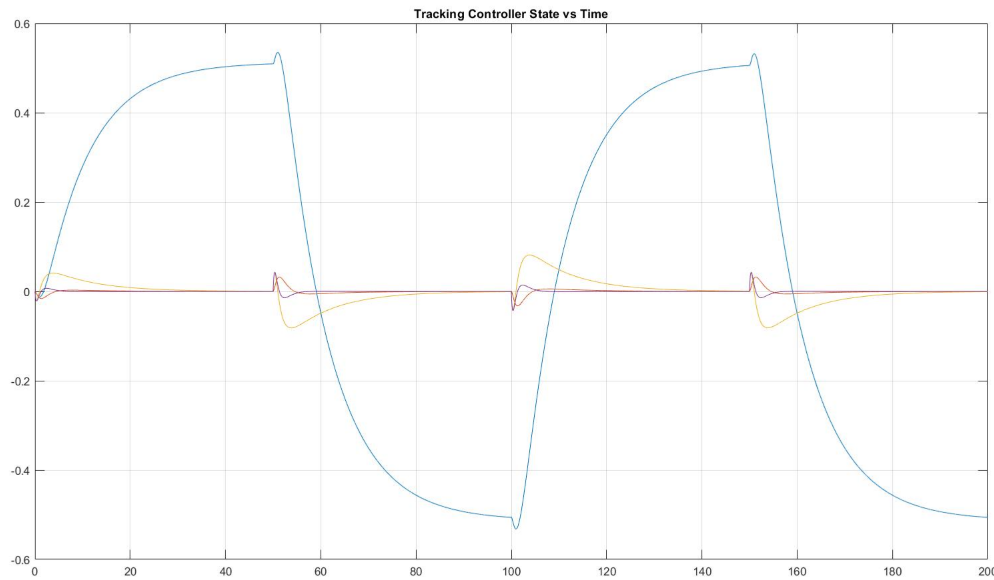




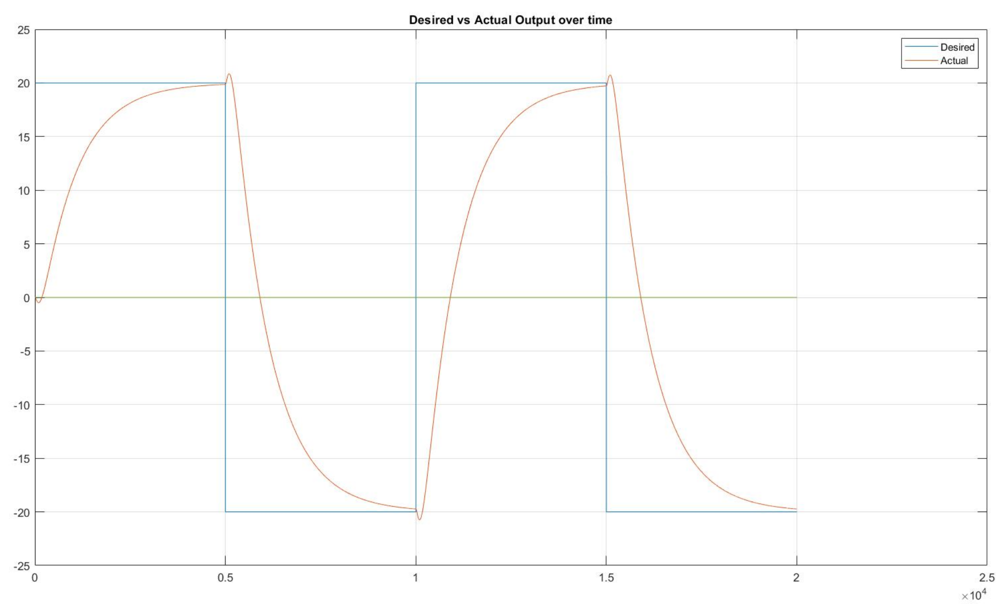


(f) **Answer:**
y = $\begin{bmatrix} 39 & 0 & 0 & 0 \end{bmatrix}$ x(t), where $\begin{bmatrix} 39 & 0 & 0 & 0 \end{bmatrix}$ = C, as 1 meter is approx 39 inches.

(g) **Answer:**



Graph of Output Comparison



Explanation:
The controller seeks to use the feedback law to meet the new input level each time there's a change of input. The small overshoot at each moment of input change is the controller reacting to a change in input state.

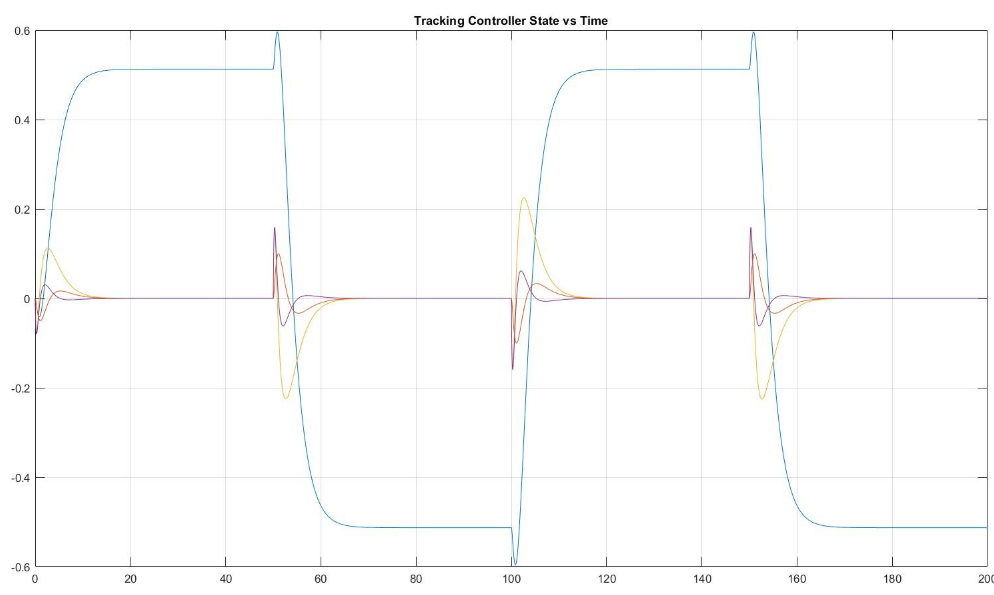
(h) **Answer:**

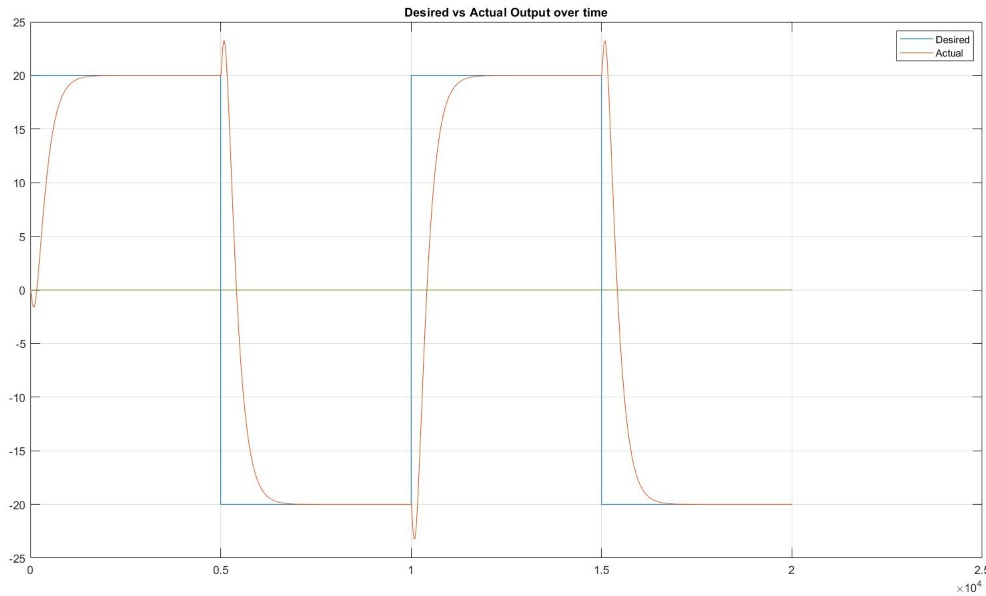
First we set $Q_1 = \begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 80 \end{bmatrix}$

We observe that this results in a more aggressive stabilization evidenced by a more negative eigenvalues. This is because the new values of Q penalizes imposes a greater cost function to deviations in $\dot{x}$(t) and $\dot{\phi}$.
Next, set R_1 =5. The lower value of R imposes a lower control cost penalty (e.g. energy is cheap or control motor is overspeced). It is observed that the most stable eigenvalue (i.e. the most negative) for the new values of Q and R are greater (more negative) than the original parameters. Hence, increasing Q selectively and decreasing R reduces time required for the system to achieve steady-state value. Notably, a more aggressive control regime results in larger overshoots.

With LQR implemented, the original eigenvalues of the system were:
λ_1 = -0.317+1.466i
λ_2 = -0.317-1.466i
λ_3 = -1.226
λ_4 = -1.093

With the new values of Q and R, the new eigenvalues were:
λ_1 = -1.414
λ_2 = 14.522
λ_3 = -9.910
λ_4 = 13.535




Desired vs Actual Output over time
Desired
Actual
25
20
15
10
5
0
-5
-10
-15
-20
-25
0
0.5
1
1.5
2
2.5
×10^4