Fall 2019 – 16-642 Manipulation, Estimation, and Control
Problem Set 1 Solutions

Due: 25 September 2019

GUIDELINES:

- You must *neatly* write up your solutions and submit all required material electronically via canvas by the start of the lecture on the due date. Include any MATLAB scripts that were used in finding the solutions.
- You are encouraged to work with other students in the class, however you must turn in your own *unique* solution.
- Late Policy: If you do not turn your problem set in on time, you can turn it in up to 48 hours later but you will lose half of the points. After 48 hours, you will receive a zero.

1. All of the following problems refer to the following linear time invariant state space system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} x(t).$$

(a) (5 points) Is the system stable? Explain your answer.

Solution:

Using matlab to check the eigenvalues of A, we get

$$\Lambda = \{7.669, -0.3345 + 0.136i, -0.3345 - 0.136i\}.$$

Since one eigenvalue has positive real part, the system is unstable.

(b) (5 points) Is the system controllable? Explain your answer.

Solution:

We build the controllability test matrix

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 7 \end{bmatrix}.$$

Since it has full rank, the system is controllable.

(c) (5 points) Let the initial state vector be

$$x_0 = x(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

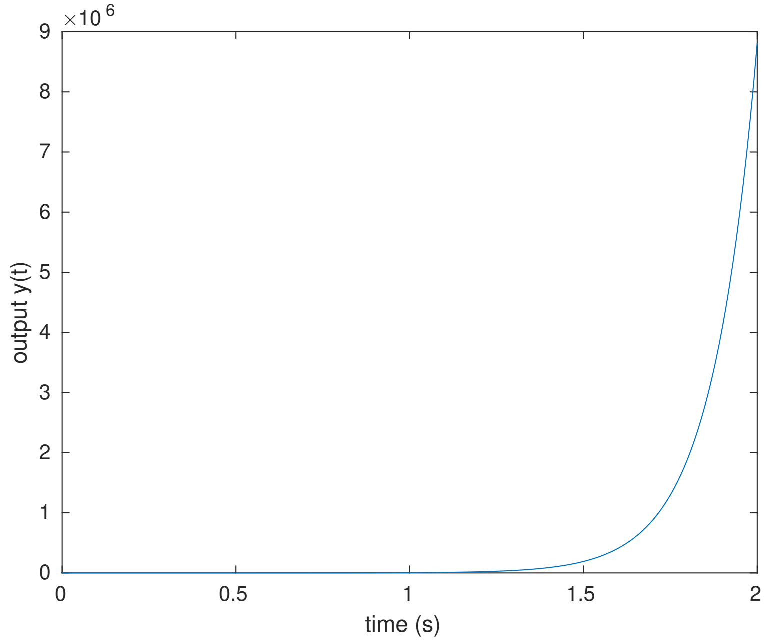
Using the MATLAB `expm` command, plot the output of the unforced system for $t \in [0, 2]$.

Solution:

The closed form solution for an unforced linear system is

$$x(t) = e^{At}x(0).$$

Note the solution goes to infinity, which is expected since the system is unstable.



(d) (5 points) Use the MATLAB `place` command to find the matrix K such that the matrix $A - bK$ contains the following set of eigenvalues:

$$\{-1+i,\ -1-i,\ -2\}$$

Solution:

Using the MATLAB `place` command gives

$$K = \texttt{place}(A, B, [-1+i; -1-i; -2]) = \begin{bmatrix} 11 & 60 & 88 \end{bmatrix}$$

(e) (5 points) Let the initial state vector be

$$x_0 = x(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

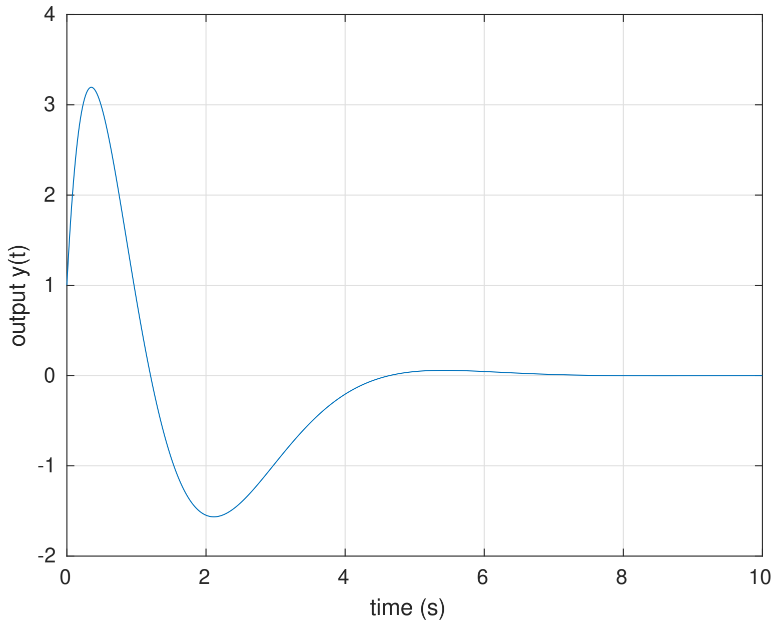
Use the MATLAB `expm` command to plot the output of the system under the feedback law $u(t) = -Kx(t)$ for $t \in [0, 10]$. Use the K found in the previous problem.

Solution:

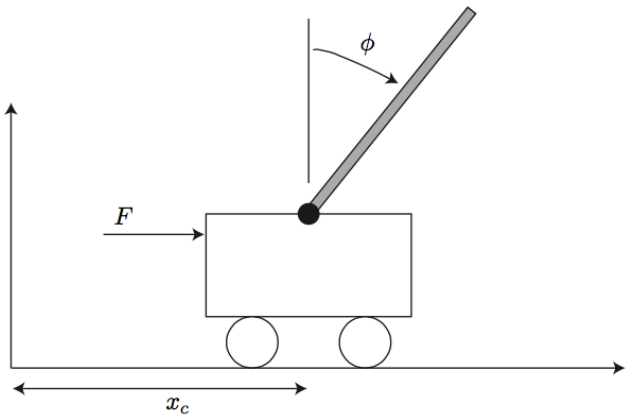
The closed form solution for the linear system under the feedback law $u(t) = -Kx(t)$ is

$$x(t) = e^{(A-BK)t}x(0).$$

Note the feedback causes the output to converge to zero.


4
3
2
1
0
-1
-2
0
2
4
6
8
10
output y(t)
time (s)

2. Consider the "pendulum on a cart" system:



The equations of motion for this system are

$$\gamma\ddot{x}_c - \beta\ddot{\phi}\cos\phi + \beta\dot{\phi}^2\sin\phi + \mu\dot{x}_c = F$$
$$\alpha\ddot{\phi} - \beta\ddot{x}_c\cos\phi - D\sin\phi = 0,$$

where α, β, γ, D, and μ are physical constants determined by the masses of the cart and pendulum, pendulum length, and friction, and F is an externally applied force. All units are mks, and angles are expressed in radians.

(a) (10 points) Define the state vector

$$x = \begin{bmatrix} x_c \\ \phi \\ \dot{x}_c \\ \dot{\phi} \end{bmatrix}$$

and the input $u = F$. Write the cart/pendulum equations of motion as a nonlinear state space equation. *hint:* It may help to convert the system into standard mechanical form as an intermediate step, and note that the matrix

$$M = \begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix}$$

is always invertible.

Solution:

We first rewrite the system in standard matrix form:

$$\begin{bmatrix} \gamma & -\beta cos\phi \\ -\beta cos\phi & \alpha \end{bmatrix}\begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} \mu & -\beta\dot{\phi}sin\phi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_c \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ -Dsin\phi \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

while noting that the input is a scalar control $u = F$. We then solve for the second derivatives $\ddot{x}_c$ and $\ddot{\phi}$ by inverting the first matrix:

$$\begin{aligned} \begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} &= \begin{bmatrix} \gamma & -\beta cos\phi \\ -\beta cos\phi & \alpha \end{bmatrix}^{-1}\left(\begin{bmatrix} F \\ 0 \end{bmatrix} - \begin{bmatrix} \mu & -\beta\dot{\phi}sin\phi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_c \\ \dot{\phi} \end{bmatrix} - \begin{bmatrix} 0 \\ -Dsin\phi \end{bmatrix}\right) \\ &= \frac{1}{\gamma\alpha - \beta^2 cos^2\phi}\begin{bmatrix} \alpha & \beta cos\phi \\ \beta cos\phi & \gamma \end{bmatrix}\left(\begin{bmatrix} F \\ 0 \end{bmatrix} - \begin{bmatrix} \mu & -\beta\dot{\phi}sin\phi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_c \\ \dot{\phi} \end{bmatrix} - \begin{bmatrix} 0 \\ -Dsin\phi \end{bmatrix}\right) \end{aligned}$$

Finally, we may perform variable replacement to rewrite in state space form:

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_c \\ \phi \\ \dot{x}_c \\ \dot{\phi} \end{bmatrix}, \quad \dot{\mathbf{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{x}_c \\ \dot{\phi} \\ \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \ddot{x}_c \\ \ddot{\phi} \end{bmatrix}$$

(b) (10 points) Describe the set of equilibrium points for the system. Describe them mathematically (i.e., with equations) and in English (i.e., what do they physically mean).

Solution:

At an equilibrium, the system state is unchanging. This implies that the state derivative vector, $\dot{\mathbf{X}}$, is zero. This means that $\dot{x}_c, \dot{\phi}, \ddot{x}_c$ and $\ddot{\phi}$ must all be zero. Plugging these values into the original dynamics equations produces the two equations $F = 0$ and $-D sin\phi = 0 \Rightarrow \phi = n\pi, n \in \mathbb{Z}$. Physically, this means that neither the cart nor the pendulum is moving (zero velocity) nor accelerating. Further, the applied force must be zero, and the pendulum must be vertically aligned, pointing straight up or straight down.

(c) (5 points) Let $\gamma = 2$, $\alpha = 1$, $\beta = 1$, $D = 1$, and $\mu = 3$. The linearized system about the equilibrium point at $x = 0$ is

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t).$$

Compute the eigenvalues of A, and use them to say whatever you can about the stability of the equilibrium point at $x = 0$ for both the linearized system and original nonlinear system.

Solution: The eigenvalues of the linearized system are

$$\Lambda = \begin{bmatrix} 0 & -3.3301 & 1.1284 & -0.7984 \end{bmatrix}$$

Since there is a positive eigenvalue, the equilibrium point $x = 0$ is unstable for both the linearized and nonlinear system.

(d) (15 points) Assume that the entire state can be measured directly (i.e., there is no need for an observer). Letting $R = 10$ and

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

use the MATLAB lqr command to find the corresponding optimal feedback control $u(t) = -K_c x(t)$.

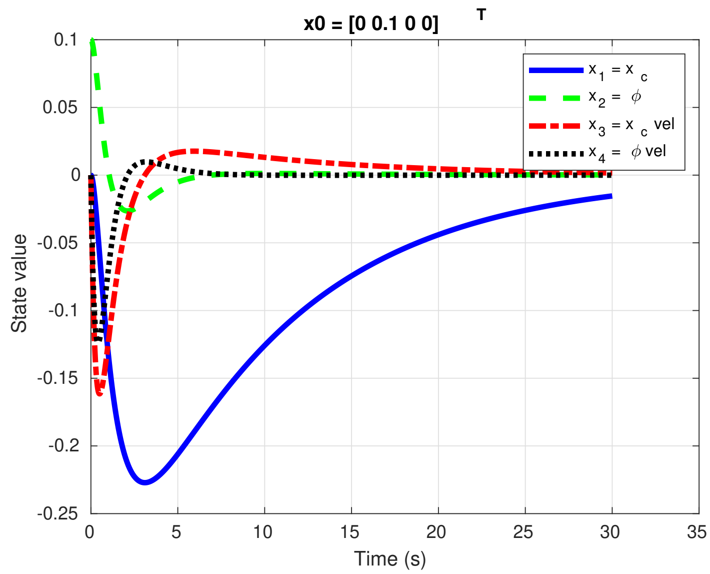
Using a timestep of $T = 0.01$ seconds and a final time of $t_f = 30$ seconds and an initial state of $x_0 = [0,\ 0.1,\ 0,\ 0]^T$, calculate and plot the state of the *linearized system* under the feedback control law above. You may either write your own 4th order Runge-Kutta routine to solve the state equation, or use the MATLAB function `ode45`. Repeat for $x_0 = [0,\ 0.5,\ 0,\ 0]^T$, $x_0 = [0\, 1.0886\, 0\, 0]^T$, and $x_0 = [0,\ 1.1,\ 0,\ 0]^T$.

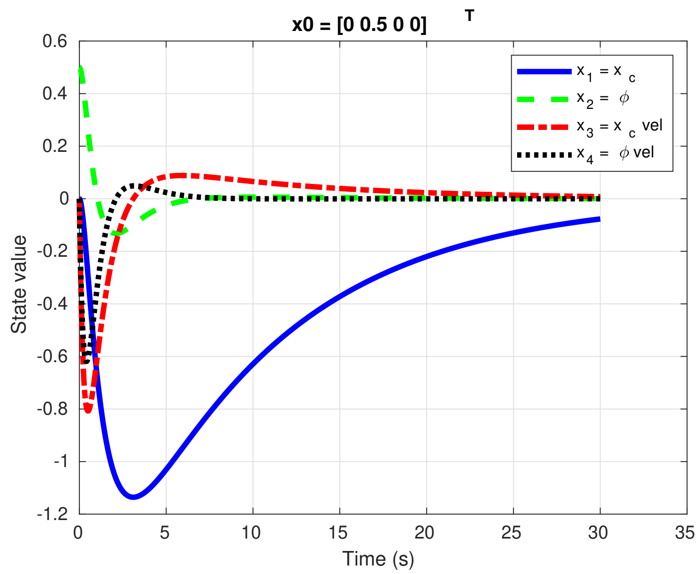
Solution:

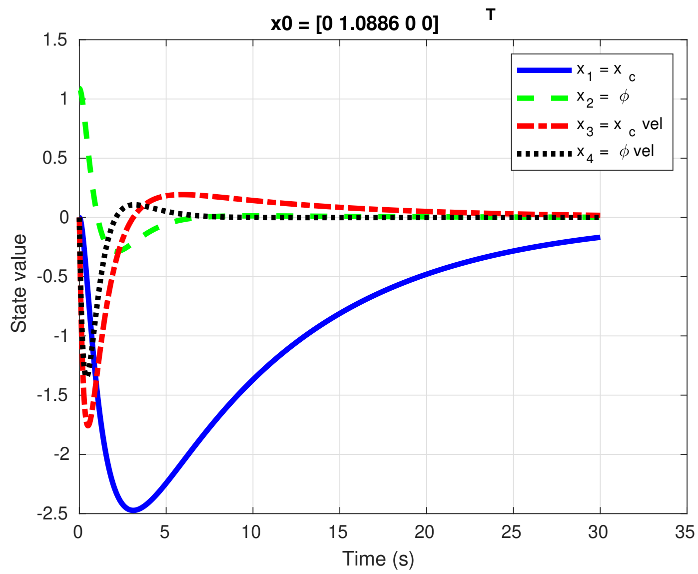
Using `lqr` we get the following K matrix:

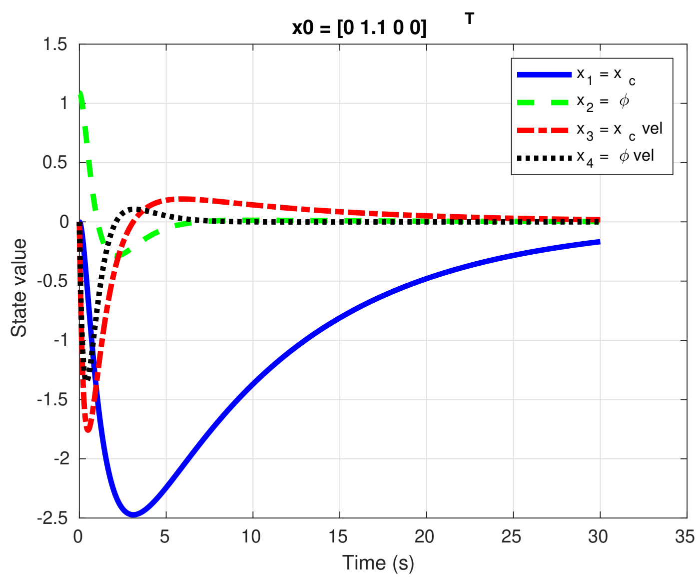
$$K = \begin{bmatrix} -0.3162 & 10.2723 & -6.7857 & 9.2183 \end{bmatrix}$$

And plots for each initial condition:


x0 = [0 0.1 0 0]T
State value
Time (s)
x1 = x c
x2 = ϕ
x3 = x c vel
x4 = ϕ vel


x0 = [0 0.5 0 0]T
State value
Time (s)
x1 = x c
x2 = ϕ
x3 = x c vel
x4 = ϕ vel



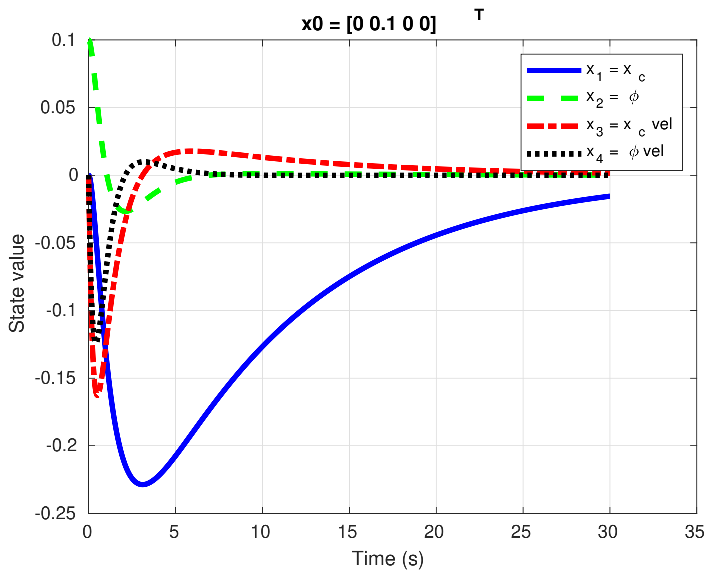


(e) (5 points) Repeat part 2d using the full nonlinear state equations in the simulation (as opposed to the linearized state equations). Explain any differences you see in the results.

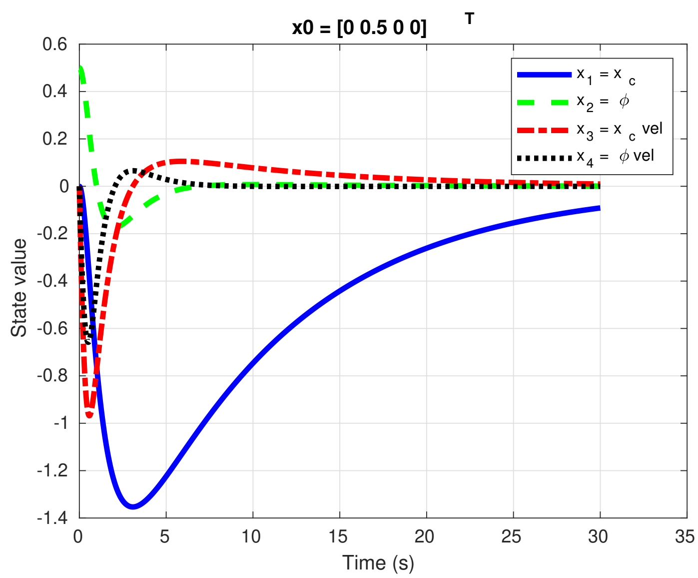
Solution:

At $x = \begin{bmatrix} 0 & 1.0886 & 0 & 0 \end{bmatrix}$, the non-linear system exhibits more oscillations than the linearized system before converging to the steady state. At $x = \begin{bmatrix} 0 & 1.1 & 0 & 0 \end{bmatrix}$, the non-linear system fully diverges from the behavior of the linearized system. In fact, the non-linear system is unstable.

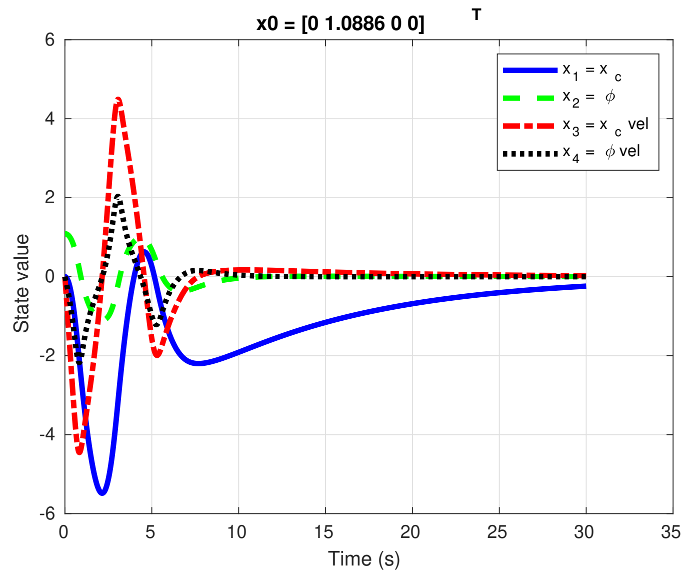
Plots of the non-linear system for each initial condition:

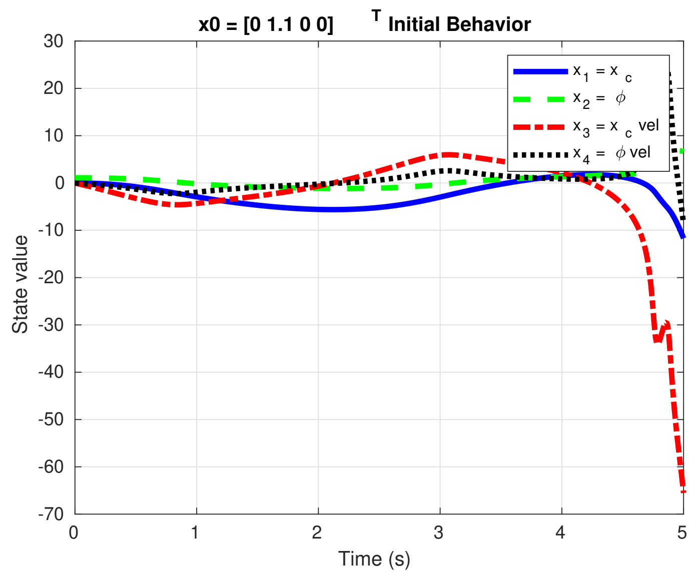


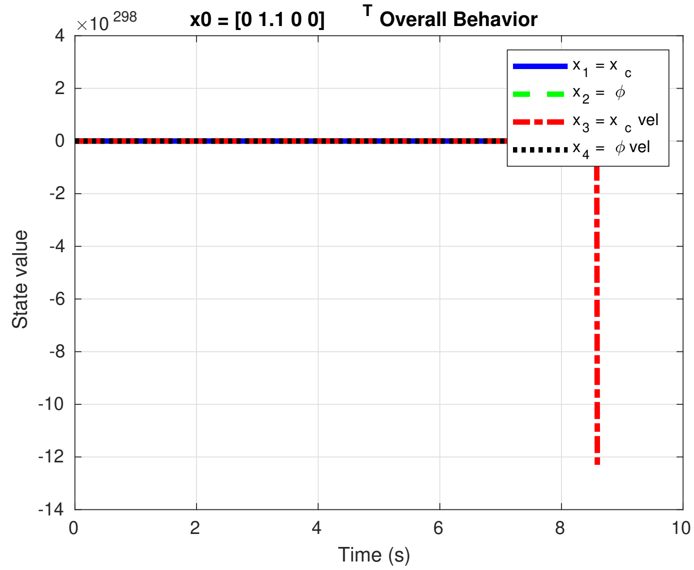
x0 = [0 0.1 0 0]
T
$x_1 = x_c$
$x_2 = \phi$
$x_3 = x_c$ vel
$x_4 = \phi$ vel
State value
Time (s)



x0 = [0 0.5 0 0]
T
$x_1 = x_c$
$x_2 = \phi$
$x_3 = x_c$ vel
$x_4 = \phi$ vel
State value
Time (s)


x0 = [0 1.0886 0 0]
T
x 1 = x c
x 2 = ϕ
x 3 = x c vel
x 4 = ϕ vel
State value
Time (s)


x0 = [0 1.1 0 0]
T Initial Behavior
x 1 = x c
x 2 = ϕ
x 3 = x c vel
x 4 = ϕ vel
State value
Time (s)



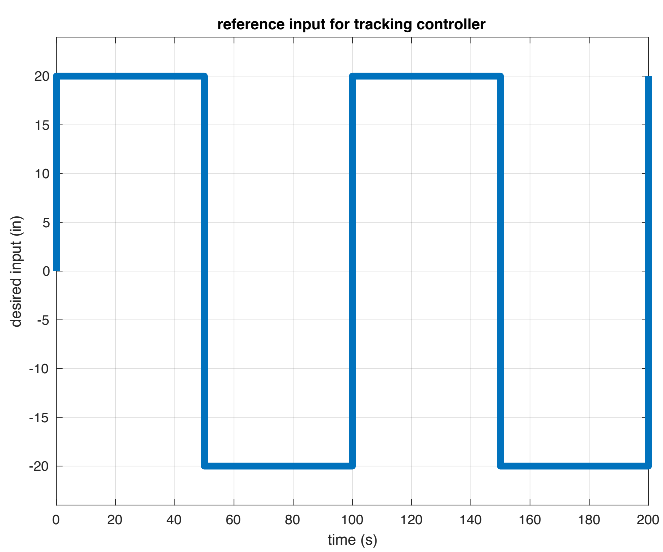
(f) (5 points) Assume you have an output y given by a sensor that measures the cart position *in inches*. Find the matrix C so that $y = Cx$.

Solution:

Since the state vector is in MKS, the measurement matrix is

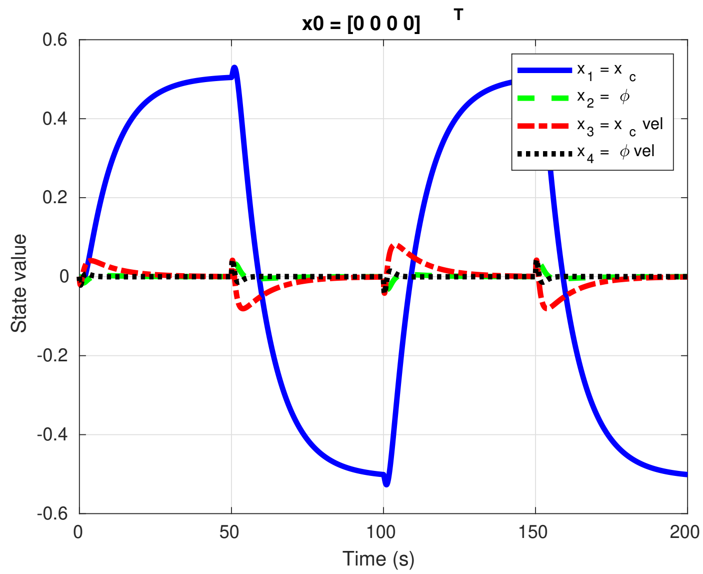
$$C = \begin{bmatrix} 39.3701 & 0 & 0 & 0 \end{bmatrix}.$$

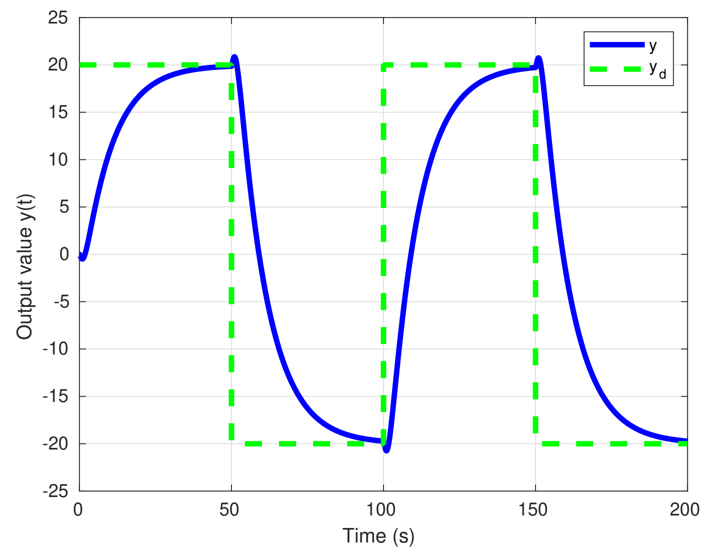
(g) (15 points) Using the LQR controller designed above, create a tracking controller that allows you to specify a desired cart position trajectory. Test this tracking controller for a desired output y_d that is a square wave as shown in the picture below. Simulate the full nonlinear dynamics using $T = 0.01$ seconds, $t_f = 200$ seconds, and $x_0 = [0,\ 0,\ 0,\ 0]^T$. Plot the state vs. time. On a separate graph, make a plot that overlays the desired and actual outputs. Explain what is going on.



Solution:

The tracking performance of the LQR controller for a square wave is shown below. We see that the system has a long rise time and no overshoot. This indicates that the system is likely overdamped with these gains.

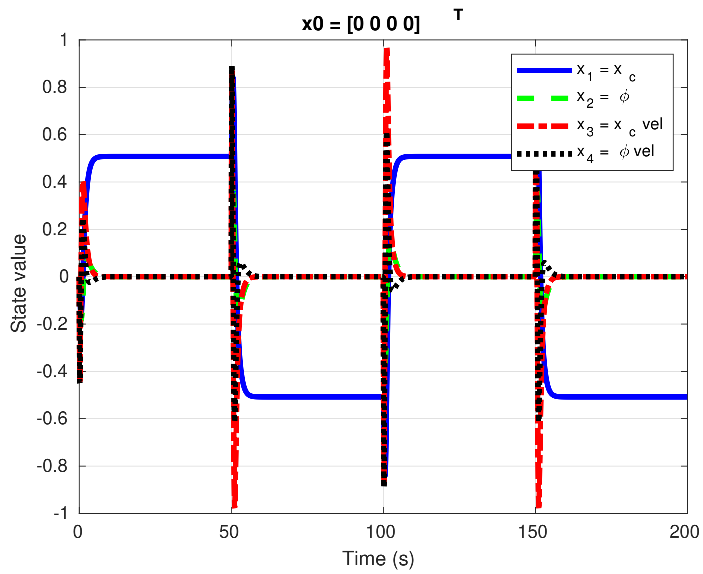


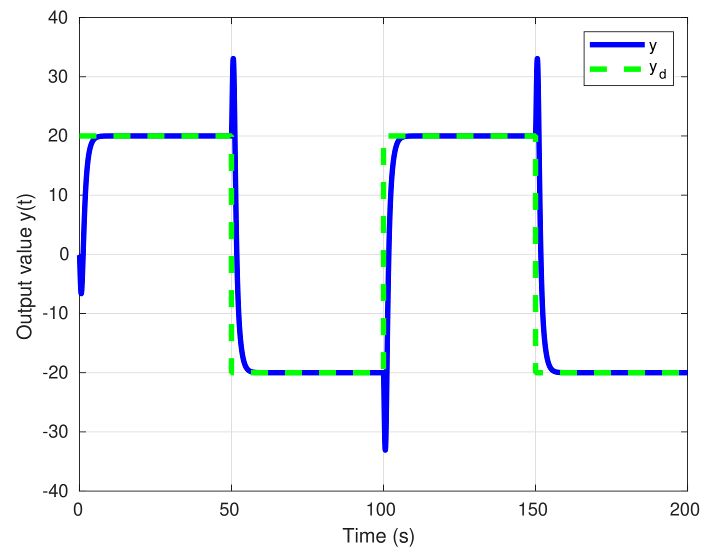


(h) (10 points) Choose new Q and R for the LQR controller to make the tracking controller better. Explain what you mean by "better", and describe your reasoning for how you changed Q and R. Simulate the system using the same parameters as above, and demonstrate your improvement by plotting the desired and actual outputs.

Solution:

We set our goal to be faster output tracking. Since the previous gains showed slow rise time in the cart position, we increased the gain on the cart position to encourage faster tracking. We used the following gains, $Q = \text{diag}([1000, 5, 1, 5])$ and $R = 10$. The simulation plots verify that our new gains accomplish faster output tracking, as desired.


x0 = [0 0 0 0] T
State value
Time (s)
x 1 = x c
x 2 = ϕ
x 3 = x c vel
x 4 = ϕ vel


Output value y(t)
Time (s)
y
y d