1 Typing Rules

$\boxed{\Theta; \Delta; \Gamma \vdash e : t}$ Typing rules for expressions

$$\frac{}{\Theta; \Delta; \cdot, x : t \vdash x : t} \quad \textsc{Ty\_Var\_Lin}$$

$$\frac{x : t \in \Delta}{\Theta; \Delta; \cdot \vdash x : t} \quad \textsc{Ty\_Var}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma', x : t \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Let}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash () : \mathbf{unit}} \quad \textsc{Ty\_Unit\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash e : \mathbf{unit} \\ \Theta; \Delta; \Gamma \vdash e' : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{let}\, () = e\, \mathbf{in}\, e' : t} \quad \textsc{Ty\_Unit\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{true} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_True}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{false} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_False}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !\mathbf{bool} \\ \Theta; \Delta; \Gamma' \vdash e_1 : t' \\ \Theta; \Delta; \Gamma' \vdash e_2 : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{if}\, e\, \mathbf{then}\, e_1\, \mathbf{else}\, e_2 : t} \quad \textsc{Ty\_Bool\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash k : \mathbf{int}} \quad \textsc{Ty\_Int\_Intro}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash el : \mathbf{elt}} \quad \textsc{Ty\_Elt\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash v : t \\ v \neq l\end{array}}{\Theta; \Delta; \cdot \vdash \mathbf{Many}\, v : !t} \quad \textsc{Ty\_Bang\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !t \\ \Theta; \Delta, x : t; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, \mathbf{Many}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Bang\_Elim}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash (e, e') : t \otimes t'} \quad \textsc{Ty\_Pair\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e_{12} : t_1 \otimes t_2 \\ \Theta; \Delta; \Gamma', a : t_1, b : t_2 \vdash e : t\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, (a, b) = e_{12}\, \mathbf{in}\, e : t} \quad \textsc{Ty\_Pair\_Elim}$$

$$\frac{\begin{array}{l}\Theta \vdash t'\ \mathsf{Type} \\ \Theta; \Delta; \Gamma, x : t' \vdash e : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{fun}\, x : t' \to e : t' \multimap t} \quad \textsc{Ty\_Lambda}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t' \multimap t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash e\ e' : t} \quad \textsc{Ty\_App}$$

$$\frac{\Theta, fc; \Delta; \Gamma \vdash e : t}{\Theta; \Delta; \Gamma \vdash \mathbf{fun}\, fc \to e : \forall fc.t} \quad \textsc{Ty\_Gen}$$

$$\frac{\begin{array}{l}\Theta \vdash f\ \mathsf{Cap} \\ \Theta; \Delta; \Gamma \vdash e : \forall fc.t\end{array}}{\Theta; \Delta; \Gamma \vdash e[f] : t[f/fc]} \quad \textsc{Ty\_Spc}$$

$$\frac{\Theta; \Delta, g : t \multimap t'; \cdot, x : t \vdash e : t'}{\Theta; \Delta; \cdot \vdash \mathbf{fix}\,(g, x : t, e : t') : !(t \multimap t')} \quad \textsc{Ty\_Fix}$$

2 Operational Semantics

$\boxed{\langle \sigma, e\rangle \to \mathit{StepsTo}}$ operational semantics

$$\frac{}{\langle \sigma, \mathbf{let}\ () = ()\ \mathbf{in}\ e\rangle \to \langle \sigma, e\rangle}\ \textsc{Op\_Let\_Unit}$$

$$\frac{}{\langle \sigma, \mathbf{let}\ x = v\ \mathbf{in}\ e\rangle \to \langle \sigma, e[x/v]\rangle}\ \textsc{Op\_Let\_Var}$$

$$\frac{}{\langle \sigma, \mathbf{if}\ (\mathbf{Many}\ \mathbf{true})\ \mathbf{then}\ e_1\ \mathbf{else}\ e_2\rangle \to \langle \sigma, e_1\rangle}\ \textsc{Op\_If\_True}$$

$$\frac{}{\langle \sigma, \mathbf{if}\ (\mathbf{Many}\ \mathbf{false})\ \mathbf{then}\ e_1\ \mathbf{else}\ e_2\rangle \to \langle \sigma, e_2\rangle}\ \textsc{Op\_If\_False}$$

$$\frac{}{\langle \sigma, \mathbf{let}\ \mathbf{Many}\ x = \mathbf{Many}\ v\ \mathbf{in}\ e\rangle \to \langle \sigma, e[x/v]\rangle}\ \textsc{Op\_Let\_Many}$$

$$\frac{e_1 = e[g/\mathbf{let}\ \mathbf{Many}\ g = \mathbf{fix}\,(g, x : t, e : t')\ \mathbf{in}\ \mathbf{fun}\ x : t \to e]}{\langle \sigma, \mathbf{let}\ \mathbf{Many}\ g = \mathbf{fix}\,(g, x : t, e : t')\ \mathbf{in}\ e'\rangle \to \langle \sigma, e'[g/\mathbf{fun}\ x : t \to e_1]\rangle}\ \textsc{Op\_Let\_Fix}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\ fc \to v)[f]\rangle \to \langle \sigma, v[fc/f]\rangle}\ \textsc{Op\_Frac\_Cap}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\ x : t \to e)\ v\rangle \to \langle \sigma, e[x/v]\rangle}\ \textsc{Op\_App}$$

$$\frac{\langle \sigma, e\rangle \to \langle \sigma', e'\rangle}{\langle \sigma, C[e]\rangle \to \langle \sigma, C[e']\rangle}\ \textsc{Op\_Context}$$

$$\frac{\langle \sigma, e\rangle \to \mathbf{err}}{\langle \sigma, C[e]\rangle \to \mathbf{err}}\ \textsc{Op\_Context\_Err}$$

$$\frac{0 \le k_1, k_2}{\langle \sigma, \mathbf{matrix}\ k_1\ k_2\rangle \to \langle \sigma \uplus \{l \mapsto_1 M_{k_1,k_2}\}, l\rangle}\ \textsc{Op\_Matrix}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_1 m_{k_1,k_2}\}, \mathbf{free}\ l\rangle \to \langle \sigma, ()\rangle}\ \textsc{Op\_Free}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, \mathbf{share}\ l\rangle \to \langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, (l, l)\rangle}\ \textsc{Op\_Share}$$

$$\frac{f \le 1}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\ l\ l\rangle \to \langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, l\rangle}\ \textsc{Op\_Unshare\_Eq}$$

$$\frac{l \ne l'}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l' \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\ l\ l'\rangle \to \mathbf{err}}\ \textsc{Op\_Unshare\_Neq}$$

$$\frac{\sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2,k_3}\}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \mathbf{gemm}\ l_1\ l_2\ l_3\rangle \to \langle \sigma' \uplus \{l_3 \mapsto_1 (m_1\ m_2 + m_3)_{k_1,k_3}\}, ((l_1, l_2), l_3)\rangle}\ \textsc{Op\_Gemm\_Match}$$

$$\frac{\begin{array}{l} k_2 \ne k_2' \\ \sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2',k_3}\} \end{array}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \mathbf{gemm}\ l_1\ l_2\ l_3\rangle \to \mathbf{err}}\ \textsc{Op\_Gemm\_Mismatch}$$

3 Interpretation

$$\mathcal{V}_k[\![\mathbf{unit}]\!] = \{(\emptyset, *)\}$$

$$\mathcal{V}_k[\![\mathbf{bool}]\!] = \{(\emptyset, true), (\emptyset, false)\}$$

$$\mathcal{V}_k[\![\mathbf{int}]\!] = \{(\emptyset, n) \mid 2^{-63} \leq n \leq 2^{63} - 1\}$$

$$\mathcal{V}_k[\![\mathbf{elt}]\!] = \{(\emptyset, f) \mid f \text{ a IEEE Float64 }\}$$

$$\mathcal{V}_k[\![f\,\mathbf{mat}]\!] = \{(\{l \mapsto_{2^{-f}} \_\}, l)\}$$

$$\begin{aligned}\mathcal{V}_k[\![!(t' \multimap t'')]\!] = &\{(\emptyset, \mathbf{Many}\, v) \mid (\emptyset, v) \in \mathcal{V}_k[\![t' \multimap t'']\!]\} \\ &\cup \{(\emptyset, \mathbf{fix}(g, x : t, e : t')) \mid \forall\, j \leq k, (\sigma', v') \in \mathcal{V}_j[\![t']\!]. \\ &\qquad (\sigma', \mathbf{let\,Many}\, g = \mathbf{fix}\,(g, x : t, e : t')\,\mathbf{in}\, g\, v) \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\mathcal{V}_k[\![!t]\!] = \{(\emptyset, \mathbf{Many}\, v) \mid \neg(\exists t', t''.\ t = t' \multimap t'') \wedge (\emptyset, v) \in \mathcal{V}_k[\![t]\!]\}$$

$$\mathcal{V}_k[\![\forall fc.\ t]\!] = \{(\sigma, \mathbf{fun}\, fc \rightarrow\ v) \mid \forall f.\ (\sigma, v[fc/f]) \in \mathcal{V}_k[\![t[fc/f]]\!]\}$$

$$\mathcal{V}_k[\![t' \otimes t'']\!] = \{(\sigma, (v', v'')) \mid \exists \sigma', \sigma''.\ (\sigma', v') \in \mathcal{V}_k[\![t]\!] \wedge (\sigma'', v'') \in \mathcal{V}_k[\![t'']\!] \wedge \sigma = \sigma' \star \sigma''\}$$

$$\begin{aligned}\mathcal{V}_k[\![t \multimap t']\!] = &\{(\sigma, \mathbf{fun}\, x : t \rightarrow e) \mid \forall j \leq k, (\sigma', v') \in \mathcal{V}_j[\![t']\!].\ \sigma \star \sigma' \text{ defined } \Rightarrow \\ &\quad (\sigma \star \sigma', (\mathbf{fun}\, x : t \rightarrow e)\, v') \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}_k[\![t]\!] = &\{(\sigma_s, e) \mid \forall\, j < k, \sigma_r.\ \sigma_s \star \sigma_r \text{ defined } \Rightarrow \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \mathbf{err}\ \vee \exists \sigma_f, e'. \\ &\quad \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \langle \sigma_f \star \sigma_r, e'\rangle \wedge (e' \text{ is a value } \Rightarrow \langle \sigma_f \star \sigma_r, e'\rangle \in \mathcal{V}_{k-j}[\![t]\!])\}\end{aligned}$$

$$\mathcal{I}_k[\![\cdot]\!]\theta = \{[]\}$$

$$\mathcal{I}_k[\![\Delta, x : t]\!]\theta = \{\delta[x \mapsto v_x] \mid \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \wedge (\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\mathcal{L}_k[\![\cdot]\!]\theta = \{(\emptyset, [])\}$$

$$\mathcal{L}_k[\![\Gamma, x : t]\!]\theta = \{(\sigma \uplus \sigma_x, \gamma[x \mapsto v_x]) \mid (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge (\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\begin{aligned}[\![\Theta; \Delta; \Gamma \vdash e : t]\!] = &\forall \theta, k, \delta, \gamma, \sigma.\ \mathrm{dom}(\Theta) = \mathrm{dom}(\theta) \wedge (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \Rightarrow \\ &\quad (\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]\end{aligned}$$

4 Soundness Proof

Proof sketch: Use the contrapositive both ways. This turns the negated existential into witnesses we can work with.

Let: $\phi(X) =$

Note: $\forall X.\ \phi(X) \subseteq X$, $\not\subset \equiv \not\subseteq \vee =$ *and* $\not\subseteq \Rightarrow \neq$

a, b, c be elements of the Martelli's semiring

$L^+ = a \cup \phi$

$L = \phi(L^+) = a \otimes (b \oplus c)$

$M^+ =$

$M = \phi(M^+) = (a \otimes b) \oplus (a \otimes c)$

Prove: Distributivity holds, i.e. $L = M$.

Suffices: Since $\oplus$ and $\otimes$ are commutative (definitions of $\oplus$ and $\otimes$ are symmetric in their arguments because $\exists x.\ \exists y.\ P(x, y) \Leftrightarrow \exists y.\ \exists x.\ P(x, y)$ and $\cup$ is commutative) it suffices to show only left-distributivity.

Proof: We show $L \subseteq M$ and $M \subseteq L$.

$\langle 1\rangle 1$. Case: $L \subseteq M$.

We show $m \notin M \Rightarrow m \notin L$ for arbitrary m.

Proof: We do this by cases on $m \in M^+$.

Suffices: Because $L \subseteq L^+$, to show $m \notin L$ it suffices to show either $m \notin L^+$ or $\exists y \in L^+.\ y \subset m$.

$\langle 2\rangle 1$. Case: $m \in M^+$.

This means that $\exists x \in a \otimes b,\ y \in a \otimes c.\ x \cup y = m$ and because $m \notin M$, we have $\exists x' \in a \otimes b,\ y \in a \otimes c.\ x' \cup y' = m' \subset m = x \cup y$. Assume, without loss of generality, they are the smallest such x' and y'. Because $\phi(X) \subseteq X$ for any X, we proceed by cases: either $x' \in a$ or $y' \in a$ or both $x' \in b$ and $y' \in c$.

$\langle 2\rangle 2$. Case: $m \notin M^+$.

This means $\forall\ x \in \phi(a \cup b),\ y \in \phi(a \cup c).\ m \neq x \cup y$.

$\langle 2\rangle 3$. **Thus, if $m \notin M$, then $m \notin L$.** Q.E.D.

$\langle 1\rangle 2$. Case: $M \subseteq L$.

We show $l \notin L \Rightarrow l \notin M$ for arbitrary l.

Suffices: Because $M \subseteq M^+$, to show $l \notin M$ it suffices to show either $l \notin M^+$ or $\exists y \in M^+.\ y \subset l$.

$\langle 2\rangle 1$. Case: $l \notin L^+$.

This means $l \notin a$ and $l \notin b \oplus c = \phi$.

We conclude from the latter, that $\forall x \in b,\ y \in c.\ x \cup y \neq l$.

We reason by cases on *why* $l \notin a$, to show that $\exists y \in M^+.\ y \subset l$ or $l \notin M^+$.

$\langle 2\rangle 2$. Case: $l \in L^+$.

Under the assumption $l \notin L$, we need only consider two cases: the rest produce the contradiction $l \in L$.

$\langle 1\rangle 3$. **Thus,** $L = M$ Q.E.D.