# 1 Typing Rules

$\boxed{\Theta; \Delta; \Gamma \vdash e : t}$ Typing rules for expressions

$$\frac{}{\Theta; \Delta; \cdot, x : t \vdash x : t} \quad \textsc{Ty\_Var\_Lin}$$

$$\frac{x : t \in \Delta}{\Theta; \Delta; \cdot \vdash x : t} \quad \textsc{Ty\_Var}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma', x : t \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Let}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash () : \mathbf{unit}} \quad \textsc{Ty\_Unit\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash e : \mathbf{unit} \\ \Theta; \Delta; \Gamma \vdash e' : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{let}\, () = e\, \mathbf{in}\, e' : t} \quad \textsc{Ty\_Unit\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{true} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_True}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{false} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_False}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !\mathbf{bool} \\ \Theta; \Delta; \Gamma' \vdash e_1 : t' \\ \Theta; \Delta; \Gamma' \vdash e_2 : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{if}\, e\, \mathbf{then}\, e_1\, \mathbf{else}\, e_2 : t} \quad \textsc{Ty\_Bool\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash k : \mathbf{int}} \quad \textsc{Ty\_Int\_Intro}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash el : \mathbf{elt}} \quad \textsc{Ty\_Elt\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash v : t \\ v \neq l\end{array}}{\Theta; \Delta; \cdot \vdash \mathbf{Many}\, v : !t} \quad \textsc{Ty\_Bang\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !t \\ \Theta; \Delta, x : t; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, \mathbf{Many}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Bang\_Elim}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash (e, e') : t \otimes t'} \quad \textsc{Ty\_Pair\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e_{12} : t_1 \otimes t_2 \\ \Theta; \Delta; \Gamma', a : t_1, b : t_2 \vdash e : t\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, (a, b) = e_{12}\, \mathbf{in}\, e : t} \quad \textsc{Ty\_Pair\_Elim}$$

$$\frac{\begin{array}{l}\Theta \vdash t'\ \mathsf{Type} \\ \Theta; \Delta; \Gamma, x : t' \vdash e : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{fun}\, x : t' \rightarrow e : t' \multimap t} \quad \textsc{Ty\_Lambda}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t' \multimap t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash e\, e' : t} \quad \textsc{Ty\_App}$$

$$\frac{\Theta, fc; \Delta; \Gamma \vdash e : t}{\Theta; \Delta; \Gamma \vdash \mathbf{fun}\, fc \to e : \forall fc.t} \quad \textsc{Ty\_Gen}$$

$$\frac{\begin{array}{l}\Theta \vdash f \ \mathsf{Cap} \\ \Theta; \Delta; \Gamma \vdash e : \forall fc.t\end{array}}{\Theta; \Delta; \Gamma \vdash e[f] : t[f/fc]} \quad \textsc{Ty\_Spc}$$

$$\frac{\Theta; \Delta, g : t \multimap t'; \cdot, x : t \vdash e : t'}{\Theta; \Delta; \cdot \vdash \mathbf{fix}\, (g, x : t, e : t') : !(t \multimap t')} \quad \textsc{Ty\_Fix}$$

## 2 Operational Semantics

$\boxed{\langle \sigma, e \rangle \to \mathit{StepsTo}}$ operational semantics

$$\frac{}{\langle \sigma, \mathbf{let}\,() = ()\,\mathbf{in}\, e \rangle \to \langle \sigma, e \rangle} \quad \textsc{Op\_Let\_Unit}$$

$$\frac{}{\langle \sigma, \mathbf{let}\, x = v\,\mathbf{in}\, e \rangle \to \langle \sigma, e[x/v] \rangle} \quad \textsc{Op\_Let\_Var}$$

$$\frac{}{\langle \sigma, \mathbf{if}\,(\mathbf{Many}\,\mathbf{true})\,\mathbf{then}\, e_1\,\mathbf{else}\, e_2 \rangle \to \langle \sigma, e_1 \rangle} \quad \textsc{Op\_If\_True}$$

$$\frac{}{\langle \sigma, \mathbf{if}\,(\mathbf{Many}\,\mathbf{false})\,\mathbf{then}\, e_1\,\mathbf{else}\, e_2 \rangle \to \langle \sigma, e_2 \rangle} \quad \textsc{Op\_If\_False}$$

$$\frac{}{\langle \sigma, \mathbf{let}\,\mathbf{Many}\, x = \mathbf{Many}\, v\,\mathbf{in}\, e \rangle \to \langle \sigma, e[x/v] \rangle} \quad \textsc{Op\_Let\_Many}$$

$$\frac{e_1 = e[g/\mathbf{let}\,\mathbf{Many}\, g = \mathbf{fix}\,(g, x : t, e : t')\,\mathbf{in}\,\mathbf{fun}\, x : t \to e]}{\langle \sigma, \mathbf{let}\,\mathbf{Many}\, g = \mathbf{fix}\,(g, x : t, e : t')\,\mathbf{in}\, e' \rangle \to \langle \sigma, e'[g/\mathbf{fun}\, x : t \to e_1] \rangle} \quad \textsc{Op\_Let\_Fix}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\, fc \to v)[f] \rangle \to \langle \sigma, v[fc/f] \rangle} \quad \textsc{Op\_Frac\_Cap}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\, x : t \to e)\, v \rangle \to \langle \sigma, e[x/v] \rangle} \quad \textsc{Op\_App}$$

$$\frac{\langle \sigma, e \rangle \to \langle \sigma', e' \rangle}{\langle \sigma, C[e] \rangle \to \langle \sigma, C[e'] \rangle} \quad \textsc{Op\_Context}$$

$$\frac{\langle \sigma, e \rangle \to \mathbf{err}}{\langle \sigma, C[e] \rangle \to \mathbf{err}} \quad \textsc{Op\_Context\_Err}$$

$$\frac{0 \leq k_1, k_2}{\langle \sigma, \mathbf{matrix}\, k_1\, k_2 \rangle \to \langle \sigma \uplus \{l \mapsto_1 M_{k_1,k_2}\}, l \rangle} \quad \textsc{Op\_Matrix}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_1 m_{k_1,k_2}\}, \mathbf{free}\, l \rangle \to \langle \sigma, () \rangle} \quad \textsc{Op\_Free}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, \mathbf{share}\, l \rangle \to \langle \sigma \uplus \{l \mapsto_{\frac{1}{2} \cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2} \cdot f} m_{k_1,k_2}\}, (l, l) \rangle} \quad \textsc{Op\_Share}$$

$$\frac{f \leq 1}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2} \cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2} \cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\, l\, l \rangle \to \langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, l \rangle} \quad \textsc{Op\_Unshare\_Eq}$$

$$\frac{l \neq l'}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2} \cdot f} m_{k_1,k_2}\} \uplus \{l' \mapsto_{\frac{1}{2} \cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\, l\, l' \rangle \to \mathbf{err}} \quad \textsc{Op\_Unshare\_Neq}$$

$$\frac{\sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m_{1\,k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m_{2\,k_2,k_3}\}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m_{1\,k_1,k_3}\}, \mathbf{gemm}\, l_1\, l_2\, l_3 \rangle \to \langle \sigma' \uplus \{l_3 \mapsto_1 (m_1\, m_2 + m_3)_{k_1,k_3}\}, ((l_1, l_2), l_3) \rangle} \quad \textsc{Op\_Gemm\_Match}$$

$$\frac{\begin{array}{l} k_2 \neq k_2' \\ \sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m_{1\,k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m_{2\,k_2',k_3}\} \end{array}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m_{1\,k_1,k_3}\}, \mathbf{gemm}\, l_1\, l_2\, l_3 \rangle \to \mathbf{err}} \quad \textsc{Op\_Gemm\_Mismatch}$$

## 3 Interpretation

$$\mathcal{V}_k[\![\textbf{unit}]\!] = \{(\emptyset, *)\}$$

$$\mathcal{V}_k[\![\textbf{bool}]\!] = \{(\emptyset, true), (\emptyset, false)\}$$

$$\mathcal{V}_k[\![\textbf{int}]\!] = \{(\emptyset, n) \mid 2^{-63} \leq n \leq 2^{63} - 1\}$$

$$\mathcal{V}_k[\![\textbf{elt}]\!] = \{(\emptyset, f) \mid f \text{ a IEEE Float64 }\}$$

$$\mathcal{V}_k[\![f\,\textbf{mat}]\!] = \{(\{l \mapsto_{2^{-f}} \_\}, l)\}$$

$$\begin{aligned}\mathcal{V}_k[\![!(t' \multimap t'')]\!] = {} & \{(\emptyset, \textbf{Many}\, v) \mid (\emptyset, v) \in \mathcal{V}_k[\![t' \multimap t'']\!]\} \\ & \cup \{(\emptyset, \textbf{fix}(g, x : t, e : t')) \mid \forall\, j \leq k, (\sigma', v') \in \mathcal{V}_j[\![t']\!]. \\ & \qquad (\sigma', \textbf{let Many}\, g = \textbf{fix}\,(g, x : t, e : t')\, \textbf{in}\, g\, v) \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\mathcal{V}_k[\![!t]\!] = \{(\emptyset, \textbf{Many}\, v) \mid \neg(\exists t', t''.\ t = t' \multimap t'') \wedge (\emptyset, v) \in \mathcal{V}_k[\![t]\!]\}$$

$$\mathcal{V}_k[\![\forall fc.\ t]\!] = \{(\sigma, \textbf{fun}\, fc \rightarrow\ v) \mid \forall f.\ (\sigma, v[fc/f]) \in \mathcal{V}_k[\![t[fc/f]]\!]\}$$

$$\mathcal{V}_k[\![t' \otimes t'']\!] = \{(\sigma, (v', v'')) \mid \exists \sigma', \sigma''.\ (\sigma', v') \in \mathcal{V}_k[\![t]\!] \wedge (\sigma'', v'') \in \mathcal{V}_k[\![t'']\!] \wedge \sigma = \sigma' \star \sigma''\}$$

$$\begin{aligned}\mathcal{V}_k[\![t \multimap t']\!] = {} & \{(\sigma, \textbf{fun}\, x : t \rightarrow e) \mid \forall j \leq k, (\sigma', v') \in \mathcal{V}_j[\![t']\!].\ \sigma \star \sigma' \text{ defined } \Rightarrow \\ & \qquad (\sigma \star \sigma', (\textbf{fun}\, x : t \rightarrow e)\, v') \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}_k[\![t]\!] = {} & \{(\sigma_s, e) \mid \forall\, j < k, \sigma_r.\ \sigma_s \star \sigma_r \text{ defined } \Rightarrow \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \textbf{err} \ \vee \exists \sigma_f, e'. \\ & \qquad \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \langle \sigma_f \star \sigma_r, e'\rangle \wedge (e' \text{ is a value } \Rightarrow \langle \sigma_f \star \sigma_r, e'\rangle \in \mathcal{V}_{k-j}[\![t]\!])\}\end{aligned}$$

$$\mathcal{I}_k[\![\cdot]\!]\theta = \{[]\}$$

$$\mathcal{I}_k[\![\Delta, x : t]\!]\theta = \{\delta[x \mapsto v_x] \mid \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \wedge (\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\mathcal{L}_k[\![\cdot]\!]\theta = \{(\emptyset, [])\}$$

$$\mathcal{L}_k[\![\Gamma, x : t]\!]\theta = \{(\sigma \uplus \sigma_x, \gamma[x \mapsto v_x]) \mid (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge (\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\begin{aligned}[\![\Theta; \Delta; \Gamma \vdash e : t]\!] = {} & \forall \theta, k, \delta, \gamma, \sigma.\ \mathrm{dom}(\Theta) = \mathrm{dom}(\theta) \wedge (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \Rightarrow \\ & \qquad (\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]\end{aligned}$$

## 4 Soundness Proof

$$\forall \Theta, \Delta, \Gamma, e, t.\ \Theta; \Delta; \Gamma \vdash e : t \Rightarrow [\![\Theta; \Delta; \Gamma \vdash e : t]\!]$$

PROOF SKETCH: Induction over the typing judgements.

ASSUME: Arbitrary $\Theta, \Delta, \Gamma, e, t$ such that $\Theta; \Delta; \Gamma \vdash e : t$.
Arbitrary $\theta, k, \delta, \gamma, \sigma$ such that:
1. $\mathrm{dom}(\Theta) = \mathrm{dom}(\theta)$
2. $(\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta$
3. $\delta \in \mathcal{I}_k[\![\Delta]\!]\theta$.

PROVE: $(\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩1. CASE: TY_VAR_LIN.
PROVE: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩2. CASE: TY_VAR.
PROVE: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩3. CASE: TY_LET.
PROVE: $(\sigma, \gamma(\delta(\mathbf{let}\, x\ =\ e\, \mathbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩4. CASE: TY_UNIT_INTRO.
PROVE: $(\sigma, \gamma(\delta(\,()\,))) \in \mathcal{C}_k[\![\theta(\mathbf{unit})]\!]$.

⟨1⟩5. CASE: TY_UNIT_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{let}\, ()\ =\ e\, \mathbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩6. CASE: TY_BOOL_TRUE.
PROVE: $(\sigma, \gamma(\delta(\mathbf{true}))) \in \mathcal{C}_k[\![\theta(\mathbf{bool})]\!]$.

⟨1⟩7. CASE: TY_BOOL_FALSE, TY_INT_INTRO, TY_ELT_INTRO.
Similar to TY_BOOL_TRUE.

⟨1⟩8. CASE: TY_BOOL_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{if}\, e\, \mathbf{then}\, e_1\, \mathbf{else}\, e_2))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩9. CASE: TY_BANG_INTRO.
PROVE: $(\sigma, \gamma(\delta(\mathbf{Many}\, e))) \in \mathcal{C}_k[\![\theta(!t)]\!]$.

⟨1⟩10. CASE: TY_BANG_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{let}\, \mathbf{Many}\, x\ =\ e\, \mathbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 1 \rangle 11.$ CASE: TY_PAIR_INTRO.
PROVE: $(\sigma, \gamma(\delta(\,(e, e')\,))) \in \mathcal{C}_k[\![\theta(t \otimes t')]\!]$.

$\langle 1 \rangle 12.$ CASE: TY_PAIR_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{let}\,(a, b) \;=\; e\,\mathbf{in}\,e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 1 \rangle 13.$ CASE: TY_LAMBDA.
PROVE: $(\sigma, \gamma(\delta(\mathbf{fun}\,x : t' \to e))) \in \mathcal{C}_k[\![\theta(t' \multimap t)]\!]$.

$\langle 1 \rangle 14.$ CASE: TY_APP.
PROVE: $(\sigma, \gamma(\delta(\,e\,e'\,))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 1 \rangle 15.$ CASE: TY_GEN.
PROVE: $(\sigma, \gamma(\delta(\mathbf{fun}\,fc \to e))) \in \mathcal{C}_k[\![\theta(\forall\, fc.\, t)]\!]$.

$\langle 1 \rangle 16.$ CASE: TY_SPC.
PROVE: $(\sigma, \gamma(\delta(e\,[f]))) \in \mathcal{C}_k[\![\theta(t\,[fc/f])]\!]$.

$\langle 1 \rangle 17.$ CASE: TY_FIX.
PROVE: $(\sigma, \gamma(\delta(\mathbf{fix}(g, x : t, e : t')))) \in \mathcal{C}_k[\![\theta(!\,(t \multimap t'))]\!]$.