1 Typing Rules

$\boxed{\Theta; \Delta; \Gamma \vdash e : t}$ Typing rules for expressions

$$\frac{}{\Theta; \Delta; \cdot, x : t \vdash x : t} \quad \textsc{Ty\_Var\_Lin}$$

$$\frac{x : t \in \Delta}{\Theta; \Delta; \cdot \vdash x : t} \quad \textsc{Ty\_Var}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma', x : t \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Let}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash () : \mathbf{unit}} \quad \textsc{Ty\_Unit\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : \mathbf{unit} \\ \Theta; \Delta; \Gamma' \vdash e' : t\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, () = e\, \mathbf{in}\, e' : t} \quad \textsc{Ty\_Unit\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{true} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_True}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{false} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_False}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !\mathbf{bool} \\ \Theta; \Delta; \Gamma' \vdash e_1 : t' \\ \Theta; \Delta; \Gamma' \vdash e_2 : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{if}\, e\, \mathbf{then}\, e_1\, \mathbf{else}\, e_2 : t} \quad \textsc{Ty\_Bool\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash k : \mathbf{int}} \quad \textsc{Ty\_Int\_Intro}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash el : \mathbf{elt}} \quad \textsc{Ty\_Elt\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash v : t \\ v \neq l\end{array}}{\Theta; \Delta; \cdot \vdash \mathbf{Many}\, v : !t} \quad \textsc{Ty\_Bang\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !t \\ \Theta; \Delta, x : t; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, \mathbf{Many}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Bang\_Elim}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash (e, e') : t \otimes t'} \quad \textsc{Ty\_Pair\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e_{12} : t_1 \otimes t_2 \\ \Theta; \Delta; \Gamma', a : t_1, b : t_2 \vdash e : t\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, (a, b) = e_{12}\, \mathbf{in}\, e : t} \quad \textsc{Ty\_Pair\_Elim}$$

$$\frac{\begin{array}{l}\Theta \vdash t'\ \mathsf{Type} \\ \Theta; \Delta; \Gamma, x : t' \vdash e : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{fun}\, x : t' \to e : t' \multimap t} \quad \textsc{Ty\_Lambda}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t' \multimap t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash e\, e' : t} \quad \textsc{Ty\_App}$$

$$\frac{\Theta, fc; \Delta; \Gamma \vdash e : t}{\Theta; \Delta; \Gamma \vdash \textbf{fun}\, fc \to e : \forall fc.t} \quad \textsc{Ty\_Gen}$$

$$\frac{\begin{array}{l}\Theta \vdash f\ \mathsf{Cap} \\ \Theta; \Delta; \Gamma \vdash e : \forall fc.t\end{array}}{\Theta; \Delta; \Gamma \vdash e[f] : t[f/fc]} \quad \textsc{Ty\_Spc}$$

$$\frac{\Theta; \Delta, g : t \multimap t'; \cdot, x : t \vdash e : t'}{\Theta; \Delta; \cdot \vdash \textbf{fix}\,(g, x : t, e : t') : !(t \multimap t')} \quad \textsc{Ty\_Fix}$$

2 Operational Semantics

$\boxed{\langle \sigma, e \rangle \to \mathit{StepsTo}}$ operational semantics

$$\frac{}{\langle \sigma, \mathbf{let}\,() = ()\,\mathbf{in}\, e \rangle \to \langle \sigma, e \rangle} \quad \textsc{Op\_Let\_Unit}$$

$$\frac{}{\langle \sigma, \mathbf{let}\, x = v\,\mathbf{in}\, e \rangle \to \langle \sigma, e[x/v] \rangle} \quad \textsc{Op\_Let\_Var}$$

$$\frac{}{\langle \sigma, \mathbf{if}\,(\mathbf{Many}\,\mathbf{true})\,\mathbf{then}\, e_1\,\mathbf{else}\, e_2 \rangle \to \langle \sigma, e_1 \rangle} \quad \textsc{Op\_If\_True}$$

$$\frac{}{\langle \sigma, \mathbf{if}\,(\mathbf{Many}\,\mathbf{false})\,\mathbf{then}\, e_1\,\mathbf{else}\, e_2 \rangle \to \langle \sigma, e_2 \rangle} \quad \textsc{Op\_If\_False}$$

$$\frac{}{\langle \sigma, \mathbf{let}\,\mathbf{Many}\, x = \mathbf{Many}\, v\,\mathbf{in}\, e \rangle \to \langle \sigma, e[x/v] \rangle} \quad \textsc{Op\_Let\_Many}$$

$$\frac{}{\langle \sigma, \mathbf{let}\,(a, b) = (v_1, v_2)\,\mathbf{in}\, e \rangle \to \langle \sigma, e[a/v_1][b/v_2] \rangle} \quad \textsc{Op\_Let\_Pair}$$

$$\frac{e_1 = e[g/\mathbf{let}\,\mathbf{Many}\, g = \mathbf{fix}\,(g, x : t, e : t')\,\mathbf{in}\,\mathbf{fun}\, x : t \to e]}{\langle \sigma, \mathbf{let}\,\mathbf{Many}\, g = \mathbf{fix}\,(g, x : t, e : t')\,\mathbf{in}\, e' \rangle \to \langle \sigma, e'[g/\mathbf{fun}\, x : t \to e_1] \rangle} \quad \textsc{Op\_Let\_Fix}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\, fc \to v)[f] \rangle \to \langle \sigma, v[fc/f] \rangle} \quad \textsc{Op\_Frac\_Cap}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\, x : t \to e)\; v \rangle \to \langle \sigma, e[x/v] \rangle} \quad \textsc{Op\_App}$$

$$\frac{\langle \sigma, e \rangle \to \langle \sigma', e' \rangle}{\langle \sigma, C[e] \rangle \to \langle \sigma, C[e'] \rangle} \quad \textsc{Op\_Context}$$

$$\frac{\langle \sigma, e \rangle \to \mathbf{err}}{\langle \sigma, C[e] \rangle \to \mathbf{err}} \quad \textsc{Op\_Context\_Err}$$

$$\frac{0 \leq k_1, k_2}{\langle \sigma, \mathbf{matrix}\, k_1\; k_2 \rangle \to \langle \sigma \uplus \{l \mapsto_1 M_{k_1,k_2}\}, l \rangle} \quad \textsc{Op\_Matrix}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_1 m_{k_1,k_2}\}, \mathbf{free}\, l \rangle \to \langle \sigma, () \rangle} \quad \textsc{Op\_Free}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, \mathbf{share}\, l \rangle \to \langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, (l, l) \rangle} \quad \textsc{Op\_Share}$$

$$\frac{f \leq 1}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\, l\; l \rangle \to \langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, l \rangle} \quad \textsc{Op\_Unshare\_Eq}$$

$$\frac{l \neq l'}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l' \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\, l\; l' \rangle \to \mathbf{err}} \quad \textsc{Op\_Unshare\_Neq}$$

$$\frac{\sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2,k_3}\}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \mathbf{gemm}\, l_1\; l_2\; l_3 \rangle \to \langle \sigma' \uplus \{l_3 \mapsto_1 (m_1\, m_2 + m_3)_{k_1,k_3}\}, ((l_1, l_2), l_3) \rangle} \quad \textsc{Op\_Gemm\_Match}$$

$$\frac{\begin{array}{l} k_2 \neq k_2' \\ \sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2',k_3}\} \end{array}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \mathbf{gemm}\, l_1\; l_2\; l_3 \rangle \to \mathbf{err}} \quad \textsc{Op\_Gemm\_Mismatch}$$

3 Interpretation

$$\mathcal{V}_k[\![\textbf{unit}]\!] = \{(\emptyset, *)\}$$

$$\mathcal{V}_k[\![\textbf{bool}]\!] = \{(\emptyset, true), (\emptyset, false)\}$$

$$\mathcal{V}_k[\![\textbf{int}]\!] = \{(\emptyset, n) \mid 2^{-63} \leq n \leq 2^{63} - 1\}$$

$$\mathcal{V}_k[\![\textbf{elt}]\!] = \{(\emptyset, f) \mid f \text{ a IEEE Float64 }\}$$

$$\mathcal{V}_k[\![f\,\textbf{mat}]\!] = \{(\{l \mapsto_{2^{-f}} \_\}, l)\}$$

$$\begin{aligned}\mathcal{V}_k[\![!(t' \multimap t'')]\!] = {} & \{(\emptyset, \textbf{Many}\, v) \mid (\emptyset, v) \in \mathcal{V}_k[\![t' \multimap t'']\!]\} \\ & \cup \{(\emptyset, \textbf{fix}(g, x : t, e : t')) \mid \forall\, j < k, (\sigma', v') \in \mathcal{V}_j[\![t']\!]. \\ & \qquad (\sigma', \textbf{let Many}\, g = \textbf{fix}\,(g, x : t, e : t')\,\textbf{in}\, g\, v) \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\mathcal{V}_k[\![!t]\!] = \{(\emptyset, \textbf{Many}\, v) \mid \neg(\exists t', t''.\ t = t' \multimap t'') \wedge (\emptyset, v) \in \mathcal{V}_k[\![t]\!]\}$$

$$\mathcal{V}_k[\![\forall fc.\ t]\!] = \{(\sigma, \textbf{fun}\, fc \rightarrow\ v) \mid \forall f.\ (\sigma, (\textbf{fun}\, fc \rightarrow\ v)\,[f]) \in \mathcal{V}_k[\![t[fc/f]]\!]\}$$

$$\mathcal{V}_k[\![t \otimes t']\!] = \{(\sigma \star \sigma', (v, v')) \mid (\sigma, v) \in \mathcal{V}_k[\![t]\!] \wedge (\sigma', v') \in \mathcal{V}_k[\![t']\!]\}$$

$$\begin{aligned}\mathcal{V}_k[\![t \multimap t']\!] = {} & \{(\sigma, \textbf{fun}\, x : t \rightarrow e) \mid \forall j < k, (\sigma', v') \in \mathcal{V}_j[\![t']\!].\ \sigma \star \sigma' \text{ defined } \Rightarrow \\ & \qquad (\sigma \star \sigma', (\textbf{fun}\, x : t \rightarrow e)\, v') \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}_k[\![t]\!] = {} & \{(\sigma_s, e) \mid \forall\, j \leq k, \sigma_r.\ \sigma_s \star \sigma_r \text{ defined } \Rightarrow \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \textbf{err}\ \vee \exists \sigma_f, e'. \\ & \qquad \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \langle \sigma_f \star \sigma_r, e'\rangle \wedge (e' \text{ is a value } \Rightarrow (\sigma_f \star \sigma_r, e') \in \mathcal{V}_{k-j}[\![t]\!])\}\end{aligned}$$

$$\mathcal{I}_k[\![\cdot]\!]\theta = \{[]\}$$

$$\mathcal{I}_k[\![\Delta, x : t]\!]\theta = \{\delta[x \mapsto v_x] \mid \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \wedge (\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\mathcal{L}_k[\![\cdot]\!]\theta = \{(\emptyset, [])\}$$

$$\mathcal{L}_k[\![\Gamma, x : t]\!]\theta = \{(\sigma \star \sigma_x, \gamma[x \mapsto v_x]) \mid (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge (\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\begin{aligned}[\![\Theta; \Delta; \Gamma \vdash e : t]\!] = {} & \forall \theta, k, \delta, \gamma, \sigma.\ \text{dom}(\Theta) = \text{dom}(\theta) \wedge (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \Rightarrow \\ & \qquad (\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]\end{aligned}$$

4 Soundness Proof

$$\forall \Theta, \Delta, \Gamma, e, t.\ \Theta; \Delta; \Gamma \vdash e : t \Rightarrow [\![\Theta; \Delta; \Gamma \vdash e : t]\!]$$

Proof sketch: Induction over the typing judgements.

Assume: 1. Arbitrary $\Theta, \Delta, \Gamma, e, t$ such that $\Theta; \Delta; \Gamma \vdash e : t$.
2. Arbitrary $\theta, k, \delta, \gamma, \sigma$ such that:
a. $\mathrm{dom}(\Theta) = \mathrm{dom}(\theta)$
b. $(\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta$
c. $\delta \in \mathcal{I}_k[\![\Delta]\!]\theta$.
3. W.l.o.g., all variables are distinct/$\mathrm{dom}(\Delta)$ and $\mathrm{dom}(\Gamma)$ are disjoint.
4. And so that over expressions $\gamma \circ \delta = \delta \circ \gamma$.
5. By construction, $\mathrm{dom}(\Delta) = \mathrm{dom}(\delta)$ and $\mathrm{dom}(\Gamma) = \mathrm{dom}(\gamma)$.
6. ??? $\mathcal{V}_k[\![\theta(t)]\!] \subseteq \mathcal{C}_k[\![\theta(t)]\!]$.
7. ??? "Stronger heap"/frame rule: $\langle \sigma, e \rangle \to^* = \langle \sigma \star \sigma_r, e \rangle \to^*$.
8. ??? $\forall \delta, \gamma, v.\ \delta(\gamma(v))$ is a value.

Prove: $(\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t')]\!]$.
Assume: Arbitrary $j \leq k$ and σ_r.
Suffices: Show whole expression either reduces to **err** or takes j steps.

$\langle 1 \rangle 1$. Case: Ty_Let.
Prove: $(\sigma, \gamma(\delta(\mathbf{let}\, x \;=\; e \,\mathbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t')]\!]$.
Suffices: $(\sigma, \mathbf{let}\, x \;=\; \gamma(\delta(e)) \,\mathbf{in}\, \gamma(\delta(e'))) \in \mathcal{C}_k[\![\theta(t')]\!]$.

$\langle 2 \rangle 1$. By induction, $[\![\Theta; \Delta; \Gamma \vdash e : t]\!]$ and $[\![\Theta; \Delta; \Gamma', x : t \vdash e' : t']\!]$.

$\langle 2 \rangle 2$. By 2b and induction on Γ', we know there exist $\sigma_{e'}$, $(\sigma_e, \gamma_e) \in \mathcal{L}_k[\![\Gamma]\!]$, such that $\sigma = \sigma_e \star \sigma_{e'}$.

$\langle 2 \rangle 3$. So, using them, θ, k, δ, and 3 we have $(\sigma_e, \gamma_e(e)) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 4$. By 3, $(\sigma_e, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 5$. By definition of $\mathcal{C}_k[\![\cdot]\!]$ and $\langle 2 \rangle 2$, we instantiate with j and $\sigma_r = \sigma_{e'}$ to conclude that $\langle \sigma, \gamma(\delta(e)) \rangle$ either reduces to **err** or another heap and expression.

$\langle 2 \rangle 6$. Case: **err**
??? By Op_Context_Err and 7 with $\sigma_{r'}$, the whole expression reduces to **err** in $j \leq k$ steps. Since $j \leq k$ and σ_r are arbitrary, $(\sigma, \gamma(\delta(\mathbf{let}\, x \;=\; e \,\mathbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t')]\!]$.

$\langle 2 \rangle 7$. Case: j steps to another heap and expression.
By Op_Context, the whole expression does the same.

$\langle 2 \rangle 8$. If it is not a value, we are done. If it is $(\sigma_{ef}, v) \in \mathcal{V}_{k-j}[\![\theta(t)]\!]$ by 8.
Suffices: $(\sigma_{ef} \star \sigma_{e'}, \mathbf{let}\, x \;=\; v \,\mathbf{in}\, \gamma(\delta(e'))) \in \mathcal{C}_{k-j}[\![\theta(t')]\!]$.
Suffices: $(\sigma_{ef} \star \sigma_{e'}, \gamma(\delta(e'))[x/v]) \in \mathcal{C}_{k-j-1}[\![\theta(t')]\!]$.

$\langle 2 \rangle 9$. Define: $\gamma_{e'}(y) = v$ if $y = x$ and $\gamma(y)$ if $y \in \mathrm{dom}(\Gamma')$.
Thus, $(\sigma_{e'}, \gamma_{e'}) \in \mathcal{L}_k[\![\Gamma', x : t]\!] \subseteq \mathcal{L}_{k-j-1}[\![\Gamma', x : t]\!]$.

$\langle 2 \rangle 10$. Instantiate induction hypothesis $[\![\Theta; \Delta; \Gamma', x : t \vdash e' : t']\!]$, with $\theta, k - j - 1, \delta, \gamma_{e'}, \sigma_{e'}$ to conclude $(\sigma_{e'}, \gamma_{e'}(\delta(e'))) \in \mathcal{C}_{k-j-1}[\![\theta(t')]\!]$.

⟨2⟩11. By 3, we have $\gamma(\delta(e'))[x/v] = \gamma_{e'}(\delta(e'))$ and by 7 we conclude $(\sigma_{ef} \star \sigma_{e'}, \gamma(\delta(e'))[x/v]) \in \mathcal{C}_{k-j-1}[\![\theta(t')]\!]$

⟨1⟩2. Case: Ty_Unit_Elim.
Prove: $(\sigma, \gamma(\delta(\mathbf{let}\,() \;=\; e\,\mathbf{in}\,e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.
Proof: Similar to Ty_Let but with Op_Let_Unit.

⟨1⟩3. Case: Ty_Bool_Elim.
Prove: $(\sigma, \gamma(\delta(\mathbf{if}\,e\,\mathbf{then}\,e_1\,\mathbf{else}\,e_2))) \in \mathcal{C}_k[\![\theta(t)]\!]$.
Proof: Similar to Ty_Let but with Op_If_{True,False}.

⟨1⟩4. Case: Ty_Pair_Elim.
Prove: $(\sigma, \gamma(\delta(\mathbf{let}\,(a,b) \;=\; e\,\mathbf{in}\,e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.
Proof: Similar to Ty_Let but with Op_Let_Pair

⟨1⟩5. Case: Ty_Bang_Intro.
Prove: $(\sigma, \gamma(\delta(\mathbf{Many}\,e))) \in \mathcal{C}_k[\![\theta(!t)]\!]$.
Suffices: $(\sigma, \mathbf{Many}\,\gamma(\delta(e))) \in \mathcal{C}_k[\![!\theta(t)]\!]$.

⟨2⟩1. By assumption of Ty_Bang_Intro, $e = v$ for some value $v \neq l$, $\Gamma = \emptyset$ and so $[\![\Theta; \Delta; \cdot \vdash v : t]\!]$ by induction.

⟨2⟩2. Suffices: $(\emptyset, \mathbf{Many}\,\delta(v)) \in \mathcal{C}_k[\![!\theta(t)]\!]$ by 3 and 2b.

⟨2⟩3. Instantiate $[\![\Theta; \Delta; \cdot \vdash v : t]\!]$ with $\theta, k, \delta, \gamma = [], \sigma = \emptyset$ to obtain $(\emptyset, \delta(v)) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨2⟩4. Instantiate $(\emptyset, \delta(v)) \in \mathcal{C}_k[\![\theta(t)]\!]$ with $j = 0$, and $\sigma_r = \emptyset$, to conclude $(\emptyset, v) \in \mathcal{V}_k[\![\theta(t)]\!]$.

⟨2⟩5. ??? By definition of $\mathcal{V}_k[\![!\theta(t)]\!]$, 8 and 6 we have $(\emptyset, \mathbf{Many}\,\delta(v)) \in \mathcal{C}_k[\![!\theta(t)]\!]$.

⟨1⟩6. Case: Ty_Bang_Elim.
Prove: $(\sigma, \gamma(\delta(\mathbf{let}\,\mathbf{Many}\,x \;=\; e\,\mathbf{in}\,e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩7. Case: Ty_Pair_Intro.
Prove: $(\sigma, \gamma(\delta(\,(e, e')\,))) \in \mathcal{C}_k[\![\theta(t \otimes t')]\!]$.

⟨1⟩8. Case: Ty_Lambda.
Prove: $(\sigma, \gamma(\delta(\mathbf{fun}\,x : t' \to e))) \in \mathcal{C}_k[\![\theta(t' \multimap t)]\!]$.

⟨1⟩9. Case: Ty_App.
Prove: $(\sigma, \gamma(\delta(\,e\,e'\,))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩10. Case: Ty_Gen.
Prove: $(\sigma, \gamma(\delta(\mathbf{fun}\,fc \to e))) \in \mathcal{C}_k[\![\theta(\forall\, fc.\, t)]\!]$.

⟨1⟩11. Case: Ty_Spc.
Prove: $(\sigma, \gamma(\delta(e\,[f]))) \in \mathcal{C}_k[\![\theta(t\,[fc/f])]\!]$.

⟨1⟩12. CASE: TY_FIX.
PROVE: $(\sigma, \gamma(\delta(\mathbf{fix}(g, x : t, e : t')))) \in \mathcal{C}_k[\![\theta(!(t \multimap t'))]\!]$.

⟨1⟩13. CASE: TY_VAR_LIN.
PROVE: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨2⟩1. $\Gamma = \{x : t\}$ by assumption of TY_VAR_LIN.

⟨2⟩2. SUFFICES: $(\sigma, \gamma(x)) \in \mathcal{C}_k[\![\theta(t)]\!]$ by 3.

⟨2⟩3. By 2b, there exist $(\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]$, such that $\sigma = \sigma_x$ and $\gamma = [x \mapsto v_x]$.

⟨2⟩4. ??? Hence, $(\sigma_x, v_x) \in \mathcal{C}_k[\![\theta(t)]\!]$, by 6.

⟨1⟩14. CASE: TY_VAR.
PROVE: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨2⟩1. $x : t \in \Delta$ and $\Gamma = \emptyset$ by assumption of TY_VAR.

⟨2⟩2. SUFFICES: $(\emptyset, \delta(x)) \in \mathcal{C}_k[\![\theta(t)]\!]$ by 3 and 2b.

⟨2⟩3. By 2c, there exists v_x such that $(\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]$.

⟨2⟩4. ??? Hence, $(\emptyset, v_x) \in \mathcal{C}_k[\![\theta(t)]\!]$, by 6.

⟨1⟩15. CASE: TY_UNIT_INTRO.
PROVE: $(\sigma, \gamma(\delta(\,()\,))) \in \mathcal{C}_k[\![\theta(\mathbf{unit})]\!]$.

⟨1⟩16. CASE: TY_BOOL_TRUE, TY_BOOL_FALSE, TY_INT_INTRO, TY_ELT_INTRO.
Similar to TY_UNIT_INTRO.