1 Typing Rules

$\boxed{\Theta; \Delta; \Gamma \vdash e : t}$ Typing rules for expressions

$$\frac{}{\Theta; \Delta; \cdot, x : t \vdash x : t} \quad \textsc{Ty\_Var\_Lin}$$

$$\frac{x : t \in \Delta}{\Theta; \Delta; \cdot \vdash x : t} \quad \textsc{Ty\_Var}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma', x : t \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Let}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash () : \mathbf{unit}} \quad \textsc{Ty\_Unit\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash e : \mathbf{unit} \\ \Theta; \Delta; \Gamma \vdash e' : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{let}\, () = e\, \mathbf{in}\, e' : t} \quad \textsc{Ty\_Unit\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{true} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_True}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{false} : \mathbf{bool}} \quad \textsc{Ty\_Bool\_False}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !\mathbf{bool} \\ \Theta; \Delta; \Gamma' \vdash e_1 : t' \\ \Theta; \Delta; \Gamma' \vdash e_2 : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{if}\, e\, \mathbf{then}\, e_1\, \mathbf{else}\, e_2 : t} \quad \textsc{Ty\_Bool\_Elim}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash k : \mathbf{int}} \quad \textsc{Ty\_Int\_Intro}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash el : \mathbf{elt}} \quad \textsc{Ty\_Elt\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash v : t \\ v \neq l\end{array}}{\Theta; \Delta; \cdot \vdash \mathbf{Many}\, v : !t} \quad \textsc{Ty\_Bang\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !t \\ \Theta; \Delta, x : t; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, \mathbf{Many}\, x = e\, \mathbf{in}\, e' : t'} \quad \textsc{Ty\_Bang\_Elim}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash (e, e') : t \otimes t'} \quad \textsc{Ty\_Pair\_Intro}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e_{12} : t_1 \otimes t_2 \\ \Theta; \Delta; \Gamma', a : t_1, b : t_2 \vdash e : t\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, (a, b) = e_{12}\, \mathbf{in}\, e : t} \quad \textsc{Ty\_Pair\_Elim}$$

$$\frac{\begin{array}{l}\Theta \vdash t'\ \mathsf{Type} \\ \Theta; \Delta; \Gamma, x : t' \vdash e : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{fun}\, x : t' \to e : t' \multimap t} \quad \textsc{Ty\_Lambda}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t' \multimap t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash e\, e' : t} \quad \textsc{Ty\_App}$$

$$\frac{\Theta, fc; \Delta; \Gamma \vdash e : t}{\Theta; \Delta; \Gamma \vdash \textbf{fun}\, fc \to e : \forall fc.t} \quad \textsc{Ty\_Gen}$$

$$\frac{\begin{array}{l}\Theta \vdash f\ \mathsf{Cap} \\ \Theta; \Delta; \Gamma \vdash e : \forall fc.t\end{array}}{\Theta; \Delta; \Gamma \vdash e[f] : t[f/fc]} \quad \textsc{Ty\_Spc}$$

$$\frac{\Theta; \Delta, g : t \multimap t'; \cdot, x : t \vdash e : t'}{\Theta; \Delta; \cdot \vdash \textbf{fix}\,(g, x : t, e : t') : !(t \multimap t')} \quad \textsc{Ty\_Fix}$$

2 Operational Semantics

$\boxed{\langle \sigma, e \rangle \to \mathit{StepsTo}}$ operational semantics

$$\frac{}{\langle \sigma, \mathbf{let}\,() = ()\,\mathbf{in}\,e \rangle \to \langle \sigma, e \rangle} \quad \text{OP\_LET\_UNIT}$$

$$\frac{}{\langle \sigma, \mathbf{let}\,x = v\,\mathbf{in}\,e \rangle \to \langle \sigma, e[x/v] \rangle} \quad \text{OP\_LET\_VAR}$$

$$\frac{}{\langle \sigma, \mathbf{if}\,(\mathbf{Many}\,\mathbf{true})\,\mathbf{then}\,e_1\,\mathbf{else}\,e_2 \rangle \to \langle \sigma, e_1 \rangle} \quad \text{OP\_IF\_TRUE}$$

$$\frac{}{\langle \sigma, \mathbf{if}\,(\mathbf{Many}\,\mathbf{false})\,\mathbf{then}\,e_1\,\mathbf{else}\,e_2 \rangle \to \langle \sigma, e_2 \rangle} \quad \text{OP\_IF\_FALSE}$$

$$\frac{}{\langle \sigma, \mathbf{let}\,\mathbf{Many}\,x = \mathbf{Many}\,v\,\mathbf{in}\,e \rangle \to \langle \sigma, e[x/v] \rangle} \quad \text{OP\_LET\_MANY}$$

$$\frac{e_1 = e[g/\mathbf{let}\,\mathbf{Many}\,g = \mathbf{fix}\,(g, x : t, e : t')\,\mathbf{in}\,\mathbf{fun}\,x : t \to e]}{\langle \sigma, \mathbf{let}\,\mathbf{Many}\,g = \mathbf{fix}\,(g, x : t, e : t')\,\mathbf{in}\,e' \rangle \to \langle \sigma, e'[g/\mathbf{fun}\,x : t \to e_1] \rangle} \quad \text{OP\_LET\_FIX}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\,fc \to v)[f] \rangle \to \langle \sigma, v[fc/f] \rangle} \quad \text{OP\_FRAC\_CAP}$$

$$\frac{}{\langle \sigma, (\mathbf{fun}\,x : t \to e)\,v \rangle \to \langle \sigma, e[x/v] \rangle} \quad \text{OP\_APP}$$

$$\frac{\langle \sigma, e \rangle \to \langle \sigma', e' \rangle}{\langle \sigma, C[e] \rangle \to \langle \sigma, C[e'] \rangle} \quad \text{OP\_CONTEXT}$$

$$\frac{\langle \sigma, e \rangle \to \mathbf{err}}{\langle \sigma, C[e] \rangle \to \mathbf{err}} \quad \text{OP\_CONTEXT\_ERR}$$

$$\frac{0 \le k_1, k_2}{\langle \sigma, \mathbf{matrix}\,k_1\,k_2 \rangle \to \langle \sigma \uplus \{l \mapsto_1 M_{k_1,k_2}\}, l \rangle} \quad \text{OP\_MATRIX}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_1 m_{k_1,k_2}\}, \mathbf{free}\,l \rangle \to \langle \sigma, () \rangle} \quad \text{OP\_FREE}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, \mathbf{share}\,l \rangle \to \langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, (l, l) \rangle} \quad \text{OP\_SHARE}$$

$$\frac{f \le 1}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\,l\,l \rangle \to \langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, l \rangle} \quad \text{OP\_UNSHARE\_EQ}$$

$$\frac{l \ne l'}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l' \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \mathbf{unshare}\,l\,l' \rangle \to \mathbf{err}} \quad \text{OP\_UNSHARE\_NEQ}$$

$$\frac{\sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2,k_3}\}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \mathbf{gemm}\,l_1\,l_2\,l_3 \rangle \to \langle \sigma' \uplus \{l_3 \mapsto_1 (m_1\,m_2 + m_3)_{k_1,k_3}\}, ((l_1, l_2), l_3) \rangle} \quad \text{OP\_GEMM\_MATCH}$$

$$\frac{\begin{array}{l} k_2 \ne k_2' \\ \sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2',k_3}\} \end{array}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \mathbf{gemm}\,l_1\,l_2\,l_3 \rangle \to \mathbf{err}} \quad \text{OP\_GEMM\_MISMATCH}$$

3 Interpretation

$$\mathcal{V}_k[\![\mathbf{unit}]\!] = \{(\emptyset, *)\}$$

$$\mathcal{V}_k[\![\mathbf{bool}]\!] = \{(\emptyset, true), (\emptyset, false)\}$$

$$\mathcal{V}_k[\![\mathbf{int}]\!] = \{(\emptyset, n) \mid 2^{-63} \le n \le 2^{63} - 1\}$$

$$\mathcal{V}_k[\![\mathbf{elt}]\!] = \{(\emptyset, f) \mid f \text{ a IEEE Float64 }\}$$

$$\mathcal{V}_k[\![f\,\mathbf{mat}]\!] = \{(\{l \mapsto_{2^{-f}} \_\}, l)\}$$

$$\begin{aligned}\mathcal{V}_k[\![!(t' \multimap t'')]\!] = {} & \{(\emptyset, \mathbf{Many}\, v) \mid (\emptyset, v) \in \mathcal{V}_k[\![t' \multimap t'']\!]\} \\ & \cup \{(\emptyset, \mathbf{fix}(g, x : t, e : t')) \mid \forall\, j \le k, (\sigma', v') \in \mathcal{V}_j[\![t']\!]. \\ & \qquad (\sigma', \mathbf{let\, Many}\, g = \mathbf{fix}\,(g, x : t, e : t')\, \mathbf{in}\, g\, v) \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\mathcal{V}_k[\![!t]\!] = \{(\emptyset, \mathbf{Many}\, v) \mid \neg(\exists t', t''.\ t = t' \multimap t'') \wedge (\emptyset, v) \in \mathcal{V}_k[\![t]\!]\}$$

$$\mathcal{V}_k[\![\forall fc.\ t]\!] = \{(\sigma, \mathbf{fun}\, fc \rightarrow\ v) \mid \forall f.\ (\sigma, v[fc/f]) \in \mathcal{V}_k[\![t[fc/f]]\!]\}$$

$$\mathcal{V}_k[\![t' \otimes t'']\!] = \{(\sigma, (v', v'')) \mid \exists \sigma', \sigma''.\ (\sigma', v') \in \mathcal{V}_k[\![t]\!] \wedge (\sigma'', v'') \in \mathcal{V}_k[\![t'']\!] \wedge \sigma = \sigma' \star \sigma''\}$$

$$\begin{aligned}\mathcal{V}_k[\![t \multimap t']\!] = {} & \{(\sigma, \mathbf{fun}\, x : t \rightarrow e) \mid \forall j \le k, (\sigma', v') \in \mathcal{V}_j[\![t']\!].\ \sigma \star \sigma' \text{ defined } \Rightarrow \\ & \qquad (\sigma \star \sigma', (\mathbf{fun}\, x : t \rightarrow e)\, v') \in \mathcal{C}_j[\![t']\!]\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}_k[\![t]\!] = {} & \{(\sigma_s, e) \mid \forall\, j < k, \sigma_r.\ \sigma_s \star \sigma_r \text{ defined } \Rightarrow \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \mathbf{err}\ \vee \exists \sigma_f, e'. \\ & \qquad \langle \sigma_s \star \sigma_r, e\rangle \rightarrow^j \langle \sigma_f \star \sigma_r, e'\rangle \wedge (e' \text{ is a value } \Rightarrow (\sigma_f \star \sigma_r, e') \in \mathcal{V}_{k-j}[\![t]\!])\}\end{aligned}$$

$$\mathcal{I}_k[\![\cdot]\!]\theta = \{[]\}$$

$$\mathcal{I}_k[\![\Delta, x : t]\!]\theta = \{\delta[x \mapsto v_x] \mid \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \wedge (\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\mathcal{L}_k[\![\cdot]\!]\theta = \{(\emptyset, [])\}$$

$$\mathcal{L}_k[\![\Gamma, x : t]\!]\theta = \{(\sigma \star \sigma_x, \gamma[x \mapsto v_x]) \mid (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge (\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\begin{aligned}[\![\Theta; \Delta; \Gamma \vdash e : t]\!] = {} & \forall \theta, k, \delta, \gamma, \sigma.\ \mathrm{dom}(\Theta) = \mathrm{dom}(\theta) \wedge (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \Rightarrow \\ & \qquad (\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]\end{aligned}$$

4 Soundness Proof

$$\forall \Theta, \Delta, \Gamma, e, t.\ \Theta; \Delta; \Gamma \vdash e : t \Rightarrow [\![\Theta; \Delta; \Gamma \vdash e : t]\!]$$

Proof sketch: Induction over the typing judgements.

Assume: 1. Arbitrary $\Theta, \Delta, \Gamma, e, t$ such that $\Theta; \Delta; \Gamma \vdash e : t$.
2. Arbitrary $\theta, k, \delta, \gamma, \sigma$ such that:
a. $\text{dom}(\Theta) = \text{dom}(\theta)$
b. $(\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta$
c. $\delta \in \mathcal{I}_k[\![\Delta]\!]\theta$.
3. W.l.o.g., all variables are distinct/dom(Δ) and dom(Γ) are disjoint.
4. And so that over expressions $\gamma \circ \delta = \delta \circ \gamma$.
5. By construction, $\text{dom}(\Delta) = \text{dom}(\delta)$ and $\text{dom}(\Gamma) = \text{dom}(\gamma)$.
6. ??? $\mathcal{V}_k[\![\theta(t)]\!] \subseteq \mathcal{C}_k[\![\theta(t)]\!]$

Prove: $(\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 1 \rangle 1$. Case: Ty_Var_Lin.
Prove: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 1$. $\Gamma = \{x : t\}$ by assumption of Ty_Var_Lin.

$\langle 2 \rangle 2$. Suffices: $(\sigma, \gamma(x)) \in \mathcal{C}_k[\![\theta(t)]\!]$ by 3.

$\langle 2 \rangle 3$. By 2b, there exist $(\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]$, such that $\sigma = \sigma_x$ and $\gamma = [x \mapsto v_x]$.

$\langle 2 \rangle 4$. ??? Hence, $(\sigma_x, v_x) \in \mathcal{C}_k[\![\theta(t)]\!]$, by 6.

$\langle 1 \rangle 2$. Case: Ty_Var.
Prove: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 1$. $x : t \in \Delta$ and $\Gamma = \emptyset$ by assumption of Ty_Var.

$\langle 2 \rangle 2$. Suffices: $(\emptyset, \delta(x)) \in \mathcal{C}_k[\![\theta(t)]\!]$ by 3 and 2b.

$\langle 2 \rangle 3$. By 2c, there exists v_x such that $(\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 4$. ??? Hence, $(\emptyset, v_x) \in \mathcal{C}_k[\![\theta(t)]\!]$, by 6.

$\langle 1 \rangle 3$. Case: Ty_Let.
Prove: $(\sigma, \gamma(\delta(\textbf{let}\, x\ =\ e\, \textbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 1$. Suffices: $(\sigma, \textbf{let}\, x\ =\ \gamma(\delta(e))\, \textbf{in}\, \gamma(\delta(e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 2$. By induction, $[\![\Theta; \Delta; \Gamma \vdash e : t]\!]$ and $[\![\Theta; \Delta; \Gamma', x : t \vdash e' : t']\!]$.

$\langle 2 \rangle 3$. By 2b and induction on Γ', we know there exist $\sigma_{e'}$, $(\sigma_e, \gamma_e) \in \mathcal{L}_k[\![\Gamma]\!]$, such that $\sigma = \sigma_e \star \sigma_{e'}$.

$\langle 2 \rangle 4$. So, using them, θ, k, δ, and 3 we have $(\sigma_e, \gamma_e(e)) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 5$. By 3, $(\sigma_e, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2 \rangle 6$. By definition of $\mathcal{C}_k[\![\cdot]\!]$ and $\langle 2 \rangle 3$, we choose a $j < k$ and $\sigma_{e'}$ to conclude that $\langle \sigma, \gamma(\delta(e)) \rangle$

either reduces to **err** or another heap and expression.

⟨2⟩7. In first case, by op sem, reduces to err as well.

⟨2⟩8. In second case, ...

⟨1⟩4. CASE: TY_UNIT_INTRO.
PROVE: $(\sigma, \gamma(\delta(\,()\,))) \in \mathcal{C}_k[\![\theta(\mathbf{unit})]\!]$.

⟨1⟩5. CASE: TY_UNIT_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{let}\,() \ = \ e\,\mathbf{in}\,e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩6. CASE: TY_BOOL_TRUE.
PROVE: $(\sigma, \gamma(\delta(\mathbf{true}))) \in \mathcal{C}_k[\![\theta(\mathbf{bool})]\!]$.

⟨1⟩7. CASE: TY_BOOL_FALSE, TY_INT_INTRO, TY_ELT_INTRO.
Similar to TY_BOOL_TRUE.

⟨1⟩8. CASE: TY_BOOL_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{if}\,e\,\mathbf{then}\,e_1\,\mathbf{else}\,e_2))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩9. CASE: TY_BANG_INTRO.
PROVE: $(\sigma, \gamma(\delta(\mathbf{Many}\,e))) \in \mathcal{C}_k[\![\theta(!t)]\!]$.

⟨1⟩10. CASE: TY_BANG_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{let}\,\mathbf{Many}\,x \ = \ e\,\mathbf{in}\,e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩11. CASE: TY_PAIR_INTRO.
PROVE: $(\sigma, \gamma(\delta(\,(e, e')\,))) \in \mathcal{C}_k[\![\theta(t \otimes t')]\!]$.

⟨1⟩12. CASE: TY_PAIR_ELIM.
PROVE: $(\sigma, \gamma(\delta(\mathbf{let}\,(a, b) \ = \ e\,\mathbf{in}\,e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩13. CASE: TY_LAMBDA.
PROVE: $(\sigma, \gamma(\delta(\mathbf{fun}\,x : t' \to e))) \in \mathcal{C}_k[\![\theta(t' \multimap t)]\!]$.

⟨1⟩14. CASE: TY_APP.
PROVE: $(\sigma, \gamma(\delta(\,e\,e'\,))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨1⟩15. CASE: TY_GEN.
PROVE: $(\sigma, \gamma(\delta(\mathbf{fun}\,fc \to e))) \in \mathcal{C}_k[\![\theta(\forall\, fc.\, t)]\!]$.

⟨1⟩16. CASE: TY_SPC.
PROVE: $(\sigma, \gamma(\delta(e\,[f]))) \in \mathcal{C}_k[\![\theta(t\,[fc/f])]\!]$.

⟨1⟩17. CASE: TY_FIX.
PROVE: $(\sigma, \gamma(\delta(\mathbf{fix}(g, x : t, e : t')))) \in \mathcal{C}_k[\![\theta(!(t \multimap t'))]\!]$.