1 Typing Rules

$\boxed{\Theta; \Delta; \Gamma \vdash e : t}$ Typing rules for expressions

$$\frac{}{\Theta; \Delta; \cdot, x : t \vdash x : t} \quad \text{TY\_VAR\_LIN}$$

$$\frac{x : t \in \Delta}{\Theta; \Delta; \cdot \vdash x : t} \quad \text{TY\_VAR}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma', x : t \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, x = e\, \mathbf{in}\, e' : t'} \quad \text{TY\_LET}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash () : \mathbf{unit}} \quad \text{TY\_UNIT\_INTRO}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : \mathbf{unit} \\ \Theta; \Delta; \Gamma' \vdash e' : t\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, () = e\, \mathbf{in}\, e' : t} \quad \text{TY\_UNIT\_ELIM}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{true} : \mathbf{bool}} \quad \text{TY\_BOOL\_TRUE}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash \mathbf{false} : \mathbf{bool}} \quad \text{TY\_BOOL\_FALSE}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !\mathbf{bool} \\ \Theta; \Delta; \Gamma' \vdash e_1 : t' \\ \Theta; \Delta; \Gamma' \vdash e_2 : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{if}\, e\, \mathbf{then}\, e_1\, \mathbf{else}\, e_2 : t} \quad \text{TY\_BOOL\_ELIM}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash k : \mathbf{int}} \quad \text{TY\_INT\_INTRO}$$

$$\frac{}{\Theta; \Delta; \cdot \vdash el : \mathbf{elt}} \quad \text{TY\_ELT\_INTRO}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \cdot \vdash v : t \\ v \neq l\end{array}}{\Theta; \Delta; \cdot \vdash \mathbf{Many}\, v : !t} \quad \text{TY\_BANG\_INTRO}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : !t \\ \Theta; \Delta, x : t; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, \mathbf{Many}\, x = e\, \mathbf{in}\, e' : t'} \quad \text{TY\_BANG\_ELIM}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash (e, e') : t \otimes t'} \quad \text{TY\_PAIR\_INTRO}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e_{12} : t_1 \otimes t_2 \\ \Theta; \Delta; \Gamma', a : t_1, b : t_2 \vdash e : t\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash \mathbf{let}\, (a, b) = e_{12}\, \mathbf{in}\, e : t} \quad \text{TY\_PAIR\_ELIM}$$

$$\frac{\begin{array}{l}\Theta \vdash t'\ \mathsf{Type} \\ \Theta; \Delta; \Gamma, x : t' \vdash e : t\end{array}}{\Theta; \Delta; \Gamma \vdash \mathbf{fun}\, x : t' \rightarrow e : t' \multimap t} \quad \text{TY\_LAMBDA}$$

$$\frac{\begin{array}{l}\Theta; \Delta; \Gamma \vdash e : t' \multimap t \\ \Theta; \Delta; \Gamma' \vdash e' : t'\end{array}}{\Theta; \Delta; \Gamma, \Gamma' \vdash e\, e' : t} \quad \text{TY\_APP}$$

$$\frac{\Theta, fc; \Delta; \Gamma \vdash e : t}{\Theta; \Delta; \Gamma \vdash \textbf{fun}\, fc \to e : \forall fc.t} \quad \textsc{Ty\_Gen}$$

$$\frac{\begin{array}{l}\Theta \vdash f \ \mathsf{Cap} \\ \Theta; \Delta; \Gamma \vdash e : \forall fc.t\end{array}}{\Theta; \Delta; \Gamma \vdash e[f] : t[f/fc]} \quad \textsc{Ty\_Spc}$$

$$\frac{\Theta; \Delta, g : t \multimap t'; \cdot, x : t \vdash e : t'}{\Theta; \Delta; \cdot \vdash \textbf{fix}\,(g, x : t, e : t') : !(t \multimap t')} \quad \textsc{Ty\_Fix}$$

2 Operational Semantics

$\boxed{\langle \sigma, e \rangle \rightarrow \mathit{StepsTo}}$ operational semantics

$$\frac{}{\langle \sigma, \textbf{let}\ () = ()\ \textbf{in}\ e \rangle \rightarrow \langle \sigma, e \rangle} \quad \text{OP\_LET\_UNIT}$$

$$\frac{}{\langle \sigma, \textbf{let}\ x = v\ \textbf{in}\ e \rangle \rightarrow \langle \sigma, e[x/v] \rangle} \quad \text{OP\_LET\_VAR}$$

$$\frac{}{\langle \sigma, \textbf{if}\ (\textbf{Many}\ \textbf{true})\ \textbf{then}\ e_1\ \textbf{else}\ e_2 \rangle \rightarrow \langle \sigma, e_1 \rangle} \quad \text{OP\_IF\_TRUE}$$

$$\frac{}{\langle \sigma, \textbf{if}\ (\textbf{Many}\ \textbf{false})\ \textbf{then}\ e_1\ \textbf{else}\ e_2 \rangle \rightarrow \langle \sigma, e_2 \rangle} \quad \text{OP\_IF\_FALSE}$$

$$\frac{}{\langle \sigma, \textbf{let}\ \textbf{Many}\ x = \textbf{Many}\ v\ \textbf{in}\ e \rangle \rightarrow \langle \sigma, e[x/v] \rangle} \quad \text{OP\_LET\_MANY}$$

$$\frac{}{\langle \sigma, \textbf{let}\ (a, b) = (v_1, v_2)\ \textbf{in}\ e \rangle \rightarrow \langle \sigma, e[a/v_1][b/v_2] \rangle} \quad \text{OP\_LET\_PAIR}$$

$$\frac{e_1 = e[g/\textbf{let}\ \textbf{Many}\ g = \textbf{fix}\,(g, x : t, e : t')\ \textbf{in}\ \textbf{fun}\ x : t \rightarrow e]}{\langle \sigma, \textbf{let}\ \textbf{Many}\ g = \textbf{fix}\,(g, x : t, e : t')\ \textbf{in}\ e' \rangle \rightarrow \langle \sigma, e'[g/\textbf{fun}\ x : t \rightarrow e_1] \rangle} \quad \text{OP\_LET\_FIX}$$

$$\frac{}{\langle \sigma, (\textbf{fun}\ fc \rightarrow v)[f] \rangle \rightarrow \langle \sigma, v[fc/f] \rangle} \quad \text{OP\_FRAC\_CAP}$$

$$\frac{}{\langle \sigma, (\textbf{fun}\ x : t \rightarrow e)\ v \rangle \rightarrow \langle \sigma, e[x/v] \rangle} \quad \text{OP\_APP}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle}{\langle \sigma, C[e] \rangle \rightarrow \langle \sigma, C[e'] \rangle} \quad \text{OP\_CONTEXT}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \textbf{err}}{\langle \sigma, C[e] \rangle \rightarrow \textbf{err}} \quad \text{OP\_CONTEXT\_ERR}$$

$$\frac{0 \leq k_1, k_2}{\langle \sigma, \textbf{matrix}\ k_1\ k_2 \rangle \rightarrow \langle \sigma \uplus \{l \mapsto_1 M_{k_1,k_2}\}, l \rangle} \quad \text{OP\_MATRIX}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_1 m_{k_1,k_2}\}, \textbf{free}\ l \rangle \rightarrow \langle \sigma, () \rangle} \quad \text{OP\_FREE}$$

$$\frac{}{\langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, \textbf{share}\ l \rangle \rightarrow \langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, (l, l) \rangle} \quad \text{OP\_SHARE}$$

$$\frac{f \leq 1}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \textbf{unshare}\ l\ l \rangle \rightarrow \langle \sigma \uplus \{l \mapsto_f m_{k_1,k_2}\}, l \rangle} \quad \text{OP\_UNSHARE\_EQ}$$

$$\frac{l \neq l'}{\langle \sigma \uplus \{l \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\} \uplus \{l' \mapsto_{\frac{1}{2}\cdot f} m_{k_1,k_2}\}, \textbf{unshare}\ l\ l' \rangle \rightarrow \textbf{err}} \quad \text{OP\_UNSHARE\_NEQ}$$

$$\frac{\sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2,k_3}\}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \textbf{gemm}\ l_1\ l_2\ l_3 \rangle \rightarrow \langle \sigma' \uplus \{l_3 \mapsto_1 (m_1\ m_2 + m_3)_{k_1,k_3}\}, ((l_1, l_2), l_3) \rangle} \quad \text{OP\_GEMM\_MATCH}$$

$$\frac{\begin{array}{l} k_2 \neq k_2' \\ \sigma' = \sigma \uplus \{l_1 \mapsto_{fc_1} m1_{k_1,k_2}\} \uplus \{l_2 \mapsto_{fc_2} m2_{k_2',k_3}\} \end{array}}{\langle \sigma' \uplus \{l_3 \mapsto_1 m1_{k_1,k_3}\}, \textbf{gemm}\ l_1\ l_2\ l_3 \rangle \rightarrow \textbf{err}} \quad \text{OP\_GEMM\_MISMATCH}$$

3 Interpretation

$$\mathcal{V}_k[\![\mathbf{unit}]\!] = \{(\emptyset, *)\}$$

$$\mathcal{V}_k[\![\mathbf{bool}]\!] = \{(\emptyset, true), (\emptyset, false)\}$$

$$\mathcal{V}_k[\![\mathbf{int}]\!] = \{(\emptyset, n) \mid 2^{-63} \leq n \leq 2^{63} - 1\}$$

$$\mathcal{V}_k[\![\mathbf{elt}]\!] = \{(\emptyset, f) \mid f \text{ a IEEE Float64 }\}$$

$$\mathcal{V}_k[\![f\,\mathbf{mat}]\!] = \{(\{l \mapsto_{2^{-f}} \_\}, l)\}$$

$$\begin{aligned} \mathcal{V}_k[\![!(t' \multimap t'')]\!] = \{(\emptyset, \mathbf{Many}\, v) \mid (\emptyset, v) \in \mathcal{V}_k[\![t' \multimap t'']\!]\} \\ \cup \{(\emptyset, \mathbf{fix}(g, x : t, e : t')) \mid \forall\, j \leq k, (\sigma', v') \in \mathcal{V}_j[\![t']\!]. \\ (\sigma', \mathbf{let\, Many}\, g = \mathbf{fix}\,(g, x : t, e : t')\, \mathbf{in}\, g\, v) \in \mathcal{C}_j[\![t']\!]\} \end{aligned}$$

$$\mathcal{V}_k[\![!t]\!] = \{(\emptyset, \mathbf{Many}\, v) \mid \neg(\exists t', t''.\ t = t' \multimap t'') \wedge (\emptyset, v) \in \mathcal{V}_k[\![t]\!]\}$$

$$\mathcal{V}_k[\![\forall fc.\ t]\!] = \{(\sigma, \mathbf{fun}\, fc \rightarrow\ v) \mid \forall f.\ (\sigma, v[fc/f]) \in \mathcal{V}_k[\![t[fc/f]]\!]\}$$

$$\mathcal{V}_k[\![t \otimes t']\!] = \{(\sigma \star \sigma', (v, v')) \mid (\sigma, v) \in \mathcal{V}_k[\![t]\!] \wedge (\sigma', v') \in \mathcal{V}_k[\![t']\!]\}$$

$$\begin{aligned} \mathcal{V}_k[\![t \multimap t']\!] = \{(\sigma, \mathbf{fun}\, x : t \rightarrow e) \mid \forall j \leq k, (\sigma', v') \in \mathcal{V}_j[\![t']\!].\ \sigma \star \sigma' \text{ defined } \Rightarrow \\ (\sigma \star \sigma', (\mathbf{fun}\, x : t \rightarrow e)\, v') \in \mathcal{C}_j[\![t']\!]\} \end{aligned}$$

$$\begin{aligned} \mathcal{C}_k[\![t]\!] = \{(\sigma_s, e) \mid \forall\, j < k, \sigma_r.\ \sigma_s \star \sigma_r \text{ defined } \Rightarrow \langle \sigma_s \star \sigma_r, e \rangle \rightarrow^j \mathbf{err}\ \vee \exists \sigma_f, e'. \\ \langle \sigma_s \star \sigma_r, e \rangle \rightarrow^j \langle \sigma_f \star \sigma_r, e' \rangle \wedge (e' \text{ is a value } \Rightarrow (\sigma_f \star \sigma_r, e') \in \mathcal{V}_{k-j}[\![t]\!])\} \end{aligned}$$

$$\mathcal{I}_k[\![\cdot]\!]\theta = \{[]\}$$

$$\mathcal{I}_k[\![\Delta, x : t]\!]\theta = \{\delta[x \mapsto v_x] \mid \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \wedge (\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\mathcal{L}_k[\![\cdot]\!]\theta = \{(\emptyset, [])\}$$

$$\mathcal{L}_k[\![\Gamma, x : t]\!]\theta = \{(\sigma \star \sigma_x, \gamma[x \mapsto v_x]) \mid (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge (\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]\}$$

$$\begin{aligned} [\![\Theta; \Delta; \Gamma \vdash e : t]\!] = \forall \theta, k, \delta, \gamma, \sigma.\ \mathrm{dom}(\Theta) = \mathrm{dom}(\theta) \wedge (\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta \wedge \delta \in \mathcal{I}_k[\![\Delta]\!]\theta \Rightarrow \\ (\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t)]\!] \end{aligned}$$

4 Soundness Proof

$$\forall \Theta, \Delta, \Gamma, e, t.\ \Theta; \Delta; \Gamma \vdash e : t \Rightarrow [\![\Theta; \Delta; \Gamma \vdash e : t]\!]$$

PROOF SKETCH: Induction over the typing judgements.

ASSUME: 1. Arbitrary $\Theta, \Delta, \Gamma, e, t$ such that $\Theta; \Delta; \Gamma \vdash e : t$.
2. Arbitrary $\theta, k, \delta, \gamma, \sigma$ such that:
a. $\mathrm{dom}(\Theta) = \mathrm{dom}(\theta)$
b. $(\sigma, \gamma) \in \mathcal{L}_k[\![\Gamma]\!]\theta$
c. $\delta \in \mathcal{I}_k[\![\Delta]\!]\theta$.
3. W.l.o.g., all variables are distinct/$\mathrm{dom}(\Delta)$ and $\mathrm{dom}(\Gamma)$ are disjoint.
4. And so that over expressions $\gamma \circ \delta = \delta \circ \gamma$.
5. By construction, $\mathrm{dom}(\Delta) = \mathrm{dom}(\delta)$ and $\mathrm{dom}(\Gamma) = \mathrm{dom}(\gamma)$.
6. ??? $\mathcal{V}_k[\![\theta(t)]\!] \subseteq \mathcal{C}_k[\![\theta(t)]\!]$.
7. ??? "Stronger heap"/frame rule: $\langle \sigma, e \rangle \to^* = \langle \sigma \star \sigma_r, e \rangle \to^*$.
8. ??? $\forall \delta, \gamma, v.\ \delta(\gamma(v))$ is a value.

PROVE: $(\sigma, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t')]\!]$.
ASSUME: Arbitrary $j < k$ ($k = 0$???) and $\sigma_{r'}$.
SUFFICES: Show whole expression either reduces to **err** or takes j steps.

$\langle 1 \rangle 1$. CASE: TY_LET.
PROVE: $(\sigma, \gamma(\delta(\textbf{let}\, x \;=\; e \,\textbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t')]\!]$.
SUFFICES: $(\sigma, \textbf{let}\, x \;=\; \gamma(\delta(e)) \,\textbf{in}\, \gamma(\delta(e'))) \in \mathcal{C}_k[\![\theta(t')]\!]$.

$\langle 2 \rangle 1$. By induction, $[\![\Theta; \Delta; \Gamma \vdash e : t]\!]$ and $[\![\Theta; \Delta; \Gamma', x : t \vdash e' : t']\!]$.

$\langle 2 \rangle 2$. By 2b and induction on Γ', we know there exist $\sigma_{e'}$, $(\sigma_e, \gamma_e) \in \mathcal{L}_k[\![\Gamma]\!]$, such that $\sigma = \sigma_e \star \sigma_{e'}$.

$\langle 2 \rangle 3$. So, using them, θ, k, δ, and 3 we have $(\sigma_e, \gamma_e(e)) \in \mathcal{C}_k[\![\theta(t')]\!]$.

$\langle 2 \rangle 4$. By 3, $(\sigma_e, \gamma(\delta(e))) \in \mathcal{C}_k[\![\theta(t')]\!]$.

$\langle 2 \rangle 5$. By definition of $\mathcal{C}_k[\![\cdot]\!]$ and $\langle 2 \rangle 2$, we instantiate with j and $\sigma_r = \sigma_{e'}$ to conclude that $\langle \sigma, \gamma(\delta(e)) \rangle$ either reduces to **err** or another heap and expression.

$\langle 2 \rangle 6$. CASE: **err**
??? By OP_CONTEXT_ERR and 7 with $\sigma_{r'}$, the whole expression reduces to **err** in $j < k$ steps. Since $j < k$ and $\sigma_{r'}$ are arbitrary, $(\sigma, \gamma(\delta(\textbf{let}\, x \;=\; e \,\textbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t')]\!]$.

$\langle 2 \rangle 7$. CASE: j steps to another heap and expression.
By OP_CONTEXT, the whole expression does the same.

$\langle 2 \rangle 8$. Regardless of whether $\gamma(\delta(e))$ is a syntactic value in $j < k$ steps, it will take at least $j + 1 \leq k$ steps to trigger OP_LET_VAR and so the whole expression cannot be a syntactic value after $j < k$ steps.

$\langle 2 \rangle 9$. Therefore, $(\sigma, \gamma(\delta(\textbf{let}\, x \;=\; e \,\textbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t')]\!]$

$\langle 1 \rangle 2$. CASE: TY_UNIT_ELIM.
PROVE: $(\sigma, \gamma(\delta(\textbf{let}\, () \;=\; e \,\textbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.
PROOF: Similar to TY_LET but with OP_LET_UNIT.

$\langle 1\rangle 3.$ Case: Ty_Bool_Elim.
Prove: $(\sigma, \gamma(\delta(\textbf{if}\, e\, \textbf{then}\, e_1\, \textbf{else}\, e_2))) \in \mathcal{C}_k[\![\theta(t)]\!]$.
Proof: Similar to Ty_Let but with Op_If_{True,False}.

$\langle 1\rangle 4.$ Case: Ty_Pair_Elim.
Prove: $(\sigma, \gamma(\delta(\textbf{let}\,(a, b) = e\, \textbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.
Proof: Similar to Ty_Let but with Op_Let_Pair

$\langle 1\rangle 5.$ Case: Ty_Bang_Intro.
Prove: $(\sigma, \gamma(\delta(\textbf{Many}\, e))) \in \mathcal{C}_k[\![\theta(!t)]\!]$.
Suffices: $(\sigma, \textbf{Many}\, \gamma(\delta(e))) \in \mathcal{C}_k[\![!\theta(t)]\!]$.

$\langle 2\rangle 1.$ By assumption of Ty_Bang_Intro, $e = v$ for some value $v \neq l$, $\Gamma = \emptyset$ and so $[\![\Theta; \Delta; \cdot \vdash v : t]\!]$ by induction.

$\langle 2\rangle 2.$ Suffices: $(\emptyset, \textbf{Many}\, \delta(v)) \in \mathcal{C}_k[\![!\theta(t)]\!]$ by 3 and 2b.

$\langle 2\rangle 3.$ Instantiate $[\![\Theta; \Delta; \cdot \vdash v : t]\!]$ with $\theta, k, \delta, \gamma = [], \sigma = \emptyset$ to obtain $(\emptyset, \delta(v)) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 2\rangle 4.$ Instantiate $(\emptyset, \delta(v)) \in \mathcal{C}_k[\![\theta(t)]\!]$ with $j = 0$, and $\sigma_r = \emptyset$, to conclude $(\emptyset, v) \in \mathcal{V}_k[\![\theta(t)]\!]$.

$\langle 2\rangle 5.$??? By $\mathcal{V}_k[\![!\theta(t)]\!]$ construction, 8 and 6 we have $(\emptyset, \textbf{Many}\, \delta(v)) \in \mathcal{C}_k[\![!\theta(t)]\!]$.

$\langle 1\rangle 6.$ Case: Ty_Bang_Elim.
Prove: $(\sigma, \gamma(\delta(\textbf{let}\, \textbf{Many}\, x = e\, \textbf{in}\, e'))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 1\rangle 7.$ Case: Ty_Pair_Intro.
Prove: $(\sigma, \gamma(\delta(\,(e, e')\,))) \in \mathcal{C}_k[\![\theta(t \otimes t')]\!]$.

$\langle 1\rangle 8.$ Case: Ty_Lambda.
Prove: $(\sigma, \gamma(\delta(\textbf{fun}\, x : t' \to e))) \in \mathcal{C}_k[\![\theta(t' \multimap t)]\!]$.

$\langle 1\rangle 9.$ Case: Ty_App.
Prove: $(\sigma, \gamma(\delta(\, e\, e'\,))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

$\langle 1\rangle 10.$ Case: Ty_Gen.
Prove: $(\sigma, \gamma(\delta(\textbf{fun}\, fc \to e))) \in \mathcal{C}_k[\![\theta(\forall\, fc.\, t)]\!]$.

$\langle 1\rangle 11.$ Case: Ty_Spc.
Prove: $(\sigma, \gamma(\delta(e\,[f]))) \in \mathcal{C}_k[\![\theta(t\,[fc/f])]\!]$.

$\langle 1\rangle 12.$ Case: Ty_Fix.
Prove: $(\sigma, \gamma(\delta(\textbf{fix}(g, x : t, e : t')))) \in \mathcal{C}_k[\![\theta(!(t \multimap t'))]\!]$.

$\langle 1\rangle 13.$ Case: Ty_Var_Lin.

PROVE: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨2⟩1. $\Gamma = \{x : t\}$ by assumption of TY_VAR_LIN.

⟨2⟩2. SUFFICES: $(\sigma, \gamma(x)) \in \mathcal{C}_k[\![\theta(t)]\!]$ by 3.

⟨2⟩3. By 2b, there exist $(\sigma_x, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]$, such that $\sigma = \sigma_x$ and $\gamma = [x \mapsto v_x]$.

⟨2⟩4. ??? Hence, $(\sigma_x, v_x) \in \mathcal{C}_k[\![\theta(t)]\!]$, by 6.

⟨1⟩14. CASE: TY_VAR.
PROVE: $(\sigma, \gamma(\delta(x))) \in \mathcal{C}_k[\![\theta(t)]\!]$.

⟨2⟩1. $x : t \in \Delta$ and $\Gamma = \emptyset$ by assumption of TY_VAR.

⟨2⟩2. SUFFICES: $(\emptyset, \delta(x)) \in \mathcal{C}_k[\![\theta(t)]\!]$ by 3 and 2b.

⟨2⟩3. By 2c, there exists v_x such that $(\emptyset, v_x) \in \mathcal{V}_k[\![\theta(t)]\!]$.

⟨2⟩4. ??? Hence, $(\emptyset, v_x) \in \mathcal{C}_k[\![\theta(t)]\!]$, by 6.

⟨1⟩15. CASE: TY_UNIT_INTRO.
PROVE: $(\sigma, \gamma(\delta(\,()\,))) \in \mathcal{C}_k[\![\theta(\mathbf{unit})]\!]$.

⟨1⟩16. CASE: TY_BOOL_TRUE, TY_BOOL_FALSE, TY_INT_INTRO, TY_ELT_INTRO.
Similar to TY_UNIT_INTRO.